Asymptotic Analysis of the Periodic Schrodinger Operator

O. A. Veliev

Dept. of Math, Fen-Ed. Fak, Dogus University.,

Acibadem, Kadikoy, Istanbul, Turkey,

e-mail: oveliev@dogus.edu.tr

Abstract

In this paper we obtain asymptotic formulas of arbitrary order for the Bloch eigenvalues and Bloch functions of the Schrodinger operator $-\Delta+q(x)$, of arbitrary dimension, with periodic, with respect to arbitrary lattice, potential $q(x)$. Moreover, we estimate the measure of the isoenergetic surfaces in the high energy region.

1 Introduction

In this paper we consider the operator

$$L(q(x))=-\Delta + q(x),\ x\in\mathbb{R}^d,\ d\geq 2 \tag{1}$$

with a periodic (relative to a lattice Ω) potential $q(x)\in W_2^s(F)$, where $s\geq s_0=\frac{3d-1}{2}(3^d+d+2)+\frac{1}{4}d3^d+d+6$, $F\equiv \mathrm{R}^d/\Omega$ is a fundamental domain of Ω. Without loss of generality it can be assumed that the measure $\mu(F)$ of F is 1 and $\int_F q(x)dx=0$. Let $L_t(q(x))$ be the operator generated in F by (1) and the conditions:

$$u(x+\omega)=e^{i(t,\omega)}u(x),\ \forall\omega\in\Omega, \tag{2}$$

where $t \in F^{\star} \equiv \mathbb{R}^d/\Gamma$ and Γ is the lattice dual to Ω, that is, Γ is the set of all vectors $\gamma \in \mathbb{R}^d$ satisfying $(\gamma, \omega) \in 2\pi Z$ for all $\omega \in \Omega$. It is well-known that (see [2]) the spectrum of the operator $L_t(q(x))$ consists of the eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq$ ....The function $\Lambda_n(t)$ is called n-th band function and its range $A_n = \{\Lambda_n(t) : t \in$ F* is called the n-th band of the spectrum $Spec(L)$ of L and $Spec(L) = \cup_{n=1}^{\infty} A_n$. The eigenfunction $\Psi_{n,t}(x)$ of $L_t(q(x))$ corresponding to the eigenvalue $\Lambda_n(t)$ is known as Bloch functions. In the case $q(x) = 0$ these eigenvalues and eigenfunctions are $\mid \gamma + t \mid^2$ and $e^{i(\gamma+t,x)}$ for $\gamma \in \Gamma$.

This paper consists of 4 section. First section is the introduction, where we describe briefly the scheme of this paper and discuss the related papers.

In papers [13-17] for the first time the eigenvalues $\mid \gamma + t^2$, for big $\gamma \in \Gamma$, were divided into two groups: non-resonance ones and resonance ones and for the perturbations of each group various asymptotic formulae were obtained. Let the potential $q(x)$ be a trigonometric polynomial

$$\sum_{\gamma \in Q} q_\gamma e^{i(\gamma,x)},$$

where $q_\gamma = (q(x), e^{i(\gamma,x)}) = \int_F q(x) e^{-i(\gamma_1,x)} dx$, and Q consists of a finite number of vectors γ from Γ. Then the eigenvalue $|\gamma + t|^2$ is called a non-resonance eigenvalue if $\gamma + t$ does not belong to any of the sets $\{x \in \mathbb{R}^d : \mid\mid x \mid^2 - \mid x + b \mid^2 \mid < \mid$ x$|^{\alpha_1}\}$, that is, if $\gamma + t$ lies far from the diffraction hyperplanes $\{x \in \mathbb{R}^d : \mid x \mid^2 = \mid x + b \mid^2\}$, where $\alpha_1 \in (0,1)$, $b \in \{b_1 + b_2 + ...b_m : b_1, b_2, ...b_m \in Q\}$, and m is fixed integer (see [15-17]).

If $q(x) \in W_2^s(F)$, then to describe the non-resonance and resonance eigenvalues $|\gamma + t|^2$ of the order of ρ^2 (written as $|\gamma + t|^2 \sim \rho^2$) for big paramater ρ we write the potential $q(x) \in W_2^s(F)$ in the form

$$q(x)=\sum_{\gamma_1\in\Gamma(\rho^{\alpha})} q_{\gamma_1}e^{i(\gamma_1, x)} + O(\rho^{-p\alpha}), \qquad (3)$$

where $\Gamma(\rho^{\alpha}) = \{\gamma \in \Gamma : 0 < \mid \gamma \mid$ ¡$\rho^{\alpha})\}$, $p = s - d$, $\alpha = \frac{1}{q}$, $q = 3^d + d + 2$, and the relation $|\gamma + t|^2 \sim \rho^2$ means that $c_1\rho < |\gamma + t| < c_2\rho$. Here and in subsequent relations we denote by c_i $(i = 1, 2, ...)$ the positive, independent of ρ constants whose exact values are inessential. Note that $q(x) \in$ $W_2^s(F)$ means that $\sum_{\gamma} \mid q_{\gamma} \mid^2 (1+ \mid \gamma \mid^{2s}) < \infty$. If $s \geq d$, then

$$\sum_{\gamma} \mid q_{\gamma} \mid< c_3, \;\; \sup \mid \sum_{\gamma \notin \Gamma(\rho^{\alpha})} q_{\gamma}e^{i(\gamma,x)} \mid\leq \sum_{\mid \gamma \mid\geq \rho^{\alpha}} \mid q_{\gamma} \mid= O(\rho^{-p\alpha}), \qquad (4)$$

i.e., (3) holds. By definition, put $\alpha_k = 3^k\alpha$ for $k = 1, 2, ...$ and introduce the sets $V_{\gamma_1}(\rho^{\alpha_1}) \equiv \{x \in \mathbb{R}^d :\mid\mid x \mid^2 - \mid x + \gamma_1 \mid^2\mid< \rho^{\alpha_1}\}$,

$$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{\gamma_1\in\Gamma(p\rho^{\alpha})} V_{\gamma_1}(\rho^{\alpha_1}), \;\; U(\rho^{\alpha_1}, p) \equiv \mathbb{R}^d\backslash E_1(\rho^{\alpha_1}, p)$$

$$E_k(\rho^{\alpha_k}, p) \equiv \bigcup_{\gamma_1,\gamma_2,...,\gamma_k \in \Gamma(p\rho^{\alpha})} (\cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})),$$

where the intersection $\cap_{i=1}^{k} V_{\gamma_i}$ in the definition of E_k is taken over $\gamma_1, \gamma_2, ..., \gamma_k$, that are

linearly independent. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain and the eigenvalue $|\gamma+t|^2 \sim \rho^2$ is called a non-resonance eigenvalue if $\gamma+t \in U(\rho^{\alpha_1}, p)$. The domains $V_{\gamma_1}(\rho^{\alpha_1})$ for $\gamma_1 \in \Gamma(p\rho^{\alpha})$ are called resonance domains and $|\gamma+t|^2$ is called a resonance eigenvalue if $\gamma+t \in V_{\gamma_1}(\rho^{\alpha_1})$.
In section 2 we prove that for each $\gamma+t \in U(\rho^{\alpha_1}, p)$ there exists an eigenvalue $\Lambda_N(t)$ of the operator $L_t(q(x))$ satisfying the following formulae

$$\Lambda_N(t) = |\gamma+t|^2 + F_{k-1}(\gamma+t) + O(|\gamma+t|^{-3k\alpha}) \tag{5}$$

for $k = 1, 2, ..., [\frac{1}{3}(p - \frac{1}{2}q(d-1))]$, where $[a]$ denotes the integer part of a, $F_0 = 0$, and F_{k-1} (for $k > 1$) is explicitly expresed by the potential $q(x)$ and eigenvalues of $L_t(0)$. Besides, we prove that if the conditions

$$|\Lambda_N(t) - |\gamma+t|^2| < \frac{1}{2}\rho^{\alpha_1},$$

$$|b(N,\gamma)| > c_4\rho^{-c\alpha}$$

hold, where $b(N,\gamma) = (\Psi_{N,t}, e^{i(\gamma+t,x)})$, $\Psi_{N,t}(x)$ is a normalized eigenfunction of $L_t(q(x))$ corresponding to $\Lambda_N(t)$, then the following statements are valid:
(a) if $\gamma+t$ is in the non-resonance domain, then $\Lambda_N(t)$ satisfies (5) for $k = 1, 2, ..., [\frac{1}{3}(p-c)]$ (see Theorem 1);
(b) if $\gamma+t \in E_s \backslash E_{s+1}$, where $s = 1, 2, ..., d-1$, then

$$\Lambda_N(t) = \lambda_j(\gamma+t) + O(|\gamma+t|^{-k\alpha}), \tag{6}$$

where λ_j is an eigenvalue of the matrix $C(\gamma+t)$ (see (26) and Theorem 2). Moreover, we prove that every big eigenvalue of the operator

$L_t(q(x))$ for all values of quasimomenta t satisfies one of these formulae. For investigation of the Bloch function, in section 3, we find the values of quasimomenta $\gamma+t$ for which the corresponding eigenvalues are simple , namely we construct the subset B of $U(\rho^{\alpha_1},p)$ with the following properties:

Pr.1. If $\gamma+t\in B$, then there exists a unique eigenvalue, denoted by $\Lambda(\gamma+t)$, of the operator $L_t(q(x))$ satisfying (5). This is a simple eigenvalue of $L_t(q(x))$. Therefore we call the set B the simple set of quasimomenta.

Pr.2. The eigenfunction $\Psi_{N(\gamma+t)}(x)\equiv\Psi_{\gamma+t}(x)$ corresponding to the eigenvalue $\Lambda(\gamma+t)$ is close to $e^{i(\gamma+t,x)}$, namely

$$\Psi_N(x)=e^{i(\gamma+t,x)}+O(\mid\gamma+t\mid^{-\alpha_1}), \tag{7}$$

$$\Psi_{\gamma+t}(x)=e^{i(\gamma+t,x)}+\Phi_{k-1}(x)+O(\mid\gamma+t\mid^{-k\alpha_1}),\ k=1,2,\dots, \tag{8}$$

where Φ_{k-1} is explicitly expresed by $q(x)$ and the eigenvalues of $L_t(0)$.

Pr.3. The set B contains the intervals $\{a+sb : s\in[-1,1]\}$ such that $\Lambda(a-b)<\rho^2$, $\Lambda(a+b)>\rho^2$, and $\Lambda(\gamma+t)$ is continuous on these intervals. Hence there exists $\gamma+t$ such that $\Lambda(\gamma+t)=\rho^2$ for $\rho\gg1$. It implies that there exist only a finite number of gaps in the spectrum of L, that is, it implies the validity of Bethe-Sommerfeld conjecture for arbitrary dimension and for arbitrary lattice.

Construction of the set B consists of two steps.

Step 1. We prove that all eigenvalues $\Lambda_N(t)\sim\rho^2$ of the operator $L_t(q(x))$ lie in the $\varepsilon_1=\rho^{-d-2\alpha}$ neighborhood of the numbers $F(\gamma+t)=\mid\gamma+t\mid^2+F_{k_1-1}(\gamma+t)$, $\lambda_j(\gamma+t)$ (see (5), (8)), where $k_1=[\frac{d}{3\alpha}]+2$. We call these numbers as the known parts of the eigenvalues. Moreover, for $\gamma+t\in U(\rho^{\alpha_1},p)$ there is $\Lambda_N(t)$ satisfying $\Lambda_N(t)=F(\gamma+t)+o(\varepsilon_1)$ (see (5))

Step 2. By eliminating the set of quasimomenta $\gamma+t$, for which the known

parts $F(\gamma+t)$ of $\Lambda_N(t)$ are situated from the known parts
$F(\gamma^{'}+t)$, $\lambda_j(\gamma^{'}+t)$ ($\gamma^{'}\neq\gamma$) of other eigenvalues at a distance less than
$2\varepsilon_1$, we construct the set B with the following properties: if
$\gamma+t\in B$, then the following conditions (called simplicity conditions
for $\Lambda_N(t)$) hold

$$\mid F(\gamma+t)-F(\gamma^{'}+t)\mid\geq 2\varepsilon_1 \tag{9}$$

for $\gamma^{'}\in K\backslash\{\gamma\}$, $\gamma^{'}+t\in$
$\mathrm{U}(\rho^{\alpha_1},p)$ and

$$\mid F(\gamma+t)-\lambda_j(\gamma^{'}+t)\mid\geq 2\varepsilon_1 \tag{10}$$

for $\gamma^{'}\in K,\gamma^{'}+t\in E_k\backslash$
$\mathrm{E}_{k+1}, j=1,2,...$, where K is the set of $\gamma^{'}\in\Gamma$
satisfying $\mid F(\gamma+t)-\mid\gamma^{'}+t\mid^2\mid<\frac{1}{3}\rho^{\alpha_1}$. Thus B is the set of $\gamma+t\in U(\rho^{\alpha_1}$
,p)$satisfying the simplicity conditions (11), (12). As a consequence of$
these conditions the eigenvalue $\Lambda_N(t)$ does not coincide with other
eigenvalues. To prove this, namely to prove the Pr.1 and (9), we show that
for
any normalized eigenfunction $\Psi_N(x)$ corresponding to $\Lambda_N(t)$
the following equality holds:

$$\sum_{\gamma'\in\Gamma\backslash\gamma}\mid b(N,\gamma^{'})\mid^2=O(\rho^{-2\alpha_1}). \tag{11}$$

For the first time in [15-17] we constructed the simple set B with the Pr.1
and Pr.3., though in those papers we emphasized the Bethe-Zommerfeld
conjecture. Note that for this conjecture and for Pr.1, Pr.3. it is enough to
prove that the left-hand side of (13) is less than $\frac{1}{4}$ (we proved
this inequality in [15-17] and as noted in Theorem 3 of [16] and in [18] the

proof of this inequality does not differ from the proof of (13)). From (9) we got (10) (see [18]) . But in those papers these results are written briefly. The enlarged variant is written in [19] which can not be used as reference. In this paper we write these results in improved and enlarged form. The main difficulty and the crucial point of papers [15-17] were the construction of the simple set B with the Pr.1.,Pr.3. This difficulty of the perturbation theory of $L(q(x))$ is of a physical nature and it is connected with the complicated picture of the chystal diffraction. If $d = 2, 3$, then $F(\gamma + t) =| \gamma + t |^2$ and the matrix $C(\gamma + t)$ corresponds to the Schrodinger operator with directional potential $q_\delta(x) = \sum_{n\in} \mathrm{Z}q_{n\delta}e^{in(\delta,x)}$ (see [16]). So for construction of the simple set B of quasimomenta we eliminated the vicinities of the diffraction planes and the sets connected with directional potential (see (11), (12)). Besides, for nonsmooth potentials $q(x) \in L_2(\mathbb{R}^2/\Omega)$, we eliminated a set, which is described in the terms of the number of states (see [15,19]). The simple sets B of quasimomenta for the first time is constructed and investigated (hence the main difficulty and the crucial point of perturbation theory of $L(q)$ is investigated) in [16] for $d = 3$ and in [15,17] for the cases:

1. $d = 2$, $q(x) \in L_2(F)$; 2. $d > 2$, $q(x)$ is a smooth potential.

