Exact controllability in projections for three-dimensional Navier–Stokes equations

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Abstract

The paper is devoted to studying controllability properties for 3D Navier–Stokes equations in a bounded domain. We establish a sufficient condition under which the problem in question is exactly controllable in any finite-dimensional projection. Our sufficient condition is verified for any torus in $\mathbb{R}^3$. The proofs are based on a development of a general approach introduced by Agrachev and Sarychev in the 2D case. As a simple consequence of the result on controllability, we show that the Cauchy problem for the 3D Navier-Stokes system has a unique strong solution for any initial function and a large class of external forces.

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Contents

0 Introduction

Let us consider the three-dimensional Navier–Stokes (NS) system

$$\dot{u} + (u, \nabla)u - \nu\Delta u + \nabla p = f(t, x), \quad \operatorname{div} u = 0, \tag{0.1}$$

where the space variables $x = (x_1, x_2, x_3)$ belong to a three-dimensional torus $\mathbb{T}^3 \subset \mathbb{R}^3$, $\nu > 0$ is the viscosity, $u = (u_1, u_2, u_3)$ and p are unknown velocity field and pressure, and $f(t, x)$ is an external force. Suppose that f is represented in the form

$$f(t, x) = h(t, x) + \eta(t, x), \tag{0.2}$$

where h is a given function and η is a control taking on values in a *finite-dimensional* subspace $E \subset L^2(\mathbb{T}^3, \mathbb{R}^3)$. Equations (0.1), (0.2) are supplemented with the initial condition

$$u(0) = u_0, \tag{0.3}$$

where $u_0 \in H^1(\mathbb{T}^3, \mathbb{R}^3)$ is a divergence-free vector field. Let us denote by H the space of functions $u \in L^2(\mathbb{T}^3, \mathbb{R}^3)$ such that $\operatorname{div} u = 0$ on $\mathbb{T}^3$. We fix an arbitrary subspace $F \subset H$ and denote by $\mathsf{P}_F : H \to H$ the orthogonal projection onto F. Problem (0.1), (0.2) is said to be *controllable in a time $T > 0$ for the projection to F* if for any initial function u_0 and any $\hat{u} \in F$ there exists an infinitely smooth control $\eta : [0, T] \to E$ such that (0.1) – (0.3) has a unique strong solution $u(t; \eta)$, which satisfies the relation

$$\mathsf{P}_F u(T; \eta) = \hat{u}. \tag{0.4}$$

One of the main results of this paper says that if the space E is sufficiently large, then for any $T > 0$ and $\nu > 0$ and any finite-dimensional subspace $F \subset H$ problem (0.1), (0.2) is controllable in time T for the projection to F.

A general approach for studying controllability of PDE's in finite-dimensional projections was introduced by Agrachev and Sarychev in the landmark article [AS05] (see also [AS06]). They considered the 2D NS system on a torus and proved that it is controllable for the projection to any finite-dimensional space F, with a control function taking on values in a fixed subspace E. We emphasise that the time of control T can be chosen arbitrarily small, and the control space E does not depend on ν and T.[1] The Agrachev–Sarychev approach is

[1] It is shown in [AS06] that the 2D Euler and Navier–Stokes equations are controllable by a control of dimension four.

based on the concept of *solid controllability* (cf. Definition 2.6 of the present paper). They construct explicitly an increasing sequence of finite-dimensional subspaces $\{E_k\}_{k\geq 0}$ such that $E_0 = E$, and the following two properties hold.

(i) There is an integer $N \geq 1$ such that the NS system is solidly controllable by an E_N-valued control.

(ii) If the NS system is solidly controllable by an E_k-valued control for some integer $k \geq 1$, then it is solidly controllable by an E_{k-1}-valued control.

These two assertions imply the required result.

In this paper, we take a slightly different viewpoint based on uniform approximate controllability.[2] Namely, we shall say that the NS system (0.1), (0.2) is *uniformly approximately controllable* (UAC) if for any constant $\varepsilon > 0$, any initial function u_0, and any compact subset $\mathcal{K}$ of the phase space there is a continuous mapping Ψ from $\mathcal{K}$ to the space of E-valued controls such that for every $\hat{u} \in \mathcal{K}$ problem (0.1) – (0.3) with $\eta = \Psi(\hat{u})$ has a unique strong solution $u(t;\eta)$, which satisfies the inequality

$$\|u(T;\eta) - \hat{u}\| < \varepsilon, \tag{0.5}$$

where $\|\cdot\|$ denotes the L^2-norm. It turns out that assertions (i) and (ii) remain valid for the 3D NS system if we replace the solid controllability by uniform approximate controllability (cf. [Shi06]). Hence, we prove that if E is sufficiently large, then problem (0.1), (0.2) is UAC by an E-valued control. The required result on exact controllability in finite-dimensional projections is a simple consequence of the above property. Indeed, let $B_F(R)$ be the closed ball in F of radius R centred at origin and let $\mathcal{K} = B_F(R)$. In this case, it follows from (0.5) that

$$\|\mathsf{P}_F u(T;\Psi(\hat{u})) - \hat{u}\| < \varepsilon \quad \text{for any } \hat{u} \in B_F(R). \tag{0.6}$$

The function $\Phi : \hat{u} \mapsto \mathsf{P}_F u(T, \Psi(\hat{u}))$ is continuous from $B_F(R)$ to F. Using the Brouwer fixed point theorem and inequality (0.6), it is easy to show (see Proposition 1.1) that $\Phi(B_F(R))$ contains the ball $B_F(R-\varepsilon)$. Since $R > 0$ is arbitrary, we conclude that (0.4) holds for any $\hat{u} \in F$ and an appropriate E-valued control function η.

In conclusion, we note that the problem of controllability and stabilisation for the Navier–Stokes and Euler equation was in the focus of attention of many researchers; for instance, see the papers [Fur95, Cor96, Cor96, CF96, Ima98, FE99, Cor99, FC99, Gla00, BS01, Fur01, Fur04, BT04] and references therein. However, the powerful techniques developed in those papers do not apply to the present setting because of the specific type of control we are interested in.

The paper is organised as follows. In Section 1, we recall a simple sufficient condition for surjectivity of continuous mapping in a finite-dimensional space and formulate two perturbative results on unique solvability of NS-type equations. Section 2 contains the formulations of the main results of this paper. We

[2] Note that the concept of uniform approximate controllability is implicitly present in the Agrachev–Sarychev argument [AS05].

also discuss some corollaries on solid controllability in finite-dimensional projection and the Cauchy problem for the 3D NS system. The proofs are presented in Section 3. Finally, in the Appendix, we prove an auxiliary result used in Section 3.

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Notation. We denote by $\mathbb{R}_+$ the half-line $[0,+\infty)$ and by J_T the interval $[0,T]$. If $s \geq 1$ and $r \geq 0$ are some integers, then we set $J_T(r,s) = [t_r, t_{r+1})$, where $t_r = rT/s$. Let $J \subset \mathbb{R}_+$ be a closed interval, let $D \subset \mathbb{R}^3$ be a bounded domain, let X be a Banach space with a norm $\|\cdot\|$, and let $\mathcal{K}$ be a metric space. We shall use the following functional spaces.

$L^p(J,X)$ is the space of measurable functions $u : J \to X$ with finite norm

$$\|u\|_{L^p(J,X)} := \left(\int_J \|u(t)\|_X^p dt \right)^{1/p}, \tag{0.7}$$

where $\|\cdot\|_X$ stands for the norm in X. If $p = \infty$, then (0.7) is replaced by

$$\|u\|_{L^\infty(J,X)} := \operatorname*{ess\ sup}_{t \in J} \|u(t)\|_X.$$

We shall write $L^p(J)$ instead of $L^p(J,\mathbb{R})$.

$L^p_{\mathrm{loc}}(\mathbb{R}_+,X)$ is the space of functions $u : R_+ \to X$ whose restriction to any interval $J \subset \mathbb{R}_+$ belongs to $L^p(J,X)$.

$C^k(J,X)$ is the space of continuous functions $u : J \to X$ that are k times continuously differentiable. In the case $k=0$, we shall write $C(J,X)$.

$C(\mathcal{K},X)$ is the space of continuous functions $u : \mathcal{K} \to X$. If $X = \mathbb{R}$, then we write $C(\mathcal{K})$.

$H^s(D,\mathbb{R}^3)$ is the space of vector functions (u_1,u_2,u_3) whose components belong to the Sobolev space of order s. In the case $s=0$, it coincides with the Lebesgue space $L^2(D,\mathbb{R}^3)$.

H, V, U, and $\mathcal{X}_T$ are standard functional spaces arising in the theory of Navier-Stokes equations; they are defined in Section 1.2.

1 Preliminaries

1.1 Image of continuous mappings

Let F be a finite-dimensional vector space with a norm $\|\cdot\|_F$, let $B_F(R)$ be the closed ball in F of radius R centred at origin, and let $\Phi : B_F(R) \to F$ be a continuous mapping. The following result is a simple consequence of the Brouwer theorem.

Proposition 1.1. *Suppose there is a constant $\varepsilon \in (0, R)$ such that*

$$\|\Phi(u) - u\|_F \leq \varepsilon \quad \text{for any } u \in B_F(R). \tag{1.1}$$

Then $\Phi(B_F(R)) \supset B_F(R-\varepsilon)$.

