On the number of vertices with a given degree in a Galton-Watson tree

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Abstract

Let $Y_k(\omega)$ ($k \geq 0$) be the number of vertices of a Galton-Watson tree ω having k children, so that $Z(\omega) := \sum_{k\geq 0} Y_k(\omega)$ is the total progeny of ω. In this paper, we shall prove various statistical properties of Z and Y_k. We first show, under a mild condition, an asymptotic expansion of $P(Z = n)$ as $n \to \infty$, improving the theorem of Otter (1949). Next, we show that $\mathcal{Y}_k(\omega) := \sum_{j=0}^{k} Y_j(\omega)$ is the total progeny of a new Galton-Watson tree which is hidden in the original tree ω. We then proceed to study the joint probability distribution of Z and $\{Y_k\}_k$, and show that as $n \to \infty$, $\{Y_k/n\}_k$ is asymptotically Gaussian under the conditional distribution $P(\cdot|Z = n)$.

1 Introduction

Let $\Pi = \{p_n\}_{n=0}^{\infty}$ be a probability distribution on non-negative integers and let $\{X_n\}_{n=0}^{\infty}$ ($X_0 = 0$) be the Galton-Watson process (or the discrete time branching process) with the offspring distribution Π. Various long time behaviors of the integer valued Markov chain $\{X_n\}$ form the main subject matter of classical treatises of Galton-Watson processes (e.g. [14], [2], [16], [26]). There, the random tree structure which is obvious in the intuitive description of the process is rather implicit, or even entirely omitted from the exposition. In this paper, we are interested in the typical shape of the random tree, which we shall call the Galton-Watson tree, obtained as the "trajectory "of the branching mechanism which gives rise to the process $\{X_n\}$. Namely a Galton-Watson tree is the random graph each of whose vertices gives birth to a random number of children, according to the probability distribution Π , and independently of each other. Two vertices are adjacent if and only if one is the parent of the other. Let Z be the total number of vertices, or the total progeny, of the Galton-Watson tree and let Y_k be the number of vertices having k children. Motivated by the classical pioneering work by Otter ([23]), which does not yet seem to have been fully appreciated, we shall investigate in some detail the asymptotic behavior of the probability distribution of Z, and of $\mathcal{Y}_k := \sum_{j=0}^{k} Y_j$, $k = 0, 1, 2, \ldots$, the number of vertices having at most k children. It will be seen that $\mathcal{Y}_k$ is the total progeny of a Galton-Watson tree which is hidden in the original tree. We shall also prove a central limit theorem for the distribution of $(Y_0, Y_1, \ldots)$ conditioned on Z. As a corollary of this theorem, the central limit theorem due to Mahmoud for vertices of uniform binary tree ([21]) will be reproduced. Our proofs are based on the analysis of generating functions, and we rely entirely upon the Lagrange inversion formula in obtaining explicit formulas for various probabilities, and upon the classical saddle point method in obtaining their asymptotic expressions.

In order to prepare a solid basis for our work, it is necessary to give a precise description of "trees"and to introduce a suitable σ-field $\mathcal{F}$ as well as a probability measure P on the space Ω of such trees. In this work we follow the convenient and elegant construction of the probability space $(\Omega, \mathcal{F}, P)$ due to Neveu ([22]). (It should be remarked, however, that an alternative construction of the probability space of a Galton-Watson tree had already been given by Otter himself [23], which turns out to be equivalent to that of Neveu (see [20]), and which even has some advantage when one considers Ω as a topological space.)

Now let U be the totality of finite sequences $u = j_1 j_2 \cdots j_n$ of strictly positive integers. We can write $U = \bigcup_{n \geq 0} \mathbf{N}^n$, where $\mathbf{N} = \{1, 2, \ldots\}$ and $\mathbf{N}^0 = \{\phi\}$, ϕ standing for the empty sequence. A tree ω is by definition a subset of U satisfying the following three conditions: (a) $\phi \in \omega$; (b) if $uj \in \omega$, then $u \in \omega$; (c) for each $u \in \omega$, there is a non-negative integer $\nu_u(\omega)$ such that $uj \in \omega \Leftrightarrow 1 \leq j \leq \nu_u(\omega)$. Here, for two sequences $u = j_1 j_2 \cdots j_n$ and $v = k_1 k_2 \cdots k_p$, we write uv for their conjunction: $uv = j_1 j_2 \cdots j_n k_1 k_2 \cdots k_p$. In particular, $uj = j_1 j_2 \cdots j_n j$ for $j \in \mathbf{N}$. Thus ω is a graph with vertices $u \in \omega$ and edges of the form (u, uj). Condition (a) says that ω always contain the special vertex ϕ, the root of ω, while condition (b) expresses the essential feature of a tree. Condition (c) says that our ω is what is called the ordered tree in combinatorics. The integer $\nu_u(\omega)$ is the number of children of the vertex $u \in \omega$, and we follow Otter in calling it the "type"of the vertex u. For notational convenience, we shall sometimes write $\nu_\omega(u)$ instead of $\nu_u(\omega)$.

For each $u \in U$, let $\Omega_u = \{\omega \in \Omega \mid \omega \ni u\}$ be the set of trees containing u as their common vertex, and let $\mathcal{F}$ be the σ-field generated by the family $\{\Omega_u \mid u \in U\}$. For $n = 1, 2, \ldots$, we let $\mathcal{F}_n$ be the σ-field generated by Ω_u's with $|u| \leq n$, $|u|$ being the length of the sequence u. For each $u \in U$, we also define the translation $T_u : \Omega_u \mapsto \Omega$ by $T_u(\omega) = \{v \in U | uv \in \omega\}$.

Now let $(\Omega^*, \mathcal{F}^*, P^*) = \prod_{u \in U}(\mathbf{Z}_+, \Pi)$ be the product over U of the discrete probability space $(\mathbf{Z}_+, \Pi)$, the σ-field $\mathcal{F}^*$ being generated by the coordinate maps ν_u^*, $u \in U$, and let the measurable map $\psi : (\Omega^*, \mathcal{F}^*) \mapsto (\Omega, \mathcal{F})$ be defined by

$$\psi(\omega^*) = \{u = j_1 \cdots j_p \mid j_{k+1} \leq \nu^*_{j_1 \cdots j_k}(\omega^*) \ , \ 0 \leq k < p\} \ .$$

If we define $P = P^* \circ \psi^{-1}$, then for each n, $\{T_u(\omega) \ ; \ u \in \omega, |u| = n\}$ are independent under the conditional distribution $P(\cdot|\mathcal{F}_n)$, and obey the same law P. Moreover if we set $X_n(\omega) := \sharp\{u \in \omega | \ |u| = n\}$, $n = 0, 1, 2, \ldots$, then $\{X_n\}_{n \geq 0}$ forms the Galton-Watson process with the offspring distribution Π such that $X_0 = 1$. See [22] for the precise statement of this proposition. Here $\sharp B$ means the cardinality of a finite set B.

In this setting, our Z and Y_k can be expressed as

$$Z(\omega) = \sum_{u \in U} 1_{\Omega_u}(\omega) \ ; \ Y_k(\omega) = \sharp\{u \in \omega \mid \nu_u(\omega) = k\} = \sum_{u \in U} 1_{\Omega_{uk} \setminus \Omega_{u(k+1)}}(\omega) \ ,$$

so that Z and Y_k are certainly $\mathcal{F}$-measurable. For $k = 0$, uk in the second formula is understood to mean $u\phi = u$.

Since we are interested in the probability distribution of Z and Y_k, we shall assume throughout this work that $p_0 > 0$, in which case we have $Z < \infty$ with positive probability. We also assume $p_0 + p_1 < 1$ in order to avoid the trivial case, which can be treated separately if necessary.

The outline of the present paper is as follows. In §2, we consider the asymptotic behavior of $P(Z = n)$ as $n \to \infty$. Under a certain condition on the generating function of Π, we shall show that $P(Z = n)$ has an asymptotic expansion as $n \to \infty$. If we take its main term, we can reproduce Otter's result (Theorem 4 of [23]). We also give some partial results when the above mentioned condition fails to hold. In §3, we show that $\mathcal{Y}_k(\omega)$ can be viewed as the total progeny of a random tree structure, which we shall call an abstract Galton-Watson tree, which is hidden in the original tree ω. We then investigate the asymptotic behavior of $P(\mathcal{Y}_k = n)$ as $n \to \infty$, for $k = 0, 1, 2, \ldots$. In §4, we give some explicit formulas for the joint distribution of Y_k, $k \geq 0$ conditioned on the event $\{Z = n\}$. We then consider its limiting behavior as $n \to \infty$ in §5, and prove a central limit theorem for Y_k's.

2 Asymptotic results for $P(Z = n)$

2.1 Preliminaries

Let $f(z) = \sum_{n=0}^{\infty} p_n z^n$ be the generating function of the offspring distribution $\Pi = \{p_n\}_{n=0}^{\infty}$, and let $\rho \geq 1$ be its radius of convergence. As announced in the introduction, we assume $p_0 > 0$ and $p_0 + p_1 < 1$, so that $f(0) > 0$ and $f(z)$ is strictly convex on $[0, \rho)$. Let further $\mathcal{P}(z) = \sum_{n=1}^{\infty} P(Z = n) z^n = \mathbf{E}[z^Z]$ be the generating function of the total progeny Z. Note that we have $Z \geq 1$ since a tree ω always contains the root ϕ. From the relation

$$Z(\omega) = 1 + \sum_{j=1}^{\nu_\phi(\omega)} \sum_{u \in U} 1_{\Omega_{ju}}(\omega) = 1 + \sum_{j=1}^{\nu_\phi(\omega)} Z \circ T_j(\omega) \tag{1}$$

and the conditional independence given $\mathcal{F}_1$ of $Z \circ T_j$, $j = 1, \ldots, \nu_\phi(\omega)$, we obtain

$$\begin{aligned} \mathcal{P}(z) &= \sum_{k=0}^{\infty} P(\nu_\phi = k) \mathbf{E}\left[z^{1 + Z\circ T_1 + \cdots + Z \circ T_k} \mid \nu_\phi = k \right] \\ &= z \sum_{k=0}^{\infty} p_k (\mathbf{E}[z^Z])^k = z f(\mathcal{P}(z)) \ . \end{aligned}$$

Thus $w = \mathcal{P}(z)$ satisfies the functional equation $w = z f(w)$. Since we are assuming $f(0) \neq 0$, we can apply Lagrange inversion formula (see e.g. [15]), to show that $P(Z = n)$, which is the n-th coefficient of the power series $\mathcal{P}(z)$, is given by

$$P(Z=n) = \frac{1}{n}\mathrm{Res}\left[\left(\frac{f(z)}{z}\right)^n\right] = \frac{1}{2\pi i n}\oint\left(\frac{f(z)}{z}\right)^n dz\ , \tag{2}$$

where $\oint$ denotes the contour integration along a circle surrounding the origin. (2) shows that $P(Z=n)$ is the coefficient of z^{n-1} of the power series $f(z)^n$ divided by n. But $f(z)^n$ is the generating function of the n-th convolution power Π^{*n} of Π, hence $nP(Z=n)$ is equal to the probability that the first generation of our Galton-Watson process consists of $n-1$ individuals when there are n individuals at the 0-th generation. In other words, if we let $\{q(n,m)\}_{n,m\geq 0}$ the transition probability of the Galton-Watson process with the offspring distribution Π, then we have

$$P(Z=n) = \frac{1}{n}q(n,n-1)\ . \tag{3}$$

By a purely probabilistic argument, Dwass [9] (see also [16]) obtained this seemingly non-trivial relation. The basic functional equation $\mathcal{P}(z) = zf(\mathcal{P}(z))$ seems to date back to the work of Hawkins and Ulam (see the footnote on page 32 of [14]), and was independently re-discovered by Good ([10]) and Otter ([23]), the latter having noted the formula (2) at the same time. This was also re-discovered by Boyd ([3]), who used it to give an alternative proof of the relation (3).

The class of distributions on natural numbers which are obtained as distributions of the total progeny of Galton-Watson trees coincides the class of "Lagrange distributions" treated systematically by Consul and Shenton ([4]). Good ([11]) generalized the Lagrange inversion formula to multi-variable functions, and discussed its relation to multi-type branching processes. See also [12].

Now let α ($\alpha \geq 1$) be the radius of convergence of the power series $\mathcal{P}(z)$ and let $a = \mathcal{P}(\alpha-)$. Then we have the following relations.

Proposition 1 *Under the condition $p_0 > 0$ and $p_0 + p_1 < 1$, we have*

(i) $a \leq \rho$*;*

(ii) *α and a are finite;*

(iii) $f(a)/a = 1/\alpha$*;*

(iv) $f'(a) \leq 1/\alpha$*;*

(v) *$f'(a) = 1/\alpha$ if $a < \rho$;*

(vi) *if, conversely, $f'(\zeta) = f(\zeta)/\zeta$ for some $0 < \zeta < \rho$, then $a < \rho$ and $\zeta = a$.*

Proof. All the statements except (vi) are proved in Lemma 4 and Theorem 3 of [23]. To show (vi), suppose $a = \rho$, in which case ρ is finite. Then as z increases from 0 to α, $\mathcal{P}(z)$ strictly increases from 0 to ρ. Hence there exists a $z_0 \in (0,\alpha)$ such that $\zeta = \mathcal{P}(z_0)$. If we put $z = z_0$ in the formula $zf(\mathcal{P}(z)) = \mathcal{P}(z)$ and in $zf'(\mathcal{P}(z))\mathcal{P}'(z) + f(\mathcal{P}(z)) = \mathcal{P}'(z)$ which is obtained by differentiating the former with respect to z, then by $f'(\zeta) = f(\zeta)/\zeta$, we get $\mathcal{P}'(z_0) + f(\zeta) = \mathcal{P}'(z_0)$, whence $f(\zeta) = 0$. But this is a contradiction because $f(z)$ is increasing and $f(0) = p_0 > 0$. Thus we must have $a < \rho$, and by (iii) and (v), $f'(z) = f(z)/z$ is satisfied also at $z = a$. But since $(zf'(z) - f(z))' = zf''(z) > 0$ on $(0, \rho)$, such z is unique, so that $\zeta = a$.

From now on, we shall refer the condition of the statement (vi) as *Condition A*. Since $(zf'(z) - f(z))|_{z=0} = -p_0 < 0$, it is obviously equivalent to the condition

$$\lim_{z\uparrow\rho}(zf'(z) - f(z)) > 0 \ . \tag{4}$$

It is also easy to see that Condition A holds if and only if the function $f(z)/z$ attains a unique minimum in $(0, \rho)$, which is equal to $f(a)/a$.

When, on the other hand, Condition A fails, we have $\rho = a < \infty$, and $f(z)/z$ is decreasing on $(0, a]$. Hence the number a is always characterized by the relation

$$\frac{f(a)}{a} = \inf_{0<z<\rho} \frac{f(z)}{z} \ , \tag{5}$$

whether or not Condition A holds.

The following proposition shows some trivial *a priori* estimate for $P(Z = n)$.

Proposition 2 *(i) Let $d = d(\Pi)$ be the greatest common divisor of all $n \geq 0$ such that $p_n > 0$. Then $P(Z = n) = 0$ if $n \not\equiv 1 \pmod d$.*

(ii) It holds that

$$\limsup_{n\to\infty} P(Z = n)^{1/n} = f(a)/a \ ,$$

where a is given by (5).

Proof. (i) This is a part of Theorem 4 of [23]. Here we give an alternative proof, to emphasize that it has nothing to do with Condition A.

In the expansion

$$\left(\frac{f(z)}{z}\right)^n = \sum_{j_1, j_2, \ldots, j_n \geq 0} p_{j_1} p_{j_2} \cdots p_{j_n} z^{j_1+j_2+\cdots+j_n-n} \ ,$$

we have $p_{j_1} p_{j_2} \cdots p_{j_n} > 0$ only when all of $j_1, j_2, \ldots, j_n$ are integer multiple of d. Hence if $n \not\equiv 1 \pmod d$, then we have

$$j_1 + j_2 + \cdots + j_n - n \neq -1$$

for any of such $j_1, j_2, \ldots, j_n$, so that the residue at the origin of $(f(z)/z)^n$ vanishes and so does $P(Z = n)$ by (2). This proves (i). The assertion (ii) is obvious from the Cauchy-Hadamard formula for the radius of convergence of power series and from (iii) of Proposition 1. Note that we have $0 < f(a)/a \leq 1$ by $1 \leq \alpha < \infty$.

2.2 Asymptotic expansion of $P(Z = n)$ as $n \to \infty$

The folowing is an improvement of Theorem 4 of [23].

