JACOBI MATRICES WITH RAPIDLY GROWING WEIGHTS HAVING ONLY DISCRETE SPECTRUM

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ABSTRACT. We establish sufficient conditions for self-adjointness on a class of unbounded Jacobi operators defined by matrices with main diagonal sequence of very slow growth and rapidly growing off-diagonal entries. With some additional assumptions, we also prove that these operators have only discrete spectrum.

1. INTRODUCTION

Let $\{b_n\}_{n\in\mathbb{N}}$ be a real sequence whose elements are given by

$$b_n = n^{\alpha}\left(1+\frac{c_n}{n}+h_n\right)\,,\quad \alpha>1\,,\quad 1+\frac{c_n}{n}+h_n\neq 0\quad \forall n\in\mathbb{N}\,, \tag{1}$$

where $\{c_n\}_{n\in\mathbb{N}}$ is bounded with $\liminf|c_n|>0$, and $\{h_n\}_{n\in\mathbb{N}}$ is such that $h_n=O(n^{-\gamma})$ when $n\to\infty$ for some $\gamma>1$.

Let $\{q_n\}_{n\in\mathbb{N}}$ be a real sequence that grows not faster than logarithmically, viz.,

$$q_n = O(\log n)\quad \text{as } n\to\infty\,. \tag{2}$$

In the Hilbert space $l_2(\mathbb{N})$, we define the Jacobi operator J as the minimal closed operator such that

$$(Je_n,e_n)=q_n\,,\quad (Je_n,e_{n+1})=(Je_{n+1},e_n)=b_n\,,\quad \forall n\in\mathbb{N}\,, \tag{3}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is the canonical basis in $l_2(\mathbb{N})$. Clearly, there are closed operators satisfying (3), thus J is correctly defined. This operator is

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unbounded and we have defined it in such a way that the semi-infinite Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} \tag{4}$$

is the matrix representation of J with respect to the basis $\{e_n\}_{n\in\mathbb{N}}$ (we refer to [2] for a discussion on matrix representation of unbounded symmetric operators).

For $\alpha > 1$ fixed, different sequences $\{c_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ define different concrete realizations of J. Throughout this paper, we use J to denote anyone of these realizations prescribed by (1)–(3).

Since the off-diagonal entries (weights) of (4) grow fast ($\alpha > 1$), J is not always self-adjoint. It is known, nevertheless, that certain conditions on the weights ensure self-adjointness when $\{q_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Indeed, the class of operators studied here is a generalization of an example suggested by Kostyuchenko and Mirzoev in [12] as an illustration of a Jacobi operator whose matrix representation does not satisfy the Carleman criterion [1] but nevertheless, under certain conditions, can be self-adjoint. Jacobi operators for which $\{c_n\}_{n\in\mathbb{N}}$ is even-periodic and $\{q_n\}_{n\in\mathbb{N}} \equiv \{h_n\}_{n\in\mathbb{N}} \equiv 0$ has been studied in [6, 14]. For these cases there are formulae in [6] for the asymptotic behavior of generalized eigenvectors and it is proven there, by means of the Subordinacy Theory [5, 10], that the spectrum of J is pure point in the self-adjoint case.

We carry out the spectral analysis of J on the basis of the asymptotic behavior of its generalized eigenvectors. These asymptotics are obtained by means of a Levinson-type theorem for discrete linear systems. We use transfer matrices to study the generalized eigenequation of J and arrive at a system that turns out to be asymptotically diagonal in the sense that it satisfies the assumptions of the Benzaid-Lutz theorem [3]. The fact that we can study the generalized eigenequation through a discrete system, for which the Benzaid-Lutz theorem yields the asymptotic behavior of the solutions, is a remarkable property of this class of operators.

We impose certain mild conditions over the average behavior of the sequence $\{c_n\}_{n\in\mathbb{N}}$ in order to ensure self-adjointness. These conditions are far less restrictive that those used in previous works and contain

them as special cases. Also, these conditions turn out to be sufficient for the absence of continuous spectrum. In passing, we note that in [6, 14] a much stronger assumption of periodicity of the sequence $\{c_n\}_{n\in\mathbb{N}}$ was required to gain spectral information from the asymptotic behavior of the generalized eigenvectors of operator J. As it will become apparent later, the sequences $\{q_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$, with the asymptotic behavior specified above, play no role concerning both self-adjointness and spectral properties of J. Unlike the methods used in [6, 14], our main results do not make use of Subordinacy Theory. Instead we mainly use general facts for Jacobi operators and Spectral Theory.

To the knowledge of the authors, the spectral analysis of Jacobi operators corresponding to matrices with rapidly growing weights has been treated in few occasions [6, 13, 14]. In contrast, literature on Jacobi matrices with slowly growing weights is more abundant (for instance, [4, 7, 8, 9, 17]). In particular, Jacobi operators with zero main diagonal and $b_n \sim n^\alpha$, with $1/2 < \alpha \leq 1$, has been studied in [7, 8, 9]. In these papers it is proven that, depending on the concrete realization, this kind of Jacobi operators may have either purely absolutely continuous spectrum or absolutely continuous spectrum with a gap where the spectrum is pure point.

This paper is organized as follows. In Section 2, we lay down the notation and some preparatory facts. In Section 3, we derive the asymptotic behavior of solutions of the discrete system associated to the generalized eigenequation for J. In Section 4, we establish sufficient conditions for J to be either self-adjoint (Theorem 4.1) or non self-adjoint (Theorem 4.2), and give examples. Finally, in Section 5, we characterize the spectrum (Corollary 5.3 and Theorem 5.5).

2. Preliminaries

In this section we introduce the notation and main concepts that will be used throughout this paper.

The elements of a scalar sequence will be denoted by Latin lowercase letters. For sequences of vectors and matrices we will use Greek lowercase letters and Latin capital letters, respectively. Besides the Hilbert space $l_2(\mathbb{N}) = l_2(\mathbb{N}, \mathbb{C})$, we shall consider the space $l_2(\mathbb{N}, \mathbb{C}^d)$ consisting of sequences of d-dimensional vectors $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \|\xi_n\|^2_{\mathbb{C}^d}$ is convergent, where $\|\cdot\|_{\mathbb{C}^d}$ denotes some norm in $\mathbb{C}^d$. Should no confusion arise, we will denote $\|\xi\|_{l_2(\mathbb{C}^d)}$ (for the norm in $l_2(\mathbb{N}, \mathbb{C}^d)$) simply as $\|\xi\|$. Keeping the notation used in the Introduction, the canonical basis in $l_2(\mathbb{N})$ is denoted by $\{e_n\}_{n\in\mathbb{N}}$, while $\{\delta^{(k)}\}_{k=1}^d$ will stand for the canonical basis in $\mathbb{C}^d$. Given a scalar sequence $\{f_n\}_{n\in\mathbb{N}}$, the standard asymptotic

notation $O(f_n)$ ($o(f_n)$) as $n \to \infty$, depending on the context, will be used to denote the n-th element of a sequence of either scalars, vectors, or matrices, whose norm behaves as $O(f_n)$ ($o(f_n)$) as $n \to \infty$. All products of square matrices will be taken in "chronological" order, i. e.,

$$\prod_{s=p}^{m} A_s = A_m A_{m-1} \dots A_p \,.$$

Our approach to the spectral analysis of J is based on the study of the asymptotics of solutions of the generalized eigenvalue equation

$$b_{n-1}x_{n-1}(\zeta) + q_n x_n(\zeta) + b_n x_{n+1}(\zeta) = \zeta x_n(\zeta)\,, \quad n > 1\,,\ \zeta \in \mathbb{C}\,. \tag{5}$$

Clearly, if a solution $x(\zeta) = \{x_n(\zeta)\}_{n\in\mathbb{N}}$ satisfies the "boundary condition"

$$q_1 x_1(\zeta) + b_1 x_2(\zeta) = \zeta x_1(\zeta) \tag{6}$$

and turns out to be in $\mathrm{Dom}(J)$, then $x(\zeta)$ is an eigenvector of J corresponding to the eigenvalue ζ. To simplify somewhat the notation, in what follows we will not indicate the dependency of x on ζ.