Proof. Let us fix any point $\hat{u} \in B_F(R-\varepsilon)$ and consider the continuous mapping

$$\Psi : B_F(R) \to F, \quad \Psi(u) = \hat{u} - \Phi(u) + u.$$

It follows from (1.1) that $\Psi(B_F(R)) \subset B_F(R)$. Therefore, by the Brouwer theorem (e.g., see Section 1.19 in [Tay97]), Ψ has a fixed point $u_0 \in B_F(R)$. Direct verification shows that $\Phi(u_0) = \hat{u}$. Thus, any point $\hat{u} \in B_F(R-\varepsilon)$ has a preimage, and we obtain the required inclusion. $\square$

1.2 Strong solutions of Navier–Stokes type equations

We first introduce some standard functional spaces arising in the theory of 3D Navier–Stokes (NS) equations. Let

$$H = \big\{u \in L^2(D, \mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } D,\ (u, \boldsymbol{n})|_{\partial D} = 0\big\},$$

where $\boldsymbol{n}$ is the outward unit normal to ∂D, and let Π be the orthogonal projection in $L^2(D, \mathbb{R}^3)$ onto the closed subspace H. We denote by $H^s = H^s(D, \mathbb{R}^3)$ the space of vector functions $u = (u_1, u_2, u_3)$ with components in the Sobolev class of order s and by $H_0^s(D, \mathbb{R}^3)$ the space of functions $u \in H^s$ vanishing on ∂D. Let $\|\cdot\|_s$ be the usual norm in H^s. In the case $s = 0$, we write $\|\cdot\|$. Define the spaces

$$V = H_0^1(D, \mathbb{R}^3) \cap H, \quad U = H^2(D, \mathbb{R}^2) \cap V$$

and endow them with natural norms.

It is well known (e.g., see [Tem79]) that the NS system is equivalent to the following evolution equation in H:

$$\dot{u} + \nu L u + B(u) = f(t), \tag{1.2}$$

where $L = -\Pi\Delta$ is the Stokes operator and $B(u) = \Pi\{(u, \nabla)u\}$ is the bilinear form resulting from the nonlinear term in the original system. Let $E \subset U$ be a finite-dimensional vector space and let $E^\perp$ be its orthogonal complement in H. Denote by $\mathsf{P} = \mathsf{P}_E$ and $\mathsf{Q} = \mathsf{Q}_E$ the orthogonal projections in H onto the subspaces E and $E^\perp$, respectively. Along with (1.2), consider the Cauchy problem

$$\dot{w} + \nu L_E w + \mathsf{Q}\big(B(w) + B(v, w) + B(w, v)\big) = f(t), \tag{1.3}$$

$$w(0) = w_0, \tag{1.4}$$

where $L_E = \mathsf{Q}L$, $B(v, w) = \Pi\{(v, \nabla)w\}$, and $v \in L^4(J_T, V)$ and $f \in L^2(J_T, E^\perp)$ are given functions. We set

$$\mathcal{X}_T = C(J_T, V) \cap L^2(J_T, U), \quad \mathcal{X}_T(E) = C(J_T, V \cap E^\perp) \cap L^2(J_T, U \cap E^\perp).$$

The following result is established in [Shi06, Section 1.4] (see Theorem 1.8).

Proposition 1.2. *For any $\nu > 0$ and $R > 0$ there are positive constants ε and C such that the following assertions hold.*

(i) *Let $\hat{v} \in L^4(J_T, H^1)$, $\hat{f} \in L^2(J_T, E^\perp)$, and $\widehat{w}_0 \in V \cap E^\perp$ be some functions such that problem (1.3), (1.4) with $v = \hat{v}$, $f = \hat{f}$, $w_0 = \widehat{w}_0$ has a solution $\widehat{w} \in \mathcal{X}_T(E)$. Suppose that*

$$\|\hat{v}\|_{L^4(J_T,H^1)} \le R, \quad \|\hat{f}\|_{L^2(J_T,E^\perp)} \le R, \quad \|\widehat{w}\|_{\mathcal{X}_T} \le R.$$

Then, for any triple (v, f, w_0) satisfying the inequalities

$$\|v - \hat{v}\|_{L^4(J_T,H^1)} \le \varepsilon, \quad \|f - \hat{f}\|_{L^2(J_T,E^\perp)} \le \varepsilon, \quad \|w_0 - \widehat{w}_0\|_V \le \varepsilon, \tag{1.5}$$

problem (1.3), (1.4) has a unique solution $w \in \mathcal{X}_T(E)$.

(ii) *Let*

$$\mathcal{R} : L^4(J_T, H^1) \times L^2(J_T, E^\perp) \times (V \cap E^\perp) \to \mathcal{X}_T(E)$$

be an operator that is defined on the set of functions (v, f, w_0) satisfying (1.5) and takes each triple (v, f, w_0) to the solution $w \in \mathcal{X}_T(E)$ of (1.3), (1.4). Then $\mathcal{R}$ is uniformly Lipschitz continuous, and its Lipschitz constant does not exceed C.

We now consider Eq. (1.3) in which E is a finite-dimensional vector space spanned by some eigenfunctions of the Stokes operator L. Namely, let $\{e_j\}$ be a complete set of normalised eigenfunctions for L, let H_N be the vector span of $\{e_j, 1 \le j \le N\}$, and let $H_N^\perp$ be the orthogonal complement of H_N in the space H. We denote by P_N and Q_N the orthogonal projections in H onto the subspaces H_N and $H_N^\perp$, respectively.

Let us consider the equation

$$\dot{w} + \nu L_N(w + v) + \mathsf{Q}_N B(w + v) = f(t), \tag{1.6}$$

where $L_N = \mathsf{Q}_N L$.

Proposition 1.3. *For any $R > 0$ and $\nu > 0$ there is an integer $N_0 \ge 1$ and a constant $C > 0$ such that the following assertions hold.*

(i) *Let $N \ge N_0$ be an integer and let functions $v \in \mathcal{X}_T$, $f \in L^2(J_T, H_N^\perp)$, and $w_0 \in H_N^\perp \cap V$ satisfy the inequalities*

$$\|v\|_{\mathcal{X}_T} \le R, \quad \|f\|_{L^2(J_T,H)} \le R, \quad \|w_0\|_V \le R. \tag{1.7}$$

Then problem (1.6), (1.4) has a unique solution $w \in \mathcal{X}_T(H_N)$.

(ii) *Let $\mathcal{S}$ be an operator that takes each triple (v, f, w_0) satisfying (1.7) to the solution $w \in \mathcal{X}_T(H_N)$ of (1.6), (1.4). Then $\mathcal{S}$ is uniformly Lipschitz continuous in the corresponding spaces, and its Lipschitz constant does not exceed C.*

Proof. The existence and uniqueness of solution is established in [Shi06] (see Proposition 1.10). The proof of (ii) is rather standard, and we only outline it.

Let $w^1, w^2 \in \mathcal{X}_T(H_N)$ be two solutions of (1.6), (1.4) that correspond to some triples (v^i, f^i, w_0^i), $i = 1, 2$. Then the function $w = w^1 - w^2 \in \mathcal{X}_T(H_N)$ is a solutions of the problem

$$\dot{w} + \nu L_N w = g(t), \quad w(0) = w_0^1 - w_0^2,$$

where we set

$$g(t) = (f^1 - f^2) - \nu L_N(v^1 - v^2) - \mathsf{Q}_N(B(w^1 + v^1) - B(w^2 + v^2)).$$

Repeating literally the argument used in Step 2 of the proof of Proposition 1.10 in [Shi06], we show that if N is sufficiently large, then

$$\|w\|_{\mathcal{X}_T} \le C_1\big(\|f^1 - f^2\|_{L^2(J_T,H)} + \|v^1 - v^2\|_{L^2(J_T,U)} + \|w_0^1 - w_0^2\|_V\big) + \tfrac{1}{2}\|w\|_{\mathcal{X}_T}, \tag{1.8}$$

where $C_1 > 0$ is a constant depending only T, R, and ν. Inequality (1.8) implies the required result. $\square$

2 Main results

2.1 Exact controllability in observed projections

Consider the controlled Navier–Stokes (NS) equations

$$\dot{u} + \nu L u + B(u) = h(t) + \eta(t), \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $h \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H)$ and $u_0 \in V$ are given functions and η is a control function with range in a finite-dimensional vector space $E \subset U$. For any $h \in L^2(J_T, H)$, $u_0 \in V$, and $T > 0$, we denote by $\Theta_T(h, u_0)$ the set of functions $\eta \in L^2(J_T, H)$ for which problem (2.1), (2.2) has a unique solution $u \in \mathcal{X}_T$. It follows from Proposition 1.2 with $E = \{0\}$ and $v \equiv 0$ that $\Theta_T(h, u_0)$ is an open subset of $L^2(J_T, H)$.

Let us fix a constant $T > 0$, a finite-dimensional space $F \subset H$, and a projection $\mathsf{P}_F : H \to H$ onto F.

Definition 2.1. Equation (2.1) with $\eta \in L^2(J_T, E)$ is said to be P_F*-controllable in time* T if for any $u_0 \in V$ and $\hat{u} \in F$ there is $\eta \in \Theta_T(h, u_0) \cap L^2(J_T, E)$ such that

$$\mathsf{P}_F u(T) = \hat{u}, \tag{2.3}$$

where $u \in \mathcal{X}_T$ denotes the solution of (2.1), (2.2).