Theorem 1 *Under Condition A, $P(Z = n)$ has an asymptotic expansion of the form*

$$P(Z = n) \ \sim \ \sum_{k=1}^{\infty} c_k \left(\frac{f(a)}{a} \right)^n n^{-k-1/2} \tag{6}$$

as we let $n \to \infty$ keeping $n \equiv 1 \pmod{d}$. Here a and d are the numbers given in Propositions 1 and 2 respectively, and

$$c_1 = d\sqrt{\frac{f(a)}{2\pi f''(a)}} \ .$$

Proof. Throughout the proof, we assume $n \equiv 1 \pmod{d}$. Condition A means that we can take the contour $|z| = a$ in the integral of (2). Noting that $f(z)$ depends only on z^d, we can rewrite (2) as

$$P(Z = n) = \frac{ad}{2\pi n} \int_{-\pi/d}^{\pi/d} \exp\{n\psi(\theta)\} e^{i\theta} d\theta \ , \tag{7}$$

where we let $\psi(\theta) = \log[f(ae^{i\theta})/ae^{i\theta}]$. From $f'(a) = f(a)/a$, it is easy to verify $\psi(0) = \log[f(a)/a]$, $\psi'(0) = 0$ and $\psi''(0) = -a^2 f''(a)/f(a) < 0$. Thus we are in a typical situation in which the Laplace method yields the asymptotic expansion (see [6] and [5]). Since the technique is standard, we shall only sketch the outline, omitting the details.

Pick a $\delta \in (0, \pi/d)$ sufficiently small so that for $|\theta| \le \delta$, we have the inequality $\mathbf{Re}\psi(\theta) \le -\eta\theta^2$ with an $\eta > 0$, and the expansion

$$\psi(\theta) = \log \frac{f(a)}{a} - K\theta^2 + \sum_{k=3}^{\infty} B_k \theta^k \ ,$$

where we have let $\psi''(0) = -2K$. In fact, if we let $\rho_k := p_k a^k / f(a)$, then $\Pi_a := \{\rho_k\}_{k \ge 0}$ is a probability distribution on $\mathbf{Z}_+$ whose characteristic function is $f(ae^{i\theta})/f(a)$. Hence if κ_k is the k-th cumulant of the distribution Π_a, then we have the relation $B_k = i^k \kappa_k / k!$.

It is easy to show $|f(ae^{i\theta})| < f(a)$ for $\theta \in [-\pi/d, \pi/d] \setminus \{0\}$, (see the lemma below) so that

$$\Delta_\delta := \sup_{\delta \le |\theta| \le \pi/d} \left| \frac{f(ae^{i\theta})}{ae^{i\theta}} \right| < \frac{f(a)}{a} \ . \tag{8}$$

Hence

$$\begin{aligned} P(Z = n) &= \frac{ad}{2\pi n} \int_{-\delta}^{\delta} \exp\{n\psi(\theta)\} e^{i\theta} d\theta + \mathcal{O}(n^{-1}\Delta_\delta^n) \\ &= \frac{ad}{2\pi n} \left(\frac{f(a)}{a} \right)^n \int_{-\delta}^{\delta} \exp(-nK\theta^2) \exp(i\theta + n\theta^3 \beta(\theta)) d\theta \\ &\quad + \mathcal{O}(n^{-1}\Delta_\delta^n) \end{aligned} \tag{9}$$

for large n, where we have set $\beta(\theta) := \sum_{k\geq 3} B_k\theta^{k-3}$.

Following [6], introduce the power series

$$P(\omega,\theta) = \exp(i\theta + \omega\beta(\theta)) = \sum_{\ell,m\geq 0} c_{\ell m}\omega^\ell\theta^m$$

and the polynomial

$$P_A(\omega,\theta) = \sum_{\ell+m\leq A} c_{\ell m}\omega^\ell\theta^m \ .$$

Then we have

$$P(\omega,\theta) - P_A(\omega,\theta) = \mathcal{O}(\omega^{A+1}) + \mathcal{O}(\theta^{A+1})$$

as $|\omega| + |\theta| \to 0$.

We now replace δ by $\tau_n := n^{-1/3}$ in the integral in (9). The resulting error can be estimated to be $\mathcal{O}(n^{-2/3}e^{-\eta n^{1/3}})$. Then we insert $P_A(n\theta^3,\theta)$ in place of $\exp(i\theta + n\theta^3\beta(\theta)) = P(n\theta^3,\theta)$. The bound of the error committed by this replacement is given by

$$\begin{aligned} & \frac{ad}{2\pi n}\left(\frac{f(a)}{a}\right)^n \times \mathcal{O}\left(\int_0^{\tau_n} \exp(-nK\theta^2)\{(n\theta^3)^{A+1} + \theta^{A+1}\}d\theta\right) \\ = \quad & \mathcal{O}\left(\frac{1}{n}\left(\frac{f(a)}{a}\right)^n n^{-1-A/2}\right) \ , \end{aligned}$$

which absorbs the other smaller error terms. Thus we have for any $A \geq 0$,

$$P(Z=n) = \frac{ad}{2\pi n}\left(\frac{f(a)}{a}\right)^n \left[\int_{-\tau_n}^{\tau_n} \exp(-nK\theta^2)P_A(n\theta^3,\theta)d\theta + \mathcal{O}(n^{-1-A/2})\right] \ . \tag{10}$$

Finally we extend the integral to the whole real line with the error which can be estimated to be $\mathcal{O}(e^{-\alpha n^{1/3}})$ for some $\alpha > 0$. Thus we arrive at

$$\begin{aligned} P(Z=n) &= \frac{ad}{2\pi n}\left(\frac{f(a)}{a}\right)^n \left[\sum_{\ell+m\leq A} c_{\ell m}n^\ell \int_{-\infty}^{\infty} \exp(-nK\theta^2)\theta^{3\ell+m}d\theta + \mathcal{O}(n^{-1-A/2})\right] \\ &= \frac{ad}{2\pi}\left(\frac{f(a)}{a}\right)^n \left[\sum_{\ell+m\leq A} \gamma_{\ell m}K^{-\frac{3\ell+m+1}{2}}n^{-1-\frac{\ell+m+1}{2}} + \mathcal{O}(n^{-2-\frac{A}{2}})\right] \ , \end{aligned} \tag{11}$$

where $\gamma_{\ell m} = 0$ if $\ell + m$ is odd, and $\gamma_{\ell m} = \Gamma((3\ell+m+1)/2)c_{\ell m}$ if $\ell + m$ is even. If we take $A = 2k-1$, $k = 1, 2, \ldots$, then the error term becomes $\mathcal{O}(n^{-k-3/2})$, while the final term in the sum is a constant multiple of $n^{-k-1/2}$. The validity of (11) for all $A > 0$ thus implies that we have the asymptotic expansion

$$P(Z=n) \sim \frac{ad}{2\pi}\sum_{k=1}^{\infty}\left(\sum_{\ell+m=2(k-1)} \gamma_{\ell m}K^{-\frac{3\ell+m+1}{2}}\right)\left(\frac{f(a)}{a}\right)^n n^{-k-\frac{1}{2}}$$

as $n \to \infty$. The coefficient for $k = 1$ is

$$\frac{ad}{2\pi}\gamma_{00}K^{-1/2} = \frac{ad}{2\pi}c_{00}\Gamma\left(\frac{1}{2}\right)\left(\frac{a^2 f''(a)}{2f(a)}\right)^{-\frac{1}{2}} = d\sqrt{\frac{f(a)}{2\pi f''(a)}} \ .$$

Lemma 1 *If $0 < p_0 < 1$, then we have $|f(re^{i\theta})| < f(r)$, given that $0 < r < \rho$ and $0 < |\theta| \leq \pi/d$.*

Proof. Let $\mathcal{A} := \{k \geq 1;\ p_k > 0\}$. Since we have, for any $k \in \mathcal{A}$, the inequality

$$|f(re^{i\theta})| \leq |p_0 + p_k r^k e^{ik\theta}| + \sum_{n \neq 0,k} p_n r^n \ ,$$

where

$$|p_0 + p_k r^k e^{ik\theta}|^2 = p_0^2 + p_k^2 r^{2k} + 2p_0 p_k r^k \cos k\theta \ ,$$

all we have to do is to verify that $\cos k\theta < 1$ for som $k \in \mathcal{A}$. When $\mathcal{A} = \{d\}$, this is obvious from the assumption on θ. When $\mathcal{A}$ contains at least two numbers, take distinct $k, k' \in \mathcal{A}$ such that $g.c.d.(k, k') = d$. Then there are integers m and m' such that $mk + m'k' = d$. Now if we had $\cos k\theta = \cos k'\theta = 1$, then $k\theta = 2\pi\ell$ and $k'\theta = 2\pi\ell'$ for som integers ℓ and ℓ'. Hence we would have $\theta d = \theta(mk + m'k') \in 2\pi\mathbf{Z}$, contradicting the assumption on θ. Thus either $\cos k\theta < 1$ or $\cos k'\theta < 1$.

2.3 Some partial results when Condition A fails

When Condition A fails to hold, we have in view of Proposition 1, $\rho = a < \infty$, and $f(\rho) = f(a) = a/\alpha < \infty$. Also we have $f'(\rho) = f'(a) \leq 1/\alpha < \infty$, namely $f'(\rho) \leq f(\rho)/\rho$. On the other hand, $f(\rho)/\rho = f(a)/a \leq 1/\alpha \leq 1$ because of $\alpha \geq 1$. In particular, $f(z) = f(x + iy)$ is of C^1 with respect to (x, y) on the closed disk $\{|z| \leq \rho\}$, hence we can take the circle $\{|z| = \rho\}$ as the contour of integration in (2), so that

$$P(Z = n) = \frac{\rho d}{2\pi n}\int_{-\pi/d}^{\pi/d} \exp(n\psi(\theta))e^{i\theta}d\theta \ ,$$

where we let $\psi(\theta) = \log\{f(\rho e^{-i\theta}/\rho e^{i\theta})\}$. We shall consider two cases separately.

2.3.1 The case $f'(\rho) = f(\rho)/\rho$.

In this case, we shall assume $f^{(4)}(\rho) < \infty$. As before, we have $\psi'(0) = 0$ and

$$-2K := \psi''(0) = \rho^2 f''(\rho)/f(\rho) < 0 \ ,$$

so that

$$\psi(\theta) = \psi(0) - K\theta^2 + i\beta\theta^3 + \phi(\theta) \ , \tag{12}$$

where $i\beta = \psi^{(3)}(0)/3!$ with a real β and $\phi(\theta) = \mathcal{O}(\theta^4)$ as $\theta \to 0$. Moreover, $\psi(0) = \log(f(\rho)/\rho)$ is the maximum value of $\mathbf{Re}\psi(\theta)$. Now we can proceed as in §2.2, to obtain

$$\begin{aligned} P(Z=n) &= \frac{\rho d}{2\pi n}\left\{\int_{-\delta}^{\delta} \exp(n\psi(\theta))e^{i\theta}d\theta + \mathcal{O}(\Delta_\delta^n)\right\} \\ &= \frac{\rho d}{2\pi n}\left\{\int_{-\tau_n}^{\tau_n} \exp(n\psi(\theta))e^{i\theta}d\theta + \mathcal{O}\left(\left(\frac{f(\rho)}{\rho}\right)^n \int_{\tau_n}^{\delta} e^{-\eta n\theta^2}d\theta\right) + \mathcal{O}(\Delta_\delta^n)\right\} , \end{aligned}$$

where δ, Δ_δ, τ_n and η have the same meaning as before. Since

$$\int_{\tau_n}^{\delta} e^{-\eta n\theta^2}d\theta = \mathcal{O}(n^{-2/3}e^{-\eta n^{1/3}}) ,$$

it absorbs the error term $\mathcal{O}(\Delta_\delta^n)$. Hence taking (12) and $\tau_n = n^{-1/3}$ into account, we can write

$$\begin{aligned} P(Z=n) &= \frac{\rho d}{2\pi n}\left(\frac{f(\rho)}{\rho}\right)^n \left\{\int_{-\tau_n}^{\tau_n} \exp\left(-Kn\theta^2 + i\beta n\theta^3 + n\phi(\theta)\right)e^{i\theta}d\theta + \mathcal{O}(n^{-2/3}e^{-\eta n^{1/3}})\right\} \\ &= \frac{\rho d}{2\pi n}\left(\frac{f(\rho)}{\rho}\right)^n \left\{\int_{-\tau_n}^{\tau_n} \exp\left(-Kn\theta^2 + i\beta n\theta^3\right)e^{i\theta}d\theta \right. \\ &\quad \left. + \mathcal{O}\left(\int_0^{\tau_n} e^{-Kn\theta^2} n\phi(\theta)d\theta\right) + \mathcal{O}\left(n^{-2/3}e^{-\eta n^{1/3}}\right)\right\} . \end{aligned}$$

From $\phi(\theta) = \mathcal{O}(\theta^4)$, we obtain the estimate

$$\int_0^{\tau_n} e^{-Kn\theta^2} n\phi(\theta)d\theta = \mathcal{O}(n^{-3/2}) .$$

On the other hand,

$$\begin{aligned} &\int_{-\tau_n}^{\tau_n} \exp\left(-Kn\theta^2 + i\beta n\theta^3 + i\theta\right)d\theta \\ &= \int_{-\tau_n}^{\tau_n} e^{-Kn\theta^2}\cos(\beta n\theta^3 + \theta)d\theta \\ &= \int_{-\tau_n}^{\tau_n} e^{-Kn\theta^2}d\theta + \mathcal{O}\left(\int_{-\tau_n}^{\tau_n} e^{-Kn\theta^2}(n^2\theta^6 + \theta^2)d\theta\right) \\ &= \frac{1}{\sqrt{n}}\int_{-\infty}^{\infty} e^{-Kz^2}dz + \mathcal{O}(n^{-3/2}) , \end{aligned}$$

thus

$$P(Z=n) = \sqrt{\frac{f(\rho)}{2\pi f''(\rho)}}\left(\frac{f(\rho)}{\rho}\right)^n n^{-3/2}(1 + \mathcal{O}(n^{-1})) .$$

2.3.2 The case $f'(\rho) < f(\rho)/\rho$.

In this case, we shall assume $f^{(k)}(\rho) < \infty$ for some $k \geq 3$. As before, $\psi(0) = \log(f(\rho)/\rho)$ is the maximum value of $\mathbf{Re}\psi(\theta)$. Moreover we have

$$\psi'(0) = i\frac{\rho f'(\rho) - f(\rho)}{f(\rho)} =: -i\beta$$

with $\beta > 0$ and

$$\psi''(0) = \frac{\rho f'(\rho)}{f(\rho)^2}(\rho f'(\rho) - f(\rho)) - \rho^2\frac{f''(\rho)}{f(\rho)} < 0\ .$$

Thus letting $-2K = \psi''(0)$, we can write

$$\psi(\theta) = \psi(0) - i\beta\theta - K\theta^2 + \phi(\theta)\ , \tag{13}$$

with ϕ a C^k function satisfying $\phi(\theta) = \mathcal{O}(\theta^3)$ and $\phi(0) = \phi'(0) = \phi''(0) = 0$.

We now estimate, with the same notation as before,

$$P(Z = n) \quad = \quad \frac{\rho d}{2\pi n}\left\{\int_{-\tau_n}^{\tau_n} \exp(n\psi(\theta))e^{i\theta}d\theta + \mathcal{O}\left(\left(\frac{f(\rho)}{\rho}\right)^n n^{-2/3}e^{-\eta n^{1/3}}\right)\right\}\ .$$

If we let

$$J_n := \int_{-\tau_n}^{\tau_n} \exp(n\psi(\theta))e^{i\theta}d\theta\ ,$$

then

$$J_n = \left(\frac{f(\rho)}{\rho}\right)^n \frac{1}{\sqrt{n}}\int_{-n^{1/6}}^{n^{1/6}} e^{-i\beta\sqrt{n}z}F_n(z)dz\ ,$$

where

$$F_n(z) := \exp\left[-Kz^2 + n\phi\left(\frac{z}{\sqrt{n}}\right) + i\frac{z}{\sqrt{n}}\right]\ .$$

Now it is easy to see that we can find constants $A, C > 0$ which are independent of n so that for $j = 0, 1, \ldots, k$ and $|z| \leq n^{1/6}$, we have

$$\left|\frac{d^j}{dz^j}F_n(z)\right| \leq Ae^{-cz^2}\ .$$

Integrating by parts k times, we obtain

$$\begin{aligned} J_n \quad &= \quad \left(\frac{f(\rho)}{\rho}\right)^n \frac{1}{\sqrt{n}}\left\{\sum_{j=1}^{k}(-1)^{j-1}\left[\frac{e^{-i\beta\sqrt{n}z}}{(-i\beta\sqrt{n})^j}\frac{d^{j-1}}{dz^{j-1}}F_n(z)\right]_{z=-n^{1/6}}^{z=n^{1/6}}\right. \\ &\qquad \left. +(-1)^k\int_{-n^{1/6}}^{n^{1/6}}\frac{e^{-i\beta\sqrt{n}z}}{(-i\beta\sqrt{n})^k}\frac{d^k}{dz^k}F_n(z)dz\right\} \\ &= \quad \left(\frac{f(\rho)}{\rho}\right)^n \frac{1}{\sqrt{n}} \times \mathcal{O}(n^{-k/2})\ , \end{aligned}$$

and consequently

$$P(Z=n)=\mathcal{O}\left(\left(\frac{f(\rho)}{\rho}\right)^n n^{-\frac{3}{2}-\frac{k}{2}}\right) ,$$

as $n\to\infty$.