Equation (5) can be written as follows

$$\xi_{n+1} = B_n(\zeta)\xi_n\,, \qquad n > 1\,, \quad \zeta \in \mathbb{C}\,, \tag{7}$$

where $\xi_n = \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$ and $B_n(\zeta) = \begin{pmatrix} 0 & 1 \\ -\frac{b_{n-1}}{b_n} & \frac{\zeta - q_n}{b_n} \end{pmatrix}$ is the so-called transfer matrix. From (7) it follows that

$$\xi_n = \prod_{k=n_0}^{n-1} B_k(\zeta)\xi_{n_0}\,, \qquad n_0 \geq 2\,. \tag{8}$$

Since clearly we have $\|x\|^2 < \|\xi\|^2 \leq 2\|x\|^2$, the asymptotic behavior (in norm) of solutions of (7) governs the asymptotic behavior of those of (5) (and vice-versa). There are various methods for obtaining the asymptotic behavior of solutions of discrete linear systems. As already mentioned, the main tools are the so-called discrete Levinson-type theorems. For the purpose of the present work we use the Levinson-type theorem proven by Benzaid and Lutz in [3]. For the sake of completeness, we state this theorem below.

Definition 2.1. Fix $k \in \mathbb{N}$ such that $k \leq d$. A sequence $Q = \{Q_n\}_{n=n_0}^{\infty}$ ($n_0 \in \mathbb{N}$) of $d \times d$ diagonal complex matrices,

$$Q_n = \mathrm{diag}\{q_n^{(j)}\}_{j=1}^{d}\,,$$

is said to satisfy the Levinson condition for k, denoted $Q \in \mathcal{L}(k)$, if for every $j = 1, \dots, d$ there exists a natural number $N \geq n_0$ such that

$q_n^{(j)} \neq 0$, $\forall n \geq N$, and for some constant $M > 1$ each j falls into one and only one of two classes I_1 or I_2, where:

(a) $j \in I_1$ if

$$\frac{\left|\prod_{i=N}^{n} q_i^{(k)}\right|}{\left|\prod_{i=N}^{n} q_i^{(j)}\right|} \to \infty \quad \text{as } n \to \infty\,, \quad \text{and}$$

$$\frac{\left|\prod_{i=n_1}^{n_2} q_i^{(k)}\right|}{\left|\prod_{i=n_1}^{n_2} q_i^{(j)}\right|} > \frac{1}{M}\,, \quad \forall\, n_2\,, n_1 \text{ such that } n_2 \geq n_1 \geq N\,.$$

(b) $j \in I_2$ if

$$\frac{\left|\prod_{i=n_1}^{n_2} q_i^{(k)}\right|}{\left|\prod_{i=n_1}^{n_2} q_i^{(j)}\right|} < M\,, \quad \forall\, n_2\,, n_1 \text{ such that } n_2 \geq n_1 \geq N\,.$$

Theorem 2.2 (Benzaid-Lutz)**.** *Let $Q = \{Q_n\}_{n=n_0}^{\infty}$ be a sequence of $d \times d$ diagonal complex matrices,*

$$Q_n = \operatorname{diag}\{q_n^{(k)}\}_{k=1}^{d}\,.$$

Consider also another sequence $R = \{R_n\}_{n=n_0}^{\infty}$ ($n_0 \in \mathbb{N}$) of $d \times d$ complex matrices. Suppose that the following conditions hold:

i. The sequence $Q \in \mathcal{L}(k)$, for all $k = 1, \ldots, d$,

ii. $\displaystyle\sum_{n=n_0}^{\infty} \frac{\|R_n\|}{|q_n^{(k)}|} < \infty$,

iii. $\det(Q_n + R_n) \neq 0$ *for all* $n \geq n_0$.

Then there exists a basis $\varphi^{(k)}$ ($k = 1 \ldots, d$) in the space of solutions of the system

$$\varphi_{n+1} = (Q_n + R_n)\varphi_n\,, \qquad n \geq n_0\,,$$

such that

$$\left\| \frac{\varphi_n^{(k)}}{\prod_{i=n_1}^{n-1} q_i^{(k)}} - \delta^{(k)} \right\| \to 0, \qquad \text{as } n \to \infty\,, \quad \text{for } k = 1, \ldots, d\,.$$

3. Asymptotic behavior of solutions

Let us define the matrices

$$A_m(\zeta) = B_{2m+1}(\zeta) B_{2m}(\zeta)\,, \quad \forall m \in \mathbb{N}\,.$$

Clearly, system (7) is equivalent to

$$\eta_{m+1} = A_m(\zeta)\eta_m\,, \quad \forall m \in \mathbb{N}\,, \tag{9}$$

where $\eta_m = \xi_{2m}$. A straightforward computation shows that

$$B_{n+1}(\zeta)B_n(\zeta) = -I + \begin{pmatrix} \frac{c_n - c_{n-1} + \alpha}{n} & 0 \\ 0 & \frac{c_{n+1} - c_n + \alpha}{n} \end{pmatrix} + O(n^{-1-\epsilon})$$

as $n \to \infty$, where I denotes the identity matrix and $\epsilon > 0$. We notice that the term $O(n^{-1-\epsilon})$ contains all the dependencies on the parameter ζ and sequences $\{q_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$. Up to the same order in n, we may write

$$B_{n+1}(\zeta)B_n(\zeta) = -\begin{pmatrix} \exp\frac{c_{n-1} - c_n - \alpha}{n} & 0 \\ 0 & \exp\frac{c_n - c_{n+1} - \alpha}{n} \end{pmatrix} \left[I + O(n^{-1-\epsilon})\right],$$

where again the dependency on ζ is contained exclusively in the term $O(n^{-1-\epsilon})$. Thus, $A_m(\zeta)$ admits the decomposition

$$A_m(\zeta) = G_m + R_m(\zeta)\,, \tag{10}$$

where, for notational convenience, we define

$$G_m := -\begin{pmatrix} \exp g_m^{(1)} & 0 \\ 0 & \exp g_m^{(2)} \end{pmatrix}$$

with

$$g_m^{(1)} := \frac{c_{2m-1} - c_{2m} - \alpha}{2m} \quad \text{and} \quad g_m^{(2)} := \frac{c_{2m} - c_{2m+1} - \alpha}{2m}\,,$$

and the residual term is given by

$$R_m(\zeta) := G_m\, O(m^{-1-\epsilon})\,.$$

The r. h. s. of this last identity must be understood as the product of G_m times a matrix that depends on ζ and whose norm behaves as specified by the last factor as $m \to \infty$.

As mentioned earlier, we want to apply Theorem 2.2 to system (9) in order to obtain the asymptotic behavior of its solutions. In view of (10), we must be certain that the sequence $\{G_m\}_{m\in\mathbb{N}}$ satisfies the Levinson condition stated in Definition 2.1. Since this is not *a priori* known, we will always assume it true from now on:

(H.1) $\{G_m\}_{m\in\mathbb{N}} \in \mathcal{L}(k)$ for $k = 1, 2$.

Indeed, there are examples of Jacobi matrices of the class discussed in this article whose associated sequences $\{G_m\}_{m\in\mathbb{N}}$ do not satisfy (H.1). However, the following lemma gives criteria for establishing when this hypothesis is fulfilled.

Lemma 3.1. *Consider an arbitrary monotone increasing sequence of natural numbers $\{n_i\}_{i\in\mathbb{N}}$ such that the sequence $\{\frac{n_{i+1}}{n_i}\}_{i\in\mathbb{N}}$ is bounded. Fix $k \in \{1,2\}$. Then $\{G_m\}_{m\in\mathbb{N}}$ is in $\mathcal{L}(k)$ if one of the following cases is met:*

i. There exist constants C_1 and C_2 such that, for all $i \in \mathbb{N}$,

$$C_1 < \sum_{s=1}^{n_i} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} < C_2\,.$$

ii. There exists $i_0 \in \mathbb{N}$ such that

$$\left\{ (-1)^k \sum_{s=1}^{n_i} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} \right\}_{i=i_0}^{\infty}$$

is a non-decreasing sequence.

iii. There exists $i_0 \in \mathbb{N}$ such that

$$\left\{ (-1)^k \sum_{s=1}^{n_i} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} \right\}_{i=i_0}^{\infty}$$

is an unbounded, monotone decreasing sequence.