To formulate the main result of this paper, we introduce some notation. For any finite-dimensional subspace $E \subset U$, we denote by $\mathcal{F}(E)$ the largest vector space $G \subset U$ such that any element $\eta_1 \in G$ is representable in the form

$$\eta_1 = \eta - \sum_{j=1}^{k} \alpha_j B(\zeta^j),$$

where $\eta, \zeta^1, \dots, \zeta^k \in E$ are some vectors and $\alpha_1, \dots, \alpha_k$ are non-negative constants. Since B is a quadratic operator, we see that $\mathcal{F}(E) \subset U$ is a well-defined finite-dimensional subspace containing E. For a finite-dimensional subspace $E \subset U$, we set

$$E_0 = E, \quad E_k = \mathcal{F}(E_{k-1}) \quad \text{for } k \geq 1, \quad E_\infty = \bigcup_{k=1}^{\infty} E_k. \tag{2.4}$$

The following theorem is the main result of this paper.

Theorem 2.2. *Let $h \in L^2_{\text{loc}}(\mathbb{R}_+, H)$ and let $E \subset U$ be a finite-dimensional subspace such that E_∞ is dense in H. Then for any $T > 0$, any finite-dimensional subspace $F \subset H$, and any projection $\mathsf{P}_F : H \to H$ onto F the Navier–Stokes system* (2.1) *with $\eta \in L^2(J_T, E)$ is P_F-controllable in time T. Moreover, the control function η can be chosen from the space $C^\infty(J_T, E)$.*

In the case of a general bounded domain, it is difficult to check whether E_∞ is dense in H. However, Theorem 2.2 remains valid for the NS equation (2.1) on a 3D torus, and it is shown in [Shi06, Section 2.3] that[3] if $E \supset H_N$ for a sufficiently large $N \geq 1$, then E_∞ contains all the eigenfunctions of L. Thus, we obtain the following result.

Corollary 2.3. *Let $\mathbb{T}^3$ be a torus in $\mathbb{R}^3$. Then there is an integer $N \geq 1$ such that if the control space E contains H_N, then for any constants $\nu > 0$ and $T > 0$, any function $h \in L^2_{\text{loc}}(\mathbb{R}_+, H)$, any finite-dimensional subspace $F \subset H$, and any projection $\mathsf{P}_F : H \to H$ onto F the Navier–Stokes system* (2.1) *on $\mathbb{T}^3$ with $\eta \in L^2(J_T, E)$ is P_F-controllable in time T, and the control function η can be chosen from the space $C^\infty(J_T, E)$.*

The proof of Theorem 2.2 is based on a property of uniform approximate controllability for (2.1). That concept is of independent interest and is discussed in the next subsection.

2.2 Uniform approximate controllability

Let us fix any $T > 0$ and $h \in L^2(J_T, H)$ and denote by $\mathcal{R}(u_0, \eta)$ an operator that is defined on the set

$$D(\mathcal{R}) = \big\{(u_0, \eta) \in V \times L^2(J_T, H) : \eta \in \Theta_T(h, u_0)\big\}$$

[3]Recall that H_N denotes the vector space spanned by the first N eigenfunctions of the Stokes operator L.

and takes each pair $(u_0, \eta) \in D(\mathcal{R})$ to the solution $u \in \mathcal{X}_T$ of problem (2.1), (2.2). Proposition 1.2 with $E = \{0\}$ and $v \equiv 0$ implies that $D(\mathcal{R})$ is an open subset of $V \times L^2(J_T, H)$, and $\mathcal{R}$ is locally Lipschitz continuous on $D(\mathcal{R})$. For any $t \in J_T$, we denote by $\mathcal{R}_t(u_0, \eta)$ the restriction of $\mathcal{R}(u_0, \eta)$ to the time t.

Let $X \subset L^2(J_T, H)$ be an arbitrary vector space, not necessarily closed. We endow X with the norm of $L^2(J_T, H)$.

Definition 2.4. Equation (2.1) with $\eta \in X$ is said to be *uniformly approximately controllable in time T* if for any initial point $u_0 \in V$, any compact set $\mathcal{K} \subset V$, and any $\varepsilon > 0$ there is a continuous function

$$\Psi : \mathcal{K} \to X \cap \Theta_T(h, u_0)$$

such that

$$\|\mathcal{R}_T(u_0, \Psi(\hat{u})) - \hat{u}\|_V < \varepsilon \quad \text{for any } \hat{u} \in \mathcal{K}. \tag{2.5}$$

The following result shows that, under the conditions of Theorem 2.2, Eq. (2.1) is uniformly approximately controllable (UAC).

Theorem 2.5. *Let $h \in L^2_{\text{loc}}(\mathbb{R}_+, H)$ and let $E \subset U$ be a finite-dimensional subspace such that E_∞ is dense in H. Then for any $T > 0$ and $\nu > 0$ the Navier–Stokes system* (2.1) *with $\eta \in C^\infty(J_T, E)$ is UAC in time T.*

Theorem 2.5 will be established in Section 3. Here we show that the exact controllability in a projection is a simple consequence of UAC; in the next subsection, we deduce some corollaries from Theorems 2.2 and 2.5.

Proof of Theorem 2.2. Let us fix a time $T > 0$, an initial point $u_0 \in V$, and a projection $\mathsf{P}_F : H \to H$ onto a finite-dimensional subspace $F \subset H$. Recall that $B_F(R)$ stands for the closed ball in F of radius R centred at origin and denote by C the norm of $\mathsf{P}_F : H \to H$. Let us fix any $R > C$ and choose $\delta > 0$ so small that

$$\sup_{\hat{u} \in B_F(R)} \|e^{-\delta L}\hat{u} - \hat{u}\| \le \tfrac{1}{2}. \tag{2.6}$$

Denote by $\mathcal{K}$ the image of $B_F(R)$ under $e^{-\delta L}$. This is a compact subset of V, and by Theorem 2.5, there is a continuous mapping[4]

$$\Psi : \mathcal{K} \to C^\infty(J_T, E) \cap \Theta_T(h, u_0)$$

such that

$$\sup_{\hat{v} \in \mathcal{K}} \|\mathcal{R}_T(u_0, \Psi(\hat{v})) - \hat{v}\|_V < \tfrac{1}{2}. \tag{2.7}$$

It follows (2.6) and (2.7) that

$$\sup_{\hat{u} \in B_F(R)} \|\mathcal{R}_T(u_0, \Psi(e^{-\delta L}\hat{u})) - \hat{u}\| < 1.$$

[4]Recall that the control space $C^\infty(J_T, E) \cap \Theta_T(h, u_0)$ is endowed with the metric generated by the norm in $L^2(J_T, H)$.

Therefore the continuous mapping

$$\Phi : B_F(R) \to F, \quad \hat{u} \mapsto \mathsf{P}_F \mathcal{R}_T(u_0, \Psi(e^{-\delta L}\hat{u})),$$

satisfies inequality (1.1) with $\varepsilon = C$. Hence, by Proposition 1.1, we have $\Phi(B_F(R)) \supset B_F(R - C)$. In particular, it follows that for any $\hat{u} \in B_F(R - C)$ there is $\eta \in C^\infty(J_T, E) \cap \Theta_T(h, u_0)$ such that $\mathsf{P}_F \mathcal{R}_T(u_0, \eta) = \hat{u}$. Since $R > C$ is arbitrary, we obtain the conclusion of Theorem 2.2. $\square$

2.3 Solid controllability and Cauchy problem for the NS system

In this subsection, we establish some corollaries of Theorems 2.2 and 2.5. Let $E \subset U$ and $F \subset H$ be finite-dimensional subspaces, let $\mathsf{P}_F : H \to H$ be a projection onto F, and let $T > 0$ be a constant.

Definition 2.6. The control system (2.1) with $\eta \in L^2(J_T, E)$ is said to be *solidly* P_F*-controllable in time* T if for any $R > 0$ and $u_0 \in V$ there is a constant $\varepsilon > 0$ and a compact set $\mathcal{C} \subset L^2(J_T, E) \cap \Theta_T(h, u_0)$ such that, for any continuous mapping $S : \mathcal{C} \to F$ satisfying the inequality

$$\sup_{\eta \in \mathcal{C}} \|S(\eta) - \mathsf{P}_F \mathcal{R}_T(u_0, \eta)\|_F \le \varepsilon, \tag{2.8}$$

we have $S(\mathcal{C}) \supset B_F(R)$.

Proposition 2.7. *Under the conditions of Theorem 2.2, for any* $T > 0$*, any finite-dimensional subspace* $F \subset H$*, and any projection* $\mathsf{P}_F : H \to H$ *onto* F*, Eq.* (2.1) *with* $\eta \in L^2(J_T, E)$ *is solidly* P_F*-controllable in time* $T > 0$.

Proof. Let us fix any constant $R > 0$, function $u_0 \in V$, and subspace $F \subset H$. As was shown in the proof of Theorem 2.2 (see Section 2.2), there is a continuous mapping $\Psi : B_F(R + 2) \to L^2(J_T, E) \cap \Theta_T(h, u_0)$ such that

$$\sup_{\hat{u} \in B_F(R+2)} \|\mathsf{P}_F \mathcal{R}_T(u_0, \Psi(\hat{u})) - \hat{u}\| \le 1. \tag{2.9}$$

Let us set $\mathcal{C} = \Psi(B_F(R + 2))$. Since $\dim F < \infty$ and Ψ is continuous, we conclude that $\mathcal{C}$ is a compact subset of $L^2(J_T, E) \cap \Theta_T(h, u_0)$. Let $S : \mathcal{C} \to F$ be an arbitrary continuous mapping such that (2.8) holds with $\varepsilon = 1$. Then it follows from (2.9) that the mapping $S \circ \Psi : B_F(R + 1) \to F$ satisfies the inequality

$$\sup_{\hat{u} \in B_F(R+2)} \|S \circ \Psi(\hat{u}) - \hat{u}\|_V \le 2.$$

Applying Proposition 1.1, we see that $S \circ \Psi(B_F(R + 2)) \supset B_F(R)$. It follows that $S(\mathcal{C}) \supset B_F(R)$. Since $R > 0$ was arbitrary, this completes the proof of Proposition 2.7. $\square$

We now show that the control function η in Theorem 2.2 can be taken from a finite-dimensional subspace. Namely, we have the following result.