For illustration, consider the case $\rho=1$, so that $f(\rho)/\rho=1$. The condition $f'(\rho)<f(\rho)/\rho$ means that our Galton-Watson tree is subcritical. What has just been proved shows that

$$P(Z=n)=\mathcal{O}(n^{-\frac{3}{2}-\frac{k}{2}})$$

if $f^{(k)}(1)<\infty$ $(k\geq 3)$. If, for example, $p_n\sim e^{-\delta n^\gamma}$ with $\delta>0$ and $0<\gamma<1$, then $f^{(k)}(1)<\infty$ for all $k\geq 3$, and consequently $P(Z=n)$ decays faster than any power of n. But still one has

$$\limsup_{n\to\infty} P(Z=n)^{1/n}=1$$

by Proposition 2.

3 The distribution of $\mathcal{Y}_k$

3.1 $\mathcal{Y}_k$ as the total progeny of a hidden Galton-Watson tree

In this section, we consider $\mathcal{Y}_k(\omega):=\sum_{j=0}^k Y_j(\omega)$, the number of those vertices of the Galton-Watson tree ω having at most k children, $\mathcal{Y}_0(\omega)$ being in particular the number of "leaves "of ω.

Let

$$Q(z)=Q_k(z):=\mathbf{E}[1_{\{\mathcal{Y}_k<\infty\}}z^{\mathcal{Y}_k}]=\sum_{n=1}^{\infty}P(\mathcal{Y}_k=n)z^n$$

be the generating function of the distribution of $\mathcal{Y}_k$. (Note that by our assumption $p_0>0$, we have $P(Y_0>0)=1$. Thus $P(\mathcal{Y}_k=0)=0$ for all $k\geq 0$.)

Since we have

$$\mathcal{Y}_k(\omega)=1_{\{\nu_\phi\leq k\}}(\omega)+\sum_{j=1}^{\nu_\phi(\omega)}(\mathcal{Y}_k\circ T_j)(\omega) ,$$

we can proceed, just as in §2.1,

$$\begin{aligned}
Q(z) &= \mathbf{E}\left[z^{1_{\{\nu_\phi\leq k\}}}\prod_{j=1}^{\nu_\phi}\left(1_{\{\mathcal{Y}_k<\infty\}}\circ T_j\right)\left(z^{\mathcal{Y}_k}\circ T_j\right)\right]\\
&= \mathbf{E}\left[z^{1_{\{\nu_\phi\leq k\}}}\mathbf{E}\left[\prod_{j=1}^{\nu_\phi}\left(1_{\{\mathcal{Y}_k<\infty\}}z^{\mathcal{Y}_k}\circ T_j\right)\middle|\mathcal{F}_1\right]\right]\\
&= \mathbf{E}\left[z^{1_{\{\nu_\phi\leq k\}}}\prod_{j=1}^{\nu_\phi}\mathbf{E}\left[1_{\{\mathcal{Y}_k<\infty\}}z^{\mathcal{Y}_k}\right]\right]
\end{aligned}$$

$$\begin{aligned} &= z\mathbf{E}\left[1_{\{\nu_\phi \le k\}} Q(z)^{\nu_\phi}\right] + \mathbf{E}\left[1_{\{\nu > k\}} Q(z)^{\nu_\phi}\right] \\ &= z\sum_{j=0}^{k} p_j Q(z)^j + f(Q(z)) - \sum_{j=0}^{k} p_j Q(z)^j \ . \end{aligned}$$

Thus $w = Q(z)$ solves the equation

$$z\sum_{j=0}^{k} p_j w^j = w - f(w) + \sum_{j=0}^{k} p_j w^j \ ,$$

or

$$z = \frac{w}{g_k(w)} \ ,$$

where we have set

$$g_k(w) = \frac{w\sum_{j=0}^{k} p_j w^j}{w - f(w) + \sum_{j=0}^{k} p_j w^j} \ .$$

For later use, let us further define

$$h_k(w) = \sum_{j=0}^{k} p_j w^j \quad \text{and} \quad \varphi_k(w) = \sum_{j>k} p_j w^{j-1} \ . \tag{14}$$

Then

$$g_k(w) = \frac{h_k(w)}{1 - \varphi_k(w)} \ . \tag{15}$$

We shall now prove that $g_k(w)$ is the generating function of a probability distribution $\Pi^{(k)} := \{p_n^{(k)}\}_{n=0}^{\infty}$ which satisfies our basic assumption, namely $p_0^{(k)} > 0$ and $p_0^{(k)} + p_1^{(k)} < 1$.

To this end, let us write

$$\begin{aligned} g_k(w) &= h_k(w)\sum_{n=0}^{\infty} \varphi_k(w)^n \\ &= \left(\frac{h_k(w)}{h_k(1)}\right) h_k(1) \sum_{n=0}^{\infty} \varphi_k(1)^n \left(\frac{\varphi_k(w)}{\varphi_k(1)}\right)^n \ . \end{aligned}$$

If we set

$$H(w) := \frac{h_k(w)}{h_k(1)} \ ;$$

$$G(\zeta) := h_k(1) \sum_{n=0}^{\infty} \varphi_k(1)^n \zeta^n \ ;$$

$$\psi(w) := \frac{\varphi_k(w)}{\varphi_k(1)} \ ,$$

then H, G, ψ are generating functions respectively of

$$P(\nu_\phi \in \bullet \mid \nu_\phi \le k)\ ;$$

$$\text{the geometric distribution with parameter } \varphi_k(1) = P(\nu_\phi > k)\ ;$$

and

$$P(\nu_\phi - 1 \in \bullet \mid \nu_\phi > k)\ .$$

Consequently, if X, N, $S_1, S_2, \ldots$ are independent integer-valued random variables, and if X [resp. N, S_j] has H [resp. G, ψ] as its generating function, then $g_k(w) = G(\psi(w))H(w)$ is the generating function of the random variable

$$\sum_{j=1}^{N} S_j + X\ .$$

Obviously $g_k(0) = p_0$ for $k \ge 1$ and $= p_0/(1-p_1)$ for $k = 0$, whence $p_0^{(k)} = g_k(0) > 0$. By

$$g_k'(w) = \frac{h_k'(w)(1-\varphi_k(w)) + h_k(w)\varphi_k'(w)}{(1-\varphi_k(w))^2}$$

and by $\varphi_k(1) + h_k(1) = 1$, we get

$$g_k'(1) = \frac{h_k'(1) + \varphi_k'(1)}{h_k(1)} = 1 + \frac{f'(1) - 1}{h_k(1)}\ .$$

This shows that $g_k'(1) < 1$ [resp. $= 1$ or > 1] if and only if $f'(1) < 1$ [resp. $= 1$ or > 1]. On the other hand, we have

$$g_k'(0) = \begin{cases} p_0p_2/(1-p_1)^2 & (k = 0) \\ p_1 + p_0p_2 & (k = 1) \\ p_1 & (k > 1) \end{cases}\ .$$

Hence for $k > 1$, $p_0^{(k)} + p_1^{(k)} = g_k(0) + g_k'(0) = p_0 + p_1 < 1$ by the assumption. Similarly for $k = 1$,

$$g_1(0) + g_1'(0) = p_0 + p_1 + p_0p_2 < 1\ .$$

To consider the case $k = 0$, let $q_2 = \sum_{n\ge 2} p_n > 0$. Then

$$\begin{aligned} p_0(1-p_1+p_2) &= (1-p_1-q_2)(1-p_1+p_2) \\ &\le (1-p_1-q_2)(1-p_1+q_2) \\ &= (1-p_1)^2 - q_2^2 < (1-p_1)^2\ , \end{aligned}$$

and hence

$$g_0(0) + g_0'(0) = \frac{p_0}{1-p_1} + \frac{p_0p_2}{(1-p_1)^2} = \frac{p_0(1-p_1+p_2)}{(1-p_1)^2} < 1\ .$$

Finally, if $t > 0$ satisfies $f(t) = t$, then $h_k(t) + t\varphi_k(t) = t$ for any $k \ge 0$, namely

$$g_k(t) = \frac{h_k(t)}{1-\varphi_k(t)} = t\ .$$

These considerations can be summerized in the following theorem:

Theorem 2 *For any $k \geq 0$, the distribution of $\mathcal{Y}_k$ is equal to that of the total progeny of the Galton-Watson tree with the offspring distribution $\Pi^{(k)} = \{p_n^{(k)}\}_{n=0}^{\infty}$. As the original $\Pi = \{p_n\}_{n=0}^{\infty}$, $\Pi^{(k)}$ satisfies the condition $p_0^{(k)} > 0$, $p_0^{(k)} + p_1^{(k)} < 1$. The new Galton-Watson tree is sub-critical [resp. critical or super-critical] if and only if the original one is so. Moreover the extinction probabilities of both Galton-Watson trees, which are solutions respectively of $f(t) = t$ and $g_k(t) = t$, coincide.*

Corollary 1 *$P(Z = \infty\ ,\ \mathcal{Y}_k < \infty) = 0$ for any $k \geq 0$.*

Proof. Obviously $\{Z < \infty\} \subset \{\mathcal{Y}_k < \infty\}$. But the probability of both events, being the extinction probability of the old or the new Galton-Watson tree respectively, are equal.

We have thus shown the existence of a Galton-Watson tree whose total progeny has the same distribution as $\mathcal{Y}_k$. This would be sufficient for most applications, but it is natural to ask if the new Galton-Watson tree can be constructed as a functional of the original tree ω, so that its total progeny *coincides* with $\mathcal{Y}_k(\omega)$. We shall now show that this is actually the case, but before doing so, we need to prepare an abstract notion of Galton-Watson tree. To begin with, let us generalize the notion of tree itself.

Definition 1 *Let $G = (V, E)$ be a directed graph, where V is a finite or countable vertex set and $E \subset V \times V$. We shall call G an abstract tree if the following five conditions hold:*

1) *For any $u \in V$, $(u, u) \notin E$. Namely G has no loops.*

2) *There is a special vertex $u_0 \in V$ such that $(u, u_0) \notin E$ for all $u \in V$. We call u_0 the "root "of G.*

3) *For any $v \in V \setminus \{u_0\}$, there is one and only one $u \in V$ such that $(u, v) \in E$. In particular, G is an oriented graph.*

4) *The (non-directed) graph $G^s = (V, E^s)$ obtained by letting $E^s := \{\{u, v\} :\ (u, v) \in E \text{ or } (v, u) \in E\}$ is connected.*

5) *For any $u \in V$, the set $W_G(u) = \{v \in V :\ (u, v) \in E\}$ is finite and totally ordered.*

Let us first show that an abstract tree $G = (V, E)$ is isomorphic to a tree $\omega \in \Omega$ in the sense of Neveu (and equivalently of Otter). From 4), we see that for an arbitrary $u \in V$, there is a path connecting u_0 and u in the graph G^s. Let $u_0u_1 \ldots u_{n-1}u_n$, where we let $u_n = u$, be such a path which has minimum length. By 1), we have $u_{i-1} \neq u_i$ for $i = 1, \ldots, n$, and by 2), we must have $(u_0, u_1) \in E$. Let k be the maximum of those $j = 1, \ldots, n$ for which $(u_{i-1}, u_i) \in E$ for all $i < j$. If $k < n$, then we would have $(u_{k-1}, u_k) \in E$ and $(u_{k+1}, u_k) \in E$, yielding $u_{k-1} = u_{k+1}$ by virtue of 3). Then $u_0u_1 \ldots u_{k-1}u_{k+2} \ldots u_n$ would also be a path in G^s connecting u_0 and u, which has length $n-2$, contradicting the minimality of n. Hence $k = n$ and we must have $(u_{i-1}, u_i) \in E$ for $i = 1, \ldots, n$. If there existed another path $u_0v_1 \ldots v_{n-1}v_n$, where $v_n = u$, connecting u_0 and u in G^s, then by the same argument, we have $(u_0, v_1) \in E$, $(v_1, v_2) \in E$,...,$(v_{n-1}, v_n) \in E$. Hence we must have $v_j = u_j$ for $j = n-1, n-2, \ldots, 1$ by 3). Thus the shortest path connecting u_0 and u in G^s is unique, and connects u_0 and u also in G. By 3), it is actually the unique path which connects u_0 and u in G. Thus for each $u \in V$, there corresponds a unique path $u_0 u_1 \ldots u_n$ ($u_n = u$) in G. Since $u_i \in W_G(u_{i-1})$ and since $W_G(u_{i-1})$ corresponds in a one-to-one way to the set $\{1, 2, \ldots, \nu_G(u_{i-1})\}$, where we set $\nu_G(v) = \sharp W_G(v)$ for $v \in V$, this path can be represented without ambiguity by a sequence of integers $j_1 j_2 \ldots j_n \in U$. If we denote $\tau_G(u) = j_1 j_2 \ldots j_n$ for $u \in V$, then $\omega = \tau_G(V) \subset U$ obviously satisfies the conditions a) to c) stated in the introduction, namely $\omega \in \Omega$ and τ_G is a graph isomorphism between two trees G and ω.

In order to randomize the notion of abstract tree, we fix a non-empty countable set S, and let $\mathbf{G}_S$ be the totality of abstract trees $G = (V, E)$ such that $V \subset S$. Define a mapping $\tau : \mathbf{G}_S \mapsto \Omega$ by letting $\tau(G) = \tau_G(V)$ for $G = (V, E) \in \mathbf{G}_S$, and let $\mathcal{G}$ be the σ-field on $\mathbf{G}_S$ generated by the mapping τ.

Now let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space. A random S-tree is by definition a mapping $G : \tilde{\Omega} \mapsto \mathbf{G}_S$ which is $\tilde{\mathcal{F}}/\mathcal{G}$-measurable. This condition is obviously equivalent to saying that the composition $\tau \circ G : \tilde{\Omega} \mapsto \Omega$ is $\tilde{\mathcal{F}}/\mathcal{F}$-measurable. When the image $P = \tilde{P} \circ (\tau \circ G)^{-1}$ is the distribution of a Galton-Watson tree, we shall say that our random S-tree $G(\cdot)$ is an abstract Galton-Watson tree.

Returning to our Galton-Watson tree $(\Omega, \mathcal{F}, P)$ with offspring distribution $\Pi = \{p_n\}_{n=0}^{\infty}$, we first note that our assumption $p_0 > 0$ implies $P(\cap_{n=1}^{\infty} \Omega_{u1_n}) = 0$ for each $u \in U$, where we denote $1_n = 11\ldots1$, the sequence consisting of n times repetition of 1. Hence if we let $\tilde{\Omega} = \cap_{u \in U} \cup_{n=1}^{\infty} (\Omega_{u1_n})^c$, then we have $\tilde{\Omega} \in \mathcal{F}$ and $P(\tilde{\Omega}) = 1$. Restricting $\mathcal{F}$ and P to $\tilde{\Omega}$, we obtain a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

For $k \geq 0$ and for each $\omega \in \tilde{\Omega}$, set

$$V_k(\omega) = \{u \in U;\ u \in \omega \text{ and } \nu_\omega(u) \leq k\} \subset U\ .$$

Then $\mathcal{Y}_k(\omega) = \sharp V_k(\omega)$. We are now going to show that by suitably defining the adjacency relation on $V_k(\omega)$, this can be turned into an abstract Galton-Watson tree whose offspring distribution is $\Pi^{(k)}$. To make the idea transparent, avoiding the notational complication, we shall give the full detail of its construction only for $k = 0$, in which case $V_0(\omega)$ is the set of endpoints, or “leaves ”of ω, and give a brief sketch for the general case $k \geq 1$.