Proof. We will prove the assertion for $k = 1$. Thus, we shall verify that both $j = 1$ and $j = 2$ fall on either one of the classes I_1 and I_2 mentioned in Definition 2.1. It is obvious that $j = 1$ belongs to I_2. In what follows we will show that $j = 2$ is also in I_2 when either (i) or (ii) happens, whereas $j = 2$ is in I_1 when (iii) occurs. Since the proof for $k = 2$ es analogous, it will be omitted.

For any sequence $\{n_i\}_{i\in\mathbb{N}}$ with $\{\frac{n_{i+1}}{n_i}\}_{i\in\mathbb{N}}$ bounded, there is a constant C such that, for any $m > n_1$,

$$\sum_{s=1}^{m} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} < \sum_{s=1}^{n_i} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} + C\,, \tag{11}$$

with $n_i < m \le n_{i+1}$. This follows from

$$\begin{aligned}\sum_{s=1}^{m} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} = &\sum_{s=1}^{n_i} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} \\ &+ \sum_{s=1+n_i}^{m} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s}\,,\end{aligned}$$

together with the fact that, due to the boundedness of the sequence $\{c_n\}_{n\in\mathbb{N}}$, there are positive constants C' and C such that

$$\sum_{s=1+n_i}^{m} \frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} < C' \log\left(\frac{m}{1+n_i}\right) < C' \log\left(\frac{n_{i+1}}{n_i}\right) < C\,.$$

Analogously, there exists another constant $\widetilde{C}$ for which

$$\sum_{s=1}^{m} \frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} > \sum_{s=1}^{n_i} \frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} + \widetilde{C}\,, \tag{12}$$

for any $m > n_1$, where $n_i < m \leq n_{i+1}$.

From (11) and (12) it follows that, for any pair of natural numbers $m_1 < m_2$ with $n_1 < m_1$,

$$\begin{aligned}\sum_{s=m_1}^{m_2} \frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} &= \left(\sum_{s=1}^{m_2}-\sum_{s=1}^{m_1}\right)\frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} \\ &< \left(\sum_{s=1}^{n_{i_2}}-\sum_{s=1}^{n_{i_1}}\right)\frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} + C - \widetilde{C}\,,\end{aligned} \tag{13}$$

where $n_{i_1} < m_1 \leq n_{i_1+1}$, $n_{i_2} < m_2 \leq n_{i_2+1}$ and moreover $n_{i_1} \leq n_{i_2}$. Now, suppose that (i) holds, then

$$\left(\sum_{s=1}^{n_{i_2}}-\sum_{s=1}^{n_{i_1}}\right)\frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} < C_2 - C_1\,,$$

which together with (13) imply $j = 2 \in I_2$. Similarly, if one assumes (ii), the same inequality (13) yields $j = 2 \in I_2$.

Let us now assume (iii). Again, on the basis of (11) and (12), it is easy to verify that, for any pair of natural numbers $m_1 < m_2$ with $n_1 < m_1$,

$$\sum_{s=m_1}^{m_2} \frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} > \left(\sum_{s=1}^{n_{i_2}}-\sum_{s=1}^{n_{i_1}}\right)\frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s} + \widetilde{C} - C\,.$$

From (iii) and this inequality it follows that, for any pair of natural numbers $m_1 < m_2$, the expression

$$\sum_{s=m_1}^{m_2} \frac{c_{2s-1}-2c_{2s}+c_{2s+1}}{2s}$$

is bounded from below. It remains to be shown that (iii) also implies that for any real number K one can find an $N \in \mathbb{N}$ such that for all $m > N$

$$\sum_{s=1}^{m} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} > K\,.$$

But this follows from the fact that, due to (iii), for any given K we can choose i_0 such that

$$\sum_{s=1}^{n_{i_0}} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} > K - \widetilde{C}\,.$$

Then, for any $m > n_{i_0}$,

$$\sum_{s=1}^{m} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} > \sum_{s=1}^{n_{i_0}} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} + \widetilde{C} > K\,.$$

This completes the proof. □

Lemma 3.2. *Assume (H.1). Then system (9) satisfies the assumptions of Theorem 2.2. That is, there exists a basis $\eta^{(k)}$ $(k = 1, 2)$ in the space of solutions of (9) such that, for $k = 1$ and $k = 2$,*

$$\left\| \eta_m^{(k)} / \prod_{s=n_1}^{m-1} (-1) \exp g_s^{(k)} - \delta^{(k)} \right\| \to 0 \tag{14}$$

as $m \to \infty$.

Proof. We only must verify the fulfillment of conditions (ii) and (iii) of Theorem 2.2.

We have

$$\frac{v_1}{2m} \le g_m^{(k)} \le \frac{v_2}{2m}$$

for $k = 1, 2$ and some constants v_1 and v_2. It follows that

$$\frac{\|R_m(\zeta)\|}{\exp g_m^{(k)}} \le r_1 m^{-1-\epsilon} \frac{\|G_m\|}{\exp g_m^{(k)}} \le r_2 m^{-1-\epsilon} \exp \frac{v_2 - v_1}{2m}$$

for some positive constants r_1 and r_2. This implies that the l. h. s. of the inequality above is summable, hence condition (ii) of Theorem 2.2 holds.

Finally, since the determinant of each transfer matrix is different from zero, the same is true for the determinant of each A_m (which equals $G_m + R_m$). Thus, condition (iii) of Theorem 2.2 also holds. □

Up to a normalization constant, statement (14) is equivalent to saying that

$$\eta_{m+1}^{(k)} = (-1)^m \exp\left(\sum_{s=1}^{m} g_s^{(k)}\right)\left[\delta^{(k)} + \tau_m^{(k)}\right] , \tag{15}$$

where $\tau_m^{(k)}$ is $o(1)$ as $m \to \infty$. We will use this expression in the sequel.

Solutions of (9) obviously depend on ζ and the specific form of the sequences $\{q_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$. However, equation (14) (or (15)) shows that the main asymptotic term does not.

Remark 1. We have the following consequence of Lemma 3.2: Let $\{m_j\}_{j\in\mathbb{N}}$ be a monotone increasing sequence of natural numbers. Recalling that $\eta_m = \xi_{2m}$, from (15) we have

$$\xi_{2m_j+2}^{(k)} = (-1)^{m_j} \exp\left(\sum_{s=1}^{m_j} g_s^{(k)}\right)\left[\delta^{(k)} + \tau_{m_j}^{(k)}\right] .$$

Moreover,

$$\xi_{2m_j+2p+2}^{(k)} = (-1)^{m_j+p} \exp\left(\sum_{s=1}^{m_j+p} g_s^{(k)}\right)\left[\delta^{(k)} + \tau_{m_j+p}^{(k)}\right]$$

for $p \in \mathbb{N}$. Thus, by splitting the exponential factor and working out residual terms, we obtain

$$\xi_{2m_j+2p+2}^{(k)} = (-1)^{p} \exp\left(\sum_{s=m_j+1}^{m_j+p} g_s^{(k)}\right)\left[\xi_{2m_j+2}^{(k)} + \phi_{j,p}^{(k)}\right] , \tag{16}$$

where the norm of $\phi_{j,p}^{(k)}$ equals the product of the norm of $\xi_{2m_j+2}^{(k)}$ times a function $o(1)$ as $j \to \infty$, uniformly in p.

