Pure Point Spectrum in the Anderson Model with Unbounded Random Potential *

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Abstract

The Anderson model with the unbounded random potential (independent random variables with identical probability disitributions of unbounded support and bounded density) is established to have only pure point spectrum (complete system of localized wave-functions) with probability one in arbitrary dimension.
The respective result is deduced via trace-class perturbation analysis as a consequence of the new original result on absence of pure singular continuous spectrum of random perturbations [G4, G6].

1 Introduction

The Anderson model was initially introduced by P.Anderson [A] in 1958 to model physical processes of spin diffusion, impurity conduction, and localization.

Rigorous study of the respective spectral properties had been of essential importance in the recent years because of valuable applications in physics. For example, the energy states of a quantum system are described in terms of self-adjoint operator (Hamilton operator) defined on the Hilbert space of the corresponding wave functions. Existence of the non-empty (pure) absolutely continuous component in the spectrum of such an operator implies non-zero value of conductivity within the certain energy zone, and metal properties in the corresponding system. Presence of only exponentially localized states (e.g. the pure point spectrum "at high disorder" established previously) means absence of diffusion and insulator properties in the respective system (Anderson localization).

The Anderson model had been intensively studied in the recent years, and it was rigorously established via different approximation schemes ([FS, FMSS, DLS, STW, D, G2]) that the respective spectrum exhibits exponential localization (i.e. it is pure point, and corresponding eigenfunctions decay exponentially at infinity) in the regions of impurity spectrum corresponding to the certain values of disorder parameter (the so-called "high disorder localization"). The conductivity spectral component (corresponding to the region of the spectrum of non-perturbed Laplace operator) was established recently ([G4, G6]) as pure absolutely continuous with probability 1 for the bounded random potentials.
At the same time, it is well-known that self-adjoint operators (on $\ell^2(\mathbb{Z}^d)$, or $\mathbb{L}^2(\mathbb{R}^d)$) with growing potential have no absolutely continuous spectrum, for example, if $\lim\limits_{|x|\to\infty} |q(x)| = \infty$, then the spectrum is pure discrete ([Gl]).

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In the following paper there is proved the analogues theorem for the multidimensional finite-difference operators with random (ergodic) potential. All the spectrum is established to be pure point with probability 1 assuming the single-site probability distribution has unbounded support and bounded density.
The respective result (Theorem 3, the main technical result involved to establish Theorem 1) is the generalization for the case $d > 1$ of the result on absence of the absolutely continuous spectrum of the one-dimensional Jacoby matrix with the unbounded diagonal potential ([SS], 1989). Extension of this result to the case of finite-difference operator of infinite order had been found by [G1]. Similar result for one-dimensional finite-difference operator on half-line with unbounded non-random potential and random boundary conditions had been for the first time rigorously established by [KMP].
Consider the Anderson tight-binding model defined by the Hamiltonian H_U (1) - (5) with the random potential (independent identically distributed random variables with single-site probability distribution of unbounded support and bounded density).

Theorem 1 (Pure point spectrum in the Anderson model with unbounded random potential)*:*

$$\sigma(H_U) \; = \; \sigma_{pp}.$$

The proof is a consequence of Theorem 3 and Theorem 2.

Theorem 3 (on absence of absolutely continuous spectrum) permits arbitrary unbounded probability distributions, and implies "absence of diffusion" in the corresponding disordered system ([A, FS]).

Theorem 2 (on absence of singular continuous spectrum, [G4, G6]) requires probability distributions to have bounded density, and implies pure point spectrum. However, the approximation method does not allow to control the decay properties of the respective eigenfunctions.

The main condition imposed to deduce a.s. absence of the singular continuous component in the spectrum of a random perturbation, is either absence of its pure point, or absence of its absolutely continuous spectral component established á priori. Specifically, the pure absolutely continuous spectrum was recently rigorously established ([G4, G6]) to exist in the intervals free of the point spectrum of a random (ergodic) Hamiltonian. By the same way, the pure point spectrum fills the intervals where the absolutely continuous spectrum is empty.

The basic result is represented via the following

Theorem 2 (The non-mixing property [G4, G6])
Consider the Anderson Hamiltonian H defined by (1)-(4).

(A) *Suppose*

$$\sigma_{pp}(H(\omega)) \cap (a,b) \; = \; \emptyset,$$

$(a,b) \subset \mathbb{R}$, with probability 1.

Then

$$\sigma_{sc}(H(\omega)) \cap (a,b) \; = \; \emptyset, \;\; \textit{and} \;\; \sigma(H(\omega)) \cap (a,b) \; \subset \; \sigma_{ac}(H)$$

(i.e. the spectrum of H in (a,b) is pure absolutely continuous), with probability 1.