To begin with, let us make $V_0(\omega)$ into an abstract tree, which we shall denote $G_0(\omega)$, in the following way. Since $\omega \in \tilde{\Omega}$, we have

$$n_0(\omega) := \max\{p \geq 0;\ 1_p \in \omega\} < \infty\ .$$

We denote $u_0(\omega) = 1_{n_0(\omega)}$, and this will play the role of the root of $G_0(\omega)$. Now if $n_0(\omega) = 0$ or $\nu_\omega(1_p) = 1$ for all $p < n_0(\omega)$, then $u_0(\omega)$ has no children and $V_0(\omega) = \{u_0(\omega)\}$. If $\nu_\omega(1_p) > 1$ for some $0 \leq p < n_0(\omega)$, then those vertices $u \in V_0(\omega)$ of the form $u = 1_p a 1_m$, where $0 \leq p < n_0(\omega)$, $1 < a \leq \nu_\omega(1_p)$ and $m = \max\{j \geq 0;\ 1_p a 1_j \in \omega\}$, are by definition children of $u_0(\omega)$. In general, $u \in V_0(\omega)$ has the form

$$u = 1_{p_1} a_1 1_{p_2} a_2 \ldots 1_{p_N} a_N 1_m =: \tilde{u} 1_m\ , \tag{16}$$

where $p_i \geq 0$, $i = 1, \ldots, N$, $1 < a_i \leq \nu_\omega(1_{p_1} a_i \ldots 1_{p_i})$, and $m = \max\{j \geq 0;\ \tilde{u}1_j \in \omega\}$. When $m = 0$ or $\nu_\omega(\tilde{u}1_j) = 1$ for all $0 \leq j < m$, then u has no children, while in case $\nu_\omega(\tilde{u}1_j) > 1$ for some $0 \leq j < m$, then u has children of the form $v = \tilde{u}1_i b 1_\ell \in V_0(\omega)$, where $0 \leq i < m$, $1 < b \leq \nu_\omega(\tilde{u}1_i)$, and $\ell = \max\{p \geq 0;\ \tilde{u}1_i b 1_p \in \omega\}$. If we let $E_0(\omega)$ be the totality of such parent-child pairs (u, v), then the resulting directed graph $G_0(\omega) = (V_0(\omega), E_0(\omega))$ obviously satisfies 1) and 2) of Definition 1. To verify 3), consider

$$v = 1_{p_1} a_1 \ldots 1_{p_N} a_N 1_i b 1_\ell =: \tilde{v} 1_i b 1_\ell \in V(\omega) \tag{17}$$

with $N \geq 0$. Since $v \in \omega$ and since ω is a tree in the sense of Neveu, we have $\tilde{v}1_i b \in \omega$. Replacing b by 1, we also have $\tilde{v}1_{i+1} \in \omega$. Hence if we let $m = \max\{p \geq 0;\ \tilde{v}1_p \in \omega\}$, then $0 < m < \infty$ and $u := \tilde{v}1_m \in V_0(\omega)$ is the uniquely determined parent of v, i.e. $(u, v) \in E_0(\omega)$.

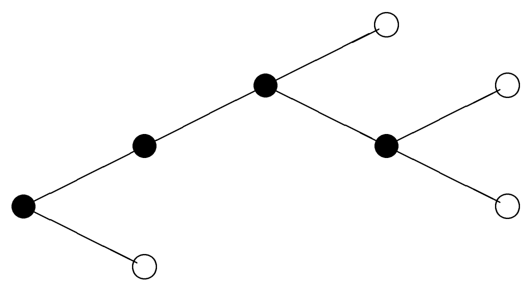
If $N > 0$ in (17), then by the same procedure we can further find the parent of u and so on. Thus starting from a vertex $v \in V_0(\omega)$ of the form (17), we can find $u_N,\ u_{N-1}, \ldots, u_1$, so that $(u_{i-1}, u_i) \in E_0(\omega)$ for $i = 1, \ldots, N+1$, where we set $u_{N+1} = v$. Since this is true for all $v \in V_0(\omega)$, we see that 4) holds, too. Finally for each $u \in V_0(\omega)$, let $W_\omega(u) \subset V_0(\omega)$ be the totality of its children. If u is of the form (16), i.e. $u = \tilde{u}1_m$, then from the consideration above, we have

$$\sharp W_\omega(u) = \sum_{i=0}^{m} \left(\nu_\omega(\tilde{u}1_i) - 1\right)_+ < \infty\ . \tag{18}$$

Moreover we can arrange members in $W_\omega(u)$ in the natural lexicographic order, under which $W_\omega(u)$ is totally ordered. Namely given two typical vertices $v = \tilde{u}1_i b 1_\ell$ and $v = \tilde{u}1_{i'} b' 1_{\ell'}$ of $W_\omega(u)$, we define $v < v'$ if $i < i'$ or if $i = i'$ and $b < b'$. Thus we have shown that $G_0(\omega) = (V_0(\omega), E_0(\omega))$ is an abstract tree for each $\omega \in \tilde{\Omega}$.

For example, consider the tree $\omega \in \Omega$ as shown in Figure 1, for which $\mathcal{Y}_0(\omega) = 4$.

Figure 1: The tree ω



Then the corresponding tree $G_0(\omega)$ is isomorphic to the one shown in Figure 2.

In Figure 3, it is shown how the tree $G_0(\omega)$ grows in the original tree ω.

Figure 2: The tree $G_0(\omega)$

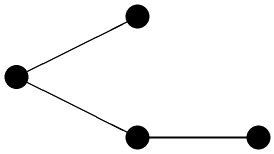
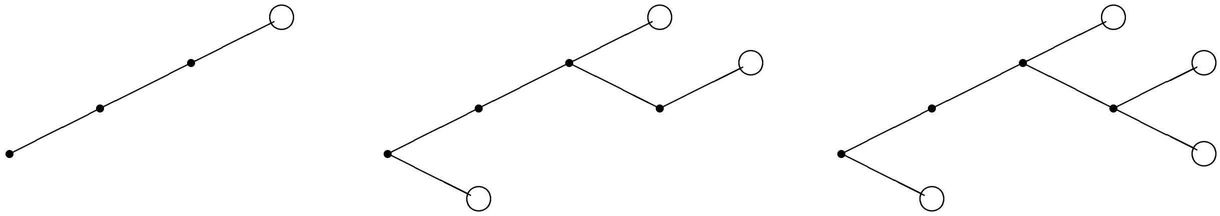
$G_0(\omega) \cong$ 

Figure 3: How $G_0(\omega)$ grows in ω



0th generation | 0th and 1st generation | 0th, 1st and 2nd generation

Thus we have constructed a mapping $G_0(\omega)$ defined for $\omega \in \tilde{\Omega}$ and taking its values in $\mathbf{G}_U$, the totality of abstract trees whose vertex sets are subsets of U. We shall now show that $G_0 : \tilde{\Omega} \mapsto \mathbf{G}_U$ is a random U-tree in the sense defined before. This amounts to showing $(\tau \circ G_0)^{-1}(\Omega_u) \in \tilde{\mathcal{F}}$ for each $u \in U$. This we shall do by induction on the length of the sequence u.

First consider the case $|u| = 1$, namely suppose $u = j$ for some $j \in \mathbf{N}$. Then

$$(\tau \circ G_0)^{-1}(\Omega_j) = \{\omega \in \tilde{\Omega};\ \sharp W_\omega(u_0(\omega)) \geq j\} .$$

Letting $m = n_0(\omega)$ and $\tilde{u} = \phi$ in (18), we can write for each $j \geq 1$,

$$\{\omega \in \tilde{\Omega};\ \sharp W_\omega(u_0(\omega)) \geq j\} = \bigcup_{n=1}^{\infty} \bigcup_{k_0+\cdots+k_{n-1}\geq j, k_i \geq 0} C_{n,k_0,\ldots,k_{n-1}} , \tag{19}$$

where we put for $n \geq 1$ and for $k_0, k_1, \ldots, k_{n-1} \geq 0$,

$$C_{n,k_0,\ldots,k_{n-1}} \quad = \quad \{\omega \in \tilde{\Omega};\ n_0(\omega) = n,\ \nu_\omega(1_i) = 1 + k_i,\ i = 0, 1, \ldots, n-1\}$$

$$= \{\omega \in \tilde{\Omega};\ \nu_\omega(1_i) = 1 + k_i,\ i = 0, 1, \ldots, n-1,\ \nu_\omega(1_n) = 0\}\ , \qquad (20)$$

which obviously belongs to $\tilde{\mathcal{F}}$, proving $(\tau \circ G_0)^{-1}(\Omega_j) \in \tilde{\mathcal{F}}$.

Suppose we have shown $(\tau \circ G_0)^{-1}(\Omega_v) \in \tilde{\mathcal{F}}$ for $v \in U$ with $|v| = n-1$, and pick a sequence $u = j_1 j_2 \ldots j_n \in U$, where $j_1 \geq 1$. Since we can write $\Omega_u = \{\nu_\phi \geq j_1\} \cap T_{j_1}^{-1}(\Omega_{j_2 \ldots j_n})$, we have, noting (19),

$$(\tau \circ G_0)^{-1}(\Omega_u) = \sum_{n=1}^{\infty} \sum_{k_0 + \cdots k_{n-1} \geq j_1} C_{n,k_0,\ldots,k_{n-1}} \cap (T_{j_1} \circ \tau \circ G_0)^{-1} (\Omega_{j_2 \ldots j_n})\ .$$

Given $n \geq 1$ and $k_0, k_1, \ldots, k_{n-1} \geq 0$ with $\sum_{i=0}^{n-1} k_i \geq j_1$, choose $\ell \geq 1$ and $1 < a \leq 1 + k_\ell$ so that $j_1 = k_0 + \cdots k_{\ell-1} + (a-1)$. Then on the set $C_{n,k_0,\ldots,k_{n-1}}$, we obviously have $T_{j_1} \circ \tau \circ G_0 = \tau \circ G_0 \circ T_{1_\ell a}$. Hence we obtain

$$(\tau \circ G_0)^{-1}(\Omega_u) = \sum_{n=1}^{\infty} \sum_{k_0 + \cdots k_{n-1} \geq j_1} C_{n,k_0,\ldots,k_{n-1}} \cap \left\{ (T_{1_\ell a})^{-1} \left((\tau \circ G_0)^{-1}(\Omega_{j_2 \ldots j_n}) \right) \right\}\ .$$

But since $C_{n,k_0,\ldots,k_{n-1}} \in \tilde{\mathcal{F}}$ as was already shown, since $(\tau \circ G_0)^{-1}(\Omega_{j_2 \ldots j_n}) \in \tilde{\mathcal{F}}$ from the assumption and since the translation $T_v : \tilde{\Omega} \cap \Omega_v \mapsto \tilde{\Omega}$ is measurable, we must have $(\tau \circ G_0)^{-1}(\Omega_u) \in \tilde{\mathcal{F}}$.

Finally, let us prove that the probability measure $Q = \tilde{P} \circ (\tau \circ G_0)^{-1}$ induced on $(\Omega, \mathcal{F})$ is the distribution of the Galton-Watson tree with the offspring distribution $\Pi^{(0)} = \{p_n^{(0)}\}_{n=0}^{\infty}$. For this purpose, we note that the following general criterion holds.

Proposition 3 *Let P be a probability distribution on $(\Omega, \mathcal{F})$ and let $\Pi = \{p_n\}_{n=0}^{\infty}$ be that on $\mathbf{Z}_+$. Then for P to be the distribution of the Galton-Watson tree with the offspring distribution Π, it is necessary and sufficient that $P(\nu_\phi = k) = p_k$ for $k \geq 0$ and that*

$$P\left(\bigcap_{j=1}^{k} T_j^{-1}(B_j) \middle|\ \nu_\phi = k \right) = \prod_{j=1}^{k} P(B_j)$$

for any $k \geq 1$ and $B_1, \ldots, B_k \in \mathcal{F}$.

Proof. For the proof of the necessity, see §3 of [22]. To show that the condition is sufficient, let us define for any finite tree $t \in \Omega$ and for any $\Lambda = (\lambda_e; e \in G_0(t)) \in \mathbf{Z}_+^{G_0(t)}$,

$$[t; \Lambda] := \{\omega \in \Omega;\ \nu_u(\omega) = \nu_u(t) \text{ for } u \in \mathcal{I}(t) \text{ and } \nu_e(\omega) = \lambda_e \text{ for } e \in G_0(t)\ \}\ ,$$

where $\mathcal{I}(t) := \{u \in t; \nu_u(t) > 0\}$ is the totality of the "inner points "of the tree t. Subsets of Ω of this form were called "neighborhoods "by Otter [23]. By the equivalence of two construction of the probability space for Galton- Watson trees respectively due to Otter and Neveu (see [20]), it is enough to verify

$$P([t; \Lambda]) = \left(\prod_{u \in \mathcal{I}(t)} p_{\nu_u(t)} \right) \left(\prod_{e \in G_0(t)} p_{\lambda_e} \right)\ . \qquad (21)$$

Given t and $\Lambda = (\lambda_e; e \in G_0(t))$ as above, let $\Lambda_j = (\lambda_e; e \in G_0(T_j t))$ for $j = 1, 2, \ldots, \nu_\phi(t)$, which altogether form a partition of Λ. Then it is easily seen that

$$[t; \Lambda] = \{\omega \in \Omega;\ \nu_\phi(\omega) = \nu_\phi(t)\} \cap \bigcap_{j=1}^{\nu_\phi(t)} T_j^{-1}([T_j t; \Lambda_j])$$

holds. Thus from the assumption,

$$\begin{aligned} P([t; \Lambda]) &= P(\nu_\phi = \nu_\phi(t)) P\left(\bigcap_{j=1}^{\nu_\phi(t)} T_j^{-1}([T_j t; \Lambda_j]) \,\middle|\, \nu_\phi = \nu_\phi(t) \right) \\ &= p_{\nu_\phi(t)} \prod_{j=1}^{\nu_\phi(t)} P([T_j t; \Lambda_j]) \ . \end{aligned}$$

Repeating this argument to rewrite $P([T_j t; \Lambda_j])$ and so on, we arrive at (21).

Let us apply this criterion to our measure Q. From (18) and (20), we see

$$Q(\nu_\phi = k) = \sum_{n=0}^{\infty} \sum_{k_0 + \cdots k_{n-1} = k, k_i \geq 0} P(\nu_\omega(1_i) = 1 + k_i,\ 0 \leq i < n,\ \text{and}\ \nu_\omega(1_n) = 0) \ .$$

But our P was defined to be the image under the mapping ψ of the product measure $P^* = \Pi^U$. Moreover on the set

$$\{\omega^* \in \Omega^*;\ \nu^*_{1_i}(\omega^*) = 1 + k_i,\ 0 \leq i < n,\ \nu^*_{1_n}(\omega^*) = 0\} \ ,$$

which is a subset of $\psi^{-1}(\Omega_{1_n})$, one has $\nu_{1_i}(\psi(\omega^*)) = \nu^*_{1_i}(\omega^*)$, $i = 0, 1, \ldots, n$. Thus we have from the independence,

$$\begin{aligned} & P(\nu_\omega(1_i) = 1 + k_i,\ 0 \leq i < n,\ \nu_\omega(i_n) = 0) \\ = \ & P^*(\nu^*_{1_i} = 1 + k_i,\ 0 \leq i < n,\ \nu^*_{1_n} = 0) \\ = \ & p_0 \prod_{i=0}^{n-1} p_{1+k_i} \\ = \ & p_0 (1 - p_0)^n \prod_{i=0}^{n-1} P(\nu_\phi - 1 = k_i |\ \nu_\phi > 0) \ , \end{aligned}$$

and hence

$$Q(\nu_\phi = k) = \sum_{n=0}^{\infty} p_0 (1 - p_0)^n \sum_{k_0 + \cdots k_{n-1} = k} \prod_{i=0}^{n-1} P(\nu_\phi - 1 = k_i |\ \nu_\phi > 0) = p_k^{(0)} \ ,$$

verifying the first part of the criterion. Next for $k \geq 1$ and $B_1, \ldots, B_k \in \mathcal{F}$, we write

$$
\begin{aligned}
& Q\left(\{\nu_\phi = k\} \cap \bigcap_{i=1}^{k} T_i^{-1}(B_i)\right) \\
= \ & P\left(\{\nu_\phi \circ \tau \circ G_0 = k\} \cap \bigcap_{i=1}^{k} (T_i \circ \tau \circ G_0)^{-1}(B_i)\right) \\
= \ & \sum_{n=1}^{\infty} \sum_{k_0 + \cdots k_{n-1} = k} P\left(C_{n,k_0,\ldots,k_{n-1}} \cap \bigcap_{\ell=0}^{k} \bigcap_{a=2}^{1+k_i} (T_{1_\ell a})^{-1}\left((\tau \circ G_0)^{-1}(B_{\ell,a})\right)\right) ,
\end{aligned}
$$

where we have written $B_{\ell,a}$ instead of B_j when $j = k_0 + \cdots + k_{\ell-1} + (a-1)$ holds. By the same reasoning as above, we have

$$
\begin{aligned}
& P\left(C_{n,k_0,\ldots,k_{n-1}} \cap \bigcap_{\ell=0}^{n-1} \bigcap_{a=2}^{1+k_i} (T_{1_\ell a})^{-1}\left((\tau \circ G_0)^{-1}(B_{\ell,a})\right)\right) \\
= \ & P^*\left(\{\nu^*_{1_i} = 1 + k_i, 0 \leq i < n, \nu^*_{1_n} = 0\} \cap \bigcap_{\ell=0}^{n-1} \bigcap_{a=2}^{1+k_i} (T^*_{1_\ell a})^{-1}\left((\tau \circ G_0 \circ \psi)^{-1}(B_{\ell,a})\right)\right) \\
= \ & \left(p_0 \prod_{i=0}^{n-1} p_{1+k_i}\right) \prod_{\ell=0}^{n-1} \prod_{a=2}^{1+k_i} P^*\left((\tau \circ G_0 \circ \psi)^{-1}(B_{\ell,a})\right) \\
= \ & P(C_{n,k_0,\ldots,k_{n-1}}) \prod_{j=1}^{k} Q(B_j) .
\end{aligned}
$$

Here $T^*_v : \Omega^* \mapsto \Omega$ is the translation on Ω^* defined by $\nu^*_v \circ T^*_u = \nu^*_{uv}$ (see [22]). Hence summing over n and k_i's, we finally obtain

$$
Q\left(\{\nu_\phi = k\} \cap \bigcap_{j=1}^{k} T_j^{-1}(B_j)\right) = Q(\nu_\phi = k) \prod_{j=1}^{k} Q(B_j) .
$$

This completes the proof that our $G_0(\omega)$ is an abstract Galton-Watson tree, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and having $\Pi^{(0)}$ as its offspring distribution.