We close this section noticing the fact that, in order to obtain the asymptotic behavior of solutions of (9), it was sufficient to assume the conditions on $\{c_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$, and $\{q_n\}_{n\in\mathbb{N}}$ given in the Introduction, with the addition of hypothesis (H.1). However, without further assumptions on $\{c_n\}_{n\in\mathbb{N}}$, these asymptotics yields no clue about the decaying behavior of these solutions. This information is necessary to determine whether the operator is self-adjoint or not and, in the latter case, to classify the spectrum (by using, for instance, Subordinacy Theory). Theses issues will be attended, in the following sections, by requiring certain regularity on average of $\{c_n\}_{n\in\mathbb{N}}$ over sequences $\{m_j\}_{j\in\mathbb{N}}$. This will give rise to nice asymptotic behavior for the norm of $\xi_{2m_j+2}^{(k)}$ from which, the asymptotic behavior of the norm of $\xi_{2m+2}^{(k)}$ will be interpolated by resorting to (16).

4. Self-adjointness

It is well known that, for Jacobi matrices, we have the discrete analogue of Weyl's limit point–limit circle theory for self-adjoint extensions of differential operators. In particular, it is true that if two linearly independent solutions of (5) are in $l_2(\mathbb{N})$, then J is not self-adjoint (limit circle case). If, however, one solution of (5) is not in $l_2(\mathbb{N})$ then $J = J^*$ (limit point case) [1, 15].

Next, we establish sufficient conditions for J to be self-adjoint.

Theorem 4.1. *Define* $w^{(1)} = \min\{0, \inf_s\{c_{2s-1} - c_{2s} - \alpha\}\}$ *and* $w^{(2)} = \min\{0, \inf_s\{c_{2s} - c_{2s+1} - \alpha\}\}$*, and assume that (H.1) is satisfied. Fix the value of* k*. Consider a monotone increasing sequence* $\{m_j\}_{j\in\mathbb{N}}$ *of natural numbers that obeys*

$$\sum_{j=1}^{\infty} \frac{m_{j+1} - m_j}{m_j} \left(\frac{m_{j+1}}{m_j}\right)^{w^{(k)}} = \infty\,. \tag{H.2}$$

Suppose now that, for this sequence, the following holds:

$$\sum_{s=1}^{m_j} g_s^{(k)} \geq -\frac{1}{2}\log(m_j) + O(1), \quad \text{when } j \to \infty\,. \tag{17}$$

Then, there exists a solution of (5) that is not in $l_2(\mathbb{N})$ *and, therefore,* $J = J^*$*.*

Proof. We first note that (H.2) implies that either

$$\sum_{j=1}^{\infty} \frac{m_{j+1} - m_j - 1}{m_j} \left(\frac{m_{j+1} - 1}{m_j + 1}\right)^{w^{(k)}} = \infty \quad \text{or} \quad \sum_{j=1}^{\infty} \frac{1}{m_j} = \infty\,.$$

We now prove the assertion under the assumption that $k = 1$. The case $k = 2$ follows analogously. We note that (17) and (15) imply

$$\left\|\xi^{(1)}_{2m_j+2}\right\|^2 \geq r_1 m_j^{-1}\,, \quad \forall j \geq j_0, \tag{18}$$

where r_1 is some positive constant and j_0 is sufficiently large. Moreover, from (16) we obtain

$$\begin{aligned}\left\|\xi^{(1)}_{2m_j+2p+2}\right\|^2 &\geq r_2 \exp\left(2\sum_{s=m_j+1}^{m_j+p} g_s^{(1)}\right) \left\|\xi^{(1)}_{2m_j+2}\right\|^2 \\ &\geq r_3 \left(\frac{m_j + p}{m_j + 1}\right)^{w^{(1)}} \left\|\xi^{(1)}_{2m_j+2}\right\|^2\,, \end{aligned} \tag{19}$$

for some positive constants r_2, r_3 and $0 < p < m_{j+1} - m_j$. Inequality (19) follows from the fact that

$$\sum_{s=m_j+1}^{m_j+p} g_s^{(1)} \geq \frac{w^{(1)}}{2} \sum_{s=m_j+1}^{m_j+p} \frac{1}{s} = \frac{w^{(1)}}{2} \log\left(\frac{m_j+p}{m_j+1}\right) + o(1)\,,$$

as $j \to \infty$.

Therefore,

$$\begin{aligned}
\left\|\xi^{(1)}\right\|^2 &\geq \sum_{m=0}^{\infty} \left\|\xi^{(1)}_{2m+2}\right\|^2 \\
&\geq \sum_{j=j_0}^{\infty} \left(\left\|\xi^{(1)}_{2m_j+2}\right\|^2 + \sum_{p=1}^{m_{j+1}-m_j-1} \left\|\xi^{(1)}_{2m_j+2p+2}\right\|^2 \right) \\
&\geq r_1 \sum_{j=j_0}^{\infty} \frac{1}{m_j} + r_1 r_3 \sum_{j=j_0}^{\infty} \frac{m_{j+1}-m_j-1}{m_j} \left(\frac{m_{j+1}-1}{m_j+1}\right)^{w^{(1)}}\,,
\end{aligned}$$

where the last inequality follows from (18) and (19). By the comment at the beginning of the proof, we conclude that $\xi^{(1)}$ is not in $l_2(\mathbb{N}, \mathbb{C}^2)$ if (H.2) is fulfilled. Since $\|\xi\| \asymp \|x\|$, it follows that $x^{(1)}$ does not belong to $l_2(\mathbb{N})$ (so J is in the limit point case). □

Remark 2. For $w^{(k)} = 0$, condition (H.2) seems to be met for any monotone increasing sequence $\{m_j\}_{j\in\mathbb{N}}$. In any case, we could not find counterexamples. However, this condition is more stringent for $w^{(k)} < 0$, although it is satisfied for sequences of fairly fast growth, for instance, $m_j \sim \exp j$ (for any $w^{(k)} < 0$) and also $m_j \sim j^j$ (for $-2 \leq w^{(k)} < 0$). Clearly, sequences of polynomial growth, like $m_j \sim j^l$ with $l \geq 1$, also satisfy (H.2) for any $w^{(k)} \leq 0$.

Remark 3. Theorem 4.1 applies quite easily to the particular case where the sequence $\{c_n\}_{n\in\mathbb{N}}$ is even-periodic (say, of minimal period $2L$). In this case $\left|\sum_{n=1}^{2L}(-1)^n c_n\right| \geq (\alpha-1)L$ implies (17) and therefore self-adjointness. As already mentioned, this model, with $\{q_n\}_{n\in\mathbb{N}} \equiv \{h_n\}_{n\in\mathbb{N}} \equiv 0$, was discussed in [6, 14].

Example 1. As a more involved example of application of Theorem 4.1, we now construct a self-adjoint operator J with no periodicity in $\{c_n\}_{n\in\mathbb{N}}$. Let the sequence $\{c_n\}_{n\in\mathbb{N}}$ be given by c_1, c_2, and c_3 as

follows:

$$\underbrace{\underbrace{c_1\,c_2\,c_1\,c_2\cdots c_1\,c_2\,c_1\,c_2}_{2p_1}\,c_2\,c_3\,c_2\,c_3\cdots c_2\,c_3\,c_2\,c_3}_{2p_2}\,c_1\,c_2\,c_1\,c_2\cdots \tag{20}$$

(we pray allowance for the abuse of notation), where $\{p_n\}_{n\in\mathbb{Z}^+}$ is a sequence with polynomial growth of the form $p_n = Cn^a$, with $a \in \mathbb{N}\setminus\{1\}$ and $C \in \mathbb{N}$. Define $m_n = p_{2n}$; this sequence satisfies (H.2).

