Gap Solitons in Periodic Discrete Nonlinear Schrödinger Equations

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Abstract

It is shown that the periodic DNLS, with cubic nonlinearity, possesses gap solutions, i. e. standing waves, with the frequency in a spectral gap, that are exponentially localized in spatial variable. The proof is based on the linking theorem in combination with periodic approximations.

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0 Introduction

In this paper we consider spatially localized standing waves for the discrete nonlinear Schrödinger equation (DNLS)

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - \sigma\chi_n|\psi_n|^2\psi_n, \quad n \in \mathbb{Z}, \tag{0.1}$$

where $\sigma = \pm 1$,

$$\Delta\psi_n = (\psi_{n+1} + \psi_{n-1} - 2\psi_n)$$

is the discrete Laplacian in one spatial dimension and given sequences ε_n and χ_n are assumed to be N-periodic in n, i. e. $\varepsilon_{n+N} = \varepsilon_n$ and $\chi_{n+N} = \chi_n$. Such solutions are often called intrinsic localized modes or breathers, but in the case under consideration we prefer the name "gap solitons" due to the obvious analogy with gap solutions in photonic crystals (see, e. g. [1, 4, 12]).

Making use the standing wave Ansatz

$$\psi_n = u_n \exp(i\omega t),$$

where u_n is a real valued sequence and $\omega \in \mathbb{R}$, we arrive at the equation

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = \sigma\chi_n|u_n|^2 u_n. \tag{0.2}$$

We impose the following boundary condition at infinity:

$$\lim_{n\to\pm\infty} u_n = 0, \tag{0.3}$$

and we are looking for nontrivial solutions, i. e. solutions that are not equal to 0 identically.

Actually, we consider a more general equation

$$Lu_n - \omega u_n = \sigma\chi_n |u_n|^2 u_n \tag{0.4}$$

with the same boundary condition (0.3). Here L is a second order difference operator

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n$$

where a_n and b_n are real valued N-periodic sequences. The operator L can be represented in the form

$$Lu_n = -(\partial^* a_n \partial) f_n + (a_{n-1} + a_n + b_n)u_n,$$

where

$$\partial u_n = u_{n+1} - u_n, \quad \partial^* u_n = u_{n-1} - u_n.$$

When $a_n \equiv 1$ and $b_n = -2 + \varepsilon_n$, we obtain equation (0.2).

We consider equation (0.4) as a nonlinear equation in the space l^2 of two-sided infinite sequences. Note that every element of l^2 automatically satisfies (0.3).

The operator L is a bounded and self-adjoint operator in l^2. Its spectrum $\sigma(L)$ has a band structure, i. e. $\sigma(L)$ is a union of a finite number of closed intervals (see, e. g., [22]). The complement $\mathbb{R} \setminus \sigma(L)$ consists of a finite number of open intervals called spectral gaps. Two of them are semi-infinite. We fix one such gap and denote it by (α, β).

Our main result is the following

Theorem 0.1 *Suppose that $\chi_n > 0$ and $\omega \in (\alpha, \beta)$. If either $\sigma = +1$ and $\beta \neq +\infty$, or $\sigma = -1$ and $\alpha \neq -\infty$, then equation (0.4) has a nontrivial solution $u \in l^2$ and, moreover, the solution u decays exponentially at infinity:*

$$|u_n| \leq Ce^{-\gamma|n|}, \quad n \in \mathbb{Z},$$

with some $C > 0$ and $\gamma > 0$. If either $\sigma = +1$ and $\beta = +\infty$, or $\sigma = -1$ and $\alpha = -\infty$, then there is no nontrivial solution in l^2.

The proof contained in Sections 1–5 is variational. Its idea is borrowed from [14] and based on so-called periodic approximations. In what follows we consider the case $\sigma = +1$. The other case reduces to the previous one if we replace L by $-L$ and ω by $-\omega$.

1 Variational setting

On the Hilbert space $E = l^2$, we consider the functional

$$J(u) = \frac{1}{2}\,(Lu - \omega u, u) - \frac{1}{4}\sum_{n=-\infty}^{+\infty} \chi_n u_n^4, \tag{1.1}$$

where $(\cdot,\cdot)$ is l^2 inner product. The corresponding norm in E is denoted by $\|\cdot\|$. The functional J is a well-defined C^1 functional on E and equation (0.4) is easily recognized as the corresponding Euler-Lagrange equation for J (remind that $\sigma = +1$). Thus, we are looking for nonzero critical points of J.

Fix an integer $k > 0$ and denote by E_k the space of all kN-periodic sequences. This is a kN-dimensional Hilbert space endowed with the inner product

$$(u, v)_k = \sum_{n=0}^{kN-1} u_n v_n, \quad u, v \in E_k,$$

and corresponding norm $\|\cdot\|_k$. On the space E_k we consider the functional

$$J_k(u) = \frac{1}{2}\,(Lu - \omega u, u)_k - \frac{1}{4}\sum_{n=0}^{kN-1} \chi_n u_n^4. \tag{1.2}$$

Due to the periodicity of coefficients the operator L acts in E_k. Critical points of J_k are exactly kN-periodic solutions of equation (0.4) with $\sigma = +1$.

