Infinite Töplitz–Lipschitz matrices and operators

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Abstract

We introduce a class of Hilbert matrices ($A_{ss'}$, $s, s' \in \mathbb{Z}^d$), which are asymptotically (as $|s|+|s'| \to \infty$) close to Hankel–Töplitz matrices. We prove that this class forms an algebra, and that flow-maps of nonautonomous linear equations with coefficients from the class also belong to it.

0 Introduction.

In this work we suggest a new algebra of infinite matrices ($A_{ss'}$, $s, s' \in \mathbb{Z}^d$) with exponential decay of elements off the diagonals. The latter means that

i) $|A_{ss'}| \le \mathrm{const}\, e^{-\gamma \min\{|s-s'|,|s+s'|\}}$

(we treat the set $\{s=-s'\}$ as the second diagonal).

It is not hard to see that matrices, satisfying i) with various $\gamma>0$, form an algebra. The matrices of our class, apart from i), satisfy two more properties. Namely, they are

ii) *asymptotically Töplitz*: matrix elements $A_{ss'}$ converge to limits when $(ss') \to \infty$ along one of the diagonals. That is, the limits

$$\lim_{t\to\infty} A_{(ta+b_1)\pm(ta+b_2)}$$

exist for all $a, b_1, b_2 \in \mathbb{Z}^d$, $a \ne 0$.

iii) *Lipschitz at infinity*: the matrix elements under the limit–sign in ii) make a Lipschitz function of the argument t^{-1} for t in the vicinity of infinity.

We call the matrices, satisfying i)-iii), the *Töplitz–Lipschitz (TL)* matrices.

A delicate point, related to this definition, is to introduce a right Lipschitz–norm to control the convergence in iii), i.e. to find right neighbourhoods of $t=\infty$, depending on a, b_1, b_2, where the Lipschitz constant is to be calculated. The neighbourhoods, suggested in our work, are complicated sets, depending on a parameter $\Lambda \ge 6$. They allow to prove the properties of TL matrices, given below, and insure that the matrices possess some additional properties, needed in applications. These sets were found by trials and errors. It may be that our choice of them is not optimal.

The set of TL matrices is bi-stratified by the exponent γ in i) and by the parameter $\Lambda \ge 6$, characterising the neighbourhoods of infinity in iii). Accordingly the space of TL matrices is given the family of norms $|||\cdot|||_{\gamma,\Lambda}$, $\gamma>0, \Lambda \ge 6$.

We prove that this space is an algebra. Moreover, multiplication of TL matrices is continuous in the sense that

$$
\begin{aligned}
&|||A\,B|||_{\gamma',\Lambda'} \le C\,|||A|||_{\gamma_1,\Lambda_1}|||B|||_{\gamma_2,\Lambda_2}\\
&\text{for }\ \gamma' \le \gamma_1\wedge\gamma_2,\ \gamma' < \gamma_1\vee\gamma_2\ \text{ and }\ \Lambda' \ge \Lambda_1\vee\Lambda_2+3
\end{aligned}
$$

(the constant C explicitly depends on γ's and Λ's). See section 2.2.

Second important property of the space of TL matrices is that it is invariant with respect to taking a non-commutative exponent. Namely, if $A(t)$ is a TL matrix, continuous it t, then the fundamental matrix of the linear differential equation

$$\dot X_s = \sum_{s'} A(t)_{ss'} X_{s'}(t)$$

also is TL. Its norms $|||\cdot|||_{\gamma,\Lambda}$ are estimated via norms of the matrix $A(t)$, see section 3.2.

The basic definition i)-iii) above admits variations. Some of them are discussed in our paper since they are needed for applications; see section 1.4 + end of section 2.1, and section 2.4.

The algebra of TL matrices contains
a) matrices $\{A_{ss'}\}$ with finitely many non-zero elements;
b) Töplitz matrices (where a matrix element $A_{ss'}$ depends on $s-s'$)
and
c) Hankel matrices (where an element $A_{ss'}$ depends on $s+s'$).
In a number of situations a perturbative infinite–dimensional problem is resolved by an iterative procedure which starts with a linear operator, satisfying a), b) or c), or with a functional whose Hessians are linear operators of this form. Then in interesting situations the iterative procedure pushes us outside the class a)-c) (since this is not an algebra), but still we often stay in the TL algebra. Controlling iteratively TL norms of the involved linear operators we get better control for the procedure. This allows to establish its convergence in a number of important situations which do not yield traditional techniques.

The infinite–dimensional KAM theory gives examples of such iterative procedures. In [EK05] we crucially use the algebra of TL matrices to prove KAM–persistence of time–quasiperiodic solutions of the linear Schrödinger equation

$$-i\dot u = \Delta u + V(x)*u;\quad u=u(t,x), x\in\mathbb{T}^d, d\ge 1;\quad V(x)=\sum \hat V(a)e^{ia\cdot x}\,,$$

under small (non-linear) Hamiltonian perturbations of the equation, for a typical potential $V(x)$.

Notations. By C, C_1 etc we denote various constants, independent of the parameters γ and Λ, by $a\vee b$ and $a\wedge b$ – maximum and minimum of real numbers a and b; for any subset $F\subset\mathbb{Z}^d$ by $\mathbf{1}_F$ we denote the indicator function of F (which equals one on F and vanishes outside F). By $\mathcal{O}(1)$ we denote various functions, bounded in modulus by one on their domains of definition.

1 Weighted l^2-spaces and infinite matrices with exponential decay of elements off diagonals.

1.1 Weighted l^2-spaces.

Let us take any real number γ such that

$$|\gamma| \leq 1,$$

fix any $m_* \geq 0$ and consider the following weighted l^2-space:

$$Y_\gamma = Y_{\gamma,m_*} = \{y = (y_s, s \in \mathbb{Z}^d) \mid \|y\|_\gamma < \infty\}.$$

Here

$$\|y\|_\gamma^2 = \|y\|_{\gamma.m_*}^2 = \sum_{s\in\mathbb{Z}^d} |y_s|^2 e^{2\gamma|s|} \langle s\rangle^{2m_* \operatorname{sgn}\gamma} \tag{1.1}$$

if $\gamma \neq 0$, where $\langle s\rangle = |s| \vee 1$ for any d-vector s. If $\gamma = 0$, then we consider two spaces, Y_{+0} and Y_{-0}. The norms in these spaces are defined by the relations (1.1), where we set $\operatorname{sgn} +0 = 1$, $\operatorname{sgn} -0 = -1$.

For any γ we have the pairing

$$Y_\gamma \times Y_{-\gamma} \to \mathbb{R}, \quad (y, y') \mapsto \sum y_s\, {y'}_s, \tag{1.2}$$

which identifies $Y_{-\gamma}$ with the space, dual to Y_γ:

$$(Y_\gamma)^* = Y_{-\gamma} \quad \forall\gamma.$$

Here and below in this subsection 'for any γ' means 'for any γ from the set $[-1,0) \cup (0,1] \cup \{+0,-0\}$'.

By Y_γ^c we denote the complexification of a space Y_γ. The pairing (1.2) extends to a complex–bilinear map $Y_\gamma^c \times Y_{-\gamma}^c \to \mathbb{C}$.

Lemma 1.1. *If $m_* > \frac{1}{2}d$, then each space Y_γ, $\gamma \in (0,1] \cup \{+0\}$, is a Banach algebra with respect to the convolution:*

$$\|p * q\|_\gamma \leq C(m_*, d)\, \|p\|_\gamma \|q\|_\gamma \quad \forall\, p, q \in Y_\gamma, \tag{1.3}$$

where $(p * q)_s = \sum_{a\in\mathbb{Z}^d} p_{s-a} q_a$.

Proof. For $\gamma = +0$ the estimate is a classical property of Sobolev spaces. If $\gamma > 0$, we note that

$$\|Q\|_{+0} = \|q\|_\gamma, \quad \|P\|_{+0} = \|p\|_\gamma,$$

where $Q_s = q_s e^{\gamma|s|}$, $P_s = p_s e^{\gamma|s|}$ for each s. Therefore,

$$\begin{aligned}
\|p * q\|_\gamma^2 &= \sum_s \langle s\rangle^{2m_*} e^{2\gamma|s|} \Big(\sum_j p_j q_{s-j}\Big)^2 \\
&= \sum_s \langle s\rangle^{2m_*} \Big(\sum_j e^{\gamma(|s|-|j|-|s-j|} P_j Q_{s-j}\Big)^2 \\
&\le \sum_s \langle s\rangle^{2m_*} \Big(\sum_j P_j Q_{s-j}\Big)^2 = \|P * Q\|_{+0}^2 \\
&\le C^2(m_*, d)\|P\|_{+0}^2 \|Q\|_{+0}^2 = C^2(m_*, d)\|p\|_\gamma^2 \|q\|_\gamma^2,
\end{aligned}$$

as stated. □

1.2 Matrices with exponential decay off the diagonals.

In $\mathbb{R}^d$ and $\mathbb{Z}^d$ we introduce a quasidistance which (with some abuse of notations) will be denoted $[x-y]$:

$$[x-y] = \min\{|x-y|, |x+y|\}.$$

Obviously, $[x-y] = [y-x]$, $[0-x] = |x|$, $[x-y] = 0$ if and only if $x = \pm y$, and

$$[x-y] = |x-y| \quad \text{iff} \quad \langle x, y\rangle \ge 0.$$

Moreover, the triangle inequality holds:

$$[x-y] \le [x-z] + [z-y]. \tag{1.4}$$

Indeed, $[x-y]$ equals the Hausdorff distance between the sets $\{x, -x\}$ and $\{y, -y\}$, so (1.4) follows from the triangle inequality for the Hausdorff distance.

Let $A = (A_{ss'},\ s, s' \in \mathbb{Z}^d)$, be an infinite matrix with complex or real entries. For $|\gamma| \le 1$ we set

$$|||A|||_\gamma = \sup_{s,s'} \{e^{\gamma[s-s']}|A_{ss'}|\},$$

and denote

$$M_\gamma = \{A \mid A \text{ is real and } |||A|||_\gamma < \infty\},$$
$$M_\gamma^c = M_\gamma \otimes \mathbb{C} = \{A \mid A \text{ is complex and } |||A|||_\gamma < \infty\}.$$

Obviously, M_γ and M_γ^c are Banach spaces. They are formed by matrices which decay exponentially outside the union of the diagonal $\{s = s'\}$ and the 'anti-diagonal' $\{s = -s'\}$. Clearly, $\|\,\mathrm{id}\,|||_\gamma = 1$ for any γ. So the identity matrix belongs to all spaces M_γ.