Having given a full account of the hidden Galton-Watson tree corresponding to $\mathcal{Y}_0$, let us now give a brief sketch of the construction of the abstract Galton-Watson tree corresponding to $\mathcal{Y}_k$ for $k \geq 1$. Again by the definition of $\tilde{\Omega}$, for each $\omega \in \tilde{\Omega}$, there is a finite $n_0(\omega)$ such that $\nu_\omega(1_i) > k$ for $i < n_0(\omega)$ and $\nu_\omega(1_{n_0(\omega)}) \leq k$. Then $u_0(\omega) := 1_{n_0(\omega)} \in V_k(\omega)$, and this vertex will play the roll of the root of the tree $G_k(\omega)$ we are after. In contrast to the case of $k = 0$, $u_0(\omega)$ can bear two types of children described below:

type I: When $n_0(\omega) \geq 1$, vertices of the form $v = 1_\ell b 1_{\ell'}$ are children of $u_0(\omega)$, where $0 \leq \ell < n_0(\omega)$, $1 < b \leq \nu_\omega(1_\ell)$, and $\ell' = 0$ if $\nu_\omega(1_\ell b) \leq k$ while

$$
\ell' = 1 + \max\{i \geq 0; \nu_\omega(1_\ell b 1_i) > k\}
$$

if $\nu_\omega(1_\ell b) > k$;

type II: when $\nu_\omega(1_{n_0(\omega)}) > 0$, then vertices of the form $v = 1_{n_0(\omega)}a1_m$ are children of $u_0(\omega)$, where $1 \le a \le \nu_\omega(1_{n_0(\omega)})$ and $m = 0$ if $\nu_\omega(1_{n_0(\omega)}a) \le k$, while

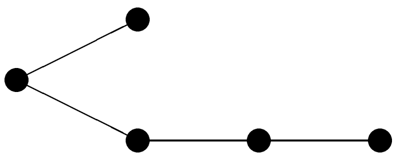
$$m = 1 + \max\{i \ge 0; \nu_\omega(1_{n_0(\omega)}a1_i) > k\}$$

if $\nu_\omega(1_{n_0(\omega)}a) > k$.

The parent-child relationship between general vertices in $V_k(\omega)$ is defined similarly, which gives $V_k(\omega)$ a structure of an abstract tree $G_k(\omega)$. It is actually an abstract Galton-Watson tree whose offspring distribution is the distribution of the independent sum of the random numbers of two types of children, namely $\Pi^{(k)}$.

For illustration, consider again the tree ω drawn in Figure 1, for which $\mathcal{Y}_1(\omega) = 5$. $G_1(\omega)$ is seen to be isomorphic to the tree depicted in Figure 4.

Figure 4: The tree $G_1(\omega)$

$G_1(\omega) \cong$ 

In Figure 5, it is shown how the tree $G_1(\omega)$ grows in ω from the 0th generation (the root) to the third generation.

3.2 Asymptotic behavior of the distribution of $\mathcal{Y}_k$

By Theorem 2, the analysis of the asymptotic behavior of $P(\mathcal{Y}_k = n)$ as $n \to \infty$ is essentially reduced to that of $P(Z = n)$ with a proper choice of offspring distribution. Namely if we let σ_k be the radius of convergence of the power series $g_k(w)$ and if we define $b_k \in (0, \sigma_k]$ by (5), that is by the relation

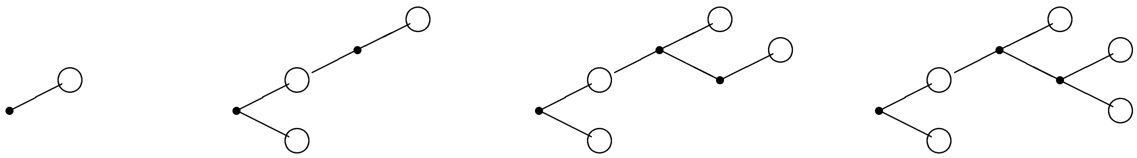
$$\inf_{0<w<\sigma_k} \frac{g_k(w)}{w} = \frac{g_k(b_k)}{b_k} ,$$

then we immediately obtain the following result.

Proposition 4 *(i) Let $d(\Pi^{(k)})$ be the greatest common divisor of all $n \ge 0$ such that $p_n^{(k)} > 0$. Then $P(\mathcal{Y}_k = n) > 0$ only for those n such that $n \equiv 1 \pmod{d(\Pi^{(k)})}$.*

(ii) $\limsup_{n\to\infty} P(\mathcal{Y}_k = n)^{1/n} = g_k(b_k)/b_k$.

Figure 5: How $G_1(\omega)$ grows in ω



0th gen. | 0th and 1st gen. | 0th, 1st and 2nd gen. | 0th, 1st, 2nd and 3rd gen.

(iii) If $0 < b_k < \sigma_k$*, then the following asymptotic expansion holds as we let* $n \to \infty$ *keeping* $n \equiv 1 \pmod{d(\Pi^{(k)})}$*:*

$$P(\mathcal{Y}_k = n) \sim \sum_{j=1}^{\infty} c_j^{(k)} \left(\frac{g_k(b_k)}{b_k}\right)^n n^{-j-1/2},$$

where

$$c_1^{(k)} = d(\Pi^{(k)}) \sqrt{\frac{g_k(b_k)}{2\pi g_k''(b_k)}} \ .$$

There is no simple relation between $d(\Pi)$ and $d(\Pi^{(k)})$. Actually $d(\Pi^{(k)})$ is not determined solely by $d(\Pi)$, but we can prove the following proposition:

Proposition 5 *$d(\Pi^{(k)})$ is given by*

$$d(\Pi^{(k)}) = g.c.d.\left[\{0 \le j \le k \mid p_j > 0\} \cup \{j \ge k \mid p_{j+1} > 0\}\right] \ .$$

Proof. Note that

$$d(\Pi^{(k)}) = \max\{\ell \ge 1 \mid g_k(e^{\frac{2\pi}{\ell}i}) = 1\} \ .$$

On the other hand, $g_k(w) = 1$ is equivalent to $h_k(w) = 1 - \varphi_k(w)$, namely to the condition

$$\Gamma_k(w) := \sum_{j=0}^{k-1} p_j w^j + (p_k + p_{k+1}) w^k + \sum_{j=k+1}^{\infty} p_{j+1} w^{j+1} = 1 \ ,$$

and therefore

$$\begin{aligned}
d(\Pi^{(k)}) &= \max\{\ell \geq 1 \mid \Gamma_k(e^{\frac{2\pi}{\ell} i}) = 1\} \\
&= \begin{cases} g.c.d.\,[\{0 \leq j \leq k \mid p_j > 0\} \cup \{k\} \cup \{j \geq k \mid p_{j+1} > 0\}] \\ \hfill \text{if } p_k + p_{k+1} > 0 \\ \\ \hfill g.c.d.\,[\{0 \leq j \leq k \mid p_j > 0\} \cup \{j \geq k \mid p_{j+1} > 0\}] \\ \hfill \text{if } p_k + p_{k+1} = 0 \ , \end{cases}
\end{aligned}$$

which gives the desired result.

Now we shall give a condition for $0 < b_k < \sigma_k$ in terms of $\Pi = \{p_n\}_{n=0}^{\infty}$ and ρ. By (15), our consideration is divided into 4 cases.

case 1. $\varphi_k(\rho-) > 1$. In this case, there is a $\sigma_k \in (0, \rho)$ such that $\varphi_k(\sigma_k) = 1$. This σ_k is the radius of convergence of the power series $g_k(w)$ and we have $g_k(\sigma_k-) = \infty$. Consequently $b_k \in (0, \sigma_k)$.

case 2. If $\varphi_k(\rho-) = 1$, then $\sigma_k = \rho < \infty$ and $g_k(\sigma_k) = \infty$. In this case also, one has $b_k \in (0, \sigma_k)$.

case 3. When $0 < \varphi_k(\rho-) < 1$, one has $\sigma_k = \rho < \infty$ and $g_k(\sigma_k-) < \infty$.

case 4. Finally when $\varphi_k(\rho-) = 0$, then $g_k(w) = f(w)$ and we trivially have $b_k = a$.

In the last two cases, both $b_k < \sigma_k$ and $b_k = \sigma_k$ are possible.

Proposition 6 *$0 < b_k < \sigma_k$ holds if and only if*

$$\lim_{z \uparrow \rho} \{(z - f(z))h_k'(z) - (1 - f'(z))h_k(z)\} > 0 \ .$$

Proof. Define

$$\Delta_k(z) := (z - f(z))h_k'(z) - (1 - f'(z))h_k(z) \ .$$

Then

$$z g_k'(z) - g_k(z) = \frac{\Delta_k(z)}{(1 - \varphi_k(z))^2} \ , \tag{22}$$

and after some computation we get

$$\begin{aligned}
\Delta_k(z) &= \sum_{\ell=0}^{k} (\ell - 1) p_\ell z^\ell + \sum_{j=k+1}^{\infty} \sum_{\ell=0}^{k} (j - \ell) p_j p_\ell z^{j+\ell-1} \\
&= \sum_{\ell=1}^{k} (\ell - 1) p_\ell z^\ell + \sum_{\ell=1}^{k} \sum_{j=k+1}^{\infty} (j - \ell) p_j p_\ell z^{j+\ell-1} + \\
&\qquad + p_0 \left(\sum_{j=k+1}^{\infty} j p_j z^{j-1} - 1 \right) \ ,
\end{aligned} \tag{23}$$

which shows in particular $\Delta_k'(z) > 0$ for $z > 0$.

From the preliminary consideration preceeding the proposition and the condition (4) (see §2) applied to $g_k(z)$, we first see that $0 < b_k < \sigma_k$ if and only if either $\varphi_k(\rho-) \geq 1$, or $\varphi_k(\rho-) < 1$ and $\lim_{z\uparrow\sigma_k} \Delta_k(z) > 0$.

Hence if $0 < b_k < \sigma_k$ and $\varphi_k(\rho-) < 1$, we have $\sigma_k = \rho$ and

$$\lim_{z\uparrow\rho} \Delta_k(z) = \lim_{z\uparrow\sigma_k} \Delta_k(z) > 0 \ .$$

On the other hand, in case $\varphi_k(\rho-) \geq 1$, we have $\varphi_k(\sigma_k) = 1$, namely

$$\sum_{j=k+1}^{\infty} p_j \sigma_k^{j-1} = 1 \ .$$

Since $p_0 + p_1 < 1$, this implies

$$\sum_{j=k+1}^{\infty} j p_j \sigma_k^{j-1} > 1 \ ,$$

so that $\Delta_k(\sigma_k) > 0$ by (23). Since $\rho \geq \sigma_k$, this implies $\Delta_k(\rho-) \geq \Delta_k(\sigma_k) > 0$.

Conversely suppose $\Delta_k(\rho-) > 0$. If, in addition, $\varphi_k(\rho-) \geq 1$, then one has $0 < b_k < \sigma_k$ as we already saw, and if $\varphi_k(\rho-) < 1$, then since $\sigma_k = \rho$,

$$\lim_{z\uparrow\sigma_k} \{z g_k'(z) - g_k(z)\} = \frac{1}{(1-\varphi_k(\rho-))^2} \Delta_k(\rho-) > 0 \ ,$$

namely $0 < b_k < \sigma_k$ by (4).

Since $\mathcal{Y}_k \uparrow Z$ as $k \to \infty$, the number $g_k(b_k)/b_k$ governing the exponential behavior of $P(\mathcal{Y}_k = n)$ is expected to increase with k. This is actually the case as the following proposition shows.

Proposition 7 *For each $k \geq 0$, we have*

$$\frac{g_k(b_k)}{b_k} \leq \frac{g_{k+1}(b_{k+1})}{b_{k+1}} \leq \frac{f(a)}{a} \ . \tag{24}$$

If $f(z)$ is critical (i.e. $f'(1) = 1$) or sub-critical ($f'(1) < 1$) with $\rho = 1$, then each member equals to 1. If $p_{k+1} = 0$, then $g_k(z) = g_{k+1}(z)$ and the first inequality is actually an equality. If, further, $p_\ell = 0$ for all $\ell > k$, then $g_k(z) = f(z)$ and we have equality also in the second inequalty. In all the other cases, the inequalities are strict.

Finally it holds that

$$\lim_{k\to\infty} \frac{g_k(b_k)}{b_k} = \frac{f(a)}{a} \ . \tag{25}$$

Proof. First note that

$$g_{k+1}(z) - g_k(z) = \frac{z(z-f(z))(h_{k+1}(z) - h_k(z))}{(z-f(z)+h_k(z))(z-f(z)+h_k(z))} \ . \tag{26}$$

For the moment, we suppose that there exists q_1 and q_2 such that $f(q_i) = q_i$, $i = 1, 2$, and that $0 < q_1 < q_2 \le \rho$. Since

$$\varphi_k(q_2) = 1 - \frac{h_k(q_2)}{q_2} < 1 \ ,$$

we have $q_2 \le \sigma_k$, thus $g_k(z)$ is well defined on $[q_1, q_2]$. But for $z \in [q_1, q_2]$, we have $f(z) \le z$, and hence $g_{k+1}(z) \ge g_k(z)$ holds on $[q_1, q_2]$ by (26). By the convexity of $f(z)/z$ and $g_k(z)/z$, and by $f(q_i)/q_i = g_k(q_i)/q_i = 1$, $i = 1, 2$, we see that a and b_k belong to $[q_1, q_2]$.

Now since we have $h_k(z) \uparrow f(z)$, we also have $g_k(z) \uparrow f(z)$ on $[q_1, q_2]$ as $k \to \infty$. By Dini's lemma, this convergence is uniform on the compact interval $[q_1, q_2]$, that is for any $\epsilon > 0$, $g_k(z)/z > (f(z)/z) - \epsilon$ holds for all $z \in [q_1, q_2]$ and for all sufficiently large k. Taking $z = b_k$ and $z = a$, we obtain

$$\frac{f(a)}{a} \ge \frac{g_k(a)}{a} \ge \frac{g_k(b_k)}{b_k} > \frac{f(b_k)}{b_k} - \epsilon \ge \frac{f(a)}{a} - \epsilon$$

for large k. This shows $\lim_{k\to\infty} g_k(b_k)/b_k = f(a)/a$.