Proposition 2.8. *Suppose that the conditions of Theorem 2.2 are fulfilled, and let X be a vector space dense in $L^2(J_T, E)$. Then for any positive constants T and R, any initial function $u_0 \in V$, any subspace $F \subset H$ with $\dim F < \infty$, and any projection $\mathsf{P}_F : H \to H$ onto F, there is a ball B in a finite-dimensional subspace $Y \subset X$ such that*

$$\mathsf{P}_F \mathcal{R}_T(u_0, B) \supset B_F(R). \tag{2.10}$$

In particular, we can take $X = C^\infty(J_T, E)$.

Proof. By Proposition 2.7, Eq. (2.1) with $\eta \in L^2(J_T, E)$ is solidly P_F-controllable in time T. Let $\varepsilon > 0$ and $\mathcal{C} \subset L^2(J_T, E) \cap \Theta_T(h, u_0)$ be the corresponding constant and compact set entering Definition 2.6. It follows from Proposition 1.2 with $E = \{0\}$ and $v \equiv 0$ that $\Theta_T(h, u_0)$ is an open subset of $L^2(J_T, E)$. Therefore there is $\delta > 0$ such that

$$O_\delta(\mathcal{C}) = \big\{\eta \in L^2(J_T, E) : \operatorname{dist}(\eta, \mathcal{C}) \le \delta\big\} \subset \Theta_T(h, u_0),$$

where we set

$$\operatorname{dist}(\eta, \mathcal{C}) = \inf_{\zeta \in \mathcal{C}} \|\eta - \zeta\|_{L^2(J_T, H)}.$$

Furthermore, since X is dense in $L^2(J_T, E)$, we can find a finite-dimensional subspace $Y \subset X$ such that

$$\sup_{\eta \in \mathcal{C}} \|P_Y \eta - \eta\|_{L^2(J_T, E)} \le \delta, \tag{2.11}$$

where P_Y denotes the orthogonal projection in $L^2(J_T, E)$ onto Y. It follows that

$$P_Y \mathcal{C} \subset O_\delta(\mathcal{C}) \subset \Theta_T(h, u_0). \tag{2.12}$$

By Proposition 1.2, the operator $\mathcal{R}(u_0, \cdot) : \Theta_T(h, u_0) \to \mathcal{X}_T$ is locally Lipschitz continuous. Therefore, taking $\delta > 0$ sufficiently small, we deduce from (2.11) and (2.12) that

$$\sup_{\eta \in \mathcal{C}} \|\mathcal{R}_T(u_0, P_Y \eta) - \mathcal{R}_T(u_0, \eta)\|_V \le \frac{\varepsilon}{C},$$

where C is the norm of P_F regarded as an operator from V to H. Thus, the mapping $S(\eta) = \mathsf{P}_F \mathcal{R}_T(u_0, P_Y \eta)$ satisfies (2.8). Hence, by Proposition 2.7, we have $\mathsf{P}_F \mathcal{R}_T(u_0, P_Y \mathcal{C}) \supset B_F(R)$. It remains to note that $P_Y \mathcal{C}$ is contained in a ball of the finite-dimensional space $Y \subset X$. $\square$

We now consider the Cauchy problem for the NS equation (1.2). Let $G \subset H$ be a closed vector space. For any $u_0 \in V$, $T > 0$, and $\nu > 0$, let $\Xi_{T,\nu}(G, u_0)$ be the set of functions $f \in L^2(J_T, G)$ for which problem (1.2), (2.2) has a unique solution $u \in \mathcal{X}_T$. If $E \subset G$ is a closed subspace, then we denote by $G \ominus E$ the orthogonal complement of E in G and by $Q(T, G, E)$ the orthogonal projection in $L^2(J_T, G)$ onto the subspace $L^2(J_T, G \ominus E)$.

Proposition 2.9. *Let $E \subset U$ be a finite-dimensional subspace such that E_∞ is dense in H and let $G \subset H$ be a closed subspace containing E. Then $\Xi_{T,\nu}(G,u_0)$ is a non-empty open subset of $L^2(J_T,G)$ such that*

$$Q(T,G,E)\Xi_{T,\nu}(G,u_0) = L^2(J_T, G \ominus E) \quad \text{for any } T>0,\ \nu>0,\ u_0 \in V.$$

Proof. The fact that $\Xi_{T,\nu}(G,u_0)$ is open follows immediately from Proposition 1.2. The other claims of the proposition are equivalent to the following property: for any $h \in L^2(J_T, G \ominus E)$ there is $\eta \in L^2(J_T,E)$ such that $h+\eta \in \Xi_{T,\nu}(G,u_0)$. This is a straightforward consequence of Theorem 2.2. $\square$

3 Proof of Theorem 2.5

3.1 Scheme of the proof

Let E be a finite-dimensional vector space and let $E_1 = \mathcal{F}(E)$ (see (2.4)). Along with Eq. (2.1), consider two other control systems:

$$\dot u + \nu L(u+\zeta(t)) + B(u+\zeta(t)) = h(t) + \eta(t), \tag{3.1}$$

$$\dot u + \nu L u + B(u) = h(t) + \eta_1(t). \tag{3.2}$$

Here η and ζ are E-valued controls and η_1 is an E_1-valued control. Let us fix a constant $\varepsilon > 0$, an initial point $u_0 \in V$, a compact set $\mathcal{K} \subset V$, and a vector space $X \subset L^2(J,H)$. Equation (2.1) with $\eta \in X$ is said to be *uniformly* $(\varepsilon, u_0, \mathcal{K})$*-controllable* if there is a continuous mapping

$$\Psi : \mathcal{K} \to X \cap \Theta_T(h,u_0)$$

such that (2.5) holds. In what follows, if ε, u_0, and $\mathcal{K}$ are fixed in advance, then the above property will be called *uniform ε-controllability*.

The concept of uniform ε-controllability for (3.1) is defined in a similar way. Namely, let $\widehat\Theta_T(h,u_0)$ be the set of pairs $(\eta,\zeta) \in L^2(J_T,H) \times L^4(J_T,H^2)$ for which problem (3.1), (2.2) has a unique solution $u \in \mathcal{X}_T$ and let $\widehat{\mathcal{R}}$ be an operator that is defined on the set

$$D(\widehat{\mathcal{R}}) = \bigl\{(u_0,\eta,\zeta) \in V \times L^2(J_T,H) \times L^4(J_T,H^2) : (\eta,\zeta) \in \widehat\Theta_T(h,u_0)\bigr\}$$

and takes each triple $(u_0,\eta,\zeta) \in D(\widehat{\mathcal{R}})$ to the solution $u \in \mathcal{X}_T$ of (3.1), (2.2). Rewriting Eq. (3.1) in the form

$$\dot u + \nu L u + B(u) + B(u,\zeta) + B(\zeta,u) = h(t) + \eta - \nu L\zeta - B(\zeta)$$

and applying Proposition 1.2 with $E = \{0\}$, we see that $D(\widehat{\mathcal{R}})$ is an open subset of $V \times L^2(J_T,H) \times L^4(J_T,H^2)$, and the operator $\widehat{\mathcal{R}}$ is locally Lipschitz continuous on $D(\widehat{\mathcal{R}})$.

Now let $\widehat{X} \subset L^2(J,H) \times L^4(J,H^2)$ be a vector space, not necessarily closed. Equation (3.1) with $(\eta,\zeta) \in \widehat{X}$ is said to be *uniformly* $(\varepsilon, u_0, \mathcal{K})$*-controllable* if there is a continuous mapping

$$\widehat{\Psi} : \mathcal{K} \to \widehat{X} \cap \widehat{\Theta}_T(h, u_0)$$

such that

$$\|\widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}(\hat{u})) - \hat{u}\|_V < \varepsilon \quad \text{for any } \hat{u} \in \mathcal{K}, \tag{3.3}$$

where $\widehat{\mathcal{R}}_t(u_0, \eta, \zeta)$ denotes the restriction of $\widehat{\mathcal{R}}(u_0, \eta, \zeta)$ to the time t.

The proof of Theorem 2.5 is based on the following three propositions (cf. Propositions 3.1 and 3.2 and Section 2.2 in [Shi06]). Let us fix a constant $\varepsilon > 0$, an initial point $u_0 \in V$, and a compact subset $\mathcal{K} \subset V$.

