FLAT LEVEL SET REGULARITY OF p-LAPLACE PHASE TRANSITIONS

ENRICO VALDINOCI, BERARDINO SCIUNZI AND VASILE OVIDIU SAVIN

Abstract. We prove a Harnack inequality for level sets of p-Laplace phase transition minimizers. In particular, if a level set is included in a flat cylinder, then, in the interior, it is included in a flatter one. The extension of a result conjectured by De Giorgi and recently proven by the third author for $p = 2$ follows.

Keywords: Ginzburg-Landau-Allen-Cahn phase transition models, De Giorgi conjecture, p-Laplacian operator, sliding methods, geometric and qualitative properties of solutions.

Subject Classification: 35J70, 35B65.

Contents

We thank Xavier Cabré for an interesting discussion. EV would like to thank also Alberto Farina for first mentioning the conjecture of De Giorgi in a pleasant meeting at the Erwin Schroedinger Institute (Vienna, 2002). EV and BS have been supported by MIUR *Variational Methods and Nonlinear Differential Equations*.

1. Introduction

Given a domain $\Omega \subseteq \mathbb{R}^N$, we define the following functional on $W^{1,p}(\Omega)$:

$$\mathcal{F}_\Omega(u) = \int_\Omega \frac{|\nabla u(x)|^p}{p} + h_0(u(x))\, dx\,.$$

Here above and in the sequel, we suppose that $1 < p < \infty$ and that $h_0 \in C^0([-1,1]) \cap C^{1,1}((-1,1))$ can be extended to a function which is C^1 in a neighborhood of $[-1,1]$. We will also assume that, for some $0 < c < 1 < C$ and some $\theta^\star \in (0,1)$, we have

$$\begin{aligned} &\text{for any } \theta \in [0,1],\ c\,\theta^p \le h_0(-1+\theta) \le C\,\theta^p \text{ and} \\ &\qquad c\,\theta^p \le h_0(1-\theta) \le C\,\theta^p, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\text{for any } \theta \in [0,\theta^*),\ h_0'(-1+\theta) \ge c\theta^{p-1} \text{ and} \\ &\qquad h_0'(1-\theta) \le -c\theta^{p-1}. \end{aligned} \tag{1.2}$$

We also assume that h_0' is monotone increasing in $(-1,-1+\theta^\star) \cup (1-\theta^\star,1)$. Quantities depending only on the constants above will be referred to as "universal constants". As a model example for a potential h_0 satisfying the conditions stated here above, one may consider

$$h_0(\zeta) := (1-\zeta^2)^p\,.$$

In the literature, h_0 is often referred to as a "double-well" potential, and its derivative as a "bi-stable nonlinearity".

In the light of (1.1) and (1.2), we have that, with no loss of generality, possibly reducing the size of $\theta^\star$, we may and do assume that

$$\begin{aligned} &\text{for any } \zeta \in [-1+\theta^\star, 1-\theta^\star], \\ &h_0(\zeta) \ge \max\nolimits_{[-1,-1+\theta^\star]\cup[1-\theta^\star,1]} h_0, \end{aligned} \tag{1.3}$$

Notice that, if $u \in W^{1,p}(\Omega)$, $|u| \leq 1$, is[1] critical for $\mathcal{F}_\Omega$, then u satisfies in the weak sense the following singular/degenrate elliptic equation of p-Laplacian type:

$$\Delta_p u(x) = h_0'(u(x)), \tag{1.5}$$

for any $x \in \Omega$. Here and in what follows, we make use of the standard notation

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

In particular, we will consider local minimizers for the functional above. We say that u is a local minimizer for $\mathcal{F}$ in the domain Ω if

$$\mathcal{F}_\Omega(u) \leq \mathcal{F}_\Omega(u+\phi),$$

for any $\phi \in C_0^\infty(\Omega)$. In the literature, it is also customary to say that u is a Class A minimizer for $\mathcal{F}$ if

$$\mathcal{F}_K(u) \leq \mathcal{F}_K(u+\phi),$$

for any compact set $K \subset \mathbb{R}^N$ and any $\phi \in C_0^\infty(K)$. That is, u is a Class A minimizer if it is a local minimizer in any domain.

The functional $\mathcal{F}$ here above has been widely studied both for pure mathematical reasons and for physical applications. For instance, this functional is a model for interfaces appearing in physical problems when two phases (the phase "close to $+1$" and the one "close to -1") coexist. On one side, the "potential" h_0 tends to drive the minima of the functional towards the "pure states" ± 1; on the other hand, the "kinetic term" $|\nabla u|^p$ prevents the system from sudden phase changes. The balance between these tendencies (or, in the physical language, the effect of the surface tension) leads interfaces of minimal solution to minimize area. The physical relevance of the interfaces and the mathematical interest arising from geometric measure theory thus motivated an extensive study of the transition layers, i.e., of the level sets of solutions of (1.5). We refer to [3], [7], [14], [23], [25], [26], [33], [27], [28], [29], [27], [28], [30] and [36] for more detailed discussions on the physical relevance of the above functional and for its relation with the theory of minimal surfaces.

The main result that will be proved in this paper is the following Harnack inequality for level sets of minimizers. Roughly speaking, such results says that, once one knows that the zero level set of a minimizer is trapped in a rectangle whose height is small enough, then, in a smaller neighborhood, it

[1] Many results of this paper are obtained without assuming $|u| < 1$, but assuming only $|u| \leq 1$. For future use, however, we recall that the condition $|u| < 1$ is fulfilled by any solution u such that $|u| \leq 1$ with $|u|$ not identically equal to ± 1, under suitable assumptions on h_0. This holds, for instance, if we suppose that

$$h_0'(-1+\theta) \leq c'\theta^{p-1} \qquad \text{and} \qquad h_0'(1-\theta) \geq -c'\theta^{p-1}, \tag{1.4}$$

for any $\theta \in [0, \theta^*)$. For the proof of this observation, see, e.g., footnote 7 in [30]. The case $p = 2$ was also dealt with in Theorem 1.1 of [17].

can be trapped in a rectangle with even smaller height. More precisely, we have the following result:

Theorem 1.1. *Let $l > 0$, $\theta > 0$. Let u be a local minimizer for $\mathcal{F}$ in*

$$\Big\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \, \Big| \, |x'| < l \, , \; |x_N| < l \Big\} .$$

Assume that $u(0) = 0$ and that

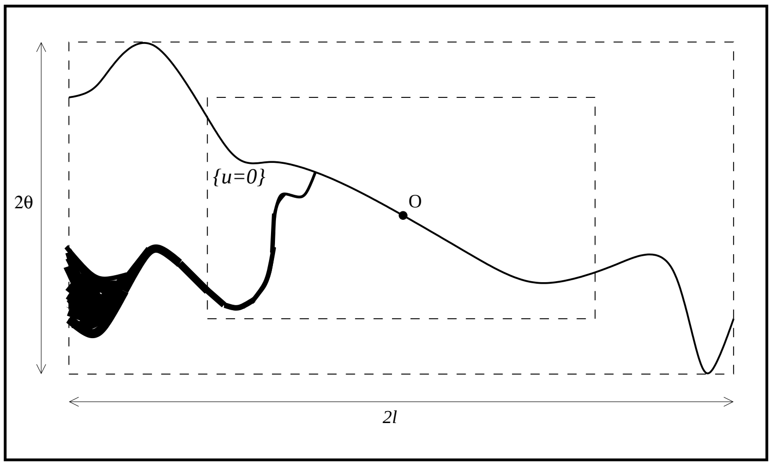
$$\{u = 0\} \subseteq \{|x_N| < \theta\} .$$

Then, there exists a universal constant $c \in (0, 1)$ so that, for any $\theta_0 > 0$ there exists $\varepsilon_0(\theta_0) > 0$ such that, if

$$\frac{\theta}{l} \le \varepsilon_0(\theta_0) \qquad \textit{and} \qquad \theta \ge \theta_0 \, ,$$

then

$$\{u = 0\} \cap \{|x'| < cl\} \subseteq \{|x_N| < (1 - c)\, \theta\} .$$



The Harnack-type result of Theorem 1.1

Theorem 1.1 is an extension of a similar result obtained in [31] for $p = 2$. Also, some results from [7], [28] and [30] will be needed in the course of the proof.

The proof of Theorem 1.1 is quite long, both because we will need some fine analysis on the measure estimates of the touching points between u and some appropriate barriers, and because some delicate details and technical points will appear in the course of the proof. Very roughly, we can say that the final target of the proof consists in deducing a measure estimates on the above mentioned contact points, which, in case the statement of Theorem 1.1 were false, would contradict the minimality of u. Such estimates will be obtained by sliding suitable barriers, constructed via the one-dimensional heteroclinic solution.

The ideas of such proof become more transparent in the easier case of a uniformly elliptic equation involving principal curvatures (of the type $\sum_{i=1}^N a_i \kappa_i = 0$): see [32].

Following the ideas of [31], several results may be deduced from Theorem 1.1. First, we deduce the following "flatness improvement" result, stating that, once a level set is trapped inside a flat cylinder, then, possibly changing coordinates, it is trapped in an even flatter cylinder in the interior. More precisely, we have:

Theorem 1.2. *Let $l > 0$, $\theta > 0$. Let u be a local minimizer for $\mathcal{F}$ in*

$$\left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,\middle|\, |x'| < l\,,\ |x_N| < l \right\}.$$

Assume that $u(0) = 0$ and that

$$\{u = 0\} \subseteq \{|x_N| < \theta\}\,.$$

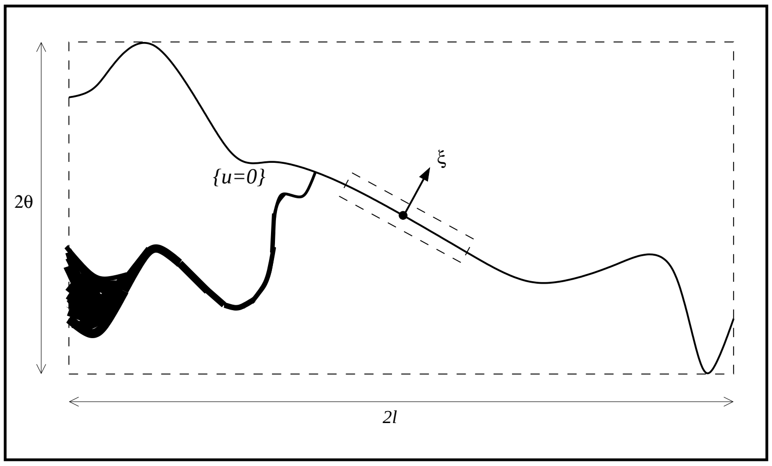
Then, there exist universal constants $\eta_1, \eta_2 > 0$, with $0 < \eta_1 < \eta_2 < 1$, such that, for any $\theta_0 > 0$, there exists $\varepsilon_1(\theta_0) > 0$ such that, if

$$\frac{\theta}{l} \leq \varepsilon_1(\theta_0) \qquad \text{and} \qquad \theta \geq \theta_0\,,$$

then

$$\begin{aligned} &\{u = 0\} \cap \big(\{|\pi_\xi\, x| < \eta_2 l\} \times \{|(x \cdot \xi)| < \eta_2 l\}\big) \subseteq \\ \subseteq\ &\Big(\{|\pi_\xi x| < \eta_2 l\} \times \{|(x \cdot \xi)| < \eta_1 \theta\}\Big) \end{aligned}$$

for some unit vector ξ.



The flatness improvement of Theorem 1.2

Several ideas related with Theorem 1.2 have been extensively used by De Giorgi and his school in the minimal surface setting (e.g., for proving smoothness and analytic regularity): see, for instance, chapters 6–8 in [20].

The extension of a result conjectured by De Giorgi in [14] for $p = 2$ also follows, namely we have the following two flatness results:

Theorem 1.3. *Let $N \le 7$. Then, level sets of Class A minimizers of $\mathcal{F}$ are hyperplanes.*

Theorem 1.4. *Let $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ be a solution of (1.5). Let h_0 fulfill the assumptions on page 2 and*[2] *(1.4). Assume that $|u| \le 1$, $\partial_N u > 0$ and*

$$\lim_{x_N \to \pm\infty} u(\cdot, x_N) = \pm 1\,.$$

Assume also that either $N \le 8$ or that $\{u = 0\}$ has at most linear growth at infinity. Then, level sets of u are hyperplanes.

Results of these type have been proved in [21] for $p = N = 2$ and [2] for $p = 2$ and $N = 3$ (and actually for any nonlinearity, see [1]). Extensions of the results in [21] and [2] to p-Laplace equations have been considered in [11]. See also [5, 13, 12, 16, 4, 6, 22] for related results. Results analogous to Theorems 1.3 and 1.4 for $p = 2$ have been recently given by the third author in [31]. In §9 here below, we will see that Theorem 1.4 is a consequence of the fact that monotone solutions of (1.5) are minimizers (see, e.g., [24] or [31]) under a ± 1-limit assumption.

This paper is organized in the following way. In §2 we construct the barriers to be used in the course of the proofs of the main results. Roughly speaking, such barriers are obtained by modifying the heteroclinic one-dimensional solutions given by the potential h_0 and by taking flat or rotational extensions. The study of the touching points between these barriers and our solution occupies §3. Particular emphasis is given to the measure of the projection of the set of "first time" touches. In §4, some covering lemmas are presented, whose proof has been deferred to the Appendix. The results of §3 and §4 are then used in §5 to obtain an estimate on the projection of the touching points between an appropriate barrier and our solution. The proofs of the main results occupy §6—§9. The Appendix contains the proof of the covering lemmas and some elementary ancillary results.

2. Modifications of the potential and of one-dimensional solutions

We now construct some barriers, which will be of use in the proof of the main results. Such barriers will be obtained by appropriate modifications on the potential h_0, which induce corresponding modifications on one-dimensional solutions.

Here and below, we fix $\overline{C}_0 > 0$, to be conveniently chosen in the following (actually, during the proof of Proposition 2.13 here below). We will also

[2] In particular, the result of Theorem 1.4 holds for $h_0(\zeta) := (1 - \zeta^2)^p$.

fix R, to be assumed suitably large (with respect to $\overline{C}_0$ and some universal constants). The first function needed in our construction is the following modification of the potential h_0 in the interval [-3/4,3/4]:

Definition 2.1. Fix $|s_0| \le 1/4$. For any $|s| \le 3/4$, we define[3]

$$\varphi_{s_0,R}(s) = \varphi(s) := \frac{h_0(s)R^p}{\left[R - \overline{C}_0(s-s_0)(\frac{p}{p-1}h_0(s))^{\frac{1}{p}}\right]^p}. \tag{2.1}$$

Note that, by construction,

$$\frac{1}{\left(\frac{p}{p-1}\varphi(s)\right)^{\frac{1}{p}}} = \frac{1}{\left(\frac{p}{p-1}h_0(s)\right)^{\frac{1}{p}}} - \frac{\overline{C}_0}{R}(s-s_0). \tag{2.2}$$

Roughly speaking, for large R, φ is close to h_0: this is the reason for which we consider φ as a modified potential in $[-3/4, 3/4]$. We now consider some properties enjoyed by φ. First of all, we estimate φ in terms of h_0 in $[-3/4,-1/2] \cup [1/2,3/4]$:

Lemma 2.2. *The following inequalities hold:*

$$\varphi(s) < h_0(s) - \frac{2\widehat{C}_0}{R} \qquad \textit{if} \quad s \in \left[-\frac{3}{4}, -\frac{1}{2}\right] \tag{2.3}$$

and

$$\varphi(s) > h_0(s) + \frac{2\widehat{C}_0}{R} \qquad \textit{if} \quad s \in \left[\frac{1}{2}, \frac{3}{4}\right], \tag{2.4}$$

provided R and $\overline{C}_0/\widehat{C}_0$ are suitably large. Also, $\widehat{C}_0$ may be taken large if so is $\overline{C}_0$.

Proof. To prove (2.3), note that for $s \in [-\frac{3}{4}, -\frac{1}{2}]$ we have $\overline{C}_0(s-s_0) \in [-\overline{C}_0, -\frac{\overline{C}_0}{4}]$. Also, from (1.1), there exists $k > 0$ such that

$$0 < k \le \inf_{\sigma \in [-3/4,-1/2] \cup [1/2,3/4]} \left(\frac{p}{p-1}h_0(\sigma)\right)^{\frac{1}{p}}. \tag{2.5}$$

[3]Notice that $R - \overline{C}_0(s-s_0)(\frac{p}{p-1}h_0(s))^{\frac{1}{p}} > 0$ for any $|s| \le 3/4$ and $|s_0| \le 1/4$, if R is large enough, thus the definition of φ is well posed.

Therefore,

$$\begin{aligned}
\varphi(s) &= \frac{h_0(s)R^p}{\Big[R-\overline{C}_0(s-s_0)\big(\frac{p}{p-1}h_0(s)\big)^{\frac{1}{p}}\Big]^p} \le \\
&\le \frac{h_0(s)R^p}{(R+\frac{\overline{C}_0}{4}k)^p} = \\
&= h_0(s) + \frac{h_0(s)R^p - h_0(s)(R+\frac{\overline{C}_0}{4}k)^p}{(R+\frac{\overline{C}_0}{4}k)^p} \le \\
&\le h_0(s) - \frac{\mathrm{const}}{R}\cdot\frac{(R+\mathrm{const}\,\overline{C}_0)^p - R^p}{R^{p-1}}\,.
\end{aligned}$$

Using the fact that

$$\lim_{x\to+\infty}\frac{(x+a)^p-x^p}{x^{p-1}} = pa\,,$$

and taking R suitably large, we get

$$\varphi(s) \le h_0(s) - \frac{\mathrm{const}\,\overline{C}_0}{R} \le h_0(s) - \frac{2\widehat{C}_0}{R}\,,$$

for $\overline{C}_0 >$ const $\widehat{C}_0$, proving (2.3). Let us now prove (2.4). Recalling (2.5) and arguing as above, for $s\in[1/2,3/4]$, we get that $\overline{C}_0(s-s_0)\in[\frac{\overline{C}_0}{4},\overline{C}_0]$ and so

$$\begin{aligned}
\varphi(s) &= \frac{h_0(s)R^p}{\Big[R-\overline{C}_0(s-s_0)\big(\frac{p}{p-1}h_0(s)\big)^{\frac{1}{p}}\Big]^p} \ge \\
&\ge \frac{h_0(s)R^p}{(R-\frac{\overline{C}_0}{4}k)^p} \ge \\
&\ge h_0(s) + \frac{\mathrm{const}}{R}\cdot\frac{R^p-(R-\mathrm{const}\,\overline{C}_0)^p}{R^{p-1}}\,.
\end{aligned}$$

Using the fact that

$$\lim_{x\to+\infty}\frac{x^p-(x-a)^p}{x^{p-1}} = pa\,,$$

we thus gather that, for R large,

$$\varphi(s) \ge h_0(s) + \frac{\mathrm{const}\,\overline{C}_0}{R} \ge h_0(s) + \frac{2\widehat{C}_0}{R}\,,$$

for $\overline{C}_0 >$ const $\widehat{C}_0$, proving (2.4). □

Now, for R large enough, we define $s_R\in(-1,\theta^*)$ as the point such that

$$h_0(s_R) = \frac{1}{R}\,. \tag{2.6}$$

From (1.1), we have that

$$c(1+s_R) \le \frac{1}{R^{1/p}} \le C(1+s_R)\,. \tag{2.7}$$

For further estimates, in the next two lemmas, we now point out some elementary bounds for s_R:

Lemma 2.3. *Let C a positive universal constant. Then, for R large (with respect to C), we have that*

$$(1+\xi)^p - (1+s_R)^p - \frac{C}{R}(\xi - s_R) > 0 \qquad \text{if} \quad s_R < \xi < 0 \tag{2.8}$$

and

$$\int_{s_R}^{0} \frac{d\xi}{\Big((1+\xi)^p - (1+s_R)^p - \frac{C}{R}(\xi - s_R)\Big)^{1/p}} \le \widetilde{C} \log R\,, \tag{2.9}$$

for a suitable constant $\widetilde{C} > 0$.

Proof. For $\xi \in [s_R, 0]$, let

$$g(\xi) := (1+\xi)^p - \frac{C}{R}(1+\xi)\,.$$

Then, by (2.7),

$$\begin{aligned} g'(\xi) &= p(1+\xi)^{p-1} - \frac{C}{R} \ge \\ &\ge p(1+s_R)^{p-1} - \frac{C}{R} \ge \\ &\ge \frac{\text{const}}{R^{(p-1)/p}} - \frac{C}{R} > 0\,, \end{aligned}$$

if R is large enough, thence $g(\xi) > g(s_R)$, proving (2.8).

In order to estimate the integral in (2.9), we introduce the notation $b = 1 + s_R$. Since, for R large, s_R is near -1, we may assume $\frac{1}{b} > 2$. We use the substitution

$$\tau = \frac{1+\xi}{b}$$

obtaining a bound for the above integral given by

$$\begin{aligned} &\int_1^{1/b} \frac{d\tau}{\big(\tau^p - 1 - \frac{C}{Rb^p} b(\tau-1)\big)^{1/p}} \le \int_1^{1/b} \frac{d\tau}{\big(\tau^p - 1 - C'b(\tau-1)\big)^{1/p}} \le \\ &\le \int_1^{2} \frac{d\tau}{\big(\tau^p - 1 - C'b(\tau-1)\big)^{1/p}} + \int_2^{1/b} \frac{d\tau}{\big(\tau^p - 1 - C'b(\tau-1)\big)^{1/p}}\,. \end{aligned}$$

where we used the fact that $b^p \approx \frac{1}{R}$ (see (2.7)).

Noticing that

$$\tau^p - 1 \ge \frac{2^p - 1}{2^p}\,\tau^p$$

if $\tau \ge 2$, and that

$$\tau^p - 1 \ge \tau - 1$$

if $\tau \geq 1$, we bound the quantity here above by

$$\text{const}\left(\int_1^2 \frac{d\tau}{((1-C'b)(\tau-1))^{1/p}} + \int_2^{1/b} \frac{d\tau}{\left(\tau^p(1-\frac{C'b(\tau-1)}{\tau^p})\right)^{\frac{1}{p}}}\right).$$

Now, if R is large, then b is small, and therefore we may assume that $(1-C'b)^{\frac{1}{p}} > \frac{1}{2}$. Thus,

$$\begin{aligned} \left(1-\frac{C'b(\tau-1)}{\tau^p}\right)^{\frac{1}{p}} &\geq \left(1-\frac{C'b}{\tau^{p-1}}\right)^{\frac{1}{p}} \geq \\ \geq\ (1-C'b)^{\frac{1}{p}} &> \frac{1}{2}. \end{aligned}$$

This yields

$$\int_{s_R}^0 \frac{d\xi}{\left((1+\xi)^p-(1+s_R)^p-\frac{C}{R}(\xi-s_R)\right)^{1/p}} \leq \text{const}\left(1+\log\frac{1}{2b}\right),$$

which proves (2.9). □

Let us now estimate how s_R varies as a function of R:

Lemma 2.4. *There exists a suitable universal constant $C>0$ so that*

$$-\frac{C}{R^{(p+1)/p}} \leq \partial_R s_R < 0.$$

Proof. Differentiating (2.6),

$$-\frac{1}{R^2} = h_0'(s_R)\,\partial_R s_R,$$

thus $\partial_R s_R < 0$ thanks to (1.2), and so, by (2.7),

$$\begin{aligned} \frac{1}{R^2} &= h_0'(s_R)\,|\partial_R s_R| \geq \\ &\geq \text{const}\,(1+s_R)^{p-1}\,|\partial_R s_R| \geq \\ &\geq \frac{\text{const}}{R^{(p-1)/p}}\,|\partial_R s_R|. \end{aligned}$$

□

We now define a modification of the potential h_0 in the whole interval $[-1,1]$ in the following way:

Definition 2.5. We define $h_{s_0,R} : [s_R, 1] \to \mathbb{R}$ by

$$h_{s_0,R}(s) := \begin{cases} h_0(s) - h_0(s_R) - \frac{\widehat{C}_0}{R}(s - s_R) \\ \qquad \text{if} \quad s_R \le s \le -\frac{1}{2} \\ \\ \varphi(s) \\ \qquad \text{if} \quad -\frac{1}{2} < s < \frac{1}{2} \\ \\ h_0(s) + h_0(s_R) + \frac{\widehat{C}_0}{R}(1 - s) \\ \qquad \text{if} \quad \frac{1}{2} \le s \le 1. \end{cases}$$

Notice that $h_{s_0,R}$ may be discontinuous at $s = \pm 1/2$. Let us now point out some easy properties enjoyed by the above potential:

Lemma 2.6. *The following inequalities hold. If $s \in [-3/4, -1/2]$, then*

$$h_{s_0,R}(s) > h_0(s) - \frac{2\widehat{C}_0}{R} > \varphi(s) .$$

If $s \in [1/2, 3/4]$, then

$$h_{s_0,R}(s) < h_0(s) + \frac{2\widehat{C}_0}{R} < \varphi(s) .$$

Proof. Let $s \in [-3/4, -1/2]$. If $\widehat{C}_0$ is taken suitably large, then, recalling Lemma 2.2, we have that

$$\begin{aligned} h_{s_0,R}(s) &= h_0(s) - \frac{1}{R} - \frac{\widehat{C}_0}{R}(s - s_R) \ge \\ &\ge h_0(s) - \frac{2\widehat{C}_0}{R} > \varphi(s) \end{aligned}$$

where we have used the fact that $s - s_R \le 1$.

In the same way, if $s \in [1/2, 3/4]$, taking $\widehat{C}_0$ suitably large, and using Lemma 2.2, we gather that

$$\begin{aligned} h_{s_0,R} &< h_0(s) + \frac{1}{R} + \frac{\widehat{C}_0}{2R} < \\ &< h_0(s) + \frac{2\widehat{C}_0}{R} < \varphi(s) . \end{aligned}$$

□

Lemma 2.7. *Let $R_1 \le R_2$ be suitably large. Then,*[4]

$$h_{s_0,R_1}(s) \le h_{s_0,R_2}(s) \qquad \textit{if} \quad s_{R_1} \le s \le s_0 \tag{2.10}$$

[4]Notice that, for $R_1 \le R_2$ suitably large, using (1.2), (2.6) and (2.7), one has

$$h_0(s_{R_1}) = \frac{1}{R_1} \ge \frac{1}{R_2} = h_0(s_{R_2}) ,$$

so that $s_{R_1} \ge s_{R_2}$.

and

$$h_{s_0,R_1}(s) \geq h_{s_0,R_2}(s) \qquad \textit{if} \quad s_0 \leq s \leq 1\,. \tag{2.11}$$

Proof. Let us prove (2.10). For this purpose, let $s \leq s_0$. Two cases are possible: either $s > -1/2$ or $s \leq -1/2$. Let us first deal with the first case. Notice that, by (2.1), $\varphi_{s_0,R}$ is increasing in R, since $-1/2 < s \leq s_0 \leq 1/4 < 1/2$, thus, from Definition 2.5 we gather that

$$\begin{aligned} h_{s_0,R_1}(s) &= \varphi_{s_0,R_1}(s) \leq \\ \leq \varphi_{s_0,R_2}(s) &= h_{s_0,R_2}(s)\,, \end{aligned}$$

which proves (2.10) if $s > -1/2$.

Let us now deal with the case $s \leq -1/2$. Fixed R_1 large and $s \in [s_R,\,-1/2]$, let us define

$$g(R) := \frac{1}{R} + \frac{\widehat{C}_0}{R}\,(s - s_R)\,.$$

By means of Lemma 2.4,

$$g'(R) \leq -\frac{1}{R^2} + \frac{\mathrm{const}\,\widehat{C}_0}{R^{2+1/p}} < 0\,,$$

thence

$$\begin{aligned} \frac{1}{R_2} + \frac{\widehat{C}_0}{R_2}\,(s - s_{R_2}) &= g(R_2) \leq \\ \leq g(R_1) &= \frac{1}{R_1} + \frac{\widehat{C}_0}{R_1}\,(s - s_{R_1})\,. \end{aligned}$$

Therefore, if $s \leq -1/2$, from Definition 2.5,

$$\begin{aligned} h_{s_0,R_1}(s) &= h_0(s) - \frac{1}{R_1} - \frac{\widehat{C}_0}{R_1}\,(s - s_{R_1}) \leq \\ &\leq h_0(s) - \frac{1}{R_2} - \frac{\widehat{C}_0}{R_2}\,(s - s_{R_2}) = \\ &= h_{s_0,R_2}(s)\,, \end{aligned}$$

thus proving (2.10).

Having completed the proof of (2.10), we now deal with the proof of (2.11). Two cases are possible: either $s < 1/2$ or $s \geq 1/2$. Let us first deal with the first case. Notice that, by (2.1) and the fact that $s_0 \leq s < 1/2$, $\varphi_{s_0,R}$ is decreasing in R. Hence, from Definition 2.5,

$$\begin{aligned} h_{s_0,R_1}(s) &= \varphi_{s_0,R_1}(s) \geq \\ \geq \varphi_{s_0,R_2}(s) &= h_{s_0,R_2}(s)\,, \end{aligned}$$

which proves (2.11) if $s < 1/2$.

Let us now deal with the case $s \geq 1/2$. In this case, by Definition 2.5,

$$h_{s_0,R_1}(s) - h_{s_0,R_2}(s) \;=\; \frac{1}{R_1} - \frac{1}{R_2} + \widehat{C}_0(1-s)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \geq 0\,,$$

proving (2.11) for $s \geq 1/2$ and thus completing the proof of Lemma 2.7. □

Let now

$$H_0(s) \;:=\; \int_0^s \frac{(p-1)^{\frac{1}{p}}}{(p\,h_0(\zeta))^{\frac{1}{p}}}\,d\zeta\,, \qquad \text{for any } s \in (-1,1). \tag{2.12}$$

Notice that the inverse of H_0 is a "one-dimensional" solution of (1.5). Indeed, if $g_0 := H_0^{-1}$, by Lemma B.3, we obtain that

$$\begin{aligned} \Delta_p g_0 &= (|g'|^{p-2}g')' = \\ &= (p-1)|g'|^{p-2}g'' = \\ &= h_0'(g_0)\,. \end{aligned}$$

We would like now to compare g_0 with all other solutions.

To this aim, using Definition 2.5 we now introduce suitable modifications of H_0 (and thus of g_0), which will be used in the course of the proof:

Definition 2.8. We define, for $s \in [s_R, 1]$,

$$H_{s_0,R}(s) \;:=\; H_0(s_0) + \int_{s_0}^s \frac{(p-1)^{\frac{1}{p}}}{(p\,h_{s_0,R}(\zeta))^{\frac{1}{p}}}\,d\zeta\,.$$

We notice that, by Lemmas B.1 and 2.3, we get

$$h_{s_0,R} > 0\,, \tag{2.13}$$

thence $H_{s_0,R}$ is well defined.

Remark 2.9. By Definition 2.8, exploiting Lemma 2.7, it follows that, if $R_1 \leq R_2$,

$$H_{s_0,R_1}(s) \leq H_{s_0,R_2}(s) \qquad \text{if} \quad s_{R_1} \leq s \leq 1\,. \tag{2.14}$$

Let us now analyze some properties of the above defined quantity:

Lemma 2.10. *Assuming R suitably large, there exists a positive constant C_1 so that the following inequalities hold:*

$$H_{s_0,R}(s_R) \geq -C_1 \log R\,, \tag{2.15}$$

$$H_{s_0,R}(1) \leq C_1 \log R\,, \tag{2.16}$$

$$\frac{d}{ds}\left(H_{s_0,R}(s)\right) > 0\,, \qquad \forall s \in (s_R, 1)\,. \tag{2.17}$$

Proof. Note that, if R is suitably big, Definition 2.1 implies that

$$\inf_{|s| \leq 1/2} \varphi(s) > 0\,. \tag{2.18}$$

Hence, recalling Lemma B.1 and (2.7), we have that

$$\begin{aligned}
H_{s_0,R}(s_R) \;&\geq\; H_0(s_0) - \int_{s_R}^{-\frac{1}{2}} \frac{\mathrm{const}\, d\zeta}{\Big((1+\zeta)^p - (1+s_R)^p - \frac{C}{R}(\zeta - s_R)\Big)^{1/p}} - \\
&\quad - \int_{-\frac{1}{2}}^{s_0} \frac{(p-1)^{\frac{1}{p}}}{(p\,\varphi(\zeta))^{\frac{1}{p}}}\, d\zeta \;\geq \\
&\geq\; -\,\mathrm{const}\, - \int_{s_R}^{0} \frac{\mathrm{const}\, d\zeta}{\Big((1+\zeta)^p - (1+s_R)^p - \frac{C}{R}(\zeta - s_R)\Big)^{1/p}} \;\geq \\
&\geq\; -C_1 \log R\,,
\end{aligned}$$

for a suitable C_1, where we used Lemma 2.3 to estimate the integral above. This proves (2.15).

We now prove (2.16). By Definition 2.5, (1.1), (2.18) and (2.6), we have that

$$\begin{aligned}
H_{s_0,R}(1) \;&\leq\; H_0(s_0) + \int_{s_0}^{\frac{1}{2}} \frac{(p-1)^{\frac{1}{p}}}{(p\,\varphi(\zeta))^{\frac{1}{p}}}\, d\zeta \,+ \int_{\frac{1}{2}}^{1} \frac{\mathrm{const}\, d\zeta}{\Big((1-\zeta)^p + \frac{1}{R}\Big)^{1/p}} \;\leq \\
&\leq\; \mathrm{const}\, + \int_{1/2}^{1} \frac{\mathrm{const}\, d\zeta}{(1-\zeta) + \frac{1}{R^{1/p}}} \;\leq \\
&\leq\; \mathrm{const}\,\left(1 + \log \frac{R^{1/p}+2}{2}\right) \;\leq \\
&\leq\; C_1 \log R\,,
\end{aligned}$$

proving (2.16). Finally, (2.17) follows from Definition 2.8 and (2.13). □

Using (2.17), we may now give the following definition:

Definition 2.11. We define $g_{s_0,R}(t) : (-\infty, H_{s_0,R}(1)] \to \mathbb{R}$ by

$$g_{s_0,R}(t) \;:=\; \begin{cases} s_R & \text{if} \quad t \leq H_{s_0,R}(s_R)\,, \\ H^{-1}_{s_0,R}(t) & \text{if} \quad H_{s_0,R}(s_R) < t < H_{s_0,R}(1)\,. \end{cases}$$

We now state some notation. Given $X \in \mathbb{R}^{N+1}$ we define $x \in \mathbb{R}^N$ and $x_{N+1} \in \mathbb{R}$ in such a way

$$X = (x, x_{N+1}) \in \mathbb{R}^N \times \mathbb{R}\,.$$

Also, we will often denote

$$x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}\,.$$

We now define a hypersurface in $\mathbb{R}^{N+1}$, which will provide a useful barrier:

Definition 2.12. Given $Y \in \mathbb{R}^{N+1}$ with $|y_{N+1}| \leq \frac{1}{4}$ and R large as above, let

$$\mathbb{S}(Y,R) := \Big\{ x \in \mathbb{R}^{N+1} \;\Big|\; x_{N+1} = g_{y_{N+1},R}\Big(H_0(y_{N+1}) + |x-y| - R\Big)\Big\}\,. \tag{2.19}$$

In the above definition, we will sometimes refer to Y as the "center" and to R as the "radius" of $\mathbb{S}$. For short, we also denote

$$g_{\mathbb{S}(Y,R)}(x) := g_{y_{N+1},R}(H_0(y_{N+1}) + |x-y| - R)\,, \tag{2.20}$$

so that (2.19) becomes

$$\mathbb{S}(Y,R) = \left\{ x \in \mathbb{R}^{N+1} \,\middle|\, x_{N+1} = g_{\mathbb{S}(Y,R)}(x) \right\}.$$

Let us now prove that $g_{\mathbb{S}(Y,R)}$ is a strict supersolution in the viscosity sense (for the definition of viscosity super/sub/solutions, see, e.g., [30]):

Proposition 2.13. *Let $Y \in \mathbb{R}^{N+1}$ with $|y_{N+1}| \le \frac{1}{4}$. Then, $g_{\mathbb{S}(Y,R)}$ is a strict supersolution of (1.5) in the viscosity sense at any $x \in \mathbb{R}^N$ for which $g_{\mathbb{S}(Y,R)}(x) \in [s_R,\, -1/2] \cap [1/2,\, 1)$.*
Moreover, there are not smooth functions touching $g_{\mathbb{S}(Y,R)}$ by below at x if $|g_{\mathbb{S}(Y,R)}(x)| = \frac{1}{2}$.

Proof. We use the notation

$$s = g_{y_{N+1},R}(t) \qquad \text{and} \qquad t = H_0(y_{N+1}) + |x-y| - R$$

In this setting, we have to prove the desired supersolution property for $s_R \le s \le -1/2$ and for $1/2 \le s < 1$.

Let us first consider the case $s = s_R$, that is $t \le H_{y_{N+1},R}(s_R)$. In this case, $g_{y_{N+1},R}(t)$ is constantly equal to s_R, thus any paraboloid touching from above must have vanishing gradient at the contact point and negative definite Hessian matrix. Thus,

$$\Delta_p(g_{y_{N+1},R}(t)) = 0 < h_0'(s_R)$$

in the viscosity sense.

Let us now consider the case $H_{y_{N+1},R}(s_R) < t < H_{y_{N+1},R}(-\frac{1}{2})$ (that is, $s_R < s < -\frac{1}{2}$): in this case, $g_{y_{N+1},R}$ is smooth so that we can compute all the derivatives in the classic sense. As a matter of fact, by Lemma B.3,

$$\begin{aligned} g'_{y_{N+1},R}(t) &= \Big(\frac{p}{p-1}\, h_{y_{N+1},R}(g_{y_{N+1},R}(t)) \Big)^{1/p} \\ g''_{y_{N+1},R}(t) &= \frac{\Big(p\, h_{y_{N+1},R}(g_{y_{N+1},R}(t)) \Big)^{(2-p)/p}}{(p-1)^{2/p}}\, h'_{y_{N+1},R}(g_{y_{N+1},R}(t))\,. \end{aligned}$$

Hence, exploiting Lemma B.2, we get

$$\Delta_p(g_{y_{N+1},R}(t)) = h'_{y_{N+1},R}(s) + \frac{N-1}{|x-y|} \Big(\frac{p}{p-1}\, h_{y_{N+1},R}(s) \Big)^{\frac{p-1}{p}}\,.$$

Notice that, from Lemma 2.10,

$$\begin{aligned} -C_1 \log R &\le H_{y_{N+1},R}(s_R) < t = \\ &= H_0(y_{N+1}) + |x-y| - R \le \\ &\le \text{const} + |x-y| - R\,, \end{aligned}$$

and therefore, if R is big enough,

$$|x-y| \geq \frac{R}{2}\,. \tag{2.21}$$

Thus, we get

$$\begin{aligned}\Delta_p(g_{y_{N+1},R}(t)) &\leq h_0'(s) - \frac{\widehat{C}_0}{R} + \frac{2(N-1)}{R}\Big(\frac{p}{p-1}\,h_0(s)\Big)^{\frac{p-1}{p}} < \\ &< h_0'(s)\end{aligned}$$

provided $\overline{C}_0$ (and so $\widehat{C}_0$) is chosen conveniently large. This proves the desired result for $H_{y_{N+1},R}(s_R) < t < H_{y_{N+1},R}(-\frac{1}{2})$.

If, on the other hand, $H_{y_{N+1},R}(\frac{1}{2}) < t < H_{y_{N+1},R}(1)$ (that is, $\frac{1}{2} < s < 1$), arguing in the same way, we get

$$\Delta_p(g_{y_{N+1},R}(t)) \leq h_0'(s) - \frac{\widehat{C}_0}{R} + \frac{2(N-1)}{R}\Big(\frac{p}{p-1}\,h_{y_{N+1},R}(s)\Big)^{\frac{p-1}{p}} < h_0'(s)\,,$$

provided that $\widehat{C}_0$ is conveniently large. This completes the proof in the case $H_{y_{N+1},R}(\frac{1}{2}) < t < H_{y_{N+1},R}(1)$.

Up to now, we have therefore proved the desired result for

$$t \in \Big(-\infty,\, H_{y_{N+1},R}(-\frac{1}{2})\Big) \cup \Big(H_{y_{N+1},R}(\frac{1}{2}),\, H_{y_{N+1},R}(1)\Big)\,,$$

that is, for

$$s \in \Big[s_R, -\frac{1}{2}\Big) \cup \Big(\frac{1}{2},\,1\Big)\,,$$

To complete the proof of the claim, we have therefore to take now into account the case $|s| = 1/2$.

Let us now deal with the case $|s| = 1/2$. Recalling Lemma 2.6, by (2.2), we have

$$\begin{aligned}\lim_{s\to-\frac{1}{2}^-} H'_{y_{N+1},R}(s) &< \lim_{s\to-\frac{1}{2}^+} H'_{y_{N+1},R}(s)\,,\\ \lim_{s\to\frac{1}{2}^-} H'_{y_{N+1},R}(s) &< \lim_{s\to\frac{1}{2}^+} H'_{y_{N+1},R}(s)\,,\end{aligned}$$

so that

$$\begin{aligned}\lim_{t\to H_{y_{N+1},R}(-\frac{1}{2})^-} g'_{y_{N+1},R}(t) &> \lim_{t\to H_{y_{N+1},R}(-\frac{1}{2})^+} g'_{y_{N+1},R}(t)\,,\\ \lim_{t\to H_{y_{N+1},R}(\frac{1}{2})^-} g'_{y_{N+1},R}(t) &> \lim_{t\to H_{y_{N+1},R}(\frac{1}{2})^+} g'_{y_{N+1},R}(t)\,.\end{aligned} \tag{2.22}$$

Let now w be a smooth function whose graph touches $\mathbb{S}(Y,R)$ by below at $(x_0, 1/2)$ (the case of touching at the $-1/2$-level is analogous). Let us consider the radial direction $\nu_0 = \frac{x_0-y}{|x_0-y|}$ and let us define, for $\theta \in \mathbb{R}$,

$$\mathfrak{g}(\theta) := g_{\mathbb{S}(Y,R)}(x_0+\theta\nu_0) - w(x_0+\theta\nu_0)\,.$$

Then, by construction, $\mathfrak{g}(0)=0\le\mathfrak{g}(\theta)$, therefore

$$\begin{aligned}0 &\le \lim_{\theta\to 0^+}\frac{\mathfrak{g}(\theta)-\mathfrak{g}(0)}{\theta}=\\ &= \lim_{\theta\to 0^+}\frac{g_{y_{N+1},R}(t_0+\theta)-g_{y_{N+1},R}(t_0)}{\theta}-\partial_{\nu_0}w(x_0)=\\ &= \lim_{t\to H_{y_{N+1},R}(1/2)^+} g'_{y_{N+1},R}(t)-\partial_{\nu_0}w(x_0)\,,\end{aligned}$$

where

$$t_0:=H_0(y_{N+1})+|x_0-y|-R=H_{y_{N+1},R}(1/2)\,.$$

By arguing in the same way, we also get that

$$\begin{aligned}0 &\ge \lim_{\theta\to 0^-}\frac{\mathfrak{g}(\theta)-\mathfrak{g}(0)}{\theta}=\\ &= \lim_{t\to H_{y_{N+1},R}(1/2)^-} g'_{y_{N+1},R}(t)-\partial_{\nu_0}w(x_0)\,.\end{aligned}$$

Thence,

$$\lim_{t\to H_{y_{N+1},R}(1/2)^-} g'_{y_{N+1},R}(t)\le\partial_{\nu_0}w(x_0)\le\lim_{t\to H_{y_{N+1},R}(1/2)^+} g'_{y_{N+1},R}(t)\,,$$

which is a contradiction with (2.22). Therefore, no smooth function may touch $g_{\mathbb{S}(Y,R)}$ by below at $\pm1/2$-level sets, showing, in particular, the claimed supersolution property. This completes the proof of Proposition 2.13. □

As a consequence of the above result, we show now that touching points between $\mathbb{S}(Y,R)$ and a subsolution of (1.5) may only occur when $|x_{N+1}|<1/2$ (and this fact will be of great help in future computations, thanks to the explicit form of the barrier in $|x_{N+1}|<1/2$):

Corollary 2.14. *Let $U\in C^1(\Omega)$ be a weak Sobolev subsolution of (1.5), with $|U|\le1$. Assume that $U\le g_{\mathbb{S}(Y,R)}$ and that $U(x^\star)=g_{\mathbb{S}(Y,R)}(x^\star)$ for some $x^\star$ in the closure of Ω. Then, either $x^\star\in\partial\Omega$ or $|g_{\mathbb{S}(Y,R)}(x^\star)|<1/2$.*

Proof. Let us assume that $x^\star\notin\partial\Omega$ We first prove that $g_{\mathbb{S}(Y,R)}(x^\star)\neq1$. We argue by contradiction, assuming $g_{\mathbb{S}(Y,R)}(x^\star)=1$. Notice that, due to Definition 2.5,

$$\begin{aligned}\lim_{s\longrightarrow1^-}\frac{dH_{y_{N+1},R}}{ds}(s) &= \lim_{s\longrightarrow1^-}\frac{1}{\left(\frac{p}{p-1}h_{y_{N+1},R}(s)\right)^{1/p}}=\\ &= \left(\frac{(p-1)R}{p}\right)^{1/p},\end{aligned}$$

thus $g'_{y_{N+1},R}(t)>0$ if $t=H_{y_{N+1},R}(1)$. Then, if

$$\nu^\star:=\frac{x^\star-y}{|x^\star-y|}$$

is the radial direction, this yields

$$\frac{\partial g_{\mathbb{S}(Y,R)}}{\partial \nu^\star}(x^\star) > 0 . \tag{2.23}$$

However, since $U \in C^1(\mathbb{R}^N)$, $U(x^\star) = 1$ and $U \leq 1$,

$$\nabla U(x^\star) = 0 . \tag{2.24}$$

Similarly, since $U \leq g_{\mathbb{S}(Y,R)}$ and $U(x^\star) = g_{\mathbb{S}(Y,R)}(x^\star)$,

$$\frac{\partial g_{\mathbb{S}(Y,R)}}{\partial \nu^\star}(x^\star) \leq \frac{\partial U}{\partial \nu^\star}(x^\star) . \tag{2.25}$$

Thus, a contradiction easily follows from (2.23), (2.24) and (2.25), showing that $g_{\mathbb{S}(Y,R)}(x^\star) \neq 1$.

Also we claim that $g_{\mathbb{S}(Y,R)}(x^\star) \notin [s_R, , -1/2] \cup [1/2, , 1)$.

Let us first show that $g_{\mathbb{S}(Y,R)}(x^\star) \neq s_R$. We argue by contradiction and assume that $g_{\mathbb{S}(Y,R)}(x^\star) = s_R$. We recall that, by Definitions 2.11 and 2.12, $g_{\mathbb{S}(Y,R)}$ is constantly equal to s_R in $B_r(y)$, with

$$r = r(R) := R + H_{y_{N+1},R}(s_R) - H_0(y_{N+1})$$

and[5] that $r > 0$ by (2.15). Then, there would be $\rho > 0$ so that

$$\Omega_\star := B_r(y) \cap B_\rho(x^\star) \subseteq \Big\{ g_{\mathbb{S}(Y,R)} = s_R \Big\} .$$

Possibly taking ρ smaller, we may assume also that

$$\Omega_\star \subseteq \Big\{ U < -1 + \theta^* \Big\} \tag{2.26}$$

and, since $x^\star \notin \partial\Omega$ by our assumption, that

(2.27) $\Omega_\star$ is contained in the interior of Ω.

Note that U cannot coincide with $g_{\mathbb{S}(Y,R)}$ in $\Omega_\star$, otherwise

$$\begin{aligned} 0 &= -\int |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi \\ &\geq \int h_0'(U)\,\varphi \\ &= \int h_0'(s_R)\,\varphi \\ &> 0 , \end{aligned}$$

for any non-negative smooth test-function φ supported in $\Omega_\star$. Then, there exists $\bar{x} \in \Omega_\star$ and $\rho' > 0$ so that $U < g_{\mathbb{S}(Y,R)}$ in the interior of $B_{\rho'}(\bar{x}) \subseteq \Omega_\star$, but there exists $\breve{x} \in \partial B_{\rho'}$ for which $U(\breve{x}) = g_{\mathbb{S}(Y,R)}(\breve{x})$. Setting

$$U^\star = s_R - U$$

[5] Proceeding as done here and exploiting (2.16), one may also prove that $g_{\mathbb{S}(Y,R)}$ reaches the value 1 well inside the ball of radius $R + R^{\frac{1}{3}}/2$ around y.

it follows

$$U^\star = g_{\mathbb{S}(Y,R)} - U > 0$$

in $B_{\rho'}(\bar{x})$ and $U^\star(\breve{x}) = 0$. Moreover,

$$-\Delta_p U^\star = \Delta_p U \geq h_0'(U)\,. \tag{2.28}$$

Hence, $h_0'(U) > 0$ in the light of (1.2) and (2.26). Thence, from (2.28), $-\Delta_p U^\star > 0$. Therefore, by Theorem B.6 (applied with $c = g = 0$),

$$\partial_\nu U(\breve{x}) = -\partial_\nu U^\star(\breve{x}) > 0\,, \tag{2.29}$$

where

$$\nu := \frac{\breve{x} - \bar{x}}{|\breve{x} - \bar{x}|}$$

is the outer normal of $B_{\rho'}(\bar{x})$ at $\breve{x}$.

On the contrary, note that $\breve{x}$ is in the interior of the domain of U thanks to (2.27), and so, since U touches $g_{\mathbb{S}(Y,R)}$ at $\breve{x}$ and $\nabla g_{\mathbb{S}(Y,R)}$ vanishes on $\{g_{\mathbb{S}(Y,R)} = s_R\}$, we have that

$$\partial_\nu U(\breve{x}) = \partial_\nu g_{\mathbb{S}(Y,R)}(\breve{x}) = 0\,,$$

against (2.29). This contradiction shows that $x^\star$ does not lie in $\{g_{\mathbb{S}(Y,R)} = s_R\}$.

Let us now prove that

$$g_{\mathbb{S}(Y,R)}(x^\star) \notin \left(s_R\,,\,-\frac{1}{2}\right)\,. \tag{2.30}$$

First, we recall that $g_{\mathbb{S}(Y,R)}$ is smooth with non-vanishing gradient in

$$\Omega_0 := \left\{ g_{\mathbb{S}(Y,R)}(x) \in \left(s_R\,,\,-\frac{1}{2}\right) \right\}\,,$$

thence it is a classical strict supersolution of (1.5) in Ω_0. This implies that U cannot coincide with $g_{\mathbb{S}(Y,R)}$ in $\{g_{\mathbb{S}(Y,R)}(x^\star) \in (s_R\,,\,-\frac{1}{2})\}$, otherwise,

$$\begin{aligned}
\int h_0'(g_{\mathbb{S}(Y,R)})\,\varphi &> -\int |\nabla g_{\mathbb{S}(Y,R)}|^{p-2} \nabla g_{\mathbb{S}(Y,R)} \cdot \nabla\varphi = \\
&= -\int |\nabla U|^{p-2}\nabla U \cdot \nabla\varphi \geq \\
&\geq \int h_0'(U)\,\varphi \\
&= \int h_0'(g_{\mathbb{S}(Y,R)})\,\varphi\,,
\end{aligned}$$

for any non-negative smooth test-function φ supported in Ω_0, which is an obvious contradiction. Therefore, since U and $g_{\mathbb{S}(Y,R)}$ do not agree in Ω_0 and $\nabla g_{\mathbb{S}(Y,R)}$ never vanishes there, we can exploit Theorem B.5 and get that $U < g_{\mathbb{S}(Y,R)}$ in Ω_0. Thence, no touching point may occur in $\{g_{\mathbb{S}(Y,R)}(x) \in (s_R\,,\,-\frac{1}{2})\}$, proving (2.30).

Also, $g_{\mathbb{S}(Y,R)}(x^\star) \neq -\frac{1}{2}$ by Proposition 2.13. The fact that $g_{\mathbb{S}(Y,R)}(x^\star) \notin [1/2\,,\,1)$ follows with similar arguments. $\square$

We will now define another hypersurface in $\mathbb{R}^{N+1}$, which will be denoted by $\widetilde{\mathbb{S}}(Y,R)$ and we investigate its relation with $\mathbb{S}(Y,R)$. While $\mathbb{S}(Y,R)$ is continuous but not smooth, $\widetilde{\mathbb{S}}(Y,R)$ will be smooth, and thus it will be easier to deal with during the calculations. Also, the two surfaces will coincide in $\{|x_{N+1}| \leq \frac{1}{2}\}$ and $\mathbb{S}(Y,R)$ will always stay below $\widetilde{\mathbb{S}}(Y,R)$. Therefore, $\widetilde{\mathbb{S}}(Y,R)$ will provide, in some sense, a sharp barrier for $\mathbb{S}(Y,R)$ which will be more explicit to treat. Let us now approach the definition of the hypersurface $\widetilde{\mathbb{S}}(Y,R)$.

Setting

$$\widetilde{H}_{s_0,R}(s) \;:=\; H_0(s) - \frac{\overline{C}_0}{2R}(s-s_0)^2 \tag{2.31}$$

then, by (2.12), we have

$$\frac{d}{ds}\left(\widetilde{H}_{s_0,R}(s)\right) = \frac{1}{(\frac{p}{p-1}h_0(s))^{\frac{1}{p}}} - \frac{\overline{C}_0}{R}(s-s_0) > 0 \tag{2.32}$$

for $|s| \leq \frac{3}{4}$, provided R is big enough. Therefore, for $|s| \leq \frac{3}{4}$ we have that $\widetilde{H}_{s_0,R}$ is strictly increasing and we can give the following

Definition 2.15. We define

$$\rho_{s_0,R}(t) : \left[\widetilde{H}_{s_0,R}(-\frac{3}{4})\,,\,\widetilde{H}_{s_0,R}(\frac{3}{4})\right] \longrightarrow \left[-\frac{3}{4},\,\frac{3}{4}\right]$$

by

$$\rho_{s_0,R}(t) := \widetilde{H}^{-1}_{s_0,R}(t)\,. \tag{2.33}$$

Moreover, given $Y \in \mathbb{R}^{N+1}$ with $|y_{N+1}| \leq \frac{1}{4}$ and R large as above, we define

$$\widetilde{\mathbb{S}}(Y,R) \;:=\; \{x \in \mathbb{R}^{N+1} \,|\, x_{N+1} = \rho_{y_{N+1},R}(H_0(y_{N+1}) + |x-y| - R)\} \tag{2.34}$$

As done on page 15, it is convenient to introduce the notation

$$g_{\widetilde{\mathbb{S}}(Y,R)}(x) := \rho_{y_{N+1},R}(H_0(y_{N+1}) + |x-y| - R)\,, \tag{2.35}$$

so that (2.34) becomes

$$\widetilde{\mathbb{S}}(Y,R) = \left\{x \in \mathbb{R}^{N+1} \,\middle|\, x_{N+1} = g_{\widetilde{\mathbb{S}}(Y,R)}(x)\right\}.$$

Notice also that, by construction, for any x for which $g_{\widetilde{\mathbb{S}}(Y,R)}$ is defined[6], we have that

$$|g_{\widetilde{\mathbb{S}}(Y,R)}(x)| \le \frac{3}{4}. \tag{2.37}$$

Moreover, a straightforward computation gives that

$$\widetilde{H}_{s_0,R_1}(s) - \widetilde{H}_{s_0,R_2}(s) = \frac{\overline{C}_0}{2}(s-s_0)^2\left(\frac{1}{R_2}-\frac{1}{R_1}\right), \tag{2.38}$$

for any $|s| \le 3/4$. Also, if x is in the domain of $g_{\widetilde{\mathbb{S}}(Y,R)}$, then x and y must be suitably far from each other, as next result points out:

Lemma 2.16. *Let $x \in \mathbb{R}^N$ be so that $g_{\widetilde{\mathbb{S}}(Y,R)}$ is well defined (that is, let x be such that (2.36) holds). Then,*

$$|x-y| \ge R - C,$$

for a suitable universal constant $C > 0$.