Then, Yu.E. Karpeshina proved (see [7-9]) the convergence of the perturbation series of two and three dimensional Schrodinger operator $L(q)$ with a wide class of nonsmooth potential $q(x)$ for a set, that is similar to B, of quasimomenta. In papers [3,4] the asymptotic formulas for the eigenvalues and Bloch function of the two and three dimensional operator $L_t(q(x))$ were obtained. In [5] the asymptotic formulae for the eigenvalues of $L_0(q(x))$ were obtained.

In section 4 we consider the geometrical aspects of the simple sets. We prove that the simple sets B has asymptotically full measure on $\mathbb{R}^d$. Moreover we construct a part of isoenergetic surfaces corresponding to ρ^2, which is smooth surfaces and has the measure asymptotically close to the measure of the Fermi surfaces $\{x \in R :| x |= \rho\}$ of the operator $L(0)$. The nonemptyness of the Fermi surfaces for $\rho \gg 1$ implies the the validity of the Bethe-Sommerfeld conjecture.

For the first time M.M. Skriganov [11,12] proved the validity of the Bethe-Sommerfeld conjecture for the Scrodinger operator for dimension $d = 2, 3$ for arbitrary lattice, for dimension $d > 3$ for rational lattice. The Skriganov's method is based on the detail investigation of the arithmetic and geometric properties of the lattice. B.E.J.Dahlberg and E.Trubowits [1] using an asymptotic of Bessel function, gave the simple proof of this conjecture for the two dimensional Scrodinger operator. Then in papers [15-17] we proved the validity of the Bethe-Sommerfeld conjecture for arbitrary lattice and for arbitrary dimension by using the asymptotic formulas and by construction of the simple set B, that is, by the method of perturbation theory. Yu.E. Karpeshina (see [7-9]) proved this conjecture for two and three dimensional Schrodinger operator $L(q)$ for a wide class of singular potentials $q(x)$, including Coulomp potential, by the method of perturbation theory. B. Helffer and A. Mohamed [6], by investigations the integrated density of states, proved the validity of the Bethe-Sommerfeld conjecture for the Scrodinger operator for $d \leq 4$ for arbitrary lattice. Recently L. Parnovski and A. V. Sobelev [10] proved this conjecture for $d \leq 4$. The method of this paper and papers [15-17] is a first and uniqie, for the present, by which the validity of the Bethe-Sommerfeld conjecture for arbitrary lattice and for arbitrary dimension is proved.

In this paper for the different types of the measures of the subset A of $\mathbb{R}^d$ we use the same notation $\mu(A)$. By $\mid A \mid$ we denote the number of elements of the set $A \subset \Gamma$ and use the following obvious fact. If $a \sim \rho$, then the number of elements of the set $\{\gamma + t : \gamma \in \Gamma\}$ satisfying $|| \gamma + t \mid -a |< 1$ is less than $c_5 \rho^{d-1}$. Therefore the number of eigenvalues of $L_t(q)$ lying in $(a^2 - \rho, a^2 + \rho)$ is less than $c_5 \rho^{d-1}$. Besides, we use the inequalities:

$$\alpha_1 + d\alpha < 1 - \alpha\,, \qquad d\alpha < \tfrac{1}{2}\alpha_d, \quad k_1 \leq \tfrac{1}{3}(p - \tfrac{1}{2}(q(d-1)),$$

$$p_1\alpha_1 \geq p\alpha, \qquad 3k_1\alpha > d + 2\alpha, \qquad \alpha_k + (k-1)\alpha < 1,$$

$$\alpha_{k+1} > 2(\alpha_k + (k-1))\alpha$$

for $k = 1, 2, ..., d$, which follow from the definitions $p = s - d$, $\alpha_k = 3^k\alpha$, $\alpha = \frac{1}{q}$, $q = 3^d + d + 2$, $k_1 = [\frac{d}{3\alpha}]+2$, $p_1 = [\frac{p}{3}] + 1$ of the numbers $p, q, \alpha_k, \alpha, k_1, p_1$.

2 Asymptotic Formulae for Eigenvalues

First we obtain the asymptotic formulas for the non-resonance eigenvalues by iteration of the formula

$(\Lambda_N - \mid \gamma + t \mid^2) b(N,\gamma) = (\Psi_{N,t}(x) q(x), e^{i(\gamma}$
+t,x)),
(12)

which is obtained from equation $-\Delta\Psi_{N,t}(x) + q(x)\Psi_{N,t}$
(x)=$\Lambda_N \Psi_{N,t}(x)$ by multiplying by $e^{i(\gamma+t,x)}$).
Introducing into (15) the expansion (3) of $q(x)$, we get

$(\Lambda_N - \mid \gamma + t \mid^2) b(N,\gamma) = \sum_{\gamma_1 \in \Gamma}$
$(\rho^\alpha) q_{\gamma_1} b(N, \gamma - \gamma_1) + O(\rho^{-p\alpha}).$
(13)

From the relations (15), (16) it follows that

b(N,$\gamma^{'}$) =
)$\Lambda_N - \mid \gamma^{'} + t \mid^2$=
$\sum_{\gamma_1 \in \Gamma(\rho^\alpha)}$
-$\mid \gamma^{'} + t \mid^2$ $+O(\rho^{-p\alpha})$
(14)

for all vectors $\gamma^{'} \in \Gamma$ satisfying the inequality

$\mid \Lambda_N - \mid \gamma^{'} + t \mid^2 \mid > \frac{1}{2}\rho$
α_1.
(15)

This inequality is called the iterability condition. If (6) holds and
$\mid \gamma + t \mid^2$ is a non-resonance eigenvalue, i.e., $\gamma + t \in$

U(ρ^{α_1}, p), then

$|| \gamma + t |^2 - | \gamma - \gamma_1 + t |^2 |> \rho^{\alpha_1}$
$, | \Lambda_N - | \gamma - \gamma_1 + t |^2 |> \frac{1}{2}$
ρ^{α_1}
(16)

for all $\gamma_1 \in \Gamma(p\rho^{\alpha})$. Hence the vector γ
-γ_1 for $\gamma + t \in U(\rho^{\alpha_1}, p)$ and $\gamma_1 \in$
$\Gamma(p\rho^{\alpha})$ satisfies (18). Therefore, in (17) one can replace
$\gamma^{'}$ by $\gamma - \gamma_1$ and write

b(N,$\gamma - \gamma_1) =$
$\sum$
$\gamma_2 \in \Gamma(\rho^{\alpha})$
$\gamma - \gamma_1 + t |^2 + O(\rho^{-p\alpha})$.

Substituting this for $b(N, \gamma - \gamma_1)$ into right-hand side of (16)
and isolating the terms containing the multiplicand $b(N, \gamma)$, we get

$(\Lambda_N - | \gamma + t |^2) b(N, \gamma) =$
$\sum$
$\gamma_1, \gamma_2 \in \Gamma(\rho^{\alpha})$
$\Lambda_N - | \gamma - \gamma_1 + t |^2 + O(\rho^{-p\alpha}) =$

$\sum$
$\gamma_1 \in \Gamma(\rho^{\alpha})$
-$\gamma_1 + t |^2 +$
$\sum$
$\gamma_1, \gamma_2 \in \Gamma(\rho^{\alpha}$
),
$\gamma_1 + \gamma_2 \neq 0$
$\Lambda_N - | \gamma - \gamma_1 + t |^2 + O(\rho^{-p\alpha})$,

since $q_{\gamma_1}q_{\gamma_2} =\mid q_{\gamma_1}\mid^2$ for
$\gamma_1+\gamma_2=0$ and the last sum is taken under the condition
$\gamma_1+\gamma_2\neq 0$. Repeating this process $p_1\equiv[$
p$_{\overline{3]+1}}$ times, i.e., in the last formula replacing $b(N,\gamma$
-$\gamma_1-\gamma_2)$ by its expression from (17) (in (17) replace
$\gamma^{'}$ by $\gamma-\gamma_1-\gamma_2)$ and isolating the
terms containing $b(N,\gamma)$ etc., we obtain

$(\Lambda_N-\mid\gamma+t\mid^2)b(N,\gamma)=A_{p_1}(\Lambda_N$
$,\gamma+t)b(N,\gamma)+C_{p_1}+O(\rho^{-p\alpha}),$
(17)

where $A_{p_1}(\Lambda_N,\gamma+t)=\sum_{k=1}^{p_1}S_k(\Lambda$
$_N,\gamma+t)$,

S$_k(\Lambda_N,\gamma+t)=$
$\sum\limits_{\gamma_1,...,\gamma_k\in\Gamma(\rho^\alpha)}$
$_2-...-\gamma_k\prod_{j=1}^{k}(\Lambda_N-\mid\gamma+t-\sum_{i=1}$
$^j\gamma_i\mid^2),$

C$_{p_1}=\sum_{\gamma_1,...,\gamma_{p_1+1}\in\Gamma(\rho^\alpha)}$
-$\gamma_1-\gamma_2-...-\gamma_{p_1+1})\prod_{j=1}^{p_1}(\Lambda$
$_N-\mid\gamma+t-\sum_{i=1}^{j}\gamma_i\mid^2).$

Here the sums for S_k and C_{p_1} are taken under the additional conditions
$\gamma_1+\gamma_2+...+\gamma_s\neq 0$ for $s=1,2,...,k$ and
$s=1,2,...,p_1$ respectively. These conditions and the inclusion γ
$_i\in\Gamma(\rho^\alpha)$ for $i=1,2,...,p_1$ imply the relation
$\sum_{i=1}^{j}\gamma_i\in\Gamma(p\rho^\alpha)$. Therefore from the second
inequality in (19) it follows that the absolute values of the denominators of
the fractions in S_k and C_{p_1} are greater than $(\frac{1}{2}$
$\rho^{\alpha_1})^k$ and $(\frac{1}{2}\rho^{\alpha_1})^{p_1}$
respectively. Hence the first inequality in (4) and $p_1\alpha_1\geq$
pα (see the fourth inequality in (14)) yield

$C_{p_1} = O(\rho^{-p_1\alpha_1}) = O(\rho^{-p\alpha}),\ S_k$
$(\Lambda_N, \gamma + t) = O(\rho^{-k\alpha_1}), \forall k = 1, 2, ..., p_1.$
(18)

Since we used only the condition (6) for Λ_N, it follows that

$S_k(a, \gamma + t) = O(\rho^{-k\alpha_1})$
(19)

for all $a \in \mathbb{R}$ satisfying $\mid a- \mid \gamma + t \mid^2 \mid <$ $\overline{12\rho^{\alpha_1}.}$ Thus finding N such that Λ_N is close to $\mid \gamma + t \mid^2$ and $b(N, \gamma)$ is not very small, then dividing both sides of (20) by $b(N, \gamma)$, we get the asymptotic formulas for Λ_N.

Theorem 1 *(a) Suppose $\gamma + t \in U(\rho^{\alpha_1}, p)$, $\mid \gamma \mid \sim \rho$. If (6) and (7) hold, then Λ_N satisfies formulas (5) for $k = 1, 2, ..., [\frac{1}{3}(p - c)]$, where*

$F_s = O(\rho^{-\alpha_1}), \forall s = 0, 1, ...,$
(20)

and $F_0 = 0$, $F_s = A_s(\mid \gamma + t \mid^2 + F_{s-1}, \gamma + t)$ for $s = 1, 2, ...$
(b) For $\gamma + t \in U(\rho^{\alpha_1}, p)$, $\mid \gamma \mid \sim \rho$ there exists an eigenvalue Λ_N of $L_t(q(x))$ satisfying (5).

Proof.
(a) To prove (5) in case $k = 1$ we divide both side of (20) by $b(N, \gamma)$ and use (7), (21). Then we obtain

$\Lambda_N- \mid \gamma + t \mid^2 = \mid \gamma + t \mid^2 + O(\rho^{-\alpha_1}).$
(21)

This and $\alpha_1 = 3\alpha$ (see the end of the introduction) imply that formula (5) for $k = 1$ holds and $F_0 = 0$. Hence (23) for $s = 0$ is also proved. Moreover, from (22), we obtain $S_k(\mid \gamma + t \mid^2$
+O$(\rho^{-\alpha_1}), \gamma + t) = O(\rho^{-\alpha_1})$ for $k = 1, 2, ...$
Therefore (23) for arbitrary s follows from the definition of F_s by induction. Now we prove (5) by induction on k. Suppose (5) holds for $k = j$, that is,
$\Lambda_N =\mid \gamma + t \mid^2 + F_{k-1}(\gamma + t) + O(\rho^{-3k\alpha})$.
Substituting this into $A_{p_1}(\Lambda_N, \gamma + t)$ in (20) and dividing both sides of (20) by $b(N, \gamma)$, we get

$\Lambda_N =\mid \gamma + t \mid^2 + A_{p_1}(\mid \gamma + t \mid^2$
+F$_{j-1} + O(\rho^{-j\alpha_1}), \gamma + t) + O(\rho^{-(p-c)\alpha}) =$

$\mid \gamma + t \mid^2 + \{A_{p_1}(\mid \gamma + t \mid^2 + F_{j-1}$
+O$(\rho^{-j\alpha_1}), \gamma + t) -$

A$_{p_1}(\mid \gamma + t \mid^2 + F_{j-1}, \gamma + t)\} + A_{p_1}(\mid$
$\gamma + t \mid^2 + F_{j-1}, \gamma + t) + O(\rho^{-(p-c)\alpha})$.

To prove (a) for $k = j + 1$ we need to show that the expression in curly brackets is equal to $O(\rho^{-(j+1)\alpha_1})$. It can be checked by using (4), (19), (23) and the obvious relation

$1\overline{\prod_{j=1}^{s}(|\gamma+t|^2+F_{j-1}+O(\rho^{-j\alpha_1}}$
)-$\mid \gamma + t - \sum_{i=1}^{s} \gamma_i \mid^2) -$

$1\overline{\prod_{j=1}^{s}(|\gamma+t|^2+F_{j-1}-|\gamma}$
+t-$\sum_{i=1}^{s} \gamma_i \mid^2)$

=$1\overline{\prod_{j=1}^{s}(|\gamma+t|^2+F_{j-1}-|\gamma}$
+t-$\sum_{i=1}^{s} \gamma_i \mid^2)(\frac{1}{1-O(\rho^{-(j+1)\alpha_1}}$
)-1)

$= O(\rho^{-(j+1)\alpha_1})$ for $s = 1, 2, ..., p_1$.
(b) Let A be the set of indices N satisfying (6). Using (15) and Bessel inequality, we obtain

$\sum_{N\notin A} \mid b(N,\gamma) \mid^2 = \sum_{N\notin A} \mid$
$_N(x), q(x)e^{i(\gamma+t,x)})\Lambda_N - \mid \gamma+t \mid^2 \mid$
$^2 = O(\rho^{-2\alpha_1})$

Hence, by the Parseval equality, we have $\sum_{N\in A} \mid b(N,\gamma$
$)\mid^2 = 1 - O(\rho^{-2\alpha_1})$. This and the inequality $\mid A \mid$
¡c$_5\rho^{d-1} = c_5\rho^{(d-1)q\alpha}$ (see the end of the introduction)
imply that there exists a number N satisfying $\mid b(N,\gamma) \mid >$
$\frac{}{12(c_5)^{-1}\rho^{-\frac{(d-1)q}{2}\alpha}}$, that is, (7) holds for
$c = \frac{(d-1)q}{2}$. Thus Λ_N satisfies (5) due to (a)
■

Theorem 1 shows that in the non-resonance case the eigenvalue of the perturbed
operator $L_t(q(x))$ is close to the eigenvalue of the unperturbed operator
$L_t(0)$. However, in theorem 2 we prove that if $\gamma + t \in \cap_{i=1}$
$^kV_{\gamma_i}(\rho^{\alpha_k})\backslash E_{k+1}$ for $k \geq 1$, where
$\gamma_1, \gamma_2, ..., \gamma_k$ are linearly independent vectors of
$\Gamma(p\rho^{\alpha})$, then the corresponding eigenvalue of $L_t(q(x))$
is close to the eigenvalue of the matrix constructed as follows. Introduce the sets:
$B_k \equiv B_k(\gamma_1, \gamma_2, ..., \gamma_k) = \{b : b = \sum_{i=1}$
$^kn_i\gamma_i, n_i \in Z, \mid b \mid < \frac{1}{2}\rho^{\frac{1}{2}}$
$\alpha_{k+1}\}$,

B$_k(\gamma + t) = \gamma + t + B_k = \{\gamma + t + b : b \in B_k\}$,
(22)

$B_k(\gamma + t, p_1) = \{\gamma + t + b + a : b \in B_k, \mid a \mid < p_1\rho^{\alpha}$
,a$\in \Gamma\}$.
Denote by $h_i + t$ for $i = 1, 2, ..., b_k$ the vectors of $B_k(\gamma$
+t,p$_1$), where
$b_k \equiv b_k(\gamma_1, \gamma_2, ..., \gamma_k)$ is the number of the
vectors of $B_k(\gamma + t, p_1)$. Define the matrix $C(\gamma + t, \gamma$
$_1, \gamma_2, ..., \gamma_k) \equiv (c_{i,j})$ by the formulas

c$_{i,i} = \mid h_i + t \mid^2,\ c_{i,j} = q_{h_i - h_j},\ \forall$
i$\neq j$,
(23)

where $i,j=1,2,...,b_k$. We consider the resonance eigenvalue $\mid \gamma+t\mid^2$ for $\gamma+t\in(\cap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))$ by using the following lemma.