Let Y_γ^{c0} be the set of vectors $y \in Y_\gamma^c$ with finitely many non-zero coefficients. Then any $A \in M_{\gamma'}^c$ defines a linear map

$$A : Y_\gamma^{c0} \to Y_\gamma^c, \quad y \to Ay, \ \ (Ay)_s = \sum A_{ss'} y_{s'}.$$

The main result of this section is that the map A is bounded in the $\|\cdot\|_\gamma$-norm if $\gamma' > |\gamma|$. Accordingly, it extends to a bounded map $A : Y_\gamma^c \to Y_\gamma^c$:

Theorem 1.2. *Let $1 \geq \gamma' > |\gamma|$. Then*

$$\|Ay\|_\gamma \leq C(d, m_*)(\gamma' - |\gamma|)^{-d-m_*} |||A|||_{\gamma'} \|y\|_\gamma, \tag{1.5}$$

for any $y \in Y_\gamma^{c0}$. Hence, A defines a bounded linear operator in Y_γ^c and (1.5) holds for each $y \in Y_\gamma^c$.

Proof. Without lost of generality we assume below that $|||A|||_{\gamma'} = 1$.

For any $s \in Z^d$ let us denote by l^s the vector

$$l^s = \{l_a^s = \delta_{s,a}, \quad a \in \mathbb{Z}^d\}. \tag{1.6}$$

Then the vectors

$$f^s = e^{-\gamma|s|} \langle s \rangle^{-m_\gamma} l^s, \quad m_\gamma = m_* \operatorname{sgn} \gamma,$$

form a Hilbert basis of the space Y_γ. In this basis the matrix of the operator A is

$$\{B_{ab} = \langle a \rangle^{m_\gamma} \langle b \rangle^{-m_\gamma} e^{\gamma(|a|-|b|)} A_{ab}\}$$

(see [HS78]). Let us first assume that $\gamma > 0$ and denote $\gamma_\Delta = \gamma' - \gamma$, $\gamma_\Delta \in (0, 1]$. For any $a \in \mathbb{Z}^d$ we have:

$$\sum_b |B_{ab}| \leq e^{\gamma|a|} \sum_{b \in \mathbb{Z}^d} \Big(\frac{\langle a \rangle}{\langle b \rangle} \Big)^{m_*} e^{-\gamma|b|} e^{-\gamma' [a-b]}.$$

To bound the r.h.s. we need

Lemma 1.3. *For any $x, y \in \mathbb{Z}^d$ we have*

$$\Big(\frac{\langle y \rangle}{\langle x \rangle} \Big)^{m_*} \leq C \gamma_\Delta^{-m_*} e^{\frac{1}{2}\gamma_\Delta [y-x]}. \tag{1.7}$$

Proof. We may assume that $|y| > |x| > 1$ since otherwise the estimate is obvious. Also, it is easy to see that the estimate for $0 \leq m_* < 1$ follows from the one for $m_* \geq 1$. Below we assume that

$$|y| > |x| > 1, \quad m_* \geq 1,$$

and prove the estimate with $C = 2^{m_*} e^{m_*(\ln m_* + 1)}$. Denoting $r = \frac{1}{2}\gamma_\Delta / m_*$ and taking logarithm of (1.7), we rewrite it as

$$\ln |y| - \ln |x| \leq -\ln r + r[y - x] + 1.$$

By the triangle inequality (1.4) (with $y = 0$ and z re-denoted as y), $[y - x] \geq |y| - |x|$. So it suffice to check that

$$\ln \frac{|y|}{|x|} \leq -\ln r + r(|y| - |x|) + 1,$$

where $|y| > |x| > 1$ and $0 < r < 1$. For $|x| < |y| \leq 2|x|$ the estimate follows from the inequality $\ln 2 < 1$. For $|y| \geq 2|x|$ it follows from the estimate $\ln t \leq \frac{1}{2}t$, valid for $t \geq 1$. ∎

Due to (1.7),

$$\sum_b |B_{ab}| \le C_1 e^{\gamma|a|} \gamma_\Delta^{-m_*} \sum_{b\in\mathbb{Z}^d} e^{-\gamma|b|-(\gamma+\frac{1}{2}\gamma_\Delta)[a-b]}.$$

To estimate the sum in the r.h.s. we use the following lemma:

Lemma 1.4. *For any $a \in \mathbb{Z}^d$ and any $0 \le \gamma \le 1$, $\gamma_\Delta \in (0,1]$ we have*

$$\sum_{b\in\mathbb{Z}^d} e^{-\gamma|b|-(\gamma+\gamma_\Delta)[a-b]} \le C_1 e^{-\gamma|a|}\gamma_\Delta^{-d}. \tag{1.8}$$

and

$$\sum_{a\in\mathbb{Z}^d} e^{\gamma|a|-(\gamma+\gamma_\Delta)|a-b|} \le C_2 e^{\gamma|b|}\gamma_\Delta^{-d}. \tag{1.9}$$

The lemma is proved in Appendix.

Due to (1.8), we have

$$\sum_b |B_{ab}| \le C\gamma_\Delta^{-d-m_*}. \tag{1.10}$$

Similar arguments show that

$$\sum_a |B_{ab}| \le Ce^{-\gamma|b|}\gamma_\Delta^{-m_*} \sum_a e^{\gamma|a|-(\gamma+\frac{1}{2}\gamma_\Delta)[a-b]} \le C\gamma_\Delta^{-d-m_*}. \tag{1.11}$$

The estimates (1.10), (1.11) and the Schur criterion (see [HS78]) imply (1.5) for the case when $\gamma > 0$; these relations with $\gamma = 0$ imply (1.5) with $\gamma = +0$.

If $\gamma \le 0$, we re-denote $\gamma = -\hat\gamma$, $\hat\gamma \ge 0$. Now $m_\gamma = -m_*$, so the Hilbert basis is formed by the vectors $f^s = l^s e^{\tilde\gamma|s|}\langle s\rangle^{m_*}$ and the matrix elements B_{ab} are

$$B_{ab} = \langle a\rangle^{-m_*}\langle b\rangle^{m_*} e^{\hat\gamma(|b|-|a|)} A_{ab}.$$

Therefore,

$$\sum_b B_{ab} \le e^{-\hat\gamma|a|} \sum_b \Big(\frac{\langle b\rangle}{\langle a\rangle}\Big)^{m_*} e^{\hat\gamma|b|-\gamma'[a-b]}$$
$$\le C_1 e^{-\hat\gamma|a|}\gamma_\Delta^{-m_*} \sum_b e^{\hat\gamma|b|-(\hat\gamma+\frac{1}{2}\gamma_\Delta)[a-b]} \le C_2\gamma_\Delta^{-d-m_*},$$

where we use (1.9) to get the last inequality. Similar

$$\sum_a B_{ab} \le C_2\gamma_\Delta^{-d-m_*}.$$

So (1.5) is also proved for $\gamma \le 0$. $\square$

1.3 Multiplication of the matrices.

The set of all matrices with exponential decay off diagonals $\cup_{0<\gamma<1} M_\gamma$ obviously is a linear space. Here we show that this is an algebra. More precisely , the following result holds:

Theorem 1.5. *Let $|\gamma| < \gamma' \le 1$. Then*

$$|||AB|||_\gamma + |||BA|||_\gamma \le C(\gamma' - |\gamma|)^{-d} |||A|||_\gamma \, |||B|||_{\gamma'}. \tag{1.12}$$

for any $A \in M^c_\gamma,\ B \in M^c_{\gamma'}$.

Proof. We start with a lemma:

Lemma 1.6. *If $|\gamma| < \gamma'$, then*

$$\sum_{s\in\mathbb{Z}^d} e^{-\gamma[a-s]-\gamma'[s-b]} < C(\gamma' - |\gamma|)^{-d} e^{-\gamma[a-b]},$$

for any $a, b \in \mathbb{Z}^d$.

Proof. Noting that

$$e^{-\gamma[a-s]-\gamma'[s-b]} < \sum_{\sigma_1,\sigma_2=\pm 1} e^{-\gamma|\sigma_1 a - s| - \gamma'|s-\sigma_2 b|},$$

we get

$$\sum_{s\in\mathbb{Z}^d} e^{-\gamma[a-s]-\gamma'[s-b]} < \sum_{\sigma_1,\sigma_2=\pm} \sum_{s\in\mathbb{Z}^d} e^{-\gamma|\sigma_1 a - s| - \gamma'|s-\sigma_2 b|}. \tag{1.13}$$

Let us fix any choice of σ_1, σ_2 and denote $a' = \sigma_1 a$, $b' = \sigma_2 b$. We have

$$\sum_s e^{-\gamma|a'-s|-\gamma'|s-b'|} = \sum_{s'} e^{-\gamma|s'|-\gamma'|(a'-b')-s'|}. \tag{1.14}$$

By (1.8) this sum is bounded by $C' e^{-\gamma|a'-b'|}(\gamma' - |\gamma|)^{-d}$. Since $|a'-b'| \ge [a-b]$, then

$$(1.14) \le C' e^{-\gamma[a-b]}(\gamma' - |\gamma|)^{-d}$$

for any choice of the signs σ_1, σ_2. Now the estimate follows. $\square$

Since

$$|(AB)_{ab}| \le \sum_s |A_{as}|\,|B_{sb}| \le |||A|||_\gamma \, |||B|||_{\gamma'} \sum_{s\in\mathbb{Z}^d} e^{-\gamma[a-s]-\gamma'[s-b]},$$

then Lemma 1.6 implies the estimate for the first term in the r.h.s. of (1.12). To estimate the second term we note that $|||A|||_\delta = |||A^t|||_\delta$ for any δ, where A^t is the transposed matrix, and that $(BA)^t = A^t B^t$. $\square$

Remark. The estimate (1.12) remain true if we replace the norm $|||A|||_\gamma$ by the simpler and more tradition norm

$$\sup_{s,s'} \{ e^{\gamma|s-s'|} |A_{ss'}| \}.$$

Proof remains the same (and even simplifies).

1.4 Infinite matrices, formed by 2×2-blocks

Let X be a Banach algebra over reals, such that its complexification $X^c = X \otimes \mathbb{C}$, is a complex Banach algebra. For an infinite matrix $A = (A_{ss'} \in X,\, s,\, s' \in \mathbb{Z}^d)$, we define its norm $|||A|||_\gamma$ by the same relation as in the section 1.3, and define the spaces $M_\gamma = M_\gamma^X$ and $M_\gamma^c = M_\gamma^{X c}$ accordingly. Straightforward analysis of arguments in section 1.3 shows that they apply to matrices from the spaces M_γ^{Xc}. Therefore the *assertion of Theorem 1.5 remains true if $A \in M_\gamma^{Xc}$ and $B \in M_{\gamma'}^{Xc}$.*

Now let X be the algebra of real 2×2-matrices, and X^c be the algebra of complex matrices. The matrices from the space $M_\gamma^c = M_\gamma^{Xc}$ naturally act on spaces $Y_{\hat\gamma}^c = Y_{\hat\gamma}^{\mathbb{R}^2 c} = Y_{\hat\gamma}^{\mathbb{C}^2}$, formed by vectors $y = (y_s \in \mathbb{C}^2, s \in \mathbb{Z}^d)$. An obvious version of Theorem 1.2 holds for this action. Let us consider the four classical matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and set

$$X_p = \mathbb{R}\sigma_0 + \mathbb{R}\sigma_2 \subset X, \quad X_q = \mathbb{R}\sigma_1 + \mathbb{R}\sigma_3 \subset X.$$

Clearly, $X_p \oplus X_q = X$. We denote by $p : X \to X_p$ the projection to X_p along X_q, and by $q : X \to X_q$ – the projection to X_q along X_p.