Proof. From (2.37), we get that

$$s := g_{\widetilde{\mathbb{S}}(Y,R)}(x) \in [-3/4,\, 3/4].$$

So, by means of (2.31) and (2.33),

$$\begin{aligned} |x-y| &= R - H_0(y_{N+1}) + H_0(s) - \frac{\overline{C}_0}{2R}(s-y_{N+1})^2 \ge \\ &\ge R - \text{const}. \end{aligned}$$

□

Let us now show that the surface $\mathbb{S}(Y,R)$ coincides with $\widetilde{\mathbb{S}}(Y,R)$ in the set $|x_{N+1}| \le \frac{1}{2}$ and that $\mathbb{S}(Y,R)$ stays below $\widetilde{\mathbb{S}}(Y,R)$ at all other points where $\widetilde{\mathbb{S}}(Y,R)$ is defined:

Lemma 2.17. *If $|s| \le 1/2$, then*

$$H_{s_0,R}(s) = \widetilde{H}_{s_0,R}(s). \tag{2.39}$$

If $\frac{1}{2} < |s| < \frac{3}{4}$, then

$$H_{s_0,R}(s) > \widetilde{H}_{s_0,R}(s). \tag{2.40}$$

Also, let $x \in \mathbb{R}^N$ be so that $g_{\widetilde{\mathbb{S}}(Y,R)}$ is well defined (that is, let x be such that (2.36) holds). Then,

$$g_{\widetilde{\mathbb{S}}(Y,R)}(x) \ge g_{\mathbb{S}(Y,R)}(x). \tag{2.41}$$

[6] I.e., for any x so that

$$H_0(y_{N+1}) + |x-y| - R \in \left[\widetilde{H}_{s_0,R}(-\frac{3}{4}),\, \widetilde{H}_{s_0,R}(\frac{3}{4})\right]. \tag{2.36}$$

Furthermore, if $|g_{\widetilde{\mathbb{S}}(Y,R)}(x)| \leq \frac{1}{2}$*, then*

$$g_{\widetilde{\mathbb{S}}(Y,R)}(x) = g_{\mathbb{S}(Y,R)}(x)\,. \tag{2.42}$$

Proof. We use the notation $s := g_{\widetilde{\mathbb{S}}(Y,R)}$. Let $|s| \leq 1/2$. By Definitions 2.5 and 2.8, recalling also (2.2), we have that

$$\begin{aligned} H_{s_0,R}(s) &= H_0(s_0) + \int_{s_0}^{s} \frac{d\zeta}{(\frac{p}{p-1}h_0(s))^{\frac{1}{p}}} - \int_{s_0}^{s} \frac{\overline{C}_0}{R}(s-s_0) = \\ &= H_0(s) - \frac{\overline{C}_0}{2R}(s-s_0)^2 = \\ &= \widetilde{H}_{s_0,R}(s)\,. \end{aligned} \tag{2.43}$$

This and (2.31) prove (2.39) and (2.42). We now prove (2.40) and (2.41). Let us consider only the case $s \in [-3/4,\, -1/2]$, the case $s \in [1/2,\, 3/4]$ being analogous. In this case, $s < s_0$, and thus, exploiting Lemma 2.6, we get

$$\begin{aligned} H_{s_0,R}(s) &= H_0(s_0) - \int_{s}^{s_0} \frac{1}{\left(\frac{p}{p-1}h_{s_0,R}\right)^{1/p}} > \\ &> H_0(s_0) - \int_{s}^{s_0} \frac{1}{\left(\frac{p}{p-1}\varphi\right)^{1/p}} = \\ &= H_0(s_0) - \int_{s}^{s_0} \frac{1}{\left(\frac{p}{p-1}h_0\right)^{1/p}} - \frac{\overline{C}_0}{2R}(s-s_0)^2 = \\ &= H_0(s) - \frac{\overline{C}_0}{2R}(s-s_0)^2\,, \end{aligned}$$

proving (2.40) and (2.41). ☐

Since, by construction, the function $g_{\widetilde{\mathbb{S}}(Y,R)}$ defined above is smooth (and, due to (2.32, its gradient never vanishes), we can compute its derivatives (and its p-Laplacian) in the classic sense. In particular, we can sharply estimate how far $g_{\widetilde{\mathbb{S}}(Y,R)}$ is from being a solution of (1.5), thanks to the following result:

Proposition 2.18. *Let* $Y \in \mathbb{R}^{N+1}$ *with* $|y_{N+1}| \leq \frac{1}{4}$*. Then, there exists a positive universal constant* $C > 0$ *such that*

$$h_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) - \frac{C}{R} \leq \Delta_p g_{\widetilde{\mathbb{S}}(Y,R)}(x) \leq h_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) + \frac{C}{R}\,,$$

for any x *for which* $g_{\widetilde{\mathbb{S}}(Y,R)}$ *is defined (i.e., for any* x *so that (2.36) holds).*

Proof. We will use the notation

$$\begin{aligned} t &:= H_0(y_{N+1}) + |x-y| - R \qquad \text{and} \\ s &:= \rho_{y_{N+1},R}(t) \;=\; g_{\widetilde{\mathbb{S}}(Y,R)}(x)\,. \end{aligned} \tag{2.44}$$

Let us note that, from (2.31) and (2.2),

$$\frac{d}{dt}\rho_{y_{N+1},R}(t)=\frac{1}{H_0'(s)-\frac{\overline{C}_0}{R}(s-y_{N+1})}=\left(\frac{p}{p-1}\varphi(s)\right)^{\frac{1}{p}}. \tag{2.45}$$

Hence, differentiating again, a straightforward calculation leads to

$$\frac{d^2}{dt^2}\rho_{y_{N+1},R}(t)=\frac{(p\,\varphi(s))^{\frac{2-p}{p}}}{(p-1)^{\frac{2}{p}}}\,\varphi'(s)\,. \tag{2.46}$$

Therefore, by Lemma B.2,

$$\Delta_p(\rho_{y_{N+1},R}(t))=\varphi'(s)+\frac{N-1}{|x-y|}\Big(\frac{p}{p-1}\,\varphi(s)\Big)^{\frac{p-1}{p}}. \tag{2.47}$$

Furthermore, note that, by differentiating (2.2), one obtains

$$-\frac{1}{p}\left(\frac{\frac{p}{p-1}\varphi'(s)}{(\frac{p}{p-1}\varphi(s))^{\frac{p+1}{p}}}\right)=-\frac{1}{p}\left(\frac{\frac{p}{p-1}h_0'(s)}{(\frac{p}{p-1}h_0(s))^{\frac{p+1}{p}}}\right)-\frac{\overline{C}_0}{R} \tag{2.48}$$

so that

$$\varphi'(s)=\left(\frac{\varphi(s)}{h_0(s)}\right)^{\frac{p+1}{p}}h_0'(s)+\frac{p\overline{C}_0}{R}\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\Big(\varphi(s)\Big)^{\frac{p+1}{p}}, \tag{2.49}$$

whence, from (2.1),

(2.50)

$$\varphi'(s)=h_0'(s)+ \\ +\frac{1}{R}\left\{h_0'(s)R\left[\left(\frac{R}{R-a}\right)^{p+1}-1\right]+p\,\overline{C}_0\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\Big(\varphi(s)\Big)^{\frac{p+1}{p}}\right\},$$

with

$$a=a(s):=\overline{C}_0\,(s-s_0)\left(\frac{p}{p-1}\,h_0(s)\right)^{1/p}.$$

Using now the fact that

$$\lim_{x\to 0^+}\left|\frac{\left(\frac{1}{1-xa}\right)^{p+1}-1}{x}\right|=|(p+1)a|<+\infty\,,$$

we obtain, if R is suitably large, that

$$\left|h_0'(s)R\left[\left(\frac{R}{R-a}\right)^{p+1}-1\right]+p\,\overline{C}_0\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\Big(\varphi(s)\Big)^{\frac{p+1}{p}}\right|\le\text{ const}\,, \tag{2.51}$$

so that, from (2.47) and the fact that $\varphi\ge 0$,

$$h_0'(s)-\frac{\text{const}}{R}\le\Delta_p(\rho_{y_{N+1},R}(t))\,,$$

which proves one side of the claimed inequality.

The other side of the inequality is obtained by arguing in the same way, making use also of Lemma 2.16. ☐

For further reference, we point out some easy calculations on the above barrier:

Lemma 2.19. *At any x for which $g_{\widetilde{\mathbb{S}}(Y,R)}$ is defined, we have that*

$$\begin{aligned}\Big(H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) - \frac{\overline{C}_0}{R}\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x) - y_{N+1}\Big)\Big)\,\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x) \;=\\ = \frac{x-y}{|x-y|}\,.\end{aligned}\tag{2.52}$$

Also, there exists a universal constant $C>0$ so that

$$\Big|H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\,\Big|\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big| - 1\Big| \;\le\; \frac{C}{R}\tag{2.53}$$

and

$$\begin{aligned}&\Big|H_0''\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\,\partial_i g_{\widetilde{\mathbb{S}}(Y,R)}(x)\partial_j g_{\widetilde{\mathbb{S}}(Y,R)}(x) +\\ &+\; H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\,\partial_{ij} g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big| \le\\ &\le\; \frac{C}{R}\,,\end{aligned}\tag{2.54}$$

provided R is large enough.

Proof. Using the notation in (2.44) and the first equality in (2.45), the claim in (2.52) easily follows. From (2.52) and (2.37), one easily gets (2.53).

Let us now prove (2.54). From (2.31) and (2.33), we have that

$$\tau = H_0\Big(\rho_{y_{N+1},R}(\tau)\Big) - \frac{\overline{C}_0}{2R}\Big(\rho_{y_{N+1},R}(\tau) - y_{N+1}\Big)^2,$$

for any τ for which $\rho_{y_{N+1},R}$ is defined; then, differentiating twice and recalling (2.37),

$$\begin{aligned}&\Big|H_0''(\rho_{y_{N+1},R}(\tau))\;\Big(\rho'_{y_{N+1},R}(\tau)\Big)^2 +\\ &+ H_0'(\rho_{y_{N+1},R}(\tau))\,\rho''_{y_{N+1},R}(\tau)\Big| \le \frac{\mathrm{const}}{R}\,.\end{aligned}\tag{2.55}$$

Furthermore, differentiating twice the relation

$$g_{\widetilde{\mathbb{S}}(Y,R)}(x) \;=\; \rho_{y_{N+1},R}\Big(H_0(y_{N+1}) + |x-y| - R\Big)\,,$$

a direct computation gives

$$\begin{aligned}&\partial_{ij} g_{\widetilde{\mathbb{S}}(Y,R)}(x) =\\ =\;& R_{ij}\rho''_{y_{N+1},R}(t) + \Big(\frac{\delta_{ij}}{|x-y|} - \frac{R_{ij}}{|x-y|}\Big)\,\rho'_{y_{N+1},R}(t)\,,\end{aligned}$$

where we define, for short,

$$R_{ij} := \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2}.$$

In other words,

$$\partial_{ij} g_{\widetilde{\mathbb{S}}(Y,R)}(x) = R_{ij} \rho''_{y_{N+1},R}(t) + S_{ij} \tag{2.56}$$

for a suitable S_{ij} satisfying $|S_{ij}| \leq \text{const}/|x-y|$. Therefore, using (2.56) and the fact that

$$\partial_i g_{\widetilde{\mathbb{S}}(Y,R)}(x) = \rho'_{y_{N+1},R}(t) \frac{x_i - y_i}{|x-y|}, \tag{2.57}$$

we get that

$$\begin{aligned}
& H_0''\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\,\partial_i g_{\widetilde{\mathbb{S}}(Y,R)}(x)\partial_j g_{\widetilde{\mathbb{S}}(Y,R)}(x) + \\
& + H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\,\partial_{ij} g_{\widetilde{\mathbb{S}}(Y,R)}(x) = \\
= \; & H_0''(\rho_{y_{N+1},R}(t))\ \big(\rho_{y_{N+1},R}(t)\big)^2\ R_{ij} + \\
& + H_0'(\rho_{y_{N+1},R}(t))\ \Big(R_{ij}\rho''_{y_{N+1},R}(t) + S_{ij}\Big) = \\
= \; & R_{ij}\Big[H_0''\Big(\rho_{y_{N+1},R}(t)\Big)\ \Big(\rho_{y_{N+1},R}(t)\Big)^2 + \\
& + H_0'\Big(\rho_{y_{N+1},R}(t)\Big)\ \rho''_{y_{N+1},R}(t)\Big] + T_{ij},
\end{aligned}$$

with $|T_{ij}| \leq \text{const}/|x-y|$. Thus, from (2.55),

$$\begin{aligned}
& \Big| H_0''\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\partial_i g_{\widetilde{\mathbb{S}}(Y,R)}(x)\partial_j g_{\widetilde{\mathbb{S}}(Y,R)}(x) + \\
& + H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\partial_{ij} g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big| \leq \\
& \leq \text{const}\left(\frac{1}{R} + \frac{1}{|x-y|}\right).
\end{aligned}$$

Therefore, (2.54) is proved thanks to Lemma 2.16. □

We now recall a result, proved in [30], concerning another barrier which will be used in the course of the proof of the main results:

Lemma 2.20. *There exist universal constants $\bar{l} > 1$ and $0 < \bar{c} \leq 1/2$, so that, if $l \geq \bar{l}$, we can find $T_l \in [\bar{c}l, l/2]$ and a nondecreasing function*

$$g_l \in C^0(-\infty, T_l) \cap C^{1,1}(-\infty, 0) \cap C^2((-\bar{c}l, T_l) \setminus \{0\})$$

which is constant in $(-\infty, -l/2]$, with $g_l' > 0$ in $[-\bar{c}l, T_l]$, satisfying $g_l(0) = 0$, $g_l(T_l) = 1$ and such that, if we define

$$\Psi^{y,l}(x) := g_l(|x-y| - l), \tag{2.58}$$

then $\Psi^{y,l}(x)$ is a strict supersolution of (1.5) in the viscosity sense, in $B_{T_l+l}(y) \setminus \partial B_l(y)$.

More precisely, g_l is constructed as follows. There exists a suitable constants $0 < c_1 < \overline{C}_1, \overline{C}_2$ so that, if we define

$$\begin{aligned}
s_l &:= e^{-\bar{c}_1 l}, \\
h_l(s) &:= \begin{cases} h_0(s) - h_0(s_l - 1) - \frac{\overline{C}_2}{l}\left((1+s)^p - s_l^p\right) \\ \qquad \text{if} \quad (s_l - 1) < s < 0 \\[2ex] h_0(s) + h_0(1 - s_l) + \frac{\overline{C}_2}{l}\Big((1-s)^p + s_l^{(p-1)}(1-s)\Big) \\ \qquad \text{if} \quad 0 \le s < 1, \end{cases} \\
H_l(s) &:= \int_0^s \frac{(p-1)^{\frac{1}{p}}}{(p\, h_l(\zeta))^{\frac{1}{p}}}\, d\zeta, \\
H_0(s) &:= \int_0^s \frac{(p-1)^{\frac{1}{p}}}{(p\, h_0(\zeta))^{\frac{1}{p}}}\, d\zeta, \qquad \text{for any } s \in (-1, 1),
\end{aligned}$$

then the following holds:

(i) *$h_l(s) > 0$ in $(s_l - 1) < s < 1$; in particular, H_l is well defined and strictly increasing for $(s_l - 1) < s < 1$ and thus we may define $g_l(t) := H_l^{-1}(t)$ for $t \in H_l(s_l - 1, 1)$;*

(ii) *$g_l(t)$ is defined to be constantly equal to $s_l - 1$ for $t \le H_l(s_l - 1)$;*

(iii) *the following estimates on H_l hold:*

$$H_l(1) \le \frac{l}{2}; \tag{2.59}$$

$$H_l(s_l - 1) \ge -\frac{l}{2}; \tag{2.60}$$

$$H_0(s) \le H_l(s) - \frac{\overline{C}_1}{l} \log(1 - |s|) \qquad \forall |s| < 1 - e^{-\frac{\bar{c}_1 l}{2}}; \tag{2.61}$$

$$H_l(1 - e^{-\frac{\bar{c}_1 l}{2}}) \ge \bar{c}\, l; \tag{2.62}$$

$$H_l(e^{-\frac{\bar{c}_1 l}{2}} - 1) \le -\bar{c}\, l. \tag{2.63}$$

A detailed proof of Lemma 2.20 is contained in [30] (see, in particular, Lemma 5.1 there).

We now point out some properties of the touching points between the barrier $\Psi^{y,l}$ and a (sub)solution of (1.5). To this end, we notice that, by construction, the radially increasing function $\Psi^{y,l}$ built in Lemma 2.20 is so that:

- $\Psi^{y,l}$ is defined in $B_{T_l+l}(y)$, and it is greater than $s_l - 1$;
- there exists $\rho_l \in [\bar{c}l, l/2]$ so that the only critical points of $\Psi^{y,l}$ are in $B_{l-\rho_l}(y)$, where $\Psi^{y,l}$ is flat;
- $\Psi^{y,l} = 0$ on $\partial B_l(y)$.

The geometry of such spheres is related with possible touching points, as next result shows:

Lemma 2.21. *Fix $y \in \mathbb{R}^N$ and let $l > 0$ be suitably large. Let u be a weak Sobolev subsolution of (1.5) in some domain Ω. Suppose that $u \in C^1(\Omega)$ and that $|u| \le 1$. Then the following results hold:*

- *If $\Psi^{y,l}$ touches the graph of u from above at some point $x^\star$ in the closure of $\Omega \cap B_{T_l+l}(y)$, then, either $x^\star \in \partial\Omega$ or $u(x^\star) = \Psi^{y,l}(x^\star) = 0$.*
- *If*
$$B_{l+T_l}(y) \subset \{x \in \Omega \mid u(x) \le -1 + \theta^*\}\,,$$
then,
$$u(x) \,<\, \Psi^{y,l}(x)\,,$$
for any $x \in B_{l+T_l}(y)$.

For the proof of Lemma 2.21, we refer to [30] (see, in particular, Lemma 6.2 and Corollary 6.4 there). We now notice that a statement analogous to Lemma 2.20 holds for a subsolution (instead of supersolution) property:

Lemma 2.22. *There exist universal constants $\bar{l} > 1$ and $0 < \bar{c} \le 1/2$, so that, if $l \ge \bar{l}$, we can find $T_l \in [\bar{c}l, l/2]$ and a nondecreasing function*

$$\widetilde{g}_l \in C^0(-T_l, +\infty) \cap C^{1,1}(0, +\infty) \cap C^2((-T_l, \bar{c}l) \setminus \{0\})$$

which is constant in $[l/2, +\infty)$, with $\widetilde{g}_l' > 0$ in $[-T_l, \bar{c}l]$, satisfying $\widetilde{g}_l(0) = 0$, $\widetilde{g}_l(-T_l) = -1$ and such that, if we define

$$\widetilde{\Psi}^{y,l}(x) \,:=\, \widetilde{g}_l(l - |x - y|)\,,$$

then $\widetilde{\Psi}^{y,l}(x)$ is a strict subsolution of (1.5) in the viscosity sense, in $B_{T_l+l}(y) \setminus \partial B_l(y)$.

Also, if we define $\widetilde{h}_0(s) := h_0(-s)$ and

$$\widetilde{H}_0(t) := \int_0^t \frac{(p-1)^{1/p}\,d\zeta}{(p\,\widetilde{h}_0(\zeta))^{1/p}}\,,$$

then, there exists a strictly increasing function $\widetilde{H}_l$ and a positive function h_l, such that

$$\widetilde{H}_l'(s) = \frac{(p-1)^{1/p}}{(p\,\widetilde{h}_l(s))^{1/p}}\,,$$

so that the following holds:

(i) *$\widetilde{h}_l(s)$ is defined and strictly positive in $-1 < s < 1 - s_l$; $\widetilde{H}_l$ is defined and strictly increasing for $-1 < s < 1 - s_l$; $\widetilde{g}_l(t) = \widetilde{H}_l^{-1}(t)$ for $t \in \widetilde{H}_l(1,\, 1 - s_l)$;*

(ii) *$\widetilde{g}_l(t)$ is constantly equal to $1 - s_l$ for $t \ge \widetilde{H}_l(1 - s_l)$;*

(iii) *the following estimates on* $\widetilde{H}_l$ *hold:*

$$\begin{aligned}
\widetilde{H}_l(-1) &\geq -\frac{l}{2}\,, \\
\widetilde{H}_l(1-s_l) &\leq \frac{l}{2}\,, \\
\widetilde{H}_0(s) &\geq \widetilde{H}_l(s) + \frac{\overline{C}_1}{l}\log(1-|s|) \qquad \forall |s| < 1 - e^{-\frac{\bar{c}_1 l}{2}}\,, \\
\widetilde{H}_l(-1+e^{-\frac{\bar{c}_1 l}{2}}) &\leq -\bar{c}\, l\,, \\
H_l(1-e^{-\frac{\bar{c}_1 l}{2}}) &\geq \bar{c}\, l\,.
\end{aligned}$$

Proof. Notice that $\widetilde{h}_0$ satisfies the same assumption as h_0, thus, we can use Lemma 2.20 with $\widetilde{h}_0$ replacing h_0: let us denote by $h_l^\sharp$, $H_l^\sharp$ and $g_l^\sharp$ the functions thus obtained via Lemma 2.20 with $\widetilde{h}_0$ replacing h_0. Then, define

$$\widetilde{h}_l(s) := h_l^\sharp(-s)\,.$$

Thus,

$$\widetilde{H}_l(s) := \int_0^s \frac{(p-1)^{1/p}}{(p\widetilde{h}_l)^{1/p}} = -H_l^\sharp(-s)$$

and therefore

$$\widetilde{g}_l(t) = -g_l^\sharp(-t)\,.$$

With this, let us show that $\widetilde{\Psi}^{y,l}$ is a strict viscosity subsolution of (1.5) outside $\partial B_l(y)$. Indeed, if ϕ is a smooth function touching $\widetilde{\Psi}^{y,l}$ from above at $x^\star \notin \partial B_l(y)$, then $\varphi(x) := -\phi(x)$ touches from below at the point $x^\star$ the strict viscosity supersolution

$$-\widetilde{\Psi}^{y,l}(x) = g_l^\sharp(|x-y|-l)\,.$$

Thus, by Lemma 2.20,

$$\Delta_p\phi = -\Delta_p\varphi > -(h_l^\sharp)'(\varphi) = \widetilde{h}_l'(-\varphi) = \widetilde{h}_l'(\phi)$$

at the point $x^\star$, which shows the desired subsolution property of $\widetilde{\Psi}^{y,l}$.

The reader may easily convince herself that $\widetilde{g}_l$ and $\widetilde{H}_l$ also enjoy the properties listed above. $\square$

We are now in the position to show that minimizers are trapped in between the functions constructed in Lemmata 2.20-2.22 (an exponential decay was also pointed out in §9 of [30]):

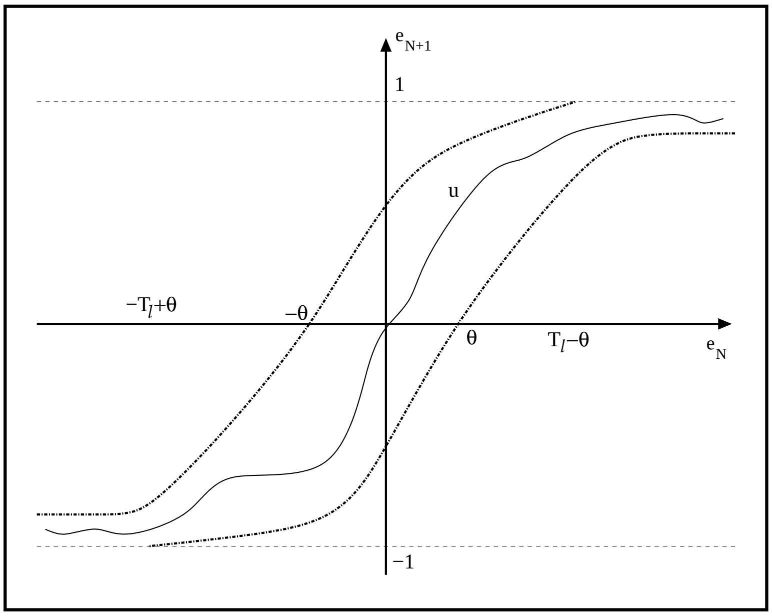
Lemma 2.23. *Let* $l>0$*,* $\theta>0$*. Let* u *be a local minimizer for* $\mathcal{F}$ *in*

$$\left\{ (x',x_N)\in\mathbb{R}^{N-1}\times\mathbb{R} \;\middle|\; |x'|<l\,,\ |x_N|<l \right\}.$$

Assume that $|u| \le 1$*, that* θ/l *is suitably small, that* $u(0) = 0$*, that* $u(x) > 0$ *if* $x_N \ge \theta$ *and that* $u(x) < 0$ *if* $x_N \le -\theta$*. Then, there exist suitable constants* κ*,* $\widetilde{\kappa}$*,* $\hat{\kappa} \in (0, 1]$ *so that*

$$\widetilde{g}_{\widetilde{\kappa}l}(x_N - \theta) \le u(x) \le g_{\kappa l}(x_N + \theta)\,,$$

at any point $x \in [-\hat{\kappa}l, \hat{\kappa}l]^N$*, provided that the functions above are defined at* x*.*



Trapping a minimizer between two barriers, as in Lemma 2.23

Proof. We proof the latter inequality, the first one being analogous. From Theorem 1.1 of [28] and the fact that $\{u = 0\} \subseteq \{|x_N| \le \theta\}$, it easily follows that

$$u(x) < -1 + \theta^\star \text{ for any } x \in \mathbb{R}^N \text{ with } x_N \le \kappa^\star \text{ and } |x'| \le l/2, \tag{2.64}$$

for some constant $\kappa^\star$ (which may depend on $\theta^\star$ and other universal constants).

Then, from (2.64) and the second item in Lemma 2.21,

$$u(x) \le \Psi^{y,\kappa l}(x)\,,$$

where $y := (0, \dots, 0, -l/2)$. Let now $e \in S^{N-1}$, with $e_N > 0$ and let us slide $\Psi^{y,\kappa l}$ in the e-direction until it touches u. Notice indeed that there exists a suitable constant $\bar{c} \in (0, 1)$ so that if

$$e_N \ge \bar{c}\,, \tag{2.65}$$

then $\Psi^{y+te,\kappa l}$ does touch u for some $t=t(e)>0$ at some point $x^\star = x^\star(e)\in [-l,l]^N$, that is

$$u(\mathfrak{x}) \le \Psi^{y+te,\kappa l}(\mathfrak{x}) = g_{\kappa l}(|\mathfrak{x}-(y+te)|-\kappa l)\,, \tag{2.66}$$

for any $\mathfrak{x}\in B_{T_{\kappa l}+\kappa l}(y+te)$, being the latter the domain where $\Psi^{y+te,\kappa l}$ is defined, and

$$u(x^\star) = \Psi^{y+te,\kappa l}(x^\star)\,.$$

In the light of the first item in Lemma 2.21, we have that

$$u(x^\star) = \Psi^{y+te,\kappa l}(x^\star) = 0$$

and so, since, by our hypotheses $\{\mathfrak{x}\mid u(\mathfrak{x})=0\}\subseteq\{|\mathfrak{x}_N|\le\theta\}$, we have that

$$|x^\star_N|\le\theta\,.$$

Let us now consider, for $d\ge 0$, the point

$$\mathfrak{x}=\mathfrak{x}(e) := x^\star + de_N\,.$$

Then,

$$\begin{aligned} |\mathfrak{x}-(y+te)| &\le |\mathfrak{x}-x^\star| + |x^\star-(y+te)| = \\ &= d+\kappa l = \\ &= \mathfrak{x}_N - x^\star_N + \kappa l \\ &\le \mathfrak{x}_N + \theta + \kappa l\,. \end{aligned}$$

Therefore, since g_l is increasing, (2.66) implies that

$$u(\mathfrak{x}) \le g_{\kappa l}(\mathfrak{x}_N+\theta)\,, \tag{2.67}$$

provided $\mathfrak{x}\in B_{T_{\kappa l}+\kappa l}(y+te)$.

With this inequality, we are now in the position to choose e here above in order to infer the desired result. We proceed in the following way: take $x=(x',x_N)\in[-\hat\kappa l,\hat\kappa l]^{N-1}\times[-\hat\kappa l,\hat\kappa l]$ and consider two cases.

If

$$\{\mathfrak{x}\in[-\hat\kappa l,\hat\kappa l]^N \mid \mathfrak{x}_N\le x_N\}\cap\{u=0\}\neq\emptyset\,, \tag{2.68}$$

take $x^\star$ so that $(x^\star)'=x'$, $u(x^\star)=0$ and $x^\star_N$ as low as possible (in particular, from (2.68), $x^\star_N\le x_N$). Let also, as above, $y:=(0,\dots,0,-l/2)$. Then, choosing

$$e := \frac{x^\star-y}{|x^\star-y|}\,,$$

we have that x and $x^\star$ here agree with $\mathfrak{x}(e)$ and $x^\star(e)$ constructed here above and that (2.65) is fulfilled provided $\hat\kappa$ is small enough. Thus, the desired result follows, in this case, from (2.67).

If, on the other hand,

$$\{\mathfrak{x}\in[-\hat\kappa l,\hat\kappa l]^N \mid \mathfrak{x}_N\le x_N\}\cap\{u=0\}=\emptyset\,, \tag{2.69}$$

notice that $x_N+\theta+\kappa l\ge\theta+(\kappa-\hat\kappa)l>0$, provided $\hat\kappa\le\kappa$ and define

$$y^\star := x-(x_N+\theta+\kappa l)e_N$$

and

$$e := \frac{y^\star - y}{|y^\star - y|}\,.$$

Notice also that $y^\star_N + \kappa l = -\theta$, hence

$$B_{\kappa l}(y^\star) \subseteq [-\kappa l, \kappa l]^{N-1} \times (-\infty, -\theta] \subseteq \{u < 0\}\,.$$

Therefore, by the first item in Lemma 2.21, we have that $\Psi^{y+te,\kappa l}$ does not touch u for $t \in [0, |y^\star - y|]$. In particular, for $t = |y^\star - y|$,

$$\begin{aligned} u(x) &\leq \Psi^{y+te,\kappa l}(x) = \\ &= \Psi^{y^\star,\kappa l}(x) = \\ &= g_{\kappa l}(x_N + \theta)\,. \end{aligned}$$

This proves the desired result also in case (2.69) holds and it completes the proof of Lemma 2.23. $\square$

Corollary 2.24. *Let $l > 0$, $\theta > 0$. Let u be a local minimizer for $\mathcal{F}$ in*

$$\Omega := \Big\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,\Big|\, |x'| < l\,,\ |x_N| < l\Big\}\,.$$

Assume that $|u| \leq 1$, that θ/l is suitably small, that $u(0) = 0$, that $u(x) > 0$ if $x_N \geq \theta$ and that $u(x) < 0$ if $x_N \leq -\theta$. Then, there exists a suitable constant $c \in (0, 1]$ so that

$$(-1 + s_{cl},\, 1 - s_{cl}) \subseteq u(\Omega)\,.$$

Proof. By Lemma 2.23 and the fact that $g_l(t) = -1 + s_l$ if $t \leq -l/2$ (recall Lemma 2.20), we have that, if $x_N \leq -3\kappa l/4$,

$$u(x) \leq g_{\kappa l}(x_N + \theta) = -1 + s_{\kappa l}\,,$$

provided θ/l small enough. Analogously, by Lemma 2.23 and the fact that $\widetilde{g}_l(t) = 1 - s_l$ if $t \geq l/2$ (recall Lemma 2.22), we get that, if $x_N \geq 3\widetilde{\kappa} l/4$,

$$u(x) \geq \widetilde{g}_{\widetilde{\kappa} l}(x_N - \theta) = 1 - s_{\widetilde{\kappa} l}\,.$$

The inequalities above and the continuity of u (see [15] or [34]) imply the desired result. $\square$

In the following, we will often slide the barriers in a given direction. More precisely, we will start from a configuration in which the barrier is above a subsolution u and then we slide the barrier until it touches the graph of u. With some poetry, we may think that the barrier is like a ship which moves forward until it touches the land u: of course, the ship will touch the land with the fore, not with the aft. Next result gives a formal justification of this fact:

Lemma 2.25. *Let $u \in C(\Omega)$. Let $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ with $|\xi| = 1$ and let $\hat{\xi} := (\xi, 0) \in \mathbb{R}^{N+1}$. Assume that $g_{\mathbb{S}(Y - t\hat{\xi}, R)} > u$ in their common domain of definition for any $t \in (0, 1]$. Assume also that $g_{\mathbb{S}(Y,R)}$ touches u from* *above at some point* $X = (x, x_{N+1})$*, that is, assume that* $g_{\mathbb{S}(Y,R)} \geq u$ *and* $g_{\mathbb{S}(Y,R)}(x) = u(x) = x_{N+1}$*. Then,*

$$(x - y) \cdot \xi \geq 0\,.$$

Proof. By construction,

$$\begin{aligned} & g_{y_{N+1},R}\Big(|x-y| + H_0(y_{N+1}) - R\Big) = \\ = \;& g_{\mathbb{S}(Y,R)}(x) = \\ = \;& u(x) \leq \\ \leq \;& g_{\mathbb{S}(Y-t\hat{\xi},R)}(x) = \\ = \;& g_{y_{N+1},R}\Big(|x-y+t\xi| + H_0(y_{N+1}) - R\Big) \end{aligned}$$

for any $t \in [0,1]$, from which

$$|x-y| \leq |x-y+t\xi|$$

for any $t \in [0,1]$. This says that the function

$$f(t) := |x-y+t\xi|^2$$

attains its minimum in the domain $[0,1]$ at $t = 0$. Thus, $f'(0) \geq 0$, which gives the desired estimate. □

3. Geometry of the touching points

This section, which is very technical, follows many of the ideas in Chapter 4 of [31] (we provide full details for the reader's facility). The main result of this section will be Proposition 3.14 here above, in which we investigate the measure theoretic properties of (an N-dimensional projection of) the set of possible touching points between a subsolution of (1.5) and the barrier $\mathbb{S}(Y,R)$ introduced here above. Roughly speaking, we will prove in Proposition 3.14 that the measure of the projection of the "first occurrence" touching points controls the measure of the projection of the centers of the corresponding surfaces.

For this scope, given

$$\xi := (\xi_1, \dots, \xi_N, 0) \in \mathbb{R}^{N+1}\,, \tag{3.1}$$

with $|\xi| = 1$, we define $\mathfrak{P}_\xi$ as the hyperplane in $\mathbb{R}^{N+1}$ orthogonal to ξ, i.e.,

$$\mathfrak{P}_\xi := \{X \in \mathbb{R}^{N+1} \mid \xi \cdot X = 0\}$$

We also denote by π_ξ the projection onto $\mathfrak{P}_\xi$, i.e.,

$$\pi_\xi(X) := X - (\xi \cdot X)\xi\,, \qquad \forall X \in \mathbb{R}^{N+1}\,.$$

With a slight abuse of notation, we will sometimes identify ξ with its N-dimensional projection, implicitly dropping the zero in the last coordinate, that is, we will write

$$\xi := (\xi_1, \dots, \xi_N) \in \mathbb{R}^N\,,$$

instead of (3.1).

Let now $u \in C^1([-C^\sharp l\,,\,C^\sharp l]^N)$ be a weak Sobolev subsolution of (1.5), with $|u| \le 1$. Here we will fix $C^\sharp$ suitably large (also, l and R are fixed and suitably large, and l/R is assumed conveniently small). Let us now define the set of first contact points as follows. Given a compact set[7] $\mathfrak{A} \subseteq \mathfrak{P}_\xi \subset \mathbb{R}^N$, let us consider $Y \in \mathfrak{A}$ and define

$$\begin{aligned}\widetilde{\mathfrak{A}} = \widetilde{\mathfrak{A}}_{\mathfrak{A}} \;:=\; & \{Y \in \mathbb{R}^{N+1} \quad \text{s.t.} \\ & \exists \hat{Y} \in \mathfrak{A},\, t_{\hat{Y}} \in \mathbb{R} \quad \text{s.t.} \quad Y = \hat{Y} + t_{\hat{Y}}\xi\,, \\ & g_{\mathbb{S}(\hat{Y}+t\xi,R)} > u \text{ for any } t < t_{\hat{Y}}, \\ & g_{\mathbb{S}(Y,R)} \ge u \\ & \text{and} \quad \exists x \in \mathbb{R}^{N+1} \quad \text{s.t.} \quad g_{\mathbb{S}(Y,R)}(x) = u(x)\}\,.\end{aligned}$$

We refer to $\widetilde{\mathfrak{A}}$ as the set of the "centers". Moreover, we define

$$\begin{aligned}\widetilde{\mathfrak{B}} = \widetilde{\mathfrak{B}}_{\mathfrak{A}} \;:=\; & \{X = (x, x_{N+1}) \in \mathbb{R}^{N+1} \quad \text{s.t.} \quad \exists Y \in \widetilde{\mathfrak{A}} \quad \text{s.t.} \\ & g_{\mathbb{S}(Y,R)} \ge u \qquad \text{and} \\ & g_{\mathbb{S}(Y,R)}(x) = u(x) = x_{N+1}\}\,.\end{aligned}$$

and we refer to $\widetilde{\mathfrak{B}}$ as the set of "first contact points". Roughly speaking, we are sliding our barriers until it touches the graph of u for the first time: the set $\widetilde{\mathfrak{B}}$ collects all such first occurrence contact points, while $\widetilde{\mathfrak{A}}$ collects the corresponding centers. In this section, we will assume that

$$\mathfrak{A} \subseteq [-C^\sharp l/2,\, C^\sharp l/2]^N \times [-1/4,\, 1/4] \tag{3.2}$$

and that

(3.3)

$$\Big\{X = (x, x_{N+1}) \in \mathbb{R}^{N+1} \text{ s.t. } \exists Y \in \mathbb{R}^{N+1} \text{ s.t. } \pi_\xi Y \in \mathfrak{A}\,,\; g_{\mathbb{S}(Y,R)} \ge u \quad \text{and} \\ g_{\mathbb{S}(Y,R)}(x) = u(x)\Big\} \cap \partial[-C^\sharp l,\, C^\sharp l]^N \;=\; \emptyset\,.$$

We also define

$$\mathfrak{B} := \pi_\xi(\widetilde{\mathfrak{B}})\,.$$

We also denote the graph of u by $\mathfrak{G}$, that is we set

$$\mathfrak{G} := \{x_{N+1} = u(x)\}\,. \tag{3.4}$$

We now show some properties of the above defined sets. First of all, from (3.3) and Corollary 2.14, we have that:

Lemma 3.1.

$$\widetilde{\mathfrak{B}} \subseteq \{X \in \mathbb{R}^{N+1} \mid |x_{N+1}| < 1/2\}\,.$$

[7]The closure of $\mathfrak{A}$ will be used to deduce closure and measurability properties of sets of interest: see, e.g., Lemma 3.2 and Proposition 3.14 here below.

We now show that the compactness property of $\mathfrak{A}$ is inherited by the other sets defined above:

Lemma 3.2. *If l/R is small enough, then $\widetilde{\mathfrak{A}}$, $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ are compact sets.*

Proof. Note that $\widetilde{\mathfrak{B}} \in [-C^\sharp l,\, C^\sharp l]^N \times [-1,1]$, hence $\widetilde{\mathfrak{B}}$ is bounded. Therefore, $\mathfrak{B}$ and $\widetilde{\mathfrak{A}}$ are also bounded. Thence, we only need to show that the above sets are closed. Let us first show that $\widetilde{\mathfrak{A}}$ is closed. For this, let $Y_k \in \widetilde{\mathfrak{A}}$ converge to some Y_∞. We need to show that $Y_\infty \in \widetilde{\mathfrak{A}}$.
For this scope, note that, since $\mathfrak{A}$ is closed and $\pi_\xi Y_k \in \pi_\xi(\widetilde{\mathfrak{A}}) \subseteq \mathfrak{A}$, we have that $\pi_\xi Y_\infty \in \mathfrak{A}$. Also, since $Y_k \in \widetilde{\mathfrak{A}}$, we have that there exists $X_k \in [-C^\sharp l,\, C^\sharp l]^N \times [-1,1]$ so that

$$\left\{ \begin{array}{c} g_{\mathbb{S}(Y_k,R)}(x_k) = u(x_k) \\ g_{\mathbb{S}(Y_k,R)} \geq u \qquad \text{and} \\ g_{\mathbb{S}(Y_k - t\xi,R)} > u\,, \end{array} \right. \tag{3.5}$$

for any $t > 0$. Of course, up to subsequence, we may assume that X_k converges to some point X_∞. Thus, passing to the limit (3.5), we obtain that

$$\left\{ \begin{array}{c} g_{\mathbb{S}(Y_\infty,R)}(x_\infty) = u(x_\infty) \\ g_{\mathbb{S}(Y_\infty,R)} \geq u \qquad \text{and} \\ g_{\mathbb{S}(Y_\infty - t\xi,R)} \geq u\,, \end{array} \right.$$

for any $t > 0$. Hence, to show that $Y_\infty \in \widetilde{\mathfrak{A}}$, we need to prove the strict inequality in the last relation here above, i.e., we need to show that

$$g_{\mathbb{S}(Y_\infty - t\xi,R)} > u \tag{3.6}$$

for any $t > 0$. We argue by contradiction: assume that there exists $x_\sharp$ so that $|x_\sharp| \leq \text{const}\, l$ and $t_\sharp > 0$, such that

$$g_{\mathbb{S}(Y_\infty - t_\sharp \xi,R)}(x_\sharp) = u(x_\sharp)\,.$$

Note that, by (3.3) and Corollary 2.14, $|g_{\mathbb{S}(Y_\infty,R)}(x_\infty)| \leq 1/2$, thus

$$|x_\infty - y_\infty| \;\geq\; \text{const}\, R\,. \tag{3.7}$$

Also, by (3.2), we get that

$$\begin{array}{rcl} |\pi_\xi(x_\infty - y_\infty)| & \leq & |x_\infty| + |\pi_\xi y_\infty| = \\ & = & |x_\infty| + |\pi_{e_N}(\pi_\xi Y_\infty)| \leq \\ & \leq & \text{const}\, l\,. \end{array}$$

Thus, from (3.7), we have that

$$\angle(x_\infty - y_\infty,\, \xi) \;\leq\; \text{const}\, l/R\,. \tag{3.8}$$

Furthermore, from Lemma 2.25,

$$(x_\infty - y_\infty) \cdot \xi \;\geq\; 0\,.$$

This, (3.8) and (3.7) say that

$$\begin{aligned}(x_\infty - y_\infty)\cdot\xi &= |(x_\infty - y_\infty)\cdot\xi| = \\ &= |x_\infty - y_\infty| \cos\angle(x_\infty - y_\infty,\, \xi) \geq \\ &\geq \ \mathrm{const}\, R\,,\end{aligned}$$

provided that l/R is small enough. For this reason,

$$\begin{aligned}(x_\sharp - y_\infty)\cdot\xi &\geq (x_\infty - y_\infty)\cdot\xi - |x_\infty| - |x_\sharp| \geq \\ &\geq \mathrm{const}\, R - \mathrm{const}\, l \geq \\ &\geq 0\,,\end{aligned} \tag{3.9}$$

if l/R is small enough. Thus, from (3.9), we deduce that

$$\begin{aligned}|x_\sharp - y_\infty + t_\sharp \xi|^2 &= |x_\sharp - y_\infty|^2 + t_\sharp^2 + 2t_\sharp \xi\cdot(x_\sharp - y_\infty) > \\ &> |x_\sharp - y_\infty|^2\,.\end{aligned}$$

As a result, we infer that

$$\begin{aligned}u(x_\sharp) &= g_{\mathbb{S}(Y_\infty - t_\sharp\xi, R)}(x_\sharp) = \\ &= g_{y_{\infty,N+1},R}\Big(|x_\sharp - y_\infty + t_\sharp\xi| + H_0(y_{\infty,N+1}) - R\Big) > \\ &> g_{y_{\infty,N+1},R}\Big(|x_\sharp - y_\infty| + H_0(y_{\infty,N+1}) - R\Big) = \\ &= g_{\mathbb{S}(Y_\infty,R)}(x_\sharp) \geq \\ &\geq u(x_\sharp)\,.\end{aligned}$$

This contradiction proves (3.6). Hence, $Y_\infty \in \widetilde{\mathfrak{A}}$ and therefore $\widetilde{\mathfrak{A}}$ is closed.

Note now that once we know that $\widetilde{\mathfrak{B}}$ is closed, it easily follows that $\mathfrak{B}$ is also closed. Thus, to end the proof of this result, we need to prove that $\widetilde{\mathfrak{B}}$ is closed. For this, let us consider a sequence $X_k \in \widetilde{\mathfrak{B}}$ so that

$$\lim_{k\to+\infty} X_k = X_\infty\,.$$

Our aim is to show that $X_\infty \in \widetilde{\mathfrak{B}}$. For this, observe that, since $X_k \in \widetilde{\mathfrak{B}}$, there exists $Y_k \in \widetilde{\mathfrak{A}}$ such that $g_{\mathbb{S}(Y_k,R)} \geq u$ and $g_{\mathbb{S}(Y_k,R)}(x_k) = u(x_k) = x_{k,N+1}$. Since we proved that $\widetilde{\mathfrak{A}}$ is compact, possibly taking subsequences, we may assume that Y_k tends to $Y_\infty \in \widetilde{\mathfrak{A}}$. Thus, passing to the limit here above we deduce that $g_{\mathbb{S}(Y_\infty,R)} \geq u$ and $g_{\mathbb{S}(Y_\infty,R)}(x_\infty) = u(x_\infty) = x_{\infty,N+1}$, with $Y_\infty \in \widetilde{\mathfrak{A}}$. This proves that $X_\infty \in \widetilde{\mathfrak{B}}$, thence $\widetilde{\mathfrak{B}}$ is closed. $\square$

Lemma 3.3. *For any* $X \in \mathbb{R}^{N+1}$ *with* $X \in \widetilde{\mathbb{S}}(Y,R)$ *and* $\nu \in S^N$, *let*[8]

$$\begin{aligned}
\nu^{\widetilde{\mathbb{S}}(Y,R)}(X) &:= \frac{\Big(-\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x),\, 1\Big)}{\sqrt{1+|\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x)|^2}} \in \mathbb{R}^{N+1}\,, \\
\omega(X,\nu) &:= \frac{R}{\overline{C}_0}\left(\frac{\nu_{N+1}}{|(\nu_1,\dots,\nu_N)|} - H_0'(x_{N+1})\right) \in \mathbb{R}, \\
\sigma(X,\nu) &:= -\frac{\overline{C}_0}{2R}\omega^2(X) + H_0(x_{N+1}) - H_0(x_{N+1}+\omega) + R \in \mathbb{R}, \\
F(X,\nu) &:= \left(x + \frac{(\nu_1,\dots,\nu_N)}{|(\nu_1,\dots,\nu_N)|}\sigma(X,\nu),\ x_{N+1} + \omega(X,\nu)\right) \in \mathbb{R}^{N+1}\,.
\end{aligned}$$

Then,

$$Y = F(X, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X))\,, \tag{3.10}$$

for any $X \in \widetilde{\mathbb{S}}(Y,R)$.

Proof. For short, we will set here $\widetilde{\mathbb{S}} := \widetilde{\mathbb{S}}(Y,R)$. Recalling Lemma B.11, we see that there exists $\sigma \in \mathbb{R}$ so that

$$x - y = \sigma\,\frac{(\nu_1^{\widetilde{\mathbb{S}}}(X),\dots,\nu_N^{\widetilde{\mathbb{S}}}(X))}{|(\nu_1^{\widetilde{\mathbb{S}}}(X),\dots,\nu_N^{\widetilde{\mathbb{S}}}(X))|}\,.$$

Using that $x_{N+1} = g_{\widetilde{\mathbb{S}}(Y,R)}(x)$, (2.31), (2.33) and (2.35), we also gather that

$$H_0(y_{N+1}) + |x-y| - R = H_0(x_{N+1}) - \frac{\overline{C}_0}{2R}(x_{N+1}-y_{N+1})^2\,.$$

Hence, if $\omega := y_{N+1} - x_{N+1}$, we gather

$$\sigma = |x-y| = H_0(x_{N+1}) - H_0(x_{N+1}+\omega) + R - \frac{\overline{C}_0}{2R}\omega^2\,.$$

This determines σ, as desired, we now need to determine ω. Using (2.2) (with $s_0 := y_{N+1}$ and $s := x_{N+1}$) and Definition 2.5 (with the fact that $|s| < 1/2$ as pointed out above), we get

$$\begin{aligned}
\omega &= -(x_{N+1} - y_{N+1}) = \\
&= \frac{R}{\overline{C}_0}\left(\frac{1}{\Big(\frac{p}{p-1}\varphi(x_{N+1})\Big)^{1/p}} - \frac{1}{\Big(\frac{p}{p-1}h_0(x_{N+1})\Big)^{1/p}}\right) = \\
&= \frac{R}{\overline{C}_0}\left(\frac{1}{\Big(\frac{p}{p-1}h_{s_0,R}(x_{N+1})\Big)^{1/p}} - \frac{1}{\Big(\frac{p}{p-1}h_0(x_{N+1})\Big)^{1/p}}\right)\,.
\end{aligned}$$

[8] Of course, $\nu^{\widetilde{\mathbb{S}}(Y,R)}(X)$ is normal to $\widetilde{\mathbb{S}}(Y,R)$ at the point X.

Thus, from Definition 2.12 and (2.42), we have that

$$\begin{aligned}\omega &= \frac{R}{\overline{C}_0}\Big(H'_{s_0,R}(x_{N+1}) - H'_0(x_{N+1})\Big) = \\ &= \frac{R}{\overline{C}_0}\Big(\frac{1}{|\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x)|} - H'_0(x_{N+1})\Big)\,,\end{aligned}$$

and, therefore,

$$\omega = \frac{R}{\overline{C}_0}\Big(\frac{\nu^{\widetilde{\mathbb{S}}}_{N+1}(X)}{|(\nu^{\widetilde{\mathbb{S}}}_1(X),\dots,\nu^{\widetilde{\mathbb{S}}}_N(X))|} - H'_0(x_{N+1})\Big)\,,$$

which determines ω. □

Corollary 3.4. *Let the notation of Lemma 3.3 holds. For $X = (x, x_{N+1}) \in \mathbb{R}^N$, with $x_{N+1} = u(x)$, let*[9]

$$\nu^u(X) \;:=\; \frac{\Big(-\nabla u(x),\,1\Big)}{\sqrt{1+|\nabla u(x)|^2}} \in \mathbb{R}^{N+1}\,, \tag{3.11}$$

Let $X, Y \in \mathbb{R}^N$ be so that

$$\begin{aligned} &g_{\mathbb{S}(Y,R)}(z) \geq u(z) \qquad \forall z \in \mathbb{R}^N\,, \\ &g_{\mathbb{S}(Y,R)}(x) = u(x) = x_{N+1}\,. \end{aligned} \tag{3.12}$$

Then, $Y = F(X, \nu^u(X))$.

Proof. By (3.12), we have that the graph of u and the surface $\mathbb{S}(Y,R)$ are tangent at the point X, therefore

$$\nu^u(X) \;=\; \nu^{\mathbb{S}(Y,R)}(X)\,,$$

hence the claim follows from Lemma 3.3. □

Lemma 3.5. *In the notation of Corollary 3.4, if $X \in \widetilde{\mathfrak{B}}$, then there exists a positive constant c so that*

$$c \leq \nu^u_{N+1}(X) \leq 1 - c\,.$$

Proof. Since, by Lemma 3.1, $|x_{N+1}| \leq 1/2$,

$$|\nabla u(x)| = |\nabla g_{\mathbb{S}(Y,R)}(x)| = \left|\Big(\frac{p}{p-1}h_{s_0,R}(x_{N+1})\Big)^{1/p}\right| \in [1/C,\,C]\,,$$

for a suitable constant C, which implies the desired claim. □

Lemma 3.6. *Let the notation of Lemma 3.3 and Corollary 3.4 hold. Let*

$$\mathcal{Y}(X) := F(X, \nu^u(X))$$

and let $D_X\mathcal{Y}$ be the differential map. Then, there exists a positive constant C such that

$$|D_X\mathcal{Y}(X)| \leq C\,,$$

[9] Note that $\nu^u(X)$ is a unit vector, normal to the graph of u at the point $X = (x, u(x))$.

for any $X \in \widetilde{\mathfrak{B}}$.

Proof. By direct inspection,

$$D_X \mathcal{Y}(X) = D_X F(X, \nu^u(X)) + D_\nu F(X, \nu^u(X))\, D_X \nu^u(X)\,. \tag{3.13}$$

On the other hand, by differentiating (3.10),

$$0 = D_X F(X, \nu^{\mathbb{S}(Y,R)}(X)) + D_\nu F(X, \nu^{\mathbb{S}(Y,R)}(X))\, D_X \nu^{\mathbb{S}(Y,R)}(X)\,. \tag{3.14}$$

Moreover, if $X \in \widetilde{\mathfrak{B}}$, then

$$\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x) = \nabla u(x)\,, \tag{3.15}$$

and so

$$\nu^u(X) = \nu^{\widetilde{\mathbb{S}}(Y,R)}(X)\,.$$

Thus, from (3.13) and (3.14), we gather that

$$D_X \mathcal{Y}(X) = D_\nu F(X, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X)) \Big(D_X \nu^u(X) - D_X \nu^{\widetilde{\mathbb{S}}(Y,R)}(X) \Big)\,,$$

for any $X \in \widetilde{\mathfrak{B}}$. By the definitions given in Lemma 3.3, one sees that

$$|D_\nu F(X, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X))| \le \text{ const}\, R\,,$$

and so we get from the above relation that

$$|D_X \mathcal{Y}(X)| \le \text{ const}\, R \Big| D_X \nu^u(X) - D_X \nu^{\widetilde{\mathbb{S}}(Y,R)}(X) \Big|\,.$$

Therefore, in the light of (3.15), Lemma B.12, Remark B.13 and Lemma 3.5,

$$|D_X \mathcal{Y}(X)| \le \text{ const}\, R \Big| D^2 u(X) - D^2 g_{\widetilde{\mathbb{S}}(Y,R)}(X) \Big|\,.$$

Also, since $g_{\widetilde{\mathbb{S}}(Y,R)}$ touches u from above at X, we have that

$$D^2 g_{\widetilde{\mathbb{S}}(Y,R)}(X) - D^2 u(X) \text{ is a non-negative definite matrix} \tag{3.16}$$

and therefore

$$\text{const} \Big| D^2 u(X) - D^2 g_{\widetilde{\mathbb{S}}(Y,R)}(X) \Big| \le \Delta \Big(g_{\widetilde{\mathbb{S}}(Y,R)} - u \Big)(X)\,.$$

Therefore, we have obtained the following estimate:

$$|D_X \mathcal{Y}(X)| \le \text{ const}\, R\, \Delta \Big(g_{\widetilde{\mathbb{S}}(Y,R)} - u \Big)(X)\,. \tag{3.17}$$

Notice now that u is C^2 near X by standard elliptic results, since $\nabla u(X) \neq 0$ thanks to (3.15). Thence, making use of (3.15), we get

$$\begin{aligned} & \Delta_p g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta_p u(X) = \\ = \ & |\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X)|^{p-2} \Big(\Delta g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta u(X) \Big) + \\ & + (p-2)\, |\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X)|^{p-4}\, \nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X) \cdot \\ & \cdot \Big[\Big(D^2 g_{\widetilde{\mathbb{S}}(Y,R)}(X) - D^2 u(X) \Big) \cdot \nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X) \Big] \end{aligned} \tag{3.18}$$

We need now to distinguish two cases. If $p \geq 2$, we use Lemma 3.5 and (3.16) in order to deduce from (3.18) that

$$\begin{aligned} &\Delta_p g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta_p u(X) \geq \\ \geq\ & |\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X)|^{p-2} \Big(\Delta g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta u(X)\Big) \geq \\ \geq\ & \text{const}\, \Big(\Delta g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta u(X)\Big)\,. \end{aligned} \tag{3.19}$$

On the other hand, if $1 < p < 2$, (3.18), Lemma 3.5 and (3.16) give that

$$\begin{aligned} &\Delta_p g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta_p u(X) \geq \\ \geq\ & |\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X)|^{p-2} \Big(\Delta g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta u(X)\Big) - \\ & - (2-p)\, |\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X)|^{p-2}\, |D^2 g_{\widetilde{\mathbb{S}}(Y,R)}(X) - D^2 u(X)| \geq \\ \geq\ & |\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(X)|^{p-2}\, [1-(2-p)] \Big(\Delta g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta u(X)\Big) \geq \\ \geq\ & \text{const}\, (p-1) \Big(\Delta g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta u(X)\Big)\,. \end{aligned} \tag{3.20}$$

In any case, for any $p \in (1, +\infty)$, (3.19) and (3.20) give that

$$\begin{aligned} &\Delta_p g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta_p u(X) \geq \\ \geq\ & \text{const}\, \Big(\Delta g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta u(X)\Big)\,. \end{aligned} \tag{3.21}$$

Furthermore, exploiting Proposition 2.18 and the fact that u is a subsolution of (1.5), we have that

$$\Delta_p g_{\widetilde{\mathbb{S}}(Y,R)}(X) - \Delta_p u(X) \leq \frac{C}{R}\,. \tag{3.22}$$

The desired result now follows from (3.17), (3.21) and (3.22). □

Recalling (3.4), we define

$$\mathfrak{S} := \mathcal{Y}(\mathfrak{G})\,.$$

The construction of $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$ easily implies the following observation:

Lemma 3.7. *$\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$ belong to Lipschitz surfaces. More precisely, $\widetilde{\mathfrak{B}}$ lies in $\mathfrak{G}$ while $\widetilde{\mathfrak{A}}$ lies in $\mathfrak{S}$.*

The next observation will say that $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$ are graphs with respect to the ξ-direction:

Lemma 3.8. *π_ξ is injective on $\widetilde{\mathfrak{A}}$ and on $\widetilde{\mathfrak{B}}$.*

Proof. π_ξ in injective on $\widetilde{\mathfrak{A}}$ by construction. Let us show that is also injective on $\widetilde{\mathfrak{B}}$. Assume, by contradiction, that $X^{(1)} \in \mathfrak{B}$ and $X^{(2)} = X^{(1)} + \tau\xi \in \mathfrak{B}$, with $\tau > 0$. Let also $Y^{(1)}$ and $Y^{(2)}$ be the corresponding centers in $\widetilde{\mathfrak{A}}$, i.e., for $i = 1, 2$, let $Y^{(i)} \in \widetilde{\mathfrak{A}}$ be so that $g_{\mathbb{S}(Y^{(i)},R)}$ touches u for the first time at $X^{(i)}$.