Lemma 1 *If* $\gamma+t\in\cap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k})\backslash E_{k+1}$, $h+t\in B_k(\gamma+t,p_1)$, $(h-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$, *then*

$$\mid\mid \gamma+t\mid^2-\mid h-\gamma^{'}-\gamma_1^{'}-\gamma_2^{'}-...-\gamma_s^{'}+t\mid^2\mid>\frac{1}{5}\rho^{\alpha_{k+1}}, \tag{24}$$

where $\gamma^{'}\in\Gamma(\rho^{\alpha})$, $\gamma_j^{'}\in\Gamma(\rho^{\alpha})$, $j=1,2,...,s$ *and* $s=0,1,...,p_1-1$.

Proof.
The inequality $p>2p_1$ (see the end of the introduction) and the conditions of the lemma 1 imply that $h-\gamma^{'}-\gamma_1^{'}-\gamma_2^{'}-...-\gamma_s^{'}+t\in B_k(\gamma+t,p)\backslash B_k(\gamma+t)$ for all $s=0,1,...,p_1-1$. It follows from the definitions of $B_k(\gamma+t,p)$, B_k that (see (25)) $h-\gamma^{'}-\gamma_1^{'}-\gamma_2^{'}-...-\gamma_s^{'}+t=\gamma+t+b+a$, where

$$\mid b\mid<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}},\mid a\mid<p\rho^{\alpha},\ \gamma+t+b+a\notin\gamma+t+B_k. \tag{25}$$

Then (27) has the form

$$\mid\mid\gamma+t+a+b\mid^2-\mid\gamma+t\mid^2\mid>\frac{1}{5}\rho^{\alpha_{k+1}}. \tag{26}$$

To prove (29) we consider two cases:
Case 1. $a\in P$, where $P=Span\{\gamma_1,\gamma_2,...,\gamma_k\}$.
Since $b\in B_k\subset P$, we have $a+b\in P$. This with the third relation in (28) imply that $a+b\in P\backslash B_k$,i.e., $\mid a+b\mid\geq$
$\frac{}{12}\ \rho^{\frac{1}{2}\alpha_{k+1}}$. Consider the orthogonal decomposition $\gamma+t=y+v$ of $\gamma+t$, where $v\in P$ and $y\perp P$.
First we prove that the projection v of any vector $x\in\cap_{i=1}$
${}^kV_{\gamma_i}(\rho^{\alpha_k})$ on P satisfies

$$\mid v\mid=O(\rho^{(k-1)\alpha+\alpha_k}). \tag{27}$$

For this we turn the coordinate axis so that $Span\{\gamma_1,\gamma_2,...,\gamma_k\}$ coincides with the span of the vectors e_1 =(1,0,0,...,0),e$_2=(0,1,0,...,0)$, ..., e_k. Then γ_s $=\sum_{i=1}^k\gamma_{s,i}e_i$ for $s=1,2,...,k$. Therefore the relation $x\in\cap_{i=1}^kV_{\gamma_i}(\rho^{\alpha_k})$ implies that

$$\sum_{i=1}^k\gamma_{s,i}x_i=O(\rho^{\alpha_k}),s=1,2,...,k;$$
$$\mathrm{x}_n=\frac{\det(b_{j,i}^n)}{\det(\gamma_{j,i})},\ n=1,2,...,k,$$

where $x=(x_1,x_2,...,x_d),\gamma_j=(\gamma_{j,1},\gamma_{j,2},...,\gamma_{j,k},0,0,...,0)$, $b_{j,i}^n=\gamma_{j,i}$ for $n\neq j$ and $b_{j,i}^n=O(\rho^{\alpha_k})$ for $n=j$. Taking into account that the determinant $\det(\gamma_{j,i})$ is a volume of the parallelepiped $\{\sum_{i=1}^kb_i\gamma_i:b_i\in[0,1],i=1,2,...,k\}$ and using $\mid\gamma_{j,i}\mid<p\rho^{\alpha}$ (since $\gamma_j\in\Gamma$ (pρ^{α})), we get the estimations

$$\mathrm{x}_n=O(\rho^{\alpha_k+(k-1)\alpha})\ ,\forall n=1,2,...,k;$$
$$\forall x\in\cap_{i=1}^kV_{\gamma_i}(\rho^{\alpha_k}). \tag{28}$$

Hence (30) holds. Therefore, using the inequalities $\mid a+b\mid\geq$

$\overline{12}\ \rho^{\frac{1}{2}\alpha_{k+1}}$ (see above), α_{k+1}
¿$2(\alpha_k + (k-1)\alpha)$ (see the seventh inequality in (14)), and the obvious equalities $(y,v) = (y,a) = (y,b) = 0$,

$$\mid \gamma + t + a + b \mid^2 - \mid \gamma + t \mid^2 = \mid a + b + v \mid^2 - \mid v \mid^2, \tag{29}$$

we obtain the estimation (29).

Case 2. $a \notin P$. First we show that

$$\mid\mid \gamma + t + a \mid^2 - \mid \gamma + t \mid^2 \mid \geq \rho^{\alpha_{k+1}}. \tag{30}$$

Suppose, to the contrary, that it does not hold. Then $\gamma + t \in V_a(\rho^{\alpha_{k+1}})$. On the other hand $\gamma + t \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_{k+1}})$ (see the conditions of Lemma 1). Therefore we have $\gamma + t \in E_{k+1}$ which contradicts the conditions of the lemma. So (33) is proved. Now, to prove (29) we write the difference $\mid \gamma + t + a + b \mid^2 - \mid \gamma + t \mid^2$ as the sum of $d_1 \equiv \mid \gamma + t + a + b \mid^2 - \mid \gamma + t + b \mid^2$ and $d_2 \equiv \mid \gamma + t + b \mid^2 - \mid \gamma + t \mid^2$. Since $d_1 = \mid \gamma + t + a \mid^2 - \mid \gamma + t \mid^2 + 2(a,b)$, it follows from the inequalities (33), (28) that $\mid d_1 \mid > \frac{2}{3}\ \rho^{\alpha_{k+1}}$. On the other hand, taking $a = 0$ in (32) we have $d_2 = \mid b + v \mid^2 - \mid v \mid^2$. Therefore (30), the first inequality in (28) and the seventh inequality in (14) imply that $\mid d_2 \mid < \frac{1}{3}\ \rho^{\alpha_{k+1}}$, $\mid d_1 \mid - \mid d_2 \mid > \frac{1}{3}\rho^{\alpha_{k+1}}$, that is, (29) holds

■

Theorem 2 *(a) Suppose* $\mid \gamma \mid \sim \rho$, $\gamma + t \in (\cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k}))\backslash E_{k+1}$, *where* $k = 1, 2, ..., d-1$. *If (6) and (7) hold, then there is an index j such that*

$$\Lambda_N = \lambda_j(\gamma + t) + O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}), \tag{31}$$

where $\lambda_1(\gamma+t)\leq\lambda_2(\gamma+t)\leq ...\leq\lambda$
$_{b_k}(\gamma+t)$ *are the eigenvalues of the matrix* $C(\gamma+t,\gamma$
$_1,\gamma_2,...,\gamma_k)$ *defined in (26).*
(b) *Every eigenvalue* $\Lambda_N(t)\sim\rho^2$ *of the operator*
$L_t(q(x))$ *satisfies either (5) or (34) for* $c=\frac{q(d-1)}{2}$.

Proof.
(a)Writing the equation (16) for all $h_i+t\in B_k(\gamma+t,p_1)$, we obtain

$(\Lambda_N-\mid h_i+t\mid^2)b(N,h_i)=\sum_{\gamma'\in}$
$\Gamma(\rho^\alpha)q_{\gamma'}b(N,h_i-\gamma^{'}$
$)+O(\rho^{-p\alpha})$
(32)

for $i=1,2,...,b_k$ (see (25) for definition of $B_k(\gamma+t,p_1)$).
It follows from (6) and lemma 1 that if $(h_i-\gamma^{'}+t)\notin$
$B_k(\gamma+t,p_1)$, then

$\mid\Lambda_N-\mid h_i-\gamma^{'}-\gamma_1-\gamma_2$
-...-$\gamma_s+t\mid^2\mid>\frac{1}{6}\rho^{\alpha_{k+1}}$,

where $\gamma^{'}\in\Gamma(\rho^\alpha),\gamma_j\in\Gamma$
(ρ^α), $j=1,2,...,s$ and $s=0,1,...,p_1-1$. Therefore, applying the formula (17) p_1 times, using (4) and $p_1\alpha_{k+1}>p_1$
$\alpha_1\geq p\alpha$ (see the fourth inequality in (14)), we see that if
$(h_i-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$, then

b(N,h$_i-\gamma^{'}$) =

$\sum_{\gamma_1,...,\gamma_{p_1-1}\in\Gamma(\rho^\alpha)}$
$q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1}}b(N,h_i-\gamma^{'}$
-$\sum_{i=1}^{p_1}\gamma_i)\prod_{j=0}^{p_1-1}(\Lambda_N-\mid$

$h_i-\gamma^{'}+t-\sum_{i=1}^{j}\gamma_i\mid^2)+$

(33)

$+O(\rho^{-p\alpha})=O(\rho^{p_1\alpha_{k+1}})+O(\rho^{-p\alpha})=O(\rho^{-p\alpha})$.

Hence (35) has the form

$$(\Lambda_N-\mid h_i+t\mid^2)b(N,h_i)=\sum_{\gamma'}q_{\gamma'}b(N,h_i-\gamma^{'})+O(\rho^{-p\alpha}),i=1,2,...,b_k,$$

where the sum is taken under the conditions $\gamma^{'}\in\Gamma(\rho^{\alpha})$ and $h_i-\gamma^{'}+t\in B_k(\gamma+t,p_1)$. It can be written in matrix form

$$(C-\Lambda_N I)(b(N,h_1),b(N,h_2),...b(N,h_{b_k}))=O(\rho^{-p\alpha}),$$

where the rigth-hand side of this system is a vector having the norm $\parallel O(\rho^{-p\alpha})\parallel=O(\sqrt{b_k}\rho^{-p\alpha})$. Now, taking into account that $\gamma+t\in\{h_i+t:i=1,2,...,b_k\}$ and (7) holds, we have

$$c_4\rho^{-c\alpha}<(\sum_{i=1}^{b_k}\mid b(N,h_i)\mid^2)^{\frac{1}{2}}\leq\parallel(C-\Lambda_N I)^{-1}\parallel\sqrt{b_k}c_6\rho^{-p\alpha},$$

$$\max_{i=1,2,...,b_k}\mid\Lambda_N-\lambda_i\mid^{-1}=\parallel(C-\Lambda_N I)^{-1}\parallel>c_4c_6^{-1}b_k^{-\frac{1}{2}}\rho^{-c\alpha+p\alpha}.$$

Since b_k is the number of the vectors of $B_k(\gamma+t,p_1)$, it follows from the definition of $B_k(\gamma+t,p_1)$ (see (25)) and the obvious relations$\mid B_k\mid= O(\rho^{\frac{k}{2}\alpha_{k+1}})$, $\mid\Gamma(p_1\rho^{\alpha})\mid= O(\rho^{d\alpha})$ and $d\alpha<$ $\frac{}{12\alpha_d}$ (see the end of introduction), we get

$$b_k = O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}}) = O(\rho^{\frac{d}{2}\alpha_d})=O(\rho^{\frac{d}{2}3^d\alpha}), \forall k = 1,2,...,d-1 \quad (34)$$

Thus formula (34) follows from (38) and (39).

(b) Let $\Lambda_N\ (t)$ be any eigenvalue of order ρ^2 of the operator $L_t(q(x))$. Denote by D the set of all vectors $\gamma\in\Gamma$ satisfying (6). From (15), arguing as in the proof of Theorem 1(b), we obtain $\sum_{\gamma\in D}\mid b(N,\gamma)\mid^2= 1-O(\rho^{-2\alpha_1})$. Since $\mid D\mid= O(\rho^{d-1})$ (see the end of the introduction), there exists $\gamma\in D$ such that $\mid b(N,\gamma)\mid> c_7\rho^{-\frac{(d-1)}{2}} = c_7\rho^{-}$ $\frac{}{(d-1)q2\alpha}$, that is, condition (7) for $c = \frac{(d-1)q}{2}$ holds.
Now the proof of (b) follows from Theorem 1 (a) and Theorem 2(a), since either $\gamma+t\in U(\rho^{\alpha_1},p)$ or $\gamma+t\in E_k$ $\backslash E_{k+1}$ for $k = 1,2,...,d-1$ (see (42) in Remark 1)
■

Remark 1 *Here we note that the non-resonance domain $U(c_8\rho^{\alpha_1},p)$ has an asymptotically full measure on $\mathbb{R}^d$ in the sence that $\frac{\mu(U\cap B(\rho))}{\mu(B(\rho))}$ tends to 1 as ρ tends to infinity, where $B(\rho)=\{x\in\mathbb{R}^d :\mid x\mid=\rho\}$. Clearly, $B(\rho)\cap V_b(c_8\rho^{\alpha_1})$ is the part of sphere $B(\rho)$, which is contained between two parallel hyperplanes $\{x :\mid x\mid^2-\mid x+b\mid^2= -c_8\rho^{\alpha_1}\}$ and $\{x :\mid x\mid^2-\mid x+b\mid^2= c_8\rho^{\alpha_1}\}$. The distance of these hyperplanes from origin is $O(\frac{\rho^{\alpha_1}}{|b|})$. Therefore, the relations $\mid\Gamma(p\rho^{\alpha})\mid= O(\rho^{d\alpha})$, and $\alpha_1+d\alpha<1-\alpha$ (see the first inequality in (14)) imply*

$$\mu(B(\rho)\cap V_b(c_8\rho^{\alpha_1})) = O(\rho^{\alpha}$$

$_1 + d - 2\overline{|b|),\ \mu(E_1\cap B(\rho))=O(\rho^{d-1-\alpha})}$
$\mu(U(c_8\rho^{\alpha_1},p)\cap B(\rho)) = (1+O(\rho^{-\alpha}$
$))\mu(B(\rho))$.