(B) *Suppose*

$$\sigma_{ac}(H(\omega)) \cap (a,b) \;=\; \emptyset,$$

$(a,b) \subset \mathbb{R}$, *with probability 1.*

Then

$$\sigma_{sc}(H(\omega)) \cap (a,b) \;=\; \emptyset, \;\; \textit{and} \;\; \sigma(H(\omega)) \cap (a,b) \;\subset\; \sigma_{pp}(H)$$

(i.e. the spectrum of H in (a,b) is pure point), with probability 1.

For the unbounded random (strongly unbounded non-random) potentials (cf. definition in Section 2), it is rigorously proved (Theorems 3, 4) absence of the absolutely continuous spectrum in multiple dimension with probability 1:

$$\sigma_{ac}(H_U) \;=\; \emptyset,$$

which, according to the basic result (Theorem 2), implies that the Anderson tight binding model (with unbounded random potential) has only pure point spectrum, with probability 1.
Example. Consider the Anderson Hamiltonian H_G on $\ell^2(\mathbb{Z}^d)$, $d \geq 2$, with random potential formed by the independent identically distributed random variables with the Gauss probability distribution:

$$g(q) \;=\; \frac{1}{\sqrt{2\pi}\sigma}\, e^{\frac{-(q-m)^2}{2\sigma^2}},$$

$g_0 \;=\; \sqrt{2\pi}\sigma$. It had been previously established ([DLS, STW], 1986), that there exist $0 \leq E_0 = E_0(g_0) < \infty$, and $0 < \overline{g}_0 < \infty$, such that $E_0 = 0$ if $g_0 > \overline{g_0}$, and

$$\sigma(H_G) \;\cap\; (\pm E_0, \pm\infty) \;\subset\; \sigma_{pp}(H_G)$$

(i.e. the impurity spectrum is pure point), with probability one.
The new Theorem 1 provides much stronger result, proving all the spectrum of H_G is pure point at arbitrary disorder $g_0 > 0$.

2 Pure point spectrum

The Anderson Hamiltonian with the unbounded potential is defined by the finite-difference operator

$$H_U(\omega) \;=\; H_0 \;+\; Q(\omega), \tag{1}$$

where H_0 is the Laplace operator

$$H_0\psi(x) \;=\; \sum_{\|x-y\|=1} (\psi(x) - \psi(y)), \;\; \psi \in \ell^2(\mathbb{Z}^d), \; x, y \in \mathbb{Z}^d, \tag{2}$$

$\|x\| = \sum_{1\leq j\leq d} |x_j|$, Q_ω is the random potential

$$Q(\omega)\psi(x) \;=\; q_\omega(x)\psi(x), \;\; \psi \in \ell^2(\mathbb{Z}^d), \;\; x \in \mathbb{Z}^d, \tag{3}$$

$\{q_\omega\}(x)_{x\in \mathbb{Z}^d}$ are independent random variables with identical probability distributions having bounded density and unbounded support:

$$dP(q) \;=\; \mathrm{Prob}\{q(0) \in dq\} \;=\; g(q)dq, \;\; g_0^{-1} = \sup_q g(q) < \infty, \tag{4}$$

$$\begin{aligned} \sup_q \,\mathrm{supp}\, dP(q) \;&=\; +\infty \quad \text{or} \\ \inf_q \,\mathrm{supp}\, dP(q) \;&=\; -\infty. \end{aligned} \tag{5}$$

The corresponding probability space

$$(\Omega, \mathbb{P}) = \prod_{j\in \mathbb{Z}^d} (\mathbb{Z}_j, dP(q_j)).$$

Operator H_U is ergodic self-adjoint operator, which spectrum $\sigma(H_U)$, as well as its corresponding pure point and (absolutely) continuous spectral components $\sigma_{pp}(H_A)$, $\sigma_{ac}(H_U)$, $\sigma_c(H_U)$, are non-random subsets of $\mathbb{R}$:

$$\sigma(H_U) \;=\; \sigma(H_0)\dot{+}\mathrm{supp}\, dP(q) \;=\; (-\infty, +\infty),$$

where $\dot{+}$ denotes the algebraic sum of subsets of $\mathbb{R}$, $\sigma(H_0)$ denotes the spectrum of the non-perturbed Laplace operator, which is pure absolutely continuous:

$$\sigma(H_0) \;=\; [0, 4d] \;=\; \sigma_{ac}.$$

Theorem 3 (Absence of absolutely continuous spectrum for unbounded random potential)*:*

$$\sigma_{ac}(H_U) \;=\; \emptyset$$

with probability 1.

Remark. Theorem 3 permits arbitrary unbounded single-site probability distributions.