Since $\xi_{N+1} = 0$,

$$x^{(1)}_{N+1} = x^{(2)}_{N+1} ,$$

therefore,

$$\begin{aligned} g_{\mathbb{S}(Y^{(1)},R)}(x^{(1)}) &= x^{(1)}_{N+1} = u(x^{(1)}) = \\ &= x^{(2)}_{N+1} = g_{\mathbb{S}(Y^{(2)},R)}(x^{(2)}) = u(x^{(2)}) . \end{aligned} \tag{3.23}$$

If now we consider

$$\bar Y := Y^{(2)} - \tau\xi$$

it follows that

$$g_{\mathbb{S}(\bar Y,R)}(x^{(1)}) = g_{\mathbb{S}(Y^{(2)}-\tau\xi,R)}(x^{(2)} - \tau\xi) = g_{\mathbb{S}(Y^{(2)},R)}(x^{(2)}) = u(x^{(1)}) . \tag{3.24}$$

On the other hand, since $Y^{(2)} \in \widetilde{\mathfrak{A}}$, $g_{\mathbb{S}(\bar Y,R)} > u$, in contradiction with (3.24). □

Given $X \in \mathbb{R}^{N+1}$, we now define

$$\Sigma_X := \{ Z \mid X \in \widetilde{\mathbb{S}}(Z,R) \} . \tag{3.25}$$

In other words, given a point X, Σ_X is the surface containing all the centers of the surfaces $\widetilde{\mathbb{S}}(\cdot,R)$ to which X belongs. Let us now investigate the properties of Σ_X

Lemma 3.9. *Let $\mathcal{Y}$ be as in Lemma 3.6. Then, $\mathcal{Y}(X) \in \Sigma_X$ for any $X \in \widetilde{\mathfrak{B}}$.*

Proof. Since $X \in \widetilde{\mathfrak{B}}$, there exists Y so that (3.12) holds. Then, by Corollary 3.4,

$$\Sigma_X \ni Y = F(X, \nu^u(X)) = \mathcal{Y}(X) .$$

□

We denote by $\nu^{\Sigma_X}(Z)$ the unit normal vector to the surface Σ_X at a point $Z \in \Sigma_X$ (in a fixed orientation). Such definition is well posed, since Σ_X is a Lipschitz surface, as we show here below. Also, we can express ν^{Σ_X} in terms of the normal to $\widetilde{\mathbb{S}}(Y,R)$, according to the following result:

Lemma 3.10. *Let $\mathcal{Y}$ be as in Lemma 3.6 and let $X \in \widetilde{\mathfrak{B}}$. Then Σ_X is a Lipschitz rotation surface with axis parallel to e_{N+1} and passing through X. Also, $\nu^{\Sigma_X}(\mathcal{Y}(X))$ belongs to the space spanned by e_{N+1} and $\nu^{\widetilde{\mathbb{S}}(\mathcal{Y}(X),R)}(X)$.*

Proof. Let us first show that Σ_X is a Lipschitz rotation surface. By (2.35), we have that $Z \in \Sigma_X$ if and only if

$$x_{N+1} = g_{\widetilde{\mathbb{S}}(Z,R)}(x) = \rho_{z_{N+1},R}(H_0(z_{N+1}) + |z - x| - R) ,$$

that is, by (2.33) and (2.31), if and only if

$$H_0(x_{N+1}) - \frac{\overline{C}_0}{2R}(x_{N+1} - z_{N+1})^2 = H_0(z_{N+1}) + |z - x| - R .$$

We now define

$$\mathfrak{H}_{x_{N+1},R}(\zeta) := H_0(\zeta) + \frac{\overline{C}_0}{2R}(x_{N+1}-\zeta)^2 .$$

Then, $\mathfrak{H}$ is strictly increasing in $[-1/2, 1/2]$; thus $Z \in \Sigma_X$ if and only if

$$z_{N+1} = \mathfrak{H}^{-1}_{x_{N+1},R}\Big(H_0(x_{N+1}) + R - |z-x|\Big) .$$

This proves the rotational symmetry and Lipschtiz properties of Σ_X.

Consequently, by Lemma B.10, we gather that $\nu^{\Sigma_X}(\mathcal{Y}(X))$ is in the space spanned by $(\pi_{e_{N+1}}\mathcal{Y}(X) - x)$ and e_{N+1}. But $\nu^{\widetilde{\mathbb{S}}(\mathcal{Y}(X),R)}(X)$ is also in the space spanned by $(\pi_{e_{N+1}}\mathcal{Y}(X) - x)$ and e_{N+1}, as follows by the radial symmetry of $\widetilde{\mathbb{S}}(\mathcal{Y}(X), R)$ and Lemma B.10 again; moreover, $\nu^{\widetilde{\mathbb{S}}(\mathcal{Y}(X),R)}(X)$ is not parallel to e_{N+1} (because $|x_{N+1}| \le 1/2$ due to Lemma 3.1 and so $\nabla g_{\mathbb{S}(\mathcal{Y}(X),R)}(x) \neq 0$). Therefore, $\nu^{\Sigma_X}(\mathcal{Y}(X))$ belongs to the space spanned by $\nu^{\widetilde{\mathbb{S}}(\mathcal{Y}(X),R)}(X)$ and e_{N+1}. □

Lemma 3.11. *There exists a positive constant C such that*

$$|\nu^{\Sigma_X}(\mathcal{Y}(X)) \cdot \xi| \le C\, |\nu^u(X) \cdot \xi| ,$$

for any $X \in \widetilde{\mathfrak{B}}$.

Proof. From Lemma 3.10, for any $X \in \widetilde{\mathfrak{B}}$, we have that

$$\nu^{\Sigma_X}(\mathcal{Y}(X)) = \alpha\hat{\nu} + \beta e_{N+1} , \tag{3.26}$$

for some $\alpha = \alpha(X)$ and $\beta = \beta(X) \in \mathbb{R}$, where we denoted

$$\hat{\nu} := \Big(\nu_1^{\widetilde{\mathbb{S}}(\mathcal{Y}(X),R)}(X), \dots, \nu_N^{\widetilde{\mathbb{S}}(\mathcal{Y}(X),R)}(X), 0\Big) .$$

Obviously, since $X \in \widetilde{\mathfrak{B}}$,

$$\hat{\nu} \;=\; \Big(\nu_1^u(X), \dots, \nu_N^u(X), 0\Big) .$$

Thence, by exploiting Lemma 3.5, we see that

$$|\hat{\nu}|^2 = 1 - |\nu_{N+1}^{\widetilde{\mathbb{S}}(\mathcal{Y}(X),R)}(X)|^2 \ge 1 - (1-c)^2 \ge c .$$

Thus, (3.26) implies that

$$\begin{aligned} c\,|\alpha| \;&\le\; |\alpha\hat{\nu}\cdot\hat{\nu}| = \\ &=\; \Big|\Big(\nu^{\Sigma_X}(\mathcal{Y}(X)) - \beta e_{N+1}\Big)\cdot\hat{\nu}\Big| = \\ &=\; \Big|\nu^{\Sigma_X}(\mathcal{Y}(X))\cdot\hat{\nu}\Big| \le \\ &\le\; 1 , \end{aligned}$$

that is

$$|\alpha| \;\le\; \text{const} .$$

Hence, being $\xi_{N+1} = 0$, (3.26) gives that

$$|\nu^{\Sigma_X}(\mathcal{Y}(X))\cdot\xi| = |\alpha\hat{\nu}\cdot\xi| \le\; \text{const}\,|\hat{\nu}\cdot\xi| =\; \text{const}\,|\nu^u\cdot\xi| .$$

□

Lemma 3.12. *Let $X \in \mathfrak{G}$, with $|x_{N+1}| \le 1/2$. Assume that $g_{\widetilde{\mathbb{S}}(Y,R)} \ge u$. Then, Y is above Σ_X (with respect to the e_{N+1}-direction).*

Proof. Assume that $(y, y^*_{N+1}) \in \Sigma_X$. We need to show that $y^*_{N+1} \le y_{N+1}$. For this purpose, note that, by construction, $X \in \widetilde{\mathbb{S}}((y, y^*_{N+1}), R)$, which says that

$$g_{\widetilde{\mathbb{S}}((y,y^*_{N+1}),R)}(x) = x_{N+1} = u(x) \le g_{\widetilde{\mathbb{S}}(Y,R)}(x)\,. \tag{3.27}$$

Fix now R, x and y. For $|\zeta| \le 1/2$, we define

$$f(\zeta) := g_{\widetilde{\mathbb{S}}((y,\zeta),R)}(x)\,.$$

Recalling (2.35) and (2.31), we have that

$$H_0(f(\zeta)) - \frac{\overline{C}_0}{2R}(\zeta - f(\zeta))^2 = H_0(\zeta) + |x - y| - R\,.$$

Differentiating with respect to ζ, we get that

$$H_0'(f(\zeta))\, f'(\zeta) \ge H_0'(\zeta) - \frac{\mathrm{const}}{R}\,.$$

In particular, since by definition $|f(\zeta)| \le 1/2$, we get that $f'(\zeta) > 0$ if R is large enough, thence f is increasing. Since, by (3.27), we have that

$$f(y^*_{N+1}) \le f(y_{N+1})\,,$$

we deduce that $y^*_{N+1} \le y_{N+1}$, as desired. □

Lemma 3.13. *Let $X \in \widetilde{\mathfrak{B}}$. Then $\widetilde{\mathfrak{A}}$ touches Σ_X from above at $\mathcal{Y}(X)$.*

Proof. First, note that

$$\mathcal{Y}(X) \in \mathcal{Y}(\widetilde{\mathfrak{B}}) \subseteq \widetilde{\mathfrak{A}}\,.$$

On the other hand, $\mathcal{Y}(X) \in \Sigma_X$ by Lemma 3.9. Thus, to end the proof of this result we need to show that $\widetilde{\mathfrak{A}}$ is above Σ_X (with respect to the e_{N+1}-direction). For this purpose, take $Y \in \widetilde{\mathfrak{A}}$. Then, by construction, $g_{\widetilde{\mathbb{S}}(Y,R)} \ge u$ (and equality holds at some point). Thus, by Lemma 3.12, Y is above Σ_X. □

The following is the main result of this section, in which a measure estimate for contact points is given:

Proposition 3.14. *Let $1/c \le l \le cR$, for a suitably small positive constant c. Assume (3.2) and (3.3). Assume also that $\mathfrak{A}$ is the closure of an open set and that*

$$\text{for any } Y \in \mathfrak{A} \text{ there exist } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^N, \text{ such that } g_{\mathbb{S}(Y+t\xi,R)}(x) \le u(x). \tag{3.28}$$

Then, denoting the N-dimensional Lebesgue measure by $\mathfrak{L}^N$, we have that

$$\mathfrak{L}^N(\mathfrak{A}) \le C\,\mathfrak{L}^N(\mathfrak{B})\,,$$

for a suitable positive universal constant C.

Proof. Note that $\mathfrak{A}$ is closed by hypothesis and so is $\mathfrak{B}$ thanks to Lemma 3.2; in particular, $\mathfrak{A}$ and $\mathfrak{B}$ are measurable sets. Also, by (3.28), $\mathfrak{A} = \pi_\xi(\widetilde{\mathfrak{A}})$. What is more, since $\mathfrak{A}$ is the closure of an open set, Lemma 3.7 and Lemma 3.8 say that $\widetilde{\mathfrak{A}}$ is a Lipschitz surface which is also a continuous[10] graph over $\mathfrak{A}$ in the ξ-direction. In particular, the normal $\nu^{\widetilde{\mathfrak{A}}}$ is well defined. Thanks to Lemma 3.13, we also have that

$$\nu^{\Sigma_X}(\mathcal{Y}(X)) = \nu^{\widetilde{\mathfrak{A}}}(\mathcal{Y}(X))$$

for any $X \in \widetilde{\mathfrak{B}}$, provided $\pi_\xi(\mathcal{Y}(X))$ lies in the interior of $\mathfrak{A}$ (up to an orientation choice). Therefore, by Lemma 3.11,

$$|\nu^{\widetilde{\mathfrak{A}}}(\mathcal{Y}(X)) \cdot \xi| \le \text{const}\, |\nu^u(X) \cdot \xi|\,, \tag{3.29}$$

where $\nu^u(X)$ is the normal to $\mathfrak{G}$ at X, for any $X \in \widetilde{\mathfrak{B}}$, provided $\pi_\xi(\mathcal{Y}(X))$ lies in the interior of $\mathfrak{A}$. We denote by $\widetilde{\mathfrak{B}}'$ this set, that is

$$\widetilde{\mathfrak{B}}' := \Big\{ X \in \widetilde{\mathfrak{B}}, \text{ with } \pi_\xi(\mathcal{Y}(X)) \text{ in the interior of } \mathfrak{A} \Big\}.$$

Also, applying the divergence theorem to (the interior of) $\widetilde{\mathfrak{A}}$, we have that

$$\mathfrak{L}^N(\mathfrak{A}) \le \int_{\widetilde{\mathfrak{A}}} |\nu^{\widetilde{\mathfrak{A}}} \cdot \xi|\,, \tag{3.30}$$

where the above is a surface integral.

For $\varepsilon > 0$, let us now define

$$\widetilde{\mathfrak{B}}_\varepsilon := \bigcup_{X \in \widetilde{\mathfrak{B}}'} B_\varepsilon(Y) \cap \mathfrak{G}\,,$$

where $\mathfrak{G}$, as in (3.4), denotes the graph of u. Then, $\widetilde{\mathfrak{B}}_\varepsilon$ is a Lipschitz surface contained in $\mathfrak{G}$. By the divergence theorem,

$$\mathfrak{L}^N\Big(\pi_\xi(\widetilde{\mathfrak{B}}_\varepsilon)\Big) = \int_{\widetilde{\mathfrak{B}}_\varepsilon} \nu^{\widetilde{\mathfrak{B}}_\varepsilon} \cdot \xi\,,$$

where $\nu^{\widetilde{\mathfrak{B}}_\varepsilon}$ is the external normal of the surface $\widetilde{\mathfrak{B}}_\varepsilon$. Obviously, up to the sign, $\nu^{\widetilde{\mathfrak{B}}_\varepsilon}$ agrees with ν^u. Sending ε to zero, we thus get

$$\mathfrak{L}^N(\mathfrak{B}) = \mathfrak{L}^N\Big(\pi_\xi(\widetilde{\mathfrak{B}})\Big) \ge \mathfrak{L}^N\Big(\pi_\xi(\widetilde{\mathfrak{B}}')\Big) = \int_{\widetilde{\mathfrak{B}}'} \nu^{\widetilde{\mathfrak{B}}_\varepsilon} \cdot \xi\,. \tag{3.31}$$

Also, by Lemma 3.8, the exterior normal $\nu^{\widetilde{\mathfrak{B}}_\varepsilon}$ in $\widetilde{\mathfrak{B}}$ has the signed assigned by the property that $\nu^{\widetilde{\mathfrak{B}}_\varepsilon} \cdot \xi \ge 0$. Therefore,

$$|\nu^u \cdot \xi| = |\nu^{\widetilde{\mathfrak{B}}_\varepsilon} \cdot \xi| = \nu^{\widetilde{\mathfrak{B}}_\varepsilon} \cdot \xi$$

[10] The continuity of $(\pi_\xi\big|_{\widetilde{\mathfrak{A}}})^{-1}$ follows from the following elementary property: if $f : K \longrightarrow f(K)$ is continuous and injective and K is compact, then f^{-1} is continuous.

in $\widetilde{\mathfrak{B}}$. Thus, by (3.31), we get that

$$\mathfrak{L}^N(\mathfrak{B}) \geq \int_{\widetilde{\mathfrak{B}}'} |\nu^u \cdot \xi| \,. \tag{3.32}$$

Also, by construction, $\mathcal{Y}$ sends $\widetilde{\mathfrak{B}}$ into $\widetilde{\mathfrak{A}}$; hence, by (3.30), the change of variables formula (see, e.g., page 99 in [18]) and Lemma 3.6, we get that

$$\begin{aligned} \mathfrak{L}^N(\mathfrak{A}) &\leq \int_{\widetilde{\mathfrak{A}}} |\nu^{\widetilde{\mathfrak{A}}}(Y) \cdot \xi| \, dY \leq \\ &\leq \int_{\widetilde{\mathfrak{B}}'} |\nu^{\widetilde{\mathfrak{A}}}(\mathcal{Y}(X)) \cdot \xi| \, |\det D\mathcal{Y}(X)| \, dX \leq \\ &\leq \text{const} \int_{\widetilde{\mathfrak{B}}'} |\nu^{\widetilde{\mathfrak{A}}}(\mathcal{Y}(X)) \cdot \xi| \, dX \,. \end{aligned} \tag{3.33}$$

Thence, from (3.32), (3.29) and (3.33), we have that

$$\begin{aligned} \mathfrak{L}^N(\mathfrak{A}) &\leq \text{const} \int_{\widetilde{\mathfrak{B}}'} |\nu^{\widetilde{\mathfrak{A}}}(\mathcal{Y}(X)) \cdot \xi| \, dX \\ &\leq \text{const} \int_{\widetilde{\mathfrak{B}}'} |\nu^u(X) \cdot \xi| \, dX \\ &\leq \mathfrak{L}^N(\mathfrak{B}) \,. \end{aligned}$$

□

4. Measure theoretic results

This section collects some measure theory lemmata, which are extensions of analogous results in [31]. These lemmata will be used in the sequel for estimating the measure of the projection of the set in which a suitable barrier touches a minimal solution of (1.5). For the reader's convenience, the proofs of the lemmata of this section are deferred to the Appendix.

Given two vectors v and w, we define $\angle(v, w)$ to be the (positive) angle between these vectors, i.e.,

$$\angle(v, w) := \left| \arccos \frac{v \cdot w}{|v| \, |w|} \right| \,.$$

By elementary geometric consideration, if $|v| = |w| = 1$,

$$|v - w| \leq \angle(v, w) \,. \tag{4.1}$$

We also define, for $l > 0$,

$$\begin{aligned} L &:= \Big\{ (x', 0, x_{N+1}) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R} \mid |x_{N+1}| \leq 1/2 \Big\} \,, \\ Q_l &:= \Big\{ (x', 0, x_{N+1}) \in L \mid |x'| \leq l \Big\} \,. \end{aligned}$$

Of course,

$$\mathfrak{L}^N(Q_l) = \text{const} \, l^{N-1} \,.$$

Also, for $X = (x_1, \dots, x_{N+1}) \in \mathbb{R}^{N+1}$, and $1 \leq i \leq N+1$, we define

$$\pi_i X := (x_1, \dots, x_{i-1}, 0, x_{i+1}, x_{N+1}) \,.$$

Also, given $X \in \mathbb{R}^{N+1}$, we will often write $X = (x', x_N, x_{N+1}) \in \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}$, i.e., the notation x' will often denote the first $(N-1)$ entries of X.

Then, with the above notation, we have the following results:

Lemma 4.1. *Let u be a solution of (1.5). Suppose that $\mathbb{S}(Y, R)$ touches the graph of u by above at $X_0 = (x_0, u(x_0)) = (x_0, g_{\mathbb{S}(Y,R)}(x_0))$. Assume that*

$$\angle\left(\frac{\nabla u(x_0)}{|\nabla u(x_0)|}, e_N\right) \leq \frac{\pi}{8}. \tag{4.2}$$

Then, there exists a universal $a_0 > 1$ so that, for any $a \geq a_0$, there exist a universal $\kappa > 1$ and a suitable $C > 1$, which depends only on a and on universal constants, such that the following holds. For any point $Z \in L \cap B_a(\pi_N X_0)$ there exists x satisfying the following properties:

- $|x - x_0| \leq \kappa a$,
- $Z = \pi_N(x, u(x))$,
- $(x - x_0) \cdot \dfrac{\nabla u(x_0)}{|\nabla u(x_0)|} \leq H_0(u(x)) - H_0(u(x_0)) + \dfrac{C}{R}$,

provided R is large enough (possibly in dependence of a).

Lemma 4.2. *Let u be a C^1-subsolution of* (1.5) *in $\{|x'| < l\} \times \{|x_N| < l\}$. Assume that $\mathbb{S}(Y_0, R)$ is above the graph of u and that $\mathbb{S}(Y_0, R)$ touches the graph of u at the point $(x_0, u(x_0))$. Suppose that*

- $|u(x_0)| < 1/2$, $|x_{0N}| < l/4$, $q := |x_0'| < l/4$.
- $\angle\left(\dfrac{\nabla u(x_0)}{|\nabla u(x_0)|}, e_N\right) \leq \dfrac{\pi}{8}$.

Then, there exist universal constants $C_1, C_2 > 1 > c > 0$ such that, if

$$q \geq C_1 \qquad \text{and} \qquad 4\sqrt[3]{R} \leq l \leq cR,$$

the following holds. Let Ξ be the set of points $(\mathfrak{x}, u(\mathfrak{x})) \in \mathbb{R}^N \times \mathbb{R}$ satisfying the following properties:

- $|\mathfrak{x}'| < q/15$, $|u(\mathfrak{x})| < 1/2$, $|\mathfrak{x} - x_0| < 2q$;
- *there exists $Y \in \mathbb{R}^N \times [-1/4, 1/4]$ such that $\mathbb{S}(Y, R/C_2)$ is above u and it touches u at $(\mathfrak{x}, u(\mathfrak{x}))$;*
- $\angle\left(\dfrac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, \dfrac{\nabla u(x_0)}{|\nabla u(x_0)|}\right) \leq \dfrac{C_1 q}{R}$;
- $(\mathfrak{x} - x_0) \cdot \dfrac{\nabla u(x_0)}{|\nabla u(x_0)|} \leq \dfrac{C_1 q^2}{R} + H_0(u(\mathfrak{x})) - H_0(u(x_0))$.

Then,

$$\mathfrak{L}^N\Big(\pi_N(\Xi)\Big) \geq cq^{N-1}.$$

Lemma 4.3. *Let $C \geq 2$, $c > 0$. Let us consider, for $k \in \mathbb{N}$, a family of sets $D_k \subseteq L$, so that $D_k \subseteq D_{k+1}$ for any $k \in \mathbb{N}$. Assume that the following properties hold, for some $l > Ca \geq 2a > c > 0$:*

(P1) $D_0 \cap Q_l \neq \emptyset$;

(P2) *for any* $Z_0 \in D_k \cap Q_{2l}$ *and any* $Z_1 \in L$, *with* $a \leq |Z_1 - Z_0| \leq 2l$, *one has that*

$$\mathfrak{L}^N\Big(D_{k+1} \cap B_{|Z_1-Z_0|/10}(Z_1)\Big) \;\geq\; c\,\mathfrak{L}^N\Big(L \cap B_{|Z_1-Z_0|}(Z_1)\Big)\,.$$

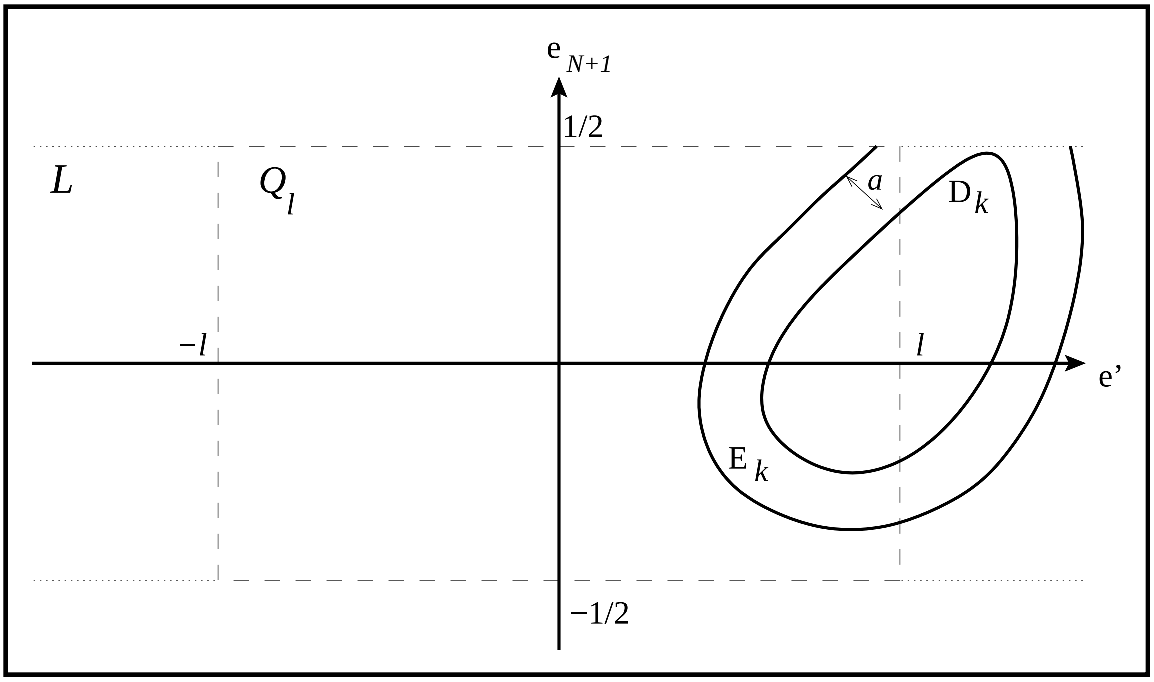
Define, for any $k \in \mathbb{N}$

$$E_k := \Big\{ Z \in L \;|\; \operatorname{dist}(Z, D_k) \leq a \Big\}\,. \tag{4.3}$$

Then, there exists $a_0 > 1 > c_0 > 0$ *universal constants and* $c^\star > 0$, *which depends only on* a, c *and universal constants, such that*

$$\mathfrak{L}^N\Big(Q_l \setminus E_k\Big) \;\leq\; (1 - c^\star)\,\mathfrak{L}^N(Q_l)\,,$$

provided $a \geq a_0$ *and* c, $C^{-1} \in (0, c_0]$.



The covering sets of Lemma 4.3

Full details of the proofs of the above lemmata will be provided in the Appendix.

5. Estimates on the measure of the projection of the contact set

We now show how to use Proposition 3.14 and the measure theoretic lemmata stated in §4 in order to deduce a measure estimate on the projection of the contact sets between barriers and minimal solutions of (1.5). To this aim, we first need an estimate on the contact sets obtained by touching u by above "for the first time", as dealt with by the following result.

Lemma 5.1. *Let C, $C' > 1$ be suitably large constants. Let $K_l := \{|x'| < C\,l\} \times \{|x_N| < C\,l\}$. Let $u \in W^{1,p}(K_l)$ be a local minimizer for $\mathcal{F}$ in K_l. Assume that $u(0) = 0$ and that $u(x) < 0$ if $x_N < -\theta$, for some $\theta > 0$. Define $R_0 := l^2/(C\theta)$. Let Ξ be the set of points $(\mathfrak{x}, u(\mathfrak{x})) \in K_l$ satisfying:*

- $|\mathfrak{x}'| \le l$, $|u(\mathfrak{x})| < 1/2$;
- *there exists $Y \in \mathbb{R}^N \times [-1/4, 1/4]$ such that $\mathbb{S}(Y, R_0)$ is above the graph of u in $\{|x'| < C'\,l\} \times \{|x_N| < C'\,l\}$ and it touches the graph of u at $(\mathfrak{x}, u(\mathfrak{x}))$;*
- $\angle\left(\dfrac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, e_N\right) \le \dfrac{8l}{R_0}$;
- $\mathfrak{x}_N \le \dfrac{\theta}{4} + H_0(u(\mathfrak{x}))$.

Then, there exists a universal constant $c > 0$ such that, for any $\theta_0 > 0$ there exists $\varepsilon_0(\theta_0) > 0$ for which, if

$$\frac{\theta}{l} \le \varepsilon_0(\theta_0)\,, \qquad \theta \ge \theta_0\,,$$

one has that

$$\mathfrak{L}^N\Big(\pi_N(\Xi)\Big) \ge c\,l^{N-1}\,.$$

Proof. Exploiting Lemma 2.23, we have that, if C is large enough,

$$u(x) \le g_l(x_N + \theta) \tag{5.1}$$

for any x so that $|x'| \le C'l$ and $|x_N| \le C'l$, with C' large if so is C. Let us define

$$R_0 := l^2/(C\theta) \tag{5.2}$$

and, for $C'' > 0$ conveniently large, let us consider the set

$$\begin{aligned} \widetilde{\mathfrak{H}} := \Big\{ & Y = (y, y_{N+1}) \in \mathbb{R}^{N+1} \text{ such that} \\ & |y'| \le l/C''\,,\ |y_{N+1}| \le 1/4 \\ & \text{and so that, if } (0,\dots,0,x_N,0) \in \mathbb{S}(Y,R_0) \text{ then } x_N \le 0 \Big\}. \end{aligned}$$

We claim that

$$\begin{aligned} & g_{\mathbb{S}(Y,R_0)}(x) > g_l(x_N+\theta) \text{ for any } Y \in \widetilde{\mathfrak{H}}, \\ & \text{provided } x \in K_l \text{ and } |x'| \in (l, C'l). \end{aligned} \tag{5.3}$$

To prove this, let $Y \in \widetilde{\mathfrak{H}}$ and define

$$\Sigma := \{g_{\mathbb{S}(Y,R_0)} = 0\} = \mathbb{S}(Y,R_0) \cap \{x_{N+1} = 0\}\,.$$

Then, the last condition in the definition of $\widetilde{\mathfrak{H}}$ reads

$$\text{if } (0,\dots,0,x_N) \in \Sigma, \text{ then } x_N \le 0. \tag{5.4}$$

Recalling the definitions on page 14 and (2.39), one sees that Σ is an $(N-1)$-dimensional sphere, namely

$$\Sigma = \{|x-y| = r\}$$

with

$$r := R_0 - H_0(y_{N+1}) - \frac{\overline{C}_0}{2R_0} y_{N+1}^2 . \tag{5.5}$$

Let us now estimate r by noticing that, if l (and therefore R_0) is suitably large, we have that

$$\begin{aligned} &\frac{2l^2}{3C\theta} = \frac{2}{3} R_0 \le R_0 - \text{ const } \le \\ &\le r \le \\ &\le R_0 + \text{ const } \le \frac{3}{2} R_0 = \frac{3l^2}{2C\theta} . \end{aligned} \tag{5.6}$$

Notice also that x in (5.3) must lie in the intersection between K_l and the domain of $g_{\mathbb{S}(Y,R_0)}$, otherwise there is nothing to prove; therefore,

$$|x-y| \le \text{ const} \,(C'l + R_0) \le \text{ const}\, R_0 ,$$

and so, by (5.6),

$$|x-y| \le \text{ const}\, r . \tag{5.7}$$

We now point out that Σ is below the hyperplane $x_N = \theta/8$, that is

$$x_N \le \theta/8, \text{ for any } x \in \Sigma. \tag{5.8}$$

In order to prove (5.8), let

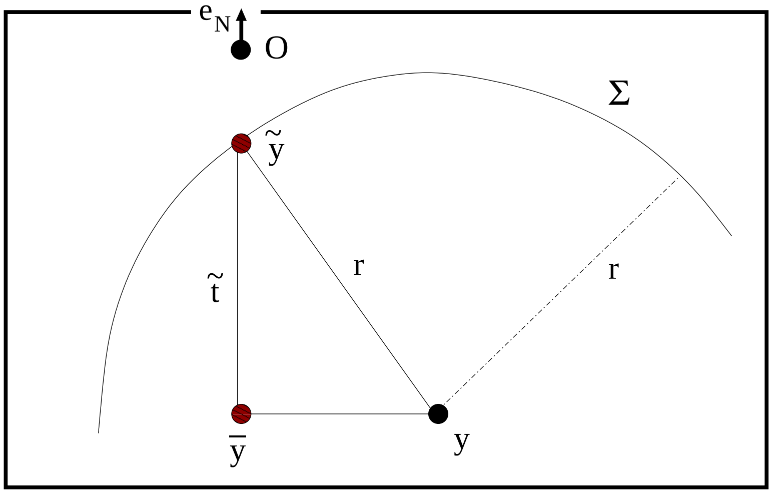
$$\bar{y} := y - y_1 e_1 - \cdots - y_{N-1} e_{N-1} = (0, \ldots, 0, y_N) ,$$

so that, by the definition of $\widetilde{\mathfrak{D}}$,

$$|y - \bar{y}| = |y'| \le \frac{l}{C''} , \tag{5.9}$$

which is less than r due to (5.6), provided θ/l is small enough. Thus, let $\tilde{t} > 0$ be so that

$$\tilde{y} := \bar{y} + \tilde{t} e_N \in \Sigma .$$



The geometry related with $\tilde{y}$

Then $\tilde{y} = (0, \ldots, 0, y_N + \tilde{t})$, thus, from (5.4),

$$y_N \le -\tilde{t}\,. \tag{5.10}$$

Also, from (5.9),

$$\tilde{t}^2 = r^2 - |y - \bar{y}|^2 \ge r^2 - \left(\frac{l}{C''}\right)^2,$$

therefore, in the light of (5.10) and (5.6),

$$\begin{aligned} y_N + r &\le r - \tilde{t} \le \\ &\le r - \sqrt{r^2 - \left(\frac{l}{C''}\right)^2} \le \\ &\le \frac{\theta}{8}, \end{aligned} \tag{5.11}$$

provided C'' is large enough, completing the proof of (5.8).

Let us now go back to the proof of (5.3). For this, we introduce the following notation: define

$$\begin{aligned} d_1(x) &:= |x - y| - r\,, \\ d_2(x) &:= x_N + \theta\,. \end{aligned}$$

Let now x be as requested in (5.3). From (5.7),

$$0 \le r + d_1(x) \le \text{const}\, r\,. \tag{5.12}$$

Also,

$$|x' - y'| \ge |x'| - |y'| \ge \frac{l}{2} - \frac{l}{C'} \ge \frac{2}{5} l\,,$$

if $C' \geq 10$, thus,

$$\begin{aligned} \Big(r+d_1(x)\Big)^2 &= |x-y|^2 = \\ &= |x'-y'|^2+|x_N-y_N|^2 \geq \\ &\geq \frac{4}{25}l^2+|x_N-y_N|^2\,; \end{aligned} \tag{5.13}$$

on the other hand, from (5.8) and the fact that $y+re_N\in\Sigma$,

$$y_N+r\leq\frac{\theta}{8}\,. \tag{5.14}$$

Thus, from (5.13) and (5.14), we infer that

$$\begin{aligned} x_N &\leq y_N+\sqrt{\Big(r+d_1(x)\Big)^2-\frac{4}{25}l^2}\leq \\ &\leq \frac{\theta}{8}-r+\sqrt{\Big(r+d_1(x)\Big)^2-\frac{4}{25}l^2}\,. \end{aligned}$$

This, (5.12) and (5.6) imply that

$$x_N\leq -2\theta+d_1(x)$$

and, therefore,

$$d_1(x)\geq x_N+2\theta=d_2(x)+\theta\,,$$

proving that

$$d_1\geq d_2+\theta \tag{5.15}$$

in $K_l\cap\{|x'|\in(l,C'l)\}$.

We now observe that

$$H_{y_{N+1},R_0}(s)-H_{y_{N+1},R_0}(0)\leq H_0(s)+\frac{2\overline{C}_0}{R_0}\,, \tag{5.16}$$

for any $s\in[s_{R_0},1]$. To prove this, recall Definition 2.8 to get

$$H_{y_{N+1},R_0}(s)-H_{y_{N+1},R_0}(0)=\int_0^s\frac{(p-1)^{\frac{1}{p}}}{(p\,h_{s_0,R}(\zeta))^{\frac{1}{p}}}\,d\zeta\,,$$

and use Definition 2.5 and (2.2) to deduce (5.16).

Therefore, from (5.16) and (5.2), if l is large enough, we get that

$$H_{y_{N+1},R_0}(s)-H_{y_{N+1},R_0}(0)<H_0(s)+\frac{\theta}{2}\,. \tag{5.17}$$

Notice now that, by (2.7), (5.2) and the definition of s_l given in Lemma 2.20, we have that

$$s_{R_0}\geq -1+\frac{\mathrm{const}}{R_0^{1/p}}=-1+\frac{\mathrm{const}\,\theta^{1/\mathrm{p}}}{l^{2/p}}>-1+s_l\,.$$

In particular, the function

$$h_{y_{N+1},R_0}(s)-h_l(s)$$

is defined for any s so that

$$s_{R_0} = \max\{s_{R_0}, -1 + s_l\} \le s \le 1 .$$

Also, if $c_* > 0$ is suitably small (possibly in dependence also of θ_0) and

$$s_{R_0} \le s \le -1 + c_*/l^{1/p} ,$$

we infer from (2.6), the definition of h_l given in Lemma 2.20 and the one of $h_{s_0,R}$ given on page 10 that

$$\begin{aligned} & h_{y_{N+1},R_0}(s) - h_l(s) = \\ = \; & -h_0(s_R) - \frac{\widehat{C}_0}{R_0}(s - s_{R_0}) + \\ & +h_0(s_l - 1) + \frac{\overline{C}_2}{l}\left((1+s)^p - s_l^p\right) \le \\ \le \; & -\frac{1}{R_0} + e^{-\operatorname{const} l} + \frac{\overline{C}_2\, c_*^p}{l^2} \le \\ \le \; & -\frac{C\theta_0}{l^2} + e^{-\operatorname{const} l} + \frac{\overline{C}_2\, c_*^p}{l^2} \le 0 , \end{aligned}$$

that is

$$h_{y_{N+1},R_0}(s) \le h_l(s) \text{ for any } s \in [s_{R_0}, -1 + c_*/l^{1/p}], \tag{5.18}$$

provided that c_* is small enough. Analogously, one can show that

$$h_{y_{N+1},R_0}(s) \ge h_l(s) \text{ for any } s \in [1 - c_*/l^{1/p}, 1]. \tag{5.19}$$

From (5.18), (5.19) and the definitions of H_{y_{N+1},R_0} and H_l (see pages 13 and 25), we deduce that the maximum of the function

$$[s_{R_0}, 1] \ni s \mapsto H_{y_{N+1},R_0}(s) - H_l(s)$$

occurs for $|s| \le 1 - c_*/l^{1/p}$. For these values of s, estimate (2.61) in Lemma 2.20 implies that

$$\begin{aligned} H_0(s) \; & \le \; H_l(s) - \frac{\operatorname{const}}{l}\log(1 - |s|) \le \\ & \le \; H_l(s) + \frac{\operatorname{const}}{l}\log\frac{l^{1/p}}{c_*} \le \\ & \le \; H_l(s) + \frac{\theta_0}{2} , \end{aligned}$$

provided l is suitably large. Thus, summarizing the above observations and using (5.17), we have that

$$\begin{aligned} & \max_{[s_{R_0},1]} \Big(H_{y_{N+1},R_0} - H_l \Big) = \\ = & \max_{[-1+c_*/l^{1/p},\,1-c_*/l^{1/p}]} \Big(H_{y_{N+1},R_0} - H_l \Big) < \\ < & \max_{[-1+c_*/l^{1/p},\,1-c_*/l^{1/p}]} \Big(H_0 - H_l \Big) + \frac{\theta}{2} + H_{y_{N+1},R_0}(0) \le \\ \le & \frac{\theta_0}{2} + \frac{\theta}{2} + H_{y_{N+1},R_0}(0)\,. \end{aligned}$$

Hence,

$$H_{y_{N+1},R_0}(s) - H_{y_{N+1},R_0}(0) < H_l(s) + \theta\,, \tag{5.20}$$

for any $s \in (s_{R_0},\,1]$. From (5.20), by inverting H_{y_{N+1},R_0}, we have

$$s \,<\, g_{y_{N+1},R_0}\Big(H_{y_{N+1},R_0}(0) + H_l(s) + \theta \Big)\,,$$

for any $s \in (s_{R_0}, 1]$ and so, for $s := g_l(d_1(x) - \theta)$, we get

$$g_l(d_1(x) - \theta) \,<\, g_{y_{N+1},R_0}\Big(d_1(x) + H_{y_{N+1},R_0}(0) \Big)\,.$$

Therefore, recalling also (5.15),

$$\begin{aligned} & g_{y_{N+1},R_0}\Big(|x-y| - r + H_{y_{N+1},R_0}(0) \Big) = \\ = \;& g_{y_{N+1},R_0}\Big(d_1(x) + H_{y_{N+1},R_0}(0) \Big) > \\ > \;& g_l(d_1(x) - \theta) \ge \\ \ge \;& g_l(d_2(x)) = \\ = \;& g_l(x_N + \theta)\,. \end{aligned}$$

This completes the proof of (5.3).

By (5.3) and (5.1), we have that

(5.21)

$$g_{\mathbb{S}(Y,R_0)}(x) \,>\, u(x) \text{ for any } Y \in \widetilde{\mathfrak{D}}, \text{ provided } |x'| \in (l, C'l) \text{ and } |x_N| \le C'l.$$

Let now $\widetilde{\Xi}$ be the set of $(\mathfrak{x}, u(\mathfrak{x}))$'s described in the statement of Lemma 5.1. Let us also define $\Xi := \pi_{e_N}\widetilde{\Xi}$ and $\mathfrak{D} := \pi_{e_N}\widetilde{\mathfrak{D}}$. Of course,

$$\mathfrak{D} \,=\, \Big\{ Y = (y', 0, y_{N+1}) \in \mathbb{R}^{N+1} \text{ such that } |y'| \le l/C''\,,\ |y_{N+1}| \le 1/4 \Big\}\,,$$

therefore

$$\mathfrak{L}^N(\mathfrak{D}) \ge \text{ const}\, l^{N-1}\,. \tag{5.22}$$

For any $Y \in \mathfrak{D}$, from (5.1) and the fact that $\mathbb{S}(Y, R_0)$ takes value 1 on the boundary of its domain of definition, we know that $\mathbb{S}(Y - te_N, R_0)$ is below the graph of u in the intersection between $\{|x'| \le C'l\} \times \{|x_N| \le C'l\}$ and the domain of definition of $\mathbb{S}(Y - te_N, R_0)$, provided that t is large enough. Also, by looking at the construction of $\bar{y}$ on page 48, the reader may easily convince herself that by decreasing t, there will be a suitable t^* for which $\mathbb{S}(Y - t^* e_N, R_0)$ touches for the first time the graph of u, say at the point $\widetilde{X}$. We denote by $\widetilde{\mathfrak{G}}$ the set of such touching points $\widetilde{X}$'s and define also $\mathfrak{G} := \pi_{e_N}\widetilde{\mathfrak{G}}$.

We claim that

$$\mathfrak{G} \subseteq \Xi\,. \tag{5.23}$$

For proving this, take any $\widetilde{X} \in \widetilde{\mathfrak{G}}$ be a touching point between $\mathbb{S}(Y - t^* e_N, R_0)$ and the graph of u, as described above. Let us observe that, since $u(0) = 0$, the first touching property of $\widetilde{X}$ implies that if $\check{X} = (0, \dots, \check{x}_N, 0) \in \mathbb{S}(Y - t^* e_N, R_0)$, then $\check{x}_N \le 0$, hence

$$\widetilde{\mathfrak{H}} \ni Y - t^* e_N =: \widetilde{Y}\,.$$

From this and (5.21), we gather that

$$|\widetilde{x}'| \le l\,. \tag{5.24}$$

We now show that

$$\widetilde{x} \text{ is in the interior of } \{|x'| \le C'l\} \times \{|x_N| \le C'l\}\ . \tag{5.25}$$

Note that, thanks to (5.24), this will be proved if we show that $|\widetilde{x}_N| < C'l$.

Let us first show that $\widetilde{x}_N > -C'l$. If, by contradiction, $\widetilde{x}_N = -C'l$, we gather from (2.7) and (5.1) that

$$\begin{aligned}
-1 + \frac{\mathrm{const}\,\theta_0^{1/p}}{l^{2/p}} &\le -1 + \frac{\mathrm{const}}{R_0^{1/p}} \le \\
&\le s_{R_0} \le \\
&\le g_{\mathbb{S}(\widetilde{Y}, R_0)}(\widetilde{x}) = \\
&= u(\widetilde{x}) \le \\
&\le g_l(\widetilde{x}_N + \theta) \le \\
&\le g_l\Big(-\frac{C'\,l}{2}\Big) = \\
&= -1 + e^{-\mathrm{const}\,l}\,,
\end{aligned}$$

which is a contradiction for large l. This shows that $\widetilde{x}_N > -C'l$ and thus we now show that $\widetilde{x}_N < C'l$, in order to complete the proof of (5.25). That $\widetilde{x}_N < C'l$ will be actually obtained from the fact that the domain of $\mathbb{S}(\widetilde{Y}, R_0)$ is below the hyperplane $\{x_N \le l/2\}$. To prove this, first note that, by (5.11), we have that

$$\widetilde{y}_N \le -r + \frac{\theta}{8}\,, \tag{5.26}$$

Also, if $\mathfrak{x}$ is in the domain of $\mathbb{S}(\widetilde{Y}, R_0)$, we have that

$$\mathfrak{x}_N \le \widetilde{y}_N + H_{\widetilde{y}_{N+1}, R_0}(1) - H_0(\widetilde{y}_{N+1}) + R_0\,.$$

Thus, (5.26), (5.5) and (2.16) yield that

$$\mathfrak{x}_N \leq \text{ const}\,(1+\log R_0) \leq \frac{l}{2}\,,$$

hence the domain of $\mathbb{S}(\widetilde{Y}, R_0)$ is below $\{x_N \leq l/2\}$ and therefore $\widetilde{x}_N < C'l$.

This ends the proof of (5.25).

Proposition 2.13 and (5.25) yield that

$$|u(\widetilde{x})| \leq 1/2\,. \tag{5.27}$$

We now notice that, from (5.27) and (2.20),

$$\begin{aligned} -\frac{1}{2} &\leq u(\widetilde{x}) = \\ &= g_{\mathbb{S}(\widetilde{Y},R_0)}(\widetilde{x}) = \\ &= g_{\widetilde{y}_{N+1},R_0}\Big(H_0(\widetilde{y}_{N+1}) + |\widetilde{x}-\widetilde{y}| - R_0\Big)\,, \end{aligned}$$

and so, by Definition 2.11,

$$H_0(\widetilde{y}_{N+1}) + |\widetilde{x}-\widetilde{y}| - R_0 \,\geq\, H_{\widetilde{y}_{N+1},R_0}(-1/2)\,,$$

from which we deduce that

$$|\widetilde{x}-\widetilde{y}| \,\geq\, R_0 - \text{ const} \,\geq R_0/2\,, \tag{5.28}$$

provided that l (and so R_0) is large enough. On the other hand, exploiting (5.24) and the definition of $\widetilde{\mathfrak{D}}$ given on page 47, we have that

$$|\widetilde{x}'-\widetilde{y}'| \leq |\widetilde{x}'| + |\widetilde{y}'| \leq l + \frac{l}{C''} \leq 2l\,.$$

Hence, from (5.28),

$$\begin{aligned} \frac{|\widetilde{x}_N-\widetilde{y}_N|^2}{|\widetilde{x}-\widetilde{y}|^2} &= 1 - \frac{|\widetilde{x}'-\widetilde{y}'|^2}{|\widetilde{x}-\widetilde{y}|^2} \geq \\ &\geq 1 - \frac{16l^2}{R_0^2}\,, \end{aligned}$$

and, therefore,

$$\begin{aligned} 1-\frac{1}{4}\left[\angle\left(\frac{\widetilde{x}-\widetilde{y}}{|\widetilde{x}-\widetilde{y}|},\,e_N\right)\right]^2 &\geq \cos^2\left[\angle\left(\frac{\widetilde{x}-\widetilde{y}}{|\widetilde{x}-\widetilde{y}|},\,e_N\right)\right] = \\ &= \frac{|\widetilde{x}_N-\widetilde{y}_N|^2}{|\widetilde{x}-\widetilde{y}|^2} \geq \\ &\geq 1-\frac{16l^2}{R_0^2}\,, \end{aligned}$$

that is

$$\angle\left(\frac{\widetilde{x}-\widetilde{y}}{|\widetilde{x}-\widetilde{y}|},\,e_N\right) \,\leq\, \frac{8l}{R_0}\,. \tag{5.29}$$

Moreover, from the touching property of $\widetilde{X}$ and (2.20), we have that

$$\begin{aligned}\angle\left(\frac{\nabla u(\widetilde{x})}{|\nabla u(\widetilde{x})|}, e_N\right) &= \angle\left(\frac{\nabla g_{\mathbb{S}(\widetilde{Y},R_0)}(\widetilde{x})}{|\nabla g_{\mathbb{S}(\widetilde{Y},R_0)}(\widetilde{x})|}, , e_N\right) = \\ &= \angle\left(\frac{\widetilde{x}-\widetilde{y}}{|\widetilde{x}-\widetilde{y}|}, e_N\right).\end{aligned}$$

Therefore, from (5.29),

$$\angle\left(\frac{\nabla u(\widetilde{x})}{|\nabla u(\widetilde{x})|}, e_N\right) \le \frac{8l}{R_0}. \tag{5.30}$$

Furthermore, recalling (2.39),

$$\begin{aligned}& H_{\widetilde{y}_{N+1},R_0}(u(\widetilde{x})) - H_{\widetilde{y}_{N+1},R_0}(0) = \\ &= H_{\widetilde{y}_{N+1},R_0}\Big(g_{\mathbb{S}(\widetilde{Y},R_0)}(\widetilde{x})\Big) - H_{\widetilde{y}_{N+1},R_0}(0) = \\ &= H_{\widetilde{y}_{N+1},R_0}\Big(g_{\mathbb{S}(\widetilde{Y},R_0)}(\widetilde{x})\Big) + \frac{\overline{C}_0}{2R_0}\widetilde{y}_{N+1}^2,\end{aligned}$$

thus, from (2.33) and (2.35),

$$\begin{aligned}& H_{\widetilde{y}_{N+1},R_0}(u(\widetilde{x})) - H_{\widetilde{y}_{N+1},R_0}(0) = \\ &= H_{\widetilde{y}_{N+1},R_0}\Big(\rho_{\widetilde{y}_{N+1},R_0}(H_0(\widetilde{y}_{N+1}) + |\widetilde{x}-\widetilde{y}| - R_0)\Big) + \\ &+ \frac{\overline{C}_0}{2R_0}\widetilde{y}_{N+1}^2 = \\ &= H_0(\widetilde{y}_{N+1}) + |\widetilde{x}-\widetilde{y}| - R_0 + \frac{\overline{C}_0}{2R_0}\widetilde{y}_{N+1}^2.\end{aligned}$$

Hence, by (5.5),

$$\begin{aligned}& H_{\widetilde{y}_{N+1},R_0}(u(\widetilde{x})) - H_{\widetilde{y}_{N+1},R_0}(0) = \\ &= |\widetilde{x}-\widetilde{y}| - r.\end{aligned} \tag{5.31}$$

We now claim that

$$\widetilde{x}_N \le H_0(u(\widetilde{x})) + \frac{\theta}{4}. \tag{5.32}$$

For proving this, we denote by $\widehat{x}$ the intersection point between the sphere $\{g_{\mathbb{S}(\widetilde{Y},R_0)} = 0\}$ and the half-line from $\widetilde{y}$ towards $\widetilde{x}$. Then, by (5.8),

$$\widehat{x}_N \le \theta/8.$$

We now distinguish two cases: either $\widehat{x}$ is inside or it is outside the sphere $\{g_{\mathbb{S}(\widetilde{Y},R_0)}=0\}$. If it is inside, then

$$\begin{aligned} |\widetilde{x}-\widetilde{y}|-r &= |\widehat{x}-\widetilde{x}| \geq \\ &\geq |\widehat{x}_N-\widetilde{x}_N| \geq \\ &\geq \widetilde{x}_N-\widehat{x}_N \geq \\ &\geq \widetilde{x}_N-\theta/8\,. \end{aligned}$$

Thus, from the latter estimate, (5.31) and (5.16), we have that

$$\begin{aligned} \widetilde{x}_N &\leq |\widetilde{x}-\widetilde{y}|-r+\frac{\theta}{8} = \\ &= H_{\widetilde{y}_{N+1}}(u(\widetilde{x}))-H_{\widetilde{y}_{N+1}}(0)+\frac{\theta}{8} \leq \\ &\leq H_0(u(\widetilde{x}))+\frac{2\overline{C}_0}{R_0}+\frac{\theta}{8}\,. \end{aligned}$$

Therefore, if l (and so R_0) is large enough, (5.32) follows in this case. Let us now deal with the case in which $\widehat{x}$ is outside the sphere $\{g_{\mathbb{S}(\widetilde{Y},R_0)}=0\}$. By (5.30), we infer in this case that $\widehat{x}_N \geq \widetilde{x}_N$ and that

$$\begin{aligned} \widehat{x}_N-\widetilde{x}_N &= |\widehat{x}_N-\widetilde{x}_N| = \\ &= |\widehat{x}-\widetilde{x}|\,\cos\Big(\angle(\widehat{x}-\widetilde{x},\,e_N)\Big) \geq \\ &\geq |\widehat{x}-\widetilde{x}|\left(1-\frac{\mathrm{const}\,l^2}{R_0^2}\right)\,. \end{aligned}$$

Therefore, (5.31) and (5.16) yield that

$$\begin{aligned} H_0(u(\widetilde{x}))+\frac{2\overline{C}_0}{R_0} &\geq |\widetilde{y}-\widetilde{x}|-r = \\ &= -|\widehat{x}-\widetilde{x}| \geq \\ &\geq \frac{\widetilde{x}_N-\widehat{x}_N}{1-\frac{\mathrm{const}\,l^2}{R_0^2}} \geq \\ &\geq \frac{\widetilde{x}_N-(\theta/8)}{1-\frac{\mathrm{const}\,l^2}{R_0^2}}\,, \end{aligned}$$

which easily implies (5.32) in this case. This completes the proof of (5.32).

Thus, in the light of (5.24), (5.27), (5.30) and (5.32), we have that $\widetilde{X}\in\widetilde{\Xi}$ and, therefore, that $\pi_{e_N}\widetilde{X}\in\Xi$, ending the proof of (5.23).

Now we exploit Proposition 3.14, applied to $\mathfrak{G}$ and $\mathfrak{O}$: from that, (5.23) and (5.22),

$$\begin{aligned} \mathfrak{L}^N(\Xi) &\geq \mathfrak{L}^N(\mathfrak{G}) \geq \\ &\geq \mathrm{const}\,\mathfrak{L}^N(\mathfrak{O}) \geq \\ &\geq \mathrm{const}\,l^{N-1}\,. \end{aligned}$$

This completes the proof of Lemma 5.1. □

The next one is the main result of this section:

Proposition 5.2. *Let C be a suitably large constant. Let $K_l := \{|x'| < C\,l\} \times \{|x_N| < C\,l\}$. Let $u \in W^{1,p}(K_l)$ be a local minimizer for $\mathcal{F}$ in K_l. Assume that $u(0) = 0$ and that $u(x) < 0$ if $x_N < -\theta$, for some $\theta > 0$. Fix $\bar{C} > 0$ and $k \in \mathbb{N}$. Let Ξ be the set of points $(\mathfrak{x},\, u(\mathfrak{x}))$ satisfying the following properties:*

- *$|\mathfrak{x}'| \le l$, $|\mathfrak{x}_{N+1}| \le 1/2$;*
- *$\mathfrak{x}_N \le \bar{C}^k\,\theta + H_0(u(\mathfrak{x}))$.*

Then, there exist positive universal constants c and $\hat{c}$ for which the following holds. For any $\theta_0 > 0$, there exists $\varepsilon_0(\theta_0) > 0$, so that, if

$$\frac{\theta}{l} \le \varepsilon_0(\theta_0)\,, \qquad \theta \ge \theta_0 \qquad \text{and} \qquad \frac{\bar{C}^k\,\theta}{l} \le \hat{c}\,,$$

then

$$\mathfrak{L}^N\Big(\pi_N(\Xi)\Big) \;\ge\; (1-(1-c)^k)\,\mathfrak{L}^N(Q_l)\,.$$

Proof. Let $R_0 := l^2/(C\theta)$, with C suitably large. For any $k \in \mathbb{N}$, let $R_k := R_0\,\bar{C}^{-k}$, where $\bar{C}$ is a positive universal constant, to be chosen suitably large in the sequel. We define $\mathfrak{D}_k \subseteq \mathbb{R}^{N+1}$ as the set of points $(\mathfrak{x},\, u(\mathfrak{x}))$ satisfying the following properties:

- $|\mathfrak{x}'| \le C\,l/2$, $|u(\mathfrak{x})| < 1/2$;
- there exists $Y \in \mathbb{R}^N \times [-1/4,\, 1/4]$ so that $\mathbb{S}(Y, R_k)$ is above the graph of u in $\{|x'| < C\,l/2\} \times \{|x_N| < C\,l/2\}$ and it touches the graph of u at $(\mathfrak{x},\, u(\mathfrak{x}))$;
- $\angle\left(\dfrac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, e_N\right) \;\le\; \dfrac{\bar{C}^k\,l}{R_0}$;
- $\mathfrak{x}_N \le \dfrac{\bar{C}^k\,\theta}{4} + H_0(u(\mathfrak{x}))$.