Noting that $\sigma_2^2 = -\operatorname{id}$, we introduce in $\mathbb{R}^2$ a complex structure by means of the operator σ_2. With respect to this structure X_p is the algebra of complex-linear operators, while X_q is the linear space of complex-antilinear operators. This observation makes it obvious that the multiplication of matrices defines the maps

$$X_p \times X_p \to X_p, \quad X_q \times X_q \to X_p, \quad X_p \times X_q \to X_q, \quad X_q \times X_p \to X_q. \tag{1.15}$$

We define the following norms $|||\cdot|||_\gamma^+$ and $|||\cdot|||_\gamma^-$ on the spaces M_γ^X and M_γ^{Xc}:

$$|||A|||_\gamma^+ = \sup_{s,s'} \{ |qA_{ss'}| e^{\gamma|s+s'|} \vee |pA_{ss'}| e^{\gamma|s-s'|} \}$$

and

$$|||A|||_\gamma^- = \sup_{s,s'} \{ |pA_{ss'}| e^{\gamma|s+s'|} \vee |qA_{ss'}| e^{\gamma|s-s'|}.$$

Clearly, $|||A|||_\gamma \le |||A|||_\gamma^\pm$ and $|||\operatorname{id}|||_\gamma^+ = 1$, $|||\operatorname{id}|||_\gamma^- = \infty$ for each γ. We set

$$M_\gamma^\pm = \{ A \in M_\gamma^X \mid |||A|||_\gamma^\pm < \infty \},$$

and define the complexified spaces $M_\gamma^{\pm c}$ accordingly.

The spaces $Y_{\hat\gamma}^{R^2}$ inherit the complex structure which we have introduced in $\mathbb{R}^2$. Let us take any $A \in M_\gamma^X, |\gamma| > \hat\gamma$, and denote by pA and qA the matrices with the elements $pA_{ss'}$ and the elements $qA_{ss'}$, respectively. Clearly, the matrix pA defines a complex-linear transformation of the space $Y_{\hat\gamma}^{R^2}$, while

qA defines its complex-antilinear transformation. We shall call pA *the complex-linear part* of A, and qA – its *complex-antilinear part*. The space M^+_γ is formed by matrices such that their complex-linear parts decay exponentially 'along the anti-diagonal $\{s=-s'\}$', while their complex-antilinear parts decay 'along the diagonal $\{s=s'\}$'. The space M^-_γ can be interpreted similar.

All assertions below are made for the real spaces $M^\pm_\gamma$. Their reformulations for the complex spaces are straightforward.

The transposition of matrices preserves the norms $|||\cdot|||^\pm_\gamma$. So it also preserves the spaces $M^\pm_\gamma$.

Denoting by Υ the matrix of complex conjugation (formed by the 2×2 blocks $\delta_{s,s'}\,\sigma_1$), we have the following isometries:

$$
\begin{aligned}
&M^\pm_\gamma \to M^\mp_\gamma, \quad A \to \Upsilon A,\\
&M^\pm_\gamma \to M^\mp_\gamma, \quad A \to A\Upsilon.
\end{aligned} \tag{1.16}
$$

Example 1.7. For any vectors $y^1, y^2 \in Y^X_\gamma$ the tensor product $y^1\otimes y^2$ is a 2×2-matrix with entries in X. If $\gamma \ge 0$, then we have

$$
|(y^1\otimes y^2)_{ss'}| \le \|y^1\|_\gamma \|y^2\|_\gamma e^{-\gamma(|s|+|s'|)}.
$$

Since $|s|+|s'|\ge |s\pm s'|$, then

$$
|||y^1\otimes y^2|||^\pm_\gamma \le \|y^1\|_\gamma\|y^2\|_\gamma \quad \text{if} \quad \gamma\ge 0. \tag{1.17}
$$

So $y^1\otimes y^2 \in M^+_\gamma \cap M^-_\gamma$ if $y^1,y^2\in Y^X_\gamma$ and $\gamma\ge 0$. $\square$

Multiplication of matrices from the spaces $M^\pm_\gamma$ agrees with Theorem 1.5, as specifies the following

Theorem 1.8. *Multiplication of infinite matrices defines the following continuous maps, where* $|\gamma| < \gamma' \le 1$*:*

$$
M^+_{\gamma'}\times M^+_\gamma \to M^+_\gamma, \quad M^-_{\gamma'}\times M^-_\gamma \to M^+_\gamma, \tag{1.18}
$$

$$
M^+_{\gamma'}\times M^-_\gamma \to M^-_\gamma, \quad M^-_{\gamma'}\times M^+_\gamma \to M^-_\gamma.
$$

Moreover, $|||AB|||^+_\gamma \le C(\gamma'-|\gamma|)^{-d}|||A|||^+_{\gamma'}|||B|||^+_\gamma$*, and similar estimates hold for other multiplications.*

Proof. To check the first relation in (1.18) we take any $|||A|||^+_{\gamma'}=1$, $\;|||B|||^+_\gamma=1$ and denote $D=AB$. Then, due to (1.15) and Lemma 1.6,

$$
\begin{aligned}
|pD_{ij}| &\le \sum_s \big(|pA_{is}|\,|pB_{sj}| + |qA_{is}|\,|qB_{sj}|\big)\\
&\le \sum_s \big(e^{-\gamma'|i-s|-\gamma|s-j|} + e^{-\gamma'|i+s|-\gamma|s+j|}\big) \le C(\gamma'-|\gamma|)^{-d}e^{-\gamma|i-j|},
\end{aligned}
$$

and similar

$$
|qD_{ij}| \le \sum_s \big(|pA_{is}|\,|qB_{sj}| + |qA_{is}|\,|pB_{sj}|\big) \le C(\gamma'-|\gamma|)^{-d}e^{-\gamma|i+j|}.
$$

So $|||D|||_\gamma^+ \le C(\gamma' - |\gamma|)^{-d}$.

The three remaining estimates follow from this one and (1.16). For example, if $A \in M_\gamma^+$ and $B \in M_\gamma^-$, then we write AB as $\Upsilon\Big((\Upsilon A)B\Big)$. We see that $AB \in M_\gamma^-$ and satisfies the desired estimate. $\square$

2 Töplitz–Lipschitz matrices and operators

2.1 Definitions and examples

In this section we study operators $A = (A_{ab}) \in M_\gamma^c$ which possess additional properties TL1 – TL3:

TL1. For any a, b_1, $b_2 \in \mathbb{Z}^d$, $a \neq 0$, the two limits

$$A^{\infty\pm}_{a,b_1,b_2} = \lim_{t\to\infty,\, ta\in\mathbb{Z}^d} A_{(ta+b_1)\pm(ta+b_2)} \tag{2.1}$$

exist and are finite.

TL2. There exists $\Lambda \ge 6$ such that

$$A(a, b_1, b_2;\ \Lambda) = \max\big(A^+(a, b_1, b_2;\ \Lambda),\ A^-(a, b_1, b_2;\ \Lambda) < \infty,$$

where

$$A^\pm(a, b_1, b_2;\ \Lambda) = \sup\{t|A_{(ta+b_1)\pm(ta+b_2)} - A^{\infty\pm}_{a,b_1,b_2}|\}\,,$$

and the supremum is taken over all $t > 0$ such that

$$|ta + b_j| \ge \Lambda(1 + |a| + |b_j|)|a|, \quad j = 1,\, 2. \tag{2.2}$$

Let us define

$$\langle A\rangle_{\gamma,\,\Lambda} = \sup e^{\gamma|b_1-b_2|}A(a,\, b_1,\, b_2;\ \Lambda)\,,$$

where the supremum is taken over all a, b_1, $b_2 \in \mathbb{Z}^d$, $a \neq 0$.

TL3. For some $|\gamma| \le 1$ we have

$$|||A|||_{\gamma,\,\Lambda} := |||A|||_\gamma + \langle A\rangle_{\gamma,\,\Lambda} < \infty\,.$$

We denote

$$M^c_{\gamma,\,\Lambda} = \{A \in M^c_\gamma \mid |||A|||_{\gamma,\,\Lambda} < \infty\}\,, \qquad M_{\gamma,\Lambda} = M^c_{\gamma,\Lambda} \cap M_\gamma.$$

These are a complex and a real Banach spaces. The norm $|||A|||_{\gamma,\,\Lambda}$ grows when γ increases or Λ decreases. So

$$M^c_{\gamma_1,\,\Lambda_1} \subset M^c_{\gamma_2,\,\Lambda_2} \quad \text{if} \quad \gamma_1 \ge \gamma_2,\ \Lambda_1 \le \Lambda_2$$

Remark. For infinite matrices whose entries are elements of a Banach algebra, the spaces $M^c_{\gamma,\Lambda}$ are defined by the same relations. All properties of the spaces

$M^c_{\gamma,\Lambda}$ which we discuss below, remain true under this generalisation, with the same proof.

Clearly, if $A \in M^c_{\gamma,\,\Lambda}$ and $s_j = ta + b_j$ satisfies (2.2) for $j = 1,\,2,$ then

$$A_{s_1 \pm s_2} = A^{\infty\pm}_{a,\,b_1,\,b_2} + t^{-1}\mathcal{O}(1)A(a,\,b_1,\,b_2;\,\Lambda). \tag{2.3}$$

Here and below $\mathcal{O}(1)$ stands for a function of $t,\ a,\ b_1$ and b_2, bounded by one in modulus if (2.2) holds. The first term in the r.h.s. may be bounded in terms of $|||A|||_{\gamma,\,\Lambda}$. Indeed, since for $t \gg 1$ we have $[(ta+b_1) \pm (ta+b_2)] = |b_1 - b_2|$, then

$$|A^{\infty\pm}_{a,b_1,b_2}| = \lim_{t\to\infty} |A_{(ta+b_1)\pm(ta+b_2)}| \le e^{-\gamma|b_1-b_2|}|||A|||_\gamma \qquad \forall\, 0 \ne a,\, b_1, b_2 \in \mathbb{Z}^d\,. \tag{2.4}$$

For $a \in \mathbb{Z}^d \setminus \{0\}$, $\Lambda \ge 6$ and $t \ge 0$ we set

$$O^a_t(\Lambda) = \bigcup_{b\in\mathbb{Z}^d} \{(ta+b) \mid |ta+b| \ge \Lambda(1+|a|+|b|)|a|\}.$$

Note that sets $O^a_{t_1}$, $O^a_{t_2}$ with $t_1 \ne t_2$ may intersect each other.