The proof follows from the result for the strongly unbounded non-random potentials (Theorem 4). Consider the operator H defined on $\ell^2(\mathbb{Z}^d)$ by (1)-(3), where

$$\limsup_{j\to\infty} \inf_{x\in\partial\Lambda_{L_j}} |q(x)| = \infty, \tag{6}$$

where $\Lambda_{L_j} = \{x \in \mathbb{Z}^d | \, \|x\| \leq L_j\}$, i.e. the potential $q(x)$ is unbounded over increasing to infinity sequence of concentric spheres $\Lambda_{L_j} \subset \Lambda_{L_{j+1}}$ of radius L_j, $j \in \mathbb{N}$.

Definition. *The potential satisfying (6) is referred in the following paper as the strongly unbounded.*

Theorem 4 (Absence of absolutely continuous spectrum for the strongly unbounded potential):

$$\sigma_{ac}(H) \,=\, \emptyset.$$

Proof of Theorem 4.

Lemma 1 (The resolvent identity)

$$\begin{aligned}(A+B-z)^{-1} \,-\, (A-z)^{-1} &= -(A-z)^{-1}\, B\,(A+B-z)^{-1} \\ &= -(A+B-z)^{-1}\, B\,(A-z)^{-1},\end{aligned} \tag{7}$$

where A, B are arbitrary linear operators with bounded resolvents, $z \notin \sigma(A) \cup \sigma(B)$.

Proof follows multiplying both sides of (7) by $(A+B-z), (A-z)$.

Lemma 2 *Consider*

$$h \,=\, h_0 \,+\, q,$$

where q is multiplication operator, and $h_0 \in \mathcal{B}$, where $\mathcal{B}$ denotes the Banach algebra of bounded operators on $\ell^2(\mathbb{Z}^d)$. Denote by $e(x)$ the unit vector in $\ell^2(\mathbb{Z}^d)$. Suppose $z \notin \sigma(h)$, $\Delta = \mathrm{dist}\{z, \sigma(h)\}$. Then

$$\|(h-z)^{-1}e(x)\| \,\leq\, \frac{\Delta + \|h_0\|}{\Delta |q(x)-z|}.$$

Proof. By the resolvent identity (Lemma 1),

$$\begin{aligned}\|(h-z)^{-1}\, e(x)\| &\leq \|(q-z)^{-1}e(x) \\ &- (h-z)^{-1}h_0(q-z)^{-1}e(x)\| \\ &\leq \frac{\Delta \,+\, \|h_0\|}{\Delta\, |q(x)-z|^{-1}}.\end{aligned}$$

Lemma 2 is proved.
Denote

$$H_n \,=\, \begin{cases} H(x,y), & x,y \in \Lambda_{L_n} \setminus \Lambda_{L_{n-1}} \\ 0, & \text{otherwise}, \end{cases}$$

$n > 1$ (finite-volume operator with the Dirichlet boundary conditions);

$$H_\Lambda \,=\, \sum_n^{\oplus} H_n; \tag{8}$$

$$H \,=\, H_\Lambda \,+\, \mathring{A}.$$

Denote

$$\mathring{A}_n(x,y) = \begin{cases} 1, & \text{if } x \in \Lambda_n, y \notin \Lambda_n, \text{ or } x \notin \Lambda_n, y \in \Lambda_n \\ 0, & \text{otherwise;} \end{cases}$$

Then

$$\begin{aligned} \mathring{A} &= \sum_n^{\oplus} \mathring{A}_n \\ &= \sum_{y \in \Lambda_{n-1}} \sum_{\substack{x \in \Lambda_n \\ \|x-y\|=1}} \langle e(y), . \rangle \, e(x). \end{aligned} \tag{9}$$

By the resolvent identity (Lemma 1), if $\Im z \neq 0$,

$$\begin{aligned} (H-z)^{-1} - (H_\Lambda - z)^{-1} &= -(H_\Lambda - z)^{-1} \, \mathring{A} \, (H-z)^{-1} \\ &= (H_\Lambda - z)^{-1} \, \mathring{A} \, (H_\Lambda - z)^{-1} \, (\mathring{A}(H-z)^{-1} - I). \end{aligned} \tag{10}$$