If $x \in \cap_{i=1}^{d} V_{\gamma_i}(\rho^{\alpha_d})$, then (31) holds for
$k = d$ and $n = 1, 2, ..., d$. Hence we have $|\ x\ | = O(\rho^{\alpha}$
$_d + (d-1)\alpha)$. It is imposible, because of $\alpha_d + (d-1)\alpha < 1$ (
see the sixth inequality in (14)) and $x \in B(\rho)$. It means that
$(\cap_{i=1}^{d} V_{\gamma_i}(\rho^{\alpha_k})) \cap B(\rho_0) = \emptyset$
for $\rho_0 \gg 1$. Thus, for $\rho_0 \gg 1$, we have

$\mathrm{R}^d \cap \{|\ x\ | > \rho_0\} = (U \cup (\cup_{s=1}^{d-1}$
$(\mathrm{E}_s \backslash E_{s+1}))) \cap \{|\ x\ | > \rho_0\}$.
(35)

Remark 2 *Here we note some properties of the known parts*
$|\ \gamma + t\ |^2 + F_k(\gamma + t)$ *(see Theorem 1)and* λ_j
$(\gamma + t)$ *(see Theorem 2) of the non-resonance and resonance eigenvalues of*
$L_t(q(x))$. *Denoting* $\gamma + t$ *by* x *, where* $|\ \gamma + t\ | \sim \rho$,
$\gamma + t \in U(\rho^{\alpha_1}, p)$, *we prove*

$\partial F_k(x)\overline{\partial x_i = O(\rho^{-2\alpha_1+\alpha}),\forall}$
i=1,2,...,d;$\forall k = 1, 2, ...$
(36)

by induction on k. *For* $k = 1$ *the formula (43) follows from (4) and*

$\partial\overline{\partial x_i(}$
$|^2) =$
$^2)^2 = O(\rho^{-2\alpha_1+\alpha})$,
(37)

where $\gamma_1(i)$ *is the i-th component of the vector* γ_1
$\in \Gamma(p\rho^{\alpha})$ *hence is equal to* $O(\rho^{\alpha})$. *Now suppose*

that (43) holds for $k = s$. *Using this and (23), replacing* $\mid x \mid^2$ *by* $\mid x \mid^2 + F_s(x)$ *in (44) and evaluting as above we obtain*

$\partial \overline{\partial x_i(}$
x-$\gamma_1 \mid^2) =$
$(x)\partial x_i(\mid x \mid^2 + F_s - \mid x - \gamma_1 \mid^2)^2$
$= O(\rho^{-2\alpha_1+\alpha})$.

This formula together with the definition of F_k *give (43) for* $k = s+1$.
Now denoting $\lambda_i(\gamma+t) - \mid \gamma + t \mid^2$ *by* $r_i(\gamma+t)$
we prove that

$\mid r_i(x) - r_i(x^{'}) \mid \leq 2\rho^{\frac{1}{2}\alpha_d} \mid$
x-$x^{'} \mid, \forall i.$
(38)

Clearly $r_1(x) \leq r_2(x) \leq ... \leq r_{b_k}(x)$ *are the eigenvalue of the matrix*
$C(x) - \mid x \mid^2 I \equiv C^{'}(x)$, *where* $C(x)$ *is defined in (26). By definition, only the diagonal elements of the matrix* $C^{'}$
(x)dependonxandtheyare
$\mid x \mid^2 - \mid x - a_i \mid^2 = 2(x, a_i) - \mid a_i \mid^2$, *where*
$a_i = h_i + t - \gamma - t$ *and* $h_i + t \in B_k(\gamma + t, p_1)$. *It follows from the definitions of* $B_k(\gamma + t, p_1)$ *(for* $k < d$ *see (25)), and* α_d *(see introduction) that* $\mid a_i \mid < \frac{1}{2}\rho$
$\frac{1}{2}\alpha_k + p_1\rho^\alpha < \rho^{\frac{1}{2}\alpha_d}$.
Using this and taking into account that $C^{'}(x) - C^{'}$
$(x^{'}) = (a_{i,j})$, *where* $a_{i,i} = 2(x - x^{'}, a_i)$,
$a_{i,j} = 0$ *for* $i \neq j$, *we obtain* $\parallel C^{'}(x) - C^{'}$
$(x^{'}) \parallel \leq 2\rho^{\frac{1}{2}\alpha_d} \mid x - x^{'}$
$\mid$ *from which follows (45).*

3 Asymptotic Formulas for Bloch Functions

In this section using the asymptotic formulas for eigenvalues and the simplicity conditions (11), (12), we prove the asymptotic formulas for the Bloch functions with a quasimomenta of the simple set B.

Theorem 3 *If $\gamma+t\in B$ and $\mid\gamma+t\mid\sim\rho$, then there exists a unique eigenvalue $\Lambda_N(t)$ satisfying (5) for $k=1,2,...,[\frac{p}{3}]$, where p is defined in (3). This is a simple eigenvalue and the corresponding eigenfunction $\Psi_{N,t}(x)$ of $L(q(x))$ satisfies (9) if $q(x)\in W_2^{s_0}(F)$, where $s_0=\frac{3d-1}{2}(3^d+d+2)+\frac{1}{4}d3^d+d+6$.*

Proof.
By Theorem 1(b) if $\gamma+t\in B\subset U(\rho^{\alpha_1},p)$, then there exists an eigenvalue $\Lambda_N(t)$ satisfying (5) for $k=1,2,...,[\frac{1}{3}(p-\frac{1}{2}q(d-1))]$. Since $k_1=[\frac{d}{3\alpha}]+2\le\frac{1}{3}(p-\frac{1}{2}q(d-1))$ (see the third inequality in (14)) formula (5) holds for $k=k_1$. Therefore using (5), the relation $3k_1\alpha>d+2\alpha$ (see the fifth inequality in (14)), and notations $F(\gamma+t)=\mid\gamma+t\mid^2+F_{k_1-1}(\gamma+t)$, $\varepsilon_1=\rho^{-d-2\alpha}$ (see Step 1 in introduction), we obtain

$$\Lambda_N(t)=F(\gamma+t)+o(\varepsilon_1). \tag{39}$$

Let Ψ_N be any normalized eigenfunction corresponding to Λ_N. Since the normalized eigenfunction is defined up to constant of modulas 1, without loss of generality it can assumed that $\arg b(N,\gamma)=0$, where $b(N,\gamma)=(\Psi_N,e^{i(\gamma+t,x)})$. Therefore to prove (9) it suffices to show that (13) holds. To prove (13) first we estimate $\sum_{\gamma'\notin K}\mid b(N,\gamma')\mid^2$ and then $\sum_{\gamma'\in K\backslash\{\gamma\}}\mid b(N,\gamma')\mid^2$, where K is defined in (11), (12). Using (46), the definition of K, and (15), we get

$$\mid\Lambda_N-\mid\gamma'+t\mid^2\mid>\frac{1}{4}\rho^{\alpha_1},\ \forall\gamma'\notin K,$$

$$\sum_{\gamma'\notin K}\mid b(N,\gamma')\mid^2=\parallel q(x)\Psi_N\parallel^2 O(\rho^{-2\alpha_1})=O(\rho^{-2\alpha_1}).$$

If $\gamma'\in K$, then by (46) and by definition of K, it follows that

$$\mid \Lambda_N - \mid \gamma^{'} + t \mid^2 \mid < \frac{1}{2}\rho^{\alpha_1}$$

(40)

Now we prove that the simplicity conditions (11), (12) imply

$$\mid b(N, \gamma^{'}) \mid \leq c_4 \rho^{-c\alpha}, \forall \gamma^{'} \in K\backslash\{\gamma\}, \tag{41}$$

where $c = p - dq - \frac{1}{4}d3^d - 3$. If for $\gamma^{'} + t \in U(\rho^{\alpha_1}, p)$ and $\gamma^{'} \in K\backslash\{\gamma\}$ the inequality in (49) is not true, then by (48) and Theorem 1(a), we have

$$\Lambda_N = \mid \gamma^{'} + t \mid^2 + F_{k-1}(\gamma^{'} + t) + O(\rho^{-3k\alpha}) \tag{42}$$

for $k = 1, 2, ..., [\frac{1}{3}(p - c)] = [\frac{1}{3}(dq + \frac{1}{4}d3^d + 3)]$. Since $\alpha = \frac{1}{q}$ and $k_1 \equiv [\frac{d}{3\alpha}] + 2 < \frac{1}{3}(dq + \frac{1}{4}d3^d + 3)$, the formula (50) holds for $k = k_1$. Therefore arguing as in the prove of (46), we get $\Lambda_N - F(\gamma^{'} + t) = o(\varepsilon_1)$. This with (46) contradicts (11). Similarly, if the inequality in (49) does not hold for $\gamma^{'} + t \in (E_k\backslash E_{k+1})$ and $\gamma^{'} \in K$, then by Theorem 2(a)

$$\Lambda_N = \lambda_j(\gamma^{'} + t) + O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}), \tag{43}$$

where $(p - c - \frac{1}{4}d3^d)\alpha = (dq + 3)\alpha > d + 2\alpha$. Hence we have

$\Lambda_N - \lambda_j(\gamma^{'} + t) = o(\varepsilon_1)$. This with (46) contradicts (12). So the inequality in (49) holds. Therefore, using $|K| = O(\rho^{d-1})$, $q\alpha = 1$, we get

$$\sum_{\gamma' \in K \backslash \{\gamma\}} | b(N, \gamma^{'})|^2 = O(\rho^{-(2c-q(d-1))\alpha}) = O(\rho^{-(2p-(3d-1)q-\frac{1}{2}d3^d-6)\alpha}). \quad (44)$$

If $s = s_0$, that is, $p = s_0 - d$ then $2p - (3d-1)q - \frac{1}{2}d3^d - 6 = 6$. Since $\alpha_1 = 3\alpha$, the equality (52) and the equality in (47) imply (13). Thus we proved that the equality (9) holds for any normalized eigenfunction Ψ_N corresponding to any eigenvalue Λ_N satisfying (5). If there exist two different eigenvalues or multiple eigenvalue satisfying (5), then there exist two orthogonal normalized eigenfunction satisfying (9), which is imposible. Therefore Λ_N is a simple eigenvalue. It follows from Theorem 1(a) that Λ_N satisfies (5) for $k = 1, 2, ..., [\frac{p}{3}]$, because the inequality (7) holds for $c = 0$ (see (9)).
■

Remark 3 *Since for $\gamma + t \in B$ there exists a unique eigenvalue satisfying (5), (46) we denote this eigenvalue by $\Lambda(\gamma + t)$. Since this eigenvalue is simple, we denote the corresponding eigenfunction by $\Psi_{\gamma+t}(x)$. By Theorem 3 this eigenfunction satisfies (9). Clearly, for $\gamma + t \in B$ there exists a unique index $N \equiv N(\gamma + t)$ such that $\Lambda(\gamma + t) = \Lambda_{N(\gamma+t)}$) and $\Psi_{\gamma+t}(x) = \Psi_{N(\gamma+t)}(x)$).*

Now we prove the asymptotic formulas of arbitrary order for $\Psi_{\gamma+t}(x)$.

Theorem 4 *If $\gamma + t \in B$ and $| \gamma + t | \sim \rho$, then the eigenfunction $\Psi_{\gamma+t}(x) \equiv \Psi_{N(\gamma+t)}(x)$ corresponding to the eigenvalue $\Lambda_N \equiv \Lambda(\gamma + t)$ satisfies formulas (10), for $k = 1, 2, ..., n$, where $n = [\frac{1}{6}(2p - (3d-1)q - \frac{1}{2}d3^d - 6)]$, $\Phi_0(x) = 0$, $\Phi_1(x) = \sum_{\gamma_1 \in \Gamma(\rho^{\alpha})}$ -$| \gamma + \gamma_1 + t |^2)$, and $\Phi_{k-1}(x)$ for $k > 2$ is a linear combination of $e^{i(\gamma+t+\gamma^{'}, x)}$ for $\gamma^{'} \in \Gamma((k-1)\rho^{\alpha}) \cup \{0\}$ with coefficients (58), (59).*

Proof.
By Theorem 3, formula (10) for $k=1$ is proved. To prove formula (10) for arbitrary $k\leq n$ we prove the following equivalent relations

$\sum_{\gamma^{\prime}\in\Gamma^c(k-1)}\mid b(N,\gamma+\gamma^{'}$
$)|^2= O(\rho^{-2k\alpha_1}),$
(45)

$\Psi_N=b(N,\gamma)e^{i(\gamma+t,x)}+\sum_{\gamma^{\prime}\in}$
$\Gamma((k-1)\rho^{\alpha})b(N,\gamma+\gamma^{'})e^{i(\gamma}$
$+\mathrm{t}+\gamma^{'},x)+H_k(x),$
(46)

where $\Gamma^c(m)\equiv\Gamma\backslash(\Gamma(m\rho^{\alpha})\cup\{0\})$
and $\parallel H_k(x)\parallel= O(\rho^{-k\alpha_1})$. The case $k=1$ is
proved due to (13). Assume that (53) is true for $k=m$. Then using (54)
for
$k=m$, and (3), we have $\Psi_N(x)(q(x))=H(x)+O(\rho^{-m\alpha_1})$,
where $H(x)$ is a linear combination of $e^{i(\gamma+t+\gamma^{'},x)}$
for $\gamma^{'}\in\Gamma(m\rho^{\alpha})\cup\{0\}$. Hence
$(H(x),e^{i(\gamma+t+\gamma^{'},x)})=0$ for $\gamma^{'}$
$\in\Gamma^c(m)$. So using (15) and the inequality in (47), we get

$\sum_{\gamma^{\prime}}\mid b(N,\gamma+\gamma^{'})\mid^2$
$=\sum_{\gamma^{\prime}}\mid$
$+\mathrm{t}+\gamma^{'},x))\Lambda_N-\mid\gamma+\gamma^{'}$
$+\mathrm{t}|^2|^2= O(\rho^{-2(m+1)\alpha_1}),$
(47)

where the sum is taken under conditions $\gamma^{'}\in\Gamma^c(m)$,
$\gamma+\gamma^{'}\notin K$. On the other hand, using α
$_1=3\alpha$, (52), and the definition of n (see Theorem 4), we get

$\sum_{\gamma^{\prime}\in K\backslash\{\gamma\}} \mid b(N,\gamma^{'}$
$)|^2= O(\rho^{-2n\alpha_1})$.