Now suppose $f(z)$ is supercritical and let q be the extinction probability: $f(q) = q$. Then we can take $q_1 = q$, $q_2 = 1$ and (24) and (25) are proved in this case. When $f(z)$ is critical, or subcritical with $\rho = 1$, we have

$$a = f(a) = b_k = g_k(b_k) = 1$$

and there is nothing to be proved. Next consider the case where $f(z)$ is sub-critical (i.e. $f'(1) < 1$), $\rho > 1$, but $f(z) < z$ for $1 < z < \rho$. In this case, we have in particular $\rho < \infty$ and $f(\rho) < \infty$, and $g_k(z) \uparrow f(z)$ holds uniformly on $[1, \rho]$. Thus the above argument can be applied to prove $g_k(b_k)/b_k \uparrow f(a)/a$.

Finally suppose that $f(z)$ is super-critical, or sub-critical with $\rho > 1$, and that $p_{k+1} > 0$. Then $h_{k+1}(z) > h_k(z)$ for $z > 0$ and hence we have $g_{k+1}(z) > g_k(z)$ on (q_1, q_2) or on $(1, \rho)$. Thus

$$\frac{g_{k+1}(b_{k+1})}{b_{k+1}} > \frac{g_k(b_{k+1})}{b_{k+1}} \ge \frac{g_k(b_k)}{b_k}$$

and the (first) inequalty is strict.

4 The joint distribution of Z and $\mathcal{Y}_k$

4.1 Generality

For complex numbers u, v_j $(j = 0, 1, \ldots)$ satisfying $|u|, |v_j| \le 1$, let

$$w := G(u; v_0, v_1, \ldots) := \mathbf{E}\left[Z < \infty;\ u^Z \prod_{j=0}^{\infty} v_j^{Y_j}\right] \tag{27}$$

be the joint probability generating function of $Z, Y_0, Y_1, \ldots$. Here $\prod_{j=0}^{\infty} v_j^{Y_j}$ is actually a finite product because of $\sum_{j=0}^{\infty} Y_j = Z < \infty$. Due to the relations (1) and

$$Y_k(\omega) = 1_{\{\nu_\phi(\omega)=k\}} + \sum_{j=1}^{\nu_\phi(\omega)} Y_k(T_j(\omega)) \ , \tag{28}$$

we can proceed, using the formula of Neveu (page 202 of [22]),

$$\begin{aligned}
w &= \mathbf{E}\left[Z < \infty;\ u^{1+\sum_{j=1}^{\nu_\phi} Z\circ T_j} \prod_{k=0}^{\infty} v_k^{1_{\{\nu_\phi=k\}}+\sum_{j=1}^{\nu_\phi} Y_k\circ T_j}\right] \\
&= u\mathbf{E}\left[\left(\prod_{j=1}^{\nu_\phi} 1_{\{Z<\infty\}} \circ T_j\right)\left(\prod_{k=0}^{\infty} v_k^{1_{\{\nu_\phi=k\}}}\right)\left\{\prod_{j=1}^{\nu_\phi}\left(u^Z \prod_{k=0}^{\infty} v_k^{Y_k}\right)\circ T_j\right\}\right] \\
&= u\mathbf{E}\left[\prod_{k=0}^{\infty} v_k^{1_{\{\nu_\phi=k\}}} \mathbf{E}\left[\prod_{j=1}^{\nu_\phi}\left(1_{\{Z<\infty\}} u^Z \prod_{k=0}^{\infty} v_k^{Y_k}\right)\circ T_j \middle| \mathcal{F}_1\right]\right] \\
&= u\mathbf{E}\left[\prod_{k=0}^{\infty} v_k^{1_{\{\nu_\phi=k\}}} \prod_{j=1}^{\nu_\phi} \mathbf{E}\left[Z<\infty;\ u^Z \prod_{k=0}^{\infty} v_k^{Y_k}\right]\right] \\
&= u\mathbf{E}\left[\prod_{k=0}^{\infty} v_k^{1_{\{\nu_\phi=k\}}} w^{\nu_\phi}\right] \\
&= u\sum_{k=0}^{\infty} p_k v_k w^k \ .
\end{aligned}$$

Thus if we define

$$\Phi(w) = \Phi(w; v_0, v_1, \ldots) := \sum_{k=0}^{\infty} p_k v_k w^k \ ,$$

then w is a solution of the equation $w = u\Phi(w)$. If we assume $v_0 \neq 0$, then $\Phi(0) = p_0 v_0 \neq 0$, and we can again apply Lagrange inversion, to obtain the formula

$$\mathbf{E}\left[Z = n;\ \prod_{k=0}^{\infty} v_k^{Y_k}\right] = \frac{1}{2\pi i n} \oint \left\{\frac{\Phi(w)}{w}\right\}^n dw \tag{29}$$

for the coefficients of the power series

$$w = \sum_{n=1}^{\infty} \mathbf{E}\left[Z = n;\ \prod_{k=0}^{\infty} v_k^{Y_k}\right] u^n \ .$$

Now if $Z(\omega) = n$, then we have $Y_k(\omega) = 0$ for $k \geq n$. Moreover $\sum_{k=0}^{n-1} Y_k(\omega) = 1 + \sum_{k=1}^{n-1} kY_k(\omega) = n$. Hence for $r \geq 1$, $0 \leq k_1 < \cdots < k_r$ and for $n_{k_j} \geq 0$ ($j = 1, \ldots, r$), the probability $P(Z = n, Y_{k_j} = n_{k_j},\ j = 1, \ldots, r)$ is non-zero only if

$$n_{k_j} = 0 \text{ for } k_j \geq n \ ;\ \mu := \sum_{j=1}^{r} n_{k_j} \leq n \text{ and } \nu := \sum_{j=1}^{r} k_j n_{k_j} < n \ , \tag{30}$$

and in such a case, we obtain from (29)

$$
\begin{aligned}
& P(Z=n\ ,\ Y_{k_j}=n_{k_j}\ ,\ j=1,\ldots,r) \\
= \ & \left(\prod_{k\geq 0}\frac{1}{n_k!}\right)\frac{\partial^\mu}{\partial v_{k_1}^{n_{k_1}}\partial v_{k_2}^{n_{k_2}}\cdots\partial v_{k_r}^{n_{k_r}}}\mathbf{E}\left[Z=n\ ;\ \prod_{k\geq 0}v_k^{Y_k}\right] \\
= \ & \frac{1}{2\pi in}\left(\prod_{k\geq 0}\frac{1}{n_k!}\right)\frac{\partial^\mu}{\partial v_{k_1}^{n_{k_1}}\partial v_{k_2}^{n_{k_2}}\cdots\partial v_{k_r}^{n_{k_r}}}\oint\left\{\frac{\Phi(z;v_0,v_1,\ldots)}{z}\right\}^n dz \\
= \ & \frac{1}{2\pi in}\left(\prod_{k\geq 0}\frac{1}{n_k!}\right)\oint\frac{1}{z^n}\frac{n!}{(n-\mu)!}\left(\prod_{j=1}^r p_{k_j}^{n_{k_j}}\right)z^\nu\tilde{f}(z)^{n-\mu}dz \\
= \ & \frac{(n-1)!}{(n-\mu)!}\left(\prod_{k\geq 0}\frac{1}{n_k!}\right)\left(\prod_{j=1}^r p_{k_j}^{n_{k_j}}\right)\frac{1}{2\pi i}\oint\frac{\tilde{f}(z)^{n-\mu}}{z^{n-\nu}}dz\ ,
\end{aligned}
$$

where we have set

$$\tilde{f}(z)=\sum_{k\neq k_j,j=1,\ldots,r}p_kz^k\ ,$$

and the partial derivatives are evaluated as $v_{k_1},\ldots,v_{k_r}\to 0$ and with $v_k=1$ for $k\neq k_j$, $j=1,\ldots,r$.

Thus, we have obtained the following result.

Proposition 8 *For (n_{k_j}) satisfying (30), we have*

$$
\begin{aligned}
& P(Z=n\ ,\ Y_{k_j}=n_{k_j}\ ,\ j=1,\ldots,r) \\
= \ & \frac{(n-1)!}{(n-\mu)!}\left(\prod_{k\geq 0}\frac{1}{n_k!}\right)\left(\prod_{j=1}^r p_{k_j}^{n_{k_j}}\right)\frac{1}{2\pi i}\oint\frac{\tilde{f}(z)^{n-\mu}}{z^{n-\nu}}dz \\
= \ & \frac{(n-1)!}{(n-\mu)!(n-\nu-1)!}\left(\prod_{k\geq 0}\frac{1}{n_k!}\right)\left(\prod_{j=1}^r p_{k_j}^{n_{k_j}}\right)\frac{1}{2\pi i}\frac{d^{n-\nu-1}}{dz^{n-\nu-1}}\tilde{f}(z)^{n-\nu}\bigg|_{z=0}\ .
\end{aligned}
\tag{31}
$$

In particular, if $(n_k)_{k\geq 0}$ are such that

$$n_k=0\ \textit{for}\ k\geq n\ ;\ \sum_{k=0}^{n-1}n_k=1+\sum_{k=1}^{n-1}kn_k=n\ , \tag{32}$$

then

$$P(Z=n\ ,\ Y_k=n_k\ ,\ k=0,1,\ldots)=(n-1)!\prod_{k=0}^{n-1}\frac{p_k^{n_k}}{n_k!}\ . \tag{33}$$

Next, let us consider the joint factorial moments of Y_k's, restricted on the event $\{Z = n\}$, which we shall define as

$$M(n; r_0, r_1, \ldots) := \mathbf{E}\left[Z = n;\ \prod_{k\geq 0} Y_k^{[r_k]}\right] ,$$

where as usual we define

$$n^{[r]} = \begin{cases} n(n-1)\cdots(n-r+1), & \text{for } r \leq n \\ 0, & \text{for } r > n \ , \end{cases}$$

for non-negative integers n and r, so that in particular $n^{[0]} = 1$ and $n^{[1]} = n$ for any $n \geq 0$, and $r_k > 0$ only for finite number of k's. More specifically, we shall assume $r_k = 0$ for $k \geq n$ in view of (30), and set $r = \sum_{k=0}^{n-1} r_k$. Now it is easy to compute, using (29),

$$\begin{aligned} M(n; r_0, r_1, \ldots) &= \frac{\partial^r}{\partial v_0^{r_0} \partial v_1^{r_1} \cdots \partial v_{n-1}^{r_{n-1}}} \mathbf{E}\left[Z = n;\ \prod_{k\geq 0} v_k^{Y_k}\right]\Bigg|_{v_0=v_1=\cdots=1} \\ &= \frac{1}{2\pi i n} \oint \frac{1}{z^n} \left. \frac{\partial^r}{\partial v_0^{r_0} \partial v_1^{r_1} \cdots \partial v_{n-1}^{r_{n-1}}} \Phi(z; v_0, v_1 \ldots)^n \right|_{v_0=v_1=\cdots=1} dz \\ &= \frac{1}{2\pi i n} \oint \frac{1}{z^n} n^{[r]} f(z)^{n-r} \prod_{k=0}^{n-1} p_k^{r_k} z^{k r_k} dz \\ &= \frac{n^{[r]}}{2\pi i n} \left(\prod_{k=0}^{n-1} p_k^{r_k}\right) \oint \frac{f(z)^{n-r}}{z^{n-s}}\ dz \ , \end{aligned} \tag{34}$$

where we have set $s = \sum_{k=0}^{n-1} k r_k$. This formula may be used in obtaining means and covariances of Y_k's under the conditional distribution $P(\bullet | Z = n)$. Moreover when Condition A holds, we can apply the analysis in §2.2 to the above formula, to obtain an asymptotic expansion of $M(n; r_0, r_1, \ldots)$ as $n \to \infty$. For example, we obtain an asymptotic expression as $n \to \infty$ for the expectation $\mathbf{E}_n(Y_k)$, variance $\mathbf{Var}_n(Y_k)$ and covariance $\mathbf{Cov}_n(Y_k, Y_{k'})$ $(k \neq k')$ of Y_k's under the conditional distribution $P(\bullet | Z = n)$ as follows:

$$\mathbf{E}_n(Y_k) = \rho_k n + \mathcal{O}(1) \ ; \tag{35}$$

$$\mathbf{Var}_n(Y_k) = \frac{\rho_k}{\kappa_2}[(1-\rho_k)\kappa_2 - (k-1)^2 \rho_k] n + \mathcal{O}(1) \ ; \tag{36}$$

$$\mathbf{Cov}_n(Y_k, Y_{k'}) = -\frac{\rho_k \rho_{k'}}{\kappa_2}[\kappa_2 + (k-1)(k'-1)] n + \mathcal{O}(1) \ , \tag{37}$$

where as before $\rho_k := p_k a^k / f(a)$, and κ_2 is the variance of the probability distribution $\Pi_a := \{\rho_k\}_{k\geq 0}$.

Let us now discuss several examples. See [1] for related topics.

4.2 Example 1: Geometric offspring distribution

Suppose $\Pi = \{p_k\}_{k=0}^{\infty}$ is the geometric distribution with parameter p, namely suppose that $p_k = (1-p)p^k$, $k = 0, 1, 2, \ldots$, for some $0 < p < 1$. In this case $f(z) = (1-p)/(1-pz)$, and we obtain

$$\begin{aligned} P(Z=n) &= \frac{1}{2\pi i n} \oint \frac{1}{z^n} \left(\frac{1-p}{1-pz}\right)^n dz \\ &= \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{1-p}{1-pz}\right)^n \Bigg|_{z=0} \\ &= (1-p)^n p^{n-1} \, \frac{1}{n} \binom{2n-2}{n-1} . \end{aligned} \tag{38}$$

For a finite tree $\omega \in \Omega$ with $Z(\omega) = n$, we thus obtain

$$P(\{\omega\} |\ Z=n) = \frac{1}{P(Z=n)} \prod_{u\in\omega} p_{\nu_u(\omega)} = \frac{1}{\frac{1}{n}\binom{2n-2}{n-1}} . \tag{39}$$

The right hand side being independent of the detailed data $\{\nu_u(\omega)\}_{u\in\omega}$ of ω, this shows that conditioned on the event $\{Z = n\}$, all trees of size n are equi-probable, which in turn implies that the total number of trees of size n is $(1/n)\binom{2n-2}{n-1}$, which is a well known fact in combinatorics.

For each integer $n \geq 1$ and for each sequence (n_k) of non-negative integers satisfying (32), we have, by (33) of Proposition 8 and (38),

$$P(Y_k = n_k,\ k \geq 0 \mid Z = n) = \frac{(n-1)! \left(\prod_{k\geq 0} \frac{(1-p)p^{kn_k}}{n_k!}\right)}{\frac{1}{n}\binom{2n-2}{n-1}(1-p)^n p^{n-1}} = \frac{\frac{1}{n}\binom{n}{n_0\ n_1\ \ldots}}{\frac{1}{n}\binom{2n-2}{n-1}} ,$$

where

$$\binom{n}{n_0 n_1 \cdots} := \frac{n!}{n_0! n_1! \cdots} \quad \text{with } \sum_{j\geq 0} n_j = n$$

is the multinomial coefficient. By the equi-probability of the trees under the condition $\{Z = n\}$, this shows that the total number of trees ω such that $Z(\omega) = n$, $Y_k(\omega) = n_k$, $k \geq 0$ is given by

$$\frac{1}{n}\binom{n}{n_0\ n_1\ \cdots} , \tag{40}$$

where (n_k) satisfies (32). This result is also well known and is attributed to Tutte [27], but it had already been stated by Otter (page 213 of [23]), though with somewhat incomplete proof.

Let us now take $r = 1$, $k_1 = 0$ and $j := n_{k_1} = \mu < n$ in (31). Then we have $\nu = 0$ and

$$\tilde{f}(z) = (1-p)\sum_{k\geq 1} (pz)^k = pzf(z) = (1-p)\frac{pz}{1-pz} ,$$

so that

$$
\begin{aligned}
& P(Z=n\ ,\ Y_0=j) \\
= \ & \frac{(n-1)!}{(n-j)!}\frac{1}{j!}(1-p)^j\frac{1}{2\pi i}\oint\frac{(pz)^{n-j}}{z^n}f(z)^{n-j}dz \\
= \ & \frac{(n-1)!}{(n-j)!j!}p^{n-j}(1-p)^n\frac{1}{(j-1)!}\left.\frac{d^{j-1}}{z^{j-1}}(1-pz)^{-(n-j)}\right|_{z=0} \\
= \ & p^{n-1}(1-p)^n\frac{1}{n}\binom{n}{j}\binom{n-2}{j-1}\ .
\end{aligned}
$$

Again by (38) and the equi-probability of trees, we see that the number $p_{n,j}$ of trees of size n with exactly j vertices having no children, namely "leaves", is given by

$$
p_{n,j}=\frac{1}{n}\binom{n}{j}\binom{n-2}{j-1}\ .
$$

This formula is also known in combimatorics. (See e.g. [8].) Note that "trees" in our sense are often called "planted plane trees" in combinatorics. Thus $p_{n,j}$ is the number of planted plane trees with j leaves.