Proposition 3.1. (extension principle) *Let $E \subset U$ be a finite-dimensional vector space. Then Eq. (2.1) with $\eta \in C^\infty(J_T, E)$ is uniformly ε-controllable if and only if so is Eq. (3.1) with $(\eta, \zeta) \in C^\infty(J_T, E \times E)$.*

Proposition 3.2. (convexification principle) *Let $E \subset U$ be a finite-dimensional subspace and let $E_1 = \mathcal{F}(E)$. Then Eq. (3.1) with $(\eta, \zeta) \in C^\infty(J_T, E \times E)$ is uniformly ε-controllable if and only if so is Eq. (3.2) with $\eta_1 \in C^\infty(J_T, E_1)$.*

Proposition 3.3. *Let $E \subset U$ be a finite-dimensional vector space such that E_∞ is dense in H. Then there is an integer $k \geq 1$ depending on ε, u_0, and $\mathcal{K}$ such that Eq. (2.1) with $\eta \in C^\infty(J_T, E_k)$ is uniformly ε-controllable.*

If Propositions 3.1 – 3.3 are established, then for any $\varepsilon > 0$, $u_0 \in V$, and $\mathcal{K} \subset V$ we first use Proposition 3.3 to find an integer $k \geq 1$ such that Eq. (2.1) with $\eta \in C^\infty(J_T, E_k)$ is uniformly $(\varepsilon, u_0, \mathcal{K})$-controllable. Combining this property with Propositions 3.2 and 3.1 in which $E = E_{k-1}$, we conclude that Eq. (2.1) with $\eta \in C^\infty(J_T, E_{k-1})$ is uniformly $(\varepsilon, u_0, \mathcal{K})$-controllable. Repeating this argument $k-1$ times, we see that the same property is true for Eq. (2.1) with $\eta \in C^\infty(J_T, E)$. Since ε, u_0, and $\mathcal{K}$ are arbitrary, this completes the proof of Theorem 2.5.

To prove the above propositions, we repeat the scheme used in [Shi06] (see Sections 2.2, 3.2, and 3.3). The important point now is that we have to follow carefully the dependence of controls on the final state $\hat{u}$. The proofs of Propositions 3.1 – 3.3 are carried out in next three subsections. Here we formulate a lemma on uniform ε-controllability; it will be used in Sections 3.2 – 3.4. As before, we fix a constant $\varepsilon > 0$, an initial point $u_0 \in V$, and a compact set $\mathcal{K} \subset V$.

Lemma 3.4. *Let $X, Y \subset L^2(J_T, H)$ be vector spaces such that X is contained in the closure of Y and Eq. (2.1) with $\eta \in X$ is uniformly ε-controllable. Then there is a finite-dimensional subspace $Y_0 \subset Y$ such that Eq. (2.1) with $\eta \in Y_0$ is uniformly ε-controllable.*

To prove this lemma, it suffices to repeat the argument used in the proof of Proposition 2.8; we shall not dwell on it. Also note that an analogue of Lemma 3.4 is true for Eq. (3.1).

3.2 Extension principle: proof of Proposition 3.1

We need to show that if Eq. (3.1) with $(\eta,\zeta)\in C^\infty(J_T,E\times E)$ is uniformly ε-controllable, then so is Eq. (2.1) with $\eta\in C^\infty(J_T,E)$. Since $C^\infty(J_T,E)$ is dense in $L^2(J_T,E)$, in view of Lemma 3.4, it suffices to establish that property for Eq. (2.1) with $\eta\in L^2(J_T,E)$.

Recall that P and Q stand for the orthogonal projection in H onto the subspaces E and $E^\perp$, respectively. Let

$$\widehat{\Psi}:\mathcal{K}\to C^\infty(J_T,E\times E)\cap\widehat{\Theta}_T(h,u_0),\quad \widehat{\Psi}(\hat u)=\bigl(\eta(t,\hat u),\zeta(t,\hat u)\bigr),$$

be an operator for which (3.3) holds. We choose any sequence of functions $\varphi_k\in C^\infty(\mathbb{R})$ with the following properties:

$$0\le\varphi_k(t)\le 1\quad\text{for all } t\in\mathbb{R}, \tag{3.4}$$
$$\varphi_k(t)=0\quad\text{for } t\le 0 \text{ and } t\ge T, \tag{3.5}$$
$$\varphi_k(t)=1\quad\text{for } \tfrac{1}{k}\le t\le T-\tfrac{1}{k}. \tag{3.6}$$

We now define a sequence of continuous mappings $\Psi_k:\mathcal{K}\to L^2(J_T,E)$ by the following rule:

- for any $\hat u\in\mathcal{K}$ and $k\ge 1$, set

$$v_k(t,\hat u)=\varphi_k(t)\zeta(t,\hat u)+\mathsf{P}\widehat{\mathcal{R}}_t(u_0,\widehat{\Psi}(\hat u)),\quad t\in J_T; \tag{3.7}$$

- denote by $w_k(\cdot,\hat u)\in\mathcal{X}_T(E)$ the solution of the problem [5]

$$\begin{aligned}\dot w+\nu L_E w+\mathsf{Q}\bigl(B(w)+B(v_k,w)+B(w,v_k)\bigr)&=f_k(t,\hat u),\\ w(0)&=\mathsf{Q}u_0,\end{aligned} \tag{3.8}$$

  where $v_k=v_k(t,\hat u)$ and

$$f_k(t,\hat u)=\mathsf{Q}\bigl(h(t)-B(v_k(t,\hat u))-\nu L v_k(t,\hat u)\bigr); \tag{3.9}$$

- denote $u_k(t,\hat u)=w_k(t,\hat u)+w_k(t,\hat u)$ and define Ψ_k by the formula

$$\Psi_k(\hat u)=\eta_k(t,\hat u):=\dot v_k+\mathsf{P}(\nu L u_k+B(u_k)-h). \tag{3.10}$$

We claim that for sufficiently large $k\ge 1$ the function Ψ_k is well defined and continuous and satisfies (2.5). Indeed, let us write

$$v(t,\hat u)=\mathsf{P}\widehat{\mathcal{R}}_t(u_0,\widehat{\Psi}(\hat u)),\quad w(t,\hat u)=\mathsf{Q}\widehat{\mathcal{R}}_t(u_0,\widehat{\Psi}(\hat u)).$$

Then $w(\cdot,\hat u)\in\mathcal{X}_T(E)$ is a solution of the problem

$$\begin{aligned}\dot w+\nu L_E w+\mathsf{Q}\bigl(B(w)+B(v,w)+B(w,v)\bigr)&=f(t,\hat u),\\ w(0)&=\mathsf{Q}u_0,\end{aligned} \tag{3.11}$$

[5] We shall show that such a solution exists for $k\gg 1$.

where $v = v(t, \hat{u})$ and $f(t, \hat{u}) = \mathsf{Q}\big(h(t) - B(v(t, \hat{u})) - \nu L v(t, \hat{u})\big)$. We wish to consider (3.8) as a perturbation of (3.11).

Since $\mathcal{K}$ is compact and $\widehat{\mathcal{R}}(u_0, \widehat{\Psi}(\cdot)) : \mathcal{K} \to \mathcal{X}_T$ is continuous, we have

$$\sup_{\hat{u} \in \mathcal{K}} \big(\|v(\cdot, \hat{u})\|_{\mathcal{X}_T} + \|w(\cdot, \hat{u})\|_{\mathcal{X}_T}\big) < \infty. \tag{3.12}$$

It follows from (3.4), (3.6), and (3.7) that

$$\sup_{\hat{u} \in \mathcal{K}} \|v_k(\cdot, \hat{u}) - v(\cdot, \hat{u})\|_{L^4(J_T, U)} \to 0 \quad \text{as } k \to \infty. \tag{3.13}$$

Combining this with standard estimates for the nonlinear term and the fact that $\dim E < \infty$, we conclude that (cf. (3.8) in [Shi06])

$$\sup_{\hat{u} \in \mathcal{K}} \|f_k(\cdot, \hat{u}) - f(\cdot, \hat{u})\|_{L^2(J_T, H)} \to 0 \quad \text{as } k \to \infty. \tag{3.14}$$

Proposition 1.2 and relations (3.12) – (3.14) imply that there is an integer $k_0 \geq 1$ such that, for any $k \geq k_0$ and $\hat{u} \in \mathcal{K}$, problem (3.8) has a unique solution $w_k(\cdot, \hat{u}) \in \mathcal{X}_T(E)$. Moreover, the function $\hat{u} \mapsto w_k(\cdot, \hat{u})$ is continuous from $\mathcal{K}$ to $\mathcal{X}_T(E)$. It follows from (3.10) that the operator Ψ_k is well defined and continuous for $k \geq k_0$.

Let us show that Ψ_k satisfies (2.5) for $k \gg 1$. Since the resolving operator associated with (3.11) is locally Lipschitz continuous (see Proposition 1.2), for any $\hat{u} \in \mathcal{K}$ and $k \geq k_0$, we have

$$\|w_k(\cdot, \hat{u}) - w(\cdot, \hat{u})\|_{\mathcal{X}_T} \leq C\big(\|f_k(\cdot, \hat{u}) - f(\cdot, \hat{u})\|_{L^2(J_T, H)} + \|v_k(\cdot, \hat{u}) - v(\cdot, \hat{u})\|_{L^4(J_T, V)}\big),$$

where $C > 0$ does not depend on k and $\hat{u}$. Combining this inequality with (3.13) and (3.14), we derive

$$\sup_{\hat{u} \in \mathcal{K}} \|w_k(\cdot, \hat{u}) - w(\cdot, \hat{u})\|_{\mathcal{X}_T} \to 0 \quad \text{as } k \to \infty. \tag{3.15}$$

Now note that, in view of (3.5) and (3.7), we have

$$\begin{aligned}
\|\mathcal{R}_T(u_0, \Psi_k(\hat{u})) - \hat{u}\|_V &\leq \|\mathcal{R}_T(u_0, \Psi_k(\hat{u})) - \widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}_k(\hat{u}))\|_V \\
&\qquad + \|\widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}_k(\hat{u})) - \hat{u}\|_V \\
&\leq \|w_k(\cdot, \hat{u}) - w(\cdot, \hat{u})\|_V + \|\widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}_k(\hat{u})) - \hat{u}\|_V.
\end{aligned}$$

Taking the supremum over $\hat{u} \in \mathcal{K}$ and using (3.3) and (3.15), we see that Ψ_k satisfies (2.5) for sufficiently large $k \geq k_0$. The proof of Proposition 3.1 is complete.