We have

$$(ta+b) \in O^a_t(\Lambda) \;\Rightarrow\; t \ge (\Lambda-1)(1+|a|+|b|) \tag{2.5}$$

and

$$O^a_t(\Lambda_1) \subset O^a_t(\Lambda_2) \quad \text{if} \quad \Lambda_1 > \Lambda_2. \tag{2.6}$$

If $s = ta + b \in O^a_t(\Lambda)$, then $|b| \le \frac{|s|}{\Lambda|a|} \le \frac{t|a|+|b|}{\Lambda|a|}$. Therefore,

$$|b| \le \frac{t}{\Lambda - 1}. \tag{2.7}$$

Let $ta + b_j \in O^a_t(\Lambda)$, $j = 1,\,2$. Then

$$|ta| \ge \Lambda(1+|a|+|b_j|)|a| - |b_j| > |b_j|(\Lambda-1),$$

and $2|ta| > (\Lambda-1)(|b_1|+|b_2|$. Hence,

$$|2ta + b_1 + b_2| > (\Lambda-2)(|b_1|+|b_2|). \tag{2.8}$$

Since $|b_1 - b_2| \le |b_1| + |b_2|$ and $\Lambda \ge 3$, then $|2ta+b_1+b_2| > |b_1-b_2|$. Therefore,

$$[(ta+b_1)-(ta+b_2)] = [(ta+b_1)-(-ta-b_2)] = |b_1-b_2| \qquad \forall (ta+b_1),\,(ta+b_2) \in O^a_t(\Lambda). \tag{2.9}$$

Similar,

$$\langle (ta+b), ta\rangle \ge |ta|(|ta|-|b|) > 0 \tag{2.10}$$

if $ta + b \in O^a_t(\Lambda)$.

Lemma 2.1. *Let $\Lambda_1 - 3 \geq \Lambda_2 \geq 6$ and $s_1 = ta + b_1 \in O_t^a(\Lambda_1)$, $s_2 = ta + b_2 \notin O_t^a(\Lambda_2)$. Then*

$$|s_1 - s_2| \geq t\left(\frac{1}{\Lambda_2 + 1} - \frac{1}{\Lambda_1 - 1}\right) > t\,\frac{\Lambda_1 - \Lambda_2 - 2}{\Lambda_1^2}. \tag{2.11}$$

Proof. Let us denote $\Lambda_1' = \Lambda_1 - 1$ and $\Lambda_2' = \Lambda_2 + 1$. Since $s_1 \in O_t^a(\Lambda_1)$ and $\Lambda_1|a| - 1 \geq \Lambda_1'|a|$, then

$$|b_1| \leq \frac{t|a| - \Lambda_1|a|(1 + |a|)}{\Lambda_1|a| - 1} \leq \frac{t - \Lambda_1(1 + |a|)}{\Lambda_1'} \leq \frac{t}{\Lambda_1'} - (1 + |a|).$$

Since $s_2 \notin O_t^a(\Lambda_2)$ and $\Lambda_2|a| + 1 \leq \Lambda_2'|a|$, then

$$|b_2| \geq \frac{t|a| - \Lambda_2|a|(1 + |a|)}{\Lambda_2|a| + 1} \geq \frac{t - \Lambda_2'(1 + |a|)}{\Lambda_2'} \geq \frac{t}{\Lambda_2'} - (1 + |a|).$$

As $|s_1 + s_2| \geq |b_2| - |b_1|$, then (2.11) follows. □

Lemma 2.2. *If under the assumptions of Lemma 2.1 $\langle a, s_2\rangle \geq 0$, then*

$$[s_1 - s_2] \geq t\Lambda_1^{-2}. \tag{2.12}$$

Proof. Due to (2.11), $|s_1 - s_2| \geq$ r.h.s. of (2.12), so we only have to estimate from below $|s_1 + s_2|$. Let us set $m = \lceil |b_1||a|^{-1}\rceil$, where $\lceil x\rceil$ stands for the smallest integer $\geq x$. We rewrite s_1 as

$$s_1 = (t - m)a + b_1', \quad b_1' = b_1 + ma.$$

Clearly,

$$\langle a, b_1'\rangle \geq 0. \tag{2.13}$$

Due to (2.7), $|b_1| \leq \frac{t}{\Lambda - 1}$. Therefore

$$m \leq \left\lceil \frac{t}{|a|(\Lambda - 1)}\right\rceil < \frac{2t}{|a|(\Lambda - 1)} \leq \frac{t}{4}$$

(we use (2.5) to get the second inequality and use that $\Lambda \geq \Lambda' + \Lambda_\Delta \geq 9$ to get the third one). Hence, $t - m \geq \frac{3}{4}t$. We have

$$|s_1 + s_2|^2 = |s_2|^2 + |(t - m)a|^2 + |b_1'|^2 + 2t\langle a,\, s_2\rangle + 2\langle b_1',\, s_2\rangle + 2t\langle b_1',\, a\rangle.$$

Since $\langle a, s_2\rangle \geq 0$ by assumption, then using (2.13) we get that

$$|s_1 + s_2|^2 \geq |s_2|^2 + |(t - m)a|^2 + |b_1'|^2 + 2\langle b_1',\, s_2\rangle = (t - m)^2|a|^2 + |s_2 + b_1'|^2.$$

Using that $t - m \geq \frac{3}{4}t$ we get (2.12). □

Elements of the space

$$M^c_{TL} = \bigcup_{|\gamma|\le 1,\, \Lambda\ge 6} M^c_{\gamma,\,\Lambda}$$

are called *Töplitz–Lipschitz matrices.*[1] The linear operators which they define are called *Töplitz–Lipschitz operators*.

Example 2.3. (Töplitz matrices, see [HS78].) Let $\mathcal{A} : \mathbb{Z}^d \to \mathbb{R}$ be a function with compact support, and

$$A_{s_1\, s_2} = \mathcal{A}(s_1 - s_2).$$

Then $A_{(ta+b_1)-(ta+b_2)} = \mathcal{A}(2ta+b_1+b_2)$ vanishes if $t \gg 1$, and $A_{(ta+b_1)\,(t\,a+b_2)} = \mathcal{A}(b_1 - b_2)$. So TL1 holds and

$$A^{\infty-}_{a,b_1,b_2} = 0, \quad A^{\infty+}_{a,b_1,b_2} = \mathcal{A}(b_1 - b_2).$$

The properties TL2 and TL3 obviously hold, so Töplitz matrices belong to all spaces $M_{\gamma,\Lambda}$.

In particular, the identity matrix is Töplitz with $\mathcal{A} = \mathbf{1}_{\{\mathbf{0}\}}$. In this case $A_{(ta+b_1)\,(ta+b_2)} - A^{\infty+}_{a,b_1,b_2} = \mathbf{1}_{\{0\}}(2ta + b_1 + b_2) = 0$ since $|2ta + b - 1 + b_2| > 0$. This inequality is obvious if $b_1 = b_2 = 0$, otherwise it follows from (2.8). So, $\langle \mathrm{id}\rangle_{\gamma,\Lambda} = 1$ for any $|\gamma| \le 1, \Lambda \ge 6$. □

Example 2.4. (Hankel matrices, see [HS78].) Let $A_{s_1 s_2} = \mathcal{A}(s_1 + s_2)$, where $\mathcal{A}$ is a function on $\mathbb{Z}^d$ with compact support. Now

$$A^{\infty+}_{a,b_1,b_2} \equiv 0, \quad A^{\infty-}_{a,b_1,b_2} = \mathcal{A}(b_1 - b_2).$$

Again, relations TL1-TL3 hold for all (admissible) γ and Λ, so the Hankel matrices also belong to all spaces $M_{\gamma,\Lambda}$. □

Example 2.5. (Compactly supported matrices). Let $A_{s_1\, s_2}$ be a matrix such that $A_{s_1\, s_2} = 0$ if $|s_1| + |s_2| \ge C_A$ for a suitable constant C_A. This is a Töplitz–Lipschitz matrix such that $\langle A\rangle_{\gamma,\,\Lambda} = 0$, if Λ is sufficiently large. □

Example 2.6. (Tensor product of vectors). Let $y^1, y^2 \in Y_\gamma = Y_{\gamma,m_*}$, where $\gamma \ge 0$ and $m_* \ge 1$. We define $A = y^1 \otimes y^2$, i.e. $A_{ss'} = y^1_s y^2_{s'}$. Then

$$|A_{ss'}| \le \|y^1\|_\gamma \|y^2\|_\gamma e^{-\gamma(|s|+|s'|)} \langle s\rangle^{-m_*} \langle s'\rangle^{-m_*}.$$

So $A^{\infty\pm}_{a,b_1,b_2} = 0$. Due to (2.2), (2.5) and (2.9), for any $ta + b_j \in O^a_t(\Lambda)$, $j = 1, 2$, $\Lambda \ge 6$, we have

$$|ta + b_j| \ge t\left(1 - \frac{1}{\Lambda - 1}\right) \ge \frac{1}{2}t, \quad t \ge 2(\Lambda - 1), \quad |ta + b_1| + |ta + b_2| \ge |b_1 - b_2|.$$

[1]It would be more consistent to call these matrices Hankel–Töplitz–Lipschitz (see below Examples 2.3, 2.4), reserving the name Töplitz–Lipschitz for the matrices, defined below in Section 2.3. Unfortunately, the former name is hardly acceptable.

So we get the estimate

$$
\begin{aligned}
|A_{(ta+b_1)\pm(ta+b_2)}| &\le \|y^1\|_\gamma \|y^2\|_\gamma 4t^{-2} e^{-\gamma(|ta+b_1|+|ta+b_2|)}\\
&\le \|y^1\|_\gamma \|y^2\|_\gamma t^{-1} 2(\Lambda-1)^{-1} e^{-\gamma|b_1-b_2|}.
\end{aligned}
$$

That is,

$$
\langle A\rangle_{\gamma,\Lambda} \le 2(\Lambda-1)^{-1}\|y^1\|_\gamma \|y^2\|_\gamma. \quad \square \tag{2.14}
$$

For matrices, formed by 2×2–blocks, the right definition of Töplitz–Lipschitz matrices in the spaces $M_\gamma^\pm$ (see section 1.4) is the following. Take any $A \in M_\gamma^+$. Then

$$
(pA)^{\infty -}_{a,b_1,b_2} \equiv 0\,, \quad (qA)^{\infty +}_{a,b_1,b_2} \equiv 0\,.
$$

Accordingly, for $A\in M_\gamma^+$ we define

$$
\begin{aligned}
\langle A\rangle^+_{\gamma,\,\Lambda} = \max\big(&\sup_{a\neq 0, b_1, b_2\in\mathbb{Z}^d} e^{\gamma|b_1-b_2|}(pA)^+(a,\,b_1,\,b_2;\,\Lambda)\,,\\
&\sup_{a\neq 0, b_1, b_2\in\mathbb{Z}^d} e^{\gamma|b_1-b_2|}(qA)^-(a,\,b_1,\,b_2;\,\Lambda)\big)\,,
\end{aligned}
$$

and

$$
|||A|||^+_{\gamma,\,\Lambda} = |||A|||^+_\gamma + \langle A\rangle^+_{\gamma,\,\Lambda}\,.
$$

Finally, we set

$$
M^{+c}_{\gamma,\,\Lambda} = \{A\in M^{+c}_\gamma \mid |||A|||^+_{\gamma,\,\Lambda} < \infty\}\,, \quad M^+_{\gamma,\,\Lambda} = M^{+c}_{\gamma,\,\Lambda}\cap M_\gamma\,.
$$

More on the spaces $M^{+c}_{\gamma,\,\Lambda}$ see in [EK05], section on the Töplitz–Lipschitz natrices.