We also set $D_k := \pi_N(\mathfrak{D}_k)$. We would like to apply Lemma 4.3 to D_k, and we therefore now prove that D_k fulfills the assumption of Lemma 4.3. For this, first of all, notice that, by Lemma 5.1,

$$D_0 \cap Q_l \neq \emptyset\,. \tag{5.33}$$

Let us now fix $Z_k \in D_k \cap Q_{2l}$. By construction, there exists $(x_k,\, u(x_k)) \in \mathfrak{D}_k$ so that $Z_k = \pi_N(x_k,\, u(x_k))$. Take also $\tilde{Z} \in L$, with $a \le |\tilde{Z} - Z_k| =: q \le 2l$, and suppose a suitably large. We claim that

$$\mathfrak{L}^N\Big(D_{k+1} \cap B_{q/10}(\tilde{Z})\Big) \;\ge\; \mathfrak{L}^N\Big(L \cap B_q(\tilde{Z})\Big)\,. \tag{5.34}$$

In order to prove the above inequality, we denote by $\hat{C} > 0$ a constant, to be suitably chosen in the sequel, and we define $\tilde{\Xi}$ as the set of points $(\mathfrak{x}, u(\mathfrak{x}))$ satisfying the following properties:

- $|\mathfrak{x}' - \tilde{z}'| \le q/15$, $|\mathfrak{x} - x_k| < 4l$, $|u(\mathfrak{x})| < 1/2$;

- there exists $Y \in \mathbb{R}^N \times [-1/4, 1/4]$ so that $\mathbb{S}(Y, R_{k+1})$ is above the graph of u in $\{|x' - \tilde z'| < C\,l/2\} \times \{|x_N| < C\,l/2\}$ and it touches the graph of u at $(\mathfrak{x}, u(\mathfrak{x}))$;
- $\angle\left(\frac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, \frac{\nabla u(x_k)}{|\nabla u(x_k)|}\right) \le \frac{\hat C \bar C^k\, l}{R_0}$;
- $(\mathfrak{x} - x_k) \cdot \frac{\nabla u(x_k)}{|\nabla u(x_k)|} \le \frac{\hat C \bar C^k\, l^2}{4R_0} + H_0(u(\mathfrak{x})) - H_0(u(x_k))$.

Notice that, by means[11] of Lemma 4.2 (applied in $\{|x' - \tilde z'| \le 8l\} \times \{|x_N| \le 8l\}$,

$$\mathfrak{L}^N(\pi_N(\tilde\Xi) \cap B_{q/10}(\tilde Z)) \ge \text{const}\, q^{N-1} \ge \text{const}\, \mathfrak{L}^N\Big(L \cap B_q(\tilde Z)\Big)\,. \tag{5.35}$$

Let us now deduce some easy properties of $\tilde\Xi$. First of all, by the definitions of $\tilde\Xi$ and $\mathfrak{D}_k$, we have that, for any $(\mathfrak{x}, u(\mathfrak{x})) \in \tilde\Xi$,

$$\begin{aligned} \angle\left(\frac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, e_N\right) &\le \angle\left(\frac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, \frac{\nabla u(x_k)}{|\nabla u(x_k)|}\right) + \angle\left(\frac{\nabla u(x_k)}{|\nabla u(x_k)|}, e_N\right) \le \\ &\le \frac{\hat C \bar C^k\, l}{R_0} + \frac{\bar C^k\, l}{R_0} \le \\ &\le \frac{\bar C^{k+1}\, l}{R_0}\,, \end{aligned} \tag{5.36}$$

provided $\bar C$ is big enough with respect to $\hat C$. Furthermore, by (4.1) and the definition of $\mathfrak{D}_k$,

$$\begin{aligned} \left|\frac{\nabla u(x_k)}{|\nabla u(x_k)|} - e_N\right| &\le \angle\left(\frac{\nabla u(x_k)}{|\nabla u(x_k)|}, e_N\right) \le \\ &\le \frac{\bar C^k\, l}{R_0}\,, \end{aligned} \tag{5.37}$$

and so

$$\left|(\mathfrak{x} - x_k) \cdot \left(\frac{\nabla u(x_k)}{|\nabla u(x_k)|} - e_N\right)\right| \le \frac{4\bar C^k\, l^2}{R_0}\,, \tag{5.38}$$

provided $|\mathfrak{x} - x_k| \le 4l$ (and observe that this condition is fulfilled by any $(\mathfrak{x}, u(\mathfrak{x})) \in \tilde\Xi$). By using the latter inequality and the definition of $\tilde\Xi$, it also

[11] We apply here Lemma 4.2 with R_k replacing what there was denoted by R. Note also that

$$\sqrt{|\tilde Z - Z_k|^2 - 4} \le |x_k' - \tilde z_k'| \le \sqrt{|\tilde Z - Z_k|^2 + 4}\,,$$

thus

$$|x_k' - \tilde z_k'| \in \left(\frac{9}{10}\,q,\ \frac{11}{10}\,q\right)\,,$$

if a is large enough. Finally, observe that, by construction,

$$\pi_N(\tilde\Xi) \subseteq B_{q/10}(\tilde Z)\,.$$

follows that

$$\begin{aligned}(\mathfrak{x}-x_k)\cdot e_N \;&\leq\; (\mathfrak{x}-x_k)\cdot\frac{\nabla u(x_k)}{|\nabla u(x_k)|}+\left|(\mathfrak{x}-x_k)\cdot\left(\frac{\nabla u(x_k)}{|\nabla u(x_k)|}-e_N\right)\right|\leq\\ &\leq\; \frac{\hat{C}\bar{C}^k\,l^2}{4R_0}+\frac{4\bar{C}^k\,l^2}{R_0}+H_0(u(\mathfrak{x}))-H_0(u(x_k))\,,\end{aligned}\tag{5.39}$$

for any $(\mathfrak{x},\,u(\mathfrak{x}))\in\tilde{\Xi}$; from (5.39), the definition of R_0 and the assumptions of Proposition 5.2, we thus deduce that

$$\mathfrak{x}_N \;\leq\; \frac{\bar{C}^{k+1}\,\theta}{4}+H_0(u(\mathfrak{x}))\,,\tag{5.40}$$

for any $(\mathfrak{x},\,u(\mathfrak{x}))\in\tilde{\Xi}$, if $\bar{C}$ is large enough. Therefore, thanks to (5.36) and (5.40), we have that $\tilde{\Xi}\subseteq\mathfrak{D}_{k+1}$. From this and (5.35), we gather (5.34), as desired. This says that the hypotheses of Lemma 4.3 are fulfilled by D_k, thus we will freely use such result in what follows.

Let now E_k be as in (4.3). From Lemma 4.1, and taking $\hat{C}$ suitably large, we deduce that, for each $Z\in E_k$ there exists $x=x(Z)$ and $x_k=x_k(Z)$ so that $(x_k,\,u(x_k))\in\mathfrak{D}_k$, $|x-x_k|\leq\hat{C}$, $Z=\pi_N(x,\,u(x))$ and

$$(x-x_k)\cdot\frac{\nabla u(x_k)}{|\nabla u(x_k)|}\;\leq\;H_0(u(x))-H_0(u(x_k))+\frac{\mathrm{const}\,\hat{C}}{R_k}\,.\tag{5.41}$$

Thus, from (5.38) and (5.41),

$$\begin{aligned}(x-x_k)\cdot e_N \;&\leq\; (x-x_k)\cdot\frac{\nabla u(x_k)}{|\nabla u(x_k)|}+\left|(x-x_k)\cdot\left(\frac{\nabla u(x_k)}{|\nabla u(x_k)|}-e_N\right)\right|\leq\\ &\leq\; H_0(u(x))-H_0(u(x_k))+\frac{\mathrm{const}\,\hat{C}}{R_k}+\frac{4\bar{C}^k\,l^2}{R_0}\,,\end{aligned}$$

which implies, thanks to the definition of R_0 and the assumptions of Proposition 5.2, that

$$x_N\leq\bar{C}^{k+1}\,\theta+H_0(u(x))\,.$$

Hence, if Ξ is as defined here above in the statement of Proposition 5.2,

$$E_k\;\subseteq\;\Xi\,.\tag{5.42}$$

Also, by Lemma 4.3,

$$\mathfrak{L}^N\Big(E_k\cap Q_l\Big)\geq\left(1-(1-c)^k\right)\mathfrak{L}^N(Q_l)\,,\tag{5.43}$$

for some $c\in(0,1)$. Thus, the claim in Proposition 5.2 follows from (5.42) and (5.43). $\square$

6. Proof of Theorem 1.1

First of all, note that u must attain both positive and negative values thanks to the density estimates in [28]. Thus, possibly replacing l by $C\,l$, we may assume that u is a local minimizer for $\mathcal{F}$ in $\{|x'| < C\,l\} \times \{|x_N| < C\,l\}$, that $u(0) = 0$ and that

$$u(x) > 0 \text{ if } x_N > \theta > 0 \text{ and } u(x) < 0 \text{ if } x_N < -\theta. \tag{6.1}$$

The strategy for proving Theorem 1.1 consists in assuming, by contradiction that there exists a point in $\{u = 0\} \cap \{|x'| < l/4\}$ close to $x_N = -\theta$. The contradiction will be, then, that the energy of u is larger than it should.

The first step in proving Theorem 1.1 is thus the following: we assume, by contradicting Theorem 1.1, that

$$\{u = 0\} \cap \{|x'| < \bar{C}^{-k_0} l/4\} \cap \{x_N < (-1 + \bar{C}^{-k_0}/4)\theta\} \neq \emptyset\,,$$

with $k_0 \in \mathbb{N}$ large and $\frac{\theta}{l}$ small (possibly in dependence of k_0). We also set

$$\begin{aligned} \Xi_0 \;:=\; & \Big\{(x, u(x)) \in \mathbb{R}^N \times \mathbb{R} \text{ s.t.} \\ & x_N \le H_0(u(x)) - \theta/2\,,\ |x' - (x^*)'| \le l/2\,,\ |u(x)| \le 1/2\Big\}\,. \end{aligned}$$

Then, we claim that

$$\mathfrak{L}^N\Big(\pi_N(\Xi_0)\Big) \;\ge\; (1 - (1 - c_0)^{k_0})\,\mathfrak{L}^N(Q_{l/2})\,, \tag{6.2}$$

for a suitable constant $c_0 > 0$. To prove this, let

$$x^* \in \{u = 0\} \cap \{|x'| < \bar{C}^{-k_0} l/4\} \cap \{x_N < (-1 + \bar{C}^{-k_0}/4)\theta\}$$

Define $\theta^* := \theta/(4\bar{C}^{k_0})$ and $v(x) := u(x + x^*)$. Notice that $v(0) = 0$ and $v(x) < 0$ if $x_N < -\theta^*$. Also v is a local minimizer for $\mathcal{F}$. Then, if we define

$$\begin{aligned} \Xi^* \;:=\; & \Big\{(z, v(z)) \in \mathbb{R}^N \times \mathbb{R} \text{ s.t.} \quad |z'| \le l/2 \\ & z_N \le H_0(v(z)) + \bar{C}^{k_0}\theta^*\,, \quad |v(z)| \le 1/2\Big\}\,, \end{aligned}$$

we deduce by Proposition 5.2 that

$$\mathfrak{L}^N\Big(\pi_N(\Xi^*)\Big) \;\ge\; (1 - (1 - c_0)^{k_0})\,\mathfrak{L}^N(Q_{l/2})\,.$$

By elementary computations, one also sees that

$$\Xi^* + (x^*, 0) \;\subseteq\; \Xi_0\,,$$

thus proving (6.2).

Let now

$$\begin{aligned} \Xi_1 \;:=\; & \Big\{(x, u(x)) \in \mathbb{R}^N \times \mathbb{R} \text{ s.t.} \\ & x_N \ge H_0(u(x)) - \theta/4\,,\ |x'| \le l/2\,,\ |u(x)| \le 1/2\Big\}\,. \end{aligned}$$

Then, we claim that

$$\mathfrak{L}^N\Big(\pi_N(\Xi_1)\Big) \;\geq\; c_1 \mathfrak{L}^N(Q_{l/2})\,, \tag{6.3}$$

for a suitable constant $c_1 > 0$, provided θ/l is suitably small. To prove (6.3), let

$$\begin{aligned} \widetilde{u}(x) = \widetilde{u}(x', x_N) \;&:=\; -u(x', -x_N)\,, \\ \widetilde{h}_0(s) \;&:\; = h_0(-s)\,. \end{aligned}$$

Then, $\widetilde{h}_0$ satisfies the same assumptions as h_0 and $\widetilde{u}$ is a local minimizer for the functional

$$\widetilde{\mathcal{F}}(v) := \int \frac{|\nabla v|^p}{p} + \widetilde{h}_0(v)\,.$$

Hence, we may apply Lemma 5.1 with h_0 replaced by $\widetilde{h}_0$, and deduce that, if

$$\begin{aligned} \widetilde{\Xi}_1 \;:=\; &\Big\{(x, \widetilde{u}(x)) \in \mathbb{R}^N \times \mathbb{R} \text{ s.t.} \\ &x_N \leq \widetilde{H}_0(\widetilde{u}(x)) + \theta/4\,,\ |x'| \leq l/2\,,\ |u(x)| \leq 1/2\Big\}\,, \end{aligned}$$

then

$$\mathfrak{L}^N\Big(\pi_N(\widetilde{\Xi}_1)\Big) \;\geq\; \text{const}\, l^{N-1}\,.$$

From this, (6.3) easily follows.

We now make some remarks on the measure properties of the above sets. First note that, by construction,

$$\pi_N(\Xi_0) \;\subseteq\; Q_{\frac{l}{2}+\frac{l}{4\bar{C}^{k_0}}}\,,$$

therefore

$$\begin{aligned} \mathfrak{L}^N\Big(\pi_N(\Xi_0) \setminus Q_{l/2}\Big) \;&\leq\; \mathfrak{L}^N\Big(Q_{\frac{l}{2}+\frac{l}{4\bar{C}^{k_0}}} \setminus Q_{l/2}\Big) \leq \\ &\leq\; \frac{\text{const}\, l^{N-1}}{\bar{C}^{(N-1)k_0}} \leq \\ &\leq\; \frac{\text{const}}{\bar{C}^{(N-1)k_0}}\, \mathfrak{L}^N(Q_{l/2})\,. \end{aligned}$$

This and (6.2), by assuming k_0 large enough, yield that

$$\begin{aligned} \mathfrak{L}^N\Big(\pi_N(\Xi_0) \cap Q_{l/2}\Big) \geq &\Big(1 - (1-c_0)^{k_0} - \frac{\text{const}}{\bar{C}^{(N-1)k_0}}\Big)\, \mathfrak{L}^N(Q_{l/2}) \geq \\ \geq &\Big(1 - \frac{c_1}{2}\Big)\, \mathfrak{L}^N(Q_{l/2})\,, \end{aligned}$$

where c_1 is the constant introduced here above. Thus,

$$\mathfrak{L}^N\Big(Q_{l/2} \setminus \big(\pi_N(\Xi_0) \cap Q_{l/2}\big)\Big) \;\leq\; \frac{c_1}{2}\, \mathfrak{L}^N(Q_{l/2})\,.$$

From this, (6.3) and the fact that $\pi_N(\Xi_1) \subseteq Q_{l/2}$, we gather that

$$\begin{aligned} c_1 \mathfrak{L}^N(Q_{l/2}) \le \mathfrak{L}^N\Big(\pi_N(\Xi_1)\Big) \le \\ &\le \mathfrak{L}^N\Big(\pi_N(\Xi_1) \cap \big(\pi_N(\Xi_0) \cap Q_{l/2}\big)\Big) + \\ &+ \mathfrak{L}^N\Big(\pi_N(\Xi_1) \setminus \big(\pi_N(\Xi_0) \cap Q_{l/2}\big)\Big) \le \\ &\le \mathfrak{L}^N\Big(\pi_N(\Xi_0) \cap \pi_N(\Xi_1)\Big) + \\ &+ \mathfrak{L}^N\Big(Q_{l/2} \setminus \big(\pi_N(\Xi_0) \cap Q_{l/2}\big)\Big) \le \\ &\le \mathfrak{L}^N\Big(\pi_N(\Xi_0) \cap \pi_N(\Xi_1)\Big) + \frac{c_1}{2}\,\mathfrak{L}^N(Q_{l/2}), \end{aligned}$$

that is

$$\mathfrak{L}^N\Big(\pi_N(\Xi_0) \cap \pi_N(\Xi_1)\Big) \ \ge\ \frac{c_1}{2}\,\mathfrak{L}^N(Q_{l/2}). \tag{6.4}$$

On the other hand,

$$\Xi_0 \cap \Xi_1 \subseteq \left\{ -\frac{\theta}{4} \le x_N - H_0(u(x)) \le -\frac{\theta}{2} \right\} = \emptyset. \tag{6.5}$$

Let now

$$\mathfrak{V} := \Big\{ Z \in Q_{l/2} \ \Big|\ \exists \tilde{x} \ne \hat{x}\,,\ \text{s.t. } Z = \pi_N\big(\tilde{x}, u(\tilde{x})\big) = \pi_N\big(\hat{x}, u(\hat{x})\big) \Big\}.$$

By (6.5), we have that

$$\mathfrak{V} \supseteq \pi_N(\Xi_0) \cap \pi_N(\Xi_1),$$

thus, due to (6.4),

$$\mathfrak{L}^N(\mathfrak{V}) \ \ge\ \mathrm{const}\, l^{N-1}. \tag{6.6}$$

With these inequalities in hand, we now start to estimate the functional, in order to show that the energy of u is too large, and hence obtaining a contradiction.

First of all, for any $x' \in \mathbb{R}^N$ with $|x'| \le l$, let us define

$$\begin{aligned} T_{x'}(x_N) &:= u(x', x_N), \\ \mathfrak{C}_{x'} &:= \{x_N \in \mathbb{R} \mid DT_{x'}(x_N) = 0\}. \end{aligned}$$

By standard regularity results (see [15] and [34]), we have that $T_{x'}$ is C^1. Hence, by Sard's Lemma,

$$\mathfrak{L}^N\Big(T_{x'}(\mathfrak{C}_{x'})\Big) \ =\ 0. \tag{6.7}$$

Thus, using that $T_{x'}$ is locally invertible on the complement of $\mathfrak{C}_{x'}$, we may write the latter set as

$$\bigcup_a J_{a,x'},$$

in such a way $T_{x'}\Big|_{J_{a,x'}}$ is a diffeomorphism. Therefore, by Young's inequality (writing q for the dual exponent of p) and by changing variable $x_{N+1} := T_{x'}(x_N)$,

$$\begin{aligned}
&\int_{J_{a,x'}} \frac{|\nabla u(x',x_N)|^p}{p} + h_0(u(x',x_N))\,dx_N \geq \\
\geq\ &\int_{J_{a,x'}} \frac{|\partial_N u(x',x_N)|^p}{p} + h_0(u(x',x_N))\,dx_N = \\
=\ &\int_{J_{a,x'}} \frac{|DT_{x'}(x_N)|^p}{p} + h_0(T_{x'}(x_N))\,dx_N \geq \\
\geq\ &\int_{J_{a,x'}} \Big(q\,h_0(T_{x'}(x_N))\Big)^{1/q} |DT_{x'}(x_N)|\,dx_N = \\
=\ &\int_{T_{x'}(J_{a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q} dx_{N+1}\,,
\end{aligned}$$

therefore,

$$\begin{aligned}
&\sum_a \int_{|x'|\leq l} \int_{T_{x'}(J_{a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q} dx_{N+1}\,dx' \leq \\
\leq\ &\sum_a \int_{|x'|\leq l} \int_{J_{a,x'}} \frac{|\nabla u(x',x_N)|^p}{p} + h_0(u(x',x_N))\,dx_N\,dx' = \\
=\ &\int_{|x'|\leq l} \int_{[-Cl,Cl]\setminus\mathfrak{C}_{x'}} \frac{|\nabla u(x',x_N)|^p}{p} + h_0(u(x',x_N))\,dx_N\,dx' \leq \\
\leq\ &\mathcal{F}_{A_l}(u)\,,
\end{aligned} \tag{6.8}$$

where $A_l := \{|x'| < l\} \times \{|x_N| < Cl\}$. Now, we notice that

$$\mathfrak{V} \subseteq \Big\{(x',0,x_{N+1}) \ \Big|\ |x'| \leq l,\ x_{N+1} \in T_{x'}(J_{a,x'}) \cap T_{x'}(J_{\hat a,x'}) \text{ for some } a \neq \hat a\Big\}\,,$$

and that

$$\mathfrak{V} \subseteq Q_l \subseteq \{|x_{N+1}| \leq 1/2\}\,,$$

hence, recalling also (6.6),

$$\begin{aligned}
&\int_{|x'|\leq l} \int_{x_{N+1}\in\bigcup_{a\neq\hat a} T_{x'}(J_{a,x'})\cap T_{x'}(J_{\hat a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q} dx_{N+1}\,dx' \geq \\
\geq\ &\int_{\mathfrak{V}} \Big(q\,h_0(x_{N+1})\Big)^{1/q} d(x',x_{N+1}) \geq \\
\geq\ &\inf_{[-1/2,\,1/2]} (q\,h_0)^{1/q}\,\mathfrak{L}^N(\mathfrak{V}) \geq \\
\geq\ &\tilde c_1\, l^{N-1} \inf_{[-1/2,\,1/2]} (q\,h_0)^{1/q} \geq \\
\geq\ &\tilde c_1\, l^{N-1} \inf_{[-1/2,1/2]} (q\,h_0)^{1/q}\,,
\end{aligned}$$

for a suitably small positive constant $\tilde{c}_1$. Therefore, we gather from the above inequality that

$$\begin{aligned}
&\sum_a \int_{|x'|\le l}\int_{T_{x'}(J_{a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' \ge \\
\ge\ & \int_{|x'|\le l}\int_{\bigcup_a T_{x'}(J_{a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' + \\
&+\int_{|x'|\le l}\int_{x_{N+1}\in\bigcup_{a\neq\hat a} T_{x'}(J_{a,x'})\cap T_{x'}(J_{\hat a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' \ge \\
\ge\ & \int_{|x'|\le l}\int_{x_{N+1}\in u(x',[-Cl,Cl]\setminus\mathfrak{C}_{x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' + \\
&+\tilde{c}_1\, l^{N-1} \inf_{[-1/2,1/2]} (q\,h_0)^{1/q}\,.
\end{aligned}$$

Thus, due to (6.7),

$$\begin{aligned}
&\sum_a \int_{|x'|\le l}\int_{T_{x'}(J_{a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' \ge \\
\ge\ & \int_{|x'|\le l}\int_{x_{N+1}\in u(x',[-l,l])} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' + \\
&+\tilde{c}_1\, l^{N-1} \inf_{[-1/2,1/2]} (q\,h_0)^{1/q}\,.
\end{aligned} \tag{6.9}$$

On the other hand, from Corollary 2.24, we get that

$$\begin{aligned}
&\int_{|x'|\le l}\int_{x_{N+1}\in u(x',[-l,l])} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' \ge \\
\ge\ & \int_{|x'|\le l}\int_{-1+s_l}^{1-s_l} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' = \\
=\ & \omega_{N-1}\, l^{N-1}\int_{-1+s_l}^{1-s_l} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,,
\end{aligned} \tag{6.10}$$

where ω_{N-1}, as usual, denotes the volume of the $(N-1)$-dimensional unit ball. From (6.9) and (6.10), we thus obtain that

$$\begin{aligned}
&\sum_a \int_{|x'|\le l}\int_{T_{x'}(J_{a,x'})} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1}\,dx' \ge \\
\ge\ & \omega_{N-1}\, l^{N-1}\int_{-1+s_l}^{1-s_l} \Big(q\,h_0(x_{N+1})\Big)^{1/q}\,dx_{N+1} + \\
&+\tilde{c}_1\, l^{N-1} \inf_{[-1/2,1/2]} (q\,h_0)^{1/q}\,,
\end{aligned}$$

and, therefore, thanks to (6.8),

$$\begin{aligned}
&\mathcal{F}_{A_l}(u) \geq \\
&\geq \omega_{N-1}\, l^{N-1} \int_{-1+s_l}^{1-s_l} \Big(q\, h_0(x_{N+1})\Big)^{1/q} dx_{N+1} + \\
&+\tilde{c}_1\, l^{N-1} \inf_{[-1/2,1/2]} (q\, h_0)^{1/q} . \qquad (6.11)
\end{aligned}$$

We now notice that, if c_2 and c_3 are positive constants, suitably small with respect to $\tilde{c}_1$, one has

$$\begin{aligned}
&\omega_{N-1}\, l^{N-1} \int_{[-1,-1+c_2]\cup[1-c_2,1]} \Big(q\, h_0(x_{N+1})\Big)^{1/q} dx_{N+1} \leq \\
&\leq 2c_2\omega_{N-1}\, l^{N-1} \sup_{[-1,1]} (q\, h_0(x_{N+1}))^{1/q} \leq \\
&\leq \frac{\tilde{c}_1}{2}\, l^{N-1} \inf_{[-1/2,1/2]} (q\, h_0)^{1/q} \qquad (6.12)
\end{aligned}$$

and

$$c_3\, l^{N-1} \leq \frac{\tilde{c}_1}{2}\, l^{N-1} \inf_{[-1/2,1/2]} (q\, h_0)^{1/q} . \qquad (6.13)$$

We now assume l big enough so that $s_l < c_2$: then, by means of (6.11), (6.12) and (6.13),

$$\begin{aligned}
\mathcal{F}_{A_l}(u) \;&\geq\; \omega_{N-1}\, l^{N-1} \int_{-1}^{1} \Big(q\, h_0(x_{N+1})\Big)^{1/q} dx_{N+1} + \\
&\quad +c_3\, l^{N-1} . \qquad (6.14)
\end{aligned}$$

This estimate will say that the energy of u is too large (thanks to the term "$c_3\, l^{N-1}$" here above), and it will provide the desired contradiction. For this, let us define the rescaled functional

$$\mathcal{F}^{\varepsilon}_{\Omega}(v) := \int_{\Omega} \frac{\varepsilon^{p-1}|\nabla v(x)|^p}{p} + \frac{1}{\varepsilon} h_0(v(x))\, dx .$$

Then, if $\varepsilon := 1/l$ and $u_\varepsilon(x) := u(x/\varepsilon)$, by scaling (6.14), we deduce that

$$\begin{aligned}
\mathcal{F}^{\varepsilon}_{A_1}(u_\varepsilon) \;&=\; \varepsilon^{N-1}\, \mathcal{F}_{A_l}(u) \geq \\
&\geq\; \omega_{N-1} \int_{-1}^{1} \Big(q\, h_0(x_{N+1})\Big)^{1/q} dx_{N+1} \,+\, c_3 . \qquad (6.15)
\end{aligned}$$

On the other hand, by §3 of [7], up to subsequences, we have that u_ε converges almost everywhere and in $L^1_{\rm loc}$ to the step function $\chi_E - \chi_{\mathbb{R}^N\setminus E}$, for a suitable set $E \subseteq \mathbb{R}^N$, and that

$$\lim_{\varepsilon\to 0^+} \mathcal{F}^{\varepsilon}_{A_1}(u_\varepsilon) = \text{Per}\,(E, A_1) \int_{-1}^{1} \Big(q\, h_0\Big)^{1/q} , \qquad (6.16)$$

where, given $A \subseteq B$, we denote the perimeter of A in B as $\mathrm{Per}\,(A, B)$ (see, for instance, [20] for full details on such definition). As a matter of fact, in our situation, the set E may be better specified, in the following way. From (6.1), there exists $\kappa > 0$ so that $u(x) \geq \kappa$ if $|x'| \leq l$ and $x_N \geq 2\theta$ and $u(x) \leq -\kappa$ if $|x'| \leq l$ and $x_N \leq -2\theta$.
Therefore, $u_\varepsilon(x) \geq \kappa$ if $|x'| \leq 1$ and $x_N \geq 2\varepsilon\theta$ and $u(x) \leq -\kappa$ if $|x'| \leq 1$ and $x_N \leq -2\varepsilon\theta$. In particular, for almost any $x \in A_1$,

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(x) \geq \kappa \text{ if } x_N > 0 \text{ and}$$
$$\lim_{\varepsilon \to 0^+} u_\varepsilon(x) \leq -\kappa \text{ if } x_N > 0.$$

This implies that $E = A_1 \cap \{x_N > 0\}$. And so $\mathrm{Per}\,(E, A_1) = \omega_{N-1}$. Therefore, from (6.16)

$$\lim_{\varepsilon \to 0^+} \mathcal{F}^\varepsilon_{A_1}(u_\varepsilon) = \omega_{N-1} \int_{-1}^{1} \Big(q\, h_0\Big)^{1/q}.$$

This contradicts (6.15) and finishes the proof of Theorem 1.1.

7. Proof of Theorem 1.2

The proof of Theorem 1.2 will be performed by compactness, by using Theorem 1.1 and a result of [30].

We fix $\theta_0 > 0$ and we assume by contradiction that there exist u_k , θ_k , l_k for which

- **(C1)** u_k is a local minimizer for $\mathcal{F}$ in $\{|x'| < l_k\} \times \{|x_N| < l_k\}$, with $u_k(0) = 0$.
- **(C2)** $\{u_k = 0\} \subseteq \{|x'| < l_k\} \times \{|x_N| < \theta_k\}$, with $\theta_k \geq \theta_0$ and $\frac{\theta_k}{l_k} \longrightarrow 0$ when $k \to \infty$,

but the thesis of Theorem 1.2 does not hold. Let us consider the following rescaling:

$$y' = \frac{x'}{l_k} \qquad ; \qquad y_N = \frac{x_N}{\theta_k} \tag{7.1}$$

say $(y', y_N) = T(x', x_N)$. Define

$$A_k := \Big\{(y', y_N) \text{ s.t. } T^{-1}(y', y_N) \in \{u_k = 0\}\Big\} = T\Big(\{u_k = 0\}\Big).$$

STEP 1: There exists a Hölder continuous function $w : \mathbb{R}^{N-1} \to \mathbb{R}$ such that: if we define

$$A_\infty := \Big\{(y', w(y')) \,, \, |y'| \leq \frac{1}{2}\Big\}$$

then, for any $\varepsilon > 0$, $A_k \cap \{|y'| \leq 1/2\}$ lies in a ε-neighborhood of A_∞, for k sufficiently large.

Proof of step 1.
Let us suppose that

$$y_0 = (y_0', y_{0\,N}) \in A_k \qquad \text{with} \qquad |y_0'| \leq 1/2\,.$$

Then, $u_k(l_k y_0', \theta_k y_{0\,N}) = 0$, and so, by means of **(C2)**, $|\theta_k y_{0\,N}| < \theta_k$; therefore, using again **(C2)**, we infer that

$$\{u_k = 0\} \subseteq \{|x_N - \theta_k y_{0\,N}| < 2\theta_k\}\,.$$

Thence, we can exploit Theorem 1.1 in the cylinder

$$\begin{aligned} &\{|x' - l_k y_0'| < \frac{l_k}{2}\} \times \{|x_N - \theta_k y_{0\,N}| < 2\theta_k\} \subseteq \\ &\subseteq \{|x'| < l_k\} \times \{|x_N| < l_k\}\,, \end{aligned} \tag{7.2}$$

and get that there exists a universal constant $\eta_0 > 0$ such that

$$\{u_k = 0\} \cap \{|x' - l_k y_0'| < \eta_0 \frac{l_k}{2}\} \subseteq \{|x_N - \theta_k y_{0\,N}| < 2(1-\eta_0)\theta_k\}\,,$$

provided

$$\frac{4\theta_k}{l_k} \leq \varepsilon_0(2\theta_0)\,,$$

where $\varepsilon_0(\cdot)$ is the one given by Theorem 1.1. Rescaling back, we get

$$A_k \cap \{|y' - y_0'| < \frac{\eta_0}{2}\} \subseteq \{|y_N - y_{0\,N}| < 2(1-\eta_0)\}\,.$$

By iterating, we get

$$A_k \cap \{|y' - y_0'| < \frac{\eta_0^m}{2}\} \subseteq \{|y_N - y_{0\,N}| < 2(1-\eta_0)^m\}\,, \tag{7.3}$$

provided

$$\frac{4\theta_k}{l_k} \leq \eta_0^{m-1} \varepsilon_0\Big(2(1-\eta_0)^{m-1}\theta_0\Big)\,. \tag{7.4}$$

We now fix $m_0 \in \mathbb{N}$ and consider $m \leq m_0$ (later on, during a limiting procedure performed on page 69, we let $m_0 \longrightarrow +\infty$). Note that, in this setting, (7.4) (and therefore (7.3)) is fulfilled for k suitably large, say $k \geq k^\star(m_0)$. We claim that $A_k \cap \{|y'| \leq 1/2\}$ is above the graph of

$$\Psi_{y_0,k}(y') = y_{0\,N} - 2(1-\eta_0)^{m_0} - \alpha|y' - y_0'|^\beta \tag{7.5}$$

where α and $\beta > 0$ depend only on η_0.
To prove this, let $(y', y_N) \in A_k \cap \{|y'| \leq 1/2\}$. Since $|y_0'| \leq \frac{1}{2}$ we have that $|y' - y_0'| \leq 1$. Now, we consider three different cases: the case $|y' - y_0'| \leq \frac{\eta_0^{m_0}}{2}$, the case $\frac{\eta_0^{m_0}}{2} \leq |y' - y_0'| \leq \frac{1}{2}$, and the case $\frac{1}{2} \leq |y' - y_0'| \leq 1$.
In case $|y' - y_0'| \leq \frac{\eta_0^{m_0}}{2}$, (7.5) follows immediately from (7.3), with $m = m_0$.
If, on the other hand, $\frac{\eta_0^{m_0}}{2} \leq |y' - y_0'| \leq \frac{1}{2}$, then we argue as follows. We first note that, in this case, there exists m with $0 \leq m \leq m_0$, such that

$$\frac{\eta_0^{m+1}}{2} \leq |y' - y_0'| \leq \frac{\eta_0^m}{2}\,. \tag{7.6}$$

Consequently, from (7.3), we have that

$$2(1-\eta_0)^m \geq |y_N - y_{0\,N}| \tag{7.7}$$

By (7.6) and the fact that $0<\eta_0<1$, we also get

$$m \leq \frac{-\ln(2|y'-y_0'|)}{\ln(\frac{1}{\eta_0})} \leq m+1\,.$$

In particular, it follows that

$$(1-\eta_0)^m \leq (1-\eta_0)^{\left(\frac{-\ln(2|y'-y_0'|)}{\ln(\frac{1}{\eta_0})}-1\right)} =$$

$$= \frac{1}{(1-\eta_0)} e^{\beta \ln(2|y'-y_0'|)} = \frac{(2|y'-y_0'|)^\beta}{(1-\eta_0)}\,,$$

where $\beta := \frac{-\ln(1-\eta_0)}{\ln(\frac{1}{\eta_0})}$.
Therefore, recalling (7.7), it follows

$$|y_N - y_{0\,N}| \leq \frac{2^{\beta+1}}{(1-\eta_0)} |y'-y_0'|^\beta$$

which is the desired result, with $\alpha := 2^{\beta+1}/(1-\eta_0)$.
Finally, eventually adding[12] a constant to α, the result also follows for the case $|y'-y_0'| \in [1/2, 1]$. This ends the proof of (7.5).

Note now that, as y_0 varies, $\Psi_{y_0,k}$ are Hölder continuous functions with Hölder modulus of continuity bounded via the function αt^β (recall that m_0 is fixed for the moment, and that α and β depend only on η_0). Therefore, if we set

$$\psi_k(y') := \sup_{\substack{|y_0'|\leq \frac{1}{2} \\ y_0\in A_k}} \Psi_{y_0,k}(y')$$

then, ψ_k is a Hölder continuous function (with Hölder modulus of continuity bounded via the function αt^β), and $A_k \cap \{|y'|\leq 1/2\}$ is above the graph of ψ_k.

Arguing in the same way, possibly taking α and β larger (depending only on η_0), we also get that, if we define

$$\Phi_{y_0,k}(y') := y_{0\,N} + 2(1-\eta_0)^{m_0} + \alpha|y'-y_0'|^\beta\,,$$

then $A_k \cap \{|y'|\leq 1/2\}$ is below the graph of $\Phi_{y_0,k}$. Arguing as above, we define

$$\phi_k(y') := \inf_{\substack{|y_0'|\leq \frac{1}{2} \\ y_0\in A_k}} \Phi_{y_0,k}(y')\,,$$

[12] Notice indeed that, by **(C2)**, we have that

$$|y_N - y_{0\,N}| \leq |y_N| + |y_{0\,N}| \leq 2\,.$$

so that ϕ_k is a Hölder continuous function (with Hölder modulus of continuity bounded via the function αt^β), and $A_k \cap \{|y'| \leq 1/2\}$ is below the graph of ϕ_k.

In particular, $A_k \cap \{|y'| \leq 1/2\}$ lies between $\psi_k(y')$ and $\phi_k(y')$ for any $k \geq k^\star(m_0)$ and, by construction

$$0 \leq \phi_k(y') - \psi_k(y') \leq 4(1-\eta_0)^{m_0}\,. \tag{7.8}$$

Also, for m_0 fixed, by Ascoli-Arzelà Theorem,, letting $k \to \infty$, it follows that $\psi_k(y')$ uniformly converges to a Hölder continuous function which depends on m_0, Say

$$\lim_{k \longrightarrow +\infty} \psi_k(y') \to w^-_{m_0}(y')\,.$$

Analogously, we find a Hölder continuous function $w^+_{m_0}$, such that

$$\lim_{k \longrightarrow +\infty} \phi_k(y') \to w^+_{m_0}(y')$$

uniformly. Also, by construction, we have that $w^-_{m_0} \leq w^+_{m_0}$ and $A_k \cap \{|y'| \leq 1/2\}$ lies between $w^-_{m_0}$ and $w^+_{m_0}$.

Let now $m_0 \to \infty$. In this case, by Ascoli-Arzelà Theorem[13], we get that there exists a Hölder continuous function w such that $w^-_{m_0}$ uniformly converges to w. By (7.8), also $w^+_{m_0}$ uniformly converges to w. Since $A_k \cap \{|y'| \leq 1/2\}$ lies between $w^-_{m_0}$ and $w^+_{m_0}$, the claim is proved.

STEP 2: The function w constructed in the first step is harmonic.

Proof of step 2.
We prove that w is harmonic in the viscosity sense. Then it follows that it is harmonic in the classic sense (see, e.g., Theorem 6.6 in [8]).
For this, let P be the quadratic polynomial

$$P(y') := \frac{1}{2} y'^T M y' + \xi \cdot y'\,.$$

Assume, by contradiction, that $\Delta P > 0$, that P touches the graph of w, say at 0 for simplicity and that P stays below it in $|y'| < 2r$, for some $r \in (0,1)$. Let now $\delta_0 > 0$ be the universal constant of Lemma 9.3 in [30] and let us define

$$\delta := \min\left\{ \left(\frac{\Delta P}{2\theta_0}\right)^{\frac{1}{2}},\ \frac{1}{2\theta_0\,\|M\|},\ \frac{1}{2\theta_0\,|\xi|},\ \left(\frac{\delta_0}{2\theta_0}\right)^{\frac{1}{2}},\ r \right\}\,.$$

Thus, δ is such that

$$\Delta P > 2\delta^2\theta_0 \qquad , \qquad \|M\| \leq \frac{1}{2\delta\theta_0} \qquad , \qquad |\xi| \leq \frac{1}{2\delta\theta_0}\,,$$

$$\delta^2\theta_0 \leq \frac{\delta_0}{2}\,. \tag{7.9}$$

[13] We remark that, by the construction of α and β above, the Hölder constants of $w^\pm_{m_0}$ depend on η_0, but are independent of m_0.

Note that, eventually replacing δ with 2δ and $P(y')$ with $P(y')-\delta|y'|^2$, we may assume, with no lose of generality, that P touches the graph of w at 0 and stays strictly below it in $|y'|<2\delta<2$. therefore, since $A_k\cap\{|y'|\le 1/2\}$ uniformly converges to the graph of w, it follows that, for k large, we find points $y_k=(y'_k,y_{k\,N})$ close to 0, such that $P(y')-K_k$ touches A_k at $(y'_k,y_{k\,N})$ and stays below it in $|y'-y'_k|\le\delta$, for an appropriate $K_k\in\mathbb{R}$. In particular, we have

$$y_{k\,N}+K_k=\frac{1}{2}y_k'^T My'_k+\xi\cdot y'_k\,. \tag{7.10}$$

Let us now consider the following translation

$$z'=y'-y'_k \qquad\qquad z_N=y_N-(y_{k\,N}+K_k)$$

Exploiting (7.10) we find a surface

$$\left\{z_N=\frac{1}{2}z'^T Mz'+\xi_k\cdot z'\right\},$$

with

$$\xi_k\ :=\ My'_k+\xi$$

that touches A_k by below at the origin and stays below it in $|z'|<\delta$. Notice also that, by construction,

$$|\xi_k|\le\frac{1}{\delta\theta_0}\,. \tag{7.11}$$

Rescaling back, we get that the surface

$$\left\{x_N=\frac{\theta_k}{l_k^2}\frac{1}{2}x'^T Mx'+\frac{\theta_k}{l_k}\xi_k\cdot x'\right\}$$

touches $\{u_k=0\}$ at the origin and stays below it, if $|x'|<\delta l_k$.
We write now the above surface in the form

$$\left\{x_N=\frac{\delta^2\theta_k}{(\delta_k l_k)^2}\frac{1}{2}x'^T Mx'+\frac{\delta^2\theta_k}{\delta_k l_k}\frac{1}{\delta}\xi_k\cdot x'\right\}$$

and we exploit[14] Lemma 9.3 in [30], thus gathering that

$$\Delta P\le\delta^2\theta_0\,,$$

against the assumption. This contradiction shows that $\Delta P\le 0$. By arguing in the same way, one may prove that $\Delta P\ge 0$ if P touches w by above, so that the claim of Step 2 on page 69 is proved.

[14]Notation remark: Lemma 9.3 in [30] is used here with $M_1:=M$, $\delta:=\delta^2\theta_0$, $\theta:=\delta^2\theta_k$ (note that $\delta\le\theta$ since $\theta_0\le\theta_k$), $l:=\delta l_k$, and $\xi:=\frac{1}{\delta}\xi_k$ (therefore $|\xi|\le\frac{1}{\delta^2\theta_0}$, thanks to (7.11)). In particular, with this setting, since $\frac{\theta_k}{l_k}\to 0$, then $\frac{\theta}{l}\to 0$. Also, since $\sigma(\delta)>0$ (where $\sigma(\delta)$ is as in Lemma 9.3 in [30]), then the condition $\frac{\theta}{l}<\sigma(\delta)$ (i.e. $\frac{\delta\theta_k}{l_k}<\sigma(\delta)$) is fulfilled for k sufficiently large.

CONCLUSION: Since w is harmonic, by standard elliptic estimates (see, e.g., Theorem 2.10 in [19]), we find a positive universal constant C, such that

$$\|D^2 w\| \le C$$

Therefore, since by construction $w(0) = 0$, by Taylor's formula, it follows that

$$|w(y') - \nabla w(0) \cdot y'| < C' \eta_2^2 \qquad \text{for} \quad |y'| < 2\eta_2 \,.$$

In particular, for η_2 sufficiently small, setting

$$\xi' := \nabla w(0)$$

we get that there exist positive constants $0 < \eta_1 < \eta_2 < 1$, for which

$$|w(y') - \xi' \cdot y'| < \frac{\eta_1}{2} \qquad \text{for} \quad |y'| < 2\eta_2 \,. \tag{7.12}$$

Now, let us consider

$$\xi_k := \frac{(\frac{\theta_k}{l_k}\xi', -1)}{\sqrt{\frac{\theta_k^2}{l_k^2}|\xi'|^2 + 1}} \,. \tag{7.13}$$

Considering the rescaling given by (7.1), elementary geometric considerations show that

$$\{|\pi_{\xi_k} x| < \eta_2 l_k\} \times \{|x \cdot \xi_k| < \eta_2 l_k\} \subset \{|x'| < 2 l_k \eta_2\} \subset \{|x'| < l_k/2\} \,. \tag{7.14}$$

Since $A_k \cap \{|y'| \le 1/2\}$ uniformly converges to the graph of w, for k sufficiently large (thanks to Step 1 on page 66), we may suppose that $A_k \cap \{|y'| \le 1/2\}$ is in a $\frac{\eta_1}{4}$-neighborhood of the graph of w. Consequently, by (7.12), taking into account the rescaling, it follows that

$$\{u_k = 0\} \cap \{|x'| \le l_k/2\} \subset \big\{|x_N - \frac{\theta_k}{l_k}\xi' \cdot x'| < \frac{3}{4}\theta_k \eta_1\big\} \,.$$

From (7.13), we thence get that

$$\{u_k = 0\} \cap \{|x'| \le l_k/2\} \subset \big\{|x \cdot \xi_k| < \frac{3}{4}\theta_k \eta_1\big\} \,,$$

which, together with (7.14), is a contradiction with the fact that u_k does not satisfies the statement of Theorem 1.2. This ends the proof of Theorem 1.2.

8. Proof of Theorem 1.3

The following Lemma 8.1 is an intermediate step towards the proof of Theorem 1.3 and it is also useful for the proof of Theorem 1.4. The proof of Lemma 8.1 is based on an iteration of Theorem 1.2.

Lemma 8.1. *Let u be a Class A minimizer for $\mathcal{F}$ in $\mathbb{R}^N$ with $u(0) = 0$. Suppose that there exist sequences of positive numbers θ_k , l_k and unit vectors ξ_k, with*

$$l_k \to \infty \qquad \textit{and} \qquad \frac{\theta_k}{l_k} \to 0 \,, \tag{8.1}$$

such that

$$\{u=0\}\cap\Big(\{|\pi_{\xi_k}x|<l_k\}\times\{|x\cdot\xi_k|<l_k\}\Big)\subset\{|x\cdot\xi_k|\le\theta_k\}. \tag{8.2}$$

Then, the 0 *level set* $\{u=0\}$ *is a hyperplane in* $\mathbb{R}^N$.

Proof. Let fix $\theta_0>0$ and $\varepsilon\le\varepsilon_1(\theta_0)$, with $\varepsilon_1(\theta_0)$ given by Theorem 1.2. We consider k so large in that

$$\frac{\theta_k}{l_k}\le\varepsilon\le\varepsilon_1(\theta_0)\,. \tag{8.3}$$

Two cases are now possible: either, for infinitely many k's $\theta_k\le\theta_0$, or for infinitely many k's $\theta_k>\theta_0$.

In the first case, we take the subsequence of k's for which $\theta_k\le\theta_0$ and we assume, by possibly extracting a further subsequence, that ξ_k converges to a suitable unit vector ξ. We consider a y-frame of coordinates in which y_N is parallel to ξ. Consequently, by (8.1) and (8.2), we deduce that, in this system of coordinates,

$$\{u=0\}\subseteq\{|y_N|\le\theta_0\}\,.$$

Thence, since θ_0 is arbitrary,

$$\{u=0\}\subseteq\{y_N=0\}\,,$$

which proves the desired result.

If, on the other hand, $\theta_k>\theta_0$ for infinitely many k's, then we fix k large enough to fulfill (8.3) and we apply Theorem 1.2 repeatedly as much as we can. More precisely, for $h\ge0$, let $l_k^{(h)}:=\eta_2^h l_k$ and $\theta_k^{(h)}:=\eta_1^h\theta_k$. Then, if $\theta_k^{(h)}>\theta_0$, we can keep applying Theorem 1.2; we stop this procedure when h is so large that

$$\theta_k^{(h)}\le\theta_0\,.$$

More precisely, we stop the iterative application of Theorem 1.2 when $h\ge1$ is so that

$$\theta_k^{(h-1)}>\theta_0\ge\theta_k^{(h)}\,.$$

For such h, we get, by construction, that

$$\theta_0\ge\theta_k^{(h)}\ge\eta_1\theta_0\,.$$

Also, by construction,

$$\frac{\theta_k^{(h)}}{l_k^{(h)}}=\left(\frac{\eta_1}{\eta_2}\right)^h\frac{\theta_k}{l_k}\le\varepsilon\,.$$

In particular,

$$l_k^{(h)}\ge\frac{\eta_1\theta_0}{\varepsilon}\,. \tag{8.4}$$

What is more, the repeated use of Theorem 1.2, has driven us to proving that, in some system of coordinates,

$$\{u=0\}\cap\Big(\{|y'|<l_k^{(h)}\}\times\{|y_N|<l_k^{(h)}\}\Big)\subseteq\{|y_N|\le\theta_k^{(h)}\},$$

that is,

$$\{u=0\}\cap\left(\left\{|y'|<\frac{\eta_1\theta_0}{\varepsilon}\right\}\times\{|y_N|<\frac{\eta_1\theta_0}{\varepsilon}\}\right)\subseteq\{|y_N|\le\theta_0\},$$

thanks to to (8.4). Therefore, letting $\varepsilon\longrightarrow 0$, it follows that

$$\{u=0\}\subset\{|y_N|\le\theta_0\}.$$

Since θ_0 was arbitrary, the lemma is proved. □

By means of Lemma 8.1, we are now in the position to complete the proof of Theorem 1.3, by arguing as follows.

Let us consider the rescaled functional

$$\mathcal{F}^\varepsilon_\Omega(v):=\int_\Omega\frac{\varepsilon^{p-1}|\nabla v(x)|^p}{p}+\frac{1}{\varepsilon}h_0(v(x))\,dx\,.$$

Then, for any $\Omega\subset\mathbb{R}^N$, $u_\varepsilon(x):=u(x/\varepsilon)$ is a local minimizer for $\mathcal{F}^\varepsilon_\Omega$.

Therefore, by §3 of [7], up to subsequences, we have that u_ε converges almost everywhere and in $L^1_{\rm loc}$ to the step function $\chi_E-\chi_{\mathbb{R}^N\setminus E}$, for a suitable set $E\subseteq\mathbb{R}^N$ with minimal perimeter.

We claim now that

$$\{u_{\varepsilon_k}=0\}\quad\text{uniformly converges to}\quad\partial E\quad\text{on compact sets.}$$

Assume that this is not true and note that in this case there exist $\delta>0$, and a point $z_0\in\mathbb{R}^N$ and points x_k, such that

$$x_k\in\{u_{\varepsilon_k}=0\}\cap B(z_0,\delta)\quad\text{with}\quad B(z_0,2\delta)\cap\partial E=\emptyset$$

Assume e.g. $B(z_0,2\delta)\subset E$ and note that in this case, exploiting the density estimate in [28], we get a contradiction with the fact that u_ε converges almost everywhere and in $L^1_{\rm loc}$ to the step function $\chi_E-\chi_{\mathbb{R}^N\setminus E}$ (in the same way we get a contradiction if $B(z_0,2\delta)\subset\mathbb{R}^N\setminus E$).

Since ∂E is a minimal surface in $\mathbb{R}^N$, and we assumed that $N\le 7$, then ∂E is a hyperplane (see, for instance, Theorem 17.3 in [20]). Also, since $u_{\varepsilon_k}(0)=0$ and $\{u_{\varepsilon_k}=0\}$ uniformly converges to ∂E, it follows that $0\in\partial E$.

This implies that, in some system of coordinates

$$\{u_{\varepsilon_k}=0\}\cap B_1\subset\{|x_N|\le\delta_k\}\tag{8.5}$$

with $\delta_k\to 0$. Rescaling back we get that

$$\{u=0\}\cap B_{\frac{1}{\varepsilon_k}}\subset\{|x_N|\le\frac{\delta_k}{\varepsilon_k}\}\tag{8.6}$$

Two cases are now possible: either δ_k/ε_k is bounded away from zero, or, up to subsequences, $\delta_k/\varepsilon_k \longrightarrow 0$. In the latter case, we pass to the limit (8.5) by sending $k \longrightarrow +\infty$, getting that $\{u = 0\}$ is a hyperplane. If, on the other hand, $\delta_k/\varepsilon_k \geq \theta_0$, for some $\theta_0 > 0$, we define

$$l_k := \frac{1}{2\varepsilon_k}, \qquad \theta_k := \frac{\delta_k}{\varepsilon_k}$$

and we observe that

$$\frac{\theta_k}{l_k} = \frac{\delta_k}{2} \longrightarrow 0\,;$$

then, it follows that the assumptions of Lemma 8.1 are fulfilled. Thus, the application of Lemma 8.1 proves that $\{u = 0\}$ is a hyperplane, which is the desired result.

9. Proof of Theorem 1.4

First, we prove the minimality of u:

Lemma 9.1. *Let h_0 satisfy* (1.1), (1.2), (1.3) *and* (1.4). *Let u be a weak Sobolev solution of (1.5) in the whole $\mathbb{R}^N$, satisfying $|u| \leq 1$, $\partial_N u > 0$ and* $\lim_{x_N \to +\infty} u = \pm 1$. *Then, u is a class A minimizer.*

Proof. Since u is strictly increasing, $|u| < 1$. Let $B \subset \mathbb{R}^N$ be a closed ball and let v be a minimizer for $\mathcal{F}_B$ with $v = u$ on ∂B. Our aim is to show that $u = v$ in B. Let us argue by contradiction and assume, say, that

$$v(x^\star) > u(x^\star)\,, \tag{9.1}$$

for some $x^\star \in B$. Possibly cutting v on the ± 1-levels (which decreases $\mathcal{F}_B$), we may and do assume that $|v| \leq 1$. More precisely, as mentioned in the footnote on page 3, by (1.4) it follows that $|v| < 1$.
Then, since $|\nabla u| \leq$ const thanks to [15] or [34], and $\lim_{x_N \to +\infty} u = \pm 1$, we deduce that

$$u(x + te_N) \geq v(x) \tag{9.2}$$

for any $x \in B$, provided that t is large enough. Indeed, to prove (9.2), let us argue by contradiction and assume that $u(x_t + te_N) < v(x_t)$ for some $x_t \in B$ and a diverging sequence of t; let also $\alpha > 0$ so that $v \leq 1 - \alpha$. Then, up to subsequence, we may assume that x_t converges to $x_\infty \in B$; but then

$$\begin{aligned} 1 &= \lim_{t \to +\infty} u(x_\infty + te_N) \leq \\ &\leq \lim_{t \to +\infty} u(x_t + te_N) + \text{const}\,|x_t - x_\infty| = \\ &= \lim_{t \to +\infty} u(x_t + te_N) \leq \\ &\leq \lim_{t \to +\infty} v(x_t) \leq \\ &\leq 1 - \alpha\,. \end{aligned}$$

This contradiction proves (9.2).

Thanks to (9.2), we thus slide $u(\cdot + te_N)$ towards the e_N-direction until we touch v from above. Say this happen at $\bar{x} \in B$ for $t = \bar{t}$. In the light of (9.1), we have that

$$\begin{aligned} u(x^\star + \bar{t}e_N) &\geq v(x^\star) > \\ &> u(x^\star)\,, \end{aligned}$$

thence, since u is strictly increasing in the e_N-direction,

$$\bar{t} > 0\,.$$

Since now $\partial_N u > 0$ we have that $\nabla u(\cdot + \bar{t}e_N) \neq 0$. Therefore, it follows that the assumptions of the Strong Comparison Principle for p-Laplace equations in [9] (see Corollary B.5 here) applies to $u(\cdot+\bar{t}e_N)$ and v and so this touching point must occur on ∂B, that is $\bar{x} \in \partial B$. Since $u = v$ on ∂B, it follows that $v(\bar{x}) = u(\bar{x})$. Consequently, since u is strictly increasing in the e_N-direction,

$$u(\bar{x}) = v(\bar{x}) = u(\bar{x} + \bar{t}e_N) > u(\bar{x})\,.$$

This contradiction shows that (9.1) cannot hold, hence $v \leq u$. Analogously, one sees that $v \geq u$, thence $v = u$. $\square$

By means of Lemma 9.1, we are now in the position to complete the proof of Theorem 1.4, by arguing as follows. With no loss of generality, we assume that $u(0) = 0$. Then, for $\varepsilon > 0$, setting $u_\varepsilon(x) := u(x/\varepsilon)$, we know from Lemma 9.1 and the results of [7] that u_ε $L^1_{\rm loc}$-converges (and thus a.e.-converges), up to subsequence, to $\chi_E - \chi_{\mathbb{R}^N\setminus E}$, for a suitable E with minimal perimeter.