For gradients of J and J_k we have the following formulas:

$$\big(\nabla J(u), v\big) = (Lu - \omega u, v) - \sum_{n=-\infty}^{+\infty} \chi_n u_n^3 v_n, \quad v \in E, \tag{1.3}$$

and

$$\big(\nabla J_k(u), v\big) = (L_k u - \omega u, v)_k - \sum_{n=0}^{kN-1} \chi_n u_n^3 v_n, \quad v \in E. \tag{1.4}$$

We denote by L_k the operator L acting in E_k. From the spectral theory of difference (Jacobi) operators (see, e. g., [22]) it follows immediately that $\sigma(L_k) \subset \sigma(L)$ and, hence, $\|L_k\| \le \|L\|$.

Let E_k^+ (respectively, E_k^-) be the positive (respectively, negative) spectral subspace of the operator $L_k - \omega$ in E_k. Similarly, we introduce the positive and negative spectral subspaces $E^+ \subset E$ and $E^- \subset E$, respectively, for the operator $L - \omega$. Let

$$\delta = \min\big[|\alpha - \omega|, |\beta - \omega|\big]$$

be the distance from ω to the spectrum $\sigma(L)$. Then

$$\pm(Lu - \omega u, u) \ge \delta\|u\|^2, \quad u \in E^\pm, \tag{1.5}$$

and

$$\pm(L_k u - \omega u, u)_k \ge \delta\|u\|_k^2, \quad u \in E_k^\pm. \tag{1.6}$$

Now we are ready to prove the nonexistence part of Theorem 0.1.

Proposition 1.1 *Suppose that $\beta = +\infty$. Then the only critical point of J (respectively, J_k) is the origin of the space E (respectively, E_k).*

Proof. We consider the case of J, the remaining case being similar.
Let $u \in E$ be a critical point of J. Then, by (1.3) and positivity of χ_n,

$$\begin{aligned} 0 &= \big(\nabla J(u), u\big) = (Lu - \omega u, u) - \sum_{n=-\infty}^{+\infty} \chi_n u_n^4 \leq \\ &\leq (Lu - \omega u, u). \end{aligned}$$

Since $\beta = +\infty$, we have that $E^+ = \{0\}$ and, by (1.5),

$$0 \leq -\delta \|u\|^2$$

which implies that $u = 0$. □

2 Technical results

To prove the existence of kN-periodic solutions, as well as to pass to the limit as $k \to \infty$, we need some preliminaries. We start with

Lemma 2.1 *For any nontrivial critical points $u^{(k)} \in E_k$ of J_k and $u \in E$ of J, with critical values $c^{(k)} = J_k(u^{(k)})$ and $c = J(u)$, we have*

$$\left\| u^{(k)} \right\|_k \leq 4\delta^{-1} \overline{\kappa} \underline{\kappa}^{-3/4} (c^{(k)})^{3/4}$$

and

$$\|u\| \leq 4\delta^{-1} \overline{\kappa} \underline{\kappa}^{-3/4} c^{3/4},$$

where $\underline{\kappa} = \min\{\chi_n\}$ and $\overline{\kappa} = \max\{\chi_n\}$.

Proof. We have

$$\begin{aligned} c &= J(u) - \frac{1}{2}\big(\nabla J(u), u\big) = \left(\frac{1}{2} - \frac{1}{4}\right) \sum_{n=-\infty}^{+\infty} \chi_n u_n^4 \geq \\ &\geq \frac{1}{4} \underline{\kappa} \, \|u\|_{l^4}^4, \end{aligned} \tag{2.1}$$

where $\|\cdot\|_{l^p}$ stands for the norm in the space l^p. Let $u^\pm$ be the orthogonal projection of u into $E^\pm$ along $E^\mp$. Then

$$\begin{aligned} 0 &= \big(\nabla J(u), u^+\big) = (Lu - \omega u, u^+) - \sum_{n=-\infty}^{+\infty} \chi_n u_n^3 u_n^+ = \\ &= (Lu^+ - \omega u^+, u^+) - \sum_{n=-\infty}^{+\infty} \chi_n u_n^3 u_n^+ \end{aligned}$$

and we obtain, using Hölder's inequality, that

$$\begin{aligned}
\delta\|u^+\|^2 &\leq \overline{\kappa}\left[\sum_{n=-\infty}^{+\infty} u_n^6\right]^{1/2}\left[\sum_{n=-\infty}^{+\infty} (u_n^+)^2\right]^{1/2} = \\
&= \overline{\kappa}\|u\|_{l^6}^3\|u^+\| \leq \overline{\kappa}\|u\|_{l^4}^3\|u^+\|.
\end{aligned}$$

Hence, by (2.1),

$$\|u^+\|^2 \leq 2^{3/2}\delta^{-1}\overline{\kappa}\underline{\kappa}^{-3/4}c^{3/4}\|u^+\|.$$

Similarly,

$$\|u^-\|^2 \leq 2^{3/2}\delta^{-1}\overline{\kappa}\underline{\kappa}^{-3/4}c^{3/4}\|u^-\|.$$

Since

$$\|u\|^2 = \|u^+\|^2 + \|u^-\|^2$$

and

$$\|u^+\| + \|u^-\| \leq 2^{1/2}\|u\|$$

the two previous inequalities imply that

$$\|u\| \leq 4\delta^{-1}\overline{\kappa}\underline{\kappa}^{-3/4}c^{3/4}.$$

The remaining case of J_k is similar. □

We need also lower estimates for nontrivial critical points and critical values.