This with (55) implies (53) for $k=m+1$. Thus (54) is also proved. Here $b(N,\gamma)$ and $b(N,\gamma+\gamma^{'})$ for $\gamma^{'}$ $\in\Gamma((n-1)\rho^{\alpha})$ can be calculated as follows. First we express $b(N,\gamma+\gamma^{'})$ by $b(N,\gamma)$. For this we apply (17) for $b(N,\gamma+\gamma^{'})$, where $\gamma^{'}\in\Gamma$ ((n-1)ρ^{α}), that is, in (17) replace $\gamma^{'}$ by $\gamma+\gamma^{'}$. Iterate it n times and every times isolate the terms with multiplicand $b(N,\gamma)$. In other word apply (17) for $b(N,\gamma+\gamma^{'})$ and isolate the terms with multiplicand $b(N,\gamma)$. Then apply (17) for $b(N,\gamma+\gamma^{'}-\gamma_1)$ when $\gamma^{'}-\gamma_1\neq 0$. Then apply (17) for $b(N,\gamma+\gamma^{'}-\sum_{i=1}^{2}\gamma_i)$ when γ $^{'}-\sum_{i=1}^{2}\gamma_i\neq 0$, etc. Apply (17) for $b(N,\gamma+\gamma^{'}-\sum_{i=1}^{j}\gamma_i)$ when γ $^{'}-\sum_{i=1}^{j}\gamma_i\neq 0$, where $\gamma_i\in\Gamma$ (ρ^{α}), $j=3,4,...,n-1$. Then using (4) and the relations $\mid\Lambda_N-\mid\gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i$ $|^2|>\frac{1}{2}\rho^{\alpha_1}$ (see (19) and take into account that $\gamma^{'}-\sum_{i=1}^{j}\gamma_i\in\Gamma(p\rho^{\alpha})$, since $p>2n$), $\Lambda_N=P(\gamma+t)+O(\rho^{-n\alpha_1})$, where $P(\gamma+t)=\mid\gamma+t\mid^2+F_{[\frac{p}{3}]}(\gamma+t)$ (see Theorem 3), we obtain

b(N,$\gamma+\gamma^{'}$) $=\sum_{k=1}^{n-1}A_k(\gamma^{'}$
)b(N,γ) $+O(\rho^{-n\alpha_1})$,
(48)

where
$A_1(\gamma^{'})=$
+t)-$\mid\gamma+\gamma^{'}+t\mid^2$=
$\mid\gamma+t\mid^2-\mid\gamma+\gamma^{'}+t\mid^2+O($
$\overline{1\rho^{3\alpha_1}),}$

A$_k(\gamma^{'})=$

$$\sum_{\gamma_1,\ldots,\gamma_{k-1}}$$

-$\gamma_1-\gamma_2-\ldots-\gamma_{k-1}\prod_{j=0}^{k-1}(P(\gamma$
+t)-$\mid\gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i\mid^2)$
=O$(\rho^{-k\alpha_1})$,

$\sum_{\gamma^*\in\Gamma((n-1)\rho^\alpha)}\mid A_1(\gamma^*$
$)|^2=O(\rho^{-2\alpha_1}),\sum_{\gamma^*\in\Gamma((n-1)\rho}$
$^\alpha)\mid A_k(\gamma^*)\mid=O(\rho^{-k\alpha_1})$
(49)

for $k>1$. Now from (54) for $k=n$ and (56), we obtain

$\Psi_N(x)=b(N,\gamma)e^{i(\gamma+t,x)}+$

$\sum_{\gamma^*\in\Gamma((n-1)\rho^\alpha)}\sum_{k=1}^{n-1}$
(A$_k(\gamma^*)b(N,\gamma)+O(\rho^{-n\alpha_1}))e^{i(\gamma}$
+t+$\gamma^*,x))+H_n(x)$.

Using the equalities $\parallel\Psi_N\parallel=1$, $\arg b(N,\gamma)=0$,
$\parallel H_n\parallel=O(\rho^{-n\alpha_1})$ and taking into account that
the functions $e^{i(\gamma+t,x)}$, $H_n(x)$, $e^{i(\gamma+t+\gamma^*}$
,x),$(\gamma^*\in\Gamma((n-1)\rho^\alpha))$ are orthogonal, we get
$1=\mid b(N,\gamma)\mid^2+\sum_{k=1}^{n-1}(\sum_{\gamma^*\in}$
$\Gamma((n-1)\rho^\alpha)\mid A_k(\gamma^*)b(N,\gamma)\mid^2$
+O$(\rho^{-n\alpha_1}))$,

b(N,γ) $=(1+\sum_{k=1}^{n-1}(\sum_{\gamma^*\in\Gamma((n-1)\rho}$
$^\alpha)\mid A_k(\gamma^*)\mid^2))^{-\frac{1}{2}}+O(\rho$
$^{-n\alpha_1}))$
(50)

(see the second equality in (57)). Thus from (56), we obtain

b(N,$\gamma+\gamma^{'}$) $= (\sum_{k=1}^{n-1} A_k(\gamma^{'}$
$))(1+\sum_{k=1}^{n-1}\sum_{\gamma^*} \mid A_k(\gamma^*) \mid$
$^2)^{-\frac{1}{2}} + O(\rho^{-n\alpha_1})$.
(51)

Consider the case $n=2$. By (58), (57), (59) we have $b(N,\gamma$
)=1+O($\rho^{-2\alpha_1}$),
$b(N,\gamma+\gamma^{'}) = A_1(\gamma^{'}) + O(\rho^{-2\alpha}$
$_1) =$
$+\gamma^{'} + t \mid^2 + O(\rho^{-2\alpha_1})$ for all γ
$^{'} \in \Gamma(\rho^{\alpha})$. These and (54) for $k=2$ imply the
formula for Φ_1
■

4 Simple Sets and Fermi Surfaces

In this section we consider the simple sets B and construct a big part of the isoenergetic surfaces corresponding to ρ^2 for big ρ. The isoenergetic surfaces of $L(q)$ corresponding to ρ^2 is the set $I_\rho(q(x)) = \{t \in F^* : \exists N, \Lambda_N(t) = \rho^2\}$. In the case $q(x)=0$ the isoenergetic surface $I_\rho(0) = \{t \in F^*$ $:\exists\gamma \in \Gamma, \mid \gamma+t \mid^2 = \rho^2\}$ is the translation of the sphere $B(\rho) = \{\gamma+t : t \in F^*, \gamma \in \Gamma, \mid \gamma+t \mid^2$ $=\rho^2\}$ by the vectors $\gamma \in \Gamma$. We call $B(\rho)$ the translated isoenergetic surfaces of $L(0)$ corresponding to ρ^2. Similarly, we call the sets $P_\rho^{'} = \{\gamma+t : \Lambda$ $(\gamma+t) = \rho^2\}$ and $P_\rho^{''} = \{t \in F^* : \exists\gamma \in \Gamma$ $,\Lambda(\gamma+t) = \rho^2\}$, where $\Lambda(\gamma+t) = \Lambda_{N(\gamma}$ +t)(t)$isdefinedinRemark3, thepartsoftranslatedisoenergeticsurfaces$ and isoenergetic surfaces of $L(q)$. In this section we construct the subsets $I_\rho^{'}$ and $I_\rho^{''}$ of P_ρ $^{'}$ and $P_\rho^{''}$ respectively and prove that the measures of these subsets are asymptotically equal to the measure of the isoenergetic surfaces $I_\rho(0)$ of $L(0)$. In other word we construct a big part (in some sense) of isoenergetic surfaces $I_\rho(q(x))$ of $L(q)$. As we see below the set $I_\rho^{''}$ is a translation of $I_\rho^{'}$ by vectors $\gamma \in \Gamma$ to F^* and the set $I_\rho^{'}$ lies in ε nieghborhood of the surface

$S_\rho = \{x \in U(2\rho^{\alpha_1}, p) : F(x) = \rho^2\}$, where $F(x)$ is defined in Step 1 of introduction. Due to (46) (replace $\gamma + t$ by x in $F(\gamma + t)$) it is natural to call S_ρ the approximated isoenergetic surfaces in the nonresonance domain. Here we construct a part of the simple set B in nieghborhood of S_ρ that containes $I_\rho^{'}$.
For this we consider the surface S_ρ. As we noted in introduction (see Step 2 and (11)) the non-resonance eigenvalue $\Lambda(\gamma + t)$ does not coincide with other non-resonance eigenvalue $\Lambda(\gamma + t + b)$ if $| F(\gamma + t) - F(\gamma + t + b) | > 2\varepsilon_1$ for $\gamma + t + b \in U(\rho^{\alpha_1}, p)$ and $b \in \Gamma\backslash\{0\}$. Therefore we eliminate

$$P_b = \{x : x, x + b \in U(\rho^{\alpha_1}, p), | F(x) - F(x + b) | < 3\varepsilon_1\} \quad (52)$$

for $b \in \Gamma\backslash\{0\}$ from S_ρ. Denote the remaining part of S_ρ by $S_\rho^{'}$. Then we consider the ε neighbourhood $U_\varepsilon(S_\rho^{'}) = \cup_{a \in S_\rho^{'}} U_\varepsilon(a)\}$ of $S_\rho^{'}$, where $\varepsilon = \frac{\varepsilon_1}{7\rho}$, $U_\varepsilon(a) = \{x \in R^d :| x - a |< \varepsilon\}$. In this set the first simplicity condition (11) holds (see Lemma 2(a)). Denote by $Tr(E) = \{\gamma + x \in U_\varepsilon(S_\rho^{'}) : \gamma \in \Gamma, x \in E\}$ and $Tr_{F^\star}(E) \equiv \{\gamma + x \in F^\star : \gamma \in \Gamma, x \in E\}$ the translations of $E \subset R^d$ into $U_\varepsilon(S_\rho^{'})$ and $F^\star$ respectively. In order that the second simplicity condition (12) holds, we discart from $U_\varepsilon(S_\rho^{'})$ the translation $Tr(A(\rho))$ of

$$A(\rho) \equiv \cup_{k=1}^{d-1}(\cup_{\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^\alpha)}(\cup_{i=1}^{b_k} A_{k,i}(\gamma_1, \gamma_2, ..., \gamma_k))), \quad (53)$$

where $A_{k,i}(\gamma_1, ..., \gamma_k) = \{x \in (\cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})\backslash E_{k+1}) \cap K_\rho : \lambda_i(x) \in (\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon$

$_1)\}$,
$\lambda_i(x)$, b_k is defined in Theorem 2 and

$$\mathrm{K}_\rho = \{x \in \mathbb{R}^d :\mid\mid x\mid^2 - \rho^2 \mid < \rho^\alpha$$
$_1\}$.
(54)

As a result we construct the part $U_\varepsilon(S_\rho^{'}$
$)\backslash Tr(A(\rho))$ of the simple set B (see Theorem 5(a)) which contains the set $I_\rho^{'}$ (see Theorem 5(c)). For this we need the following lemma.

Lemma 2 *(a) If $x \in U_\varepsilon(S_\rho^{'})$ and $x + b \in$ $U(\rho^{\alpha_1}, p)$, where $b \in \Gamma$, then*

$$\mid F(x) - F(x + b) \mid > 2\varepsilon_1,$$
(55)

where $\varepsilon = \frac{\varepsilon_1}{7\rho}, \varepsilon_1$
$=\rho^{-d-2\alpha}$, $F(x) = \mid x\mid^2 + F_{k_1-1}(x)$, $k_1 = [$
$\overline{d3\alpha]+2,}$ hence for $\gamma + t \in U_\varepsilon(S_\rho^{'}$
$)thesimplisitycondition(11)holds.$
(b) If $x \in U_\varepsilon(S_\rho^{'})$, then $x + b \notin$
$\mathrm{U}_\varepsilon(S_\rho^{'})$ for all $b \in \Gamma$.
(c)If E is a bounded subset of $\mathbb{R}^d$, then $\mu(Tr(E)) \leq$
$\mu(E)$.
(d) If $E \subset U_\varepsilon(S_\rho^{'})$, then μ
$(\mathrm{Tr}_{F^\star}(E)) = \mu(E)$.
Proof.
(a) If $x \in U_\varepsilon(S_\rho^{'})$, then there exists a point a in $S_\rho^{'}$ such that $x \in U_\varepsilon(a)$. Since $S_\rho^{'} \cap P_b = \emptyset$ (see (60) and the definition of $S_\rho^{'}$), we have

$$\mid F(a) - F(a + b) \mid \geq 3\varepsilon_1$$
(56)

On the other hand, using (43) and the obvious relations $\mid x \mid < \rho+1$, $\mid x-a \mid < \varepsilon$, $\mid x+b-a-b \mid$ ¡ε, we obtain

$$\mid F(x)-F(a) \mid < 3\rho\varepsilon, \mid F(x+b)-F(a+b) \mid < 3\rho\varepsilon \tag{57}$$

These inequalities together with (64) give (63), since $6\rho\varepsilon$ ¡ε_1.

(b) If x and $x+b$ lie in $U_\varepsilon(S_\rho^{'})$, then there exist points a and c in $S_\rho^{'}$ such that $x \in$ $U_\varepsilon(a)$ and $x+b \in U_\varepsilon(c)$. Repeating the proof of (65), we get $\mid F(c)-F(x+b) \mid < 3\rho\varepsilon$. This, the first inequality in (65), and the relations $F(a)=\rho^2$, $F(c)=\rho^2$ (see the definition of S_ρ) give $\mid F(x)-F(x+b) \mid < \varepsilon_1$, which contradicts (63).

(c) Clearly, for any bounded set E there exist only a finite number of vectors $\gamma_1,\gamma_2,...,\gamma_s$ such that $E(k) \equiv$ $(E+\gamma_k) \cap U_\varepsilon(S_\rho^{'}) \neq \emptyset$ for $k=1,2,...,s$ and $Tr(E)$ is the union of the sets $E(k)$. For $E(k)-\gamma_k$ we have the relations $\mu(E(k)-\gamma_k)=\mu(E(k))$, $E(k)-\gamma_k \subset E$. Moreover, by (b) $(E(k)-\gamma_k)\cap(E(j)-\gamma_j)=\emptyset$ for $k \neq j$. Therefore (c) is true.