Since $\partial_N u > 0$ and $\lim_{x_N\to+\infty} u = \pm 1$, we have that the zero level set of u is a graph in the e_N-direction; more precisely, there exists $\gamma : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ so that

$$\{u < 0\} \;=\; \{x_N < \gamma(x')\}\,.$$

By scaling, we thus deduce that

$$\{u_\varepsilon < 0\} \;=\; \{x_N < \gamma_\varepsilon(x')\}\,, \tag{9.3}$$

with

$$\gamma_\varepsilon(x') \;:=\; \varepsilon\gamma(x'/\varepsilon)\,.$$

We now claim that

$$\chi_{\{x_N<\gamma_\varepsilon(x')\}} \text{ converges in } L^1_{\rm loc} \text{ to } \chi_{\mathbb{R}^N\setminus E}. \tag{9.4}$$

Indeed: we know that u_ε converges to $\chi_E - \chi_{\mathbb{R}^N\setminus E}$ in $\mathbb{R}^N \setminus Z$, for a suitable set Z with $\mathfrak{L}^N(Z) = 0$; thus, if $x \in E \setminus Z$,

$$\lim_{\varepsilon\longrightarrow 0^+} u_\varepsilon(x) = \chi_E(x) - \chi_{\mathbb{R}^N\setminus E}(x) = 1\,,$$

therefore

$$\lim_{\varepsilon\longrightarrow 0^+} \chi_{\{u_\varepsilon<0\}}(x) = 0 = \chi_{\mathbb{R}^N\setminus E}(x)\,,$$

while if $x \in (\mathbb{R}^N \setminus E) \setminus Z$ then

$$\lim_{\varepsilon \longrightarrow 0^+} u_\varepsilon(x) = \chi_E(x) - \chi_{\mathbb{R}^N \setminus E}(x) = -1\,,$$

therefore

$$\lim_{\varepsilon \longrightarrow 0^+} \chi_{\{u_\varepsilon < 0\}}(x) = 1 = \chi_{\mathbb{R}^N \setminus E}(x)\,.$$

This shows that $\chi_{\{u_\varepsilon < 0\}}$ converges almost everywhere to $\chi_{\mathbb{R}^N \setminus E}$. Thus, (9.4) follows from the Dominated Convergence Theorem and (9.3).

In the light of (9.4) and Lemma 16.3 of [20], we have that $\mathbb{R}^N \setminus E$ is a subgraph of a measurable function which is the a.e.-limit of γ_ε up to subsequences, and which may attain the values $\pm\infty$; that is, there exists $\gamma_\star : \mathbb{R}^{N-1} \longrightarrow [-\infty, +\infty]$ in such a way that

$$\gamma_\star(x) = \lim_{\varepsilon \to 0^+} \gamma_\varepsilon(x) \tag{9.5}$$

for almost any x, up to subsequence, and

$$\mathbb{R}^N \setminus E \,=\, \{x_N < \gamma_\star(x')\}\,.$$

Since ∂E is a minimal perimeter, we have that $\gamma_\star$ is a quasi-solution of the minimal surface equation, according to Definition 16.1 of [20].

We now prove that ∂E is a hyperplane. We distinguish two cases, according to our hypotheses. If $N \leq 8$, we have that γ_* is an entire quasi-solution of the minimal surface equation in a space with dimension less or equal than 7; therefore, by Theorem 17.8 and Remark 17.9 of [20], we have that $\partial E = \{x_N = \gamma_\star(x')\}$ is a hyperplane. If, on the other hand, $\{u = 0\}$ has at most linear growth, then

$$\{u = 0\} \,\subseteq\, \{|x_N| \leq K\,(|x'| + 1)\}\,,$$

for a suitable $K > 0$. This says that

$$\begin{aligned} \{x_N = \gamma_\varepsilon(x')\} \;&=\; \{u_\varepsilon = 0\} \subseteq \\ &\subseteq\; \{|x_N| \leq K\,(|x'| + \varepsilon)\} \subseteq \\ &\subseteq\; \{|x_N| \leq K\,(|x'| + 1)\}\,, \end{aligned}$$

i.e., $|\gamma_\varepsilon(x')| \leq K\,(|x'| + 1)$. Thus, by means of (9.5), we gather that

$$|\gamma_\star(x')| \leq K\,(|x'| + 1)\,, \tag{9.6}$$

thus $\gamma_\star$ is locally bounded. Hence, $\gamma_\star$ is a solution of the minimal surface equation (see [20], page 183). Therefore, ∂E is a hyperplane thanks to (9.6) and Theorem 17.6 of [20].

In any case, we have proved that ∂E is a hyperplane, thence

$$\partial E \,=\, \Big\{\xi \cdot x = 0\Big\}\,,$$

for a suitable $\xi \in \mathbb{R}^N$ with $|\xi| = 1$. Thanks to [28], we know that $\{u_\varepsilon = 0\}$ $L^\infty_{\rm loc}$-converges ∂E, hence, for any $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ as small as we wish, so that

$$B_2 \cap \{u_{\varepsilon_k} = 0\} \subseteq \Big\{|\xi \cdot x| \le 1/k\Big\}.$$

By scaling back the variables, we thence obtain that

$$\{u = 0\} \cap \Big(\{|x \cdot \xi| \le 1/\varepsilon_k\} \times \{|\pi_\xi x| \le 1/\varepsilon_k\}\Big) \subseteq \Big\{|\xi \cdot x| \le 1/(k\varepsilon_k)\Big\}.$$

We now invoke Lemma 8.1, used here with $\xi_k := \xi$, $l_k := 1/\varepsilon_k$, $\theta_k := 1/(k\varepsilon_k)$, and we infer that $\{u = 0\}$ is a hyperplane. This completes the proof of Theorem 1.4.

Appendix A. Proof of the measure theoretic results

A.1. **Proof of Lemma 4.1.** By the hypotheses of the lemma and (2.41), $\widetilde{\mathbb{S}}(Y, R)$ touches the graph of u by above at X_0. Notice also that, in the notation of Lemma 3.3, since $Y = F(X_0, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X_0))$,

$$y = x_0 + \frac{\Big(\nu_1^{\widetilde{\mathbb{S}}(Y,R)}(X_0), \dots, \nu_N^{\widetilde{\mathbb{S}}(Y,R)}(X_0)\Big)}{\Big|\Big(\nu_1^{\widetilde{\mathbb{S}}(Y,R)}(X_0), \dots, \nu_N^{\widetilde{\mathbb{S}}(Y,R)}(X_0)\Big)\Big|}\, \sigma(X_0, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X_0)),$$

and thus

$$\begin{aligned}
(\mathrm{A.1}) \qquad & |(x_0 - y) \cdot e_N| = \\
&= \frac{\Big|\nu_N^{\widetilde{\mathbb{S}}(Y,R)}(X_0)\Big|}{\Big|\Big(\nu_1^{\widetilde{\mathbb{S}}(Y,R)}(X_0), \dots, \nu_N^{\widetilde{\mathbb{S}}(Y,R)}(X_0)\Big)\Big|} \Big|\sigma(X_0, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X_0))\Big| = \\
&= \frac{|\partial_N u(x_0)|}{|\nabla u(x_0)|} \Big[R + \\
&\quad + H_0(x_0 \cdot e_{N+1}) - H_0\Big(x_0 \cdot e_{N+1} + \\
&\quad + \omega(X_0, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X_0))\Big) - \frac{\overline{C}_0}{2R}\omega^2(X_0, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X_0))\Big].
\end{aligned}$$

Also, from Lemma 3.3,

$$\left[-\frac{1}{4}, \frac{1}{4}\right] \ni y_{N+1} = x_{N+1} + \omega\big(X_0, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X_0)\big)$$

and so

$$\Big|\omega\big(X_0, \nu^{\widetilde{\mathbb{S}}(Y,R)}(X_0)\big)\Big| \le \frac{1}{4} + \frac{1}{2} < 1. \tag{A.2}$$

Therefore, from (A.1), (A.2) and (4.2),

$$|(x_0 - y) \cdot e_N| \ge \text{ const}\, R,$$

provided $\overline{C}_0$ is large enough, and therefore

$$|x_N - y_N| \ge \text{ const}\, R, \tag{A.3}$$

for any $X \in B_{3a}(X_0)$, if R is large enough.

We now point out that, if $X \in B_{3a}(X_0)$,

$$\left|\partial_N g_{\widetilde{\mathbb{S}}(Y,R)}(x)\right| \geq \text{const} > 0 . \tag{A.4}$$

In order to prove the above inequality, first notice that, by a direct computation, using (2.35), (2.33) and (2.37), one gets that

$$\left|\partial_N g_{\widetilde{\mathbb{S}}(Y,R)}(x)\right| \geq \text{const}\, \frac{|x_N - y_N|}{|x-y|} . \tag{A.5}$$

Also, if x is in the domain of $g_{\widetilde{\mathbb{S}}(Y,R)}$, then

$$|x-y| \leq \text{const}\, R .$$

The latter inequality, together with (A.5) and (A.3), ends the proof of (A.4).

Let now

$$\begin{aligned} &\mathcal{R}(x_1,\dots,x_{N-1},x_{N+1}) := \\ := \; &\Big[\Big(H_0(x_{N+1}) - \frac{\overline{C}_0}{2R}(x_{N+1}-y_{N+1})^2 + R - H_0(y_{N+1})\Big)^2 - \\ &-\sum_{j=1}^{N-1}|x_i-y_i|^2\Big]^{1/2} . \end{aligned}$$

Let also $\pi_N : \mathbb{R}^{N+1} \longrightarrow \{x_N = 0\}$ be the natural projection, i.e.,

$$\pi_N(x_1,\dots,x_{N+1}) := (x_1,\dots,x_{N-1},0,x_{N+1}) .$$

We now show that

$$\pi_N\Big|_{\widetilde{\mathbb{S}}(Y,R)\cap B_{3a}(X_0)} \text{ is a diffeomorphism.} \tag{A.6}$$

For proving this, take any

$$X = \Big(x, g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) \in \widetilde{\mathbb{S}}(Y,R) \cap B_{3a}(X_0) ,$$

and consider

$$\pi_N(X) = \Big(x_1,\dots,x_{N-1},0,g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) .$$

Then, by (2.31) and (2.33),

$$\pi_N^{-1}\Big(x_1,\dots,x_{N-1},0,g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) = X ,$$

with x_N so that

$$|x_N - y_N| = \mathcal{R}(x_1,\dots,x_{N-1},x_{N+1}) ,$$

where $\mathcal{R}$ has been defined here above. From (A.3), we get that $\mathcal{R} \geq \text{const}\, R > 0$, thus $\mathcal{R}$ is smooth for $X \in B_{3a}(X_0)$. Also, using again (A.3), we see that, either $x_N > y_N$ or $x_N < y_N$ for any $X \in \widetilde{\mathbb{S}}(Y,R) \cap B_{3a}(X_0)$: that is, in the first case,

$$\begin{aligned} & \pi_N^{-1}\Big(x_1,\dots,x_{N-1},0,g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) = \\ = \;& \Big(x_1,\dots,x_{N-1},\, y_N+\mathcal{R}(x_1,\dots,x_{N-1},x_{N+1}),\, x_{N+1}\Big)\,, \end{aligned}$$

while, in the second case,

$$\begin{aligned} & \pi_N^{-1}\Big(x_1,\dots,x_{N-1},0,g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big) = \\ = \;& \Big(x_1,\dots,x_{N-1},\, y_N-\mathcal{R}(x_1,\dots,x_{N-1},x_{N+1}),\, x_{N+1}\Big)\,. \end{aligned}$$

In any case, (A.6) is proved.

Let now $\pi_{N+1} : \mathbb{R}^{N+1} \longrightarrow \{x_{N+1}=0\}$ be the natural projection, i.e.,

$$\pi_{N+1}(x_1,\dots,x_{N+1}) \;:=\; (x_1,\dots,x_N,0)\,.$$

Notice that, if $X = (x, g_{\widetilde{\mathbb{S}}(Y,R)}(x)) \in \widetilde{\mathbb{S}}(Y,R)$, then $\pi_{N+1}(X) = (x,0)$ and so

$$\pi_{N+1}^{-1}(x,0) \;=\; \Big(x, g_{\widetilde{\mathbb{S}}(Y,R)}(x)\Big)\,,$$

showing that

$$\pi_{N+1}\Big|_{\widetilde{\mathbb{S}}(Y,R)\cap B_{3a}(X_0)} \text{ is a diffeomorphism.} \tag{A.7}$$

Thus, we define

$$T \;:=\; \pi_{N+1}\Big|_{\widetilde{\mathbb{S}}(Y,R)\cap B_{3a}(X_0)} \circ \pi_N^{-1}\Big|_{\widetilde{\mathbb{S}}(Y,R)\cap B_{3a}(X_0)}$$

Let also introduce the following domains:

$$\begin{aligned} O_1 \;&:=\; T\Big(\{x_N=0\}\cap\{|x_{N+1}|<3/4\}\cap B_{a+2}(\pi_N(X_0))\Big) \\ O_2 \;&:=\; T\Big(\{x_N=0\}\cap\{|x_{N+1}|<5/8\}\cap B_{a+1}(\pi_N(X_0))\Big)\,. \end{aligned}$$

Of course, $x_0 \in O_2 \subset O_1$ and, more precisely, by (A.6) and (A.7),

$$\operatorname{dist}(O_2, \partial O_1) \;\geq\; \text{const}\,. \tag{A.8}$$

Let us now notice that, in the light of Proposition 2.18,

$$\begin{aligned} \Delta_p g_{\widetilde{\mathbb{S}}(Y,R)} - \Delta_p u \;&\leq\; \Big(h_0'(g_{\widetilde{\mathbb{S}}(Y,R)}) - h_0'(u)\Big) + \frac{\text{const}}{R} \leq \\ &\leq\; \Lambda\Big((g_{\widetilde{\mathbb{S}}(Y,R)} + \frac{1}{R}) - u\Big)\,. \end{aligned}$$

Hence, if $X \in O_1$,

$$-\Delta_p u + \Lambda u \leq -\Delta_p\Big(g_{\widetilde{\mathbb{S}}(Y,R)} + \frac{1}{R}\Big) + \Lambda\left(g_{\widetilde{\mathbb{S}}(Y,R)} + \frac{1}{R}\right),$$

where $\Lambda := \sup_{[-3/4,\,3/4]} |h_0'|$. Hence, from the Harnack-type comparison inequality for[15] p-Laplacian (see, for instance [9], [35] or [10]), we get that

$$\sup_{O_2} (g_{\widetilde{\mathbb{S}}(Y,R)} - u) \le \frac{C'}{R}\,, \tag{A.9}$$

for a suitable $C' > 1$, which may also depend on a.

Fix now $Z \in L \cap B_a(\pi_N X_0)$. Then, from (A.7), there exists $X^{(1)} = (x^{(1)}, x^{(1)}_{N+1}) \in \widetilde{\mathbb{S}}(Y,R) \cap B_{3a}(X_0)$ so that

$$\pi_N \Big|_{\widetilde{\mathbb{S}}(Y,R)\cap B_{3a}(X_0)} (X^{(1)}) = Z\,,$$

that is

$$X^{(1)} = Z + t^{(1)} e_N \tag{A.10}$$

for some $t^{(1)} \in \mathbb{R}$, and

$$X^{(1)} \in \widetilde{\mathbb{S}}(Y,R)\,. \tag{A.11}$$

Also, by (A.6),

$$\begin{aligned} |X^{(1)} - X_0| &\le \left| \left(\pi_N \Big|_{\widetilde{\mathbb{S}}(Y,R)\cap B_{3a}(X_0)} \right)^{-1} (Z - \pi_N X_0) \right| \le \\ &\le \text{const}\, |Z - \pi_N X_0| \le \\ &\le \text{const}\, a\,. \end{aligned} \tag{A.12}$$

Moreover, from (A.4), we have that, for any $t \ge 0$,

$$\begin{aligned} g_{\widetilde{\mathbb{S}}(Y,R)}(x^{(1)} + te_N) &\ge \text{const}\, t + g_{\widetilde{\mathbb{S}}(Y,R)}(x^{(1)}) = \\ &= \text{const}\, t + x^{(1)}_{N+1} = \\ &= \text{const}\, t + z_{N+1}\,. \end{aligned}$$

Therefore, from (A.9),

$$u(x^{(1)} + te_N) > z_{N+1}\,, \tag{A.13}$$

provided $t \ge C''/R$, for a suitable $C'' > 1$, which may also depend on a. Analogously,

$$u(x^{(1)} - te_N) < z_{N+1}\,, \tag{A.14}$$

provided $t \ge C''/R$. From (A.13) and (A.14), we deduce the existence of $t^{(2)} \in [-C''/R,\, C''/R]$ so that

$$u(x^{(1)} + t^{(2)} e_N) = z_{N+1}\,.$$

[15] We recall that the gradient of u does not vanish in the region we are considering, so that the assumptions needed in [9, 35, 10] are fulfilled.

Let us define $X^{(2)} := X^{(1)} + t^{(2)} e_N$. The point $x^{(2)} = x^{(2)}(Z)$ will be the one satisfying the thesis of Lemma 4.1, as we are now going to show. Notice that, by construction,

$$\left|X^{(1)} - X^{(2)}\right| \leq \frac{C''}{R} \qquad \text{and} \tag{A.15}$$
$$x^{(2)}_{N+1} = z_{N+1} = u(x^{(2)})\,.$$

In particular,

$$\begin{aligned} \pi_N\Big(x^{(2)},\, u(x^{(2)})\Big) &= Z \\ \text{and } \ |x^{(2)} - x_0| &\leq \\ &\leq |x^{(1)} - x_0| + \frac{C''}{R} \leq \\ &\leq \text{const}\, a + \frac{C''}{R} \leq \\ &\leq \text{const}\, a\,, \end{aligned} \tag{A.16}$$

thanks to (A.12).

We now show that

$$\Big(x^{(1)} - x_0\Big) \cdot \frac{\nabla u(x_0)}{|\nabla u(x_0)|} \;\leq\; H_0(z_{N+1}) - H_0(u(x_0)) + \frac{C'''}{R}\,, \tag{A.17}$$

for some $C''' > 0$ which may depend on a. To prove (A.17), let us define

$$w(x) := H_0(g_{\widetilde{\mathbb{S}}(Y,R)}(x))$$

and notice that, from (A.10) and (A.11),

$$z_{N+1} = x^{(1)}_{N+1} = g_{\widetilde{\mathbb{S}}(Y,R)}(x^{(1)})$$

and so, since X_0 is a point where the graph of u and $\widetilde{\mathbb{S}}(Y,R)$ touches,

$$\begin{aligned} & H_0(z_{N+1}) - H_0(u(x_0)) = \\ = \;& H_0\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x^{(1)})\Big) - H_0\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\Big) = \\ = \;& w(x^{(1)}) - w(x_0) \geq \\ \geq \;& \nabla w(x_0) \cdot (x^{(1)} - x_0) - \text{const}\, |D^2 w(\xi)| |x^{(1)} - x_0|^2\,, \end{aligned}$$

for some ξ lying on the segment joining $x^{(1)}$ and x_0. Notice now that, by the definition of w and the fact that X_0 is a point of touching between the graph of u and $\widetilde{\mathbb{S}}(Y,R)$,

$$\begin{aligned}
&\nabla w(x_0)\cdot(x^{(1)}-x_0) = \\
=\ & H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\Big)\,\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\cdot(x^{(1)}-x_0) = \\
=\ & H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\Big)\,\Big|\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\Big|\frac{\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)}{\Big|\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\Big|}\cdot(x^{(1)}-x_0) = \\
=\ & H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\Big)\,\Big|\nabla g_{\widetilde{\mathbb{S}}(Y,R)}(x_0)\Big|\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\cdot(x^{(1)}-x_0)\,,
\end{aligned}$$

hence, from (2.53) and the fact that $X^{(1)}\in B_{3a}(X_0)$,

$$\nabla w(x_0)\cdot(x^{(1)}-x_0)\geq\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\cdot(x^{(1)}-x_0)-\frac{\text{const}}{R}\,a\,.$$

Also, a direct computation and (2.54) imply that

$$\begin{aligned}
\partial_{ij}w &= H_0''\Big(g_{\widetilde{\mathbb{S}}(Y,R)}\Big)\partial_i g_{\widetilde{\mathbb{S}}(Y,R)}\partial_j g_{\widetilde{\mathbb{S}}(Y,R)}+ \\
&\quad +H_0'\Big(g_{\widetilde{\mathbb{S}}(Y,R)}\Big)\partial_{ij}g_{\widetilde{\mathbb{S}}(Y,R)}\leq \\
&\leq \frac{\text{const}}{R}\,.
\end{aligned}$$

Collecting the estimates above and recalling that $X^{(1)}\in B_{3a}(X_0)$, the claim in (A.17) now easily follows.

Now, by means of (A.15) and (A.17),

$$\begin{aligned}
&\Big(x^{(2)}-x_0\Big)\cdot\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\leq \\
\leq\ & \Big(x^{(1)}-x_0\Big)\cdot\frac{\nabla u(x_0)}{|\nabla u(x_0)|}+\Big|x^{(1)}-x^{(2)}\Big|\leq \\
\leq\ & H_0(z_{N+1})-H_0(u(x_0))+\frac{C''''}{R} = \\
=\ & H_0(u(x^{(2)}))-H_0(u(x_0))+\frac{C''''}{R}\,,
\end{aligned}$$

for a suitable $C''''>1$, which may depend on a. This, together with (A.12) and (A.16), completes the proof of Lemma 4.1.

A.2. **Proof of Lemma 4.2.** The proof of Lemma 4.2 relies on an auxiliary result, namely Lemma A.1 here below, which may be seen as a rotation of the desired claim (see below (A.20)) plus a Lipschitz property on level sets[16].

[16] The very rough idea underneath the proof of Lemma 4.2 goes as follows. First, Lemma A.1 provides a result which looks like a rotation of Lemma 4.2, and which possesses a uniform Lipschitz graph property for level sets. The proof of Lemma 4.2 will then ended by rotating back to the configuration in Lemma A.1: the Lipschitz property will take into account the error done in such rotation.

For stating Lemma A.1, we need to introduce the following notation. Given $R>0$ and $Y=(y,y_{N+1})\in\mathbb{R}^N\times[-1/4,\,1/4]$, we define $\Sigma(Y,R)$ as the zero level set of $\mathbb{S}(Y,R)$, that is:

$$\Sigma(Y,R)\;:=\;\mathbb{S}(Y,R)\cap\{x_{N+1}=0\}\;=\;\{g_{\mathbb{S}(Y,R)}=0\}\,.$$

By the definitions on page 14 and (2.39), we have that $\Sigma(Y,R)$ is an $(N-1)$-dimensional sphere, namely

$$\Sigma(Y,R)=\{x\in\mathbb{R}^N\mid |x-y|=r\}$$

with[17]

$$r\;=\;r(Y,R)\;:=\;R-H_0(y_{N+1})-\frac{\overline{C}_0}{2R}y_{N+1}^2\,. \tag{A.18}$$

The study of the geometry of such spheres is indeed linked with the study of the level sets of $\mathbb{S}(Y,R)$, via the following observation. If $s=g_{\mathbb{S}(Y,R)}(x)=g_{\widetilde{\mathbb{S}}(Y,R)}(x)\in(-1/2,\,1/2)$, then, by (2.31) and Definition 2.15,

$$H_0(y_{N+1})+|x-y|-R=H_0(s)-\frac{\overline{C}_0}{2R}(s-y_{N+1})^2\,,$$

hence, if $|s|<1/2$, the signed distance between the s-level set of $g_{\mathbb{S}(Y,R)}$ and $\Sigma(Y,R)$ is given by

$$H_0(s)+\frac{\overline{C}_0\,s}{2R}(2y_{N+1}-s)\,. \tag{A.19}$$

Given $x\in\mathbb{R}^N$, we now define

$$T_{Y,R}x=T_{Y,R}(x)$$

as the intersection point between $\Sigma(Y,R)$ and the half-line from y going through x. With this, we can now deal with the above mentioned auxiliary result:

Lemma A.1. *Let $\breve{C}>1$ be a suitably large constant. Let u be a C^1-subsolution of* (1.5) *in $\{|x'|<l\}\times\{|x_N|<l\}$. Assume that $\mathbb{S}(Y,R)$ is above the graph of u in $\{|x'|\le\breve{C}q\}\times\{|x_N|\le l/2\}$ and that $\mathbb{S}(Y,R)$ touches the graph of u at the point $(x_0,\,u(x_0))$. Suppose that*

The idea for proving Lemma A.1 is that we would like to replace the estimates on the touching point set Ξ with estimates on a suitable *first occurrence* touching point set $\breve{\Xi}$, i.e., with a set obtained by translating in the e_N direction an appropriate barrier until it touches the graph of u. This strategy will present two advantages. First, the first occurrence touching property will easily imply the Lipschitz property for level sets of $\breve{\Xi}$ (which, as mentioned above, is needed for deducing Lemma 4.2 from Lemma A.1). Second, measure estimates for $\breve{\Xi}$ can be directly deduced from Proposition 3.14. For performing this, however, a technical difficulty arises: indeed, in order to be able to replace Ξ with $\breve{\Xi}$, a "tiny" improvement of the assumptions of Lemma A.1 will be needed, namely (A.29) here below. Unfortunately, the proof of this detail is non trivial, and it will take several pages.

[17]Obviously, $r\sim R$ if R is large.

- $|u(x_0)| < 1/2$, $|x_{0N}| < l/4$, $|x_0'| < l/4$.
- $\angle\left(\dfrac{\nabla u(x_0)}{|\nabla u(x_0)|}, e_N\right) \le \dfrac{\pi}{8}$.

Assume also that

$$
\begin{aligned}
& T_{Y,R}x_0 \in \{|x'| = q\} \cap \{x_N = 0\} \qquad \textit{and} \\
& y = -e_N\sqrt{r^2 - q^2} \qquad \textit{with} \\
& r = r(Y,R) \;=\; R - H_0(y_{N+1}) - \frac{\overline{C}_0}{2R} y_{N+1}^2 \,.
\end{aligned} \tag{A.20}
$$

Then, there exist universal constants C_1, $C_2 > 1 > c > 0$ *such that, if*

$$
q \ge C_1 \qquad \textit{and} \qquad 4\sqrt[3]{R} \le l \le cR\,, \tag{A.21}
$$

the following holds. Let Ξ *be the set of points* $(\mathfrak{x}, u(\mathfrak{x})) \in \mathbb{R}^N \times \mathbb{R}$ *satisfying the following properties:*

- $|\mathfrak{x}'| < q/15$, $|u(\mathfrak{x})| < 1/2$, $|\mathfrak{x} - x_0| < \breve{C}q$;
- *there exists* $\hat{Y} \in \mathbb{R}^{N+1}$ *such that* $\mathbb{S}(\hat{Y}, R/C_2)$ *is above* u *and it touches* u *at* $(\mathfrak{x}, u(\mathfrak{x}))$;
- $\angle\left(\dfrac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, \dfrac{\nabla u(x_0)}{|\nabla u(x_0)|}\right) \le \dfrac{C_1\, q}{R}$;
- $(\mathfrak{x} - x_0)\cdot\dfrac{\nabla u(x_0)}{|\nabla u(x_0)|} \le \dfrac{C_1\, q^2}{R} + H_0(u(\mathfrak{x})) - H_0(u(x_0))$.

Then,

$$
\mathfrak{L}^N\Big(\pi_N(\Xi)\Big) \;\ge\; cq^{N-1}\,. \tag{A.22}
$$

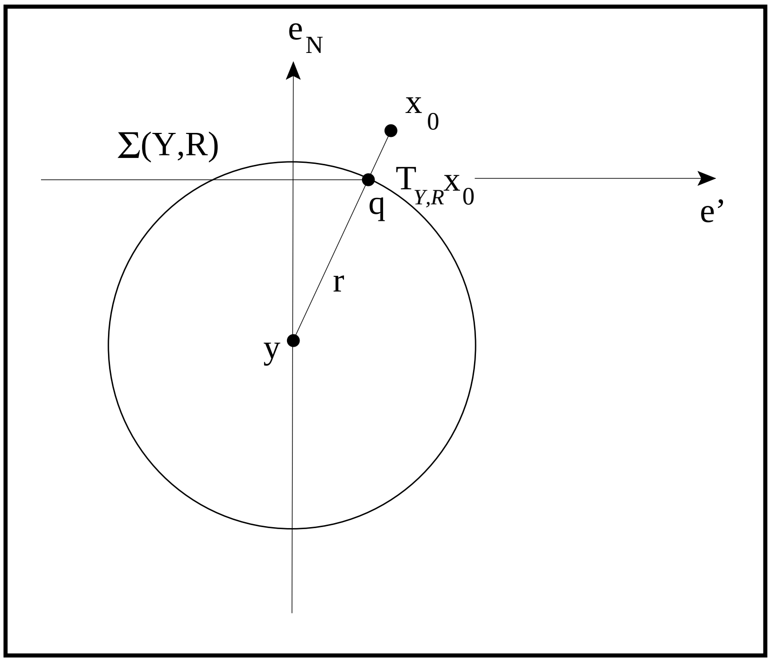
More precisely, for any $s \in (-1/2, 1/2)$, *there exists a set* $\Xi_s \subseteq \Xi \cap \{x_{N+1} = s\}$, *which is contained in a Lipschitz graph in the* e_N*-direction, with Lipschitz constant less than* 1, *and so that, if*

$$
\breve{\Xi} := \bigcup_{s\in(-1/2,\,1/2)} \Xi_s\,,
$$

we have[18]

$$
\mathfrak{L}^N\Big(\pi_N(\breve{\Xi})\Big) \;\ge\; cq^{N-1}\,. \tag{A.23}
$$

[18]Of course, (A.23) and the fact that $\breve{\Xi} \subseteq \Xi$ imply (A.22).



The geometry of (A.20)

Proof. For further reference, let us point out some geometric features linking x_0 with its projection $T_{Y,R}x_0$. First of all, by (A.19),

$$\left|T_{Y,R}x_0 - x_0\right| \leq \text{ const} \,. \tag{A.24}$$

Also, by construction,

$$\sin\Big(\angle\big(x_0 - y,\, e_N\big)\Big) \;=\; \frac{q}{r}\,. \tag{A.25}$$

Thus, (A.24) and (A.25) yield that

$$\begin{aligned} |x_0'| &\leq \left|\big(T_{Y,R}x_0 - x_0\big)'\right| + \left|\big(T_{Y,R}x_0\big)'\right| \leq \\ &\leq \left|T_{Y,R}x_0 - x_0\right| \sin\Big(\angle\big(x_0,\, e_N\big)\Big) + q \leq \\ &\leq \frac{\text{const}\, q}{R} + q\,. \end{aligned} \tag{A.26}$$

Let us also observe that, if $|\mathfrak{x}'| \leq l/8$ and $\mathfrak{x}_N \in [-l/2,\, -l/4]$, then

$$|\mathfrak{x} - y| \leq r - \frac{l}{8} + \text{ const } \leq R - \text{ const } \sqrt[3]{R}$$

and thus, exploiting (2.15),

$$
\begin{aligned}
u(\mathfrak{x}) &\le g_{\mathbb{S}(Y,R)}(\mathfrak{x}) \le \\
&\le g_{y_{N+1},R}\Big(\operatorname{const}(1-\sqrt[3]{R})\Big) = \\
&= s_R\,.
\end{aligned}
\tag{A.27}
$$

More precisely, we have that

$$
u(\mathfrak{x}) < s_R \tag{A.28}
$$

for any $\mathfrak{x}$ so that $|\mathfrak{x}'| \le l/8$ and $\mathfrak{x}_N \in [-l/2,\,-l/4]$: indeed, if not, by (A.27) there would be a point for which the graph of u touches the s_R-level from below and then a contradiction follows by applying Theorem B.6 to the function $s_R - u$ (see, e.g., the argument on page 19).

We now fix $C_* > 1$, to be chosen conveniently large. The first step of the proof of Lemma A.1 consists in proving the existence of suitable $Y_* \in \mathbb{R}^{N+1}$ and $R_* > R/\operatorname{const}$, so that $\pi_{e_N} Y_* = \pi_{e_N} Y$ and $\mathbb{S}(Y_*, R_*)$ touches the graph of u from above, in the region $\{|x'| \le \breve{C} q\}$, at the point $(x_*,\, u(x_*))$, with $|x'_*| \le \breve{C} q$ and

$$
T_{Y_*,R_*} x_* \in \Big\{ x_N \le \frac{\operatorname{const} q^2}{R} \Big\} \times \Big\{ |x'| < \frac{q}{C_*} \Big\}\,. \tag{A.29}
$$

We will prove (A.29) by iteration. Namely, we will set $Y_0 := Y$, $R_0 := R$ and, for any $k \in \mathbb{N}$, we will inductively find $Y_{k+1} \in \mathbb{R}^{N+1}$ and $R_{k+1} > R_k/4$, so that $\pi_{e_N} Y_{k+1} = \pi_{e_N} Y_k$ and $\mathbb{S}(Y_{k+1}, R_{k+1})$ touches the graph of u from above at the point $(x_{k+1},\, u(x_{k+1}))$, with $|x'_{k+1}| \le \breve{C} q$ and

$$
T_{Y_{k+1},R_{k+1}} x_{k+1} \in \Big\{ x_N \le \frac{\operatorname{const} q^2}{R_k} \Big\} \times \Big\{ |x'| < \eta q \Big\}\,, \tag{A.30}
$$

for some $\eta \in (0,1)$. Since (A.29) follows by iterating (A.30) a finite number of times, we focus now on the proof of (A.30). More precisely, we will proof the first step in (A.30), i.e., the step with $k = 0$, since the others are analogous. The proof of (A.30) is actually quite non trivial, and it will take several pages (it will be ended on page 119).

For proving (A.30), let us begin by noticing that, if $|s| < 1/2$, from Definition 2.5 and (2.1) we get that

$$
\begin{aligned}
|h'_{s_0,R}(s) - h'_0(s)| &= |\varphi'_{s_0,R}(s) - h'_0(s)| \le \\
&\le |h'_0(s)| \left(\frac{|R^p - (\star)^p|}{(\star)^p} + \frac{\overline{C}_0\, |s - s_0|\, (p/(p-1))^{1/p}}{\Big(h_0(s)\Big)^{(p-1)/p} (\star)^{p+1}} \right)\,,
\end{aligned}
$$

where used the short hand notation

$$
\star := R - \overline{C}_0 (s - s_0) \left(\frac{p}{p-1} h_0(s) \right)^{1/p}\,.
$$

Therefore,

$$|h'_{s_0,R}(s) - h'_0(s)| \le \frac{\text{const}}{R}, \text{ for any } |s| < 1/2. \tag{A.31}$$

Moreover, from Definition 2.5,

$$h'_{s_0,R}(s) = h'_0(s) - \frac{\widehat{C}_0}{R}, \text{ for any } s \in (s_R, -1/2) \cup (1/2, 1). \tag{A.32}$$

and, by construction, $\widehat{C}_0$ may be taken large if so is $\overline{C}_0$. We now fix a small parameter $\epsilon \in (0, 1/2]$ and a large parameter $\gamma > 1$ and we define

$$\omega := 2^{-1/(\gamma+2)} \text{ and } \eta := 1 - \epsilon(1 - \omega). \tag{A.33}$$

By construction, $\omega \in (0, 1)$ and $\eta \in (1/2, 1)$. Also, $\eta > \omega$ and, for γ large, ω and η are close to 1. For any $t > 0$, set also

$$\tilde{\psi}(t) := \frac{1}{\gamma}\left(\frac{1}{t^\gamma} - 1\right)$$

and, for any $z' \in \mathbb{R}^{N-1} \setminus \{0\}$,

$$\psi(z') := \tilde{\psi}(|z'|).$$

We now consider the graph

$$\mathfrak{G} := \left\{ X \in \mathbb{R}^N \mid x_N = \frac{q^2}{\sqrt{r^2 - q^2}} \psi(x'/q) \right\}.$$

Since $\mathfrak{G}$ is strictly concave, while $\Sigma(\cdot, \cdot)$ is strictly convex, one sees that $\mathfrak{G}$ touches $\Sigma(Y, R)$ from above when $|x'| = q$. Analogously, if

$$\begin{aligned} y_\omega &:= -\left[\frac{q^2}{\gamma\sqrt{r^2 - q^2}}\left(1 - \frac{1}{\omega^\gamma}\right) + \omega^{\gamma+2}\sqrt{r^2 - q^2}\right] e_N \qquad \text{and} \\ r_\omega &:= \omega^{\gamma+2}\sqrt{r^2 + q^2(\omega^{-2\gamma-2} - 1)}, \end{aligned}$$

one sees that $\mathfrak{G}$ touches the $(N-1)$-dimensional sphere $\partial B_{r_\omega}(y_\omega)$ from above when $|x'| = \omega q$. Notice that, by construction,

$$r_\omega \ge \omega^{\gamma+2} r = r/2, \tag{A.34}$$

and, more precisely, $r_\omega \in [r/2, \omega r]$ and $y_{\omega N} \ge y_N$. Also, from the fact that

$$\left|\sqrt{1 + \tau} - 1 - \frac{\tau}{2}\right| \le \tau^2, \tag{A.35}$$

provided $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently small, one sees that $y_N + r < y_{\omega N} + r_\omega$, if r/q and γ are suitably large. Notice also that, by construction,

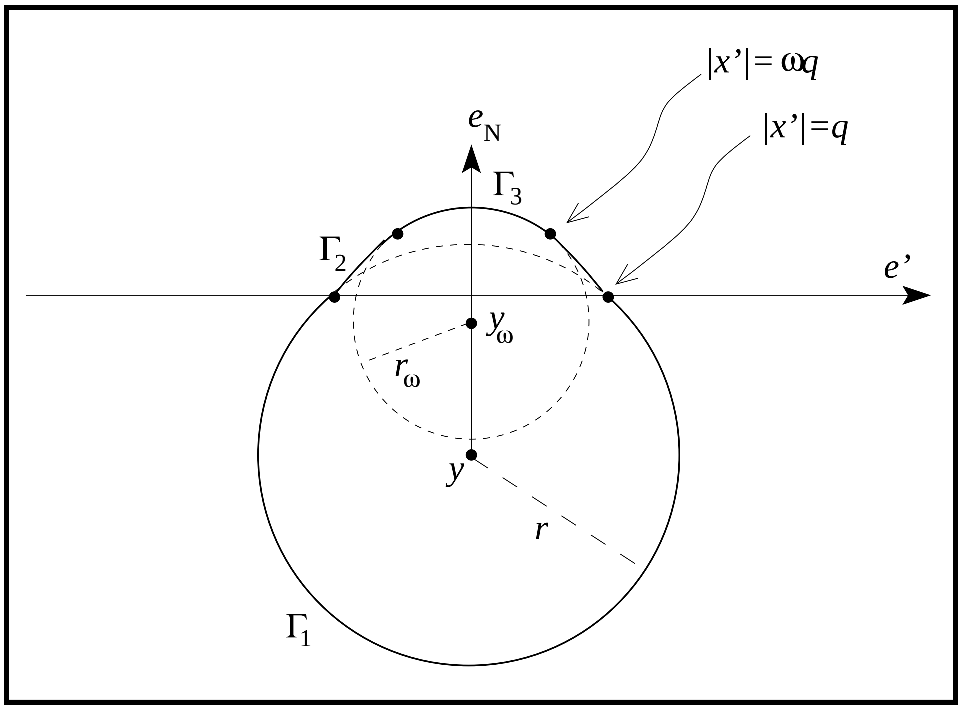
$$\begin{aligned} |y_\omega - y| &= \frac{1}{2}\sqrt{r^2 - q^2} + \frac{q^2}{\gamma\sqrt{r^2 - q^2}}\left(\frac{1}{\omega^\gamma} - 1\right) \in \\ &\in \left[\frac{1}{2}\sqrt{r^2 - q^2}, \frac{1}{2}\sqrt{r^2 - q^2} + \frac{2q^2}{\gamma r}\right], \end{aligned} \tag{A.36}$$

if γ and r/q are large enough. Thus, we now consider the surface

$$\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

defined in this way: we take

$$\begin{array}{rcl} \Gamma_1 & := & \Sigma(Y,R) \cap \{x_N < 0\} \\ \Gamma_2 & := & \mathfrak{G} \cap \{|x'| \in [\omega q,\, q]\} \qquad \text{and} \\ \Gamma_3 & := & \partial B_{r_\omega}(y_\omega) \cap \{|x'| < \omega q\} \cap \{x_N > 0\}\,. \end{array}$$



The Barbapapa-like surface Γ

In the sequel, we will often speak about points "inside (or outside) Γ", with the obvious meaning of points "inside (or outside) the bounded region whose boundary is Γ".

Also, by the above mentioned touching properties between $\mathfrak{G}$, $\Sigma(Y,R)$ and $\partial B_{r_\omega}(y_\omega)$, we have that Γ is a $C^{1,1}$ closed hypersurface in $\mathbb{R}^N$. We denote by d_Γ the signed distance to Γ, with the convention that d_Γ is positive in the exterior of Γ and negative in the interior.

The rough idea for proving (A.30) consists now in trying to find contact points whose projection on the zero level set of the corresponding barrier lies close[19] to Γ_3, and then use the geometry Γ_3, which is quite transparent.

With this (long-term) target in mind, let us now make some computation on the mutual distances of the above surfaces.

First, we show that, for any $a \in \partial B_{r_\omega}(y_\omega)$ and any $b \in \Gamma_1$,

$$|a-b| \geq \text{ const}\,(1-\omega)\,\frac{q^2}{r}\,. \tag{A.37}$$

For the proof of this, consider a^* in $\partial B_{r_\omega}(y_\omega)$ and b^* in the closure of Γ_1 so that $|a^*-b^*|$ attains the minimum value of $|a-b|$, when a and b vary as above. By its minimizing property, b^*-a^* is orthogonal to $\partial B_{r_\omega}(y_\omega)$ at a^*, thence,

$$|a^*-b^*| = |b^*-y_\omega| - r_\omega\,. \tag{A.38}$$

We notice that b^* cannot lie in $\{x_N<0\}$, otherwise its minimizing properties would imply that b^*-a^* is orthogonal to Γ_1 (and thence to $\partial B_r(y)$) at b^* and therefore, by these orthogonality properties, the line passing through a^* and b^* should also contain y and y_ω, which is impossible. This says that $b^*_N=0$ and $|(b^*)'|=q$. This and (A.38) yield that

$$\begin{aligned}
&|a^*-b^*| = \\
&= \sqrt{y^2_{\omega,N}+q^2} - r_\omega = \\
&= |y_{\omega,N}|\sqrt{1+\left(\frac{q}{y_{\omega,N}}\right)^2} - r_\omega\,.
\end{aligned}$$

Thus, from (A.35) and (A.36),

$$\begin{aligned}
&|a^*-b^*| \geq \\
\geq\ & |y_{\omega,N}| + \frac{q^2}{2\,|y_{\omega,N}|} - r_\omega - \text{ const}\,\frac{q^4}{r^3} \geq \\
\geq\ & \frac{1}{2}\sqrt{r^2-q^2} + \frac{q^2}{\sqrt{r^2-q^2}+\frac{4q^2}{\gamma r}} - \frac{1}{2}\sqrt{r^2+q^2(4\omega^2-1)} - \text{ const}\,\frac{q^4}{r^3} \geq \\
\geq\ & -\frac{q^2}{r}\omega^2 + \frac{q^2}{r}\,\frac{1}{1+\text{ const}\,q^2/r^2} - \text{ const}\,\frac{q^4}{r^3} \geq \\
\geq\ & \frac{q^2}{r}(1-\omega^2) - \text{ const}\,\frac{q^4}{r^3}\,.
\end{aligned}$$

This proves (A.37).

We now define the following hypersurface in $\mathbb{R}^{N+1}$:

$$\Psi := \Big\{X\in\mathbb{R}^{N+1}\ |\ x_{N+1} = g_{y_{N+1},R}\Big(d_\Gamma(x)+H_{y_{N+1},R}(0)\Big)\Big\}\,.$$

[19]This is the reason also for introducing $\Sigma(Y_\omega,R_1)$ later on (see page 100).

For the sake of simplicity, we will set

$$g_\Psi(x) := g_{y_{N+1},R}\Big(d_\Gamma(x) + H_{y_{N+1},R}(0)\Big)\,,$$

so that $\Psi = \{x_{N+1} = g_\Psi(x)\}$.

Notice that

$$\text{the sets } \{d_\Gamma \text{ attained on } \Gamma_1\} \text{ and } \{d_\Gamma \text{ attained on } \Gamma_3\} \text{ are open.} \tag{A.39}$$

We prove the first claim in (A.39), the other one being analogous. Namely, we show that

$$\text{the set where } d_\Gamma \text{ is attained on } \Gamma_2 \cup \Gamma_3 \text{ is closed.} \tag{A.40}$$

Indeed, take x_k in the above set, with x_k converging to x_∞ as k goes to $+\infty$: then, since $\Gamma_2 \cup \Gamma_3$ is closed, there exist $z_k \in \Gamma_2 \cup \Gamma_3$ so that

$$|x_k - z_k| = d_{\Gamma_2 \cap \Gamma_3}(x_k) \le |x_k - w|\,,$$

for any $w \in \Gamma_1$. Since $\Gamma_2 \cup \Gamma_3$ is compact, we may and do assume that z_k converges to $z_\infty \in \Gamma_2 \cup \Gamma_3$ and, therefore,

$$|x_\infty - z_\infty| \le |x_\infty - w|\,,$$

for any $w \in \Gamma_1$. Hence,

$$d_{\Gamma_2 \cup \Gamma_3}(x_\infty) \le |x_\infty - z_\infty| \le d_{\Gamma_1}(x_\infty)\,.$$

This proves (A.40) and thus (A.39).

In analogy with (A.39) the reader will easily convince herself that

$$\text{the set where } d_\Gamma \text{ is attained on } \Gamma_2 \cap \{|x'| \in (\omega q,\, q)\} \text{ is open.} \tag{A.41}$$

Let us now show some properties of Ψ. First of all, Ψ coincides with $\mathbb{S}(Y,R)$ at any points for which d_Γ is realized on Γ_1; more precisely,

$$\text{if } d_\Gamma(x) = d_{\Gamma_1}(x), \text{ then } g_\Psi(x) = g_{\mathbb{S}(Y,R)}(x). \tag{A.42}$$

Indeed, if $d_\Gamma(x) = d_{\Gamma_1}(x)$, by some geometric considerations and (2.39), we have that

$$\begin{aligned} g_\Psi(x) &= g_{y_{N+1},R}\Big(d_{\Gamma_1}(x) + H_{y_{N+1},R}(0)\Big) = \\ &= g_{y_{N+1},R}\Big(d_{\Gamma_1}(x) - \frac{\overline{C}_0}{2R}\, y_{N+1}^2\Big) = \\ &= g_{y_{N+1},R}\Big(|x-y| - r - \frac{\overline{C}_0}{2R}\, y_{N+1}^2\Big)\,. \end{aligned}$$

This, (A.18) and (2.20) end the proof of (A.42).

In the light of (A.42), (A.39) and Proposition 2.13, we deduce that

$$\begin{aligned} &\Delta_p g_\Psi(x) < h_0'(g_\Psi(x)) \text{ in the viscosity sense} \\ &\text{at any } x \in \mathbb{R}^N \text{ for which} \\ &d_\Gamma(x) \text{ is attained on } \Gamma_1 \text{ and} \\ &g_\Psi(x) \in [s_R,\, -1/2] \cap [1/2,\, 1). \end{aligned} \tag{A.43}$$

Furthermore, in an appropriate system of coordinates (see §14.6 of [19] for details on the distance function),

$$\nabla g_\psi(x) \;=\; g'_{y_{N+1},R}(\star)\, e_N$$

and $D^2 g_\psi(x)$ is the $N\times N$ diagonal matrix with the following entries on the diagonal:

$$\frac{\kappa_1}{\kappa_1 d_\Gamma(x)-1}\, g'_{y_{N+1},R}(\star),\ldots,\frac{\kappa_{N-1}}{\kappa_{N-1} d_\Gamma(x)-1}\, g'_{y_{N+1},R}(\star),\; g''_{y_{N+1},R}(\star)\,.$$

Here above, we denoted

$$\star \;:=\; d_\Gamma(x) + H_{y_{N+1},R}(0)\,,$$

while, as standard, $\kappa_1,\ldots,\kappa_{N+1}$ represent the principal curvatures of Γ at the point where $d_\Gamma(x)$ is realized. Hence, from Definitions 2.8 and 2.11 and Lemma B.3, we get that

$$\nabla g_\Psi(x) \;=\; \left(\frac{p}{p-1}h_{y_{N+1},R}(\sharp)\right)^{1/p} e_N\,, \tag{A.44}$$

where we denoted $\sharp := g_{y_{N+1},R}(\star)$, and that $D^2 g_\psi(x)$ may be represented as the $N\times N$ diagonal matrix with the following entries on the diagonal:

$$\begin{aligned}
&\frac{\kappa_1}{\kappa_1 d_\Gamma(x)-1}\left(\frac{p}{p-1}h_{y_{N+1},R}(\sharp)\right)^{1/p}\\
&\vdots\\
&\frac{\kappa_{N-1}}{\kappa_{N-1} d_\Gamma(x)-1}\left(\frac{p}{p-1}h_{y_{N+1},R}(\sharp)\right)^{1/p}\\
&\frac{(p h_{y_{N+1},R}(\sharp))^{(2-p)/p}}{(p-1)^{2/p}}\, h'_{y_{N+1},R}(\sharp)\,.
\end{aligned} \tag{A.45}$$

From (A.44) and (A.45),

$$\begin{aligned}
\Delta_p g_\Psi \;&=\; h'_{y_{N+1},R}+\\
&+\;\left(\frac{p}{p-1}h_{y_{N+1},R}\right)^{\frac{p-1}{p}}\sum_{i=1}^{N-1}\frac{\kappa_i}{\kappa_i d_\Gamma-1}\,,
\end{aligned} \tag{A.46}$$

outside $\{|g_\Psi|=1/2\}\cup\{\nabla g_\Psi=0\}$, where we dropped the $\sharp$-dependence for the sake of simplicity. Let us now compute the principal curvatures $\kappa_i^{\Gamma_2}$ of the hypersurface Γ_2: exploiting Lemma B.14, we have that

$$\begin{aligned}
-\kappa_1^{\Gamma_2}=\cdots=-\kappa_{N-2}^{\Gamma_2} \;&=\; \frac{q^{\gamma+2}}{|x'|\sqrt{(r^2-q^2)|x'|^{2\gamma+2}+q^{2\gamma+4}}}\geq 0\\
\kappa_{N-1}^{\Gamma_2} \;&=\; \frac{(\gamma+1)\,q^{\gamma+2}\,(r^2-q^2)\,|x'|^{2\gamma+1}}{\Big((r^2-q^2)\,|x'|^{2\gamma+2}+q^{2\gamma+4}\Big)^{3/2}}\geq 0\,.
\end{aligned} \tag{A.47}$$

In particular, since $|x'| \geq \omega q$ on Γ_2, we infer from the above relations that

$$-\kappa_i^{\Gamma_2} \leq \frac{1}{r_\omega}\,, \tag{A.48}$$

for $i = 1, \ldots, N-2$. Thus, we deduce from (A.34), (A.48) and (A.18) that

$$-\kappa_i^{\Gamma_2} \leq \frac{3}{R}\,, \tag{A.49}$$

for $i = 1, \ldots, N-2$, if R is large enough. Furthermore, since $q \geq |x'| \geq \omega q$ on Γ_2, (A.47) gives

$$\begin{aligned} \kappa_{N-1}^{\Gamma_2} &\geq \frac{(\gamma+1)\,(r^2-q^2)\,\omega^{2\gamma+1}}{r^3} \geq \\ &\geq \frac{9}{10}\,\frac{(\gamma+1)\omega^{2\gamma+1}}{r} \geq \\ &\geq \frac{9}{40}\,\frac{(\gamma+1)}{r}\,, \end{aligned}$$

$$\kappa_{N-1}^{\Gamma_2} \geq \text{const}\,\frac{(\gamma+1)}{R}\,. \tag{A.50}$$

Analogously, one sees that

$$\kappa_{N-1}^{\Gamma_2} \leq \text{const}\,\frac{(\gamma+1)}{R}\,. \tag{A.51}$$

We now claim that

$$\begin{aligned} &\Delta_p g_\Psi(x) \leq h_0'(g_\Psi(x)) - \frac{\text{const}\,\gamma}{R} \\ &\text{at any } x \in \mathbb{R}^N \text{ for which } |g_\Psi(x)| \neq 1/2, \\ &\nabla g_\Psi(x) \neq 0, \\ &|d_\Gamma(x)| \leq 2\sqrt{R} \text{ and } d_\Gamma \text{ is attained on } \Gamma_2 \cap \{|x'| \in (\omega q,\, q)\}. \end{aligned} \tag{A.52}$$

To prove this, take x as requested here above: then, thanks to (A.49), (A.51) and the fact that $|d_\Gamma(x)| \leq 2\sqrt{R}$, that

$$\left|\kappa_{N-1}^{\Gamma_2} d_\Gamma(x)\right| + \sum_{i=1}^{N-2} \left|\kappa_i^{\Gamma_2} d_\Gamma(x)\right| \leq \frac{\text{const}}{\sqrt{R}}$$

and so, if R is large enough,

$$\sum_{i=1}^{N-1} \frac{-\kappa_i^{\Gamma_2}}{1-\kappa_i^{\Gamma_2} d_\Gamma(x)} \leq -2\sum_{i=1}^{N-2} \kappa_i^{\Gamma_2} - \frac{\kappa_{N-1}^{\Gamma_2}}{2}\,. \tag{A.53}$$

Hence, using the regularity of the functions involved in our domain, (A.46), (A.53), (A.31) and (A.32),

$$\begin{aligned}
\Delta_p g_\Psi &= h'_{y_{N+1},R} + \left(\frac{p}{p-1} h_{y_{N+1},R}\right)^{\frac{p-1}{p}} \sum_{i=1}^{N-1} \frac{\kappa_i^{\Gamma_2}}{\kappa_i^{\Gamma_2} d_\Gamma - 1} \le \\
&\le h'_{y_{N+1},R} + \left(\frac{p}{p-1} h_{y_{N+1},R}\right)^{\frac{p-1}{p}} \left(-2\sum_{i=1}^{N-2} \kappa_i^{\Gamma_2} - \frac{\kappa_{N-1}^{\Gamma_2}}{2}\right) \le \\
&\le h'_0 + \frac{\text{const}}{R} + \\
&\quad + \left(\frac{p}{p-1} h_{y_{N+1},R}\right)^{\frac{p-1}{p}} \left(-2\sum_{i=1}^{N-2} \kappa_i^{\Gamma_2} - \frac{\kappa_{N-1}^{\Gamma_2}}{2}\right) .
\end{aligned}$$

We thus deduce, by (A.49) and (A.50), that

$$\begin{aligned}
\Delta_p g_\Psi &\le h'_0 + \frac{\text{const}}{R} + \\
&\quad + \left(\frac{p}{p-1} h_{y_{N+1},R}\right)^{\frac{p-1}{p}} \left(\frac{6(N-2)}{R} - \frac{\text{const}\,(\gamma+1)}{2R}\right) .
\end{aligned}$$

Thus, if γ is large enough,

$$\Delta_p g_\Psi \le h'_0 + \frac{\text{const}}{R} - \text{const}\left(\frac{p}{p-1} h_{y_{N+1},R}\right)^{\frac{p-1}{p}} \frac{(\gamma+1)}{4R} .$$

Therefore, since we are evaluating $h_{y_{N+1},R}$ at $g_\Psi(x) \in [-1/2, 1/2]$,

$$\Delta_p g_\Psi \le h'_0 + \frac{\text{const}}{R} - \frac{\text{const}\,(\gamma+1)}{R} ,$$

which proves (A.52), if γ is chosen to be conveniently large.

We now show that

$$\begin{aligned}
&\Delta_p g_\Psi(x) < h'_0(g_\Psi(x)) \text{ in the viscosity sense} \\
&\text{at any } x \in \mathbb{R}^N \text{ for which} \qquad\qquad \text{(A.54)} \\
&d_\Gamma \text{ is attained on } \Gamma_2.
\end{aligned}$$

For proving this, we first point out that we may assume

$$g_\Psi(x) > s_R \tag{A.55}$$

Indeed, if $g_\Psi(x) = s_R$, arguing as in Proposition 2.13

$$\Delta_p g_\Psi(x) = 0 < h'_0(s_R) = h'_0(g_\Psi(x)) ,$$

in the viscosity sense, giving the desired claim.

For proving (A.54), we may also assume

$$\nabla g_\Psi(x) \neq 0 , \tag{A.56}$$

otherwise, we would have $g_\Psi(x) = s_R$ and we go back to (A.55). We may also assume that

$$|d_\Gamma(x)| \le \sqrt{R}\ : \tag{A.57}$$

indeed, if $|d_\Gamma(x)| \ge \sqrt{R}$, we have that

$$|d_\Gamma(x) + H_{y_{N+1},R}(0)| \ge \sqrt{R}/2 > C_1 \log R$$

and so, by Lemma 2.10, either

$$d_\Gamma(x) + H_{y_{N+1},R}(0) > H_{y_{N+1},R}(1)\,,$$

in which case, due to Definition 2.11, $g_\Psi(x)$ is not even defined, or

$$d_\Gamma(x) + H_{y_{N+1},R}(0) < H_{y_{N+1},R}(s_R)\,,$$

in which case g_Ψ is constantly equal to s_R in a neighborhood of x and thus we go back again to (A.55). Also, in the proof of (A.54), we can restrict ourselves to the case in which

$$|g_\Psi| \neq 1/2\,, \tag{A.58}$$

since, by Proposition 2.13, no smooth function can touch g_Ψ from below at level $\pm 1/2$. With these further (non restrictive) assumptions, we now deal with the proof of (A.54).