Lemma 2.2 *Under the notation of Lemma 2.1*

$$\|u\|^2 \geq 2^{-1/2}\delta\overline{\kappa}^{-1},$$

$$c \geq \frac{1}{8}\,\delta^2\overline{\kappa}^{-2}\underline{\kappa},$$

$$\left\|u^{(k)}\right\|_k^2 \geq 2^{-1/2}\delta\overline{\kappa}^{-1}$$

and

$$c^{(k)} \geq \frac{1}{8}\,\delta^2\overline{\kappa}^{-2}\underline{\kappa}.$$

Proof. Consider the case of J, the other case being similar.

Since $J'(u) = 0$, then $\big(J'(u), v\big) = 0$ for any $v \in E$. Taking $v = u^+$, the orthogonal projection of u onto E^+, we have as in the proof of Lemma 2.1

$$\delta\|u^+\|^2 \leq \sum_{n\in\mathbb{Z}} \chi_n u_n^3 u_n^+.$$

Using the Hölder inequality with $p = 4/3$ and $p' = 4$, we obtain

$$\delta\|u^+\| \leq \overline{\kappa}\,\|u\|_{l^4}^3\|u^+\|_{l^4} \leq \overline{\kappa}\,\|u\|^3\|u^+\|.$$

Similarly, taking $v = u^-$, the orthogonal projection of u onto E^-, we obtain that

$$\delta \|u^-\|^2 \le \overline{\kappa}\, \|u\|^3 \|u^-\|.$$

Combining the last two inequalities, we have that

$$\delta \|u\|^2 \le \overline{\kappa}\, \|u\|^3 \left(\|u^+\| + \|u^-\|\right) \le 2^{1/2} \overline{\kappa}\, \|u\|^4.$$

Hence

$$\|u\|^2 \ge 2^{-1/2} \delta \overline{\kappa}^{-1}.$$

The bound for c follows immediately from the last inequality and Lemma 2.1. □

3 Existence of periodic solutions

In this section we prove the existence of nontrivial kN-periodic solutions to equation (0.4). Actually, we show that the functional J_k possesses a nontrivial critical point. Moreover, we derive some uniform in k bounds for the critical points and corresponding critical values. The proof relies upon the standard linking theorem (see, Appendix B).

Recall that a sequence $v^{(j)} \in E_k$ is called a *Palais-Smale sequence* for J_k at level b if $J_k(v^{(j)}) \to b$ and $J_k'(v^{(j)}) \to 0$ as $j \to \infty$.

Lemma 3.1 *The functional J_k satisfies the so-called Palais-Smale condition, i. e. every Palais-Smale sequence contains a convergent subsequence.*

Proof. Since the space E_k is finite dimensional, it is enough to show that every Palais-Smale sequence is bounded. Moreover, replacing L by $L + \omega_0$ and ω by $\omega + \omega_0$, with some ω_0, we can assume that $L \gg 1$, i. e.

$$(Lv, v)_k \ge \|v\|_k^2, \quad v \in E_k,$$

and $\omega > 0$.

Let $v^{(j)}$ be a Palais-Smale sequence at some level b. Fix $\beta \in \left(\frac{1}{4}, \frac{1}{2}\right)$. For j large, we have

$$\begin{aligned}
b + 1 + \beta \left\|v^{(j)}\right\|_k &\ge J_k(v^{(j)}) - \beta \left(J_k'(v^{(j)}), v^{(j)}\right) = \\
&= \left(\frac{1}{2} - \beta\right) \left(Lv^{(j)}, v^{(j)}\right)_k - \left(\frac{1}{2} - \beta\right) \omega \left\|v^{(j)}\right\|_k^2 + \\
&\quad + \left(\beta - \frac{1}{4}\right) \sum_{n=0}^{k} \chi_n \left(v^{(j)}\right)^4 \ge \\
&\ge \left(\frac{1}{2} - \beta\right) \left\|v^{(j)}\right\|_k^2 - \left(\frac{1}{2} - \beta\right) \omega \left\|v^{(j)}\right\|_k^2 + \\
&\quad + \left(\beta - \frac{1}{4}\right) \underline{\kappa} \left\|v^{(j)}\right\|_k^4.
\end{aligned}$$