2.2 Multiplication of the Töplitz–Lipschitz matrices

The space M^c_{TL} is linear. In this section we show that M^c_{TL} also is an algebra. The corresponding result follows from

Theorem 2.7. *Let $A\in M^c_{\gamma',\Lambda'}$ and $B\in M^c_{\gamma',\Lambda}$ for some $\gamma'>0$, $\gamma\in[-1,1]$ and $\Lambda'\ge 6$. Then*

$$
|||AB|||_{\gamma,\,\Lambda} \le C|||A|||_{\gamma',\Lambda'}\,|||B|||_{\gamma,\Lambda'}\gamma_\Delta^{-d-1}\Lambda^2, \tag{2.15}
$$

$$
|||BA|||_{\gamma,\,\Lambda} \le C|||A|||_{\gamma',\Lambda'}\,|||B|||_{\gamma,\Lambda'}\gamma_\Delta^{-d-1}\Lambda^2, \tag{2.16}
$$

provided that

$$
\gamma' \ge |\gamma| + 2\gamma_\Delta,\;\; \Lambda\ge\Lambda'+3\,, \tag{2.17}
$$

where $\gamma_\Delta>0$. Moreover,

$$
(AB)^{\infty+}_{a,\,b_1b_2} = \sum_c (A^{\infty+}_{a,\,b_1,\,c}B^{\infty+}_{a,\,c,\,b_2} + A^{\infty-}_{a,\,b_1,\,c}B^{\infty-}_{-a,\,-c,\,-b_1}), \tag{2.18}
$$

and the elements $(AB)^{\infty-}_{a,\,b_1,\,b_2}$ are given by similar formulas.

Remark. Note that in difference with Theorem 1.5, now we estimate the product AB in a norm, which is strictly weaker than both the norm, used for A and the norm, used for B.

Proof. We may assume that

$$|||A|||_{\gamma',\Lambda'} = |||B|||_{\gamma,\Lambda'} = 1\,, \tag{2.19}$$

We have to estimate the matrix elements $(AB)_{s_1 \pm s_2}$, where

$$s_j = ta + b_j \in O_t^a(\Lambda), \quad j = 1,\, 2. \tag{2.20}$$

Since the cases $s_1 + s_2$ and $s_1 - s_2$ are similar, we restrict ourselves to the sign plus.

We have

$$(AB)_{s_1 s_2} = \sum_{s\in\mathbb{Z}^d} D_s, \quad D_s = A_{s_1 s} B_{s s_2}.$$

Let us denote by Σ_a^+ and Σ_a^- the half-spaces

$$\Sigma_a^+ = \{s \mid |s - ta| \le |s + ta|\} = \{s \mid \langle s, a\rangle \ge 0\}, \tag{2.21}$$

$$\Sigma_a^- = \{s \mid |s - ta| > |s + ta|\} = \{s \mid \langle s, a\rangle < 0\}, \tag{2.22}$$

By (2.10)

$$O_t^a := O_t^a(\Lambda') \subset \Sigma_a^+, \quad O_t^{-a} := O_t^{-a}(\Lambda') \subset \Sigma_a^-.$$

Then

$$(AB)_{s_1 s_2} = \sum_{s\in O_t^a} D_s + \sum_{s\in O_t^{-a}} D_s + \sum_{s\in\Sigma_t} D_s, \tag{2.23}$$

where $\Sigma_t = \mathbb{Z}^d \setminus (O_t^a \cup O_t^{-a}) = (\Sigma_a^+ \setminus O_t^a) \cup (\Sigma_a^- \setminus O_t^{-a})$. Writing $s \in O_t^a$ as $s = at + c$ and using (2.3), (2.19) we get

$$D_s = A^{\infty+}_{a,\, b_1,\, c} B^{\infty+}_{a,\, c,\, b_2} + 3t^{-1}\mathcal{O}(1) e^{-\gamma'|b_1 - c| - \gamma|c - b_2|}$$

(note that $t > 1$ due to (2.5)). Hence,

$$\sum_{s\in O_t^a} D_s = \sum_{\{c \mid at + c \in O_t^a\}} A^{\infty+}_{a,\, b_1,\, c} B^{\infty+}_{a,\, c,\, b_2} + C\gamma_\Delta^{-d} t^{-1} \mathcal{O}(1) e^{-\gamma|b_1 - b_2|}. \tag{2.24}$$

Let $c \in \mathbb{Z}^d$ be such that $at + c \notin O_t^a$. Then, by Lemma 2.1,

$$|c - b_j| \ge t\Lambda^{-2},\ j = 1,\, 2. \tag{2.25}$$

Therefore, using (2.4) and Lemma 1.4 we get

$$\begin{aligned} \Bigg| \sum_{\{c \mid at + c \notin O_t^a\}} A^{\infty+}_{a,\, b_1,\, c} B^{\infty+}_{a,\, c,\, b_2} \Bigg| &\le e^{-\gamma_\Delta t \Lambda^{-2}} \sum_c e^{-(|\gamma| + \gamma_\Delta)|b_1 - c| - \gamma|c - b_2|} \\ &\le C t^{-1} \gamma_\Delta^{-1} \Lambda^2 \gamma_\Delta^{-d} e^{-\gamma|b_1 - b_2|}\,. \end{aligned} \tag{2.26}$$

So we can replace in (2.24) the summation over $\{c \mid at + c \in O_t^a\}$ by the summation over $\mathbb{Z}^d$:

$$\sum_{s\in O_t^a} D_s = \sum_{c\in\mathbb{Z}^d} A^{\infty+}_{a,\,b_1,\,c} B^{\infty+}_{a,\,c,\,b_2} + C_1 t^{-1}\mathcal{O}(1)\gamma_\Delta^{-d-1}\Lambda^2 e^{-\gamma|b_1-b_2|}. \tag{2.27}$$

Similar,

$$\sum_{s\in O_t^{-a}} D_s = \sum_{c\in\mathbb{Z}^d} A^{\infty-}_{a,\,b_1,\,c} B^{\infty-}_{-a,\,-c,\,-b_2} + C_1 t^{-1}\mathcal{O}(1)\gamma_\Delta^{-d-1}\Lambda^2 e^{-\gamma|b_1-b_2|}. \tag{2.28}$$

It remains to estimate the third sum in (2.23). Let us consider a term D_s with $s \in \Sigma_t$. We assume first that $s \in \Sigma_a^+ \setminus O_t^a$ and write it as $s = at + c$. By Lemma 2.2, $[s - s_1] \geq t\Lambda^2$. This estimate and (2.19), (2.17) imply that

$$|D_s| \leq e^{-\gamma'[s_1-s]-\gamma[s-s_2]} \leq e^{-\gamma_\Delta t\Lambda^{-2}} e^{-(|\gamma|+\gamma_\Delta)[s_1-s]-\gamma[s-s_2]}.$$

Estimating similar the terms D_s with $s \in \Sigma_a^- \setminus O_t^{-a}$ and using Lemma 1.6 we get that

$$\begin{aligned} \Big|\sum_{s\in\Sigma_t} D_s\Big| &\leq e^{-\gamma_\Delta t\Lambda^{-2}} \sum_{s\in\mathbb{Z}^d} e^{-(|\gamma|+\gamma_\Delta)[s_1-s]-\gamma[s-s_2]} \\ &= Ct^{-1}\mathcal{O}(1)\gamma_\Delta^{-d-1} e^{-\gamma[s_1-s_2]}\Lambda^2. \end{aligned} \tag{2.29}$$

Due to (2.27), (2.28) and (2.29) the limit $(AB)^{\infty+}_{a,\,b_1,\,b_2}$ exists and is given by the formulas (2.18). Moreover,

$$(AB)_{s_1 s_2} = (AB)^{\infty+}_{a,\,b_1,\,b_2} + C_1 t^{-1}\mathcal{O}(1)\gamma_\Delta^{-d-1}\Lambda^2 e^{-\gamma|b_1-b_2|}\,.$$

So

$$\langle AB\rangle_{\gamma,\,\Lambda} \leq C|||A|||_{\gamma',\Lambda'}\,|||B|||_{\gamma,\Lambda'}\,\gamma_\Delta^{-d-1}\Lambda^2\,.$$

This estimate and (1.12) imply (2.15).

The estimate (2.16) follows from (2.15) since $\langle C^t\rangle_{\gamma,\,\Lambda} = \langle C\rangle_{\gamma,\,\Lambda}$ and $|||C^t|||_\gamma = |||C|||_\gamma$ for any matrix C and its transposed matrix C^t, and since $(AB)^t = B^t A^t$. ∎

Relations (2.18) (and their analogy for the limits $(AB)^{\infty-}_{a,\,b_1,\,b_2}$) immediately imply the following corollaries:

Corollary 2.8. *If $A^{\infty\pm}_{a,\,b_1,\,b_2}$ and $B^{\infty\pm}_{a,\,b_1,\,b_2}$ are independent of a, then $(AB)^{\infty\pm}_{a,\,b_1,\,b_2}$ are independent of a as well.*

Corollary 2.9. *If $A^{\infty\pm}_{a,\,b_1,b_2}$ and $B^{\infty\pm}_{a,\,b_1,\,b_2}$ depend on b_1, b_2 only through $b_1 - b_2$, then $(AB)^{\infty\pm}_{a,\,b_1,\,b_2}$ also possesses this property.*

Theorem 2.7′. *Assertions of Theorem 2.7 remain true if the spaces $M^c_{\gamma,\Lambda}$ are replaced by $M^{+c}_{\gamma,\Lambda}$ and the norms $|||\cdot|||_{\gamma,\Lambda}$ – by the norms $|||\cdot|||^+_{\gamma,\Lambda}$.*

Proof. Let us write A and B as $A = pA + qA$ and $B = pB + qB$. To estimate, say, the norm of $(pA)(pB)$ we repeat the proof of Theorem 2.7, simplifying it a bit. Namely, we write

$$\big((pA)(pB)\big)_{s_1 s_2} = \sum_{s \in O_t^a} D_s + \sum_{s \notin O_t^a} D_s \,.$$

Any $s = at + c \notin O_t^a$ satisfies the estimate (2.25). So we can replace the second sum by (2.27) and proceed as in the proof of Theorem 2.7 (omitting the sum (2.28)). Other three terms in AB can be estimates similar. □

2.3 A version of the construction.

Let $\mathcal{M}^c{}_\gamma$ be the set of complex matrices with exponential decay of elements off the diagonal. I.e., $A \in \mathcal{M}^c_\gamma$ if

$$|A|_\gamma := \sup_{s,s'} e^{\gamma|s-s'|} |A_{ss'}| < \infty.$$

Clearly, $\mathcal{M}^c_\gamma \subset M^c_\gamma$. If $A \in \mathcal{M}^c_\gamma$, then $A^{\infty-}_{a,b_1,b_2} \equiv 0$. For any $\Lambda \geq 6$ we define the spaces $\mathcal{M}^c_{\gamma,\Lambda}$ as in TL1–TL3, but removing the '-' case. That is, now

$$A(a, b_1, b_2; \Lambda) = \sup\{t|A_{(ta+b_1)\,(ta+b_2)} - A^{\infty+}_{a,b_1,b_2}|\},$$

etc. We define the norm in $\mathcal{M}^c_{\gamma,\Lambda}$ as $|\cdot|_\gamma + \langle\cdot\rangle_{\gamma,\Lambda}$, where the quasi-norm $\langle\cdot\rangle_{\gamma,\Lambda}$ is defined as in TL3 with the case '-' being dropped.