4.3 Example 2: Binomial offspring distribution

Next consider the case where Π is the binomial distribution $B(d,p)$, so that $f(z)=(1-p+pz)^d$. Applying (2), we can easily compute

$$
P(Z=n)=\frac{1}{n}\binom{dn}{n-1}(1-p)^{dn-n+1}p^{n-1}\ .
$$

On the other hand, for a tree ω of size n,

$$
\begin{aligned}
P(\{\omega\}) &= \prod_{u\in\omega}p_{\nu_u(\omega)}=\prod_{u\in\omega}\binom{d}{\nu_u(\omega)}p^{\nu_u(\omega)}(1-p)^{d-\nu_u(\omega)} \\
&= p^{n-1}(1-p)^{dn-n+1}\prod_{u\in\omega}\binom{d}{\nu_u(\omega)}\ ,
\end{aligned}
$$

and hence

$$
P(\{\omega\}\mid Z=n)=\frac{1}{\frac{1}{n}\binom{dn}{n-1}}\prod_{u\in\omega}\binom{d}{\nu_u(\omega)}\ . \tag{41}
$$

These results have the following interpretation. Consider the bond percolation on the rooted d-ary tree. Here we mean by the rooted d-ary tree an element $T\in\Omega$ such that $\nu_u(T)=d$ for all $u\in T$, and we suppose that each bond, or edge, of the graph T is "open" with a certain probability $0<p<1$ and is "closed" with probability $1-p$, independently from each other. We then call a percolation cluster on T a subset γ of T consisting of those vertices $u \in T$ which are connected to the root ϕ through open bonds. A percolation cluster γ thus obtained is a "tree"which is different from our Galton-Watson tree defined in §1 in the sense that each vertex $u \in \gamma$ can have k children, among d possible ones, in $\binom{d}{k}$ different ways. (When $d = 2$, the percolation clusters thus obtained is often called "uniform binary trees", see e.g. [21].) Each realization γ of the percolation cluster is an abstract tree in the sense of Definition 1, and hence is isomorphic to a tree $\omega \in \Omega$. Denote this correspondence by $\omega = \Psi(\gamma)$. When ω is finite, there are exactly $\prod_{u\in\omega} \binom{d}{\nu_u(\omega)}$ percolation clusters which correspond to the same ω.

Now let $\mathcal{C}_n$ be the totality of percolation clusters of size n. From what we have seen above,

$$\sharp\mathcal{C}_n = \sum_{\omega\in\Omega;Z(\omega)=n} \sharp\Psi^{-1}(\{\omega\}) = \sum_{\omega\in\Omega;Z(\omega)=n} \prod_{u\in\omega} \binom{d}{\nu_u(\omega)} .$$

On the other hand, we have

$$1 = \sum_{\omega\in\Omega;Z(\omega)=n} P(\{\omega\} \mid Z = n) = \frac{1}{\frac{1}{n}\binom{dn}{n-1}} \sum_{\omega\in\Omega;Z(\omega)=n} \prod_{u\in\omega} \binom{d}{\nu_u(\omega)} ,$$

thus

$$\sharp\mathcal{C}_n = \frac{1}{n}\binom{dn}{n-1} .$$

If we let $E_u(\gamma)$ be the number of outward edges issued from the vertex u of a percolation cluster $\gamma \in \mathcal{C}_n$, then it is clear that

$$X_k(\gamma) := \sharp\{u \in \gamma |\ E_u(\gamma) = k\} = Y_k(\Psi(\gamma)) .$$

Thus if we are given $(n_k)_{k\geq 0}$ which is subject to the condition (32), then it holds that

$$\begin{aligned}
&\sharp\{\gamma \in \mathcal{C}_n \mid X_k(\gamma) = n_k\ ,\ k \geq 0\} \\
= &\sum_{\omega\in\Omega;\ Z(\omega)=n\ ,\ Y_k(\omega)=n_k\ ,\ k\geq 0} \sharp\Psi^{-1}(\{\omega\}) \\
= &\ \sharp\{\omega \in \Omega \mid Z(\omega) = n,\ Y_k(\omega) = n_k,\ k \geq 0\} \times \prod_{\nu=0}^{d} \binom{d}{\nu}^{n_\nu} \\
= &\ \frac{1}{n}\binom{n}{n_0 n_1 \cdots} \prod_{\nu=0}^{d} \binom{d}{\nu}^{n_\nu} .
\end{aligned}$$

This shows that if Q_n is the uniform distribution on $\mathcal{C}_n$, then from (40) and the result of the previous subsection,

$$\begin{aligned}
Q_n(X_k = n_k,\ k \geq 0) &= \frac{\frac{1}{n}\binom{n}{n_0 n_1\cdots} \prod_{\nu=0}^{d} \binom{d}{\nu}^{n_\nu}}{\frac{1}{n}\binom{nd}{n-1}} \\
&= P(Y_k = n_k\ ,\ k \geq 0 \mid Z = n) . \qquad (42)
\end{aligned}$$

Next consider the special case $d = 2$ and let $r = 1$, $k_1 = 0$ and $j := n_{k_1} = \mu < n$ in (31). Then $\nu = 0$ and $\tilde{f}(z) = pz(pz + 2q)$ with $q = 1 - p$, and we have

$$\begin{aligned} P(Z = n, Y_0 = j) &= \frac{(n-1)!}{(n-j)!} \frac{1}{j!} q^{2j} \frac{1}{2\pi i} \oint \frac{1}{z^n} (pz)^{n-j} (pz + 2q)^{n-j} dz \\ &= \frac{1}{n} \binom{n}{j} 2^{n+1-2j} q^{n+1} p^{n-1} \binom{n-j}{j-1} , \end{aligned}$$

and hence

$$Q_n(X_0 = j) = P(Y_0 = j | Z = n) = \frac{1}{b_n} \; 2^{n+1-2j} \; \frac{1}{n} \binom{n}{j} \binom{n-j}{j-1} , \tag{43}$$

where $b_n = \binom{2n}{n-1}/n$ is the n-th Catalan number. Note that from (32), we have

$$Y_0 + Y_1 + Y_2 = n \quad ; \quad Y_1 + 2Y_2 = n - 1 ,$$

so that in particular $n - j \geq j - 1$ when $Y_0 = j$. Moreover these relations imply

$$Q_n(X_2 = j) = Q_n(X_0 = j + 1) \tag{44}$$

and

$$Q_n(X_1 = j) = \begin{cases} Q_n\left(X_0 = \frac{n+1-j}{2}\right) & \text{if } n - j \text{ is odd} \\ \\ \qquad\qquad 0 & \text{if } n - j \text{ is even} . \end{cases} \tag{45}$$

Explicit formulas for $Q_n(X_i = j)$, $i = 0, 1, 2$, were first given by Mahmoud in [21]. But his expression consisted of very complicated alternating sum. Later Prodinger [24] computed these sums, to obtain formulas (43), (44) and (45). See also [13], [19] and [25].

Also we can apply (34) to compute the factorial moments $\mathbf{E}_{Q_n}[X_i^{[r]}]$, $i = 0, 1, 2$, reproducing Theorem 3 of [24]. Thus for example,

$$\begin{aligned} \mathbf{E}_{Q_n}\left[X_0^{[r]}\right] &= \mathbf{E}\left[Z = n \; ; \; Y_0^{[r]}\right] \Big/ P(Z = n) \\ &= \frac{1}{P(Z = n)} \frac{n^{[r]}}{2\pi i n} q^{2r} \oint \frac{f(z)^{n-r}}{z^n} dz \\ &= \frac{1}{P(Z = n)} \; n^{[r]} q^{n+1} p^{n-1} \binom{2n - 2r}{n - 1} \\ &= \frac{1}{b_n} n^{[r]} \binom{2n - 2r}{n - 1} . \end{aligned}$$

4.4 Example 3: Power series offspring distribution

As is seen in the above examples, the conditional distribution $P(\{\omega\} \mid Z = n)$ does not depend on the parameter p. This is generally true when the offspring distribution is a power series distribution. Namely, let $B(\theta) = \sum_{k \geq 0} b_k \theta^k$ be a power series with non-negative coefficients and with positive radius of convergence R. Assuming $b_0 > 0$ and $b_k > 0$ for some $k \geq 2$, we let

$$p_k(\theta) = \frac{b_k \theta^k}{B(\theta)}\ ,\ \ k = 0, 1, 2, \dots$$

for $0 < \theta < R$. Then the probability distribution $\Pi(\theta) = \{p_k(\theta)\}_{k=0}^{\infty}$ satisfies the basic assumption $p_0(\theta) > 0$, $p_0(\theta) + p_1(\theta) < 1$, and we have the following proposition.

Proposition 9 *Let P_θ be the probability measure for the Galton-Watson tree with offspring distribution $\Pi(\theta)$. Then, the total progeny Z is a sufficient statistics for the statistical model $(\Omega, \mathcal{F}, \{P_\theta(\bullet | Z < \infty)\}_{\theta \in (0,R)})$ in the sense that for any $n \geq 1$, the conditional distribution $P_\theta(\bullet | Z = n)$ does not depend on θ.*

Proof. The generating function of $\Pi(\theta)$ is given by $B(\theta z)/B(\theta)$. Hence by (2),

$$P(Z = n) = \frac{1}{2\pi i n} \oint \frac{1}{z^n} \left(\frac{B(\theta z)}{B(\theta)} \right)^n dz = \frac{1}{2\pi i n} \frac{\theta^{n-1}}{B(\theta)^n} \oint \frac{B(w)^n}{w^n} dw\ .$$

On the other hand, for a tree ω with $Z(\omega) = n$,

$$P(\{\omega\}) = \prod_{u \in \omega} p_{\nu_u(\omega)}(\theta) = \frac{\theta^{n-1}}{B(\theta)^n} \prod_{u \in \omega} b_{\nu_u(\omega)}\ ,$$

so that by dividing, the factors involving θ cancel out in $P(\{\omega\} | Z = n)$.

5 The weak law of large numbers and the central limit theorem for the distribution of $\{Y_k\}$, conditioned on Z

In this section, we prove two conditional limit theorems for $\{Y_k\}$, namely the assertions (i) and (ii) of Theorem 3 below. Although the assertion (i), the weak law of large numbers, is actually due to Otter (Theorem 6 in [23]), we shall give an alternative proof, which will be completed almost simultaneously with the proof of the central limit theorem (ii).

Theorem 3 *Suppose Condition A holds. Then the following limit theorems hold, where we let $n \to \infty$ keeping $n \equiv 1 \pmod{d(\Pi)}$:*

(i) the weak law of large numbers: for any $k \geq 0$,

$$P\left(\frac{Y_k}{n} \in \bullet \ \middle|\ Z = n \right) \longrightarrow \delta_{\rho_k}(\bullet)\ ,$$

where $\delta_b(\cdot)$ is the point mass concentrated on b. Here $\{\rho_k\}_{k \geq 0}$ is the probability distribution on $\mathbf{Z}_+$ defined by $\rho_k = p_k a^k / a^k$.

(ii) the central limit theorem: for any finite subset A of $\mathbf{Z}_+$,

$$P\left(\left\{\sqrt{n}\left(\frac{Y_k}{n}-\rho_k\right)\right\}_{k\in A}\in\bullet\;\middle|\;Z=n\right)\longrightarrow N_A(0,V_A)\ ,$$

where $N_A(0,V_A)$ is the $|A|$-dimensional Gaussian distribution with mean 0, and with the covariance matrix V_A determined from the quadratic form

$$V(\{t_k\}) \;=\; \sum_{k\in A}t_k^2\rho_k-\left(\sum_{k\in A}t_k\rho_k\right)^2-\frac{1}{\sum_{k\geq 2}k(k-1)\rho_k}\left(\sum_{k\in A}kt_k\rho_k-\sum_{k\in A}t_k\rho_k\right)^2 .$$

Here V_A is strictly positive definite if and only if $A\subset S$ and $|S\setminus A|\geq 2$, where we let $S:=\{k\in\mathbf{Z}_+;p_k>0\}$.

Remark. The special case $A=\{0\}$ of the central limit theorem is known to combinatorialists. See e.g. Theorem 12 of [8].

On the other hand, under the condition that $f(z)$ be entire, Dembo, Mörters and Sheffield ([7]) proved a general large deviation principle which reduces to the large deviation principle corresponding to the law of large numbers (i). (Set $\mathcal{X}$ = singleton in Theorem 2.2 of [7].)

For other types of conditional limit theorems considered on the event $\{Z=n\}$, see e.g. [17] and [18].

Proof. Take a real sequence $\{t_k\}_{k\geq 0}$, where $t_k\neq 0$ for only finitely many k's, and let $v_k=e^{\zeta t_k}$ with $\zeta\in\mathbf{C}$ in the formula (29). Then we obtain

$$\mathbf{E}\left[Z=n;\ \exp\left(\zeta\sum_{k\geq 0}t_kY_k\right)\right]=\frac{1}{2\pi in}\oint\exp\{nh(z,\zeta)\}dz\ , \tag{46}$$

where we let

$$h(z,\zeta):=\log\left\{\frac{\Phi(z,\zeta)}{z}\right\}\quad\text{and}\quad\Phi(z,\zeta):=\sum_{k\geq 0}p_ke^{\zeta t_k}z^k\ .$$

If we further define

$$\begin{aligned}M(z,\zeta)\;&:=\;\sum_{k\geq 1}kp_ke^{\zeta t_k}z^k-\sum_{k\geq 0}p_ke^{\zeta t_k}z^k\\&=\;z\frac{\partial}{\partial z}\Phi(z,\zeta)-\Phi(z,\zeta)\ ,\end{aligned}$$

then we get

$$\frac{\partial}{\partial z}h(z,\zeta)=\frac{M(z,\zeta)}{z\Phi(z,\zeta)}\ .$$

In order to apply the saddle point approximation to the integral in (46), we need to know the zero of the function $\partial h/\partial z$, or equivalently we need to solve the equation $M(z,\zeta)=0$

with respect to z. Since $\Phi(z,0)=f(z)$, we have $M(a,0)=af'(a)-f(a)=0$. On the other hand, we have $(\partial M/\partial z)(a,0)=af''(a)>0$. Hence by the implicit function theorem, there is a unique analytic function $a(\zeta)$, which is defined in a neighbourhood of the origin, such that $a(0)=a$ and that $M(a(\zeta),\zeta)=0$.

Now if we take the contour $|z|=|a(\zeta)|$ in the integral in (29), and if we assume $n\equiv 1 \pmod{d(\Pi)}$, the as in (7), we obtain

$$\mathbf{E}\left[Z=n;\ \prod_{k\geq 0}\exp(\zeta t_k Y_k)\right]=\frac{a(\zeta)d}{2\pi n}\int_{-\pi/d}^{\pi/d}\exp\{n\psi(\theta,\zeta)\}e^{i\theta}d\theta\ , \tag{47}$$

where we have set

$$\psi(\theta,\zeta):=h(a(\zeta)e^{i\theta},\zeta)=\log\left[\frac{\Phi(a(\zeta)e^{i\theta},\zeta)}{a(\zeta)e^{i\theta}}\right]\ .$$

For this function $\psi(\theta,\zeta)$, the following assertion can be easily verified:

I As $\zeta\to 0$ in $\mathbf{C}$, $\psi(\theta,\zeta)$, as well as all its derivatives with respect to θ, converges uniformly to those of

$$\psi(\theta):=h(ae^{i\theta},0)=\log\left[\frac{f(ae^{i\theta})}{ae^{i\theta}}\right]\ .$$

II In particular,

$$\psi(0,\zeta)=\log\left[\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right]\longrightarrow\log\left[\frac{f(a)}{a}\right]\ ,$$

as $\zeta\to 0$.

III

$$\frac{\partial}{\partial\theta}\psi(\theta,\zeta)\bigg|_{\theta=0}=ia(\zeta)\frac{\partial h}{\partial\zeta}(a(\zeta),\zeta)=0\ .$$

IV One has

$$-K(\zeta):=\frac{\partial^2}{\partial\theta^2}\psi(\theta,\zeta)\bigg|_{\theta=0}=-a(\zeta)^2\frac{\partial^2 h}{\partial z^2}(a(\zeta),\zeta)\ ,$$

and as $\zeta\to 0$,

$$-K(\zeta)\ \longrightarrow -\frac{a^2f''(a)}{f(a)}=-K<0\ .$$

V By I, we can write

$$\psi(\theta,\zeta)=\psi(0,\zeta)-\frac{1}{2}K(\zeta)\theta^2+\theta^3\beta(\theta,\zeta)\ ,$$

where $\beta(\theta,\zeta)$ is analytic and uniformly bounded in (θ,ζ).