3.3 Convexification principle: proof of Proposition 3.2

We need to prove that if Eq. (3.2) with $\eta_1 \in C^\infty(J_T, E_1)$ is uniformly ε-controllable, then so is Eq. (3.1) with $(\eta, \zeta) \in L^2(J_T, E) \times L^4(J_T, E)$; the converse assertion is obvious in view of Proposition 3.1. Let us outline the main idea.

Let $\Psi_1 : \mathcal{K} \to L^2(J_T, E_1)$ be a continuous mapping such that

$$\hat{\varepsilon} := \sup_{\hat{u}\in\mathcal{K}} \|\mathcal{R}_T(u_0, \Psi_1(\hat{u})) - \hat{u}\|_V < \varepsilon. \tag{3.16}$$

By definition, the function $u_1(t, \hat{u}) = \mathcal{R}_t(u_0, \Psi_1(\hat{u}))$ satisfies Eq. (3.2) in which $\eta_1 = \eta_1(\cdot, \hat{u}) := \Psi_1(\hat{u})$. We wish to approximate $u_1(t, \hat{u})$ by a solution $u(t, \hat{u})$ of Eq. (3.1) with some functions $\eta(\cdot, \hat{u}), \zeta(\cdot, \hat{u}) \in L^\infty(J_T, E)$. This approximation should be such that

$$\sup_{\hat{u}\in\mathcal{K}} \|u(T, \hat{u})) - u_1(T, \hat{u})\|_V \le \varepsilon - \hat{\varepsilon}, \tag{3.17}$$

and the mapping $\hat{u} \mapsto (\eta(\cdot, \hat{u}), \zeta(\cdot, \hat{u}))$ is continuous as an operator from $\mathcal{K}$ to the space $L^2(J_T, E) \times L^4(J_T, E)$.

To construct $u(t, \hat{u})$, one could try to apply the argument used in Section 3.3 of [Shi06] for approximating individual solutions. Unfortunately, it does not work because it is difficult to ensure that the resulting control functions η and ζ continuously depend on $\hat{u}$. To overcome this difficulty, we first approximate $\eta_1(t, \hat{u})$ by a family of piecewise constant controls $\tilde{\eta}_1(t, \hat{u})$ with range in the convex envelope of a finite set not depending on $\hat{u}$ (cf. Section 12.3 in [AS05]). We next repeat the scheme of [Shi06] to construct an approximation for solutions $\tilde{u}_1(t, \hat{u})$ corresponding to $\tilde{\eta}_1(t, \hat{u})$. A difficult point of the proof is to follow the dependence of the control functions on $\hat{u}$. In what follows, we shall omit the tilde from the notation.

The realisaion of the above scheme is divided into several steps. We begin with a generalisation of the concept of uniform approximate controllability.

Step 1. Let $A = \{\eta_1^l, l = 1, \dots, m\} \subset E_1$ be a finite set. For any integer $s \ge 1$, denote by $P_s(J_T, A)$ the set of functions $\eta_1 \in L^2(J_T, E_1)$ satisfying the following properties:

- there are non-negative functions $\varphi_l \in L^\infty(J_T)$, $l = 1 \dots, m$, such that

$$\sum_{l=1}^m \varphi_l(t) = 1, \quad \eta_1(t) = \sum_{l=1}^m \varphi_l(t)\eta_1^l \quad \text{for } 0 \le t < T;$$

- the functions φ_l are representable in the form

$$\varphi_l(t) = \sum_{r=0}^{s-1} c_{lr} I_{r,s}(t) \quad \text{for } 0 \le t < T,$$

  where $c_{lr} \ge 0$ are some constants and $I_{r,s}$ denotes the indicator function of the interval $J_T(r, s) = [t_r, t_{r+1})$ with $t_r = rT/s$.

The set $P_s(J_T, A)$ is endowed with the metric

$$d_P(\eta_1, \zeta_1) = \sum_{l=1}^m \|\varphi_l - \psi_l\|_{L^\infty(J_T)}, \quad \eta_1, \zeta_1 \in P_s(J_T, A),$$

where $\{\varphi_l\}$ and $\{\psi_l\}$ are the families of functions corresponding to η_1 and ζ_1, respectively.

Recall that we have fixed a constant $\varepsilon > 0$, an initial point $u_0 \in V$, and a compact set $\mathcal{K} \subset V$. We shall say that Eq. (3.2) with $\eta_1 \in P_s(J_T, A)$ is *uniformly ε-controllable* if there is a continuous[6] mapping $\Psi_1 : \mathcal{K} \to P_s(J_T, A)$ such that $\Psi(\hat{u}) \in \Theta_T(h, u_0)$ for any $\hat{u} \in \mathcal{K}$, and (3.16) holds. A proof of the following lemma is based on a standard argument of the control theory and is given in the Appendix.

Lemma 3.5. *Let us assume that Eq.* (3.2) *with $\eta \in C^\infty(J_T, E_1)$ is uniformly ε-controllable. Then there is a finite set $A = \{\eta_1^l, l = 1, \dots, m\} \subset E_1$ and an integer $s \geq 1$ such that Eq.* (3.2) *with $\eta \in P_s(J_T, A)$ is uniformly ε-controllable.*

Let $\Psi_1 : \mathcal{K} \to P_s(J_T, A)$ be the function constructed in Lemma 3.5. We write

$$\Psi_1(\hat{u}) = \eta_1(t, \hat{u}) = \sum_{l=1}^m \varphi_l(t, \hat{u})\eta_1^l.$$

The definition of the space $P_s(J_T, A)$ and of its metric imply that the functions φ_l have the form

$$\varphi_l(t, \hat{u}) = \sum_{r=0}^{s-1} c_{lr}(\hat{u}) I_{r,s}(t), \tag{3.18}$$

where $c_{lr} : \mathcal{K} \to \mathbb{R}$ are non-negative continuous functions such that

$$\sum_{l=1}^m c_{lr}(\hat{u}) = 1 \quad \text{for any } \hat{u} \in \mathcal{K}.$$

Since $\eta_1^l \in \mathcal{F}(E)$, by Lemma 3.3 in [Shi06], there are vectors $\eta^l, \zeta^{1l}, \dots, \zeta^{kl} \in E$ and non-negative constants $\lambda_{1l}, \dots, \lambda_{kl}$ such that $\sum_j \lambda_{jl} = 1$ and

$$B(u) - \eta_1^l = \sum_{j=1}^k \lambda_{jl} \big(B(u_1 + \zeta^{jl}) + \nu L \zeta^{jl} \big) - \eta^l \quad \text{for } u \in V.$$

It follows that the function $u_1(\cdot) = \mathcal{R}(u_0, \Psi_1(\hat{u}))$ is a solution of the equation

$$\partial_t u_1 + \nu L u_1 + \sum_{j=1}^k \sum_{l=1}^m \lambda_{jl} \varphi_l(t, \hat{u}) \big(B(u_1 + \zeta^{jl}) + \nu L \zeta^{jl} \big) = h(t) + \sum_{j=1}^k \sum_{l=1}^m \lambda_{jl} \varphi_l(t, \hat{u}) \eta^l. \tag{3.19}$$

[6] We emphasise that, in contrast to Definition 2.4 in which the vector space X is endowed with the norm of $L^2(J_T, H)$, the mapping Ψ_1 is required to be continuous with respect to the topology of $P_s(J_T, A)$, which is stronger than that of $L^2(J_T, H)$.

Indexing the pairs (j, l) by a single sequence $i = 1, \dots, q$, we can write (3.19) as

$$\partial_t u_1 + \nu L \left(u_1 + \sum_{i=1}^{q} \psi_i(t, \hat{u}) \zeta^i \right) + \sum_{i=1}^{q} \psi_i(t, \hat{u}) B(u_1 + \zeta^i) = h(t) + \eta(t, \hat{u}). \quad (3.20)$$

Here $\zeta^i \in E$, $i = 1, \dots, q$, are some vectors, $\eta(\cdot, \hat{u})$ denotes the sum on the right-hand side of (3.19), and

$$\psi_i(t, \hat{u}) = \sum_{r=0}^{s-1} d_{ir}(\hat{u}) I_{r,s}(t), \quad (3.21)$$

where $d_{ir} \in C(\mathcal{K})$ are non-negative functions such that $\sum_i d_{ir} \equiv 1$.