Straightforward analysis of the proof of Theorem 2.7 shows that its assertions remain true for matrices from the spaces $\mathcal{M}^c_{\gamma,\Lambda}$. Evoking the Remark in Section 1.3 we get that

$$|AB|_{\gamma,\Lambda} + |BA|_{\gamma,\Lambda} \leq C\gamma_\Delta^{-d-1}\Lambda^2 |A|_{\gamma',\Lambda'} |B|_{\gamma,\Lambda'}, \tag{2.30}$$

provided that (2.17) holds.

Similar, the assertions of Theorem 3.4 below remains true if we replace the spaces $M^c_{\dots}$ by $\mathcal{M}^c_{\dots}$, and the norms $|||\cdot|||_{\dots}$ by $|\cdot|_{\dots}$.

2.4 Double Töplitz–Lipschitz spaces.

We define the space $M(M^c_{\gamma,\Lambda})$ as the set of all matrices $A \in M^c_{\gamma,\Lambda}$ such that the limiting matrices $A^{\infty+}_a$ and $A^{\infty-}_a$ with the elements $A^{\infty\pm}_{a,b_1,b_2}$ belong to the space $\mathcal{M}^c_{\gamma,\Lambda}$ for all $a \neq 0$, and

$$\big|\, |||A|||_{\gamma,\Lambda} \big| := |||A|||_{\gamma,\Lambda} + \max_{\pm} \sup_{a\neq 0} \langle A^{\infty\pm}_a \rangle_{\gamma,\Lambda} < \infty. \tag{2.31}$$

Elements of the space $M(M^c_{TL}) = \cup_{|\gamma|\leq 1, \Lambda \geq 6} M(M^c_{\gamma,\Lambda})$ are called *double Töplitz–Lipschitz matrices.*

Example 2.10. Let $y^1, y^2 \in Y_\gamma$ and $A = y^1 \otimes y^2$ (see Example 2.6). Then $A \in M(M^c_{\gamma,\Lambda})$ for each Λ since $A_a^{\infty\pm} = 0$. $\square$

Remark. A natural generalisation of the construction above allows to define the spaces $M^c_{\gamma_1,\Lambda_1}(M^c_{\gamma,\Lambda})$. Theorems 2.11 and 3.8, dealing with the double Töplitz–Lipschitz matrices, admit obvious reformulations for the matrices from these new spaces.

Theorem 2.11. *If $A \in M(M^c_{\gamma',\Lambda'}), B \in M(M^c_{\gamma,\Lambda'})$ and (2.17) holds, then*

$$\big| \,|||AB|||_{\gamma,\Lambda}\big| + \big| \,|||BA|||_{\gamma,\Lambda}\big| \le C_1 \gamma_\Delta^{-d-1}\Lambda^2 \big| \,|||A|||_{\gamma',\Lambda'}\big| \times \big| \,|||B|||_{\gamma,\Lambda'}\big|. \tag{2.32}$$

Proof. The assertion follows from Theorem 2.7 and (2.18), (2.30). $\square$

Finally, we note that we can define in a similar way the spaces of n-times Töplitz–Lipschitz matrices. As for the cases $n = 1$ and $n = 2$, these spaces are invariant with respect to multiplication of matrices as well as with respect to taking the flow-maps of the nonautonomous linear systems with coefficients from these spaces.

3 Differential equations with the Töplitz–Lipschitz coefficients

3.1 Equations with coefficients in M^c_γ.

For some $0 < \gamma' \le 1$, let $A(t) \in M^c_{\gamma'}$, $t \ge 0$, be a linear operator which continuously depends on t, and

$$|||A(t)|||_{\gamma'} \le \mu \quad \forall\, t, \tag{3.1}$$

where $0 < \mu \le 1$. We consider the linear equation

$$\dot y(t) = A(t) y(t), \tag{3.2}$$

and denote by $S^{t_2}_{t_1}$ the corresponding flow-maps:

$$S^{t_2}_{t_1} y(t_1) = y(t_2),$$

where $y(t)$ is a solution of (3.2). If $t_1 = 0$, we abbreviate $S^t_0 = S^t$.

By Theorem 1.2, $A(t)$ is a bounded linear operator in any space Y^c_γ, $0 < \gamma < \gamma'$, continuous in t. So $S^t_{t_1}$ is a bounded linear operator in each space Y^c_γ as above, C^1-smooth in t. Let us denote $\tilde S(t) = S^t_{t_1} - \mathrm{id}$. The operator $\tilde S$ satisfies the operator equation

$$\frac{d}{dt}\tilde S(t) = A(t)(\tilde S(t) + \mathrm{id}), \quad \tilde S(t_1) = 0. \tag{3.3}$$

By Theorem 1.5, the map $S \to A(t)S$ defines a bounded linear operator in the space M^c_γ. Accordingly, we will view (3.3) as a linear differential equation in this matrix space.

The function $t \mapsto \|\tilde{S}(t)\|\|_\gamma$ is Lipschitz as a composition of two Lipschitz maps. So it is differentiable at almost every point. At such a point t, by the triangle inequality we have

$$\big|\|\tilde{S}(t+\varepsilon)\|\|_\gamma - \|\tilde{S}(t)\|\|_\gamma\big| \leq \|\tilde{S}(t+\varepsilon) - \tilde{S}(t)\|\|_\gamma.$$

Dividing this relation by ε and sending ε to zero we get that

$$\big|\frac{d}{dt}\|\|\tilde{S}(t)\|\|_\gamma\big| \leq \|\|\frac{d}{dt}\tilde{S}(t)\|\|_\gamma \quad \text{a.e.}$$

Due to (3.1), (3.3), the last inequality and Theorem 1.5 we have

$$\big|\frac{d}{dt}\|\|\tilde{S}(t)\|\|_\gamma\big| \leq \|\|A(t)\big(\tilde{S}(t) + \mathrm{id}\big)\|\|_\gamma \leq \mu + C\mu\gamma_\Delta^{-d}\|\|\tilde{S}(t)\|\|_\gamma, \quad \gamma_\Delta = \gamma' - \gamma.$$

Since $\tilde{S}(t_1) = 0$, then the Granwall inequality implies the estimate

$$\|\|\tilde{S}(t)\|\|_\gamma \leq C^{-1}\gamma_\Delta^d\Big(\exp\big(C\mu\gamma_\Delta^{-d}|t-t_1|\big) - 1\Big).$$

Assuming that $C\mu\gamma_\Delta^{-d} \leq 1$ and $|t - t_1| \leq 1$, we have $\|\|\tilde{S}(t)\|\|_\gamma \leq C_1\mu|t - t_1|$.

We have got the following result:

Theorem 3.1. *Let a linear operator $A(t) \in M^c_{\gamma'}$ be continuous in t and satisfies* (3.1)*, and let $\gamma \in [0, \gamma')$ satisfies*

$$C\mu(\gamma' - \gamma)^{-d} \leq 1, \tag{3.4}$$

where C is a sufficiently large constant. Then for any $t_1, t_2 \geq 0$ such that $|t_2 - t_1| \leq 1$ we have

$$\|\|S^{t_2}_{t_1} - \mathrm{id}\,\|\|_\gamma \leq C\mu|t_2 - t_1|.$$

Repeating the arguments above and using the remark to Theorem 1.5, we get the following result which we will need later:

Theorem 3.2. *Let us assume that $|A_{ss'}(t)| \leq \mu e^{-\gamma'|s-s'|}$ for all s and s', where $\gamma' > \gamma > 0$ and* (3.4) *holds. Then for $|t_2 - t_1| \leq 1$ we have*

$$|(S^{t_2}_{t_1})_{ss'} - \delta_{s,s'}| \leq C\mu|t_1 - t_2|e^{-\gamma|s-s'|} \quad \forall\, s, s'.$$

Let X be the algebra of $m \times m$ real matrices ($m \geq 2$), and X^c be the algebra of complex matrices. Let $A(t) \in M^{Xc}_\gamma$ be a linear operator which continuously depends on t. Then it defines the linear differential equation (3.2), where $y(t)$ is an element of the space of sequences ($y_s \in \mathbb{C}^m$, $s \in \mathbb{Z}^d$). Again, it is easy to see that the arguments above apply to prove that *the assertions of Theorem 3.1 remain true if $A(t) \in M^{Xc}_{\gamma'}$, and $S^{t_2}_{t_1}$ is an operator in M^{Xc}_γ.* If $A(t)$ is a matrix, formed by 2×2-blocks (see section 1.4), we can specify the result:

Theorem 3.3. *Let in equation* (3.1) *the operator* $A(t)$ *is such that* $|||A(t)|||^+_{\gamma'} \le \mu$ *for all* t*, for some* $0 < \gamma' \le 1$*. Let* $\gamma \in [0, \gamma')$ *satisfies* $C\mu(\gamma' - \gamma)^{-d} \le 1$*, for a sufficiently large constant* C*. Then for* $t_1, t_2 \ge 0$ *such that* $|t_1 - t_2| \le 1$ *we have*

$$|||S^{t_2}_{t_1} - \mathrm{id}\,|||^+_\gamma \le C_1 \mu |t_2 - t_1|.$$

Proof. It is sufficient to repeat the proof of Theorem 3.1, using Theorem 1.8 instead the Theorem 1.5. □

3.2 Equations with the Töplitz–Lipschitz coefficients

Let $A(t) \in M^c_{\gamma',\,\Lambda'}$, where $\gamma' \in (0, 1]$, $\Lambda' \ge 3$, be a linear operator, continuous in t and such that

$$|||A(t)|||_{\gamma',\,\Lambda'} \le \mu \quad \forall t. \tag{3.5}$$

We continue the study of equation (3.2). Due to Theorem 1.4 the flow-maps S^t, corresponding to this equation, are bounded linear operators in any space M^c_γ, $0 \le \gamma < \gamma'$. Now we show that they also are bounded in the Töplitz–Lipschitz norms $|||\cdot|||_{\gamma,\,\Lambda}$, where $\Lambda > \Lambda'$:

Theorem 3.4. *Let the operator* $A(t)$ *satisfies* (3.5)*, and*

$$\gamma \ge 0,\ \gamma_\Delta > 0,\ \gamma' = \gamma + 2\gamma_\Delta\,; \quad \Lambda' \ge 6\,,\ \Lambda \ge \Lambda' + 6\,. \tag{3.6}$$

Then for t_1, t_2 *such that* $|t_1 - t_2| \le 1$ *we have*

$$|||S^{t_2}_{t_1} - \mathrm{id}\,|||_{\gamma,\,\Lambda} \le C\mu |t_1 - t_2| \gamma_\Delta^{-d-1} \Lambda^2,$$

provided that

$$C_1 \mu \gamma_\Delta^{-d} \le 1 \tag{3.7}$$

for a sufficiently large constant C_1*.*

Proof. To simplify notations we assume that $t_1 = 0$ and re-denote t_2 by t, $|t| \le 1$. Now we cannot repeat the simple proof of Theorem 3.1 since the assertion of Theorem 2.7 on multiplication of Töplitz–Lipschitz matrices is weaker than that of Theorem 1.5 (see the remark to the former theorem). Instead we estimate directly the quasinorm $\langle S^t - \mathrm{id}\rangle_{\gamma,\Lambda}$. To do this we have to study the matrix elements

$$(S^t - \mathrm{id})_{\pm s_1\, s_2}\,,\ s_j = \tau a + b_j \in O^a_\tau(\Lambda),\ \tau \gg 1.$$

Let us consider a solution $y(t)$ of (3.2) such that

$$y(0) = y_0 = l^{s_2},$$

(the vectors l^s are defined in (1.6)). Then

$$y(t) = (y_s(t),\, s \in \mathbb{Z}^d),\ \ y_s(t) = (S^t)_{ss_2}. \tag{3.8}$$

As $|t|\le 1$, then by Theorem 3.1 we have $|y_s(t)-\delta_{s,\,s_2}|\le C\mu e^{-(\gamma+\gamma_\Delta)[s-s_2]}$. In particular, since $s_1,s_2\in O^a_\tau(\Lambda)$, then

$$\begin{aligned}\sum_{s\notin O^a_\tau(\Lambda')\cup O^{-a}_\tau(\Lambda')}\Big|A(t)_{\pm s_1 s}y_s\Big| &\le C\mu\sum_{s\notin O^a_\tau(\Lambda')\cup O^{-a}_\tau(\Lambda')} e^{-\gamma'[\pm s_1-s]-(\gamma+\gamma_\Delta)[s-s_2]}\\ &\le C\mu e^{-\tau\gamma_\Delta\Lambda^{-2}}\sum_{s\in\mathbb{Z}^d}e^{-(\gamma+\gamma_\Delta)([s_1-s]+[s-s_2])}\\ &\le C\mu\gamma_\Delta^{-d}e^{-\tau\gamma_\Delta\Lambda^{-2}}e^{-\gamma|b_1-b_2|}=:\Phi(\tau,b_1,b_2).\end{aligned}\tag{3.9}$$

Here we used Lemmas 2.2 and 1.6 to get, respectively, the second and the third inequality, and used (2.9) to replace $[s_1-s_2]$ by $|b_1-b_2|$.

We define sets Σ^+ and Σ^- by relations (2.21) and (2.22) with t replaced by τ, and represent any $s\in\mathbb{Z}^d$ in the form

$$s=\begin{cases}\tau a+b & \text{if}\quad s\in\Sigma^+,\\ -\tau a-b & \text{if}\quad s\in\Sigma^-.\end{cases}$$

Noting that $\Sigma^+=-\Sigma^-\cup\Sigma^0$, where $\Sigma^0=\{s\mid\langle s,a\rangle=0\}$, we define $\mathbf{y}(t)=\{\mathbf{y}_s(t)\in\mathbb{R}^2,\ s\in\Sigma^+\}$ as follows:

$$\mathbf{y}_s(t)=\begin{cases}(y_s(t),y_{-s}(y))^t, & s\in-\Sigma^-,\\ (y_s(t),0)^t, & s\in\Sigma^0.\end{cases}$$

We denote by $\mathbf{A}=\{\mathbf{A}_{s_1s_2},\ s_1,s_2\in\Sigma^+\}$ the matrix, formed by the 2×2-blocks

$$\mathbf{A}_{s_1s_2}=\begin{pmatrix}A_{s_1s_2} & A_{s_1-s_2}\\ A_{-s_1s_2} & A_{-s_1-s_2}\end{pmatrix},$$

and re-write the equation for $y(t)$ as

$$\dot{\mathbf{y}}_{s_1}=\sum_{s\in\Sigma^+}\mathbf{A}_{s_1s}\mathbf{y}_s,\quad s_1\in\Sigma^+.$$

By the assumptions TL2 and TL3 there exist block-matrices, formed by 2×2-blocks, $A^\infty(a)=\{A^\infty(a)_{b_1b_2}\}$ and $A^\Delta(\tau,a)=\{A^\Delta(\tau,a)_{b_1b_2}\}$, such that

$$\mathbf{A}_{s_1s_2}=A^\infty(a)_{b_1b_2}+\tau^{-1}A^\Delta(\tau,a)_{b_1b_2},$$

and

$$\Big|A^\infty(a)_{b_1b_2}\Big|,\ \Big|A^\Delta(\tau,a)_{b_1b_2}\Big|\le\langle A\rangle_{\gamma',\Lambda'}e^{-\gamma'|b_1-b_2|}\le\mu e^{-\gamma'|b_1-b_2|},\tag{3.10}$$

if $s_1,s_2\in O^a_\tau(\Lambda')$ (cf. (2.3)).

Let us set

$$\mathbf{z}_{s_1}=\sum_{s\in\Sigma^+\setminus O^a_\tau(\Lambda')}\mathbf{A}_{s_1s}\mathbf{y}_s.$$

By (3.9) we have

$$|\mathbf{z}_{s_1}(t)| \le \Phi(\tau, b_1, b_2). \tag{3.11}$$

Using the introduced notations we write the equation for $\mathbf{y}$ as

$$\dot{\mathbf{y}}_{s_1} = \sum_{s=\tau a + c \in O^a_\tau(\Lambda')} \Big(A^\infty(a)_{b_1 c} + \tau^{-1} A^\Delta(\tau, a)_{b_1 c} \Big) \mathbf{y}_s + \mathbf{z}_{s_1}, \quad \mathbf{y}(0) = l^{s_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let us seek the solution $\mathbf{y}$ of this equation as an ansatz

$$\mathbf{y}_{\tau a + b_1}(t) = \mathbf{y}^0_{b_1}(t) + \tau^{-1} \mathbf{y}^1_{b_1}(t), \tag{3.12}$$

where $\mathbf{y}^0_{b_1} = \mathbf{y}^0(a)_{b_1}$, $\mathbf{y}^1_{b_1} = \mathbf{y}^1(a,\tau)_{b_1}$, and $\mathbf{y}^0 = \{\mathbf{y}^0_b(t), b \in \mathbb{Z}^d\}$ satisfies

$$\dot{\mathbf{y}}^0 = A^\infty(a) \mathbf{y}^0, \quad \mathbf{y}^0(0) = l^{b_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.13}$$

Due to (3.10), (3.5) and Theorem 3.2,

$$\Big| \mathbf{y}^0_c(t) - \delta_{c,b_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big| \le C\mu |t| e^{-\gamma |c - b_2|}, \quad c, b_2 \in \mathbb{Z}^d \tag{3.14}$$

(as before, $|t| \le 1$).

For $\mathbf{y}^1$ we get the equation

$$\begin{aligned} \dot{\mathbf{y}}^1_{b_1} = & \sum_c \Big(A^\infty(a)_{b_1 c} + \tau^{-1} A^\Delta(\tau, a)_{b_1 c} \Big) \mathbf{1}_{O^a_\tau(\Lambda')}(\tau a + c) \mathbf{y}^1_c \\ & + \sum_c A^\Delta(\tau, a)_{b_1 c} \mathbf{1}_{O^a_\tau(\Lambda')}(\tau a + c)\, \mathbf{y}^0_c \; + \tau \mathbf{z}_{s_1}, \quad \mathbf{y}^1(0) = 0. \end{aligned} \tag{3.15}$$

Due to (3.10), (3.14) and (1.9),

$$\Big| \sum_c A^\Delta(\tau, a)_{b_1 c} \mathbf{1}_{O^a_\tau(\Lambda')}(\tau a + c) \mathbf{y}^0_c \Big| \le C\mu e^{-\gamma |b_1 - b_2|} \gamma_\Delta^{-d},$$

and due to (3.11),

$$|\tau \mathbf{z}_{s_1}(t)| \le C\mu \gamma_\Delta^{-d-1} \Lambda^2 e^{-\gamma |b_1 - b_2|}.$$

So denoting by $\mathfrak{z}_{b_1}(t)$ the sum of the last two terms in the r.h.s. of equation (3.15), we have from the last two estimates that

$$|\mathfrak{z}_{b_1}(t)| \le C\mu \Lambda^2 \gamma_\Delta^{-d-1} e^{-\gamma |b_1 - b_2|}. \tag{3.16}$$

We re-write (3.15) as

$$\dot{\mathbf{y}}^1_b = \sum_c \Big(\big(A^\infty(a)_{bc} + \tau^{-1} A^\Delta(\tau, a)_{bc} \big) \mathbf{1}_{O^a_\tau(\Lambda')}(\tau a + c) \Big) \mathbf{y}^1_c + \mathfrak{z}_b, \quad \mathbf{y}^1(0) = 0. \tag{3.17}$$

Let $\mathcal{S}_{t_1}^{t_2}$ be the flow-maps of the homogeneous version of this equation (i.e., of equation (3.17) with $\mathfrak{Z}=0$). Then due to (3.10), (3.7) and Theorem 3.2,

$$\left|\left(\mathcal{S}_{t_1}^{t_2}-\mathrm{id}\right)_{b_1 b_2}\right| \le C\mu|t|e^{-(\gamma+\gamma_\Delta)|b_1-b_2|}.$$

Since by (3.17) $\mathbf{y}^1(t)=\int_0^t \mathcal{S}_\theta^t \mathfrak{Z}(\theta)\,d\theta$, then due to (3.16), the remark to Theorem 1.5 and (3.7) we have

$$|\mathbf{y}_{b_1}^1(t)| \le C\mu|t|(1+\mu\gamma_\Delta^{-d})\gamma_\Delta^{-d-1}\Lambda^2 e^{-\gamma|b_1-b_2|} \le C_1\mu|t|\gamma_\Delta^{-d-1}\Lambda^2 e^{-\gamma|b_1-b_2|}. \tag{3.18}$$

By (3.8), $(S^t-\mathrm{id})_{\pm s_1 s_2}=y_{\pm s_1}(t)-\delta_{\pm s_1,s_2}$. Using (3.12), (3.14) and (3.18) we get that

$$\begin{pmatrix} (S^t-\mathrm{id})_{s_1 s_2} \\ (S^t-\mathrm{id})_{-s_1 s_2} \end{pmatrix} = \mathbf{y}^0(a)_{b_1}(t)-\delta_{b_1,b_2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tau^{-1}\mathbf{y}^1(a,\tau)_{b_1}(t), \tag{3.19}$$

where

$$\left|\mathbf{y}^0(a)_{b_1}(t)-\delta_{b_1,b_2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|,\ \left|\mathbf{y}^1(a,\tau)_{b_1}(t)\right| \le C\mu|t|\gamma_\Delta^{-d-1}\Lambda^2 e^{-\gamma|b_1-b_2|},$$

if $s_1,s_2\in O_\tau^a(\Lambda')$. So the quasinorm $\langle S^t-\mathrm{id}\rangle_{\gamma,\Lambda}$ is bounded by $C\mu|t|\gamma_\Delta^{-d-1}\Lambda^2$. This estimate and Theorem 3.1 imply the desired upper bound for $|||S_{t_1}^{t_2}-\mathrm{id}|||_{\gamma,\Lambda}$.