Since $a^2 \le K(\varepsilon) + \varepsilon a^4$, where $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and $\omega > 0$, we can choose ε so small that the third term above absorbs the second one up to a constant. Hence

$$b + 1 + \beta \left\| v^{(j)} \right\|_k \ge \left(\frac{1}{2} - \beta \right) \left\| v^{(j)} \right\|_k^2 + C \left\| v^{(j)} \right\|_k^4 - C_0,$$

with $C > 0$ and $C_0 > 0$. This implies immediately that the sequence $\left\| v^{(j)} \right\|_k$ is bounded and the proof is complete. $\square$

Now let us check that the functional J_k possesses the linking geometry (see Appendix B), with $Y = E_k^-$ and $Z = E_k^+$. Remind that we consider the case when $\beta \neq +\infty$ and, hence, $E_k^+ \neq \{0\}$. Fix two constants $\varrho > r > 0$ and choose $z^k \in E_k^+$ as follows.

Let $z \in E^+$ be an arbitrary unit vector. We set

$$z^k = \frac{1}{\|P_k^+ S_k z\|_k} P_k^+ S_k z \in E_k^+$$

(see Appendix A for the definition of operator S_k). The vector z^k is well-defined at least for sufficiently large k, say, $k \ge k_0$. If $k < k_0$, we choose z^k to be an arbitrary unit vector in E_k^+. Now we set

$$S = \left\{ v \in E_k^+ \ : \ \|v\|_k = r \right\}$$

and

$$M = \left\{ v = y + t z^k \ : \ \|v\|_k \le \varrho, \, t \ge 0, \, y \in E_k^- \right\}.$$

Let M_0 be the boundary of the set M,

$$M_0 = \left\{ v = y + t z^k \ : \ y \in E_k^-, \ \|v\|_k = \varrho \text{ and } t \ge 0, \ \text{or } \|y\|_k \le \varrho \text{ and } t = 0 \right\}.$$

Lemma 3.2 *For $\varrho > 0$ large enough*

$$J_k(v) \le 0, \quad v \in M_0,$$

while

$$J_k(v) \ge \frac{\delta}{4} r^2,$$

provided $r^2 \le \overline{\kappa}^{-1} \delta$. Moreover, there exists a constant $C > 0$ independent of k and such that

$$J_k(v) \le C, \quad v \in M. \tag{3.1}$$

Proof. For $v \in E_k^+$ we have

$$\begin{aligned} J_k(v) &= \frac{1}{2} (L_k v - \omega v, v)_k - \frac{1}{4} \sum_{n=0}^{kN-1} \chi_n v_n^4 \ge \\ &\ge \frac{\delta}{2} \|v\|_k^2 - \frac{\overline{\kappa}}{4} \|v\|_k^4. \end{aligned}$$

This implies that, for $r^2 \le (\overline{\kappa})^{-1}\delta$,

$$J_k \ge \frac{\delta}{4}\, r^2 \quad \text{ on } S.$$

Now let us consider J_k on M. Since $E_k^{\pm}$ are mutually orthogonal spectral subspaces of L_k,

$$\begin{aligned} J_k(y+tz) &= \frac{1}{2}\,(L_k y - \omega y, y)_k + \frac{t^2}{2}\,(L_k z^k - \omega z^k, z^k)_k - \\ &- \frac{1}{4}\sum_{n=0}^{kN-1} \chi_n (y_n + t z_n^k)^4. \end{aligned}$$

Hence,

$$J_k(y+tz) \le -\frac{\delta}{2}\,\|y\|_k^2 + \frac{t^2}{2}\big((L_k-\omega)z^k, z^k\big)_k - \frac{1}{4}\,\underline{\kappa}\,\|y+tz^k\|_{l^4,k}^4.$$

Consider the subspace $X = E_k^- \oplus \mathbb{R}z^k \subset E_k$ endowed with the norm $\|\cdot\|_{l^4,k}$. The map $y + tz^k \mapsto tz^k$ is a bounded projector onto $\mathbb{R}z$. Since its norm is not less that 1, we see that

$$\|y+tz^k\|_{l^4,k} \ge \|tz^k\|_{l^4,k}.$$

Hence,

$$J_k(y+tz^k) \le -\frac{\delta}{2}\,\|y\|_k^2 + \frac{t^2}{2}\big((L_k-\omega)z^k, z^k\big)_k - \frac{1}{4}\,\underline{\kappa} t^4 \|z^k\|_{l^4,k}^4. \tag{3.2}$$

We have that

$$\big((L_k-\omega)z^k, z^k\big) \le a_0,$$

where $a_0 = \|L_k - \omega\|$, and, by (A.2) and Lemma A.1,

$$\lim_{k\to\infty} \|z^k\|_{l^4,k}^4 = \|P^+ z\|_{l^4}^4 = \|z\|_{l^4}^4.$$

Inequality (3.2) implies that

$$J_k(y+tz^k) \le -\frac{\delta}{2}\,\|y\|_k^2 + \frac{a_0}{2}\,t^2 - a_1 t^4 \le \frac{a_0}{2}\,t^2 - a_1 t^4, \tag{3.3}$$

with some $a_1 > 0$ independent of k. Therefore, for all ϱ large we have that $J_k \le 0$ on M_0. Moreover,

$$\sup_M J_k(v) \le C = \max_{t>0}\left(\frac{a_0}{2}\,t^2 - c_1\,t^4\right),$$

with $C > 0$ independent of k. The proof is complete. $\square$

Now we are ready to prove the existence of periodic solutions. Remind that, without loss of generality, we consider the case $\sigma = +1$.