Take x as in (A.54) and consider a sequence x_k converging to x, so that $d_\Gamma(x_k)$ is attained in $\Gamma_2 \cap \{|x'| \in (\omega q, q)\}$. Due to (A.58), $\nabla g_\Psi(x)$ is well defined and it is nonzero thanks to (A.56). From this, (A.58) and (A.57) we may also assume that $|g_\Psi(x_k)| \neq 1/2$, $\nabla g_\Psi(x_k)$ is well defined and nonzero and $|d_\Gamma(x_k)| \le 2\sqrt{R}$. Then, the hypotheses of (A.52) are fulfilled by x_k: hence, from (A.52) and the continuity of the functions involved, which is assured by (A.58), we have that

$$\begin{aligned} \Delta_p g_\Psi(x) &= \lim_{k\to+\infty} \Delta_p g_\Psi(x_k) \le \\ &\le \lim_{k\to+\infty} h_0'(g_\Psi(x_k)) - \frac{\mathrm{const}\,\gamma}{R} = \\ &= h_0'(g_\Psi(x)) - \frac{\mathrm{const}\,\gamma}{R}\,, \end{aligned}$$

thus proving (A.54).

We now prove that

$$\begin{aligned} &\Delta_p g_\Psi(x) < h_0'(g_\Psi(x)) \text{ in the viscosity sense} \\ &\text{at any } x \in \mathbb{R}^N \text{ for which } |g_\Psi(x)| \ge 1/2 \text{ and} \\ &d_\Gamma(x) \text{ is attained on } \Gamma_3. \end{aligned} \tag{A.59}$$

To prove the above claim, notice that, as remarked here above, we may restrict ourselves to the case in which (A.58) and (A.57) hold. Also, since Γ_3 is a portion of sphere,

$$\kappa_1^{\Gamma_3} = \cdots = \kappa_{N-1}^{\Gamma_3} = -\frac{1}{r_\omega} < 0\,,$$

so, from (A.57), we get that, for $i = 1, \ldots, N-1$,

$$\left|\kappa_i^{\Gamma_3} d_\Gamma\right| \le \frac{\text{const}}{\sqrt{R}}$$

which is small; therefore, from (A.32),

$$\begin{aligned} &\Delta_p g_\Psi = \\ =\;& h'_{y_{N+1},R} + \left(\frac{p}{p-1} h_{y_{N+1},R}\right)^{\frac{p-1}{p}} \sum_{i=1}^{N-1} \frac{\kappa_i^{\Gamma_3}}{\kappa_i^{\Gamma_3} d_\Gamma - 1} \le \\ \le\;& h'_{y_{N+1},R} + \text{const} \sum_{i=1}^{N-1} |\kappa_i^{\Gamma_3}| \le \\ \le\;& h'_{y_{N+1},R} + \frac{\text{const}}{R} \le \\ \le\;& h'_0 - \frac{\widehat{C}_0}{R} + \frac{\text{const}}{R} < \\ <\;& h'_0 , \end{aligned}$$

provided that $\widehat{C}_0$ is chosen suitably large, thus proving (A.59).

In the light of (A.43), (A.54) and (A.59), we deduce that

$$\begin{aligned} &g_\Psi \text{ is a strict supersolution of (1.5) everywhere} \\ &\text{possibly except the set } \{|g_\Psi| < 1/2\} \cap \Big\{ d_\Gamma \text{ realized on } \Gamma_1 \cup \Gamma_3 \Big\}. \end{aligned} \tag{A.60}$$

We now show that, if $\breve{C}$ is large enough, then[20]

$$\begin{aligned} &\text{if } x \in [-l, l]^N \text{ is in the domain of } \mathbb{S}(Y,R) \text{ and } |x'| \ge \sqrt[4]{\breve{C}}\, q, \text{ then} \\ &|d_{\Gamma_1}(x)| < \min\{|d_{\Gamma_2}(x)|,\, |d_{\Gamma_3}(x)|\}\,. \end{aligned} \tag{A.61}$$

To prove (A.61), we proceed as follows. Let $\nu(p)$ be the outer unit normal at a given $p \in \Gamma$. By convexity and concavity considerations, one sees that, for $p \in \Gamma_2 \cup \Gamma_3$, the minimum of $\nu_N(p)/|\nu'(p)|$ is attained for $|p'| = \omega q$, thus

$$\min_{p \in \Gamma_2 \cup \Gamma_3} \frac{\nu_N(p)}{|\nu'(p)|} = \frac{\sqrt{r^2 - q^2}}{2\omega q}\,. \tag{A.62}$$

Take now x in the domain of $\mathbb{S}(Y,R)$ with $|x'| \ge \sqrt[4]{\breve{C}}\, q$. Let $p \in \Gamma$ be a point realizing $d_\Gamma(x)$. Then, $x - p$ is orthogonal to Γ at p. Let assume, by contradicting (A.61), that $p \in \Gamma_2 \cup \Gamma_3$; then,

$$x = p + \alpha\nu(p)\,,$$

[20] Notational remark: we have that $|d_S(x)| := \inf_{y \in S} |x - y|$ is the absolute distance. Then, (A.61) says in particular that, if $|x'| \ge \sqrt[4]{\breve{C}}\, q$, $d_\Gamma(x)$ is realized on Γ_1.

for some $\alpha > 0$. Thus, from (A.62),

$$\begin{aligned} \sqrt[4]{\breve{C}}q - q \le |x'| - |p'| \le \\ \le \quad |x' - p'| \le \frac{2\alpha\omega q\, \nu_N(p)}{\sqrt{r^2 - q^2}}\,, \end{aligned}$$

hence

$$\alpha\nu_N(p) \ge \text{ const } \sqrt[4]{\breve{C}}\, r\,.$$

Therefore,

$$x_N = p_N + \alpha\nu_N(p) \ge \text{ const } \sqrt[4]{\breve{C}}\, r\,,$$

from which

$$x_N - y_N \ge \text{ const } \sqrt[4]{\breve{C}}\, r\,,$$

proving that x is outside the domain of $\mathbb{S}(Y,R)$, if $\breve{C}$ is large enough, in contradiction with our assumptions. This proves (A.61).

In the light of (A.61), we already know that, if $|x'| \ge \breve{C}q$ and $\breve{C}$ is large enough, then

$$|d_{\Gamma_1}(x)| < |d_{\Gamma_2\cup\Gamma_3}(x)|\,.$$

We now improve this estimate by showing that, if $x \in [-l,l]^N$ is in the domain of $\mathbb{S}(Y,R)$ and $|x'| \ge \breve{C}q$, then

$$|d_{\Gamma_2\cup\Gamma_3}(x)| - |d_{\Gamma_1}(x)| \ge \text{ const } \frac{\breve{C}^2 q^2}{R}\,. \tag{A.63}$$

To prove (A.63), we may and do assume that

$$|d_{\Gamma_1}(x)| \ge \breve{C}q/2\,, \tag{A.64}$$

since

$$|d_{\Gamma_2\cup\Gamma_3}(x)| \ge \breve{C}q - q$$

and so (A.63) would be obviously fulfilled if (A.64) were violated. Also,

$$|x - y| \ge x_N - y_N \ge -l + \sqrt{r^2 - q^2} \ge \frac{r}{2}\,,$$

if r is large enough. This says that

$$\frac{|x-y|}{|d_{\partial B_r(y)}(x)|} \ge \frac{1}{2}\,. \tag{A.65}$$

Indeed, if x is inside $B_r(y)$, the above estimate gives that

$$\frac{|x-y|}{|d_{\partial B_r(y)}(x)|} \ge \frac{|x-y|}{r} \ge \frac{1}{2}\,,$$

while if x is outside $B_r(y)$

$$\frac{|x-y|}{|d_{\partial B_r(y)}(x)|} = \frac{|x-y|}{|x-y| - r} \ge 1\,,$$

which proves (A.65). Let now x be so that $|x'| \geq \breve{C} q$ and let $p \in \Gamma_2 \cup \Gamma_3$. Let us define the cone

$$\mathfrak{C} := \left\{ x \in \mathbb{R}^N \ \text{ s.t. } \frac{|x' - y'|}{|x_N - y_N|} \leq \frac{\sqrt{\breve{C}}\, q}{r} \right\} . \tag{A.66}$$

Notice that, since x is in the domain of $\mathbb{S}(Y,R)$ and $|x'| \geq \breve{C} q$, then

$$\frac{|x' - y'|}{|x_N - y_N|} \geq \frac{\breve{C} q}{\text{const}\, R} , \tag{A.67}$$

if $\breve{C}$ is chosen large enough. This shows also that x is outside the cone $\mathfrak{C}$. On the contrary, by construction, $\Gamma_2 \cup \Gamma_3$ lies inside $\mathfrak{C}$. Thus, we can denote by $\hat{p}$ the intersection between the segment joining x and p with $\partial\mathfrak{C}$. Let also $\tilde{p} \in \Gamma_1$ be the projection of $\hat{p}$ on Γ. By construction, $\tilde{p} \in \partial\mathfrak{C} \cap \partial B_r(y)$, thus

$$\frac{|\tilde{p}' - y'|}{|\tilde{p}_N - y_N|} = \frac{\sqrt{\breve{C}}\, q}{r} \tag{A.68}$$

and so

$$\begin{aligned} \frac{\breve{C} q^2 + r^2}{\breve{C} q^2} |\hat{p}'|^2 &= |\hat{p}'|^2 + \frac{r^2}{\breve{C} q^2} |\hat{p}'|^2 = \\ &= |\hat{p}' - y'|^2 + \frac{r^2}{\breve{C} q^2} |\hat{p}' - y'|^2 = \\ &= |\hat{p}' - y'|^2 + |\hat{p}_N - y_N|^2 \ = \ r^2 . \end{aligned}$$

From the latter estimate, we get that

$$|\hat{p}'| \ \geq \ \sqrt[4]{\breve{C}}\, q ,$$

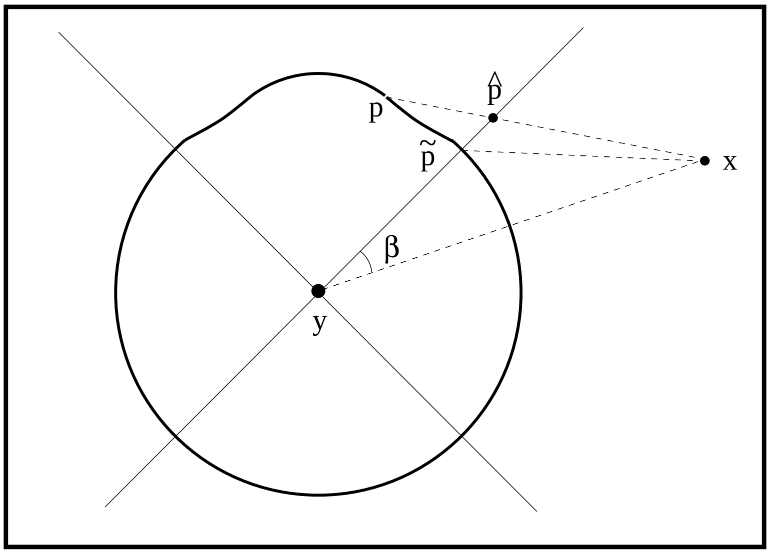
if r and $\breve{C}$ are large enough; therefore, in the light of (A.61), we know that

$$|\hat{p} - \tilde{p}| = |d_\Gamma(\hat{p})| \leq |\hat{p} - p|$$

and, therefore,

$$|x - p| = |x - \hat{p}| + |\hat{p} - p| \geq |x - \hat{p}| + |\hat{p} - \tilde{p}| \geq |x - \tilde{p}| . \tag{A.69}$$

Let now β be the angle between $x - y$ and $\tilde{p} - y$.



Geometry around the cone $\mathfrak{C}$

By (A.67) and (A.68), we have that

$$\beta \geq \text{const}\,\frac{\breve{C}q}{R},$$

provided $\breve{C}$ is large enough, therefore

$$\begin{aligned} |x-\tilde{p}|^2 &= |y-\tilde{p}|^2 + |x-y|^2 - 2\,|y-\tilde{p}|\,|x-y|\,\cos\beta \geq \\ &\geq \Big(|x-y| - |y-\tilde{p}|\Big)^2 + \text{const}\,|y-\tilde{p}|\,|x-y|\,\beta^2 = \\ &= \Big((r+d_\Gamma(x)) - r\Big)^2 + \text{const}\,r\,|x-y|\,\beta^2 \geq \\ &\geq \Big(d_\Gamma(x)\Big)^2 + \text{const}\,r\,|x-y|\,\frac{\breve{C}^2q^2}{R^2}. \end{aligned} \tag{A.70}$$

Also, $d_{\partial B_r(y)}(x)$ must be attained on Γ_1, since the straight lines normal to $\partial B_r(y) \setminus \Gamma_1$ are contained in $\mathfrak{C}$, while x is outside $\mathfrak{C}$. Therefore, from (A.61),

$$|d_\Gamma(x)| = |d_{\Gamma_1}(x)| = |d_{\partial B_r(y)}(x)|, \tag{A.71}$$

if $x \in [-l,l]^N$ with $|x'| \geq \check{C}q$. Hence, from (A.71), (A.70), (A.69) and (A.65), we gather that

$$\begin{aligned}
\text{(A.72)} \qquad \left(\frac{d_{\Gamma_2\cup\Gamma_3}(x)}{d_{\Gamma_1}(x)}\right)^2 &\geq \frac{|x-\tilde{p}|^2}{\Big(d_{\Gamma_1}(x)\Big)^2} \geq \\
&\geq 1 + \text{const}\,\frac{r\,|x-y|}{\Big(d_{\Gamma_1}(x)\Big)^2}\,\frac{\check{C}^2q^2}{R^2} \geq \\
&\geq 1 + \text{const}\,\frac{|x-y|}{\Big(d_{\Gamma_1}(x)\Big)^2}\,\frac{\check{C}^2q^2}{R} \geq \\
&\geq 1 + \text{const}\,\frac{1}{|d_{\Gamma_1}(x)|}\,\frac{\check{C}^2q^2}{R}\,.
\end{aligned}$$

Let now

$$\epsilon^* := \text{const}\,\frac{1}{|d_{\Gamma_1}(x)|}\,\frac{\check{C}^2q^2}{R}\,,$$

so that, by (A.64), $\epsilon^* \in [0,\ \text{const}\,\check{C}q/R]$ is small and so, from (A.72), we have that

$$\begin{aligned}
|d_{\Gamma_2\cup\Gamma_3}(x)| - |d_{\Gamma_1}(x)| &\geq |d_{\Gamma_1}(x)|\left(\sqrt{1+\epsilon^*}-1\right) \geq \\
&\geq \text{const}\,d_{\Gamma_1}(x)\,\epsilon^* \geq \\
&\geq \text{const}\,\frac{\check{C}^2q^2}{R}\,,
\end{aligned}$$

thus proving (A.63).

With similar arguments, we now show that, if $x \in [-l,l]^N$ with $|x'| \geq \check{C}q$, then

$$\text{(A.73)} \qquad d_{\Sigma(Y_\omega,R_\omega)}(x) - d_{\Sigma(Y,R)}(x) \geq \text{const}\,(1-\omega)\,\frac{q^2}{R}\,.$$

Indeed, if x is as above, then, by (A.67), x lies outside the cone $\mathfrak{C}$ defined in (A.66) and, by the arguments above (A.71), $d_{\Sigma(Y,R)}(x)$ is attained on Γ_1, say at a point p, and so, from (A.61),

$$d_\Gamma(x) = d_{\Gamma_1}(x) = d_{\Sigma(Y,R)}(x)\,.$$

Let also p' be the point on $\Sigma(Y_\omega,R_\omega)$ realizing $d_{\Sigma(Y_\omega,R_\omega)}(x)$. We distinguish different cases: first, if x is outside Γ (and so outside $\Sigma(Y_\omega,R_\omega)$), we take $\hat{p} \in \Gamma$ on the segment joining x and p' (it may happen that $\hat{p} = p'$). Then, if $\hat{p} \in \Gamma_1$, we have that

$$\begin{aligned}
&d_{\Sigma(Y_\omega,R_\omega)}(x) - |d_\Gamma(x)| = \\
&= |x-p'| - |x-p| \geq \\
&\geq |x-p'| - |x-\hat{p}| = \\
&= |p'-\hat{p}|\,.
\end{aligned}$$

Therefore, if x is outside Γ and $\hat p \in \Gamma_1$, by using (A.37), one proves (A.73) in this case; if, conversely, if x is outside Γ and $\hat p \in \Gamma_2 \cup \Gamma_3$, we have that

$$\begin{aligned} & d_{\Sigma(Y_\omega,R_\omega)}(x) - |d_\Gamma(x)| = \\ = \;& |x-p'| - |d_{\Gamma_1}(x)| = \\ = \;& |x-\hat p| + |\hat p - p'| - |d_{\Gamma_1}(x)| \geq \\ \geq \;& |x-\hat p| - |d_{\Gamma_1}(x)| \geq \\ \geq \;& |d_{\Gamma_2\cup\Gamma_3}(x)| - |d_{\Gamma_1}(x)|, \end{aligned}$$

and so (A.73) follows from (A.63) in this case.

If, on the other hand, x is inside Γ but outside $\Sigma(Y_\omega, R_\omega)$, we have that

$$\begin{aligned} & d_{\Sigma(Y_\omega,R_\omega)}(x) - d_\Gamma(x) = \\ = \;& |d_{\Sigma(Y_\omega,R_\omega)}(x)| + |d_\Gamma(x)| = \\ = \;& |x-p'| + |x-p| \geq \\ \geq \;& |p'-p|, \end{aligned}$$

thus (A.37) gives (A.73) in this case. Finally, if x is inside $\Sigma(Y_\omega, R_\omega)$ (and therefore inside Γ), take $\breve p \in \Sigma(Y_\omega, R_\omega)$ on the segment joining p and x, and observe that

$$\begin{aligned} & d_{\Sigma(Y_\omega,R_\omega)}(x) - d_\Gamma(x) = \\ = \;& -|d_{\Sigma(Y_\omega,R_\omega)}(x)| + |d_\Gamma(x)| \geq \\ \geq \;& -|x-\breve p| + |x-p| = \\ = \;& |\breve p - p|, \end{aligned}$$

and thus (A.73) follows from (A.37) also in this case. This ends the proof of (A.73).

Notice now that, by means of (A.61) and (A.42) it also follows that

$$\text{if } |x'| \geq \breve C q, \text{ then } g_\Psi(x) = g_{\mathbb{S}(Y,R)}(x). \tag{A.74}$$

We now define

$$\begin{aligned} Y_\omega &:= (y_\omega, y_{N+1}), \\ R_1 &:= r_\omega + H_0(y_{N+1}) + \frac{5\,\overline{C}_0}{R}, \\ r_1 &:= R_1 - H_0(y_{N+1}) - \frac{\overline{C}_0}{2R_1} y_{N+1}^2, \\ \Sigma(Y_\omega, R_1) &:= \mathbb{S}(Y_\omega, R_1) \cap \{x_{N+1} = 0\}. \end{aligned}$$

By definition,

$$\begin{aligned} & R_1 \leq \omega r + \text{const} \leq \\ & \leq \omega R + \text{const} < R \end{aligned} \tag{A.75}$$

and

$$r_1 - r_\omega \in \left[\frac{3\overline{C}_0}{R}, \frac{5\overline{C}_0}{R}\right]. \tag{A.76}$$

Furthermore, arguing as done on page 83, we have that

$$\Sigma(Y_\omega, R_1) = \partial B_{r_1}(y_\omega)\,; \tag{A.77}$$

thus, by (A.76), we infer that

$$\begin{aligned}&\Sigma(Y_\omega, R_1) \text{ stays at distance greater}\\ &\text{than } 3\overline{C}_0/R \text{ outside } \partial B_{r_\omega}(y_\omega) \supset \Gamma_3.\end{aligned} \tag{A.78}$$

Recalling the definition of η given in (A.33), we now show that

$$\partial B_{r_\omega}(y_\omega) \cap \{|x'| \geq \eta q\} \text{ is at distance at least } \frac{8\overline{C}_0}{R} \text{ inside } \Gamma. \tag{A.79}$$

The proof of this fact will take some pages (it will end on page 107). To prove (A.79), let us take $w \in \partial B_{r_\omega}(y_\omega) \cap \{|x'| \geq \eta q\}$ and let us observe that w is inside Γ: if not, the fact that $w \in \partial B_{r_\omega}(y_\omega)$ implies that

$$w \in \Gamma_3 \subseteq \{|x'| \leq \omega q\}\,,$$

which would contradict the fact that $|w'| \geq \eta q$. Hence, knowing now that w is inside Γ, we now estimate its distance to Γ. For this scope, take $z_1 \in \Gamma_1$, $z_2 \in \Gamma_2$ and $z_3 \in \Gamma_3$.

Let also $p = p_w$ one of the points in the closure of Γ_1 which is as close as possible to w. Of course,

$$|z_1 - w| \;\geq\; |p - w|\,. \tag{A.80}$$

If $p_N = 0$, we have that

$$\begin{aligned}
|w-p| &\geq |w_N - p_N| = \\
&= |w_N| \geq \\
&\geq |y_{\omega N}| - |w_N - y_{\omega N}| = \\
&= |y_{\omega N}| - \sqrt{r_\omega^2 - |w'|^2} \geq \\
&\geq |y_{\omega N}| - \sqrt{r_\omega^2 - \eta^2 q^2} = \\
&= \frac{1}{2}\sqrt{r^2-q^2} - \frac{q^2}{\gamma\sqrt{r^2-q^2}}\left(\frac{1}{\omega^\gamma} - 1\right) - \\
&\quad - \sqrt{\frac{1}{4}\left(r^2 + q^2\left(\frac{1}{\omega^{2\gamma+2}} - 1\right)\right) - \eta^2 q^2} = \\
&= \frac{1}{2}\sqrt{r^2-q^2} - \frac{q^2}{\gamma\sqrt{r^2-q^2}}\left(2\omega^2 - 1\right) - \\
&\quad - \frac{1}{2}\sqrt{r^2 - q^2\left(1 - 4\omega^2 + 4\eta^2\right)}\,.
\end{aligned}$$

Let us now fix an auxiliary small parameter $\delta > 0$. From the estimate here above, (A.33) and (A.35),we deduce, if γ and r/q are large enough[21] and $p_N = 0$, that

$$\begin{aligned}
|w-p| &\geq \frac{q^2}{r}\left[\eta^2 - \omega^2 - \frac{(1+\delta)\,(2\omega^2-1)}{\gamma+2} - \text{const}\left(\frac{q}{r}\right)^2\right] = \\
&= \frac{q^2}{r}\left[(\eta+\omega)\,(1-\epsilon)\,(1-\omega) - \frac{(1+\delta)\,(2\omega^2-1)\,\log(1/\omega)}{\log 2} - \right.\\
&\quad \left. - \text{const}\left(\frac{q}{r}\right)^2\right] \geq \\
&\geq \frac{q^2}{r}\left[(\eta+\omega)\,(1-\epsilon)\,(1-\omega) - \frac{(1+2\delta)\,(2\omega^2-1)\,(1-\omega)}{\log 2} - \right.\\
&\quad \left. - \text{const}\left(\frac{q}{r}\right)^2\right] = \\
&= \frac{q^2\,(1-\omega)}{r}\left[(\eta+\omega)\,(1-\epsilon) - \frac{(1+2\delta)\,(2\omega^2-1)}{\log 2} - \right.\\
&\quad \left. - \text{const}\left(\frac{q}{r}\right)^2\right].
\end{aligned}$$

When ϵ and δ are close to 0 and η and ω are close to 1, the quantity

$$(\eta+\omega)\,(1-\epsilon) - \frac{(1+2\delta)\,(2\omega^2-1)}{\log 2}$$

is close to

$$\check{c} := 2 - \frac{1}{\log 2} > 0\,,$$

hence the above estimate implies that, if $p_N = 0$,

$$|w-p| \geq \frac{\check{c}\,(1-\omega)\,q^2}{2r}\,. \tag{A.81}$$

On the other hand, if $p_N \neq 0$, its minimizing property gives that $w - p$ is orthogonal to $\partial B_r(y)$ at p, which says that the line joining p and w passes through y and, therefore,

$$|p-w| \geq r - |w-y|\,. \tag{A.82}$$

Let now

$$\beta = \beta_w = \angle(w - y_\omega,\, e_N)\,.$$

Of course,

$$\angle(w-y_\omega,\, y-y_\omega) = \pi - \beta\,. \tag{A.83}$$

Elementary trigonometry gives that

$$\eta q \leq |w'| = r_\omega\,|\sin\beta|\,,$$

[21] Notation remark: "log" here and in what follows is intended as the *natural* logarithm.

that is

$$|\cos\beta| \le \sqrt{1-\left(\frac{\eta q}{r_\omega}\right)^2}\,. \tag{A.84}$$

In the light of (A.83), (A.84) and (A.35), we thus gather that

$$\begin{aligned}
|w-y|^2 &= |(w-y_\omega)-(y-y_\omega)|^2 = \\
&= |w-y_\omega|^2+|y-y_\omega|^2- \\
&\quad -2\,|w-y_\omega|\,|y-y_\omega|\cos\Big(\angle(w-y_\omega,\,y-y_\omega)\Big) = \\
&= r_\omega^2+|y-y_\omega|^2+2r_\omega|y-y_\omega|\,\cos\beta \le \\
&\le r_\omega^2+|y-y_\omega|^2+2r_\omega|y-y_\omega|\sqrt{1-\left(\frac{\eta q}{r_\omega}\right)^2} \le \\
&\le r_\omega^2+|y-y_\omega|^2+ \\
&\quad +2r_\omega|y-y_\omega|\left[1-\frac{1}{2}\left(\frac{\eta q}{r_\omega}\right)^2+\,\mathrm{const}\left(\frac{q}{r_\omega}\right)^4\right] = \\
&= \Big(r_\omega+|y-y_\omega|\Big)^2-r_\omega\,|y-y_\omega|\left(\frac{\eta q}{r_\omega}\right)^2+\,\mathrm{const}\,\frac{|y-y_\omega|\,q^4}{r_\omega^3}\,.
\end{aligned}$$

We now perform a modification of the argument done for the case $p_N=0$: we fix a small parameter $\delta>0$, to be appropriately chosen, and we observe that, if $1-\omega$, $1/\gamma$ and q/r are small enough (possibly in dependence of δ), then the above estimate, (A.36) and (A.35) yield that

$$
\begin{aligned}
&|w-y|^2 \le \\
\le\; & \Big(r_\omega + |y-y_\omega|\Big)^2 - \frac{\eta^2 q^2\,|y-y_\omega|}{r_\omega} + \text{const}\,\frac{q^4}{r^2} \le \\
\le\; & \Big(\frac12\sqrt{r^2+q^2(4\omega^2-1)} + \frac12\sqrt{r^2-q^2} + \frac{(1+\delta)(\omega^{-\gamma}-1)\,q^2}{\gamma r}\Big)^2 - \\
& -\frac{\eta^2 q^2\sqrt{r^2-q^2}}{\sqrt{r^2+q^2(4\omega^2-1)}} + \text{const}\,\frac{q^4}{r^2} = \\
=\; & r^2\Big(\frac12\sqrt{1+\frac{q^2(4\omega^2-1)}{r^2}} + \frac12\sqrt{1-\frac{q^2}{r^2}} + \frac{(1+\delta)(\omega^{-\gamma}-1)\,q^2}{\gamma r^2}\Big)^2 - \\
& -\eta^2 q^2\sqrt{1-\frac{4\omega^2 q^2}{r^2+q^2(4\omega^2-1)}} + \text{const}\,\frac{q^4}{r^2} \le \\
\le\; & r^2\Big(1+\frac{q^2}{2r^2}\,(2\omega^2-1) + \frac{(1+\delta)(\omega^{-\gamma}-1)\,q^2}{\gamma r^2}\Big)^2 - \\
& -\eta^2 q^2 + \text{const}\,\frac{q^4}{r^2} \le \\
\le\; & r^2\Big(1+\frac{q^2}{r^2}\,(2\omega^2-1) + \frac{2(1+\delta)(\omega^{-\gamma}-1)\,q^2}{\gamma r^2}\Big) - \eta^2 q^2 + \text{const}\,\frac{q^4}{r^2} = \\
=\; & r^2\Big[1+\frac{q^2}{r^2}\,(2\omega^2-1) + \frac{2(1+\delta)(\omega^{-\gamma}-1)\,q^2}{\gamma r^2} - \frac{\eta^2 q^2}{r^2} + \text{const}\,\frac{q^4}{r^4}\Big] \le \\
\le\; & r^2\Big[1+\frac{q^2}{r^2}\,(2\omega^2-1) + \frac{2(1+2\delta)(\omega^{-\gamma}-1)\,q^2}{(\gamma+2) r^2} - \frac{\eta^2 q^2}{r^2} + \text{const}\,\frac{q^4}{r^4}\Big] = \\
=\; & r^2\Big[1+\frac{q^2}{r^2}\,(2\omega^2-1) + \frac{2(1+2\delta)(\omega^{-\gamma}-1)\,\log(1/\omega)\,q^2}{(\log 2)\,r^2} - \frac{\eta^2 q^2}{r^2} + \\
& +\text{const}\,\frac{q^4}{r^4}\Big] \le \\
\le\; & r^2\Big[1+\frac{q^2}{r^2}\,(2\omega^2-1) + \frac{2(1+3\delta)(\omega^{-\gamma}-1)(1-\omega)q^2}{(\log 2)\,r^2} - \frac{\eta^2 q^2}{r^2} + \\
& +\text{const}\,\frac{q^4}{r^4}\Big] = \\
=\; & r^2\Big[1+\frac{q^2}{r^2}\,(2\omega^2-1-\eta^2) + \frac{2(1+3\delta)(\omega^{-\gamma}-1)(1-\omega)q^2}{(\log 2)\,r^2} + \\
& +\text{const}\,\frac{q^4}{r^4}\Big].
\end{aligned}
$$

Note also that, by (A.33),

$$\begin{aligned} 2\omega^2-1-\eta^2 &= \omega^2-1+\omega^2-\eta^2 = \\ &= (\omega-1)(\omega+1)+(\omega-\eta)(\omega+\eta) = \\ &= (\omega-1)(\omega+1)+(\omega-1)(1-\epsilon)(\omega+\eta) = \\ &= (\omega-1)\,\varpi\,, \end{aligned}$$

where

$$\varpi := \omega+1+(1-\epsilon)(\omega+\eta)\,.$$

Thence, from the estimates here above,

$$\begin{aligned} &|w-y|^2 \le \\ \le\ & r^2\Big[1-\frac{(1-\omega)\,\varpi\,q^2}{r^2}+\frac{2(1+3\delta)(\omega^{-\gamma}-1)(1-\omega)q^2}{(\log 2)\,r^2}+\text{const}\,\frac{q^4}{r^4}\Big] = \\ =\ & r^2\Big[1-\frac{(1-\omega)\,\varpi\,q^2}{r^2}+\frac{2(1+3\delta)(2\omega^2-1)(1-\omega)q^2}{(\log 2)\,r^2}+\text{const}\,\frac{q^4}{r^4}\Big] = \\ =\ & r^2\Big[1-\frac{(1-\omega)\,q^2}{r^2}\,\Theta+\text{const}\,\frac{q^4}{r^4}\Big]\,, \end{aligned}$$

where we used the short hand notation

$$\Theta := \varpi - \frac{2(1+3\delta)(2\omega^2-1)}{\log 2}\,.$$

Note that when η and ω are close to 1, and ϵ and δ are small, then ϖ is close to 4, and thus Θ is close to

$$4-\frac{2}{\log 2} > 0\,.$$

Hence, for some small $\check{c}>0$,

$$|w-y| \le r^2\Big[1-2\check{c}\frac{q^2(1-\omega)}{r^2}+\text{const}\,\frac{q^4}{r^4}\Big]$$

and so

$$|w-y| \le r^2\Big[1-\check{c}\frac{q^2(1-\omega)}{r^2}\Big]\,.$$

Therefore, by (A.35),

$$\begin{aligned} r-|w-y| &\ge r\Big(1-\sqrt{1-\check{c}\frac{q^2(1-\omega)}{r^2}}\Big) \ge \\ &\ge \frac{\text{const}\,(1-\omega)\,q^2}{r}\,, \end{aligned}$$

in case $p_N \neq 0$. In the light of the latter estimate, (A.81) and (A.80), we deduce that

$$|z_1-w| \ge \frac{\text{const}\,q^2}{r}\,. \tag{A.85}$$

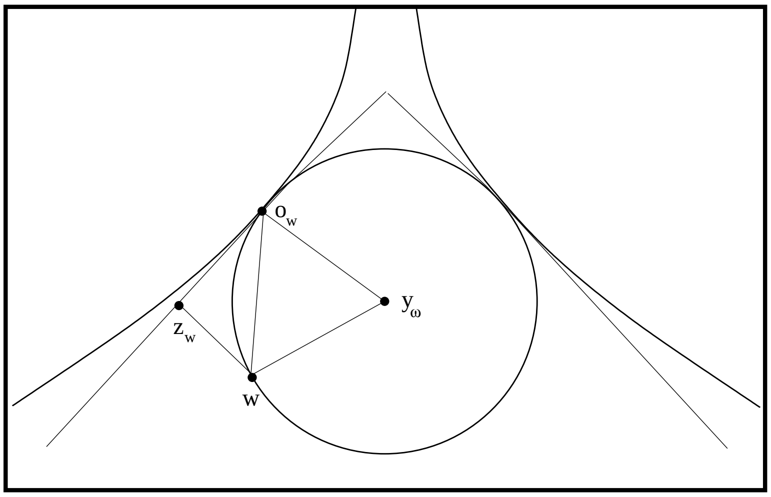
This will control the distance of w from Γ_1; we will now control the distance from Γ_2 and Γ_3, in order to complete the proof of (A.79). For this, let us consider the cone $\mathcal{C}$ which is tangent to $\partial B_{r_\omega}(y_\omega)$ (from outside) and to Γ_2 (from inside) at $\{|x'| = \omega q\}$. Let also Π be the two-dimensional plane spanned by the vectors $w - y_\omega$ and e_N and passing through y_ω, i.e.,

$$\Pi := \Big\{ y_\omega + (w - y_\omega)s + e_N t\,,\ \ s, t \in \mathbb{R} \Big\}\,.$$

Let $z_w \in \mathcal{C}$ be the projection of w on $\mathcal{C} \cap \Pi$ and let o_w be the projection of w on $\{|x'| = \omega q\} \cap \partial B_{r_\omega}(y_\omega) \cap \Pi$. By (A.33), we have that

$$|w - o_w| \geq |w'| - |o'_w| \geq (\eta - \omega)q \geq \frac{(1-\omega)\,q}{2}\,. \tag{A.86}$$

Let also $2\alpha_w$ be the angle between $w - y_\omega$ and $o_w - y_\omega$. Then, by Eucledian geometry, the angle between $z_w - o_w$ and $w - o_w$ is α_w.



Euclidean geometry of our projections

Therefore,

$$\begin{aligned}
|w - o_w|^2 &= |(w - y_\omega) - (o_w - y_\omega)|^2 = \\
&= |w - y_\omega|^2 + |y_\omega - o_w|^2 - 2\,|w - y_\omega|\,|y_\omega - o_w|\,\cos(2\alpha_w) = \\
&= 2r_\omega^2(1 - \cos(2\alpha_w)) = \\
&= 4r_\omega^2 \sin^2 \alpha_w
\end{aligned}$$

and so,

$$|w - o_w| \;=\; 2r_\omega \sin \alpha_w\,. \tag{A.87}$$

From (A.87) and (A.86),

$$\begin{aligned} 2r_\omega\,|w - z_w| &= 2r_\omega\,|w - o_w|\,\sin\alpha_w = \\ &= |w - o_w|^2 \geq \\ &\geq \frac{(1-\omega)^2\,q^2}{4}\,. \end{aligned}$$

This says that the distance from w to $\mathcal{C}$ is, at least, $(1-\omega)^2\,q^2/8r_\omega$ and so

$$|z_2 - w| \geq \frac{(1-\omega)^2\,q^2}{8r_\omega}\,. \tag{A.88}$$

Moreover,

$$\begin{aligned} &|z_3 - w| \geq |z_3' - w'| \geq \\ &\geq |w'| - |z_3| \geq \eta q - \omega q \geq \frac{(1-\omega)\,q}{2}\,. \end{aligned} \tag{A.89}$$

Then, the proof of (A.79) follows from (A.85), (A.88) and (A.89), by assuming $q \geq \text{const}\,\overline{C}_0$.

We infer from (A.77), (A.79) and (A.76) that

$$\Sigma(Y_\omega, R_1) \cap \{|x'| \geq \eta q\} \text{ is at distance at least } \frac{3\overline{C}_0}{R} \text{ inside } \Gamma. \tag{A.90}$$

Notice also that, by construction and recalling (A.35), one has

$$\begin{aligned} y_{\omega N} + r_1 &\leq -\frac{1}{2}\sqrt{r^2 - q^2} + \frac{\text{const}\,q^2}{r} + r_\omega + \frac{\text{const}\,\overline{C}_0}{R} = \\ &= \frac{1}{2}\left(\sqrt{r^2 + (4\omega^2 - 1)\,q^2} - \sqrt{r^2 - q^2}\right) + \\ &\quad + \frac{\text{const}\,q^2}{r} + \frac{\text{const}\,\overline{C}_0}{R} \leq \\ &\leq \frac{\text{const}\,q^2}{r} + \frac{\text{const}\,\overline{C}_0}{R} \leq \\ &\leq \frac{\text{const}\,q^2}{R}\,, \end{aligned}$$

thence,

$$\Sigma(Y_\omega, R_1) \subseteq \left\{ x_N \leq \frac{\text{const}\,q^2}{R} \right\}\,. \tag{A.91}$$

We now investigate the mutual position of Ψ and $\mathbb{S}(Y_\omega, R_1)$. For this, we start by claiming that

$$\begin{aligned}&\text{The region } \Big\{X \in \Psi \text{ where } |x_{N+1}| < 1/2 \text{ and}\\ &d_\Gamma(x) \text{ is realized on } \Gamma_3\Big\} \text{ is above } \mathbb{S}(Y_\omega, R_1)\\ &\text{(where above means with respect to the } e_{N+1}\text{-direction).}\end{aligned} \tag{A.92}$$

To prove (A.92), note that, if $d_\Gamma(x)$ is realized on Γ_3, then $x - y_\omega$ is orthogonal to $\Gamma_3 \subset \partial B_{r_\omega}(y_\omega)$, therefore, recalling (A.76) and (2.39),

$$\begin{aligned} d_\Gamma(x) &= d_{\Gamma_3}(x) = d_{\Sigma(Y_\omega,R_1)}(x) + r_1 - r_\omega \geq \\ &\geq d_{\Sigma(Y_\omega,R_1)}(x) + \frac{3\overline{C}_0}{R} = \\ &= |x - y_\omega| - r_1 + \frac{3\overline{C}_0}{R} = \\ &= |x - y_\omega| - R_1 + H_0(y_{N+1}) + \frac{\overline{C}_0}{2R_1}\, y_{N+1}^2 + \frac{3\overline{C}_0}{R} = \\ &= |x - y_\omega| - R_1 + H_0(y_{N+1}) + \frac{\overline{C}_0}{2R_1}\, y_{N+1}^2 + \frac{3\overline{C}_0}{R} - \\ &\qquad - H_{y_{N+1},R}(0) - \frac{\overline{C}_0}{2R} y_{N+1}^2 . \end{aligned}$$

This, recalling (2.39) and (2.38), gives that

$$\begin{aligned} &H_{y_{N+1},R_1}\Big(g_\Psi(x)\Big) - \frac{\overline{C}_0}{2}\Big(g_\Psi(x) - y_{N+1}\Big)^2 \left(\frac{1}{R} - \frac{1}{R_1}\right) = \\ =\; &\widetilde{H}_{y_{N+1},R_1}\Big(g_\Psi(x)\Big) - \frac{\overline{C}_0}{2}\Big(g_\Psi(x) - y_{N+1}\Big)^2 \left(\frac{1}{R} - \frac{1}{R_1}\right) = \\ =\; &\widetilde{H}_{y_{N+1},R}\Big(g_\Psi(x)\Big) = \\ =\; &H_{y_{N+1},R}\Big(g_\Psi(x)\Big) = \\ =\; &H_{y_{N+1},R}\Big(g_{y_{N+1},R}\Big(d_\Gamma(x) + H_{y_{N+1},R}(0)\Big)\Big) = \\ =\; &d_\Gamma(x) + H_{y_{N+1},R}(0) \geq \\ \geq\; &|x - y_\omega| - R_1 + H_0(y_{N+1}) + \frac{\overline{C}_0}{2R_1}\, y_{N+1}^2 + \frac{3\overline{C}_0}{R} - \frac{\overline{C}_0}{2R} y_{N+1}^2 . \end{aligned}$$

Therefore,

$$\begin{aligned} H_{y_{N+1},R_1}\Big(g_\Psi(x)\Big) &\geq |x - y_\omega| - R_1 + H_0(y_{N+1}) + \\ &+ \frac{3\overline{C}_0}{R} - \frac{\overline{C}_0}{2R} y_{N+1}^2 - \frac{\overline{C}_0}{2R_1}\Big(g_\Psi(x) - y_{N+1}\Big)^2 . \end{aligned} \tag{A.93}$$

Note however that $R_1 \sim R/2$ for large R, and so, in particular, we may assume that $R_1 \geq R/3$. Thence,

$$\begin{aligned} & \frac{3\overline{C}_0}{R} - \frac{\overline{C}_0}{2R} y_{N+1}^2 - \frac{\overline{C}_0}{2R_1} \Big(g_\Psi(x) - y_{N+1} \Big)^2 \geq \\ \geq \;& \frac{3\overline{C}_0}{R} - \frac{\overline{C}_0}{2R} \left(\frac{1}{4}\right)^2 - \frac{3\overline{C}_0}{2R} \left(\frac{3}{4}\right)^2 > \\ > \;& 0 . \end{aligned}$$

This and (A.93) yield that

$$H_{y_{N+1},R_1}\Big(g_\Psi(x)\Big) \; > \; |x - y_\omega| - R_1 + H_0(y_{N+1}) ,$$

and so

$$g_\Psi(x) > g_{y_{N+1},R_1}\Big(|x - y_\omega| - R_1 + H_0(y_{N+1})\Big) = g_{\mathbb{S}(Y_\omega,R_1)}(x) ,$$

which proves (A.92).

We now show that

$$\begin{aligned} &\text{The region } \Big\{ X \in \mathbb{S}(Y_\omega, R_1) \text{ where } |x_{N+1}| < 1/2 \text{ and} \\ &d_{\Sigma(Y_\omega,R_1)}(x) \text{ is realized at a point } z \text{ with } |z'| \geq \eta q \Big\} \\ &\text{is strictly above } \Psi \text{ (in the } e_{N+1}\text{-direction).} \end{aligned} \tag{A.94}$$

In order to prove (A.94), take $X \in \mathbb{S}(Y_\omega, R_1)$, with $|x_{N+1}| < 1/2$ and assume that

$$\sigma \; := \; \mathrm{dist}\Big(x, \Sigma(Y_\omega, R_1)\Big)$$

is attained at $z \in \Sigma(Y_\omega, R_1)$ and $|z'| \geq \eta q$. Then, by construction,

$$x_{N+1} \; = \; g_{\mathbb{S}(Y_\omega,R_1)}(x) ,$$

thus from (2.20) and Definition 2.11,

$$|x - y_\omega| \; = \; H_0(x_{N+1}) - \frac{\overline{C}_0}{2R_1} (x_{N+1} - y_{N+1})^2 + R_1 - H_0(y_{N+1}) .$$

Therefore,

$$\begin{aligned} \mathrm{dist}\Big(x, \Sigma(Y_\omega, R_1)\Big) \; &= \; |x - y_\omega| - r_1 = \\ = \; H_0(x_{N+1}) - \frac{\overline{C}_0}{2R_1} &(x_{N+1} - y_{N+1})^2 + R_1 - H_0(y_{N+1}) - r_1 , \end{aligned}$$

which implies that

$$\sigma \; \leq \; H_{y_{N+1},R}(x_{N+1}) + \frac{\overline{C}_0}{2} \left(\frac{1}{R} + \frac{1}{R_1} \right) .$$

Hence,

$$g_{\mathbb{S}(Y_\omega,R_1)}(x) = x_{N+1} \geq g_{y_{N+1},R} \left(\sigma - \frac{\overline{C}_0}{2} \left(\frac{1}{R} + \frac{1}{R_1} \right) \right) . \tag{A.95}$$

We consider $\breve{y} \in \Gamma$ on the half-line from y_ω towards z. Note that, by construction, y_ω, z, x and $\breve{y}$ lie on the same half-line. Let also $\tilde{y} \in \Gamma$ be the point realizing $d_\Gamma(x)$. By means of (A.90), we have that

$$|z - \breve{y}|\,,\ |z - \tilde{y}| \geq \frac{3\overline{C}_0}{R}\,. \tag{A.96}$$

We claim that this implies that

$$\sigma \,\geq\, d_\Gamma(x) + \frac{3\overline{C}_0}{R}\,. \tag{A.97}$$

To prove this, we distingush three cases, according to the mutual position of x, z and $\breve{y}$. Namely, if x is outside Γ and outside $\Sigma(Y_\omega, R_1)$, we have that

$$\begin{aligned} \sigma - d_\Gamma(x) &= |x-z| - |d_\Gamma(x)| \geq \\ &\geq |x-z| - |x-\breve{y}| = \\ &= |\breve{y} - z|\,, \end{aligned}$$

and so (A.97) follows from (A.96) in this case; if, on the other hand, x is inside Γ but outside $\Sigma(Y_\omega, R_1)$,

$$\begin{aligned} \sigma - d_\Gamma(x) &= |x-z| + |d_\Gamma(x)| = \\ &= |x-z| + |x-\tilde{y}| \geq \\ &\geq |\tilde{y} - z|\,, \end{aligned}$$

which gives (A.97) via (A.96) in this case. The case in which x is outside Γ and inside $\Sigma(Y_\omega, R_1)$ does not hold: indeed, if x is inside $\Sigma(Y_\omega, R_1)$, since, due to (A.90), z is inside Γ, then x lies, in this case, on the segment between y_ω and a point inside Γ (that is, z itself): thus, since Γ is star-shaped with respect to y_ω, x is inside Γ. Let us finally consider the case in which x is inside $\Sigma(Y_\omega, R_1)$ and inside Γ. In this case, note that $\tilde{y}$ cannot lie on $\Sigma(Y_\omega, R_\omega)$ (that is, $\tilde{y} \notin \Gamma_3$), otherwise $\tilde{y}$ would be on the radius from y_ω to z and so

$$\tilde{y} \,=\, y_\omega + \frac{r_\omega}{r_1}\,(z - y_\omega)$$

which gives that

$$|\tilde{y}'| = \frac{r_\omega}{r_1}\,|z'| \geq \frac{r_\omega}{r_1}\,\eta q > \omega q\,,$$

if R is large, thanks to (A.76), in contrast with the fact that $\tilde{y} \in \Gamma_3$.

Also, if x is inside $\Sigma(Y_\omega, R_1)$ and inside Γ and $\tilde{y}$ lies on Γ_1, then note that it must be outside $\Sigma(Y_\omega, R_1)$, thanks to (A.37) and (A.76). Thus, we take $\tilde{z} \in \Sigma(Y_\omega, R_1)$ on the segment joining x and $\tilde{y}$ and we define

$$\tilde{w} \,:=\, y_\omega + \frac{r_\omega}{r_1}\,(\tilde{z} - y_\omega)\,.$$

Note that $\tilde{w} \in \Sigma(Y_\omega, R_\omega)$ and therefore, by (A.37),

$$|\tilde{w} - \tilde{y}| \,\geq\, \mathrm{const}\,(1-\omega)\,\frac{q^2}{r}\,.$$

Furthermore, by (A.76),

$$\begin{aligned} |\tilde{w} - \tilde{z}| &= \left|\frac{r_\omega}{r_1}(\tilde{z} - y_\omega) - (\tilde{z} - y_\omega)\right| = \\ &= \left|\frac{r_\omega}{r_1} - 1\right| r_1 = \\ &= |r_\omega - r_1| \leq \frac{5\,\overline{C}_0}{R}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma - d_\Gamma(x) &= -|x - z| + |d_\Gamma(x)| \geq \\ &\geq -|x - \tilde{z}| - |x - \tilde{y}| = \\ &= |\tilde{y} - \tilde{z}| \geq \\ &\geq |\tilde{y} - \tilde{w}| - \frac{5\,\overline{C}_0}{R} \geq \\ &\geq \text{const}\,(1 - \omega)\,\frac{q^2}{R} - \frac{5\,\overline{C}_0}{R}\,, \end{aligned}$$

which proves (A.97) also in this case, by taking q conveniently large (possibly in dependence of $1/(1-\omega)$ and $\overline{C}_0$).

Then, to end the proof of (A.97), we only need to consider the case in which x is inside $\Sigma(Y_\omega, R_1)$ and inside Γ and $\tilde{y}$ lies on Γ_2. For this, let us define

$$\tilde{x} = x - \frac{6\overline{C}_0}{R}\,\frac{x - y_\omega}{|x - y_\omega|}$$

and observe that

$$r_1 \sin\Big(\angle(z - y_\omega,\, e_N)\Big) = |z'| \geq \eta q\,,$$

thus, recalling (A.76),

$$\begin{aligned} \angle(\tilde{x} - y_\omega,\, e_N) &= \angle(x - y_\omega,\, e_N) = \\ &= \angle(z - y_\omega,\, e_N) \geq \\ &\geq \frac{\eta q}{r_1} - \text{const}\,\frac{q^2}{r^2} \geq \\ &\geq \frac{\eta q}{r_\omega} - \text{const}\,\frac{q^2}{r^2}\,. \end{aligned}$$

Also, by the minimality property of $\tilde{y}$, $x - \tilde{y}$ is orthogonal to Γ at $\tilde{y}$ and so, by Lemma B.10, we have that y_ω, x, $\tilde{x}$, z and $\tilde{y}$ lie on the same plane. Let us denote this plane by Π_0. Let also ξ_0 be the point on the common boundary of Γ_2 and Γ_3 on the side where $\tilde{y}$ lies and let ℓ be the straight line lying in Π_0 and tangent to Γ at ξ_0. By construction, $|\xi_0'| = \omega q$, thus

$$r_\omega\,\sin\Big(\angle(\xi_0 - y_\omega,\, e_N)\Big) = \omega q$$

and thence

$$\angle(\tilde{x}-y_\omega,\, \xi_0-y_\omega) \geq \frac{(\eta-\omega)q}{r_\omega} - \operatorname{const}\frac{q^2}{r^2} = \frac{(1-\epsilon)\,(1-\omega)\,q}{r_\omega} - \operatorname{const}\frac{q^2}{r^2}\,,$$

from which

$$\angle(\tilde{x}-y_\omega,\, \xi_0-y_\omega) \;\geq\; \operatorname{const}\frac{(1-\omega)\,q}{r}\,.$$

We now estimate

$$\min_{\xi\in\ell}|\xi-\tilde{x}|\,.$$

For this, note that the point $\xi^*\in\ell$ attaining such minimum must be so that $\tilde{x}-\xi^*$ is orthogonal to ℓ at ξ^* and so the points ξ_0, ξ^*, $\tilde{x}$ and y_ω form a right trapezoid lying on the plane Π_0. Elementary trigonometry on this right trapezoid gives that

$$\begin{aligned}
\min_{\xi\in\ell}|\xi-\tilde{x}| &= |\xi^*-\tilde{x}| = \\
&= |\xi_0-y_\omega| - |\tilde{x}-y_\omega|\cos\Big(\angle(x-y_\omega,\, e_N)\Big) = \\
&= r_\omega - |\tilde{x}-y_\omega|\cos\Big(\angle(x-y_\omega,\, e_N)\Big) \geq \\
&\geq r_\omega - |\tilde{x}-y_\omega|\left(1-\frac{\operatorname{const}(1-\omega)^2\,q^2}{r^2}\right) \geq \\
&\geq r_\omega - |x-y_\omega|\left(1-\frac{\operatorname{const}(1-\omega)^2\,q^2}{r^2}\right) - \frac{7\overline{C}_0}{R}\,.
\end{aligned}$$

With this estimate, we now complete the proof of (A.97) in the case in which x is inside $\Sigma(Y_\omega, R_1)$ and inside Γ and $\tilde{y}$ lies on Γ_2 by arguing as follows. Since $\tilde{x}\in\Sigma(Y_\omega,R_\omega)$ thanks to (A.76), we can take $\xi\in\ell$ on the segment joining $\tilde{y}$ to $\tilde{x}$. Then,

$$\begin{aligned}
\sigma - d_\Gamma(x) &= -|x-z| + |x-\tilde{y}| = \\
&= |x-y_\omega| - r_1 + |x-\tilde{y}| \geq \\
&\geq |x-y_\omega| - r_1 + |\tilde{x}-\tilde{y}| - \frac{6\overline{C}_0}{R} \geq \\
&\geq |x-y_\omega| - r_1 + |\tilde{x}-\xi| - \frac{6\overline{C}_0}{R} \geq \\
&\geq |x-y_\omega| - r_\omega + |\tilde{x}-\xi| - \frac{11\overline{C}_0}{R} \geq \\
&\geq |x-y_\omega|\,\frac{\operatorname{const}(1-\omega)^2 q^2}{r^2} - \frac{18\overline{C}_0}{R}\,.
\end{aligned}$$

Since $x\in[-l,l]^N$, $|x-y_\omega|\geq \operatorname{const} r$, therefore the above estimate yields the proof of (A.97) in the case in which x is inside $\Sigma(Y_\omega,R_1)$ and inside Γ and $\tilde{y}$ lies on Γ_2. This ends the proof of (A.97).

Then, (A.97), the fact that $R_1 \geq R/4$, (2.39) and (A.95) imply that

$$\begin{aligned}
& g_{\mathbb{S}(Y_\omega,R_1)}(x) \geq \\
\geq\ & g_{y_{N+1},R}\left(d_\Gamma(x) + \frac{3\overline{C}_0}{R} - \frac{\overline{C}_0}{2}\left(\frac{1}{R}+\frac{1}{R_1}\right)\right) = \\
=\ & g_{y_{N+1},R}\Big(d_\Gamma(x) + H_{y_{N+1},R}(0) + \frac{\overline{C}_0}{2R} y_{N+1}^2 + \\
& + \frac{3\overline{C}_0}{R} - \frac{\overline{C}_0}{2}\left(\frac{1}{R}+\frac{1}{R_1}\right)\Big) \geq \\
\geq\ & g_{y_{N+1},R}\left(d_\Gamma(x) + H_{y_{N+1},R}(0) + \frac{\overline{C}_0}{2R} y_{N+1}^2 + \frac{3\overline{C}_0}{R} - \frac{5\overline{C}_0}{2R}\right) > \\
>\ & g_{y_{N+1},R}\big(d_\Gamma(x) + H_{y_{N+1},R}(0)\big) = \\
=\ & g_\Psi(x)\,.
\end{aligned}$$

This ends the proof of (A.94).

We now observe that

$$\begin{array}{l} \mathbb{S}(Y_\omega,R_1)\cap\{|x'|\geq \check{C}q\} \\ \text{is strictly above } \Psi \text{ (in the } e_{N+1}\text{-direction).} \end{array} \tag{A.98}$$

To prove this, take x in the domain here above and notice that, recalling (A.74), it is enough to show that

$$g_{\mathbb{S}(Y_\omega,R_1)}(x) > g_{\mathbb{S}(Y,R)}(x)\,. \tag{A.99}$$

Hence, we will focus on the proof of (A.99). Note that, by (A.73)

$$\begin{aligned}
& |x-y_\omega| - |x-y| + R - R_1 = \\
=\ & d_{\Sigma(Y_\omega,R_\omega)}(x) + r_\omega - d_{\Sigma(Y,R)}(x) - r + R - R_1 = \\
=\ & d_{\Sigma(Y_\omega,R_\omega)}(x) - d_{\Sigma(Y,R)}(x) + \frac{\overline{C}_0}{2R} y_{N+1}^2 - \frac{5\overline{C}_0}{2R} \geq \\
\geq\ & d_{\Sigma(Y_\omega,R_\omega)}(x) - d_{\Sigma(Y,R)}(x) - \frac{5\overline{C}_0}{2R} \geq \\
\geq\ & \text{const}\,\frac{q^2}{R} - \frac{5\overline{C}_0}{2R} > 0\,,
\end{aligned}$$

if q is large enough. Thus, recalling (A.75) and (2.14), for any $\tau \in [s_{R_1}, 1]$,

$$|x-y_\omega| - |x-y| + R - R_1 > H_{y_{N+1},R_1}(\tau) - H_{y_{N+1},R}(\tau)\,. \tag{A.100}$$

Let now

$$\tau := g_{y_{N+1},R}\Big(H_0(y_{N+1}) + |x-y| - R\Big) = g_{\mathbb{S}(Y,R)}(x)\,.$$

If $\tau < s_{R_1}$, then (A.99) holds true and we are done; if $\tau \geq s_{R_1}$, we infer from (A.100) that

$$\begin{aligned}
H_{y_{N+1},R_1}(\tau) &< |x-y_\omega| - |x-y| + R - R_1 + H_{y_{N+1},R}(\tau) = \\
&= |x-y_\omega| - R_1 + H_0(y_{N+1})\,,
\end{aligned}$$

hence,

$$\tau \ < \ g_{y_{N+1},R_1}\Big(|x-y_\omega| - R_1 + H_0(y_{N+1})\Big) = g_{\mathbb{S}(Y_\omega,R_1)}(x)\,.$$

From this and the definition of τ, (A.99) follows. We have therefore completed the proof of (A.98).