□

The relation (3.19), satisfied by the matrix S^t, implies

Proposition 3.5. *The limit $(S^t-\mathrm{id})_{a,b_1,b_2}^{\infty+}$ is given by the first component of the vector $\mathbf{y}^0(a)_{b_1}(t)$ minus δ_{b_1,b_2}, while the limit $(S^t-\mathrm{id})_{a,b_1,b_2}^{\infty-}$ is given by the second component of $\mathbf{y}^0(a)_{b_1}(t)$, provided that we replace a, b_1 and b_2 by $-a, -b_1$ and $-b_2$, respectively.*

Since $\mathbf{y}^0(a)(t)$ is a solution of linear equation (3.13), then we have the following corollaries:

Corollary 3.6. *If $A(t)_{a,\,b_1,\,b_2}^{\infty\pm}$ is independent of a, then for any t_1,t_2 the limits $(S_{t_1}^{t_2})_{a,\,b_1,\,b_2}^{\infty\pm}$ are independent of a as well.*

Corollary 3.7. *If $A(t)_{a,\,b_1,b_2}^{\infty\pm}$ depends on b_1, b_2 only through b_1-b_2, then $(S_{t_1}^{t_2})_{a,\,b_1,\,b_2}^{\infty\pm}$ also possesses this property.*

For Töplitz–Lipschitz matrices, formed by 2×2–blocks, we have

Theorem 3.4′. *The assertion of Theorem 3.4 remain true if we replace the norms $|||\cdot|||_{\gamma,\Lambda}$ by the norms $|||\cdot|||_{\gamma,\Lambda}^{+}$.*

Proof. Let us write $A(t)=pA(t)+qA(t)$, and denote by $\{S_{t_1}^{pt_2}\}$ (by $\{S_{t_1}^{qt_2}\}$) the flow–maps, corresponding to the operator $pA(t)$ ($qA(t)$). Repeating the proof of Theorem 3.4 (cf. the proof of Theorem 2.7′), we see that the maps $S^{pt_2}_{t_1}$ and $S^{qt_2}_{t_1}$ satisfy the desired estimate. Writing $S^{t_2}_{t_1}$ in terms of the maps S^{pt}_{τ} and S^{qt}_{τ} using the Trotter formula, e.g.

$$S^1_0 = \lim_{N\to\infty} \prod_{j=0}^{N-1} \left(S^{p\,(j+1)N^{-1}}_{jN^{-1}} \cdot S^{q\,(j+1)N^{-1}}_{jN^{-1}} \right),$$

we see that the maps $S^{t_2}_{t_1}$ also satisfy the estimate. □

Now let us consider equation (3.2) with a double Töplitz–Lipschitz operator.

Theorem 3.8. *Let the operator $A(t) \in M(M^c_{\gamma'\Lambda'})$ is continuous in t and satisfies $\left| \| |A(t)| \|_{\gamma',\Lambda'} \right| \le \mu$. Then for t_1, t_2 such that $|t_1 - t_2| \le 1$ we have*

$$\left| \| |S^{t_2}_{t_1} - \mathrm{id}\, | \|_{\gamma,\Lambda} \right| \le C\mu |t_1 - t_2| \gamma_\Delta^{-d-1} \Lambda^2,$$

provided that (3.6) *and* (3.7) *hold.*

Proof. Due to Theorem 3.4, we only have to estimate the limiting matrices $(S^{t_2}_{t_1} - \mathrm{id})^{\infty\pm}_a$. By Proposition 3.5 the matrices $(S^t_0)^{\infty\pm}_a$ are submatrices of the matrix of flow-maps of the limiting equation (3.13). So the needed estimate follows from Theorem 3.2. □

4 Appendix: Estimates for certain Laplace transforms.

In this appendix we prove inequalities (1.8) and (1.9) by majorasing their l.h.s.'s by some Laplace integrals and next estimating these integrals. We start with the first inequality.

Since $0 \le \gamma, \gamma_\Delta \le 1$, then

$$\sum_b e^{\gamma|b| - (\gamma+\gamma_\Delta)[a-b]} \le C \int_{\mathbb{R}^d} e^{-\gamma|z| - (\gamma+\gamma_\Delta)[a-z]}\, dz\,.$$

Denoting $K = \frac{\gamma+\gamma_\Delta}{\gamma} > 1$ and arguing as in (1.13) (with $a = 0$), we estimate the r.h.s. by

$$2C \int_{\mathbb{R}^d} e^{-\gamma(|a-z| + K|z|)}\, dz =: 2C\varphi(\gamma)\,.$$

Next, denoting by $\Phi(z)$ the function $\Phi(z) = |a - z| + K|z|$, and denoting by $\Phi_*(dz)$ the push-forward of the Lebesgue measure dz under the mapping $\Phi : \mathbb{R}^d \to \mathbb{R}_+$, we see that

$$\varphi(\gamma) = \int_0^\infty e^{-\gamma t} \big(\Phi_*(dz)\big)\,(dt)\,.$$

I.e., φ is the Laplace transform of the measure $\Phi_*(dz)$. Considering the distribution function F,

$$F(\tau) = \operatorname{mes} Q_\tau, \qquad Q_\tau = \{z \mid \Phi(z) \le \tau\}\,,$$

we write this measure as $\Phi_*(dz) = dF(\cdot)$. So

$$\varphi(\gamma) = \int_0^\infty e^{-\gamma\tau}\, dF(\tau) = \gamma \int_0^\infty e^{-\gamma\tau} F(\tau)\, d\tau\,. \tag{4.1}$$

Let us first assume that $a \ne 0$ and $\gamma > 0$. Then $L := |a| \ge 1$. Introducing in $\mathbb{R}^d$ a coordinate system such that the first orth is parallel to a, for any $z = (x, y) \in R^d = \mathbb{R} \times \mathbb{R}^{d-1}$ we have

$$\Phi(z) = K|z| + |z - a| = Kr + |(x - L, y)| = Kr + \sqrt{r^2 + L^2 - 2Lx}, \quad r = |z|.$$

Now let us write the set Q_τ as

$$Q_\tau = \bigcup_{0 \le t \le \tau} \mathcal{O}_t, \qquad \mathcal{O}_t = \{z \mid \Phi(z) = t\}.$$

A hypersurface $\mathcal{O}_t$ is formed by points $z = (x, y)$, satisfying $\sqrt{r^2 + L^2 - 2Lx} = t - Kr$. For any such a point z we have

$$t \ge Kr\,, \tag{4.2}$$

and

$$2Lx = r^2 + L^2 - (t - Kr)^2\,.$$

Since $-r \le x \le r$, then

$$-2Lr \le r^2 + L^2 - (t - Kr)^2 \le 2Lr.$$

The second inequality implies $(r - L)^2 \le (t - Kr)^2$, and using (4.2) we get that

$$-(t - Kr) \le r - L \le t - Kr.$$

By the first of these inequalities, $Kr - r \le t - L$. Or

$$r \le \frac{t - L}{K - 1}, \tag{4.3}$$

since $K > 1$. Therefore $\mathcal{O}_t = \emptyset$ if $t < L$. Accordingly,

$$F(\tau) = 0 \quad \text{if} \quad \tau < L.$$

If $\tau \ge L$, then (4.3) implies that $Q_\tau \subset \{|z| \le (\tau - L)/(K - 1)\}$. So

$$F(\tau) = \operatorname{mes} Q_\tau \le \varkappa_d\, (K - 1)^{-d} (\tau - L)^d\,,$$

where $\varkappa_d$ is the volume of a 1-ball in R^d. Due to this estimate and (4.1) we have

$$\begin{aligned}\varphi(\gamma) &\leq \varkappa_d\, \gamma (K-1)^{-d} \int_L^\infty e^{-\gamma\tau}(\tau - L)^d\, d\tau \\ &= \varkappa_d\, \gamma (K-1)^{-d} e^{-\gamma L} \int_0^\infty e^{-\gamma x} x^d\, dx \\ &= \varkappa_d\, \gamma^{-d} (K-1)^{-d} e^{-\gamma L} \int_0^\infty e^{-y} y^d\, dy\,.\end{aligned}$$

That is

$$\varphi(\gamma) \leq C_d \gamma_\Delta^{-d} e^{-\gamma|a|}.$$

This estimate was proved for $a \neq 0$ and $\gamma > 0$. By continuity it also holds for $\gamma = 0$. So (1.8) is proved if $a \neq 0$.

If $a = 0$, then the l.h.s. in (1.8) is bounded by

$$\begin{aligned}C \int_{\mathbb{R}^d} e^{-\gamma|z| - (\gamma+\gamma_\Delta)|z|}\, dz =& C_1 \int_0^\infty r^{d-1} e^{-(2\gamma+\gamma_\Delta)r}\, dr \\ =& C_1 (2\gamma + \gamma_\Delta)^{-d} \int_0^\infty x^{d-1} e^{-x}\, dx,\end{aligned}$$

and (1.8) also holds.

To prove (1.9) we first estimate the l.h.s. by the integral

$$C_1 \int e^{\gamma|z-b| - (\gamma+\gamma_\Delta)|z|}\, dz = C_1 \int e^{-\gamma(K|z| - |z-b|)}\, dz =: C_1 \psi(\gamma)\,.$$

Denoting

$$\Phi(z) = K|z| - |z-b|, \quad F(\tau) = \operatorname{mes}\{z \mid \Phi(z) \leq \tau\}\,,$$

we see that $\Phi \geq -|b|$ and that

$$\psi(\gamma) = \gamma \int_{-|b|}^\infty e^{-\gamma\tau} F(\tau)\, d\tau\,.$$

Analysis of this Laplace integral is similar to that of (4.1) an we omit it.

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