(d) Now let $E \subset U_\varepsilon(S_\rho^{'})$. Then by (b) the set E can be devided into a finite number of the pairwise disjoint sets $E_1,E_2,...,E_n$ such that there exist the vectors γ_1 $,\gamma_2,...,\gamma_n$ satisfying $(E_k+\gamma_k) \subset F^\star$, $(E_k+\gamma_k)\cap(E_j+\gamma_j) \neq \emptyset$ for $k,j=1,2,...,n$ and $k \neq j$. Using $\mu(E_k+\gamma_k)=\mu(E_k)$, we get the proof of (d), because $Tr_{F^\star}(E)$ and E are union of the pairwise disjoint sets $E_k+\gamma_k$ and E_k for $k=1,2,...,n$ respectively

■

Theorem 5 (a) *The set $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ is a subset of B. For every connected open subset E of U_ε $(S_\rho^{'})\backslash Tr(A(\rho)$ there exists a unique index N such that $\Lambda_N(t)=\Lambda(\gamma+t)$ for $\gamma+t \in E$, where $\Lambda(\gamma+t)$ is defined in Remark 3. Moreover,*

$\partial\overline{\partial t_j \Lambda(\gamma+t)=\frac{\partial}{\partial}}$
$t_j \mid \gamma+t \mid^2 + O(\rho^{1-2\alpha_1}), \forall j=1,2,...,d.$
(58)

(b) For the part $V_\rho \equiv S_\rho^{'}\backslash$
$U_\varepsilon(Tr(A(\rho)))$ *of the approximated isoenergetic surface*
S_ρ *the following holds*

$\mu(V_\rho) > (1 - c_9\rho^{-\alpha}))\mu(B(\rho)).$
(59)

Moreover, $U_\varepsilon(V_\rho)$ *lies in the subset* U_ε
$(S_\rho^{'})\backslash Tr(A(\rho))$ *of the simple set* B.
(c) The isoenergetic surface $I(\rho)$ *contains the set* I_ρ
$''$*, which consists of the smooth surfaces and has the measure*

$\mu(I_\rho^{''}) = \mu(I_\rho^{'}) > (1 - c_{10}$
$\rho^{-\alpha})\mu(B(\rho)),$
(60)

where $I_\rho^{'}$ *is a part of the translated isoenergetic surfaces of* $L(q)$*, which is contained in the subset* $U_\varepsilon(S_\rho$
$')\backslash Tr(A(\rho))$ *of the simple set* B*. In particular the number* ρ^2 *for* $\rho \gg 1$ *lies in the spectrum of* $L(q)$*, that is, the number of the gaps in the spectrum of* $L(q)$ *is finite, where* $q(x) \in W_2^{s_0}(\mathbb{R}^d/\Omega)$, $d \geq 2$, $s_0 = \frac{3d-1}{2}$
$(3^d + d + 2) + \frac{1}{4}d3^d + d + 6$, *and* Ω *is an arbitrary lattice.*

Proof.
(a) To prove that $U_\varepsilon(S_\rho^{'})\backslash$
Tr(A(ρ)) $\subset B$ we need to show that for each point $\gamma+t$ of $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ the simplicity conditions (11), (12) hold and $U_\varepsilon(S_\rho^{'}$
$)\backslash Tr(A(\rho)) \subset U(\rho^{\alpha_1}, p)$. By lemma 2(a), the

condition (11) holds. Now we prove that (12) holds too. Since $\gamma + t \in$ $U_\varepsilon(S_\rho^{'})$, there exists $a \in S_\rho^{'}$ $such that \gamma + t \in U_\varepsilon(a)$. The inequality (65) and equality $F(a) = \rho^2$ imply

$$F(\gamma + t) \in (\rho^2 - \varepsilon_1, \rho^2 + \varepsilon_1) \tag{61}$$

for $\gamma + t \in U_\varepsilon(S_\rho^{'})$. On the other hand $\gamma + t \notin Tr(A(\rho))$. It means that for any $\gamma^{'} \in \Gamma$, we have $\gamma^{'} + t \notin A(\rho)$. If $\gamma^{'} \in K$ and $\gamma^{'} + t \in E_k \backslash E_{k+1}$, then by definition of K (see introduction) the inequality $\mid F(\gamma$ +t)-$\mid \gamma^{'} + t \mid^2 \mid < \frac{1}{3}\rho^{\alpha_1}$ holds.
This and (69) imply that $\gamma^{'} + t \in (E_k \backslash E_{k+1}) \cap$ K_ρ (see (62) for the definition of K_ρ). Since $\gamma^{'} + t \notin A(\rho)$, we have $\lambda_i(\gamma^{'}$ +t)$\notin (\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1)$ for $\gamma^{'} \in K$ and $\gamma^{'} + t \in E_k \backslash$ E_{k+1}. Therefore (12) follows from (69). Moreover, it is clear that the inclusion $S_\rho^{'} \subset U(2\rho^{\alpha_1}, p)$ (see definition of S_ρ and $S_\rho^{'}$) implies that $U_\varepsilon(S_\rho^{'}) \subset U(\rho^{\alpha_1}, p)$. Thus $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)) \subset B$.
Now let E be a connected open subset of $U_\varepsilon(S_\rho^{'}$ $) \backslash Tr(A(\rho) \subset B$. By Theorem 3 and Remark 3 for $a \in$ $E \subset U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)$ there exists a unique index $N(a)$ such that $\Lambda(a) = \Lambda_{N(a)}(a)$, $\Psi_a(x) = \Psi_{N(a),a}(x)$, $\mid (\Psi_{N(a),a}(x), e^{i(a,x)}) \mid$ $^2 > \frac{1}{2}$ and $\Lambda(a)$ is a simple eigenvalue. On the other hand, for fixed N the functions $\Lambda_N(t)$ and $(\Psi_{N,t}(x), e^{i(t,x)})$ are continuous in neighborhood of a if $\Lambda_N(a)$ is a simple eigenvalue. Therefore for each $a \in E$ there exists a neighborhood $U(a) \subset E$ of a such that $\mid (\Psi_{N(a),y}(x), e^{i(y,x)}) \mid$ $^2 > \frac{1}{2}$, for $y \in U(a)$. Since for $y \in E$ there is a unique integer $N(y)$ satisfying $\mid (\Psi_{N(y),y}(x), e^{i(y,x)}) \mid^2 >$ $\frac{1}{2},$ we have $N(y) = N(a)$ for $y \in U(a)$. Hence we proved that

$$\forall a \in E, \exists U(a) \subset E : N(y) = N(a), \forall y \in U(a).$$

(62)

Now let a_1 and a_2 be two points of E , and let $C \subset E$ be the arc that joins these points. (Note that the open connected subset of $\mathbb{R}^d$ is arcwise connected). Let $U(y_1), U(y_2), ..., U(y_k)$ be a finite subcover of the open cover $\cup_{a\in C} U(a)$ of the compact C, where $U(a)$ is a neighborhood of a satisfying (70). By (70), we have $N(y) = N(y_i) = N_i$ for $y \in U(y_i)$. Clearly, if $U(y_i)\cap$ U(y_j) $\neq \emptyset$, then $N_i = N(z) = N_j$, where $z \in U(y_i)\cap$ U(y_j). Thus $N_1 = N_2 = ... = N_k$ and $N(a_1) = N(a_2)$. To calculate the partial derivatives of the function $\Lambda(\gamma$ +t)=$\Lambda_N(t)$ we write the operator L_t in the form $-\triangle$ -(2it,∇) + (t, t). Then, it is clear that

$\partial\overline{\partial t_j \Lambda_N(t) = 2t_j(\Phi_{N,t}}$
(x),$\Phi_{N,t}(x)) - 2i(\frac{\partial}{\partial x_j}\Phi_{N,t}(x), \Phi$
$_{N,t}(x))$,

$\Phi_{N,t}(x) = \sum_{\gamma'\in\Gamma} b(N, \gamma^{'}$
)e$^{i(\gamma^{'},x)}$,

where $\Phi_{N,t}(x) = e^{-i(t,x)}\Psi_{N,t}(x)$. If $\mid \gamma^{'}$
$|\geq 2\rho$, then using
$\Lambda_N \equiv \Lambda(\gamma + t) = \rho^2 + O(\rho^{-\alpha})$, (see (46),
(69)), and the obvious inequality
$\mid \Lambda_N - \mid \gamma^{'} - \gamma_1 - \gamma_2 - ... - \gamma$
$_k + t \mid^2 \mid > c_{11} \mid \gamma^{'} \mid^2$ for $k = 0, 1, ..., p$,
where $\mid \gamma_1 \mid < \frac{1}{4p} \mid \gamma^{'} \mid$, and
itarating (17) p times by using the decomposition $q(x) = \sum_{|\gamma}$
$_1 \mid < \frac{1}{4p} \mid \gamma^{'} \mid q_{\gamma_1} e^{i(\gamma}$
$_1, x) + O(\mid \gamma^{'} \mid^{-p})$, we get

b(N,$\gamma^{'}) = \sum_{\gamma_1,\gamma_2,...}$
q$_{\gamma_1} q_{\gamma_2} ... q_{\gamma_p} b(N, \gamma^{'}$
-$\sum_{i=1}^{p}\gamma_i)\prod_{j=0}^{p-1}(\Lambda_N - \mid \gamma$
$^{'} - \sum_{i=1}^{j}\gamma_i + t \mid^2) + O(\mid \gamma^{'}$
$\mid^{-p})$,

b(N,$\gamma^{'}$) $= O(\mid \gamma^{'} \mid^{-p})$,
$\forall \mid \gamma^{'} \mid \geq 2\rho$

By (74) the series in (72) can be differentiated term by term. Hence

-i($\partial\overline{\partial x_j \Phi_{N,t}, \Phi_{N,t}) = \sum_{\gamma}}$
$^{'} \in \Gamma \gamma^{'}(j) \mid b(N, \gamma^{'}$
$) \mid^2 = \gamma(j) \mid b(N, \gamma) \mid^2 + \Sigma_1 + \Sigma_2,$
(63)

where $\Sigma_1 = \sum_{|\gamma'| \geq 2\rho} \gamma^{'}$
(j)$\mid b(N, \gamma^{'}) \mid^2$, $\Sigma_2 = \sum_{|}$
$\gamma^{'} \mid < 2\rho, \gamma^{'} \neq \gamma \gamma^{'}$
(j)$\mid b(N, \gamma^{'}) \mid^2$.
By (13), $\sum_2 = O(\rho^{-2\alpha_1 + 1})$, $\gamma(j) \mid b(N, \gamma$
$) \mid^2 = \gamma(j)(1 + O(\rho^{-2\alpha_1})$, and by (74), $\sum_1$
=O($\rho^{-2\alpha_1}$). Therefore (71) and (75) imply (66).
(b) To ptove the inclusion $U_\varepsilon(V_\rho) \subset$
$U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$ we need to show
that if $a \in V_\rho$, then $U_\varepsilon(a) \subset U_\varepsilon$
(S$_\rho^{'}$)$\backslash Tr(A(\rho))$. This is clear, because the
relation $a \in V_\rho \subset S_\rho^{'}$ implies that
$U_\varepsilon(a) \subset U_\varepsilon(S_\rho^{'})$ and the
relation $a \notin U_\varepsilon(Tr(A(\rho)))$ implies that U_ε
(a)$\cap Tr(A(\rho)) = \emptyset$. To prove (67) first we estimate the measure
of $S_\rho, S_\rho^{'}, U_{2\varepsilon}(A(\rho))$, namely we prove

$$\mu(S_\rho) > (1 - c_{12}\rho^{-\alpha})\mu(B(\rho)),$$

$$\mu(S_\rho^{'}) > (1 - c_{13}\rho^{-\alpha})\mu(B(\rho)),$$

$$\mu(U_{2\varepsilon}(A(\rho))) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon$$

(see below, Estimations 1, 2, 3). The estimation (67) of the measure of the set V_ρ is done in Estimation 4 by using Estimations 1, 2, 3.
(c) In Estimation 5 we prove the formula (68). The Theorem is proved
■

In Estimations 1-5 we use the notations: $G(+i,a)=\{x\in G, x_i>a\}$, $G(-i,a)=\{x\in G, x_i<-a\}$, where $x=(x_1,x_2,...,x_d), a>0$. It is not hard to verify that for any subset G of $U_\varepsilon(S_\rho^{'}) \cup U_{2\varepsilon}(A(\rho))$, that is, for all considered sets G in these estimations, and for any $x\in G$ the followings hold

$$\rho-1<\mid x\mid<\rho+1,\ G\subset(\cup_{i=1}^{d}(G(+i,\rho d^{-1})\cup G(-i,\rho d^{-1})) \qquad (64)$$

Indeed, if $x\in S_\rho^{'}$, then $F(x)=\rho^2$ and by definition of $F(x)$ (see Lemma 2) and (23) we have $\mid x\mid=\rho+O(\rho^{-1-\alpha_1})$. Hence the inequalities in (79) hold for $x\in U_\varepsilon(S_\rho^{'})$. If $x\in A(\rho)$, then by definition of $A(\rho)$ (see (61), (62)), we have $x\in K_\rho$, and hence $\mid x\mid=\rho+O(\rho^{-1+\alpha_1})$. Thus the inequalities in (79) hold for $x\in U_{2\varepsilon}(A(\rho))$ too. The inclusion in (79) follows from these inequalities.
If $G\subset S_\rho$, then by (43) we have $\frac{\partial F(x)}{\partial x_k}>0$ for $x\in G(+k,\rho^{-\alpha})$. Therefore to calculate the measure of $G(+k,a)$ for $a\geq\rho^{-\alpha}$ we use the formula

$$\mu(G(+k,a))=\int\limits_{\Pr_k(G(+k,a))}\left(\frac{\partial F}{\partial x_k}\right)^{-1}\mid grad(F)\mid dx_1...dx_{k-1}dx_{k+1}...dx_d, \qquad (65)$$

where $\Pr_k(G)\equiv\{(x_1,x_2,...,x_{k-1},x_{k+1},x_{k+2},...,x_d):x\in G\}$ is the projection of G on the hyperplane $x_k=0$. Instead of $\Pr_k(G)$ we write $\Pr(G)$ if k is unambiguous. If D is $m-$dimensional subset of $\mathbb{R}^m$, then to estimate $\mu(D)$, we use the formula

$$\mu(D)=\int\limits_{\Pr_k(D)}\mu(D(x_1,...x_{k-1},x_{k+1},...,x_m))dx_1...dx_{k-1}dx_{k+1}...dx_m, \qquad (66)$$

where $D(x_1, ...x_{k-1}, x_{k+1}, ..., x_m) = \{x_k : (x_1, x_2$
,...,$x_m) \in D\}$.

ESTIMATION 1. Here we prove (76) by using (80). During this estimation the set

S_ρ is redenoted by G. If $x \in G$, then $x \notin V_b(\rho$
$^{\alpha_1})$ for all $b \in \Gamma(p\rho^\alpha)$. Since the rotation does
not change the measure, we choose the coordinate axis so that the direction of
a fixed $b \in \Gamma(p\rho^\alpha)$ coinsides with the direction of
$(1, 0, 0, ..., 0)$, that is, $b = (b_1, 0, 0, ..., 0), b_1 > 0$. Then the relations
$x \notin V_b(\rho^{\alpha_1})$, $\mid b \mid < p\rho^\alpha, \alpha$
$_1 = 3\alpha$ imply that $\mid x_1 \mid > a$, where
$a = (\rho^{\alpha_1} - b_1^2)(2b_1)^{-1} > \rho^\alpha$. Therefore
$G = G(+1, a) \cup G(-1, a)$. Now we estimate $\mu(G(+1, a))$ by using (80) for
$k = 1$ and the relations

$\partial F_{\overline{\partial x_1 > \rho^\alpha, (\partial}}$
$\mathrm{F}_{\overline{\partial x_1)^{-1} |grad(F)| = \frac{|x|}{x_1}}}$
$+\mathrm{O}(\rho^{-2\alpha})$,

$\Pr(G(+1, a)) \supset \Pr(A(+1, 2a))$,

where $x \in G(+1, a), a > \rho^\alpha$, $A = B(\rho) \cap U(3\rho^{\alpha_1}, p)$.
Here (82) follows from (43). Now we prove (83). If $(x_2, ..., x_d) \in$
$\Pr_1(A(+1, 2a))$, then by definition of $A(+1, 2a)$ there exists x_1 such
that

$\mathrm{x}_1 > 2a > 2\rho^\alpha$, $x_1^2 + x_2^2 + ... + x_d^2 = \rho$
$^2, \mid \sum_{i \geq 1}(2x_i b_i - b_i^2) \mid \geq 3\rho^{\alpha_1}$
(67)

for all $(b_1, b_2, ..., b_d) \in \Gamma(p\rho^\alpha)$. Therefore for
$h = \rho^{-\alpha}$ we have
$(x_1 + h)^2 + x_2^2 + ... + x_q^2 > \rho^2 + \rho^{-\alpha}, (x_1$
-h$)^2 + x_2^2 + ... + x_q^2 < \rho^2 - \rho^{-\alpha}$. This and (23) give
$F(x_1 + h, x_2, ..., x_d) > \rho^2$, $F(x_1 - h, x_2, ..., x_d) < \rho^2$.
Since F is a continuous function there is $y_1 \in (x_1 - h, x_1 + h)$ such
that (see (84))

$y_1 > a, F(y_1, x_2, ..., x_d) = \rho^2, \mid 2y_1 b_1 - b_1^2$
$+\sum_{i\geq 2}(2x_i b_i - b_i^2) \mid > \rho^{\alpha_1},$
(68)

because the expression under the absolute value in (85) differ from the expression under the absolute value in (84) by $2(y_1 - x_1)b_1$, where $\mid y_1 - x_1 \mid < h = \rho^{\alpha}$, $b_1 < p\rho^{\alpha}$, $\mid 2(y_1 - x_1)b_1 \mid < 2p\rho^{2\alpha} < \rho^{\alpha_1}$. The relations in (85) means that $(x_2, ..., x_d) \in \Pr G(+1, a)$. Hence (83) is proved. Now (80), (82), and the obvious relation $\mu(\Pr G(+1, a)) = O(\rho^{d-1})$ (see the inequalities in (79)) imply that

$$\mu(G(+1, a)) = \int\limits_{\Pr(G(+1,a))} \frac{|x|}{x_1} dx_2 dx_3 ... dx_d + O(\rho^{-\alpha})\mu(B(\rho)) \geq$$

$$\int\limits_{\Pr(A(+1,2a))} \frac{|x|}{x_1} dx_2 dx_3 ... dx_d - c_{14}\rho^{-\alpha}\mu(B(\rho)) =$$

$$\mu(A(+1, 2a)) - c_{14}\rho^{-\alpha}\mu(B(\rho)).$$

Similarly, $\mu(G(-1, a)) \geq \mu(A(-1, 2a)) - c_{14}\rho^{-\alpha}\mu(B(\rho))$.
Therefore using the relations $G = G(-1, a) \cup G(+1, a))$, $A = A(-1, 2a) \cup A(+1, 2a)$, $\mu(A)) = (1 + O(\rho^{-\alpha}))\mu(B(\rho))$ (see (41)) we obtain (76).