Notice that the sequences $\{m_n\}_{n\in\mathbb{Z}^+}$ and $\{c_n\}_{n\in\mathbb{N}}$ are such that

$$\sum_{s=m_{n-1}+1}^{m_n}\frac{c_{2s-1}-c_{2s}}{s} = \sum_{s=p_{2n-2}+1}^{p_{2n-1}}\frac{c_1-c_2}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}}\frac{c_2-c_3}{s}.$$

Furthermore,

$$\begin{aligned}
&\sum_{s=p_{2n-2}+1}^{p_{2n-1}}\frac{c_1-c_2}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}}\frac{c_2-c_3}{s}\\
&\quad=\frac{1}{2}\Bigg(\sum_{s=p_{2n-2}+1}^{p_{2n-1}}\frac{c_1-c_2}{s} + \sum_{s=p_{2n-2}+1}^{p_{2n-1}}\frac{c_2-c_3}{s+p_{2n-1}-p_{2n-2}}\\
&\qquad+\sum_{s=2p_{2n-1}-p_{2n-2}+1}^{p_{2n}}\frac{c_2-c_3}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}}\frac{c_1-c_2}{s-p_{2n-1}+p_{2n-2}}\\
&\qquad-\sum_{s=p_{2n-1}+1}^{p_{2n}-p_{2n-1}+p_{2n-2}}\frac{c_1-c_2}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}}\frac{c_2-c_3}{s}\Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{s=m_{n-1}+1}^{m_n}\frac{c_{2s-1}-c_{2s}}{s}\\
&\quad=\frac{c_1-c_3}{2}\sum_{s=m_{n-1}+1}^{m_n}\frac{1}{s}\\
&\qquad+\frac{c_2-c_3}{2}\left(\sum_{s=p_{2n-2}+1}^{p_{2n-1}}\frac{1}{s^2}\left(\frac{p_{2n-2}-p_{2n-1}}{1+\frac{p_{2n-1}-p_{2n-2}}{s}}\right)+\sum_{s=2p_{2n-1}-p_{2n-2}+1}^{p_{2n}}\frac{1}{s}\right)\\
&\qquad+\frac{c_1-c_2}{2}\left(\sum_{s=p_{2n-1}+1}^{p_{2n}}\frac{1}{s^2}\left(\frac{p_{2n-1}-p_{2n-2}}{1+\frac{p_{2n-2}-p_{2n-1}}{s}}\right)+\sum_{s=p_{2n-1}+1}^{p_{2n}-p_{2n-1}+p_{2n+2}}\frac{1}{s}\right).
\end{aligned} \tag{21}$$

Now, as a consequence of the polynomial growth of p_n, we have

$$(p_{2n-1} - p_{2n-2}) \sum_{s=m_{n-1}+1}^{m_n} \frac{1}{s^2} = O(n^{-2})$$

and

$$\sum_{s=2p_{2n-1}-p_{2n-2}+1}^{p_{2n}} \frac{1}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}-p_{2n-1}+p_{2n+2}} \frac{1}{s} = O(n^{-2})$$

as $n \to \infty$. Taking into account these asymptotic formulae and (21), one can write

$$\begin{aligned} \sum_{s=m_{n-1}+1}^{m_n} g_s^{(1)} &= \frac{1}{2} \sum_{s=m_{n-1}+1}^{m_n} \frac{c_{2s-1} - c_{2s} - \alpha}{s} \\ &= \frac{c_1 - c_3 - \alpha}{2} \sum_{s=m_{n-1}+1}^{m_n} \frac{1}{s} + O(n^{-2}) \,, \end{aligned} \tag{22}$$

as $n \to \infty$. Analogously,

$$\sum_{s=m_{n-1}+1}^{m_n} g_s^{(2)} = \frac{c_3 - c_1 - \alpha}{2} \sum_{s=m_{n-1}+1}^{m_n} \frac{1}{s} + O(n^{-2}) \,. \tag{23}$$

We have deduced (23) by switching over c_1 and c_3 in all the calculations that yield (22). From (22) and (23) we conclude that Lemma 3.1 holds true, then so does Theorem 4.1 whenever (17) is fulfilled.

On the basis of (22) we obtain

$$\sum_{1}^{m_j} g_s^{(1)} = \frac{c_1 - c_3 - 2\alpha}{4} \log(m_j) + O(1) \,, \qquad \text{as } m_j \to \infty \,.$$

Making this complies with (17) yields that if

$$c_1 - c_3 \geq 2(\alpha - 1) \,, \tag{24}$$

then $J = J^*$. The same argument applied to (23) implies that J is self-adjoint if we have (24) with c_1 and c_3 interchanged. Thus, the condition

$$|c_1 - c_3| \geq 2(\alpha - 1) \,, \tag{25}$$

upon c_1 and c_3 is sufficient for self-adjointness. We note the fact that the value of c_2 plays no role in this example. The computation done above shows also that this construction may be generalized to the cases where $\{p_j\}_{j \in \mathbb{Z}^+}$ grows faster than polynomially, with possibly different conditions imposed upon c_1, c_2 and c_3 to ensure self-adjointness.

We now establish sufficient conditions for J to be non-selfadjoint.

Theorem 4.2. *Define v as the largest among 0, $\sup_s\{c_{2s} - c_{2s+1} - \alpha\}$ and $\sup_s\{c_{2s-1} - c_{2s} - \alpha\}$, and assume that (H.1) holds. Consider a monotone increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ that satisfies*

$$\sum_{j=1}^{\infty} \frac{m_{j+1} - m_j}{m_j^{\beta}} \left(\frac{m_{j+1}}{m_j}\right)^{v} < \infty\,, \tag{H.3}$$

for some $\beta > 1$. Now suppose that the inequality

$$\sum_{s=1}^{m_j} g_s^{(k)} \leq -\frac{\beta}{2}\log(m_j) + O(1) \quad \textit{when } j \to \infty, \tag{26}$$

holds for both $k = 1$ and $k = 2$. Then all solutions of (5) are in $l_2(\mathbb{N})$ and, therefore, $J \neq J^*$.

Proof. We note first that (16) and (26) imply

$$\left\|\xi^{(k)}_{2+2m_j}\right\|^2 < r_0 m_j^{-\beta}\,, \quad \forall j \geq j_1,$$

where r_0 is some positive constant. Without loss we may assume $j_1 = 1$ to avoid unnecessary complications. Also, (16) implies

$$\begin{aligned}\left\|\xi^{(k)}_{2m_j+2p+2}\right\|^2 &\leq r_1 \exp\left(2\sum_{s=m_j+1}^{m_j+p} g_s^{(k)}\right)\left\|\xi^{(k)}_{2m_j+2}\right\|^2 \\ &\leq r_2\left(\frac{m_j+p}{m_j+1}\right)^{v}\left\|\xi^{(k)}_{2m_j+2}\right\|^2,\end{aligned}$$

for $0 < p < m_{j+1} - m_j$ and some positive constants r_1 and r_2, where the last inequality follows from an argument analogous to the one yielding (19). Finally, we note that the transfer matrices $B_n(\zeta)$ are uniformly norm-bounded with respect to the index n. Let $r_3 > 0$ denote the square of such a bound.

Now, bearing these facts in mind, we have

$$\begin{aligned}\left\|\xi^{(k)}\right\|^2 &\leq r_3 \sum_{m=m_1}^{\infty} \left\|\xi^{(k)}_{2m+2}\right\|^2 \\ &= r_3 \sum_{j=1}^{\infty} \sum_{p=0}^{m_{j+1}-m_j-1} \left\|\xi^{(k)}_{2m_j+2p+2}\right\|^2 \\ &\leq r_3 \sum_{j=1}^{\infty} \left\|\xi^{(k)}_{2m_j+2}\right\|^2 \left[1 + r_2 \sum_{j=1}^{m_{j+1}-m_j-1} \left(\frac{m_j+p}{m_j+1}\right)^v\right] \\ &\leq r_1 r_3 \sum_{j=1}^{\infty} \frac{1}{m_j^\beta} + r_1 r_2 r_3 \sum_{j=1}^{\infty} \frac{1}{m_j^\beta} \sum_{p=1}^{m_{j+1}-m_j-1} \left(\frac{m_j+p}{m_j+1}\right)^v .\end{aligned}$$

The first term in the last inequality is clearly convergent, since $m_j \geq j$ and $\beta > 1$. The second term is also convergent since (H.3) is fulfilled.