Theorem 3.1 *Suppose that $\chi_n > 0$ and $\omega \in (\alpha, \beta)$, with $\beta \neq +\infty$. Then for every $k \geq 1$ equation (0.4), with $\sigma = +1$, has a nontrivial kN-periodic solution $u^{(k)}$. Moreover, we have the following bounds:*

$$J_k(u^{(k)}) \leq C,$$

$$\|u^{(k)}\|_k \leq C,$$

where $C > 0$ is independent of k.

Proof. The existence follows immediately from the standard linking theorem (see Appendix B). Indeed, to apply that theorem we need two things: (a) the Palais-Smale condition and (b) the linking geometry. The Palais-Smale condition is verified in Lemma 3.1, while Lemma 3.2 means exactly that the functional J_k possesses the linking geometry.

Moreover, the linking theorem states that the corresponding critical value satisfies

$$J_k\big(u^{(k)}\big) \leq \sup_M J_k(v).$$

Therefore, the upper bounds for the critical points and critical values follow from Lemma 3.2 and Lemma 2.1. $\square$

4 Existence of localized solutions

The following result gives the existence of solution $u \in l^2$ in Theorem 0.1.

Theorem 4.1 *Under assumptions of Theorem 3.1 equation (0.4) possesses a nontrivial solution $u \in l^2$.*

Proof. Consider the sequence $u^{(k)} = \{u_n^{(k)}\}$ of kN-periodic solutions found in Theorem 3.1.

First we claim that there exists $\delta_0 > 0$ and $n_k \in \mathbb{Z}$ such that

$$\big|u_{n_k}^{(k)}\big| \geq \delta_0. \tag{4.1}$$

Indeed, if not, then $u^{(k)} \to 0$ in l^∞. Hence, $v^{(k)} = R_k u^{(k)} \to 0$ in l^∞. By Theorem 3.2, $\|v^{(k)}\|_{l^2} = \|u^{(k)}\|_k$ is bounded. Now the following simple inequality

$$\|v\|_{l^p}^p \leq \|v\|_{l^\infty}^{p-2} \|v\|_{l^2}^2,$$

where $p > 2$, shows that $v^{(k)} \to 0$ in all l^p, $p > 2$. Hence, $\|u^{(k)}\|_{l_k^p} \to 0$ for all $p > 2$. As in the beginning of proof of Lemma 2.1, we have that for the corresponding critical value $c^k = J_k(u^{(k)})$ that

$$0 < c^k = \frac{1}{4} \sum_{n \in Q_k} \chi_n \big[u_n^{(k)}\big]^4 \leq \frac{\overline{\kappa}}{4} \|u^{(k)}\|_{l_k^4}^4 \to 0.$$

However, this contradicts Lemma 2.2 and we obtain (4.1).

Due to periodicity of coefficients, $\{u^{(k)}_{n+N}\}$ is also a solution of (0.4). Hence, making such shifts, we can assume that $0 \le n_k \le N-1$ in (4.1). Moreover, passing to a subsequence of k's, we can even assume that $n_k = n_0$ in independent of k.

Next we extract a subsequence, still denoted by $u^{(k)}$, such that $u^{(k)}_n \to u_n$ for every $n \in \mathbb{Z}$. Inequality (4.1) implies that $|u_{n_0}| \ge \delta_0$ and, hence, $u = \{u_n\}$ is a nonzero sequence. It is not difficult to show that equation (0.4) possesses point-wise limits and, therefore, u is a solution of that equation.

Finally we have that

$$\sum_{n=-a}^{a} \left|u^{(k)}_n\right|^2 \le \|u^{(k)}\|_k \le C$$

for any fixed $a \in \mathbb{Z}$ and k large enough. Passing to the limit, we have that

$$\sum_{n=-a}^{a} |u_n|^2 \le C.$$

Since a is arbitrary, $u \in l^2$. □

Remark 4.1 Consider equation (0.4) with small nonlinearity, i. e.

$$Lu_n - \omega u_n = \sigma\lambda\chi_n|u_n|^2u_n,$$

where $\lambda > 0$ is a small parameter (all other data, including ω, are fixed). Then Lemma 2.1 shows that for the small solution $u = u_\lambda \in l^2$ obtained in Theorem 0.1 we have that

$$\|u_\lambda\|^2 \ge c\lambda^{-1},$$

with $c > 0$. This means that the solution u_λ bifurcates from infinity. Actually, one can scale out λ and obtain the same conclusion directly.