VI By III, IV and V, we can choose positive constants γ, δ and η so that

$$\mathbf{Re}\psi(\theta,\zeta)\leq\mathbf{Re}\psi(0,\zeta)-\eta\theta^2$$

holds for $|\theta|\leq\delta$ and $|\zeta|\leq\gamma$.

VII From I,II and VI, we infer that if $|\zeta|$ is sufficiently small, then

$$|\Phi(a(\zeta)e^{i\theta},\zeta)/a(\zeta)| = \exp(\mathbf{Re}\psi(\theta,\zeta))$$

takes its maximum value at $\theta = 0$ and that if $0 < |\theta| \le \pi/d$,

$$\left|\frac{\Phi(a(\zeta)e^{i\theta},\zeta)}{a(\zeta)}\right| < \left|\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right| .$$

VIII Let Δ_δ be defined by (8). From I and VI, if we take $0 < \epsilon < \frac{1}{3}\{(f(a)/a) - \Delta_\delta\}$, then for ζ sufficiently near to 0, we have

$$\Delta_{\zeta,\delta} := \sup_{\delta\le|\theta|\le\pi/d}\left|\frac{\Phi(a(\zeta)e^{i\theta},\zeta)}{a(\zeta)}\right| < \Delta_\delta + \frac{\epsilon}{3} < \frac{f(a)}{a} - \frac{\epsilon}{3} < \left|\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right| .$$

Now we can perform an analysis similar to that in §2.2. First we rewrite (47) as

$$\begin{aligned}
&\mathbf{E}\left[Z = n;\ \exp\left(\sum_{k\ge0}\zeta t_k Y_k\right)\right] \\
= &\frac{a(\zeta)d}{2\pi n}\left(\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right)^n \int_{-\delta}^{\delta}\exp\left\{-\frac{n}{2}K(\zeta)\theta^2 + n\theta^3\beta(\theta,\zeta) + i\theta\right\}d\theta \\
&+\mathcal{O}\left(\frac{1}{n}\Delta_{\zeta,\delta}^n\right) . \qquad (48)
\end{aligned}$$

If we introduce

$$P_\zeta(\omega,\theta) := \exp(i\theta + \omega\beta(\theta,\zeta)) = \sum_{\ell,m\ge0} c_{\ell m}(\zeta)\omega^\ell\theta^m$$

and

$$Q_\zeta(\omega,\theta) := c_{00}(\zeta) + c_{10}(\zeta)\omega + c_{01}(\zeta)\theta ,$$

then we have

$$|P_\zeta(\omega,\theta) - Q_\zeta(\omega,\theta)| = \mathcal{O}(|\omega|^2) + \mathcal{O}(|\theta|^2) ,$$

where the estimate $\mathcal{O}(\cdot)$ is valid uniformly in ζ sufficiently near to 0. Now replace δ in the limit of integration in (48) by $\tau_n := n^{-1/3}$, and then replace $\exp(i\theta + n\theta^3\beta(\theta,\zeta))$ by $Q_\zeta(n\theta^3,\theta)$. The error committed by these procedures are estimated as in §2.2 and we obtain

$$\begin{aligned}
\mathbf{E}\left[Z = n;\ \exp\left(\sum_{k\ge0}\zeta t_k Y_k\right)\right] &= \frac{a(\zeta)d}{2\pi n}\left(\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right)^n \times \\
&\times\left[\sum_{\ell+m\le1} c_{\ell m}(\zeta)n^\ell\int_{-\tau_n}^{\tau_n}\exp(-\frac{n}{2}K(\zeta)\theta^2)\theta^{3\ell+m}d\theta + \mathcal{O}(n^{-3/2})\right] .
\end{aligned}$$

Finally we extend the integration to $(-\infty, \infty)$, with a small error which is absorbed in the term $\mathcal{O}(n^{-3/2})$. We thus arrive at

$$\begin{aligned}&\mathbf{E}\left[Z = n;\ \exp\left(\sum_{k\geq 0} \zeta t_k Y_k\right)\right] \\ &= \frac{a(\zeta)d}{n}\left(\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right)^n \sqrt{\frac{1}{2\pi n K(\zeta)}}\ (1+\mathcal{O}(n^{-3/2}))\ .\end{aligned}$$

Note that the integrals for $\ell + m = 1$ vanish because of symmetry, and that the estimate $\mathcal{O}(n^{-3/2})$ of the error is uniform in ζ.

Combining this result with Theorem 1, or rather Otter's theorem, we get

$$\mathbf{E}\left[\exp\left(\sum_{k\geq 0}\zeta t_k Y_k\right)\middle| Z = n\right] = \frac{a(\zeta)\left(\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right)^n \sqrt{\frac{1}{K(\zeta)}}\ (1+\mathcal{O}(n^{-3/2}))}{a\left(\frac{f(a)}{a}\right)^n \sqrt{\frac{f(a)}{a^2 f''(a)}}(1+\mathcal{O}(n^{-3/2})}\ .$$

If we take a sequence $\zeta_n \to 0$, then $K(\zeta_n) \to a^2 f''(a)/f(a)$ and $a(\zeta_n) \to a$. Hence we finally obtain

$$\mathbf{E}\left[\exp\left(\sum_{k\geq 0}\zeta_n t_k Y_k\right)\middle| Z = n\right] \sim \left\{\frac{a}{f(a)}\frac{\Phi(a(\zeta_n),\zeta_n)}{a(\zeta_n)}\right\}^n\ , \tag{49}$$

where $n \to \infty$ keeping $n \equiv 1 \pmod{d(\Pi)}$.

Now let us take $\zeta_n = -1/n$ and suppose $t_k \geq 0$. Noting $f'(a) = f(a)/a$, it is easy to compute

$$\log \Phi\left(a(-1/n), -1/n\right) = \log f(a) - \frac{1}{n}\left\{\frac{1}{f(a)}\sum_{k\geq 0} p_k t_k a^k + \frac{a'(0)}{a}\right\} + \mathcal{O}\left(n^{-2}\right)$$

and

$$\log a(-1/n) = \log a - \frac{1}{n}\frac{a'(0)}{a} + \mathcal{O}(n^{-2})\ .$$

Thus we can conclude from (49)

$$\mathbf{E}\left[\exp\left(-\sum_{k\geq 0} t_k \frac{Y_k}{n}\right)\middle| Z = n\right] \longrightarrow \exp\left\{-\sum_{k\geq 0}\frac{p_k a^k}{f(a)} t_k\right\}\ ,$$

completing the proof of (i).

Let us remark that $\{\rho_k\}_{k\geq 0} := \{p_k a^k/f(a)\}_{k\geq 0}$ is a probability distribution on $\mathbf{Z}_+$ with mean 1:

$$\sum_{k\geq 1} k\rho_k = \frac{1}{f(a)}\sum_{k\geq 1} k p_k a^k = \frac{af'(a)}{f(a)} = 1\ .$$

In order to prove the assertion (ii), we expand $\log[\Phi(a(\zeta),\zeta)]$ and $\log a(\zeta)$ to the second order terms for small ζ, to obtain

$$
\begin{aligned}
&\log\left[\frac{a}{f(a)}\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right] \\
= \quad & \frac{1}{f(a)}\left(\frac{\partial\Phi}{\partial\zeta}(a,0)\right)\zeta + \\
&+\left[\frac{1}{2f(a)}\left(f''(a)a'(0)^2+\frac{\partial^2\Phi}{\partial\zeta^2}(a,0)+2\frac{\partial^2\Phi}{\partial\zeta\partial z}(a,0)a'(0)\right)\right. \\
&\left.-\frac{1}{2f(a)^2}\left(f'(a)a'(0)+\frac{\partial\Phi}{\partial\zeta}(a,0)\right)^2+\frac{1}{2}\frac{a'(0)^2}{a^2}\right]\zeta^2+\mathcal{O}(\zeta^3)\ .
\end{aligned}
$$

On the other hand, differentiating $M(a(\zeta),\zeta)=0$, we get

$$
a'(0)=-\frac{1}{af''(a)}\left\{a\frac{\partial^2\Phi}{\partial\zeta\partial z}(a,0)-\frac{\partial\Phi}{\partial\zeta}(a,0)\right\}\ ,
$$

which, inserted in the above formula, gives

$$
\begin{aligned}
&\log\left[\frac{a}{f(a)}\frac{\Phi(a(\zeta),\zeta)}{a(\zeta)}\right] \\
= \quad & \frac{1}{f(a)}\left(\frac{\partial\Phi}{\partial\zeta}(a,0)\right)\zeta + \\
&+\left[-\frac{f''(a)}{2f(a)}a'(0)^2+\frac{1}{2f(a)}\frac{\partial^2\Phi}{\partial\zeta^2}(a,0)-\frac{1}{2f(a)^2}\left(\frac{\partial\Phi}{\partial\zeta}(a,0)\right)^2\right]\zeta^2+\mathcal{O}(\zeta^3)\ .
\end{aligned}
$$

Letting $t_k\in\mathbf{R}$ and $\zeta=i/\sqrt{n}$ in (49), we finally obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\mathbf{E}\left[\exp\left\{i\sum_{k\geq 0}t_k\sqrt{n}\left(\frac{Y_k}{n}-\frac{p_ka^k}{f(a)}\right)\right\}\middle|\ Z=n\right] \\
= \quad & \lim_{n\to\infty}\exp\left\{-i\sqrt{n}\sum_{k\geq 0}t_k\frac{p_ka^k}{f(a)}\right\}\left\{\frac{a}{f(a)}\frac{\Phi(a(i/\sqrt{n}),i/\sqrt{n})}{a(i/\sqrt{n})}\right\}^n \\
= \quad & \lim_{n\to\infty}\exp\left\{-n\frac{1}{f(a)}\left(\frac{\partial\Phi}{\partial\zeta}(a,0)\right)\frac{i}{\sqrt{n}}\right\}\left\{\frac{a}{f(a)}\frac{\Phi(a(i/\sqrt{n}),i/\sqrt{n})}{a(i/\sqrt{n})}\right\}^n \\
= \quad & \exp\left[\frac{f''(a)}{2f(a)}a'(0)^2-\frac{1}{2f(a)}\frac{\partial^2\Phi}{\partial\zeta^2}(a,0)+\frac{1}{2f(a)^2}\left(\frac{\partial\Phi}{\partial\zeta}(a,0)\right)^2\right] \\
= \quad & \exp\left[-\frac{1}{2}V(\{t_k\})\right]\ ,
\end{aligned}
$$

where we have set

$$V(\{t_k\}) = \frac{1}{f(a)}\sum_{k\geq 0} p_k t_k^2 a^k - \frac{1}{f(a)^2}\left(\sum_{k\geq 0} p_k t_k a^k\right)^2 - \frac{1}{a^2 f(a) f''(a)}\left(\sum_{k\geq 1} k p_k t_k a^k - \sum_{k\geq 0} p_k t_k a^k\right)^2 .$$

In terms of ρ_k introduced above, this can be expressed as

$$V(\{t_k\}) = \sum_{k\geq 0} t_k^2\rho_k - \left(\sum_{k\geq 0} t_k\rho_k\right)^2 - \frac{1}{\sum_{k\geq 2} k(k-1)\rho_k}\left(\sum_{k\geq 1} k t_k\rho_k - \sum_{k\geq 0} t_k\rho_k\right)^2 .$$

Now let X be a $\mathbf{Z}_+$-valued random variable with distribution $\{\rho_k\}$ and let the function $T : \mathbf{Z}_+ \mapsto \mathbf{R}$ be defined by $T(k) = t_k$. Then $\mathbf{E}[X] = 1$ and

$$\sum_{k\geq 2} k(k-1)\rho_k = \mathbf{E}[X(X-1)] = \mathbf{E}[(X-1)^2] > 0$$

because the distribution $\{p_k\}$ is not degenerate to a point by our basic assumption, so that

$$\begin{aligned} &\left(\sum_{k\geq 2} k(k-1)\rho_k\right) V(\{t_k\}) \\ =\ & \mathbf{E}[(X-1)^2]\left\{\mathbf{E}[T(X)^2] - \mathbf{E}[T(X)]^2\right\} - \{\mathbf{E}[XT(X)] - \mathbf{E}[T(X)]\}^2 \\ =\ & \mathbf{E}[(X-1)^2]\mathbf{E}[\{T(X) - \mathbf{E}[T(X)]\}^2] - \mathbf{E}[(X-1)\{T(X) - \mathbf{E}[T(X)]\}]^2 , \end{aligned}$$

which is actually non-negative by Schwarz's inequality applied to random variables $X-1$ and $T(X) - \mathbf{E}[T(X)]$.

Now suppose $A \subset S$, $A \neq \emptyset$ and $|S \setminus A| \geq 2$. Suppose further that for some $\lambda \in \mathbf{R}$, the relation

$$T(X) - \mathbf{E}[T(X)] = \lambda(X-1), \quad \text{a.s.}$$

holds, where we assume $T(k) = t_k = 0$ for $k \notin A$. Then if $\ell \in S \setminus A$, one has $-\mathbf{E}[T(X)] = \lambda(X-1)$. But in case $|S \setminus A| \geq 2$, this is possible only if $\lambda = 0$. Thus the inequality must be strict in the above mentioned Schwarz's inequality, unless $t_k = 0$ for all $k \in A$. Thus the limiting Gaussian distribution of $\{\sqrt{n}(n^{-1}Y_k - \rho_k)\}_{k\in A}$ conditioned on $\{Z = n\}$ is non-degenerate.

Conversely suppose $A \setminus S \neq \emptyset$. Since $Y_k = 0$, a.s. for $k \notin S$, the distribution of $\{Y_k\}_{k\in A}$ is already degenerate. If, on the other hand, $A = S \setminus \{\ell\}$ for some $\ell \in S$, then since the relations $\sum_{k\in S} Y_k = Z$ and $\sum_{k\in S} kY_k = Z - 1$ yields

$$\sum_{k\in S\setminus\{\ell\}} (\ell - k)Y_k = (\ell - 1)Z + 1 ,$$

the conditional distribution $P(\{\sqrt{n}(n^{-1}Y_k - \rho_k)\}_{k\in A} \in \bullet | Z = n)$ is concentrated on a hyperplane of dimension $\leq |A| - 2$. Hence its limiting Gaussian distribution is degenerate. This is also the case when $A = S$. The proof of Theorem 3 is now complete.

As an application of Theorem 3, we consider the following question treated by Mahmoud [21]. Let $\mathcal{C}_n$ be the totality of uniform binary trees of size n. (See §4.2 for the definition.) Let further $X_i(\gamma)$, $i = 0, 1, 2$, be the number of vertices of the tree $\gamma \in \mathcal{C}_n$ which has i children. As we saw in §4.2, we have

$$Q_n(\{\gamma \in \mathcal{C}_n;\ X_i(\gamma) = n_i, i = 0, 1, 2\}) = P(Y_i = n_i, i = 0, 1, 2 |\ Z = n)\ , \tag{50}$$

where Q_n is the uniform distribution on $\mathcal{C}_n$ and P is the distribution of the Galton-Watson tree whose offspring distribution is the binomial distribution $\mathrm{Bin}(2, p)$ with $0 < p < 1$. The right hand side being independent of p, we can take $p = 1/2$, in which case we have $a = f(a) = 1$ and $\rho_i = p_i a^i / f(a) = 1/4$ for $i = 0, 2$ while $\rho_1 = p_1 a^1 / f(a) = 1/2$. Moreover $f'(a) = f''(a) = 1/2$. We can now apply (ii) of Theorem 3 to this P, and in view of (50), we obtain

$$Q_n\left(\left(\frac{X_0 - n/4}{\sqrt{n}}, \frac{X_1 - n/2}{\sqrt{n}}, \frac{X_2 - n/4}{\sqrt{n}}\right) \in \bullet\right) \longrightarrow N\left(0, \frac{1}{16}C\right), \quad n \to \infty \tag{51}$$

where

$$C = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}\ ,$$

which is precisely Theorem 2 of [21], because as a special case of (35), we have

$$\mathbf{E}_{Q_n}[X_k] = \rho_k n + \mathcal{O}(1)$$

as $n \to \infty$ with $(\rho_0, \rho_1, \rho_2) = (1/4, 1/2, 1/4)$. Note that $d(\Pi) = 1$ and $S = A = \{0, 1, 2\}$ in our case, so that the matrix C is naturally degenerate by virtue of the second part of (ii). We also remark that we can apply (36) and (37) to show that the covariance matrix of (X_0, X_1, X_2) under the distribution Q_n is asymptotically equal to $\frac{n}{16}C$, as in Equation (2.5) of [21].

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