Thus, we have proved that $\xi^{(1)}$ and $\xi^{(2)}$ belong to $l_2(\mathbb{N}, \mathbb{C}^2)$. It follows (by the argumentation given in the proof of Theorem 4.1) that $x^{(1)}$ and $x^{(2)}$ belong to $l_2(\mathbb{N})$. Moreover, they are linearly independent as a consequence of the linear independency of $\xi^{(1)}$ and $\xi^{(2)}$. The lack of self-adjointness now follows immediately. □

Remark 4. Theorem 4.2 does not allow to obtain the converse of Theorem 4.1.

Remark 5. A fairly large set of sequences $\{m_j\}_{j\in\mathbb{N}}$ satisfies condition (H.3), ranging from those of linear growth, i. e. $m_j \sim j$, up to those of very fast growth like $m_j \sim j^j$.

As we have mentioned previously, Jacobi operators of the form proposed in the present work has been already studied for the case of even-periodic sequences $\{c_n\}_{n\in\mathbb{N}}$. No results are known (to the authors anyway) concerning the odd-periodic case, although it has been suggested that, in this case, these operators would not be self-adjoint [14]. Theorem 4.2 allows us to prove that assertion easily.

Corollary 4.3. *Let $\{c_n\}_{n\in\mathbb{N}}$ be an odd-periodic sequence. Then the operator J, given by the matrix representation (4) with entries defined by (1) and (2), is not self-adjoint.*

Proof. Let T be the period of the sequence. Define $m_j = jT$ for $j \in \mathbb{N}$. Clearly, $\{m_j\}_{j\in\mathbb{N}}$ satisfies (H.3) for any $\beta > 1$ and $v \geq 0$. Thus, it only remains to verify (H.1) and (26) for $k = 1, 2$.

We have,

$$\begin{aligned}\sum_{s=1}^{m_j} g_s^{(1)} &= \frac{1}{2}\sum_{s=1}^{m_j}\frac{c_{2s-1}-c_{2s}-\alpha}{s}\\ &= \frac{1}{2}\sum_{n=1}^{j}\sum_{s=m_{n-1}+1}^{m_n}\frac{c_{2s-1}-c_{2s}}{s} - \frac{\alpha}{2}\sum_{s=1}^{m_j}\frac{1}{s}\\ &= \frac{1}{2}\sum_{n=1}^{j}\sum_{s=1}^{T}\frac{c_{2s-1}-c_{2s}}{s+(j-1)T} - \frac{\alpha}{2}\log(m_j)+O(1)\,, \qquad (27)\end{aligned}$$

where the first term in the last equality follows from the periodicity of $\{c_n\}_{n\in\mathbb{N}}$. Now, for $j \geq 2$,

$$\begin{aligned}\sum_{s=1}^{T}\frac{c_{2s-1}-c_{2s}}{s+(j-1)T} &= \sum_{s=1}^{T}\frac{c_{2s-1}-c_{2s}}{(j-1)T}\\ &\quad + \sum_{s=1}^{T}(c_{2s-1}-c_{2s})\left[\frac{1}{s+(j-1)T}-\frac{1}{(j-1)T}\right].\end{aligned}$$

The first term above equals zero because of the odd-periodicity of the sequence, while the second term is $O(j^{-2})$. Therefore, the first term in (27) is also $O(1)$ thus yielding the expected inequality (with $\beta=\alpha$) for $\sum_{s=1}^{h_m} g_s^{(1)}$. A similar computation shows that also $\sum_{s=1}^{h_m} g_s^{(2)}$ fulfills the same inequality. Finally, it is clear that case (i) of Lemma 3.1 holds. □

The proof of Corollary 4.3 tells us that an odd-periodic sequence $\{c_n\}_{n\in\mathbb{N}}$ produces a non-self-adjoint operator because one can find a suitable sequence $\{m_j\}_{j\in\mathbb{N}}$ for which, the contribution of $c_{2s-1}-c_{2s}$ to the l. h. s. of (26) is nearly canceled (and the same occurs for $c_{2s}-c_{2s+1}$). This idea may be used to construct non-selfadjoint Jacobi operators out of certain non-periodic sequences. As a matter of fact, the following example is defined essentially in the same way as the one discussed previously, having exactly one modification.

Example 2. Consider a sequence $\{c_n\}_{n\in\mathbb{N}}$ defined by two given numbers c_1 and c_2 arranged as follows:

$$\underbrace{\underbrace{c_1\,c_2\,c_1\,c_2\cdots c_1\,c_2\,c_1\,c_2}_{2p_1}\,c_2\,c_1\,c_2\,c_1\cdots c_2\,c_1\,c_2\,c_1\,c_1\,c_2\,c_1\,c_2\cdots}_{2p_2}\,, \qquad (28)$$

where $\{p_j\}_{j\in\mathbb{Z}^+}$ is a sequence with polynomial growth of the form $p_j = Cj^a$, with $a \in \mathbb{N}\setminus\{1\}$ and $C \in \mathbb{N}$. That is, we use the sequence defined

by (20) with $c_3 = c_1$. Alternatively, we may think of this sequence as one obtained from a two-periodic sequence by inserting dislocations at sites $2p_j + 1$. We claim that the Jacobi operator defined by (28) is not self-adjoint. To prove it, we first define $m_j = p_{2j}$. This sequence satisfies (H.3) for any $\beta > 1$ as a straightforward computation shows.

We can use some of the computations of Example 1. Thus, from (22) and (23) it follows straightforwardly that, for $k = 1, 2$,

$$\sum_{s=m_{n-1}+1}^{m_n} g_s^{(k)} = -\frac{\alpha}{2} \sum_{s=m_{n-1}+1}^{m_n} \frac{1}{s} + O(n^{-2}) \,, \qquad \text{as } m_j \to \infty \,.$$

This yields

$$\sum_{s=1}^{m_j} g_s^{(k)} = -\frac{\alpha}{2} \log(m_j) + O(1)$$

for $k = 1, 2$. From this equation it follows that this example satisfies case (i) of Lemma 3.1 so (H.1) is satisfied. Also, (26) holds true. Hence, we have verified that this example fulfills all the hypotheses of Theorem 4.2.

We note that the case $p_m \sim m$ can not be included here since the summability of

$$\sum_{j=1}^{\infty} \sum_{s=m_{j-1}+1}^{m_j} \frac{c_{2s-1} - c_{2s}}{s}$$

is no longer ensured. This is to be expected because, for $p_m \sim m$, the sequence $\{c_n\}_{n\in\mathbb{N}}$ is again even-periodic so the associated Jacobi operator may be self-adjoint (see Remark 3).

Other examples of non-self-adjoint Jacobi operators (of the class discussed in this work) could be devised along similar ideas. We think that the example given above is of some interest because it is derived from an operator defined by an even-periodic sequence and therefore self-adjoint (provided that $|c_1 - c_2| \geq \alpha - 1$, although this condition plays no role in our example) [6, 14].

5. Discreteness of spectrum

In this section we show that, if the assumptions of Theorem 4.1 are fulfilled and $\{m_j\}_{j\in\mathbb{N}}$ obeys certain further conditions, then J has only discrete spectrum.

We first prove an easy consequence of inequality (17).

Lemma 5.1. *If (17) is true for $k = 1$, then*

$$\sum_{s=1}^{m_j} g_s^{(2)} < -\frac{\alpha}{2}\log(m_j) + O(1) \quad \textit{when } j \to \infty\,. \tag{29}$$

An analogous assertion holds if the roles of $k = 1$ and $k = 2$ are interchanged.