We now consider the domain $\{|x'| \le \check{C}q\} \times \{|x_N| \le l/2\}$ and we slide Ψ from $-\infty$ in the e_N direction, until we touch u for the first time in such domain (this must happen since $|u(x_0)| < 1/2$): say, for fixing the notations, that for some $\beta \in \mathbb{R}$, $\Psi - \beta e_N$ touches for the first time the graph of u by above at a point Z. Notice that, by the hypotheses of Lemma A.1 and (A.42), we have that

$$u(x_0) = g_{\mathbb{S}(Y,R)}(x_0) = g_\Psi(x_0)\,, \tag{A.101}$$

therefore we have that $\beta \ge 0$. More precisely, it holds that

$$\beta > 0\,. \tag{A.102}$$

Indeed, if, by contradiction, $\beta = 0$, (A.101) says that Ψ touches the graph of u from above at x_0 (which is, by construction, an interior point): thence, by (A.60), $d_\Gamma(x_0)$ must be realized on $\Gamma_1 \cup \Gamma_3$. But, by construction, $d_\Gamma(x_0)$ is realized at $T_{Y,R}(x_0) \in \Gamma_2$: this contradiction proves (A.102).

Note now that, if $\mathfrak{x}$ is in the domain of Ψ, then, by (2.16), we have that

$$d_\Gamma(\mathfrak{x}) \le H_{y_{N+1},R}(1) - H_{y_{N+1},R}(0) \le \text{ const}\,(1+\log R) \le \text{ const}\,(1+\log l)\,.$$

So, if $\mathfrak{x}_N \ge 1$, the fact that Γ is below $\{x_N = 1\}$ implies that

$$d_\Gamma(\mathfrak{x}) \ge \mathfrak{x}_N - 1$$

and so

$$\mathfrak{x}_N \le \text{ const}\,(1+\log l)\,.$$

This says that the domain of Ψ is below the hyperplane

$$\Big\{x_N = \text{ const}\,(1+\log l)\Big\}$$

with respect to the e_N-direction. Since $\beta \ge 0$, also the domain of $\Psi - \beta e_N$ is below the hyperplane

$$\Big\{x_N = \text{ const}\,(1+\log l)\Big\}\,. \tag{A.103}$$

What is more, since $|z_N| \le l$, by (A.21), we gather that

$$z_N \ge -l \ge -cR \ge y_N\,, \tag{A.104}$$

provided c in (A.21) is small enough. Thence, from (A.102) and (A.104),

$$\begin{aligned} |z+\beta e_N - y|^2 &= |z-y|^2 + \beta^2 + 2\beta\,(z_N - y_N) > \\ &> |z-y|^2\,, \end{aligned}$$

that is

$$|z-y| < |z+\beta e_N - y|\,. \tag{A.105}$$

We now prove that

$$|z'| < \check{C} q\,. \tag{A.106}$$

Indeed, if (A.106) were false, then (A.74) and (A.105) would yield that

$$\begin{aligned}
u(z) &= g_\Psi(z+\beta e_N) = \\
&= g_{\mathbb{S}(Y,R)}(z+\beta e_N) = \\
&= g_{y_{N+1},R}\Big(|z+\beta e_N - y| - R + H_{y_{N+1},R}(0)\Big) > \\
&> g_{y_{N+1},R}\Big(|z-y| - R + H_{y_{N+1},R}(0)\Big) = \\
&= g_{\mathbb{S}(Y,R)}(z) \geq \\
&\geq u(z)\,.
\end{aligned}$$

This contradiction proves (A.106).

By means of (A.106), (A.103) and (A.28), we have that z lies in the interior of $\{|x'| \leq \check{C} q\} \times \{|x_N| \leq l/2\}$.

We now claim that

$$|z_{N+1}| < 1/2 \text{ and } d_\Gamma(z+\beta e_N) \text{ is realized on } \Gamma_3. \tag{A.107}$$

Indeed, since Ψ touches by above $u+\beta e_N$ at $\overline{z} := z+\beta e_N$, (A.60) implies that

$$1/2 > |\overline{z}_{N+1}| = |z_{N+1}|$$

and that $d_\Gamma(\overline{z})$ is realized on $\Gamma_1 \cup \Gamma_3$. To prove (A.107), we thus have to show that $d_\Gamma(\overline{z})$ is not realized on Γ_1. If, by contradiction, $d_\Gamma(\overline{z})$ were realized on Γ_1, from (A.42) and (A.105), we would have that

$$\begin{aligned}
u(z) &= g_\Psi(z+\beta e_N) = \\
&= g_{\mathbb{S}(Y,R)}(z+\beta e_N) = \\
&= g_{y_{N+1},R}\Big(|z+\beta e_N - y| - R + H_{y_{N+1},R}(0)\Big) > \\
&> g_{y_{N+1},R}\Big(|z-y| - R + H_{y_{N+1},R}(0)\Big) = \\
&= g_{\mathbb{S}(Y,R)}(z) \geq \\
&\geq u(z)\,.
\end{aligned}$$

This contradiction completes the proof of (A.107).

We now claim that

$$|Y - Y_\omega| - \beta \geq 0\,. \tag{A.108}$$

In order to get this, first observe that if $\beta \leq 5l$, then (A.108) follows by noticing that $|Y - Y_\omega| \geq \text{const}\, r$ and by taking l/R suitably small. Therefore, we restrict ourselves to the proof of (A.108) under the additional assumption that

$$\beta \geq 5l\,. \tag{A.109}$$

For this scope, note that (A.109) implies that

$$(z+\beta e_N)_N \geq \beta - l \geq 4l\,,$$

hence, recalling (A.107),

$$d_{\Gamma_3}(z+\beta e_N) = d_\Gamma(z+\beta e_N) > 0\,. \tag{A.110}$$

Let us observe now that, by (A.107),

$$\begin{aligned} & g_{y_{N+1},R}\Big(H_0(y_{N+1})+|z-y|-R\Big) = \\ = \;& g_{\mathbb{S}(Y,R)}(z) \geq \\ \geq \;& u(z) = \\ = \;& g_\Psi(z+\beta e_N) = \\ = \;& g_{y_{N+1},R}\Big(H_0(y_{N+1})+d_\Gamma(z+\beta e_N)\Big) = \\ = \;& g_{y_{N+1},R}\Big(H_0(y_{N+1})+d_{\Gamma_3}(z+\beta e_N)\Big)\,, \end{aligned}$$

thence

$$|z-y|-R \geq d_{\Gamma_3}(z+\beta e_N)\,. \tag{A.111}$$

If we now take any $x \in \Gamma_3$, we have that $x_N \leq l$ and thus

$$(z+\beta e_N - x)_N \geq \beta - 2l\,.$$

This and (A.110) imply that

$$d_{\Gamma_3}(z+\beta e_N) \geq \beta - 2l\,.$$

For this reason, recalling (A.111), we deduce that

$$\begin{aligned} \beta &\leq 2l + d_{\Gamma_3}(z+\beta e_N) \leq \\ &\leq 2l + |z-y| - R = \\ &= 2l + \mathrm{dist}(z, \partial B_r(y)) + r - R \leq \\ &\leq 2l + |z - T_{y,R}x_0| + r - R \leq \\ &\leq 5l\,. \end{aligned}$$

This ends the proof of (A.108), by taking l/R suitably small.

Let us now consider, for $t \ge 0$, the surface $\mathbb{S}(Y + te_N, R_1)$. For $t = 0$, this surface is above $\mathbb{S}(Y, R)$ since, thanks to (A.75) and (2.14),

$$\begin{aligned} g_{\mathbb{S}(Y,R)}(x) &= g_{y_{N+1},R}\Big(H_0(y_{N+1}) + |x - y| - R\Big) \le \\ &\le g_{y_{N+1},R}\Big(H_0(y_{N+1}) + |x - y| - R_1\Big) \le \\ &\le g_{y_{N+1},R_1}\Big(H_0(y_{N+1}) + |x - y| - R_1\Big) = \\ &= g_{\mathbb{S}(Y,R_1)}(x) . \end{aligned}$$

Since $\mathbb{S}(Y, R)$ is above the graph of u by our hypotheses in $\{|x'| \le \check{C}q\} \times \{|x_N| \le l/2\}$, we thus deduce that $\mathbb{S}(Y + te_N, R_1)$ is, for $t = 0$, above the graph of u in $\{|x'| \le \check{C}q\} \times \{|x_N| \le l/2\}$. Hence, we may increase t till we touch the graph of u in $\{|x'| \le \check{C}q\} \times \{|x_N| \le l/2\}$. In order to fix the notation, say this happen for $t = t_1 \ge 0$ and let X_1 be the above mentioned touching point. Set also

$$Y_1 := Y + t_1 e_N .$$

Let also

$$\tilde{r}_1 := R_1 + H_{y_{N+1},R_1}(-1/2) - H_0(y_{N+1}) , \tag{A.112}$$

so that

$$\Big\{g_{\mathbb{S}(Y_1,R_1)} = -\frac{1}{2}\Big\} = \partial B_{\tilde{r}_1}(y_1) .$$

Then, since $u(x_0) > -1/2$ by assumption, the first touching property of X_1 implies that x_0 is above $\partial B_{\tilde{r}_1}(y_1)$. In particular, recalling (A.26), since

$$y_1 + (x_0', 0) + \sqrt{\tilde{r}_1^2 - |x_0'|^2}\, e_N \in \partial B_{\tilde{r}_1}(y_1) ,$$

we have, using (A.24), that

$$\begin{aligned} y_{1,N} + \sqrt{\tilde{r}_1^2 - 4q^2} &\le y_{1,N} + \sqrt{\tilde{r}_1^2 - |x_0'|^2} \le \\ &\le x_{0,N} \le \\ &\le \text{const} . \end{aligned} \tag{A.113}$$

By construction, the domain of $\mathbb{S}(Y_1, R_1)$ lies below the hyperplane

$$\Big\{x_N = y_{1,N} + R_1 - H_0(y_{N+1}) + H_{y_{N+1},R_1}(1)\Big\} ,$$

thus, (A.112), (A.113) and (2.16) imply that the domain of $\mathbb{S}(Y_1, R_1)$ lies below the hyperplane $\Big\{x_N = \text{const}\,(1 + \log R_1)\Big\}$ and therefore below the hyperplane

$$\Big\{x_N = \text{const}\,(1 + \log l)\Big\} . \tag{A.114}$$

Notice also that, when $Y + te_N = Y_\omega - \beta e_N$, (A.92) and (A.107) imply that

$$\begin{aligned} g_{\mathbb{S}(Y+te_N,R_1)}(z) &= g_{\mathbb{S}(Y_\omega-\beta e_N,R_1)}(z) = \\ &= g_{\mathbb{S}(Y_\omega,R_1)}(z+\beta e_N) \leq \\ &\leq g_\Psi(z+\beta e_N) = \\ &= u(z)\,. \end{aligned}$$

This and (A.108) say that

$$0 \leq t_1 \leq |Y - Y_\omega| - \beta\,. \tag{A.115}$$

We also have that

$$X_1 \in \{|x'| < \breve{C}q\}\,. \tag{A.116}$$

Indeed, if (A.116) were false, then $|x'_\sharp| \geq \breve{C}q$, where

$$x_\sharp := x_1 + |Y_\omega - Y_1|\, e_N\,. \tag{A.117}$$

Thus, in the light of (A.98), we would get that

$$g_{\mathbb{S}(Y_\omega,R_1)}(x_\sharp) > g_\Psi(x_\sharp)\,.$$

Therefore,

$$\begin{aligned} u(x_1) &= g_{\mathbb{S}(Y_1,R_1)}(x_1) = \\ &= g_{\mathbb{S}(Y_\omega-|Y_\omega-Y_1|e_N,R_1)}(x_1) = \\ &= g_{\mathbb{S}(Y_\omega-\hat\beta e_N,R_1)}(x_1) = \\ &= g_{\mathbb{S}(Y_\omega,R_1)}(x_1+\hat\beta e_N) = \\ &= g_{\mathbb{S}(Y_\omega,R_1)}(x_\sharp) > \\ &> g_\Psi(x_\sharp) = \\ &= g_\Psi(x_1+\hat\beta e_N) \geq \\ &\geq u(x_1)\,. \end{aligned} \tag{A.118}$$

This contradiction proves (A.116).

Notice also that $x_{1,N} > -l/4$, otherwise, from (A.116), (A.27), (A.75) and Lemma 2.4, we would get

$$s_{R_1} > s_R = u(x_1) = g_{\mathbb{S}(Y_1,R_1)}(x_1) \geq s_{R_1}\,,$$

which is, of course, a contradiction.

Therefore, recalling also (A.116) and (A.114), we have that x_1 is in the interior of the domain $\{|x'| \leq \breve{C}q\} \times \{|x_N| \leq l/2\}$, thence, by means of Proposition 2.13, $|u(x_1)| < 1/2$.

We now claim that

$$T_{Y_1,R_1}(x_1) \in \{|x'| < \eta q\} \cap \{x_N < \operatorname{const} q^2/R\}\,, \tag{A.119}$$

where $T_{\cdot,\cdot}$ was introduced on page 83 and η in (A.33). To prove (A.119), first observe that, by (A.115), y_1 is below $y_\omega - \beta e_N$ in the e_N-direction, and so $\Sigma(Y_1, R_1)$ is below $\Sigma(Y_\omega - \beta e_N, R_1)$ (and, a fortiori, below $\Sigma(Y_\omega, R_1)$) in the e_N-direction. Thence, by (A.91),

$$T_{Y_1,R_1}(x_1) \in \Sigma(Y_1, R_1) \subseteq \{x_N < \operatorname{const} q^2/R\}. \tag{A.120}$$

This gives a first step towards the proof of (A.119); we now show that

$$T_{Y_1,R_1}(x_1) \in \{|x'| < \eta q\}. \tag{A.121}$$

Assume, by contradiction, that (A.121) is false. Then, by translating in the e_N-direction, we have that

$$T_{Y_\omega,R_1}(x_\sharp) \in \{|x'| \geq \eta q\}, \tag{A.122}$$

where $x_\sharp$ is the one defined in (A.117). Let now

$$\begin{aligned} \hat\beta &:= |Y - Y_\omega| - t_1 = \\ &= |Y_\omega - Y_1|. \end{aligned}$$

Then, $x_\sharp = x_1 + \hat\beta e_N$ and, in the light of (A.115), $\hat\beta \geq \beta$. The latter inequality and the first touching property of β implies that $\Psi - \hat\beta e_N$ is above the graph of u, that is

$$g_\Psi(x + \hat\beta e_N) \geq u(x), \tag{A.123}$$

for any x in the domains of definition of the above functions. With this information, we now derive the desired contradiction. Indeed, from (A.122) and (A.94), we get that

$$g_{\mathbb{S}(Y_\omega,R_1)}(x_\sharp) > g_\Psi(x_\sharp).$$

Thence, using also the touching property of X_1 and (A.123), repeating the argument in (A.118) verbatim, one obtains the contradiction which ends the proof of (A.121). Thus, (A.121) and (A.120) end the proof of (A.119).

The fact that $R_1 \geq R_0/4$, (A.116) and (A.119) end the proof of (A.30) (in case $k = 0$, the other steps being analogous) and, therefore, the proof of (A.29).

With (A.29) in hand, we now complete the proof of Lemma A.1 (and this will still take some effort, the proof ending on page 134). For this, let us make some estimates on the point x_* found in (A.29).

Since x_* is a touching point between the barrier and u, we have that $|u(x_*)| \leq 1/2$ thanks to Corollary 2.14 and, therefore,

$$|\nabla u(x_*)| = |\nabla g_{\mathbb{S}(Y_*,R_*)}(x_*)| \neq 0.$$

We now show that

$$\left| T_{Y_*,R_*} x_* - x_* + \frac{\nabla u(x_*)}{|\nabla u(x_*)|} H_0(u(x_*)) \right| \leq \frac{\operatorname{const}}{R}. \tag{A.124}$$

Note that (A.124) is obviously fulfilled if $T_{Y_*,R_*} x_* = x_*$, since, in this case

$$H_0(u(x_*)) = H_0(g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)) = H_0(g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(T_{Y_*,R_*} x_*)) = H_0(0) = 0.$$

Hence, we focus on the proof of (A.124) under the assumption $T_{Y_*,R_*}x_* \neq x_*$. For this scope, first notice that $x_* - y_*$ is, by construction, parallel to $x_* - T_{Y_*,R_*}x_*$, that is

$$\frac{x_* - y_*}{|x_* - y_*|} = \frac{x_* - T_{Y_*,R_*}x_*}{|x_* - T_{Y_*,R_*}x_*|}\,. \tag{A.125}$$

Also, from (A.19),

$$\Big|x_* - T_{Y_*,R_*}x_*\Big| \leq \text{ const}\,. \tag{A.126}$$

In the light of (2.57) and (A.125),

$$\frac{\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)}{|\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)|} = \frac{x_* - y_*}{|x_* - y_*|} = \frac{x_* - T_{Y_*,R_*}x_*}{|x_* - T_{Y_*,R_*}x_*|}\,. \tag{A.127}$$

Let us now define $W(x) := H_0(g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x))$. By the touching property of x_*, the fact that $T_{Y_*,R_*}x_* \in \Sigma(Y_*,R_*)$ and (A.127), we have that

$$\begin{aligned}
&\Big|T_{Y_*,R_*}x_* - x_* + \frac{\nabla u(x_*)}{|\nabla u(x_*)|}\,H_0(u(x_*))\Big| = \\
&= \Big|T_{Y_*,R_*}x_* - x_* + \frac{\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)}{|\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)|}\,W(x_*)\Big| = \\
&= \Big|\frac{\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)}{|\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)|}\,\Big[|T_{Y_*,R_*}x_* - x_*| + W(x_*)\Big]\Big| = \\
&= \Big||T_{Y_*,R_*}x_* - x_*| + W(x_*)\Big| = \\
&= \Big||T_{Y_*,R_*}x_* - x_*| + W(x_*) - W(T_{Y_*,R_*}x_*)\Big|\,.
\end{aligned}$$

Therefore, a second order Taylor expansion of W, (2.54), (A.126), (A.127) and (2.53) give

$$\begin{aligned}
&\Big|T_{Y_*,R_*}x_* - x_* + \frac{\nabla u(x_*)}{|\nabla u(x_*)|}\,H_0(u(x_*))\Big| \le \\
\le\ &\Big||T_{Y_*,R_*}x_* - x_*| - \\
&-\nabla W(x_*)(T_{Y_*,R_*}x_* - x_*)\Big| + \\
&+\mathrm{const}\,|D^2 W|\,|T_{Y_*,R_*}x_* - x_*|^2 \le \\
\le\ &\Big||T_{Y_*,R_*}x_* - x_*| - \\
&-H_0'(g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*))\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)\cdot(T_{Y_*,R_*}x_* - x_*)\Big| + \\
&+\frac{\mathrm{const}}{R} \le \\
\le\ &\Big|\frac{T_{Y_*,R_*}x_* - x_*}{|T_{Y_*,R_*}x_* - x_*|}\Big[T_{Y_*,R_*}x_* - x_* - \\
&-H_0'(g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*))\,|\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)|\cdot(T_{Y_*,R_*}x_* - x_*)\Big]\Big| + \\
&+\frac{\mathrm{const}}{R} = \\
=\ &\Big|T_{Y_*,R_*}x_* - x_* - \\
&-H_0'(g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*))\,|\nabla g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)|\cdot(T_{Y_*,R_*}x_* - x_*)\Big| + \\
&+\frac{\mathrm{const}}{R} \le \\
\le\ &\Big|T_{Y_*,R_*}x_* - x_* - (T_{Y_*,R_*}x_* - x_*)\Big| + \frac{\mathrm{const}}{R}\,,
\end{aligned}$$

that is (A.124).

We now show that

$$\angle\left(\frac{\nabla u(x_*)}{|\nabla u(x_*)|},\, e_N\right) \le \frac{\mathrm{const}\,q}{R}\,. \tag{A.128}$$

For proving this, let us note that

$$\begin{aligned}
|T_{Y_*,R_*}x_* - y_*| &= r(Y_*,R_*) = \\
&= R_* - H_0(y_{N+1}) - \frac{\overline{C}_0}{2R}\,y_{N+1}^2 \ge \\
&\ge\ \mathrm{const}\,R_*\,,
\end{aligned} \tag{A.129}$$

hence,

$$\begin{aligned}\sin\left[\angle(T_{Y_*,R_*}x_*-y_*,\,e_N)\right] &= \\ &= \frac{|(T_{Y_*,R_*}x_*-y_*)'|}{|T_{Y_*,R_*}x_*-y_*|} = \\ &= \frac{|(T_{Y_*,R_*}x_*)'|}{|T_{Y_*,R_*}x_*-y_*|} \le \\ &\le \frac{\mathrm{const}\, q}{R}\,.\end{aligned} \tag{A.130}$$

Also, in analogy with (A.127), we have that

$$\frac{\nabla u(x_*)}{|\nabla u(x_*)|} = \frac{g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)}{|g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*)|} = \frac{T_{Y_*,R_*}x_*-y_*}{|T_{Y_*,R_*}x_*-y_*|}\,,$$

which, together with (A.130), proves (A.128).

We now observe that, thanks to (A.124) and (A.128), we have

$$\begin{aligned}& x_*\cdot e_N \le \\ &\le (T_{Y_*,R_*}x_*)\cdot e_N + \frac{\nabla u(x_*)}{|\nabla u(x_*)|}\cdot e_N\, H_0(u(x_*)) + \frac{\mathrm{const}}{R} \le \\ &\le (T_{Y_*,R_*}x_*)\cdot e_N + H_0(u(x_*)) + \frac{\mathrm{const}\, q^2}{R^2} + \frac{\mathrm{const}}{R}\,.\end{aligned} \tag{A.131}$$

Let now consider the set

$$\begin{aligned}\Xi_* \;:=\; & \Big\{ \bar Y = (\bar y', 0, \bar y_{N+1}) \in \mathbb{R}^{N+1} \;\text{ s.t.} \\ & |(\bar y - T_{Y_*,R_*}x_*)'| \le c\, q\,,\;\; |\bar y_{N+1}| \le 1/4 \Big\}\end{aligned} \tag{A.132}$$

and let

$$\breve R := cR \tag{A.133}$$

with $c>0$ suitably small. Recalling the definition of $r(\cdot,\cdot)$ given in (A.18), we also denote $\breve r := r(\breve Y, \breve R)$.

For any $\bar Y \in \Xi_*$, let us slide $\mathbb{S}(\bar Y, \breve R)$ from $-\infty$ in the e_N direction, until it touches the graph of u by above for the first time, and let $\breve Y$ denote the center of the corresponding barrier: more explicitly, $\breve Y = \bar Y - t e_N$, for some $t \in \mathbb{R}$ and $\mathbb{S}(\breve Y, \breve R)$ touches the graph of u from above for the first time coming from $-\infty$ in the e_N direction. Say, also, to fix notations, that such touching occurs at some point $\breve X$. We will denote by $\breve \Xi$ the set collecting all the points $\breve X$ which lie in the interior of the domain of u, when $\bar Y$ varies in Ξ_*

We know from (A.29) that

$$\Big(T_{Y_*,R_*}x_*\Big)_N \le C_\sharp q^2/R \tag{A.134}$$

for some suitably large constant $C_\sharp$. We now claim that, if $\bar{Y} \in \Xi_*$, then

$$\left(\left(T_{Y_*,R_*}x_*\right)', 2C_\sharp\,\frac{q^2}{R}\right) \text{ is outside } \Sigma(\breve{Y}, \breve{R}). \tag{A.135}$$

The proof of (A.135) (which is pretty long and will be completed only on page 126) is by contradiction. If (A.135) were false, Lemma B.7 would imply that

$$\begin{aligned} &\Sigma(\breve{Y}, \breve{R}) \cap \{|(x - T_{Y_*,R_*}x_*)'| \le cq\} \cap \{x_N \ge \breve{y}_N\} \\ &\text{is above the hyperplane } \{x_N = 3C_\sharp q^2/(2R)\}, \end{aligned} \tag{A.136}$$

provided c is small enough[22]. We now show that

$$\begin{aligned} &d_{\Sigma(\breve{Y},\breve{R})}(x_*) \le \\ &\le H_0(u(x_*)) + \text{const}\,\frac{q}{R} - \frac{C_\sharp\, q^2}{2R}\,. \end{aligned} \tag{A.137}$$

To prove this (and some effort will be needed), we distinguish two cases: either x_* is in the exterior or it is in the interior of $\Sigma(\breve{Y}, \breve{R})$.

Let us first assume that x_* is not in the interior of $\Sigma(\breve{Y}, \breve{R})$. We first point out that

$$x_* \cdot e_N \;\ge\; \frac{3C_\sharp q^2}{2R}\,. \tag{A.138}$$

For this, note that, by (A.126),

$$\left|\left(x_* - T_{Y_*,R_*}x_*\right)'\right| \;\le\; cq$$

if q is large enough. This, (A.136) and the assumption that x_* is not in the interior of $\Sigma(\breve{Y}, \breve{R})$ would imply that either (A.138) holds or x_* is below $\Sigma(\breve{Y}, \breve{R})$. Thus, to complete the proof of (A.138), we show now that the latter possibility cannot hold. Indeed, note that the first touching property of $\mathbb{S}(\breve{Y}, \breve{R})$ and Lemma 2.25 imply that

$$\breve{y} \cdot e_N \le \breve{x} \cdot e_N$$

while (A.19) says that

$$|\breve{x} - \breve{y}| \;\ge\; \text{const}\,\breve{R}\,.$$

Furthermore, in analogy with (A.128), we have that

$$\angle\big(\breve{x} - \breve{y},\, e_N\big) \;\le\; \frac{\text{const}\, q}{R} \tag{A.139}$$

[22] The reader will indeed notice that, due to (A.132),

$$\breve{y} + re_N \;\in\; \Sigma(\breve{Y}, \breve{R}) \cap \{|(x - T_{Y_*,R_*}x_*)'| \le cq\} \cap \{x_N \ge y_N\} \;\ne\; \emptyset\,.$$

Then, the above estimates yield that

$$
\begin{aligned}
(x_* - \breve{y}) \cdot e_N \geq (\breve{x} - \breve{y}) \cdot e_N - \operatorname{const} l = \\
= \left|(\breve{x} - \breve{y}) \cdot e_N\right| - \operatorname{const} l \geq \\
\geq \frac{1}{2}\left|\breve{x} - \breve{y}\right| - \operatorname{const} l \geq \\
\geq \operatorname{const} \breve{R} - \operatorname{const} l\,,
\end{aligned}
\tag{A.140}
$$

thus showing that $(x_* - \breve{y}) \cdot e_N > 0$ and thus that x_* is not below $\Sigma(\check{Y}, \breve{R})$. This ends the proof of (A.138).

With this, we now go back to the proof of (A.137) when x_* is not in the interior of $\Sigma(\check{Y}, \breve{R})$. By means of (A.136), we have that the point $\Big((T_{Y_*,R_*} x_*)',\, 3C_\sharp q^2/(2R)\Big)$ lies inside $\Sigma(\check{Y}, \breve{R})$, thus

$$
d_{\Sigma(\check{Y},\breve{R})}(x_*) \leq \left| x_* - \left(\Big(T_{Y_*,R_*} x_*\Big)', \frac{3C_\sharp q^2}{2R}\right)\right|\,,
$$

and therefore, recalling (A.138),

$$
d_{\Sigma(\check{Y},\breve{R})}(x_*) \leq |(x_* - T_{Y_*,R_*} x_*)'| + x^* \cdot e_N - \frac{3C_\sharp\, q^2}{2R}.
$$

We recall that, by (A.128),

$$
\angle\left((x_* - T_{Y_*,R_*}),\, e_N\right) = \angle\left(\frac{\nabla u(x_*)}{|\nabla u(x_*)|},\, e_N\right) \leq \frac{\operatorname{const} q}{R}\,,
$$

and so, by means of (A.126), we have that

$$
\left|\Big((T_{Y_*,R_*} x_*) - x_*\Big)'\right| \leq \operatorname{const} \frac{q}{R}\,. \tag{A.141}
$$

Therefore, recalling (A.126) and taking into account (A.131),

$$
d_{\Sigma(\check{Y},\breve{R})}(x_*) \leq \operatorname{const} \frac{q^2}{R} + T_{Y_*,R_*} x_* \cdot e_N + H_0(u(x_*)) - \frac{3C_\sharp\, q^2}{2R}.
$$

Thence, if x_* is not in the interior of $\Sigma(\check{Y}, \breve{R})$, (A.137) follows by (A.134) and by taking $C_\sharp$ suitably large. If otherwise x_* is in the interior of $\Sigma(\check{Y}, \breve{R})$, that is, if

$$
d_{\Sigma(\check{Y},\breve{R})}(x_*) < 0\,, \tag{A.142}
$$

in order to prove (A.137), we argue as follows. We first note that, in this case,

$$
u(x_*) \leq g_{\widetilde{\mathbb{S}}(\check{Y},\breve{R})}(x_*) < 0
$$

and consequently

$$
g_{\widetilde{\mathbb{S}}(Y_*,R_*)}(x_*) = u(x_*) < 0\,,
$$

so that x_* is also in the interior of $\Sigma(Y_*, R_*)$. Therefore, recalling (A.128), it follows that

$$
x_* \cdot e_N < (T_{Y_*,R_*} x_*) \cdot e_N\,. \tag{A.143}
$$

We now deduce that, when x_* is in the interior of $\Sigma(\breve{Y},\breve{R})$, one has that

$$-d_{\Sigma(\breve{Y},\breve{R})}(x_*) \geq \min\left\{\frac{cq}{4}, \frac{3C_\sharp q^2}{2R} - x_* \cdot e_N\right\}. \tag{A.144}$$

To prove (A.144), take $x \in \Sigma(\breve{Y},\breve{R})$. Obviously, we may and do assume that

$$|x| \leq \text{const}\, l\,, \tag{A.145}$$

otherwise

$$|x_* - x| \geq \text{const}\, l$$

and so (A.144) trivially follows from (A.142). There are now two possibilities: either x is above the hyperplane $\{x_N = 3C_\sharp q^2/(2R)\}$ or the converse. If x is above the hyperplane $\{x_N = 3C_\sharp q^2/(2R)\}$, then

$$|x - x_*| \geq x_N - x_* \cdot e_N \geq \frac{3C_\sharp q^2}{2R} - x_* \cdot e_N\,,$$

which, together with (A.142), proves (A.144) in case x is above the hyperplane $\{x_N = 3C_\sharp q^2/(2R)\}$. If, on the other hand, x is not above the hyperplane $\{x_N = 3C_\sharp q^2/(2R)\}$, by (A.136), we have that either $|(x - T_{Y_*,R_*}x_*)'| \geq cq$ or $x_N \leq \breve{y}_N$. However, the latter cannot hold, since, by Lemma 2.16, (A.145), (A.139) and (A.140), we have that

$$\begin{aligned} x_N - \breve{y}_N &= |\breve{x}_N - \breve{y}_N| + x_N - \breve{x}_N \geq \\ &\geq \frac{1}{2}|\breve{x} - \breve{y}| + x_N - \breve{x}_N \geq \\ &\geq \text{const}\, R - \text{const}\, l > 0\,. \end{aligned}$$

Therefore, from these considerations, we have that if x is not above the hyperplane $\{x_N = 3C_\sharp q^2/(2R)\}$, then $|(x - T_{Y_*,R_*}x_*)'| \geq cq$. For this reason, recalling (A.141),

$$\begin{aligned} \big|(x - x_*)'\big| &\geq \Big|\big(x - (T_{Y_*,R_*}x_*)\big)'\Big| - \Big|\big((T_{Y_*,R_*}x_*) - x_*\big)'\Big| \geq \\ &\geq cq - \frac{\text{const}\, q}{R} \geq \frac{cq}{2}\,, \end{aligned}$$

which, together with (A.143) and (A.134), yields the proof of (A.144) also when x is not above the hyperplane $\{x_N = 3C_\sharp q^2/(2R)\}$. This ends the proof of (A.144).

We now exploit (A.131), (A.134) and (A.144) to get that

$$-d_{\Sigma(\breve{Y},\breve{R})}(x_*) \geq -H_0(u(x_*)) - \text{const}\,\frac{q}{R} + \frac{C_\sharp q^2}{2R}\,.$$

This ends the proof of (A.137) also in the case in which x_* is in the interior of $\Sigma(\breve{Y},\breve{R})$.

Having completed the proof of (A.137), we now deduce the contradiction that will finish the proof of (A.135). To this end, note that, by (A.19), letting $s_* := u(x_*)$, we have that the signed distance between the s_*-level set of $g_{\mathbb{S}(\check{Y},\check{R})}$ and $\Sigma(\check{Y},\check{R})$ is bigger than

$$H_0(u(x_*)) - \frac{\text{const}}{R} .$$

Thus, from (A.137), by taking q and $C_\sharp$ suitably large, we have that

$$d_{\Sigma(\check{Y},\check{R})}(\mathfrak{x}) \,>\, d_{\Sigma(\check{Y},\check{R})}(x_*) ,$$

for any $\mathfrak{x}$ so that $g_{\mathbb{S}(\check{Y},\check{R})}(\mathfrak{x}) = s_*$. Therefore, x_* is strictly in the interior of the s_*-level set of $g_{\mathbb{S}(\check{Y},\check{R})}$, that is

$$g_{\mathbb{S}(\check{Y},\check{R})}(x_*) < s_* = u(x_*) .$$

This contradicts the fact that $g_{\mathbb{S}(\check{Y},\check{R})}$ touches u by above and, thus, yields the proof of (A.135).

In the light of (A.135), some elementary trigonometry implies that $y+\check{r}e_N$ is below the hyperplane $\{x_N = 4C_\sharp q^2/R\}$ and therefore

$$\Sigma(\check{Y},\check{R}) \text{ is below } \{x_N = 4C_\sharp q^2/R\}. \tag{A.146}$$

Let us now fix a small constant $c_* > 0$. If q/R is assumed to be small enough (possibly in dependence of $c_*/C_\sharp$), then (A.135) and Corollary B.9 imply that

$$\Sigma(\check{Y},\check{R}) \cap \{|x'| \geq c_* q\} \text{ is below } \{x_N = 0\}. \tag{A.147}$$

We now show that the above geometric considerations imply that

$$\begin{aligned} &\text{outside } \{|x'| \leq c_* q\} \times \{x_N > 0\},\\ &\Sigma(\check{Y},\check{R}) \text{ is at distance greater than } q^2/(4R)\\ &\text{in the interior of } \Sigma(Y,R). \end{aligned} \tag{A.148}$$

For the proof of (A.148), first notice that

$$\check{y}_N - \check{r} \,\geq\, -\text{const}\,\check{r} \,\geq\, y_N \tag{A.149}$$

if c in (A.133) is chosen suitably small. Let us now consider the surface

$$\Sigma_t(\check{Y},\check{R}) \,:=\, \Sigma(\check{Y},\check{R}) + te_N$$

with $t \geq 0$. By (A.149), it follows that, given $\mathfrak{x} \in \Sigma(\check{Y},\check{R})$, then $\mathfrak{x}_N \geq y_N$ and so

$$|(\mathfrak{x} + te_N) - y| \geq |\mathfrak{x} - y| .$$

Hence,

$$d_{\Sigma(Y,R)}(\mathfrak{x}) \leq d_{\Sigma(Y,R)}(\mathfrak{x} + te_N) . \tag{A.150}$$

Recalling (A.147), we now select the first $t \geq 0$ for which

$$\mathfrak{S} \,:=\, \Sigma_t(\check{Y},\check{R}) \cap \{|x'| = c_* q\} \cap \{x_N = 0\} \,\neq\, \emptyset . \tag{A.151}$$

We also denote by $\mathfrak{S}^-$ the portion of $\Sigma_t(\breve{Y},\breve{R})$ which is below $\{x_N = 0\}$ (with respect to the e_N-direction), that is the portion of $\Sigma_t(\breve{Y},\breve{R})$ which is below $\mathfrak{S}$. The choice in (A.151) implies that

$$\breve{y}_N + t + \breve{r} \geq 0 \tag{A.152}$$

and, by (A.132) and (A.29), that

$$\begin{aligned} |(\breve{y} + te_N)'| &= |\breve{y}'| \leq \\ &\leq \left|(\breve{y} - T_{Y_*,R_*}x_*)'\right| + \left|(T_{Y_*,R_*}x_*)'\right| \leq \\ &\leq cq + \frac{q}{C_*} = \\ &= \tilde{c}_* q\,, \end{aligned} \tag{A.153}$$

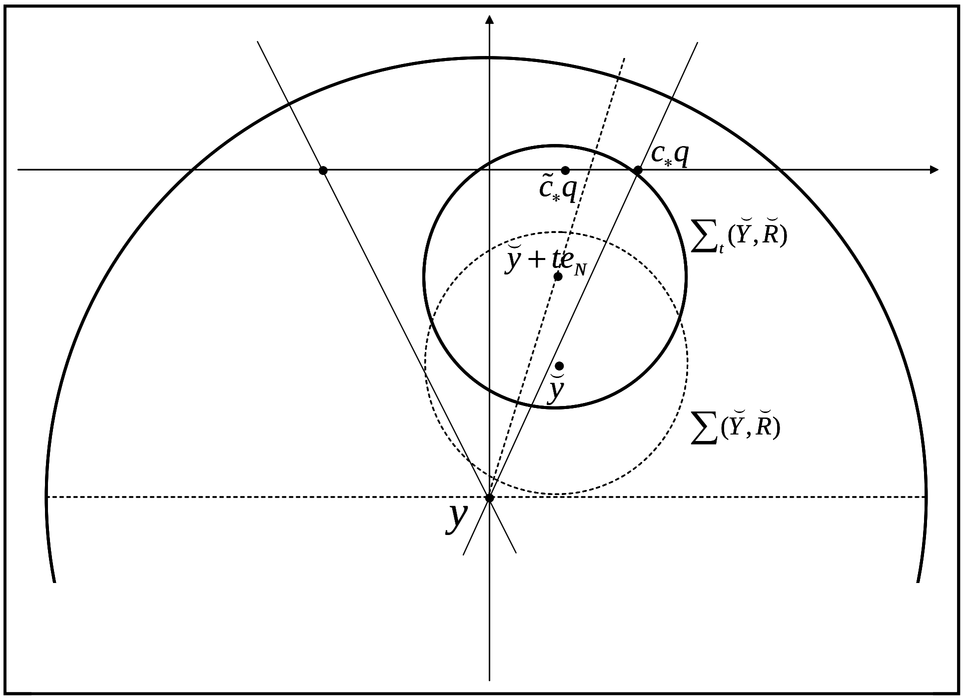
where

$$\tilde{c}_* := c + \frac{1}{C_*} \tag{A.154}$$

is a positive constant, which is small if c and $1/C_*$ are small.

We now show that, with this construction and taking $\tilde{c}_*$ small enough, we have that $\breve{y} + te_N$ is above the cone $\mathfrak{C}_*$ defined as

$$\mathfrak{C}_* := \left\{ x \in \mathbb{R}^N \ \text{ s.t. } |x'| = \frac{c_* q}{\sqrt{r^2 - q^2}}\, |x_N + \sqrt{r^2 - q^2}| \right\}.$$



The proof of (A.148)

Indeed, by (A.152) and (A.153),

$$\begin{aligned}
&\frac{c_* q}{\sqrt{r^2-q^2}}\left(\breve{y}_N + t + \sqrt{r^2-q^2}\right) \geq \\
&\geq \frac{c_* q}{\sqrt{r^2-q^2}}\left(\sqrt{r^2-q^2} - \breve{r}\right) = \\
&= c_* q\left(1 - \frac{\breve{r}}{\sqrt{r^2-q^2}}\right) \geq \\
&\geq \frac{c_* q}{2} \\
&> \tilde{c}_* q \geq \left|(\breve{y}+te_N)'\right|
\end{aligned}$$

if q/R, $\breve{r}/r$, c and $1/C_*$ (and thus $\tilde{c}_*$) are small enough (recall (A.133) and (A.154)).

This fact, (A.151) and some elementary geometric considerations yield that if $x \in \mathfrak{S}^-$, and $x_\sharp \in \mathfrak{S}$, then

$$\left|d_{\Sigma(Y,R)}(x)\right| \geq \left|d_{\Sigma(Y,R)}(x_\sharp)\right| . \tag{A.155}$$

By construction, since $c_* < 1$, if $x_\sharp \in \mathfrak{S}$, then $x_\sharp$ lies inside $\Sigma(Y,R)$ (see the figure on page 84); thus,

$$d_{\Sigma(Y,R)}(x_\sharp) < 0 . \tag{A.156}$$

Also, from (A.151) and the fact that $\breve r < r$, if $x \in \mathfrak{S}^-$, then x is also inside $\Sigma(Y,R)$ and therefore

$$d_{\Sigma(Y,R)}(x) < 0\,. \tag{A.157}$$

Thus, from (A.155), (A.156) and (A.157),

$$d_{\Sigma(Y,R)}(x) \le d_{\Sigma(Y,R)}(x_\sharp)\,, \tag{A.158}$$

for $x \in \mathfrak{S}^-$ and $x_\sharp \in \mathfrak{S}$.

Also, since $y' = 0$ by its definition, we gather that, if $x_\sharp \in \mathfrak{S}$, then

$$|x_\sharp - y|^2 = |y_N|^2 + |x'_\sharp|^2 = r^2 - q^2 + c_*^2 q^2$$

and therefore

$$d_{\Sigma(Y,R)}(x_\sharp) = |x_\sharp - y| - r = \sqrt{r^2 - q^2 + c_*^2 q^2} - r\,. \tag{A.159}$$

Thus, taking $x \in \mathfrak{S}^-$ and $x_\sharp \in \mathfrak{S}$, making use of (A.158) and (A.159), and recalling also (A.35), we have that

$$\begin{aligned} -d_{\Sigma(Y,R)}(x) \ge{} & - d_{\Sigma(Y,R)}(x_\sharp) = \\ ={}& r - r\sqrt{1 - \frac{q^2}{r^2} + \frac{c_*^2 q^2}{r^2}} \ge \\ \ge{}& r - r\left[1 - \frac{q^2}{2r^2} + \frac{c_*^2 q^2}{2r^2} + \Big(- \frac{q^2}{r^2} + \frac{c_*^2 q^2}{r^2}\Big)^2\right] \ge \\ \ge{}& \frac{q^2}{2r}\Big(1 - c_*^2 - \text{const}\,\frac{q^2}{R^2}\Big) > \\ >{}& \frac{q^2}{4R}\,, \end{aligned} \tag{A.160}$$

provided c_* and q/R are sufficiently small.

To complete the proof of (A.148), take now $\mathfrak{x} \in \Sigma(\breve Y, \breve R)$ outside $\{|x'| \le c_* q\} \times \{x_N > 0\}$ and let $x := \mathfrak{x} + t e_N$ Then $x \in \mathfrak{S}^-$ by the choice in (A.151), thus, from (A.150) and (A.160), we gather that

$$-d_{\Sigma(Y,R)}(\mathfrak{x}) \ \ge\ -d_{\Sigma(Y,R)}(x) \ >\ \frac{q^2}{4R}\,.$$

This ends the proof of (A.148).

Let us now prove that

$$\begin{gathered}\text{the } s\text{-level surface of } g_{\mathbb{S}(Y,R)} \text{ is at distance} \\ \text{greater than } H_{y_{N+1},\breve R}(s) \text{ from } \Sigma(Y,R).\end{gathered} \tag{A.161}$$

In order to prove this, take any $\hat x$ in the s-level surface of $g_{\mathbb{S}(Y,R)}$, that is assume that

$$g_{y_{N+1},R}(H_0(y_{N+1}) + |\hat x - y| - R) = s\,.$$

Let also $\bar x \in \Sigma(Y,R)$, that is

$$g_{y_{N+1},R}(H_0(y_{N+1}) + |\bar x - y| - R) = 0\,,$$

and assume that $\bar{x}$ lies on the half-line from y towards $\hat{x}$. Then,

$$\begin{aligned} d_{\Sigma(Y,R)}(\hat{x}) &= |\hat{x}-y| - |\bar{x}-y| = \\ &= (H_0(y_{N+1}) + |\hat{x}-y| - R) - (H_0(y_{N+1}) + |\bar{x}-y| - R) = \\ &= H_{y_{N+1},R}(s) - H_{y_{N+1},R}(0). \end{aligned}$$

Hence, by the fact that $\breve{R} \le R$, (2.14) and (2.39), we get that

$$\begin{aligned} d_{\Sigma(Y,R)}(\hat{x}) &\ge H_{y_{N+1},\breve{R}}(s) - H_{y_{N+1},R}(0) = \\ &= H_{y_{N+1},\breve{R}}(s) + \frac{\overline{C}_0}{2R}\, y_{N+1}^2 \ge \\ &\ge H_{y_{N+1},\breve{R}}(s), \end{aligned}$$

and this ends the proof of (A.161).

We now show that

$$\text{if } |s| \le 1/2, \text{ the } s\text{-level surface of } g_{\mathbb{S}(\breve{Y},\breve{R})} \text{ is at distance less than } H_{y_{N+1},\breve{R}}(s) + \text{const}\, \overline{C}_0/(2R) \text{ from } \Sigma(\breve{Y},\breve{R}). \tag{A.162}$$

Indeed, we argue as in the proof of (A.161), by taking now $\hat{x}$ in the s-level surface of $g_{\mathbb{S}(\breve{Y},\breve{R})}$ and $\bar{x} \in \Sigma(\breve{Y},\breve{R})$, with $\bar{x}$ lying on the half-line from y towards $\hat{x}$, and by arguing as above, we have that

$$\begin{aligned} d_{\Sigma(\breve{Y},\breve{R})}(\hat{x}) &= |\hat{x}-\breve{y}| - |\bar{x}-\breve{y}| = \\ &= (H_0(y_{N+1}) + |\hat{x}-\breve{y}| - \breve{R}) - (H_0(y_{N+1}) + |\bar{x}-\breve{y}| - \breve{R}) = \\ &= H_{\breve{y}_{N+1},\breve{R}}(s) - H_{\breve{y}_{N+1},\breve{R}}(0) = \\ &= H_{\breve{y}_{N+1},\breve{R}}(s) + \frac{\overline{C}_0}{2\breve{R}}\, \breve{y}_{N+1}^2 \le \\ &\le H_{\breve{y}_{N+1},\breve{R}}(s) + \text{const}\, \frac{\overline{C}_0}{2R}. \end{aligned}$$

By the assumption that $|s| \le 1/2$ and (2.39), we may now estimate the quantity $H_{\breve{y}_{N+1},\breve{R}}(s)$ here below with $H_{y_{N+1},\breve{R}}(s) + \overline{C}_0/R$, and this ends the proof of (A.162).

We now deduce from the above estimates that

$$\begin{aligned} &\text{at any } x \text{ for which} \\ &|g_{\mathbb{S}(\breve{Y},\breve{R})}(x)| \le 1/2 \text{ and} \\ &d_{\Sigma(\breve{Y},\breve{R})}(x) - d_{\Sigma(Y,R)}(x) \ge \text{const}\, \overline{C}_0/R, \\ &\text{we have that} \\ &g_{\mathbb{S}(\breve{Y},\breve{R})}(x) > g_{\mathbb{S}(Y,R)}(x). \end{aligned} \tag{A.163}$$

To prove this, take x as in (A.163) here above and let

$$\begin{aligned} \breve{s} &:= g_{\mathbb{S}(\breve{Y},\breve{R})}(x) \in [-1/2,\ 1/2] \\ s &:= g_{\mathbb{S}(Y,R)}(x). \end{aligned} \tag{A.164}$$

Then, by (A.161) and (A.162),

$$
\begin{aligned}
& d_{\Sigma(\breve{Y},\breve{R})}(x) < H_{y_{N+1},R}(\breve{s}) + \mathrm{const}\,\overline{C}_0/R \qquad \text{and} \\
& d_{\Sigma(Y,R)}(x) > H_{y_{N+1},R}(s) - \mathrm{const}\,\overline{C}_0/R\,,
\end{aligned} \tag{A.165}
$$

hence, for the assumption in (A.163),

$$
H_{y_{N+1},R}(s) < H_{y_{N+1},R}(\breve{s})\,,
$$

which proves (A.163) via the monotonicity of $H_{y_{N+1},R}$ and (A.164).

Next, notice that, thanks to (A.148) and (A.163), we have that

$$
\begin{aligned}
& \mathbb{S}(\breve{Y},\breve{R}) \cap \{|x_{N+1}| \le 1/2\} \cap \{|x'| > c_* q\} \\
& \text{is strictly above } \mathbb{S}(Y,R),
\end{aligned} \tag{A.166}
$$

provided q is large enough. Hence, since $u \le g_{\mathbb{S}(Y,R)}$, (A.166) implies that

$$
\begin{aligned}
& \mathbb{S}(\breve{Y},\breve{R}) \cap \{|x_{N+1}| \le 1/2\} \cap \{|x'| > c_* q\} \\
& \text{is strictly above the graph of } u.
\end{aligned} \tag{A.167}
$$

We now apply the previous considerations for deducing some properties of the contact points $\breve{X}$ and of the contact point set $\breve{\Xi}$ (recall the notation on page 122). First of all,

$$
|u(\breve{x})| \; < \; 1/2\,, \tag{A.168}
$$

thanks to Corollary 2.14. This and (A.167) imply that

$$
|\breve{x}| \; \le \; c_* q\,, \tag{A.169}
$$

while (A.146) yields

$$
T_{\breve{Y},\breve{R}}\breve{x} \in \Sigma(\breve{Y},\breve{R}) \subseteq \{x_N < 4C_\sharp q^2/R\}\,. \tag{A.170}
$$

Also, from (A.19) and (A.168), we get that

$$
\Big| |\breve{x}-\breve{y}| - r \Big| \; \le \; \mathrm{const}\,,
$$

and, consequently, $|\breve{x}-\breve{y}| \ge r/2$, if r is large. From this circumstance, recalling the touching properties of $\breve{x}$, (2.57), and (A.169), we gather that

$$
\begin{aligned}
\sin\left(\angle\Big(\frac{\nabla u(\breve{x})}{|\nabla u(\breve{x})|}\,,\, e_N \Big) \right) \;&=\; \sin\left(\angle\Big(\frac{\nabla g_{\mathbb{S}(\breve{Y},\breve{R})}(\breve{x})}{|\nabla g_{\mathbb{S}(\breve{Y},\breve{R})}(\breve{x})|}\,,\, e_N \Big) \right) = \\
&=\; \sin\left(\angle\Big(\frac{\breve{x}-\breve{y}}{|\breve{x}-\breve{y}|}\,,\, e_N \Big) \right) = \\
&=\; \frac{|\breve{x}'-\breve{y}'|}{|\breve{x}-\breve{y}|} \le \\
&\le\; \mathrm{const}\,\frac{q}{r}\,.
\end{aligned}
$$

Thence,

$$\angle\left(\frac{\nabla u(\breve{x})}{|\nabla u(\breve{x})|}\,,\ e_N\right)\ \le\ \mathrm{const}\,\frac{q}{R}\,. \tag{A.171}$$

Analogously, if x_0 is the point in the statement of Lemma A.1, one sees that

$$\angle\left(\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\,,\ e_N\right)\ \le\ \mathrm{const}\,\frac{q}{R}\,. \tag{A.172}$$

In particular, from (A.171) and (A.172) it follows that

$$\angle\left(\frac{\nabla u(\breve{x})}{|\nabla u(\breve{x})|}\,,\ \frac{\nabla u(x_0)}{|\nabla u(x_0)|}\right)\ \le\ \mathrm{const}\,\frac{q}{R}\,. \tag{A.173}$$

In addition, from Proposition 3.14,

$$\mathfrak{L}^N\Big(\pi_N(\breve{\Xi})\Big)\ \ge\ \mathrm{const}\,\mathfrak{L}^N(\Xi_*)\ \ge\ \mathrm{const}(cq)^{N-1}\,. \tag{A.174}$$

What is more, arguing as in the proof of (A.124), one can see that

$$\begin{aligned}\Big|T_{\breve{Y},\breve{R}}\breve{x}-\breve{x}+\frac{\nabla u(\breve{x})}{|\nabla u(\breve{x})|}\,H_0(u(\breve{x}))\Big| &\le \frac{\mathrm{const}}{R}\qquad\text{and}\\ \Big|T_{Y,R}x_0-x_0+\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\,H_0(u(x_0))\Big| &\le \frac{\mathrm{const}}{R}\,.\end{aligned} \tag{A.175}$$

Hence, from (A.175), (A.170) and the fact that $T_{Y,R}x_0\cdot e_N=0$ (recall the assumptions in Lemma A.1), we have that

$$\begin{aligned}(\breve{x}-x_0)\cdot e_N \le{}& \Big(T_{\breve{Y},\breve{R}}\breve{x}-T_{Y,R}x_0+\\ &+\frac{\nabla u(\breve{x})}{|\nabla u(\breve{x})|}\,H_0(u(\breve{x}))-\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\,H_0(u(x_0))\Big)\cdot e_N+\\ &+\frac{\mathrm{const}}{R}\le\\ &\le\Big(\frac{\nabla u(\breve{x})}{|\nabla u(\breve{x})|}\,H_0(u(\breve{x}))-\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\,H_0(u(x_0))\Big)\cdot e_N+\\ &+\frac{4\,C_\sharp\,q^2}{R}+\frac{\mathrm{const}}{R}\,,\end{aligned}$$

and so, thanks to (A.168), (A.171) and (A.172),

$$\begin{aligned}&(\breve{x}-x_0)\cdot e_N\le\\ &\le H_0(u(\breve{x}))-H_0(u(x_0))+\frac{\mathrm{const}\,q^2}{R^2}+\frac{4\,C_\sharp\,q^2}{R}+\frac{\mathrm{const}}{R}\le\\ &\le H_0(u(\breve{x}))-H_0(u(x_0))+\frac{5\,C_\sharp\,q^2}{R}\,,\end{aligned} \tag{A.176}$$

by assuming $C_\sharp$, q and R/q suitably large.

Moreover, in the light of (A.26), (A.169) and (A.176), and recalling that $|u(\breve{x})| < 1/2$ and $|u(x_0)| < 1/2$, we have

$$|\breve{x}-x_0| \le |(\breve{x}-x_0)'|+|(\breve{x}-x_0)_N| \le (1+2c_*)q+|(\breve{x}-x_0)_N| \le \breve{C}q\,, \tag{A.177}$$

where $\breve{C}$ is a conveniently large constant as in the statement of the Lemma. By means of (A.176) and (A.172), we conclude that

$$(\breve{x}-x_0)\cdot\frac{\nabla u(x_0)}{|\nabla u(x_0)|} \le H_0(u(\breve{x}))-H_0(u(x_0))+\frac{6\,C_\sharp\,q^2}{R}\,. \tag{A.178}$$

Let us now consider the set Ξ defined in the statement of Lemma A.1 and take c_* conveniently small: then, the reader may easily convince herself that, thanks to (A.169), (A.168), (A.177), (A.173) and (A.178), we have that

$$\Xi \supseteq \breve{\Xi};$$

the proof of (A.22)–(A.23) is thus ended by means of (A.174).