Prove that

$$(H-z)^{-1} - (H_\Lambda - z)^{-1} \in \mathcal{B}_1, \tag{11}$$

where $\mathcal{B}_1$ denotes the normed space of trace-class operators on $\ell^2(\mathbb{Z}^d)$, then by Theorem 16, ch.10 ([Ka]),

$$\sigma_{ac}(H) = \sigma_{ac}(H_\Lambda) = 0. \tag{12}$$

Since $H_n \in \mathcal{B}_0$ (H_n is of finite rank), (8) implies $(H_\Lambda - z)^{-1} \in \mathcal{B}_0$, where $\mathcal{B}_0$ denotes the Banach space of compact operators on $\ell^2(\mathbb{Z}^d)$ which is a closed linear space with respect to the $\mathcal{B}$- norm.
Since $\mathcal{B}_1$ is a closed ideal in $\mathcal{B}$, and $(\mathring{A}(H-z)^{-1} - I) \in \mathcal{B}$, (9) implies that it is sufficient to prove

$$(H_\Lambda - z)^{-1} \mathring{A} (H_\Lambda - z)^{-1} \in \mathcal{B}_1.$$

Denote by $\|.\|_1$ the trace norm. If $A \in \mathcal{B}$ is of finite rank, then

$$\|A\|_1 \leq \operatorname{rank}(A) \, \|A\|.$$

It follows by Lemma 2 and (8)-(10)

$$\begin{aligned} & \|(H_\Lambda - z)^{-1} \mathring{A} (H_\Lambda - z)^{-1}\|_1 \\ \leq & \sum_n \sum_{\substack{\|x-y\|=1 \\ x \in \Lambda_n \\ y \in \Lambda_{n-1}}} |\langle (H_n - \overline{z})^{-1} e(y), . \rangle| \, \|(H_n - z)^{-1} e(x)\| \\ \leq & \sum_n \frac{2d(\Delta + 2d)}{\Delta^2 \inf\limits_{x \in \partial \Lambda_{L_n}} |q(x) - z|} < \infty, \end{aligned} \tag{13}$$

since condition (6) implies that it is possible to choose $\{x_n\}_{n \in \mathbf{N}}$ such that

$$\sum_n \frac{1}{|q(x_n)|} < \infty.$$

Hence (13) imply (11) and (12). Theorem 4 is proved. □

Proof of Theorem 3. Denote as before by $(\Omega, \mathcal{S}, \mathbb{P})$ the probability space of realizations of the random potential (3)-(5), where $\mathcal{S}$ denotes the σ- algebra of $\mathbb{P}$- measurable subsets of Ω. Consider the sequence $L_{n+1} > L_n > 0$, $n \in \mathbb{N}$, and denote

$$\Omega(b, \Lambda_l) \;=\; \{q \in \mathcal{A}_Q | \inf_{x \in \partial\Lambda_l} |q(x)| \geq b\} \;\in\; \mathcal{S},$$

$$\Omega_n \;=\; \bigcup_{\Lambda_{L_n} \in \Lambda_{L_{n+1}}} \Omega(b_n, \Lambda_{L_n}) \;\in\; \mathcal{S}.$$

Then

$$\mathbb{P}\{\Omega(b, \Lambda_l)\} \;=\; dP\{(\pm b, \pm\infty)\}^{|\partial\Lambda_l|}, \tag{14}$$

$$\mathbb{P}\{\Omega_n\} \;\geq\; \frac{L_{n+1}}{L_n}\, \mathbb{P}\{\Omega(b_n, \Lambda_{L_n})\}. \tag{15}$$

Condition (6) implies that it is possible to choose $b_n > 0$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} b_n \;=\; \infty,$$
$$dP\{(\pm b_n, \pm\infty)\} \;\neq\; 0.$$

So choose L_n, $n \in \mathbb{N}$:

$$L_{n+1} \;\geq\; \frac{1}{|n| dP\{(\pm b_n, \pm\infty)\}^{C_d L_n^{d-1}}}\, L_n.$$

Then by (14), (15):

$$\sum_{n\to\infty} \mathbb{P}\{\Omega_n\} \;=\; \infty. \tag{16}$$

It follows by Borel-Kantelli lemma via (16) that

$$\mathbb{P}\{\overline{\Omega} = \bigcap_{n\geq 1} \bigcup_{k\geq n} \Omega_n\} \;=\; 1. \tag{17}$$

(17) implies the potential Q_ω satisfying (5) is strongly unbounded with probability 1. Theorem 4 implies

$$\sigma_{ac}(H_U) \;=\; \emptyset$$

holds with probability 1. Theorem 3 is proved. □

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VG would like to request excuse for not answering to the e-mail correspondence could had arrived to his previous e-mail address (grinshpun@ilt.kharkov.ua): mentioned e-mail box was closed, and permit for entrance to the host-keeping institution was denied (not prolonged) by institute for low temperature physics, kharkov ("ukraine") on December 31 (1999), when the following paper had been under preparation for submission for publication by author.

VG had had no opportunity to present his described in part personal research results at ICMP XIII, the wrong reference in [ICMP2000].

His wrong postal address in the IAMP (international association of Mathematical Physics since 1972) internet database had not been corrected since 1997.

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