ESTIMATION 2 Here we prove (77). For this we estimate the measure of the set $S_\rho \cap P_b$ by using (80). During this estimation the set $S_\rho \cap P_b$ is redenoted by G. We choose the coordinate axis so that the direction of b coincides with the direction of $(1, 0, 0, ..., 0)$, i.e.,

$b=(b_1,0,0,...,0)$ and $b_1>0$. It follows from the definitions of S_ρ, P_b and $F(x)$ (see the beginning of this section, (60), and Lemma 2(a)) that if $(x_1,x_2,...,x_d)\in G$ then

$$x_1^2+x_2^2+...+x_d^2+F_{k_1-1}(x)=\rho^2,$$

$$(x_1+b_1)^2+x_2^2+x_3^2+...+x_d^2+F_{k_1-1}(x+b)$$
$$=\rho^2+h,$$

where $h\in(-3\varepsilon_1,3\varepsilon_1)$. Subtracting (86) from (87) and using (23), we get

$$(2x_1+b_1)b_1=O(\rho^{-\alpha_1}). \tag{69}$$

This and the inequalities in (79) imply

$$\mid b_1\mid<2\rho+3,\ x_1=\tfrac{b_1}{2}+O(\rho^{-\alpha_1}$$
$$b_1^{-1}),\mid x_1^2-(\tfrac{b_1}{2})^2\mid=O(\rho^{-\alpha_1}). \tag{70}$$

Consider two cases. Case 1: $b\in\Gamma_1$, where $\Gamma_1=\{b\in \Gamma:\mid\rho^2-\mid\frac{b}{2}\mid^2\mid<3d\rho^{-2\alpha}\}$. In this case using the last equality in (89), (86), (23), and taking into account that $b=(b_1,0,0,...,0)$, $\alpha_1=3\alpha$, we obtain

$$x_1^2=\rho^2+O(\rho^{-2\alpha}),\mid x_1\mid=\rho+O(\rho^{-2\alpha}$$
$$-1),x_2^2+x_3^2+...+x_d^2=O(\rho^{-2\alpha}). \tag{71}$$

Therefore $G\subset G(+1,a)\cup G(-1,a)$, where $a=\rho-\rho^{-1}$. Using (80), the obvious relation $\mu(\Pr_1(G(\pm1,a))=O(\rho^{-(d-1)\alpha})$ (see (90)) and taking into account that the expression under the integral in (80) for $k=1$ is equal to $1+O(\rho^{-\alpha})$ (see (82) and (90)), we get

$\mu(G(\pm 1,a))=O(\rho^{-(d-1)\alpha})$. Thus $\mu(G)=O(\rho^{-(d-1)\alpha})$.
Since $\mid\Gamma_1\mid=O(\rho^{d-1})$, we have

$\mu(\cup_{b\in\Gamma_1}(S_\rho\cap P_b)=O(\rho^{-(d-1)\alpha}$
+d-1)=O($\rho^{-\alpha}$)$\mu(B(\rho))$.
(72)

Case 2: $\mid\rho^2-\mid\frac{b}{2}\mid^2\mid\geq 3d\rho^{-2\alpha}$.
Repeating the proof of (90), we get

$\mid x_1^2-\rho^2\mid>2d\rho^{-2\alpha},\ \sum_{k=2}^d x_k$
$^2>d\rho^{-2\alpha},\ \max_{k\geq 2}\mid x_k\mid>\rho^{-\alpha}$.
(73)

Therefore $G\subset\cup_{k\geq 2}(G(+k,\rho^{-\alpha})\cup G(-k,\rho^{-\alpha}$
$)).Nowweestimate\mu(G(+d,\rho^{-\alpha}))$ by using (80). Redenote by
D the set $\mathrm{Pr}_d\,G(+d,\rho^{-\alpha})$. If $x\in G(+d,\rho^{-\alpha})$,
then according to (86) and (43) the under integral expression in (80) for
$k=d$ is $O(\rho^{1+\alpha})$. Therefore the first equality in

$\mu(D)=O(\varepsilon_1\mid b\mid^{-1}\rho^{d-2}),\ \mu(G(+d,\rho$
$^{-\alpha}))=O(\rho^{d-1+\alpha}\varepsilon_1\mid b\mid^{-1})$
(74)

implies the second equality in (93). To prove the first equality in (93) we
use (81) for $m=d-1$ and $k=1$ and prove the relations $\mu(\mathrm{Pr}_1$
D)=O(ρ^{d-2}),

$\mu(D(x_2,x_3,...,x_{d-1}))<6\varepsilon_1\mid b\mid^{-1}$
(75)

for $(x_2,x_3,...,x_{d-1})\in\mathrm{Pr}_1\,D$. First relation follows from the
inequalities in (79)). So we need to prove (94). If $x_1\in D(x_2$
,x$_3,...,x_{d-1})$ then (86) and (87) holds. Subtructing (86) from (87), we

get

$$2x_1b_1 + (b_1)^2 + F_{k_1-1}(x-b) - F_{k_1-1}(x) = h, \tag{76}$$

where $x_2, x_3, ..., x_{d-1}$ are fixed . Hence we have two equations (86) and (95) with respect two unknown x_1 and x_d. Using (43), the implicit function theorem, and the inequalities $\mid x_d \mid > \rho^{-\alpha}$,$\alpha_1 > 2\alpha$ from (86), we obtain

$$x_d = f(x_1),\ \frac{df}{dx_1} = -2x_1 + O(\rho^{-2\alpha}$$
$$_1 + \alpha)\overline{{}_{2x_d+O(\rho^{-2\alpha_1+\alpha})=-\frac{x_1}{x_d}}}$$
$$+O(\rho^{-\alpha_1}). \tag{77}$$

Substituting this in (95), we get

$$2x_1b_1 + b_1^2 + F_{k_1-1}(x_1+b_1, x_2, ..., x_{d-1}, f(x_1))-F_{k_1-1}(x_1, ..., x_{d-1}, f) = h. \tag{78}$$

Using (43), (96), the first equality in (89), and $x_d > \rho^{-\alpha}$ we see that the absolute value of the derivative (w.r.t. x_1) of the left-hand side of (97) satisfies
$\mid 2b_1 + O(\rho^{-2\alpha_1+\alpha})(1+ \mid \frac{df}{dx_1} \mid) \mid =$
$\mid 2b_1 + O(\rho^{-2\alpha_1+\alpha})(1+ \mid \frac{x_1}{x_d}$
$\mid) + O(\rho^{-3\alpha_1+\alpha}) \mid > b_1$. Therefore from (97) by implicit function theorem, we get $\mid \frac{dx_1}{dh} \mid < \frac{1}{\mid}$
b| . This inequality and relation $h \in (-3\varepsilon_1, 3\varepsilon$
$_1)$ imply (94). Thus (93) is proved. In the same way we get the same estimation for $G(+k, \rho^{-\alpha})$ and $G(-k, \rho^{-\alpha})$ for $k \geq 2$. Hence
$\mu(S_\rho \cap P_b) = O(\rho^{d-1+\alpha}\varepsilon_1 \mid b \mid^{-1})$,
for $b \notin \Gamma_1$. Since $\mid b \mid < 2\rho + 3$ (see (89)) and
$\varepsilon_1 = \rho^{-d-2\alpha}$, taking into account that the number of the vectors of Γ satisfying $\mid b \mid < 2\rho + 3$ is $O(\rho^d)$, we obtain
$\mu(\cup_{b \notin \Gamma_1}(S_\rho \cap P_b)) = O(\rho^{2d-1+\alpha}$

$\varepsilon_1) = O(\rho^{-\alpha})\mu(B(\rho))$. This, (91) and (76) give the proof of (77).

ESTIMATION 3. Here we prove (78). Denote $U_{2\varepsilon}(A_{k,j}(\gamma_1,\gamma_2,...,\gamma_k))$ by G, where $\gamma_1,\gamma_2,...,\gamma_k \in \Gamma(p\rho^{\alpha}), k \leq d-1$, and $A_{k,j}$ is defined in (61). We turn the coordinate axis so that $Span\{\gamma_1,\gamma_2,...,\gamma_k\} = \{x = (x_1, x_2, ..., x_k, 0,0,...,0) : x_1, x_2, ..., x_k \in \mathbb{R}\}$. Then by (31), we have $x_n = O(\rho^{\alpha_k+(k-1)\alpha})$ for $n \leq k$, $x \in G$. This, (79), and $\alpha_k + (k-1)\alpha < 1$ (see the sixth inequality in (14)) give $G \subset (\cup_{i>k}(G(+i,\rho d^{-1}) \cup G(-i,\rho d^{-1}))$, $\mu(\Pr_i(G(+i,\rho d^{-1}))) = O(\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)})$ for $i > k$. Now using this and (81) for $m = d$, we prove that

$$\mu(G(+i,\rho d^{-1})) = O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)}), \forall i > k. \tag{79}$$

For this we redenote by D the set $G(+i,\rho d^{-1})$ and prove that

$$\mu((D(x_1,x_2,...x_{i-1},x_{i+1},...x_d)) \leq (42d^2+4)\varepsilon \tag{80}$$

for $(x_1,x_2,...x_{i-1},x_{i+1},...x_d) \in \Pr_i(D)$ and $i > k$. To prove (99) it is sufficient to show that if both $x = (x_1, x_2, ..., x_i, ...x_d)$ and $x' = (x_1, x_2, ..., x_i', ..., x_d)$ are in D, then $| x_i - x_i' | \leq (42d^2+4)\varepsilon$. Assume the converse. Then $| x_i - x_i' | > (42d^2+4)\varepsilon$. Without loss of generality it can be assumed that $x_i' > x_i$. So $x_i' > x_i > \rho d^{-1}$ (see definition of D). Since x and x' lie in the 2ε neighborhood of $A_{k,j}$, there exist points a and a' in $A_{k,j}$ such that $| x - a | < 2\varepsilon$ and $| x' - a' | < 2\varepsilon$. It follows from the definitions of the points x, x', a, a' that the following inequalities hold:

$$\rho d^{-1} - 2\varepsilon < a_i < a_i', \ a_i'$$

-$a_i > 42d^2\varepsilon$,

$(a_i^{'})^2 - (a_i)^2 > 2(\rho d^{-1} - 2\varepsilon$
$)(a_i^{'} - a_i)$,

$|| a_s | - | a_s^{'} ||< 4\varepsilon, \forall$
$s \neq i$.

On the other hand for points of $A_{k,j}$ the inequalities in (79) hold, that is, we have $| a_s |< \rho + 1, | a_s^{'} |< \rho + 1$.
Therefore these inequalities and the inequalities in (100) imply $||$
$a_s |^2 - | a_s^{'} |^2|< 12\rho\varepsilon$ for $s \neq$
$i, and hence$
$\sum_{s\neq i} || a_s |^2 - | a_s^{'} |^2$
$|< 12d\rho\varepsilon < \frac{2}{7}\rho d^{-1}(a_i^{'} - a_i)$,

$|| a |^2 - | a^{'} |^2|> \frac{3}{2}\rho d^{-1} |$
$a_i^{'} - a_i |$.
(81)

Now using the inequality (45), the obvious relation $\frac{1}{2}\alpha_d < 1$
(see the end of the introduction), the notations $r_j(x) = \lambda$
$_j(x)- | x |^2$ (see Remark 2), $\varepsilon_1 = 7\rho\varepsilon$ (
see Lemma 2(a)), and (101), (100), we get
$| r_j(a) - r_j(a^{'}) |< \rho^{\frac{1}{2}\alpha_d} |$
$a-a^{'} |< \frac{1}{2}\rho d^{-1} | a_i^{'} - a_i |$,
$| \lambda_j(a) - \lambda_j(a^{'}) |\geq|| a |^2 - |$
$a^{'} |^2| - | r_j(a) - r_j(a^{'}) |>$
$\rho d^{-1} | a_i^{'} - a_i |> 42d\rho\varepsilon > 6\varepsilon$
$_1$.
The obtained inequality $| \lambda_j(a) - \lambda_j(a^{'}$
$)|> 6\varepsilon_1$ controdicts with inclutions $a \in A_{k,j}$,
$a^{'} \in A_{k,j}$, since by definition of $A_{k,j}$ (see (61)) both
$\lambda_j(a)$ and $\lambda_j(a^{'})$ lie in $(\rho^2$
$-3\varepsilon_1, \rho^2 + 3\varepsilon_1)$. Thus (99), hense (98) is
proved. In the same way we get the same formula for $G(-i, \frac{\rho}{d})$. So
$\mu(U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, ..., \gamma_k$
$)))=O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+d-1-k})$. Now taking into
account that $U_{2\varepsilon}(A(\rho))$ is union of $U_{2\varepsilon}$
$(A_{k,j}(\gamma_1, \gamma_2, ..., \gamma_k)$ for $k = 1, 2, .., d-1$;

$j=1,2,...,b_k(\gamma_1,\gamma_2,...,\gamma_k)$, and γ
$_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^{\alpha})$ (see (61)) and
using that $b_k=O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}})$ (see (39)) and
the number of the vectors $(\gamma_1,\gamma_2,...,\gamma_k)$ for
$\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^{\alpha})$ is
$O(\rho^{dk\alpha})$, we obtain

$\mu(U_{2\varepsilon}(A(\rho)))=O(\varepsilon\rho^{d\alpha+\frac{k}{2}}$
$\alpha_{k+1}+dk\alpha+k(\alpha_k+(k-1)\alpha)+d-1-k)$.