Proof. We have

$$\sum_{s=1}^{m_j} g_s^{(2)} = -\sum_{s=1}^{m_j} g_s^{(1)} + \sum_{s=1}^{m_j} \frac{c_{2s-1} - c_{2s+1}}{2s} - \alpha \sum_{s=1}^{m_j} \frac{1}{s}\,. \tag{30}$$

Furthermore,

$$\sum_{s=1}^{m_j} \frac{c_{2s-1} - c_{2s+1}}{2s} = \sum_{s=2}^{m_j} \frac{c_{2s-1}}{2}\left(\frac{1}{s} - \frac{1}{s-1}\right) + \frac{c_1}{2} - \frac{c_{2m_j+1}}{2m_j}\,.$$

Thus, the second term in (30) is $O(1)$ as $j \to \infty$ because the sequence $\{c_n\}_{n\in\mathbb{N}}$ is bounded and $(s^{-1} - (s-1)^{-1})$ is $O(s^{-2})$. Since moreover the last term in (30) equals $\alpha\log(m_j)$ up to a term $O(1)$, we obtain

$$\begin{aligned}\sum_{s=1}^{m_j} g_s^{(2)} &= -\sum_{s=1}^{m_j} g_s^{(1)} - \alpha\log(m_j) + O(1)\\ &\le \frac{1}{2}\log(m_j) - \alpha\log(m_j) + O(1)\\ &< -\frac{\alpha}{2}\log(m_j) + O(1)\,,\end{aligned}$$

where the last inequality holds since $\alpha > 1$. □

Theorem 5.2. *Suppose that the hypotheses of Theorem 4.1 are satisfied for $k = 1$ (then $x^{(1)}$ is not in $l_2(\mathbb{N})$). In addition, assume that $\{m_j\}_{j\in\mathbb{N}}$ satisfies*

$$\sum_{j=1}^{\infty} \frac{m_{j+1} - m_j}{m_j^{\alpha}} \left(\frac{m_{j+1}}{m_j}\right)^{\check{v}} < \infty\,, \tag{H.4}$$

with $\check{v} = \max\{0, \sup_s\{c_{2s} - c_{2s+1} - \alpha\}\}$. Then $x^{(2)}$ is in $l_2(\mathbb{N})$. The assertion holds also if we interchange $k = 1$ and $k = 2$ and take $\check{v} = \max\{0, \sup_s\{c_{2s-1} - c_{2s} - \alpha\}\}$.

Proof. By Lemma 5.1 we have (29) and then, taking into account (H.4) and reasoning as in Theorem 4.2, we obtain that $\xi^{(2)}$ is in $l_2(\mathbb{N}, \mathbb{C}^2)$. □

Remark 6. In view of Remarks 2 and 5, the set of sequences $\{m_j\}_{j\in\mathbb{N}}$ that fulfill both (H.2) and (H.4) is far from being empty.

Corollary 5.3. *J has only pure point spectrum whenever the assumptions of Theorem 5.2 are met.*

Proof. Under the assumptions of the previous theorem we have that $x^{(1)} \notin l_2(\mathbb{N})$ and $x^{(2)} \in l_2(\mathbb{N})$. This behavior of the basis $\{x^{(1)}, x^{(2)}\}$ in the space of solutions of (5) is the same for any $\zeta \in \mathbb{C}$. Thus, we always have a subordinate solution for the generalized eigenequation of the self-adjoint operator J. By applying Subordinacy Theory [10, Theorem 3], we conclude that the spectrum is pure point. □

Remark 7. Notice that a sequence $\{m_j\}_{j\in\mathbb{N}}$ may satisfy (H.2) but not (H.4). That is, Theorems 4.1 and 5.2 not necessarily hold true simultaneously. However, in account of what is said in Remark 6, that happens for a large set of Jacobi operators J.

We notice that (H.2) along with (29) imply a certain correlation between the slow decay of $x^{(1)}$ and the fast decay of $x^{(2)}$ (or vice-versa). We shall use this to show that the resolvent $(J-\zeta I)^{-1}$ is Hilbert-Schmidt. Unfortunately, (17) is not good enough for that purpose. The next technical result accounts for what is needed.

Lemma 5.4. *Suppose that for a certain monotone increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ one of the following holds:*

i. For any natural i greater than some $i_0 \in \mathbb{N}$,

$$\sum_{s=m_i+1}^{m_{i+1}} g_s^{(1)} \geq -\frac{1}{2}\log\left(\frac{m_{i+1}}{m_i}\right) + f_i\,,$$

where $\sum_{i=j}^{l-1} f_i$ is bounded for any j and l ($i_0 \leq j < l$).

ii. For any natural numbers j and l, where $i_0 \leq j < l$ and $i_0 \in \mathbb{N}$,

$$\sum_{s=m_j+1}^{m_l} \frac{c_{2s} - c_{2s+1}}{s}$$

has an upper bound.

Then,

$$\exp\left(\sum_{s=m_j+1}^{m_l} g_s^{(2)}\right) \leq r\left(\frac{m_l}{m_j}\right)^{-\frac{\alpha}{2}}\,,$$

for some constant r. A similar result holds if the roles of $k=1$ and $k=2$ are interchanged and $c_{2s} - c_{2s+1}$ is substituted by $c_{2s-1} - c_{2s}$ in (ii)

Proof. Let us assume (i) first. From (30), it follows that

$$\exp\left(\sum_{s=m_j+1}^{m_l} g_s^{(2)}\right) = \exp\left(-\sum_{s=m_j+1}^{m_l} g_s^{(1)} + \sum_{s=m_j+1}^{m_l} \frac{c_{2s-1}-c_{2s+1}}{2s} - \alpha \sum_{s=m_j+1}^{m_l} \frac{1}{s}\right)$$

As in the proof of Lemma 5.1, one has

$$\sum_{s=m_j+1}^{m_l} \frac{c_{2s-1}-c_{2s+1}}{2s} = \sum_{s=m_j+2}^{m_l} \frac{c_{2s-1}}{2}\left(\frac{1}{s}-\frac{1}{s-1}\right) + \frac{c_{2m_j+1}}{2m_j+2} - \frac{c_{2m_l+1}}{2m_l},$$

where the first term above is clearly $O(m_j^{-1})$. Therefore, there is a constant r_1 such that, for any pair $j<l$,

$$\exp\left(\sum_{s=m_j+1}^{m_l} \frac{c_{2s-1}-c_{2s+1}}{2s}\right) < r_1\,.$$

We thus conclude that there exists a constant r_2 such that

$$\exp\left(\sum_{s=m_j+1}^{m_l} g_s^{(2)}\right) \le r_2 \left(\frac{m_l}{m_j}\right)^{-\frac{\alpha}{2}} \exp\left(\sum_{i=j}^{l-1} f_i\right)$$

which proves our assertion.

In the case where (ii) holds, we have the result straightforwardly. □

Remark 8. Condition (i) of last lemma clearly implies (17).

The results proven above allow us to give conditions ensuring the discreteness of J's spectrum when it is self-adjoint. To simplify proofs, we will assume that the conditions of Theorem 4.1 are satisfied for $k=1$ (so $\xi^{(1)}$ is not in $l_2(\mathbb{N},\mathbb{C}^2)$).

Theorem 5.5. *Assume (H.1). Consider a monotone increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ that complies with (H.2) and*

$$\sum_{j=1}^{\infty} \frac{m_{j+1}-m_j}{m_j^{\alpha}} \left(\frac{m_{j+1}}{m_j}\right)^{\hat{v}} < \infty\,, \tag{H.5}$$

where $\hat{v} = \max\{0, \sup_s\{c_{2s}-c_{2s+1}-\alpha\}, \sup_s\{c_{2s-1}-c_{2s}-\alpha\}\}$. Suppose moreover the fulfillment of either condition (i) of Lemma 5.4, or condition (ii) of the same lemma and (17). Then the resolvent

$$(J-\zeta I)^{-1}\,, \qquad \zeta\in\mathbb{C}\setminus\sigma(J),$$

is Hilbert-Schmidt and, therefore, the spectrum of J is discrete.

Proof. We first note that (H.5) implies (H.4) since $\hat{v} \geq \check{v}$. Thus, the hypotheses of the theorem (along with Remark 8) imply that both Theorems 4.1 and 5.2 hold true. Thence, there exists a basis $\{x^{(1)}, x^{(2)}\}$ in the space of solutions of (5), with one element in $l_2(\mathbb{N})$ and the other outside it. Let $x^{(2)}$ be the element in $l_2(\mathbb{N})$.

In what follows, $r_1, r_2, \ldots, r_{12}$ will stand for certain suitable constants that will appear in estimations along the way.