5 Exponential decay and nonexistence result

To complete the proof of Theorem 0.1, we have to show that the solution obtained decays exponentially fast. Actually, we have

Theorem 5.1 *Under assumptions of Theorem 3.1, let $u \in l^2$ be a solution of equation (0.4). Then u satisfies*

$$|u_n| \le Ce^{-\gamma|n|}, \quad n \in \mathbb{Z},$$

with some $C > 0$ and $\gamma > 0$.

Proof. Let $v_n = -\sigma\chi_n|u_n|^2$. Then

$$\widetilde{L}u_n - \omega u_n = 0, \tag{5.1}$$

where

$$\widetilde{L}u_n = Lu_n + v_n u_n.$$

Since $\lim_{|n|\to\infty} v_n = 0$, the multiplication by v_n is a compact operator in l^2. Hence,

$$\sigma_{ess}(\widetilde{L}) = \sigma_{ess}(L),$$

where σ_{ess} stands for the essential spectrum. Now (5.1) means that $u = \{u_n\}$ in an eigenfunction that corresponds to the eigenvalue of finite multiplicity $\omega \notin \sigma_{ess}(\widetilde{L})$ of the operator $\widetilde{L}$. Therefore, the result follows from the standard theorem on exponential decay for such eigenfunctions (see, e. g. [22]). □

The following result shows that if $\omega \in \sigma(L)$, then equation (0.4) has no well-decaying (e. g., exponentially fast) nontrivial solution.

Theorem 5.2 *Suppose that $\omega \in \sigma(L)$ and u is a solution of (0.4) such that $|n|^{1/2}u_n \in l^2$. The $u_n \equiv 0$.*

Proof. We have that u satisfies (5.1) and this means that ω is an eigenvalue embedded into $\sigma_{ess}(\widetilde{L})$. The potential $v = \{v_n\}$ satisfies $|n|v_n \in l^1$ and, by Theorem 7.11, [22], $\sigma_{ess}(L)$ is absolutely continuous, hence, contains no embedded eigenvalues. □

6 An extension of main result

Consider the following equation

$$Lu_n - \omega u_n = \sigma f_n(u_n) \tag{6.1}$$

which is more general that (0.4). Here the operator L is of the same form as above, ω belongs to some spectral gap (α, β) of L and $\sigma = \pm 1$. The nonlinearity $f_n(u)$ is supposed to satisfy the following assumptions.

(i) *The function $f_n(u)$ is continuous in $u \in \mathbb{R}$ and depends periodically in n, with period N.*

(ii) *There exist $p > 2$ and $c > 0$ such that*

$$0 \le f_n(u) \le c|u|^{p-1}$$

near $u = 0$.

(iii) *There exists $\mu > 2$ such that*

$$0 < \mu F_n(u) \equiv \int_0^u f_n(t)\,dt \le f_n(u)u, \quad u \ne 0.$$

Arguing as in the proof of Theorem 0.1, with corresponding modifications (see [14] for a similar result for continuum periodic nonlinear Schrödinger equations), one can obtain the following result.

Theorem 6.1 *Under assumptions (i)–(iii) suppose that either* $\sigma = +1$ *and* $\beta \neq +\infty$*, or* $\sigma = -1$ *and* $\alpha \neq -\infty$*. Then equation (6.1) has a nontrivial exponentially decaying solution. If either* $\sigma = +1$ *and* $\beta = +\infty$*, or* $\sigma = -1$ *and* $\alpha = -\infty$*, then there is no nontrivial solution in* l^2*.*

The most important example is the power nonlinearity

$$f_n(u) = \chi_n |u|^{p-2} u, \tag{6.2}$$

with $p > 2$. If $p = 4$, we obtain the cubic nonlinearity considered in Theorem 0.1

Theorem 6.1 can be extended immediately to the case of equation (6.1) on $\mathbb{Z}^d$, $d \geq 1$, where

$$Lu_n = \sum_{m \in \mathbb{Z}^d} a(n,m)\, u_m.$$

Here $a(n,m)$ satisfies

$$a(n+N, m+N) = a(n,m)$$

for some $N = (N_1, N_2, \dots, N_d)$ and

$$a(n,m) = 0$$

whenever $|n - m| \geq a_0 > 0$. The nonlinearity f_n, $n \in \mathbb{Z}^d$, must satisfy (*i*)–(*iii*) and the periodicity as assumption

$$f_{n+N}(u) = f_n(u),$$

with the same N as above.

7 Concluding remarks

In the past decade, localized solutions of DNLS has become a topic of intense research. Much of this work concerns the standard constant coefficient cubic DNLS and has been summarized in reviews [7, 8, 10]. Constant coefficient DNLS with general power nonlinearity (6.2) ($\chi_n \equiv 1$) is considered in [24].

Certainly, DNLS with periodic coefficients is not less important. In this case a new phenomenon appears. While the spectrum of $-\Delta$ consists of a single closed integral, in the spectrum of periodic operator L finite gaps typically open up. The corresponding DNLS may have standing wave solutions with carrier frequency in such a gap. Theorems 0.1 and 6.1 give rigorous results of this type.