Step 2. We now approximate $u_1(t, \cdot)$ by solutions of Eq. (3.1). To this end, we first assume that there is only one interval of constancy, that is, $s = 1$. In this case, the sums in (3.18) and (3.21) contain only one term, and Eq. (3.20) takes the form

$$\partial_t u_1 + \nu L \left(u_1 + \sum_{i=1}^{q} d_i(\hat{u}) \zeta^i \right) + \sum_{i=1}^{q} d_i(\hat{u}) B(u_1 + \zeta^i) = h(t) + \eta(t, \hat{u}), \quad (3.22)$$

where $d_i \in C(\mathcal{K})$ and $\eta \in C(\mathcal{K}, L^2(J_T, E))$. Let us fix an integer $k \geq 1$ and, following a classical idea in the control theory, define a sequence of continuous mappings $\zeta_k : \mathcal{K} \to L^4(J_T, H^2)$ as

$$\zeta_k(t, \hat{u}) = \zeta(kt/T, \hat{u}), \quad (3.23)$$

where $\zeta(\cdot, \hat{u})$ is a 1-periodic function on $\mathbb{R}$ such that

$$\zeta(s, \hat{u}) = \zeta^i \quad \text{for } 0 \leq s - (d_1(\hat{u}) + \dots + d_{i-1}(\hat{u})) < d_i(\hat{u}),\ i = 1, \dots, q. \quad (3.24)$$

It is easy to see that $\{\zeta_k(\cdot, \hat{u}), \hat{u} \in \mathcal{K}, k \geq 1\}$ is a bounded subset in $L^\infty(J_T, E)$. Let us rewrite (3.22) in the form

$$\partial_t u_1 + \nu L(u_1 + \zeta_k(t, \hat{u})) + B(u_1 + \zeta_k(t, \hat{u})) = h(t) + \eta(t, \hat{u}) + f_k(t, \hat{u}), \quad (3.25)$$

where $f_k(t, \hat{u}) = f_{k1}(t, \hat{u}) + f_{k2}(t, \hat{u})$,

$$f_{k1}(t, \hat{u}) = \nu L \left(\zeta_k(t, \hat{u}) - \sum_{i=1}^{q} d_i(\hat{u}) \zeta^i \right), \quad (3.26)$$

$$f_{k2}(t, \hat{u}) = B(u_1(t, \hat{u}) + \zeta_k(t, \hat{u})) - \sum_{i=1}^{i} d_i(\hat{u}) B(u_1(t, \hat{u}) + \zeta^i). \quad (3.27)$$

We shall need the following result, which will be proved in the next steps. Denote by $B_V(u, r)$ the closed ball in V of radius r centred at u.

Lemma 3.6. *For any $\varepsilon_0 > 0$ there is an integer $k_0 \geq 1$ and a constant $\delta_0 > 0$ such that for any $k \geq k_0$, $\hat{u} \in \mathcal{K}$, and $v_0 \in B_V(u_0, \delta_0)$, the problem*

$$\partial_t v + \nu L(v + \zeta_k(t, \hat{u})) + B(v + \zeta_k(t, \hat{u})) = h(t) + \eta(t, \hat{u}), \quad v(0) = v_0 \quad (3.28)$$

has a unique solution $v_k(\cdot, \hat{u}) \in \mathcal{X}_T$, which satisfies the inequality

$$\|v_k(\cdot, \hat{u}) - u_1(\cdot, \hat{u})\|_{C(J_T, V)} \leq \varepsilon_0. \quad (3.29)$$

In particular, taking $\varepsilon_0 = \hat{\varepsilon}$, where $\hat{\varepsilon}$ is the constant in (3.16), and defining the operator

$$\widehat{\Psi}_k : \mathcal{K} \to L^2(J_T, E) \times L^4(J_T, E), \quad \hat{u} \mapsto (\eta_k(\cdot, \hat{u}), \zeta_k(\cdot, \hat{u})),$$

we conclude that $\widehat{\Psi}_k(\hat{u}) \in \widehat{\Theta}_T(h, v_0)$ for $v_0 \in B_V(u_0, \varepsilon)$ and $k \geq k_0$, and

$$\sup_{\hat{u} \in \mathcal{K}} \|\widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}_k(\hat{u})) - u_1(T, \hat{u})\|_V \leq \hat{\varepsilon} \quad \text{for } k \geq k_0.$$

Combining this with (3.16), we obtain

$$\sup_{\hat{u} \in \mathcal{K}} \|\widehat{\mathcal{R}}_T(u_0, \widehat{\Psi}_k(\hat{u})) - \hat{u}\|_V < \varepsilon \quad \text{for } k \geq k_0.$$

Hence, Eq. (3.1) with $(\eta, \zeta) \in L^2(J_T, E) \times L^4(J_T, E)$ is uniformly ε-controllable.

Step 3. We now prove Lemma 3.5. Literal repetition of the arguments in [Shi06, Section 3.3] (see Step 2) shows that the required assertion will be established if we prove the convergence

$$\sup_{\hat{u} \in \mathcal{K}} \big(\|K_\nu f_k(\cdot, \hat{u})\|_{C(J_T, V)} + \|B(K_\nu f_k(\cdot, \hat{u}))\|_{L^2(J_T, H)}\big) \to 0 \quad \text{as } k \to \infty, \quad (3.30)$$

where

$$K_\nu f(t) = \int_0^t e^{-\nu(t-s)L} f(s)\, ds.$$

Furthermore, in view of the calculations of Steps 3–4 in [Shi06, Section 3.3], it suffices to show that

$$\sup_{\hat{u} \in \mathcal{K}} \|F_k(\cdot, \hat{u})\|_{C(J_T, H)} \to 0 \quad \text{as } k \to \infty, \quad (3.31)$$

where $F_k(t, \hat{u}) = \int_0^t f_k(s, \hat{u})\, ds$. To this end, we first note that[7]

$$\|F_k(\cdot, \hat{u})\|_{C(J_T, H)} \to 0 \quad \text{as } k \to \infty \text{ for any } \hat{u} \in \mathcal{K}. \quad (3.32)$$

Suppose now that we have proved the uniform equicontinuity of the family of mappings

$$\mathbf{f}_k : \mathcal{K} \to L^1(J_T, H), \quad \hat{u} \mapsto f_k(\cdot, \hat{u}). \quad (3.33)$$

[7] See Step 5 in [Shi06, Section 3.3].

In this case, the family $\left\{\hat{u} \mapsto \int_0^{\cdot} f_k(s,\hat{u})\,ds, k \geq 1\right\}$ is uniformly equicontinuous from $\mathcal{K}$ to $C(J_T, V)$. Combining this property with (3.32), we arrive at (3.31).

Step 4. We now show that (3.33) is uniformly equicontinuous. The explicit formulas (3.26) and (3.27) and standard estimates for the bilinear form B show that it suffices to prove that the function $\hat{u} \mapsto \zeta_k(\cdot, \hat{u})$ is uniformly equicontinuous from $\mathcal{K}$ to $L^4(J_T, U)$. It follows from (3.23) and (3.24) that

$$\begin{aligned}\|\zeta_k(\cdot,\hat{u}_1) - \zeta_k(\cdot,\hat{u}_2)\|_{L^4(J_T,U)}^4 &= \int_0^T \|\zeta(kt/T,\hat{u}_1) - \zeta(kt/T,\hat{u}_2)\|_U^4 dt \\ &= T\int_0^1 \|\zeta(s,\hat{u}_1) - \zeta(s,\hat{u}_2)\|_U^4 ds \\ &\leq C\sum_{i=1}^q |d_i(\hat{u}_1) - d_i(\hat{u}_2)|,\end{aligned}$$

where $\hat{u}_1, \hat{u}_2 \in \mathcal{K}$ are arbitrary points and $C > 0$ is a constant depending only on T, q, and $\max_i \|\zeta^i\|_U$. Since the functions d_i are uniformly continuous on the compact set $\mathcal{K}$, we obtain the required result. This completes the proof of Proposition 3.2 in the case $s = 1$.

Step 5. We now consider the case of any $s \geq 2$. Let us set $I_r = [t_r, t_{r+1}]$ and $\mathcal{X}^r = C(I_r, V) \cap L^2(I_r, U)$. For any $r = 0, \dots, s-1$, we denote by $\Theta^r(h, u_0)$ the set of functions $(\eta, \zeta) \in L^2(I_r, H) \times L^4(I_r, H^2)$ for which Eq. (3.1) has a unique solution $u \in \mathcal{X}^r$ satisfying the initial condition

$$u(t_r) = u_0. \tag{3.34}$$

Introduce the set

$$\mathcal{D}^r = \left\{(u_0, \eta, \zeta) \in V \times L^2(I_r, H) \times L^4(I_r, H^2) : (\eta, \zeta) \in \Theta^r(h, u_0)\right\}$$

and define an operator $S^r : \mathcal{D}^r \to V$ that takes each triple $(u_0, \eta, \zeta) \in \mathcal{D}^r$ to $u(t_{r+1})$, where $u \in \mathcal{X}^r$ is the solution of (3.1), (3.34). It follows from Proposition 1.2 that the operator S^r is locally Lipschitz continuous.

We now define positive constants β_r, $r = 0, \dots, s$, and continuous operators $\Psi^r : \mathcal{K} \to L^2(I_r, E) \times L^4(I_r, E)$, $r = 0, \dots, s-1$, by the following rule:

- set $\beta_s = \hat{\varepsilon}$, where $\hat{\varepsilon}$ is the constant in (3.16);
- if β_{r+1} is constructed for some $r \leq s-1$, then apply Lemma 3.6 with $\varepsilon_0 = \beta_{r+1}$ to the interval I_r and denote by δ_0 and k_0 the corresponding parameters;
- set $\beta_r = \delta_0$ and $\Psi^r = \widehat{\Psi}_{k_0}$.

The construction implies that, for any $v_0 \in B_V(u_1(t_r), \beta_r)$, $r = 0, \dots, s-1$, and $\hat{u} \in \mathcal{K}$, we have

$$\Psi^r(\hat{u}) \in \Theta^r(h, v_0), \quad \|S^r(v_0, \Psi^r(\hat{u})) - u_1(t_{r+1}, \hat{u})\|_V \leq \beta_{r+1}. \tag{3.35}$$

Let us define an operator $\widehat{\Psi}:\mathcal{K}\to L^2(J_T,E)\times L^4(J_T,E)$ as

$$\widehat{\Psi}(\hat{u})(t)=\Psi^r(\hat{u})(t)\quad\text{for } t\in I_r,\ r=0,\dots,s-1.$$

It follows from (3.35) that

$$\widehat{\Psi}(\hat{u})\in\Theta_T(h,u_0),\quad \|\mathcal{R}_T(u_0,\widehat{\Psi}(\hat{u}))-u_1(T,\hat{u})\|_V\le\beta_s=\hat{\varepsilon}\quad\text{for } \hat{u}\in\mathcal{K}.$$

Comparing this with (3.16), we obtain (2.5). It remains to note that since the functions Ψ^r are continuous, so is $\widehat{\Psi}$. This completes the proof of Proposition 3.2.