Let us denote by $P_n(\zeta)$ the n-th orthogonal polynomial of the first kind associated to J. Define the sequence $u = \{u_n\}_{n\in\mathbb{N}}$ in $l_2(\mathbb{N})$ by

$$u = (J - \zeta I)^{-1} e_1 \, .$$

According to [11, 16], the Green function is given by

$$G_{m\,n}(\zeta) = \begin{cases} P_m(\zeta) u_n(\zeta) & \text{when } m \leq n \, , \\ P_n(\zeta) u_m(\zeta) & \text{otherwise} \, . \end{cases} \tag{31}$$

The sequence $\{P_n(\zeta)\}_{n\in\mathbb{N}}$ is not square-summable because $J = J^*$ [1, 15]. Since it belongs to the space of solutions of (5), it may be expressed as a linear combination of the basis $\{x^{(1)}, x^{(2)}\}$. However, the solution $u = \{u_n\}_{n\in\mathbb{N}}$ of (5) must be proportional to $x^{(2)}$. Noticing that $\|\xi_m\|^2 = |x_{m-1}|^2 + |x_m|^2 > |x_m|^2$, we have (dependencies on ζ are dropped from now on)

$$|G_{mn}| < \begin{cases} r_0 \left\|\xi_m^{(1)}\right\| \left\|\xi_n^{(2)}\right\| + r_1 \left\|\xi_m^{(2)}\right\| \left\|\xi_n^{(2)}\right\| & \text{when } m \leq n \, , \\ r_0 \left\|\xi_m^{(2)}\right\| \left\|\xi_n^{(1)}\right\| + r_1 \left\|\xi_m^{(2)}\right\| \left\|\xi_n^{(2)}\right\| & \text{otherwise} \, . \end{cases}$$

Since $\xi_m = B_{m-1}\xi_{m-1}$ and B_n is uniformly norm-bounded in n, we have $\|\xi_{2m+1}\| \leq r_2 \|\xi_{2m}\|$. It thus follows that

$$\begin{aligned} \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} |G_{mn}|^2 < & \sum_{k_0 \leq k \leq j}^{\infty} \left(r_3 \left\|\xi_{2k+2}^{(1)}\right\| \left\|\xi_{2j+2}^{(2)}\right\| + r_4 \left\|\xi_{2k+2}^{(2)}\right\| \left\|\xi_{2j+2}^{(2)}\right\| \right)^2 \\ & + \sum_{j_0 \leq j < k}^{\infty} \left(r_3 \left\|\xi_{2k+2}^{(2)}\right\| \left\|\xi_{2j+2}^{(1)}\right\| + r_4 \left\|\xi_{2k+2}^{(2)}\right\| \left\|\xi_{2j+2}^{(2)}\right\| \right)^2 , \end{aligned} \tag{32}$$

for some large m_0, n_0, k_0 and j_0.

Let us consider any term in the first sum above. Since $k = m_q + p$ and $j = m_t + l$ for some m_q, m_t, $0 \leq p < m_{q+1} - m_q$ and $0 \leq l < m_{t+1} - m_t$, these terms may be written as

$$r_3 \left\|\xi_{2m_q+2p+2}^{(1)}\right\| \left\|\xi_{2m_t+2l+2}^{(2)}\right\| + r_4 \left\|\xi_{2m_q+2p+2}^{(2)}\right\| \left\|\xi_{2m_t+2l+2}^{(2)}\right\| . \tag{33}$$

From (16), it follows that

$$\begin{aligned}\left\|\xi^{(k)}_{2m_q+2p+2}\right\| &\leq r_5 \exp\left(\sum_{s=m_q+1}^{m_q+p} g_s^{(k)}\right)\left\|\xi^{(k)}_{2m_q+2}\right\| \\ &\leq r_6\left(\frac{m_q+p}{m_q}\right)^{\frac{\hat{v}}{2}}\left\|\xi^{(k)}_{2m_q+2}\right\|\end{aligned}$$

holds for $k=1,2$. Thus, (33) is bounded from above by

$$r_7\left(\frac{m_q+p}{m_q}\right)^{\frac{\hat{v}}{2}}\left(\frac{m_t+l}{m_t}\right)^{\frac{\hat{v}}{2}}\left(\left\|\xi^{(1)}_{2m_q+2}\right\|\left\|\xi^{(2)}_{2m_t+2}\right\|+\left\|\xi^{(2)}_{2m_q+2}\right\|\left\|\xi^{(2)}_{2m_t+2}\right\|\right).$$

Let us now look at the sum between parentheses in the last expression. On one hand, since Lemma 5.1 holds, we have

$$\left\|\xi^{(2)}_{2m_q+2}\right\|\left\|\xi^{(2)}_{2m_t+2}\right\| \leq r_8\, m_q^{-\frac{\alpha}{2}} m_t^{-\frac{\alpha}{2}}.$$

On the other hand, since $k \leq j$ implies $m_q \leq m_t$,

$$\begin{aligned}\left\|\xi^{(1)}_{2m_q+2}\right\|\left\|\xi^{(2)}_{2m_t+2}\right\| &\leq r_9 \exp\left(\sum_{s=1}^{m_q} g_s^{(1)}+\sum_{s=1}^{m_q} g_s^{(2)}\right)\exp\left(\sum_{s=m_q+1}^{m_t} g_s^{(2)}\right) \\ &\leq r_9 \exp\left[-\alpha\log(m_q)+O(1)\right]\left(\frac{m_t}{m_q}\right)^{-\frac{\alpha}{2}} \\ &\leq r_{10}\, m_q^{-\frac{\alpha}{2}} m_t^{-\frac{\alpha}{2}},\end{aligned}$$

where the second inequality above follows from the proof of Lemma 5.1 along with Lemma 5.4. Therefore, we have

$$r_3\left\|\xi^{(1)}_{2m_q+2p+2}\right\|\left\|\xi^{(2)}_{2m_t+2l+2}\right\|+r_4\left\|\xi^{(2)}_{2m_q+2p+2}\right\|\left\|\xi^{(2)}_{2m_t+2l+2}\right\| \leq r_{11}\, m_q^{-\frac{\alpha}{2}} m_t^{-\frac{\alpha}{2}}.$$

Similar estimates show that the same bound holds for the terms of the second sum in (32). By rearranging sums, we obtain

$$\begin{aligned}&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}|G_{mn}|^2 \\ &< r_{11}\sum_{m=q_0}^{\infty}\sum_{p=0}^{m_{q+1}-m_q-1} m_q^{-\alpha}\left(\frac{m_q+p}{m_q}\right)^{\hat{v}}\sum_{t=t_0}^{\infty}\sum_{l=0}^{m_{t+1}-m_t-1} m_t^{-\alpha}\left(\frac{m_t+l}{m_t}\right)^{\hat{v}} \\ &< r_{12}\sum_{q=q_0}^{\infty}\frac{m_{q+1}-m_q}{m_q^{\alpha}}\left(\frac{m_{q+1}}{m_q}\right)^{\hat{v}}\sum_{t=t_0}^{\infty}\frac{m_{t+1}-m_t}{m_t^{\alpha}}\left(\frac{m_{t+1}}{m_t}\right)^{\hat{v}} < \infty,\end{aligned}$$

since (H.5) holds.

Finally, we note that both

$$\sum_{m=1}^{m_0}\sum_{n=n_0}^{\infty}|G_{mn}|^2 \quad \text{and} \quad \sum_{m=m_0}^{\infty}\sum_{n=1}^{n_0}|G_{mn}|^2$$

are finite because of the square-summability of $\{u_n\}_{n\in\mathbb{N}}$ and the way it enters in (31). Therefore, we conclude that $\mathrm{Tr}\,(G^*G)<\infty$. □

Example 1 (revisited)**.** We know that the Jacobi operator of this example is self-adjoint whenever it satisfies (25). The sequence $\{m_n\}_{n=0}^{\infty}$ defined in this example obeys (H.5), disregarding the actual values of c_1, c_2 and c_3 as a short computation shows. By looking at (22), and taking into account (25), it is clear that (i) of Lemma 5.4 holds. Therefore, by Theorem 5.5, this self-adjoint Jacobi operator has only discrete spectrum.

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