Therefore to prove (78), it remains to show that
$d\alpha+\frac{k}{2}\alpha_{k+1}+dk\alpha+k(\alpha_k+(k-1)\alpha)+d-1-k\leq q$
d-1-α or

(d+1)$\alpha+\frac{k}{2}\alpha_{k+1}+dk\alpha+k(\alpha_k+(k-1)\alpha)\leq k$
(82)

for $1\leq k\leq d-1$. Dividing both side of (102) by $k\alpha$ and using
$\alpha_k=3^k\alpha$, $\alpha=\frac{1}{q}$, $q=3^d+d+2$ (see the end of
the introduction) we see that (102) is equivalent to $\frac{d+1}{k}$
$+3^{k+1}\overline{2+3^k+k-1\leq 3^d+2.}$ The left-hand side of this
inequality gets its maximum at $k=d-1$. Therefore we need to show that
$\frac{d+1}{d-1}+\frac{5}{6}3^d+d\leq 3^d+4$, which follows from the
inequalities $\frac{d+1}{d-1}\leq 3, d<\frac{1}{6}3^d+1$ for $d\geq 2$.
ESTIMATION 4. Here we prove (67). During this estimation we denote by
G the
set $S_{\rho}^{'}\cap U_{\varepsilon}(Tr(A(\rho))$. Since V_{ρ}
$=S_{\rho}^{'}\backslash G$ and (77) holds, it is enough to prove that
$\mu(G)=O(\rho^{-\alpha})\mu(B(\rho))$. For this we use (79) and prove
$\mu(G(+i,\rho d^{-1}))=O(\rho^{-\alpha})\mu(B(\rho))$ for $i=1,2,...,d$ by
using (80) (the same estimation for $G(-i,\rho d^{-1})$ can be proved in the
same way). By (43), if $x\in G(+i,\rho d^{-1})$, then the under integral
expression in (80) for $k=i$ and $a=\rho d^{-1}$ is less than $d+1$. Therefore
it is sufficient to prove

$\mu(\Pr(G(+i,\rho d^{-1}))=O(\rho^{-\alpha})\mu(B(\rho))$
(83)

Clearly, if $(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \mathrm{Pr}_i(G(+i, \rho$ d$^{-1}))$, then $\mu(U_\varepsilon(G)(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d)) \geq$ 2ε and by (81), it follows that

$$\mu(U_\varepsilon(G)) \geq 2\varepsilon\mu(\mathrm{Pr}(G(+i, \rho d^{-1})). \quad (84)$$

Hence to prove (103) we need to estimate $\mu(U_\varepsilon(G))$. For this we prove that

$$\mathrm{U}_\varepsilon(G) \subset U_\varepsilon(S_\rho^{'}), U_\varepsilon(\mathrm{G}) \subset U_{2\varepsilon}(Tr(A(\rho))), U_\varepsilon(G) \subset \mathrm{Tr}(\mathrm{U}_{2\varepsilon}(A(\rho))). \quad (85)$$

The first and second inclusions follow from $G \subset S_\rho^{'}$ and $G \subset U_\varepsilon(Tr(A(\rho)))$ respectively (see definition of G). Now we prove the third inclusion in (105). If $x \in U_\varepsilon$ (G),$then by the second inclusion of$ (105)$there exists$ b$such that$ b$\in$ Tr(A(ρ)), $\mid x - b \mid < 2\varepsilon$. Then by the definition of $Tr(A(\rho))$ there exist $\gamma \in \Gamma$ and $c \in A(\rho)$ such that $b = \gamma + c$. Therefore $\mid x - \gamma - c \mid = \mid x - b \mid < 2\varepsilon$, $x - \gamma \in U_{2\varepsilon}(c) \subset U_{2\varepsilon}(A(\rho))$. This together with $x \in U_\varepsilon(G) \subset U_\varepsilon(S_\rho$ $')$ (see the first inclusion of (105)) give $x \in$ Tr(U$_{2\varepsilon}(A(\rho))$) (see the definition of $Tr(E)$ in the beginning of this section), i.e., the third inclusion in (105) is proved. The third inclusion, Lemma 2(c), and (78) imply that $\mu(U_\varepsilon(G)) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon$. This and (104) imply the proof of (103)$\diamondsuit$

ESTIMATION 5 Here we prove (68). Divide the set $V_\rho \equiv V$ into pairwise disjoint subsets $V^{'}(\pm 1, \rho d^{-1}) \equiv V(\pm 1, \rho$ d$^{-1})$, $V^{'}(\pm i, \rho d^{-1}) \equiv V(\pm i, \rho d^{-1}) \backslash$ $(\cup_{j=1}^{i-1}(V(\pm j, \rho d^{-1})))$, for $i = 2, 3, ..., d$. Take any point $a \in V^{'}(+i, \rho d^{-1}) \subset S_\rho$ and consider the function

$F(x)$ (see Lemma 2(a)) on the interval $[a-\varepsilon e_i, a+\varepsilon$ e$_i]$, where $e_1=(1,0,0,...,0)$, $e_2=(0,1,0,...,0)$, ... By the definition of S_ρ we have $F(a)=\rho^2$. It follows from (43) and the definition of $V^{'}(+i,\rho d^{-1})$ that ∂ F(x)$\overline{\partial x_i>\rho d^{-1}}$ for $x\in[a-\varepsilon$ e$_i, a+\varepsilon e_i]$. Therefore

F(a-$\varepsilon e_i)<\rho^2-c_{15}\varepsilon_1,\ F(a+\varepsilon$ e$_i)>\rho^2+c_{15}\varepsilon_1.$
(86)

Since $[a-\varepsilon e_i, a+\varepsilon e_i]\in U_\varepsilon(a)\subset$ U$_\varepsilon(V_\rho)\subset U_\varepsilon(S_\rho^{'}$ $)\backslash Tr(A(\rho))$ (see Theorem 5(b)), it follows from Theorem 5(a) that there exists index N such that $\Lambda(y)=\Lambda_N(y)$ for $y\in U_\varepsilon(a)$ and $\Lambda(y)$ satisfies (46)(see Remark 3). Hence (106) implies that

$\Lambda(a-\varepsilon e_i)<\rho^2, \Lambda(a+\varepsilon e_i)>\rho^2.$
(87)

Moreover it follows from (66) that the derivative of $\Lambda(y)$ with respect to i-th coordinate is positive for $y\in[a-\varepsilon e_i$,a+$\varepsilon e_i]$. So $\Lambda(y)$ is a continuous and increasing function in $[a-\varepsilon e_i, a+\varepsilon e_i]$. Therefore (107) implies that there exists a unique point $y(a,i)\in[a-\varepsilon$ e$_i, a+\varepsilon e_i]$ such that $\Lambda(y(a,i))=\rho^2$. Define $I_\rho^{'}(+i)$ by $I_\rho^{'}(+i)=\{y(a,i): a\in$ V$^{'}(+i,\rho d^{-1})\}$). In the same way we define I_ρ $^{'}(-i)=\{y(a,i): a\in V^{'}(-i,\rho d^{-1})\}$ and put $I_\rho^{'}=\cup_{i=1}^{d}(I_\rho^{'}(+i)\cup I_\rho$ $^{'}(-i))$. To estimate the measure of $I_\rho^{'}$ we compare the measure of $V^{'}(\pm i,\rho d^{-1})$ with the measure of $I_\rho^{'}(\pm i)$ by using the formula (80) and the obvious relations

Pr$(V^{'}(\pm i,\rho d^{-1}))=$ Pr$(I_\rho^{'}(\pm i))$,

$\mu(\Pr(I_\rho^{'}(\pm i))) = O(\rho^{d-1}),$
(88)

$(\partial F \overline{\partial x_i)^{-1}|grad(F)|-(\partial}$
$\Lambda \overline{\partial x_i)^{-1}|grad(\Lambda)|=O(\rho^{-2\alpha_1}).}$
(89)

Here the first equality in (108) follows from the definition of I_ρ
$^{'}(\pm i)$. The second equality in (108) follows from the
inequalities in (79), since $I_\rho^{'} \subset U_\varepsilon$
($S_\rho^{'}$). Formulas (43), (66) imply (109). Clearly, using
(108), (109), and (80) we get $\mu(V^{'}(\pm i, \rho d^{-1}$
$))$-$\mu(I_\rho^{'}(\pm i)) = O(\rho^{d-1-2\alpha_1})$. On the other
hand if
$y \equiv (y_1, y_2, ..., y_d) \in I_\rho^{'}(+i) \cap I_\rho$
$^{'}(+j)$ for $i < j$ then there are $a \in V^{'}(+i, \rho$
$d^{-1})$ and $a^{'} \in V^{'}(+j, \rho d^{-1})$ such that
$y = y(a, i) = y(a^{'}, j)$ and $y \in [a - \varepsilon e_i$
$,a+\varepsilon e_i]$, $y \in [a^{'} - \varepsilon e_j$
$,a^{'} + \varepsilon e_j]$. These inclusions imply that ρ
$d^{-1} - \varepsilon \leq y_i \leq \rho d^{-1}$. Therefore $\mu(\Pr_j(I_\rho$
$^{'}(+i) \cap I_\rho^{'}(+j))) = O(\varepsilon\rho^{d-2})$.
This equality, (80) and (66) imply that $\mu((I_\rho^{'}(+i)\cap$
$I_\rho^{'}(+j))) = O(\varepsilon\rho^{d-2})$ for all i and j. Similarly
$\mu((I_\rho^{'}(+i) \cap I_\rho^{'}(-j))) = O(\varepsilon$
$\rho^{d-2})$ for all i and j. Thus

$\mu(I_\rho^{'}) = \sum_i \mu(I_\rho^{'}(+i)) + \sum_i$
$\mu(I_\rho^{'}(-i)) + O(\varepsilon\rho^{d-2}) =$

$\sum_i \mu(V^{'}(+i, \rho d^{-1}$
$))+\sum_i \mu(V^{'}(-i, \rho d^{-1})) + O(\rho^{d-1-2\alpha_1}) =$
$\mu(V_\rho) + O(\rho^{-2\alpha_1})\mu(B(\rho))$. This and (67) yeild the
inequality (68) for $I_\rho^{'}$. Now we define I_ρ
$^{''}$ as follows. If $\gamma + t \in I_\rho^{'}$ then

$\Lambda(\gamma+t)=\rho^2$, where $\Lambda(\gamma+t)$ is a unique eigenvalue satisfying (5) (see Remark 3). Since $\Lambda(\gamma+t)=\mid\gamma+t\mid^2+O(\rho^{-\alpha_1})$ (see (5) and (23)), for fixed t there exist only a finite number of vectors $\gamma_1,\gamma_2,...,\gamma_s\in\Gamma$ satisfying $\Lambda(\gamma_k+t)=\rho^2$. Hence $I_\rho^{'}$ is the union of pairwise disjoint subsets $I_{\rho,k}^{'}\equiv\{\gamma_k+t\in I_\rho^{'}:\Lambda(\gamma_k+t)=\rho^2\}$ for $k=1,2,...s$. The translation $I_{\rho,k}^{''}=I_{\rho,k}^{'}-\gamma_k=\{t\in F^*:\gamma_k+t\in I_{\rho,k}^{'}\}$ of $I_{\rho,k}^{'}$ is a part of the isoenergetic surfaces I_ρ of $L(q(x))$. Put $I_\rho^{''}=\cup_{k=1}^{s}I_{\rho,k}^{''}$. If $t\in I_{\rho,k}^{''}\cap I_{\rho,m}^{''}$ for $k\neq m$, then $\gamma_k+t\in I_\rho^{'}\subset U_\varepsilon(S_\rho^{'})$ and $\gamma_m+t\in U_\varepsilon(S_\rho^{'})$, which contradict Lemma 2(b). So $I_\rho^{''}$ is union of the pairwise disjoint subsets $I_{\rho,k}^{''}$ for $k=1,2,...s$. Thus $\mu(I_\rho^{''})=\sum_k\mu(I_{\rho,k}^{''})=\sum_k\mu(I_{\rho,k}^{'})=\mu(I_\rho^{'})>(1-c_{10}\rho^{-\alpha}))\mu(B(\rho))\diamondsuit$

References

[1] B.E.J. Dahlberg, E. Trubuwits, A Remark on two Dimensional Periodic Potential, Comment. Math. Helvetica 57, 130-134, (1982).

[2] M.S.P. Eastham, "The Spectral Theory of Periodic Differential Equations," Scotting Academic Press, Edinburgh, 1973.

[3] J. Feldman, H. Knorrer, E. Trubowitz, The Perturbatively Stable Spectrum of the Periodic Schrödinger Operator, Invent. Math. 100 (1990), 259-300.

[4] J. Feldman, H. Knorrer, E. Trubowitz, The Perturbatively Unstable Spectrum of the Periodic Schrödinger Operator, Comment. Math. Helvetica 66 (1991), 557-579.

[5] L. Friedlanger, On the Spectrum of the Periodic Problem for the Schrödinger Operator, Communucation in Partial Differential Equations 15 (1990),1631-1647.

[6] B. Helffer, A. Mohamed, Asymptotics of the Density of States for the Schrodinger Operator with Periodic Potential, Duke Math.J. 92, 1-60 (1998).

[7] Yu. E. Karpeshina, Pertubation Theory Formulae for the Schrodinger Operator with a Non-smooth Periodic Potential, Math. USSR_Sb. 71 (1992), 101-123.

[8] Yu. E. Karpeshina, Pertubation Series for the Schrodinger Operator with a Periodic Potential near Planes of Diffraction, Communication in Analysis and Geometry 4 (1996), 3, 339-413.

[9] Yu. E. Karpeshina, Pertubation Theory for the Schrodinger Operator with a Periodic Potential, Lecture Notes in Math., 1663 (1997).

[10] L. Parnovski, A.V. Sobolev, On the Bethe-Sommerfeld Conjecture for the Polyharmonic Operator, Duke Math.J. 107, 209-238 (2001).

[11] M. Skriganov, Geometrical and Arithmetical Methods in the Spectral Theory of the Multidimensional Periodic Operators, Proc. Steklov Math. Inst. 171 (1984).

[12] M. Skriganov, The Spectrum Band Structure of the Three-dimensional Schrödinger Operator with Periodic Potential, Invent. Math. 80,107-121 (1985).

[13] O.A. Veliev, On the Spectrum of the Schrodinger Operator with Periodic Potential, Dokl. Akad. Nauk SSSR 268 (1983), no 6, 1289-1292.

[14] O.A. Veliev and S.A. Molchanov, Structure of the Spectrum of the Periodic Schrödinger Operator on a Euclidean torus, Funktsional. Anal. i Prilozhen 19 (1985), no 3, 86-87.

[15] O.A. Veliev, Asymptotic Formulas for the Eigenvalues of the Multidimensional Schrödinger Operator and Periodic Differential Operators, Preprint Inst. Phys.AN Azerb. SSR no 156, 1985.

[16] O.A. Veliev, Asymptotic Formulas for the Eigenvalues of the Periodic Schrödinger Operator and the Bethe-Sommerfeld Conjecture, Funktsional. Anal. i Prilozhen. 21(1987), no 2, 1-15.

[17] O.A. Veliev, The Spectrum of Multidimensional Periodic Operators, Teor. Funktsi. i Funktsional. Anal. i Prilozhen (49) (1988), 17-34.

[18] O.A. Veliev, Asymptotic Formulas for Block Function and Their Appl., J. Spectral Theory of Oper. and its Appl., Elm, Baku, 1989, 59-76.

[19] O.A. Veliev, Asymptotic Formulas for the Eigenvalues and Periodic Differential Operators, Doctor of Sciences Thesis, Razmadze Math. Inst., Tbilisi, 1989.