3.4 Proof of Proposition 3.3

Step 1. We first show that if the integer $N\ge 1$ is sufficiently large, then Eq. (2.1) with $\eta\in L^2(J_T,H_N)$ is uniformly ε-controllable. To this end, we fix a (small) constant $\delta>0$ and define a family of functions

$$v_N(t,\hat{u})=T^{-1}\mathsf{P}_N\big(te^{-\delta L}\hat{u}+(T-t)e^{-tL}u_0\big),\quad 0\le t\le T. \tag{3.36}$$

It is easy to see that

$$K_\delta:=\sup_{\hat{u},N}\|v_N(\cdot,\hat{u})\|_{\mathcal{X}_T}<\infty\quad\text{for any } \delta>0, \tag{3.37}$$

$$c_\delta:=\sup_{\hat{u},N}\|v_N(T,\hat{u})-\mathsf{P}_N\hat{u}\|_V\to 0\quad\text{as } \delta\to 0, \tag{3.38}$$

where the supremums are taken over $N\ge 1$ and $\hat{u}\in\mathcal{K}$. We now choose a constant $\delta>0$ so small that

$$c_\delta\le\frac{\varepsilon}{3}. \tag{3.39}$$

Consider the Cauchy problem

$$\dot{w}+\mathsf{Q}_N L(w+v_N)+\mathsf{Q}_N B(w+v_N)=\mathsf{Q}_N h(t),\quad w(0)=\mathsf{Q}_N u_0. \tag{3.40}$$

Proposition 1.3 and inequality (3.37) imply that there is an integer $N_\delta\ge 1$ not depending on $\hat{u}\in\mathcal{K}$ such that for any $N\ge N_\delta$ problem (3.40) has a unique solution $w_N(\cdot,\hat{u})\in\mathcal{X}_T(N_H)$. It follows that the function $u_N(t,\hat{u})=v_N+w_N$ belongs to $\mathcal{X}_T$ for any $N\ge N_\delta$ and satisfies Eq. (2.1) with

$$\eta(t)=\eta_N(t,\hat{u}):=\dot{v}_N+\mathsf{P}_N(Lu_N+B(u_N)-h). \tag{3.41}$$

The required assertion will be established if we prove the following two claims:

(a) For any $N\ge N_\delta$, the function $\Psi:\hat{u}\mapsto\eta_N(\cdot,\hat{u})$ is continuous from $\mathcal{K}$ to $L^2(J_T,H)$.

(b) We have

$$\sup_{\hat{u}\in\mathcal{K}}\|w_N(T,\hat{u})\|_V\to 0\quad\text{as } N\to\infty. \tag{3.42}$$

Indeed, the very construction of Ψ implies that

$$\Psi(\hat{u}) \in \Theta_T(h, u_0), \quad \mathcal{R}(u_0, \Psi(\hat{u})) = u_N.$$

Furthermore, it follows from (3.38), (3.39), and (3.42) that if $N \geq N_\delta$ is sufficiently large, then

$$\begin{aligned} \sup_{\hat{u} \in \mathcal{K}} \|\mathcal{R}_T(u_0, \Psi_N(\hat{u})) - \hat{u}\|_V &\leq c_\delta + \sup_{\hat{u} \in \mathcal{K}} \bigl(\|w_N(T, \hat{u})\|_V + \|\mathsf{Q}_N \hat{u}\|_V\bigr) \\ &\leq \frac{2\varepsilon}{3} + \sup_{\hat{u} \in \mathcal{K}} \|\mathsf{Q}_N \hat{u}\|_V. \end{aligned} \tag{3.43}$$

Since $\mathcal{K} \subset V$ is compact, the second term on the right-hand side of (3.43) can be made smaller than $\frac{\varepsilon}{3}$ by choosing a sufficiently large $N \geq N_\delta$.

Step 2. Let us prove (a) and (b). Since $\delta > 0$, it follows from (3.36) that the function $\hat{u} \mapsto v_N(\cdot, \hat{u})$ is continuous from $\mathcal{K}$ to $\mathcal{X}_T$. By Proposition 1.3, the solution $w_N \in \mathcal{X}_T(H_N)$ of problem (3.40) continuously depends on $v_N \in \mathcal{X}_T$. The continuity of Ψ follows now from (3.41) and (3.36).

The proof of (b) literally repeats the argument used in [Shi06] (see the proof of (2.12)), and therefore we omit it.

Step 3. We now show that if $k \geq 1$ is sufficiently large, then Eq. (2.1) with $\eta \in C^\infty(J_T, E_k)$ is uniformly ε-controllable. To this end, we use Lemma 3.4.

Let us denote by $Y \subset L^2(J_T, H)$ the union of the vector spaces $C^\infty(J_T, E_k)$, $k \geq 1$. Since E_∞ is dense in H, we conclude that $L^2(J_T, H_N)$ is contained in the closure of Y for any $N \geq 1$. By Lemma 3.4, there is a finite-dimensional subspace $Y_0 \subset Y$ such that Eq. (2.1) with $\eta \in Y_0$ is uniformly ε-controllable. Since $\{C^\infty(J_T, E_k)\}_{k \geq 1}$ is an increasing sequence of subspaces, we see that $Y_0 \subset C^\infty(J_T, E_k)$ for a sufficiently large $k \geq 1$. This completes the proof of Proposition 3.3.

4 Appendix: proof of Lemma 3.5

Let d be the dimension of E_1 and let $\mathcal{E} = \{e_1, \dots, e_d\}$ be a basis in E_1. We endow E_1 with a scalar product $(\cdot, \cdot)$ for which $\mathcal{E}$ is an orthonormal system. Let $\Psi_1 : \mathcal{K} \to C^\infty(J_T, E_1)$ be a continuous operator satisfying (3.16). In view of Lemma 3.4 (in which $X = Y = C^\infty(J_T, E_1)$), we can assume without loss of generality that $\Psi_1(\hat{u}) \in Y_0$ for any $\hat{u} \in \mathcal{K}$, where $Y_0 \subset C^\infty(J_T, E_1)$ is a finite-dimensional subspace. Let us set $\eta_1(\cdot, \hat{u}) = \Psi_1(\hat{u})$ and write

$$\eta_1(t, \hat{u}) = \sum_{l=1}^{d} \zeta_l(t, \hat{u}) e_l, \tag{4.1}$$

where $\zeta_l(t, \hat{u}) = (\eta_l(t, \hat{u}), e_l)$. Since all the norms in the finite-dimensional space Y_0 are equivalent, what has been said implies that $\zeta_l \in C(J_T \times \mathcal{K})$ for $l = 1, \dots, d$. Let

$$M = \max_{l,t,\hat{u}} |\zeta_l(t, \hat{u})|,$$

where the maximum is taken over $l = 1, \dots, d$ and $(t, \hat{u}) \in J_T \times \mathcal{K}$. We now set $m = 2d$,

$$\eta_1^l = dMe_l \quad \text{for } l = 1, \dots, d, \qquad \eta_1^l = -dMe_l \quad \text{for } l = d+1, \dots, m.$$

In this case, we can rewrite (4.1) in the form

$$\eta_1(t, \hat{u}) = \sum_{l=1}^{m} \tilde{\zeta}_l(t, \hat{u}) \eta_1^l,$$

where $\tilde{\zeta}_l \in C(J_T \times \mathcal{K})$, $l = 1, \dots, m$, are non-negative functions whose sum is equal to 1.

For any integer $s \geq 1$, let us set

$$\varPsi_1^s(\hat{u}) = \sum_{l=1}^{m} \psi_{ls}(t, \hat{u}) \eta_1^l,$$

where $\psi_{ls}(t, \hat{u}) = \tilde{\zeta}_l(rT/s, \hat{u})$ for $t \in J_T(r, s)$. It is clear that $\varPsi_1^s(\cdot)$ is a continuous function from $\mathcal{K}$ to $P_s(J_T, A)$, where $A = \{\eta_1^l, l = 1, \dots, m\}$. Furthermore, since $\mathcal{K} \subset V$ is compact, it is not difficult to show that

$$\sup_{\hat{u} \in \mathcal{K}} \|\varPsi_1^s(\hat{u}) - \varPsi_1(\hat{u})\|_{L^2(J_T, H)} \to 0 \quad \text{as } s \to \infty.$$

Proposition 1.2 now implies that $\varPsi_1^s(\hat{u}) \in \Theta_T(h, u_0)$ for any $\hat{u} \in \mathcal{K}$ and sufficiently large s, and we have

$$\sup_{\hat{u} \in \mathcal{K}} \|\mathcal{R}_T(u_0, \varPsi_1^s(\hat{u})) - \mathcal{R}_T(u_0, \varPsi_1(\hat{u}))\|_V \to 0 \quad \text{as } s \to \infty.$$

Combining this with (3.16), we conclude that Eq. (3.2) with $\eta_1 \in P_s(J_T, A)$ is uniformly ε-controllable.

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