To finish the proof of Lemma A.1, we need now to check the *Lipschitz graph* property of $\Xi_s := \breve{\Xi}\cap\{x_{N+1}=s\}$, for any $|s|<1/2$. The graph property is a straightforward consequence of the *first occurrence* touching point property, hence we focus on the Lipschitz estimate. For this, take $\breve{X},\ \bar{X}\in\Xi_s$; then,

$$s=u(\bar{x})=g_{\mathbb{S}(\bar{Y},\breve{R})}(\bar{x})=g_{\mathbb{S}(\breve{Y},\breve{R})}(\breve{x})=u(\breve{x})\,,$$

for suitable $\breve{Y},\bar{Y}\in\mathbb{R}^{N+1}$, so that $\pi_N\breve{Y},\ \pi_N\bar{Y}\in\Xi_*$. We will prove that

$$|\breve{x}_N-\bar{x}_N| \le \frac{\mathrm{const}\,q}{R}|(\breve{x}-\bar{x})'|\,, \tag{A.179}$$

which implies the desired Lipschitz estimate (with constant $\mathrm{const}\,q/R<1$). With no loss of generality, we may and do assume that

$$\bar{x}_N > \breve{x}_N\,. \tag{A.180}$$

Let us define

$$\bar{r} \;:=\; \breve{R}+H_{\bar{y}_{N+1},\breve{R}}(s)-H_0(\bar{y}_{N+1})\,,$$

so that

$$\{g_{\mathbb{S}(\bar{Y},\breve{R})}=s\} \;=\; \partial B_{\bar{r}}(\bar{y})\,.$$

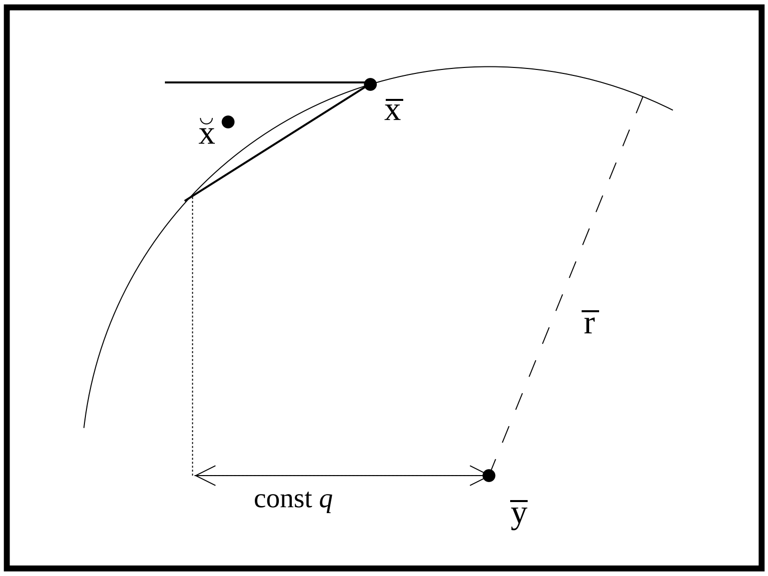
By (2.57) and (A.171), we have that

$$|\bar{x}'-\bar{y}'| \le \mathrm{const}\,q\,, \tag{A.181}$$

and therefore, from (A.181), we get that

$$\begin{aligned}|\breve{x}'-\bar{y}'| &\le |\breve{x}'-x_0'|+|x_0'-\bar{x}'|+|\bar{x}'-\bar{y}'| \le\\ &\le \mathrm{const}\,q\,,\end{aligned} \tag{A.182}$$

which is less then $\bar{r}$, if q/R is small enough. Thus, by the first occurrence touching property of $g_{\mathbb{S}(\bar{Y},\breve{R})}$, $\breve{x}$ must be above $\{g_{\mathbb{S}(\bar{Y},\breve{R})}=s\}$ (with respect to the e_N direction). This, (A.182) and (A.180) imply that $\breve{x}$ is trapped inside a cone with vertex in $\bar{x}$ and slope bounded by const $q/\bar{r}$, which gives (A.179). This ends the proof of Lemma A.1. □



The cone trapping $\breve{x}$

By a rotation/translation argument, we deduce from Lemma A.1 the following

Lemma A.2. *Fix $o \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$, with $|\xi| = 1$. Let $\breve{C} > 1$ be a suitably large constant. Let u be a C^1-subsolution of* (1.5) *in*

$$\{x \in \mathbb{R}^N \ \ s.t. \ |(x-o)\cdot\xi| < l \ \ and \ |(x-o) - ((x-o)\cdot\xi)\xi| < l\}\,.$$

Assume that $\mathbb{S}(Y,R)$ is above u in

$$\{x \in \mathbb{R}^N \ \ s.t. \ |(x-o)\cdot\xi| \le 1/2 \ \ and \ |(x-o) - ((x-o)\cdot\xi)\xi| \le \breve{C}l\}\,.$$

and that $\mathbb{S}(Y,R)$ touches the graph of u at the point $(x_0,\, u(x_0))$.

Suppose that $|u(x_0)| < 1/2$, $|(x_0-o)\cdot\xi| < l/4$, $|(x_0-o)-((x_0-o)\cdot\xi)\xi| < l/4$.

Assume also that

$$T_{Y,R}x_0 \in \{|(x-o) - ((x-o)\cdot\xi)\xi| = q\} \cap \{(x-o)\cdot\xi = 0\} \qquad and$$

$$y = o - \sqrt{r^2 - q^2}\,\xi \qquad with$$

$$r = r(Y,R) \ = \ R - H_0(y_{N+1}) - \frac{\overline{C}_0}{2R}y^2_{N+1}\,.$$

Then, there exist universal constants C_1, $C_2 > 1 > c > 0$ such that, if

$$q \ge C_1 \qquad and \qquad l \le cR\,,$$

the following holds. Let Ξ be the set of points $(\mathfrak{x}, u(\mathfrak{x})) \in \mathbb{R}^N \times \mathbb{R}$ satisfying the following properties:

- $|(\mathfrak{x} - o) \cdot \xi| < q/15$, $|u(\mathfrak{x})| < 1/2$, $|\mathfrak{x} - x_0| < \breve{C} q$;
- *there exists* $\hat{Y} \in \mathbb{R}^{N+1}$ *such that* $\mathbb{S}(\hat{Y}, R/C_2)$ *is above* u *and it touches* u *at* $(\mathfrak{x}, u(\mathfrak{x}))$;
- $\angle\left(\dfrac{\nabla u(\mathfrak{x})}{|\nabla u(\mathfrak{x})|}, \dfrac{\nabla u(x_0)}{|\nabla u(x_0)|}\right) \le \dfrac{C_1\, q}{R}$;
- $(\mathfrak{x} - x_0) \cdot \dfrac{\nabla u(x_0)}{|\nabla u(x_0)|} \le \dfrac{C_1\, q^2}{R} + H_0(u(\mathfrak{x})) - H_0(u(x_0))$.

Then,

$$\mathfrak{L}^N\Big(\pi_\xi(\Xi)\Big) \ \ge\ c q^{N-1}.$$

More precisely, for any $s \in (-1/2, 1/2)$, there exists a set $\Xi_s \subseteq \Xi \cap \{x_{N+1} = s\}$, which is contained in a Lipschitz graph in the ξ-direction, with Lipschitz constant less than 1, and so that, if

$$\breve{\Xi} := \bigcup_{s \in (-1/2, 1/2)} \Xi_s,$$

we have that

$$\mathfrak{L}^N\Big(\pi_\xi(\breve{\Xi})\Big) \ \ge\ c q^{N-1}.$$

With Lemma A.2 in hand, we can now complete the proof of Lemma 4.2, by arguing as follows.

Let us now consider a vector $\xi \in \mathbb{R}^N$ so that $|\xi| = 1$ and

$$\vartheta := \angle(\xi, T_{Y_0,R} x_0 - y_0) = \arcsin \frac{q}{r},$$

being x_0, Y_0 and R the ones in the statement of Lemma 4.2. We also denote $r = r(Y_0, R)$, according to the definition of $r(\cdot,\cdot)$ given in (A.18).

Observe that, by the assumptions of Lemma 4.2 and (2.57),

$$\begin{aligned}\angle(\xi, e_N) &\le \angle(\xi, T_{Y_0,R} x_0 - y_0) + \angle(T_{Y_0,R} x_0 - y_0, e_N) = \\ &= \vartheta + \angle\Big(\frac{\nabla u(x_0)}{|\nabla u(x_0)|}, e_N\Big) \le \\ &\le \text{const}\,\frac{q}{R} + \frac{\pi}{8} \le \frac{\pi}{6}.\end{aligned} \tag{A.183}$$

Define

$$o := y_0 + \sqrt{r^2 - q^2}\,\xi.$$

We will think o as the origin. Then,

$$|T_{Y_0,R} x_0 - o| = |y_0 - T_{Y_0,R} x_0| \sin\vartheta = r \cdot \frac{q}{r} = q \tag{A.184}$$

and so

$$
\begin{aligned}
|T_{Y_0,R}x_0 - y_0|^2 &= r^2 = \\
&= |o - y_0|^2 + q^2 = \\
&= |o - y_0|^2 + |T_{Y_0,R}x_0 - o|^2 \,.
\end{aligned} \tag{A.185}
$$

Then, (A.185) says that the triangle with vertices in y_0, o and $T_{Y_0,R}x_0$ is a right triangle in o, that is

$$
(T_{Y_0,R}x_0 - o) \cdot \xi \;=\; 0\,. \tag{A.186}
$$

Also, if

$$
\check{q} \;:=\; |(T_{Y_0,R}x_0 - o) - ((T_{Y_0,R}x_0 - o)\cdot\xi)\xi|\,,
$$

then (A.183) and (A.19) give that $q \sim \check{q}$. Then, the hypotheses of Lemma A.2 being fulfilled (with $\check{q}$ replacing q, possibly scaling l to $\mathrm{const}\, l$) thanks to (A.183), (A.184) and (A.186). Thence, we deduce that,

$$
\mathfrak{L}^N(\pi_\xi(\Xi)) \;\geq\; \mathrm{const}\, q^{N-1}\,. \tag{A.187}
$$

Also, in the light of the Lipschitz graph property in Lemma A.2, we have that $\Xi \supseteq \breve{\Xi}$, for an appropriate set $\breve{\Xi}$, with

$$
\breve{\Xi} \cap \{x_{N+1} = s\} \;=\; \Xi_s \;=\; F_s(\Xi_s^\sharp)\,,
$$

for any $|s| < 1/2$, for some

$$
\Xi_s^\sharp \subseteq \{x\cdot\xi = 0\} \cap \{x_{N+1} = s\}
$$

and $|F_s|_{\mathrm{Lip}} \leq 1$. Note that the $\Xi_s^\sharp$'s are all disjoint $(N-1)$-dimensional sets, lying on $\{x_{N+1} = s\}$.

Given a, $b \in \Xi_s^\sharp$, let now $a' := F_s(a)$ and $b' := F_s(b)$. Then, the fact that F_s gives a Lipschitz graph in the ξ-direction with $|F_s|_{\mathrm{Lip}} \leq 1$ implies that

$$
\angle\big(a' - b',\, \xi\big) \;\geq\; \frac{\pi}{4}\,.
$$

Therefore, by means of (A.183),

$$
\angle\big(a' - b',\, e_N\big) \;\geq\; \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}\,.
$$

This says that $\pi_N\Big|_{\Xi_s}$ is invertible and that its inverse is a Lipschitz function, with Lipschitz constant bounded by $1/\tan(\pi/12)$. Therefore, if

$$
G_s \;:=\; \pi_\xi \circ \left(\pi_N\Big|_{\Xi_s}\right)^{-1},
$$

we have that G_s is a Lipschitz function whose range is $\Xi_s^\sharp$, with

$$
|G_s|_{\mathrm{Lip}} \;\leq\; \frac{1}{\tan(\pi/12)} \;\leq\; \mathrm{const}\,.
$$

Hence, using the change of variables formula (see, e.g., page 99 in [18]), we deduce that

$$
\begin{aligned}
\mathrm{const}\, q^{N-1} &\leq \mathfrak{L}^N\Big(\pi_\xi(\breve{\Xi})\Big) = \\
&= \mathfrak{L}^N\Big(\bigcup_{s\in(-1/2,1/2)} \Xi_s^\sharp\Big) = \\
&= \int_{-1/2}^{1/2} \mathfrak{L}^{N-1}(\Xi_s^\sharp)\, ds = \\
&= \int_{-1/2}^{1/2} \int_{\Xi_s^\sharp} dy\, ds \leq \\
&\leq \int_{-1/2}^{1/2} \int_{G_s^{-1}(\Xi_s^\sharp)} |\det G_s'(x)|\, dx\, ds \leq \\
&\leq \mathrm{const} \int_{-1/2}^{1/2} \int_{G_s^{-1}(\Xi_s^\sharp)} dx\, ds = \\
&= \mathrm{const} \int_{-1/2}^{1/2} \mathfrak{L}^{N-1}\big(G_s^{-1}(\Xi_s^\sharp)\big)\, ds = \\
&= \mathrm{const}\, \mathfrak{L}^N\Big(\bigcup_{s\in(-1/2,1/2)} G_s^{-1}(\Xi_s^\sharp)\Big) = \\
&= \mathrm{const}\, \mathfrak{L}^N\Big(\bigcup_{s\in(-1/2,1/2)} \pi_N\big(F_s(\Xi_s^\sharp)\big)\Big) = \\
&= \mathrm{const}\, \mathfrak{L}^N\Big(\bigcup_{s\in(-1/2,1/2)} \pi_N(\Xi_s)\Big) \leq \\
&\leq \mathrm{const}\, \mathfrak{L}^N\big(\pi_N(\Xi)\big)\,,
\end{aligned}
$$

completing the proof of Lemma 4.2.

A.3. **Proof of Lemma 4.3.** Let $F_k \subseteq E_k$ be the closed set defined as

$$F_k := \{Z \in L \mid \mathrm{dist}\,(Z, D_k \cap Q_{l+a}) \leq a\}\,.$$

If $Q_l \setminus F_k = \emptyset$,

$$\mathfrak{L}^N(Q_l \setminus E_k) \leq \mathfrak{L}^N(Q_l \setminus F_k) = 0\,,$$

proving the claim, hence we may and do assume that $Q_l \setminus F_k \neq \emptyset$. Let now $Z \in Q_l \setminus F_k$ and take $Z_* \in F_k$ be so that

$$\mathrm{dist}\,(Z, F_k) = |Z - Z_*| =: r\,. \tag{A.188}$$

We use the notation $Z = (z', 0, z_{N+1}) \in L$ and we claim that

$$r \leq l + |z'| - \frac{a}{2}\,. \tag{A.189}$$

To prove this, we may assume that $r \geq a$, otherwise the claim is proved, and we proceed as follows. First of all, notice that, from (P1) in the statement of Lemma 4.3, we have that there exists $\breve{Z} \in D_k \cap Q_l$. Let $Z_\sharp$ lie on the segment joining Z and $\breve{Z}$, at distance a from $\breve{Z}$. Then,

$$|Z - Z_\sharp| = |Z - \breve{Z}| - a$$

and, therefore, since $\breve{Z} \in Q_l$,

$$\begin{aligned} |Z - Z_\sharp| &\leq \sqrt{(z' - \breve{z}')^2 + 1} - a \leq \\ &\leq \sqrt{(|z'| + |\breve{z}'|)^2 + 1} - a \leq \\ &\leq \sqrt{(|z'| + l)^2 + 1} - a \,. \end{aligned} \tag{A.190}$$

Also, by construction, $Z_\sharp \in F_k$ and thus

$$r \leq |Z - Z_\sharp| \,. \tag{A.191}$$

The proof of (A.189) now follows from (A.190) and (A.191) by taking $a > 2$.

Notice now that, since $Z \in Q_l$, (A.189) implies that

$$r \leq 2l - \frac{a}{2} \,. \tag{A.192}$$

We now claim that

$$\mathfrak{L}^N \Big(F_{k+1} \cap Q_l \cap B_r(Z) \Big) \geq \bar{c}\, \mathfrak{L}^N \Big(Q_l \cap B_r(Z) \Big) \,, \tag{A.193}$$

for a suitable $\bar{c} \in (0,1)$, which may depend on the quantity c introduced in (P2) during the statement of Lemma 4.3.

We now begin with the proof of (A.193), which will be completed on page 142.

Since $Z_* \in F_k$, there exists $Z_0 \in D_k \cap Q_{l+a}$ be so that $|Z_* - Z_0| \leq a$. We point out that, in fact,

$$|Z_* - Z_0| = a \,. \tag{A.194}$$

Indeed, if, by contradiction, $|Z_* - Z_0| < a$, we have that $B_d(Z_*) \cap L \subseteq F_k$ for some $d > 0$, from which it would exists $\hat{Z} \in F_k$ so that $|\hat{Z} - Z| < |Z_* - Z|$, that contradicts the definition of Z_* and proves (A.194).

Also, from (A.188) and (A.194),

$$|Z - Z_0| \leq |Z - Z_*| + |Z_* - Z_0| = r + a \,. \tag{A.195}$$

We notice that, in fact,

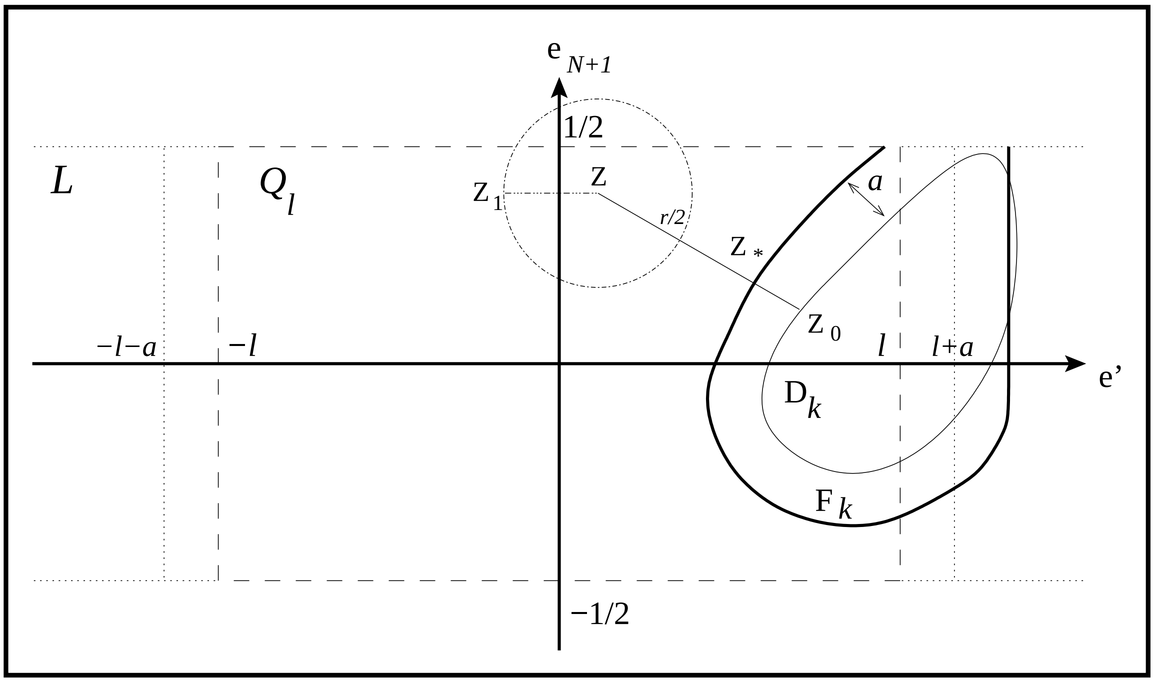
$$|Z - Z_0| = r + a \,. \tag{A.196}$$

Indeed, if it holded that $|Z - Z_0| < r + a$, take $\tilde{Z} \in \partial B_a(Z_0)$ on the segment joining Z and Z_0: then $\tilde{Z} \in F_k$, since $Z_0 \in D_k \cap Q_{l+a}$, and

$$|Z - \tilde{Z}| = |Z - Z_0| - a < r \,,$$

which contradicts the definition of r and proves (A.196).

Notice that, thanks to (A.195) and (A.196), we have that Z_* belongs to the segment joining Z and Z_0.



The proof of Lemma 4.3

To continue with the proof of (A.193), we need now to distinguish two cases: either $a \le 2r$ or $a > 2r$. Let us first deal with the case $a \le 2r$. We claim that there exists $Z_1 \in Q_l$ so that

$$|Z - Z_1| = \frac{r}{2} \qquad \text{and} \qquad B_{r/2}(Z_1) \cap L \subseteq Q_l \,. \tag{A.197}$$

To prove (A.197), we need to distinguish two sub-cases. If $z' = 0$, take $Z_1 := Z + re_1/2$. Then,

$$|z_{1,N+1}| = |z_{N+1}| \le \frac{1}{2}$$

and, recalling (A.192),

$$|z_1'| = \frac{r}{2} < l \,,$$

showing that $Z_1 \in Q_l$ in this case. Also, exploiting (A.189), if $W = (w', 0, w_{N+1}) \in B_{r/2}(Z_1)$,

$$|w'| \le r \le l \,,$$

proving (A.197) in this sub-case. If, on the other hand, $z' \neq 0$, take

$$Z_1 := Z - \frac{r\, z'}{2\,|z'|} \,;$$

thus,

$$|z_{1,N+1}| = |z_{N+1}| \le \frac{1}{2}$$

and, since $Z \in Q_l$, recalling also (A.189), we have that

$$\begin{aligned} |z_1'| &= \left|\frac{z'}{|z'|}\Big(|z'| - \frac{r}{2}\Big)\right| = \\ &= \left||z'| - \frac{r}{2}\right| = \\ &= \max\left\{|z'| - \frac{r}{2}, \frac{r}{2} - |z'|\right\} \leq \\ &\leq \max\left\{l - \frac{r}{2}, \frac{l}{2}\right\} < \\ &< l\,, \end{aligned}$$

which shows that $Z_1 \in Q_l$ in this case.

Let also $W = (w', 0, w_{N+1}) \in B_{r/2}(Z_1) \cap L$; then, we have that

$$\begin{aligned} |w'| &\leq |z_1'| + \frac{r}{2} = \\ &= \left|\frac{z'}{|z'|}\Big(|z'| - \frac{r}{2}\Big)\right| + \frac{r}{2} = \\ &= \left||z'| - \frac{r}{2}\right| + \frac{r}{2} = \\ &= \max\left\{|z'| - \frac{r}{2}, \frac{r}{2} - |z'|\right\} + \frac{r}{2} = \\ &= \max\left\{|z'|, r - |z'|\right\}. \end{aligned}$$

Hence, using that $Z \in Q_l$ and (A.189), we deduce from the above that

$$|w'| \leq l\,,$$

proving (A.197) in this sub-case.

This completes the proof of (A.197).

As a consequence of (A.197), we immediately infer from the fact that $B_{r/2}(Z_1) \cap L \subseteq Q_l$ that

$$\hat{W} := Z_1 + \frac{z_1'}{|z_1'|}\,\frac{r}{2} \in Q_l$$

and therefore

$$l \geq |\hat{w}'| = \left||z_1'| + \frac{r}{2}\right| = |z_1'| + \frac{r}{2}\,,$$

that is

$$|z_1'| \leq l - \frac{r}{2}\,. \tag{A.198}$$

We show that this yields that

$$|z_1' - z_0'| \leq 2l - 2a\,. \tag{A.199}$$

Indeed, if $r > 2(l - 3a)/3$, (A.198) and the fact that $Z_0 \in Q_{l+a}$ imply that

$$|z_1' - z_0'| \leq |z_1'| + |z_0'| \leq 2l - \frac{r}{2} + a \leq \frac{5l}{3} + 2a\,,$$

which proves (A.199) in this case; if, on the other hand, $r \le 2(l-3a)/3$, we have, by (A.196) and (A.197), that

$$|z_1' - z_0'| \le |z_1' - z'| + |z' - z_0'| \le |Z_1 - Z| + |Z - Z_0| = \frac{r}{2} + r + a \le l - 2a$$

in this case. This ends the proof of (A.199).

We now complete the proof of (A.193) in the case $a \le 2r$, by arguing as follows. We notice that, by construction, and recalling (A.196) and (A.197),

$$\begin{aligned} a + \frac{r}{2} &= |Z - Z_0| - |Z_1 - Z| \le \\ &\le |Z_1 - Z_0| \le \\ &\le |Z_1 - Z| + |Z - Z_0| = \\ &= \frac{r}{2} + r + a \le \\ &\le 5r\,. \end{aligned} \tag{A.200}$$

Furthermore,[23] using (A.199), we have that

$$\begin{aligned} |Z_1 - Z_0| &\le \sqrt{|z_1' - z_0'|^2 + 1} \le \\ &\le \sqrt{(2l - 2a)^2 + 1} \le \\ &\le 2l\,. \end{aligned} \tag{A.201}$$

Now notice that, from (A.200),

$$\mathfrak{L}^N\Big(D_{k+1} \cap B_{r/2}(Z_1)\Big) \ge \mathfrak{L}^N\Big(D_{k+1} \cap B_{|Z_1 - Z_0|/10}(Z_1)\Big)\,. \tag{A.202}$$

Furthermore, by (A.197), $B_{r/2}(Z_1) \cap L$ contains a circular sector of height 1 of a ball of radius $r/2$, while, on the other hand, $B_r(Z) \cap Q_l$ is contained in a circular sector of height 1 of a ball of radius r. Therefore,

$$\begin{aligned} \mathfrak{L}^N\Big(B_{r/2}(Z_1) \cap L\Big) &\ge \operatorname{const} r^{N-1} \ge \\ &\ge \mathfrak{L}^N\Big(B_r(Z) \cap Q_l\Big)\,. \end{aligned} \tag{A.203}$$

Also, by construction, $B_{r/2}(Z_1) \subseteq B_r(Z)$, $D_{k+1} \subseteq F_{k+1} \cap Q_{l+a}$ and $L \cap B_{r/2}(Z_1) \subseteq Q_l$, therefore,

$$\mathfrak{L}^N\Big(F_{k+1} \cap Q_l \cap B_r(Z)\Big) \ge \mathfrak{L}^N\Big(D_{k+1} \cap B_{r/2}(Z_1)\Big)\,. \tag{A.204}$$

Finally, from (P2) of Lemma 4.3 (which may be used thanks to (A.201) and (A.200)), we have that

$$\begin{aligned} \mathfrak{L}^N\Big(D_{k+1} \cap B_{|Z_1 - Z_0|/10}(Z_1)\Big) &\ge c\,\mathfrak{L}^N\Big(L \cap B_{|Z_1 - Z_0|}(Z_1)\Big) \ge \\ &\ge c\,\mathfrak{L}^N\Big(L \cap B_{r/2}(Z_1)\Big)\,, \end{aligned} \tag{A.205}$$

[23]The reader will notice that (A.201) is needed in order to use property (P2) of Lemma 4.3 here in the sequel.

where, in the latter estimate, we used again (A.200). Then, (A.193) easily follows in this case from (A.204), (A.202), (A.205) and (A.203).

Let us now deal with the case in which $a > 2r$. In this case,

$$\frac{r+a}{10} < a\,. \tag{A.206}$$

Since $Z \in L$ and $a \geq$ const > 0,

$$\mathfrak{L}^N\Big(B_{r+a}(Z) \cap L\Big) \geq \text{const } > 0\,.$$

In particular, by (P2) of Lemma 4.3, (A.196) and (A.206), we infer from this that

$$\begin{aligned} \mathfrak{L}^N\Big(B_{(r+a)/10}(Z) \cap D_{k+1}\Big) &= \mathfrak{L}^N\Big(B_{|Z-Z_0|/10}(Z) \cap D_{k+1}\Big) \geq \\ &\geq \text{const}\, \mathfrak{L}^N\Big(B_{|Z-Z_0|}(Z) \cap L\Big) = \\ &= \text{const}\, \mathfrak{L}^N\Big(B_{r+a}(Z) \cap L\Big) \geq \\ &\geq \text{const } > 0 \end{aligned}$$

thence, there exists

$$Z_\sharp \in D_{k+1} \cap B_{(r+a)/10}(Z)\,.$$

Also, the fact that $Z \in Q_l$ and (A.206) give that

$$D_{k+1} \cap B_{(r+a)/10}(Z) \subseteq Q_{l+a}\,,$$

thence $Z_\sharp \in Q_{l+a}$. Hence, the fact that $Z_\sharp \in B_{(r+a)/10}(Z)$ implies that

$$Q_l \cap B_r(Z) \subseteq Q_l \cap B_a(Z_\sharp)\,, \tag{A.207}$$

while the fact that $Z_\sharp \in D_{k+1}$ implies that

$$Q_l \cap B_a(Z_\sharp) \subseteq F_{k+1}\,. \tag{A.208}$$

Then, from (A.207) and (A.208),

$$Q_l \cap B_r(Z) \subseteq F_{k+1}$$

and, therefore,

$$F_{k+1} \cap Q_l \cap B_r(Z) = Q_l \cap B_r(Z)\,.$$

This proves (A.193) in this case (with $\bar{c} = 1$).

Having completed the proof of (A.193) we now take a finite overlapping cover $\mathfrak{C}$ of $Q_l \setminus F_k$ with balls of radius r, in order to end the proof of Lemma 4.3. Thus, using such cover,

$$\begin{aligned} \mathfrak{L}^N\Big(F_{k+1} \cap (Q_l \setminus F_k)\Big) &= \mathfrak{L}^N\Big(F_{k+1} \cap Q_l \cap (Q_l \setminus F_k)\Big) \geq \\ &\geq \text{const} \sum_{B_r(Z) \in \mathfrak{C}} \mathfrak{L}^N\Big(F_{k+1} \cap Q_l \cap B_r(Z)\Big)\,. \end{aligned}$$

Then, using (A.193), we deduce from the above that

$$\begin{aligned}
\mathrm{const}\,\bar{c}^{-1}\,\mathfrak{L}^N\Big(F_{k+1}\cap(Q_l\setminus F_k)\Big) &\geq \sum_{B_r(Z)\in\mathfrak{C}}\mathfrak{L}^N\Big(Q_l\cap B_r(Z)\Big)\geq\\
&\geq \mathfrak{L}^N\Big(Q_l\cap\Big(\bigcup_{B_r(Z)\in\mathfrak{C}}B_r(Z)\Big)\Big)\geq\\
&\geq \mathfrak{L}^N\Big(Q_l\cap(Q_l\setminus F_k)\Big)=\\
&= \mathfrak{L}^N(Q_l\setminus F_k)\,. \qquad\text{(A.209)}
\end{aligned}$$

Furthermore, since $D_k\subseteq D_{k+1}$, we have that $F_k\subseteq F_{k+1}$ and so

$$Q_l\setminus F_{k+1}\subseteq\Big(Q_l\setminus F_k\Big)\setminus\Big(F_{k+1}\cap(Q_l\setminus F_k)\Big)\,. \qquad\text{(A.210)}$$

Hence, by using (A.210) and (A.209),

$$\begin{aligned}
\mathfrak{L}^N(Q_l\setminus F_{k+1}) &\leq \mathfrak{L}^N(Q_l\setminus F_k)-\mathfrak{L}^N\Big(F_{k+1}\cap(Q_l\setminus F_k)\Big)\leq\\
&\leq (1-\hat{c})\,\mathfrak{L}^N(Q_l\setminus F_k)\,,
\end{aligned}$$

for a suitable $\hat{c}$. Therefore, iterating the above estimate,

$$\mathfrak{L}^N(Q_l\setminus F_k)\leq(1-\hat{c})^k\,\mathfrak{L}^N(Q_l)\,.$$

This completes the proof of Lemma 4.3 since $E_k\supseteq F_k$ by construction.

Appendix B. Summary of elementary lemmata

We collect here some lemmata that are in use during the proofs of the main results. We will skip the proofs of most of these lemmata, since they are quite elementary (a detailed proof of them, however, may be found in [30]).

Lemma B.1. *For any $0\leq s\leq t\leq\theta^\star$,*

$$h_0(-1+t)-h_0(-1+s)\geq c(t^p-s^p)\,,$$

for a suitable universal constant $c>0$.

Lemma B.2. *Let U be an open subset of $\mathbb{R}$. Let $g\in C^2(U)$ and assume that g has no critical points. Define*

$$\Psi^{y,l}(x) := g(|x-y|-l) \qquad\text{(B.1)}$$

Then, for $t=|x-y|-l\in U$ and $x\neq y$, we have

$$\Delta_p(\Psi^{y,l}(x))=(p-1)g''(t){g'}^{(p-2)}(t)+{g'}^{(p-1)}(t)\frac{N-1}{|x-y|} \qquad\text{(B.2)}$$

Lemma B.3. *Let $I\ni 0$ be an interval of $\mathbb{R}$ and let $h\in C^1(I)$ satisfy $h(s)>0$ for any $s\in I$. Let*

$$H(s) := \int_0^s\frac{(p-1)^{1/p}\,d\zeta}{(p\,h(\zeta))^{1/p}}\,,\qquad\forall s\in I\,.$$

Define also g as the inverse of H, that is $g(t) := H^{-1}(t)$ for any $t \in H(I)$. Then, $g \in C^2(H(I))$ and

$$\begin{aligned} g'(t) &= \Big(\frac{p}{p-1}\,h(g(t))\Big)^{1/p} \\ g''(t) &= \frac{\Big(p\,h(g(t))\Big)^{(2-p)/p}}{(p-1)^{2/p}}\,h'(g(t))\,, \end{aligned}$$

for any $t \in H(I)$.

We recall now the maximum and comparison principles needed for our purposes. First of all, in [9] (see in particular Theorem 1.4 there) the following result was obtained:

Theorem B.4 (Strong Comparison Principle I)**.** *Let Ω be an open (not necessarily bounded nor connected) subset of $\mathbb{R}^N$, $\Lambda \in \mathbb{R}$ and $u, v \in C^1(\Omega)$ satisfy*

$$-\Delta_p u + \Lambda u \le -\Delta_p(v) + \Lambda v, \qquad u \le v \text{ in } \Omega. \tag{B.3}$$

Define $Z_{u,v} = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}$ if $p \neq 2$, $Z_{u,v} = \emptyset$ if $p = 2$. If $x_0 \in \Omega \setminus Z_{u,v}$ and $u(x_0) = v(x_0)$, then $u = v$ in the connected component of $\Omega \setminus Z_{u,v}$ containing x_0.

An easy consequence of the above result is the following one (see §3 in [30] for further details):

Corollary B.5 (Strong Comparison Principle II)**.** *Let Ω be an open (not necessarily bounded nor connected) subset of $\mathbb{R}^N$, and $u, v \in C^1(\Omega)$ satisfy*

$$-\Delta_p u + f(u) \le -\Delta_p(v) + f(v), \qquad u \le v \text{ in } \Omega\,, \tag{B.4}$$

with f locally Lipschitz continuous. Define $Z_{u,v} = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}$ if $p \neq 2$, $Z_{u,v} = \emptyset$ if $p = 2$. If $x_0 \in \Omega \setminus Z_{u,v}$ and $u(x_0) = v(x_0)$, then $u = v$ in the connected component of $\Omega \setminus Z_{u,v}$ containing x_0.

As well known, the "dangerous" points in dealing with p-Laplace operators are the ones in which the gradient vanishes, due to lack of ellipticity. Next result, proved in [37] (see also [30] for details), will help us in dealing with this circumstance.

Theorem B.6 (Strong Maximum Principle and Hopf's Lemma)**.** *Let Ω be an open connected (not necessarily) bounded set in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \ge 0$ in Ω, weakly solves*

$$-\Delta_p u + cu^q = g \ge 0 \quad \text{in} \quad \Omega$$

with $q \ge p-1$, $c \ge 0$ and $g \in L^\infty_{loc}(\Omega)$. If u is not identically zero, then $u > 0$ in Ω. Moreover, for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such that u is C^1 in a neighborhood of $\Omega \cup \{x_0\}$ and $u(x_0) = 0$, we have that $\frac{\partial u}{\partial s} > 0$ for any inward directional derivative.

Following are some easy result on the geometry of Euclidean spheres. Though elementary, we give full details of their proofs, in order to take care of the constants involved.

Lemma B.7. *Let $r > q \geq 0$. Fix $\epsilon > 0$ and let c_1 and c_2 be non-negative and so that*

$$\max\{c_1,\ c_2\} \leq \min\{\sqrt{\epsilon}/3,\ 1/2\}\,. \tag{B.5}$$

Suppose[24] *that $z \in B_r(y) \subset \mathbb{R}^N$, with $|z'-y'| \leq c_1 q$. Then,*

$$\begin{aligned} &\partial B_r(y) \cap \Big\{|x'-z'| \leq c_2 q\Big\} \cap \Big\{x_N \geq y_N\Big\} \subseteq \\ &\subseteq \Big\{x_N \geq z_N - \frac{\epsilon\, q^2}{r}\Big\}\,. \end{aligned} \tag{B.6}$$

Proof. Take $w \in \partial B_r(y) \cap \Big\{|x'-z'| \leq c_2 q\Big\} \cap \Big\{x_N \geq y_N\Big\}$. For any $x \in \mathbb{R}^N$, let

$$\hat{x} := \frac{x-y}{r}\,.$$

Then, by construction,

$$\begin{gathered} \hat{z} \in B_1(0)\,, \\ |\hat{z}'| \leq \frac{c_1\, q}{r} \qquad \text{and} \\ \hat{w} \in \partial B_1(0) \cap \Big\{|\hat{x}'-\hat{z}'| \leq \frac{c_2 q}{r}\Big\} \cap \Big\{\hat{x}_N \geq 0\Big\}\,. \end{gathered}$$

Let also $t \geq 0$ so that

$$\hat{b} := \hat{z} + t e_N \in \partial B_1(0)\,.$$

We claim that

$$\hat{w}_N \geq \hat{b}_N - \frac{\epsilon\, q^2}{r^2}\,. \tag{B.7}$$

To prove this, first note that, if $\hat{w}_N \geq \hat{b}_N$, (B.7) is obvious; hence, we may assume that

$$\hat{w}_N < \hat{b}_N\,. \tag{B.8}$$

Also, if $\hat{b}_N \leq 0$, (B.7) would follow from the fact that $\hat{w}_N \geq 0$, thus we may also suppose that

$$\hat{b}_N > 0\,. \tag{B.9}$$

Also,

$$\begin{aligned} 1 &= |\hat{b}'|^2 + |\hat{b}_N|^2 = \\ &= |\hat{z}'|^2 + |\hat{b}_N|^2 \leq \\ &\leq \frac{c_1^2 q^2}{r^2} + |\hat{b}_N|^2 \leq \\ &\leq c_1^2 + |\hat{b}_N|^2\,, \end{aligned}$$

[24] Notation remark: in Lemma B.7, for definiteness, the balls are assumed to be closed.

which, together with (B.9), implies that

$$\hat{b}_N \geq \frac{1}{2}\,, \tag{B.10}$$

thanks to (B.5). Furthermore,

$$\begin{aligned} |\hat{z}'|^2 + |\hat{b}_N|^2 &= |\hat{b}'|^2 + |\hat{b}_N|^2 = \\ &= |\hat{b}|^2 = \\ &= 1 = \\ &= |\hat{w}|^2 = \\ &= |\hat{w}'|^2 + |\hat{w}_N|^2 \leq \\ &\leq \Big(|\hat{z}'| + |\hat{w}' - \hat{z}'|\Big)^2 + |w_N|^2\,, \end{aligned}$$

and so, by (B.5)

$$\begin{aligned} |\hat{b}_N|^2 - |\hat{w}_N|^2 &\leq |\hat{w}' - \hat{z}'|^2 + 2\,|\hat{z}'|\,|\hat{w}' - \hat{z}'| \leq \\ &\leq (c_2^2 + 2c_1c_2)\,\frac{q^2}{r^2} \leq \\ &\leq \frac{\epsilon\, q^2}{2r^2}\,. \end{aligned}$$

From this, the fact that $\hat{w}_N \geq 0$, (B.8) and (B.10), we conclude that

$$\frac{\epsilon\, q^2}{2r^2} \geq (\hat{b}_N + \hat{w}_N)\,(\hat{b}_N - \hat{w}_N) \geq \frac{1}{2}\,(\hat{b}_N - \hat{w}_N)\,,$$

which proves (B.7).

By using (B.7), we gather that

$$\begin{aligned} w_N - z_N &= r\,(\hat{w}_N - \hat{z}_N) = \\ &= r\,(\hat{w}_N - \hat{b}_N + t) \geq \\ &\geq r\,(\hat{w}_N - \hat{b}_N) \geq -\frac{\epsilon\, q^2}{r}\,, \end{aligned}$$

which gives (B.6). □

Corollary B.8. *Let $r > q \geq 0$ and fix*

$$\kappa \in \left(0\,,\; \frac{r}{10q}\right]\,. \tag{B.11}$$

Let us suppose that $v = (v', \kappa q^2/r) \in \mathbb{R}^N$ is above $B_r(y)$ with respect to the e_N direction. Let us assume also that $|v' - y'| \leq cq$, with

$$c \leq \min\left\{\frac{\sqrt{\kappa}}{3},\, \frac{1}{2}\right\}. \tag{B.12}$$

Then,

$$\partial B_r(y) \cap \Big\{|x' - y'| \geq 4\kappa q\Big\} \subseteq \{x_N < 0\}\,.$$

Proof. Take $w \in \partial B_r(y) \cap \Big\{ |x' - y'| \geq 4\kappa q \Big\}$. Let also $t \geq 0$ be so that

$$p := v - t e_N \in \partial B_r(y) \text{ with } p_N \geq y_N. \tag{B.13}$$

Note that

$$|p' - y'| \, = \, |v' - y'| \, \leq \, cq \, . \tag{B.14}$$

Also, by our assumptions,

$$\begin{aligned} w_N &\leq y_N + |w_N - y_N| = \\ &= y_N + \sqrt{r^2 - |w' - y'|^2} \leq \\ &\leq y_N + \sqrt{r^2 - 16\kappa^2 q^2} \, . \end{aligned} \tag{B.15}$$

Let $z := y + r e_N$. We now apply Lemma B.7 with $\epsilon := \kappa$ and $c_1 := c_2 := c$, recalling (B.12). Indeed, by (B.13), (B.14) and Lemma B.7,

$$\begin{aligned} p \in & \partial B_r(y) \cap \Big\{ |x' - z'| \leq c_2 q \Big\} \cap \Big\{ x_N \geq y_N \Big\} \subseteq \\ & \subseteq \Big\{ x_N \geq z_N - \frac{\kappa\, q^2}{r} \Big\} = \\ & = \Big\{ x_N \geq y_N + r - \frac{\kappa\, q^2}{r} \Big\} . \end{aligned} \tag{B.16}$$

Exploiting (B.15) and (B.16), we get

$$\begin{aligned} w_N &\leq p_N - r + \frac{\kappa q^2}{r} + \sqrt{r^2 - 16\kappa^2 q^2} \leq \\ &\leq v_N - r + \frac{\kappa q^2}{r} + \sqrt{r^2 - 16\kappa^2 q^2} = \\ &= \frac{2\kappa q^2}{r} - r + \sqrt{r^2 - 16\kappa^2 q^2} \leq \\ &\leq \frac{2\kappa q^2}{r} - r + r - \frac{8\kappa^2 q^2}{r} \, < \, 0 \, , \end{aligned}$$

which is the desired result. □

Corollary B.9. *Let $a > 0$, $r > q \geq 0$ so that*

$$\frac{q}{r} \, \leq \, \frac{K}{a^2} \tag{B.17}$$

and

$$\frac{q}{r} \, \leq \, \frac{a}{8K} \, . \tag{B.18}$$

Let us suppose that $v = (v', Kq^2/r) \in \mathbb{R}^N$ is above $B_r(y)$ with respect to the e_N direction. Let us assume also that $|v' - y'| \leq \hat{c} q$, with

$$\hat{c} \, \leq \, \min \Big\{ \frac{\sqrt{K}}{3}, \, \frac{2K}{a} \Big\} \, . \tag{B.19}$$

Then,

$$\partial B_r(y) \cap \Big\{|x'-y'| \geq aq\Big\} \subseteq \{x_N < 0\} . \tag{B.20}$$

Proof. Let $\bar{q} := 4Kq/a$, $\kappa := a^2/(16K)$, $c := a\hat{c}/(4K)$. Note that $\bar{q} < r$ due to (B.18). What is more, $v = (v', \kappa\bar{q}^2/r)$ and $|v'-y'| \leq c\bar{q}$. Also, (B.11) and (B.12) are fulfilled thanks to (B.17) and (B.19). Thus, by Corollary B.8 (applied with $\bar{q}$ instead of q),

$$\partial B_r(y) \cap \Big\{|x'-y'| \geq 4\kappa\bar{q}\Big\} \subseteq \{x_N < 0\} ,$$

which is (B.20). ☐

We now point out some observations on rotation hypersurfaces in $\mathbb{R}^{N+1}$. First, the normal of a rotation surface is in the space[25] generated by the radial direction and e_{N+1}, as showed by the next result:

Lemma B.10. *Fix $y \in \mathbb{R}^N$. Let $f \in C^1(\mathbb{R},\mathbb{R})$ and define*

$$\Phi(x) := f(|x-y|) .$$

Let $\nu(x)$ be a normal vector to the surface $\{x_{N+1} = \Phi(x)\}$ at the point $(x,\Phi(x))$. Then, if $x \neq y$, $\nu(x)$ belongs to the space spanned by $x-y$ and e_{N+1}. If $x = y$ and $f'(0) = 0$, the same result holds.

Proof. Assume $x \neq y$. By construction,

$$\nu(x) = \alpha\Big(\nabla\Phi(x),\, -1\Big) ,$$

for some $\alpha \in \mathbb{R}$. Therefore,

$$\nu(x) \;=\; \frac{\alpha\, f'(|x-y|)}{|x-y|}\,(x-y) - \alpha e_{N+1} ,$$

thus proving the claim if $x \neq y$.

If, on the other hand, $x = y$ and $f'(0) = 0$, then $\nu(x) = \alpha\, e_{N+1}$, for some $\alpha \in \mathbb{R}$, hence completing the proof of the claim. ☐

The next result will relate the “center” of a rotation hypersurface with the normal at any point:

Lemma B.11. *Fix $y \in \mathbb{R}^N$. Let $f \in C^1(\mathbb{R},\mathbb{R})$ and define*

$$\Phi(x) := f(|x-y|) .$$

Let us define the following hypersurface in $\mathbb{R}^{N+1}$:

$$\Sigma := \Big\{ \Big(x, \Phi(x)\Big) \;\Big|\; x \in \mathbb{R}^N \Big\} .$$

[25] And, in fact, this property characterizes the rotation surfaces, as pointed out to us by Rajko Quarta Marcon, an undergraduate in Tor Vergata.

Let us consider the normal $\nu(x)$ at a point $\Big(x, \Phi(x)\Big) \in \Sigma$, given by

$$\nu(x) \;=\; \Big(\nu_1(x), \ldots, \nu_{N+1}(x)\Big) \;:=\; \frac{\Big(-\nabla\Phi(x),\; 1\Big)}{\sqrt{1+|\nabla\Phi(x)|^2}}\,.$$

Then, for any $x \in \mathbb{R}^N \setminus \{y\}$, the vectors

$$x - y \qquad \textit{and} \qquad \Big(\nu_1(x), \ldots, \nu_N(x)\Big)$$

are parallel.

Proof. If $(\nu_1(x), \ldots, \nu_N(x)) = 0$, there is nothing to prove, so we may assume $(\nu_1(x), \ldots, \nu_N(x)) \neq 0$. For this reason,

$$\nabla\Phi(x) = f'\Big(|x-y|\Big)\,\frac{x-y}{|x-y|}\,,$$

thus

$$f'\Big(|x-y|\Big) \neq 0\,.$$

Let

$$a(x) := \frac{|x-y|\,\sqrt{1+|\nabla\Phi(x)|^2}}{f'\Big(|x-y|\Big)} \;\in\; \mathbb{R}\,.$$

Then,

$$a(x)\,\Big(\nu_1(x), \ldots, \nu_N(x)\Big) = x - y\,,$$

proving the claim. □

Next result is an explicit computation on the differential of the unit normal of a hypersurface (up to a sign, such quantity is sometimes referred to as Second Fundamental Form or Shape Operator):

Lemma B.12. *Let $\Psi \in C^1(\mathbb{R}^N, \mathbb{R})$. Let Σ be the hypersurface defined by*

$$\Sigma := \Big\{ \Big(x, \Psi(x)\Big) \;\Big|\; x \in \mathbb{R}^N \Big\}\,.$$

Let $X = X(x) := (x, \Psi(x))$ and consider the unit normal to Σ at the point X, given by

$$\hat\nu(x) \;:=\; \frac{\Big(-\nabla\Psi(x),\; 1\Big)}{\sqrt{1+|\nabla\Psi(x)|^2}} \;\in\; S^N\,.$$

For $X = (x, \Psi(x))$, let also

$$\nu(X) = \nu(x, \Psi(x)) := \hat\nu(x)\,.$$

Let[26] $D_X\nu : T_X\Sigma \longrightarrow \mathbb{R}^{N+1}$ *be the differential map. Then, for any* $W = (w, w_{N+1}) \in T_X\Sigma$,

$$D_X\nu(X)[W] = \tag{B.21}$$

$$= \begin{pmatrix} \dfrac{-(1+|\nabla\Psi(x)|^2)\partial_{1j}\Psi(X)w_j + \partial_1\Psi(X)\,\partial_k\Psi(X)\,\partial_{kj}\Psi(X)\,w_j}{(1+|\nabla\Psi(x)|^2)^{3/2}} \\ \vdots \\ \dfrac{-(1+|\nabla\Psi(x)|^2)\partial_{Nj}\Psi(X)w_j + \partial_N\Psi(X)\,\partial_k\Psi(X)\,\partial_{kj}\Psi(X)\,w_j}{(1+|\nabla\Psi(x)|^2)^{3/2}} \\ \\ \dfrac{-\partial_k\Psi(X)\,\partial_{kj}\Psi(X)\,w_j}{(1+|\nabla\Psi(x)|^2)^{3/2}} \end{pmatrix}.$$

Proof. Let $W \in T_X\Sigma$. Then, for some $v \in \mathbb{R}^N$,

$$W = \frac{d}{dt}\Big(x+tv,\, \Psi(x+tv)\Big)\Big|_{t=0} = \Big(v,\, \nabla\Psi(x)\cdot v\Big),$$

that is $w_{N+1} = \nabla\Psi(x)\cdot w$, or, equivalently

$$T_X\Sigma = \{(w, \nabla\Psi(x)\cdot w) \mid w \in \mathbb{R}^N\}. \tag{B.22}$$

Because of this,

$$\begin{aligned} D_X\nu(X)[W] &= \frac{d}{dt}\nu\Big(x+tw,\, \Psi(x+tw)\Big)\Big|_{t=0} = \\ &= \frac{d}{dt}\hat\nu(x+tw)\Big|_{t=0}, \end{aligned}$$

from which a straightforward calculation gives the claim. □

Remark B.13. In relation with Lemma B.12 above, we notice that, since $\nu(\Sigma) \subseteq S^N$ and $T_{\nu(X)}S^N = T_X\Sigma$, we may think $D_X\nu$ as a linear map from $T_X\Sigma$ to itself. Using (B.21) and (B.22), the reader will easily convince herself that $D_X\nu : T_X\Sigma \longrightarrow T_X\Sigma$ may thus be represented in matrix form as

$$\Big(D_X\nu\Big)_{ij} = \frac{-(1+|\nabla\Psi|^2)\partial_{ij}\Psi + \partial_i\Psi\partial_k\Psi\partial_{kj}\Psi}{(1+|\nabla\Psi|^2)^{3/2}} \tag{B.23}$$

for $1 \le i, j \le N$ (where, of course, the summation over the index k is understood here above).

With this, we now point out an explicit computation of the curvatures of the rotation surfaces:

Lemma B.14. *Let* $\Phi \in C^2((0,+\infty), \mathbb{R})$ *and*

$$\Sigma := \Big\{\Big(x,\, \Phi(|x|)\Big) \mid x \in \mathbb{R}^N\Big\}.$$

[26]As standard, given a manifold M and a point $X \in M$, we denote by T_XM the tangent space at X. Also, as usual, $S^N := \{X \in \mathbb{R}^{N+1} \mid |X| = 1\}$.

Then, the principal curvatures[27] *of* Σ *are given by*

$$\begin{aligned}\kappa_1 &= \ldots = \kappa_{N-1} = \frac{\Phi'}{|x|\sqrt{1+(\Phi')^2}}\\ \kappa_N &= \frac{\Phi''}{(1+|\Phi'|^2)^{3/2}}.\end{aligned}$$

Proof. We set $\Psi(x) := \Phi(|x|)$ and, after some easy computation, we infer from (B.23) that

$$\begin{aligned}\Big(D_X\nu\Big)_{ij} &= (1+(\Phi')^2)^{-3/2}\Big[-\frac{(1+(\Phi')^2)\Phi'\,\delta_{ij}}{|x|}+\\ &+\frac{(1+(\Phi')^2)\Phi'-|x|\Phi''}{|x|^3}\,x_ix_j\Big].\end{aligned}\tag{B.24}$$

Notice now that, up to rotation, we may assume that the point $X = (x, \Phi(|x|)$, in which we compute the principal curvatures, is of the form $x = |x|\,e_N$; hence, from (B.24)

$$\begin{aligned}\Big(D_X\nu\Big)_{ij} &= (1+(\Phi')^2)^{-3/2}\Big[-\frac{(1+(\Phi')^2)\Phi'\,\delta_{ij}}{|x|}+\\ &+\frac{(1+(\Phi')^2)\Phi'-|x|\Phi''}{|x|}\,\delta_{iN}\delta_{jN}\Big],\end{aligned}$$

and so the desired claim easily follows. □

References

[1] G. Alberti, L. Ambrosio, X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. 65 (2001), no. 1-3, 9–33.

[2] L. Ambrosio, X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. 13 (2000), no. 4, 725–739.

[3] S. S. Antman, *Nonuniqueness of equilibrium states for bars in tension*, J. Math. Anal. Appl. (1973), 333–349.

[4] M. T. Barlow, R. F. Bass, C. Gui, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. 53 (2000), no. 8, 1007–1038.

[5] H. Berestycki, L. A. Caffarelli, L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 25 (1997), no. 1-2, 69–94.

[6] H. Berestycki, F. Hamel, R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. 103 (2000), no. 3, 375–396.

[7] G. Bouchitté, *Singular perturbations of variational problems arising from a two-phase transition model*, Appl. Math. Optim. 21 (1990), no. 3, 289–314.

[8] L.A. Caffarelli, X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995.

[27] As standard, the principal curvature of a surface Σ at the point $\bar X$ are here defined as the eigenvalues of $-D_X\nu(\bar X) : T_{\bar X}\Sigma \longrightarrow T_{\bar X}\Sigma$.

[9] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré. Analyse non linéaire, 15(4), 1998, 493 – 516.

[10] L. Damascelli and B. Sciunzi, *Harnack Inequalities, Maximum and Comparison Principles, and Regularity of positive solutions of m-Laplace equations*, to appear in Calc. Var. Partial Differential Equations,

[11] D. Danielli, N. Garofalo, *Properties of entire solutions of non-uniformly elliptic equations arising in geometry and in phase transitions*, Calc. Var. Partial Differential Equations 15 (2002), no. 4, 451–491.

[12] A. Farina, *One-dimensional symmetry for solutions of quasilinear equations in $\mathbb{R}^2$*, Bollettino UMI 8 6-B (2003), 685–692.

[13] A. Farina, *Rigidity and one-dimensional symmetry for semilinear elliptic equations in the whole of $\mathbb{R}^N$ and in half spaces*, Adv. Math. Sci. Appl. 13 (2003), no. 1, 65–82.

[14] E. De Giorgi, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna, 1979.

[15] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. 7 (1983), no. 8, 827–850.

[16] J. Dolbeault, R. Monneau, *On a Liouville type theorem for isotropic homogeneous fully nonlinear elliptic equations in dimension two*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), no. 1, 181–197.

[17] Y. Du, L. Ma, *Some Remarks on De Giorgi conjecture*, Proc. Amer. Math. Soc., 131(2002), 2415–2422.

[18] L.C. Evans, R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[19] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[20] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.

[21] N.A. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. 311 (1998), no. 3, 481–491.

[22] N. A. Ghoussoub, C. Gui, *On the De Giorgi conjecture in dimensions* 4 *and* 5, Ann. Math. 157 (2003), no. 1, 313–334.

[23] M.E. Gurtin, *On a theory of phase transitions with interfacial energy*, Arch. Rational Mech. Anal. (1985), no. 3, 187–212.

[24] D. Jerison, R. Monneau, *The existence of a symmetric global minimizer on $\mathbb{R}^{n-1}$ implies the existence of a counter-example to a conjecture of De Giorgi in $\mathbb{R}^n$*, C. R. Acad. Sci. Paris 333 (2001), no. 5, 427–431.

[25] L. Modica, S. Mortola, *Un esempio di Γ^--convergenza*, Boll. Un. Mat. Ital. B (5) 14 (1977), no. 1, 285–299.

[26] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. 98 (1987), no. 2, 123–142.

[27] A. Petrosyan, E. Valdinoci, *Geometric properties of Bernoulli-type minimizers*, Interfaces Free Bound. 7 (2005), 55–78.

[28] A. Petrosyan, E. Valdinoci, *Density estimates for a degenerate/singular phase-transition model*, to appear in SIAM J. Math. Anal.

[29] J. S. Rowlinson, *Translation of J. D. van der Waals' "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density"*, J. Statist. Phys. (1979), no. 2, 197–244.

[30] B. Sciunzi, E. Valdinoci, *Mean curvature properties for p-Laplace phase transitions*, to appear in J. Eur. Math. Soc. (JEMS).

[31] V. O. Savin, *Phase Transitions: Regularity of Flat Level Sets*, Ph.D. thesis at the University of Texas at Austin, 2003.

[32] V. O. Savin, manuscript, 2004.
[33] P. Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Ration. Mech. Anal. 101 (1988), no. 3, 209–260.
[34] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations 51 (1984), no. 1, 126–150.
[35] P. Tolksdorf, *On the dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations (1983), no. 7, 773–817.
[36] E. Valdinoci, *Plane-like minimizers in periodic media: jet flows and Ginzburg-Landau-type functionals*, J. Reine Angew. Math 574 (2004), 147–185.
[37] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 1984, 191 – 202.

⋆ ⋆ ⋆

Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 1, I-00133 Roma (Italy)
E-mail address: sciunzi, valdinoci@mat.uniroma2.it

Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA
E-mail address: osavin@math.berkeley.edu