Note that in the case of periodic DNLS, and even for more general equations, with ω below or above the spectrum (depending on σ), such solutions are shown to exist in [18]. In that paper the so-called Nehari manifold approach is employed, together with a discrete version of concentration compactness principle. For such values of ω the quadratic part of the corresponding functional is positive or negative definite. This fact simplifies the situation considerably. When ω lies in a finite gap, the quadratic part of the functional is strictly indefinite. This suggests us to use the linking theorem combined with periodic approximations. Such approach was used before in [13, 14, 17, 15, 16, 21]. Another approach to localized solutions of periodic DNLS, based on the generalized linking theorem (see, e. g. [25]) will be discussed elsewhere.

A Operator L

Let l denote the vector spaces of all two sided complex valued sequences $u = \{u_n\}_{n\in\mathbb{Z}}$ and $l^p \subset l$, $1 \le p \le \infty$, the subspace of all p-summable (bounded if $p = \infty$) sequences. Endowed with the standard norm $\|\cdot\|_{l^p}$, l^p is a Banach spaces (Hilbert space when $p = 2$).

Let $N \ge 1$ be a given integer and a_n and b_n two real valued sequences. The formula

$$Lu_n = a_m u_{n+1} + a_{n-1}u_{n-1} + b_n u_n \tag{A.1}$$

defines a linear operator acting in the space l. Moreover, L is a bounded linear operator in the space l^p, $1 \le p \le \infty$, and this is a self-adjoint operator in l^2. The space E_k of all kN-periodic sequences is a finite dimensional subspace of l^∞ invariant with respect to L. The restriction of L to E_k is denote by L_k.

Let

$$Q_k = \left\{ n \in \mathbb{Z} \ : \ -\left[\frac{kN}{2}\right] \le n \le kN - \left[\frac{kN}{2}\right] - 1 \right\},$$

where $[x]$ stands for the integer part of x. For a sequence $u = \{u_n\} \in l$ we set

$$R_k u_n = \begin{cases} u_n & \text{if} \quad n \in Q_k, \\ 0 & \text{if} \quad n \notin Q_k \end{cases}$$

and denote by $S_k u_n$ a unique sequence that belongs to E_k and such that

$$S_k u_n = u_n \quad \text{if } n \in Q_k.$$

Thus, R_k is a "*cut off*" operator, while S_k is "*periodization*" operator. For $u \in E_k$ we set

$$\|u\|_{l^p_k} = \|R_k u\|_{l^p}.$$

For any fixed k, $\|\cdot\|_{l^p_k}$, $1 \le p \le \infty$, form a family of equivalent norms on E_k, $\|\cdot\|_{l^2_k} = \|\cdot\|_k$ is the standard Euclidean norm on E_k and $(R_k u, R_k v) = (u, v)_k$ is the standard inner product on E_k.

Obviously,

$$\|R_k u\|_{l^p} \le \|u\|_{l^p}$$

and

$$\|S_k u\|_{l^p_k} \le \|u\|_{l^p}$$

for all $u \in l^p$. The following identities are easy to verify: for every $u \in l^p$, $1 \le p < \infty$,

$$\lim_{k\to\infty} \|R_k u\|_{l^p} = \lim_{k\to\infty} \|S_k u\|_{l^p_k} = \|u\|_{l^p}, \tag{A.2}$$

$$\lim_{k\to\infty} \|LR_k u\|_{l^p} = \lim_{k\to\infty} \|R_k L u\|_{l^p_k} = \|Lu\|_{l^p}, \tag{A.3}$$

$$\lim_{k\to\infty} \|L_k S_k u\|_{l^p_k} = \lim_{k\to\infty} \|S_k L u\|_{l^p_k} = \|Lu\|_{l^p}. \tag{A.4}$$

Let $\lambda \in \mathbb{C} \setminus \sigma(L)$. Then the operator $L - \lambda$ is invertible in $E = l^2$ and for every $u = \{u_n\} \in l^2$

$$(L-\lambda)^{-1} u_n = \sum_{m\in\mathbb{Z}} G(n,m;\lambda)\, u_m, \tag{A.5}$$

where $G(\cdot,\cdot;\lambda)$ is the so-called *Green function*. Due to the periodicity assumption,

$$G(\cdot + N, \cdot + N; \lambda) = G(\cdot,\cdot;\lambda) \tag{A.6}$$

(periodicity of the Green function along the diagonal). Moreover, the Green function possesses an exponential bound of the form

$$\bigl|G(n,m;\lambda)\bigr| \le C e^{-\mu|m-n|}, \quad n,m \in \mathbb{Z}, \tag{A.7}$$

where $C > 0$ and $\mu > 0$ can be chosen independent of λ as λ ranges over any compact subset of $\mathbb{C}\setminus\sigma(L)$ (see, e. g. [22]). Representation (A.5) and inequality (A.7) imply that for every $\lambda \in \mathbb{C} \setminus \sigma(L)$ the operator $(L-\lambda)^{-1}$ is a bounded linear operator in l^p for every $p \in [1,\infty]$. Moreover, the space E_k is invariant with respect to $(L-\lambda)^{-1}$ and, for every $u \in E_k$,

$$\bigl\|(L_k - \lambda)^{-1} u\|_{l^p_k} \le C_p \|u\|_{l^p_k},$$

with $C_p > 0$ independent of k. In particular, $\sigma(L_k) \subset \sigma(L)$.

Let (α,β) be a spectral gap of L. Denote by P^+ and P^- the spectral projectors in $E = l^2$ that correspond to the parts of $\sigma(L)$ lying in $(-\infty,\alpha]$ and $[\beta,+\infty)$, respectively. Note that these are orthogonal projectors. The associated spectral subspaces are denoted by E^+ and E^-, respectively. Also, we note that $\sigma(L_k) \cap (\alpha,\beta) = \emptyset$. We denote by P_k^+ and P_k^- the spectral projectors in E_k that correspond to $\sigma(L_k)\cap(-\infty,\alpha]$ and $\sigma(L_k)\cap[\beta,+\infty)$, respectively, and set $E_k^\pm = P_k^\pm E_k$ for the spectral subspaces.

The projectors $P^\pm$ are given by the Riesz formula

$$P\pm = -\frac{1}{2\pi i}\int_{C_\pm} (L-\lambda)^{-1} d\lambda,$$

where C_+ (resp., C_-) is a contour in $\mathbb{C}\setminus\sigma(L)$ encircling $\sigma(L)\cap[\beta,+\infty)$ (resp., $\sigma(L)\cap(-\infty,\alpha]$) so that $\sigma(L)\cap(-\infty,\alpha]$ (resp., $\sigma(L)\cap[\beta,+\infty)$) lies outside. Therefore, $P^\pm$ is of the form

$$P^\pm u_n = \sum_{m\in\mathbb{Z}} K(m,n) u_m, \tag{A.8}$$

where

$$K(m,n) = -\frac{1}{2\pi i}\int_{C_\pm} G(m,n;\lambda)\, d\lambda.$$

Hence

$$\big|K(m,n)\big| \le Ce^{-\mu|m-n|}, \quad m,n \in \mathbb{Z}, \tag{A.9}$$

with some $C > 0$ and $\mu > 0$. Note that the right hand part of (A.8) is a bounded linear operator in all spaces l^p, $1 \le p \le \infty$. Moreover, the operators $P_k^\pm$ are exactly the restrictions of $P^\pm$ from l^∞ to E_k and, therefore, are given by the same formula (A.8). Also it is easy to verify the operator

$$B = (L-\omega)P^\pm = P^\pm(L-\omega)$$

has a representation of the form

$$Bu_n = \sum_{m\in\mathbb{Z}} K(m,n)u_m, \tag{A.10}$$

with $K(m,n)$ (not the same as in (A.8)) satisfying (A.9).

Lemma A.1 *Let B be an operator of the form (A.10) satisfying (A.9). Then for any $u \in l^2$*

$$\lim_{k\to\infty}(BS_ku, S_ku)_k = (Bu,u).$$

The proof is contained in the proof of Theorem 3, [5].

B Linking

Here we recall the so-called linking theorem (see [20, 25]). Let $X = Y \oplus Z$ be a Banach space decomposed into the direct sum of two closed subspaces Y and Z, with $dim Y < \infty$. Let $\varrho > r > 0$ and let $z \in Z$ be a fixed vector, $\|z\| = 1$. Define

$$M = \big\{u = y + \lambda z \;:\; y \in Y, \|u\| \le \varrho, \lambda \ge 0\big\}$$

and

$$N = \big\{u \in Z \;:\; \|u\| = r\big\}.$$

Denote by $M_0 = \partial M$ the boundary of M, i. e.

$$M_0 = \big\{u = y + \lambda z \;:\; y \in Y, \|u\| = \varrho \text{ and } \lambda \ge 0, \text{ or } \|u\| \le \varrho \text{ and } \lambda = 0\big\}.$$

Consider a C^1 functional φ on E and suppose that φ satisfies the Palais-Smale condition, i. e. any sequence $u^{(j)} \in E$ such that $\varphi\big(u^{(j)}\big)$ is convergent and $\varphi'\big(u^{(j)}\big) \to 0$ contains a convergent subsequence. Suppose also that

$$\beta = \inf_{u\in N}\varphi(u) > \alpha = \sup_{u\in M_0}\varphi(u). \tag{B.1}$$

The last assumption means that φ possesses the so-called linking geometry.

Let

$$\Gamma = \big\{\gamma \in C(M;E) \;:\; \gamma = \text{id on } M_0\big\}.$$

Then

$$c = \inf_{\gamma\in\Gamma} \sup_{u\in M} \varphi\big(\gamma(u)\big)$$

is a critical value of φ and

$$\beta \le c \le \sup_{u\in M} \varphi(u). \tag{B.2}$$

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