Lowest energy states in nonrelativistic QED: atoms and ions in motion

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Abstract

Within the framework of nonrelativisitic quantum electrodynamics we consider a single nucleus and N electrons coupled to the radiation field. Since the total momentum P is conserved, the Hamiltonian H admits a fiber decomposition with respect to P with fiber Hamiltonian $H(P)$. A stable atom, resp. ion, means that the fiber Hamiltonian $H(P)$ has an eigenvalue at the bottom of its spectrum. We establish the existence of a ground state for $H(P)$ under (i) an explicit bound on P, (ii) a binding condition, and (iii) an energy inequality. The binding condition is proven to hold for a heavy nucleus and the energy inequality for spinless electrons.
Keywords Ground state, binding energy, infrared photons

1 Introduction

An atom, resp. ion, consists of a nucleus with mass m_{n} and charge Ze and N electrons with mass m_{e} and charge $-e$. Within Schrödinger quantum mechanics the atom is described by the Hamiltonian

$$h_N = -\frac{1}{2m_{\mathrm{n}}}\Delta_0 - \sum_{j=1}^{N}\frac{1}{2m_{\mathrm{e}}}\Delta_j + \sum_{1\le i<j\le N}\frac{e^2}{4\pi|x_i - x_j|} - \sum_{j=1}^{N}\frac{Ze^2}{4\pi|x_j - x_0|}, \tag{1}$$

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where the units are such that $\hbar = 1$. Here $x_0 \in \mathbb{R}^3$ is the position of the nucleus, $x_j \in \mathbb{R}^3$ the one of the j-th electron, Δ_j, $j = 0, \ldots, N$, the corresponding Laplacian, and $m_{\mathrm{n}}, m_{\mathrm{e}}, Z > 0$. h_N is regarded as an operator in $L^2(\mathbb{R}^{3(N+1)})$. For the moment we ignore the electron spin and Fermi statistics. h_N commutes with the total momentum

$$P_{\mathrm{tot}} = \sum_{j=0}^{N} p_j, \quad p_j = -\mathrm{i}\nabla_j. \tag{2}$$

Hence, trivially, h_N has purely continuous spectrum. To investigate the stability of the atom one has to first transform to atomic coordinates, see [9]. Then h_N is written as the direct integral

$$h_N = \int_{\mathbb{R}^3}^{\oplus} h(P)\,\mathrm{d}P. \tag{3}$$

$h(P)$ is the Hamiltonian at fixed total momentum P and has the form

$$h(P) = \frac{1}{2m_{\mathrm{tot}}} P^2 + \tilde{h} \tag{4}$$

with $m_{\mathrm{tot}} = m_{\mathrm{n}} + N m_{\mathrm{e}}$. $\tilde{h}$ is independent of P and acts on $L^2(\mathbb{R}^{3N})$, its precise form can be found in Equation (48). The stability of an atom is thus reduced to prove that $\tilde{h}$ has an eigenvalue at the bottom of its spectrum. By a famous result of Zhislin [24], see also [36] such a property holds provided $N < Z+1$. The existence of negatively charged ions is a much more tricky business. We refer to [5] for a survey. Note that a stable atom can move at any speed, since the center of mass kinetic energy is proportional to P^2.

The Coulomb interaction between the charges results from the coupling to the Maxwell field and, in a full quantum theory, also the electromagnetic field has to be quantized. While ultimately such a path leads to relativistic QED, for the present paper we settle at the nonrelativistic version, which has only electrons, nuclei, and photons as elementary objects. Our task is to understand, within the framework of nonrelativistic QED, the stability of atoms and ions *in motion*.

We have to add to (1) the field degrees of freedom and the coupling of the charged particles to the field. For the present study we consider a single nucleus with spin 0 and N spin 1/2 electrons respecting Fermi statistics, which results in the Hamiltonian

$$\begin{aligned} H = & \frac{1}{2m_{\mathrm{n}}}\big(-\mathrm{i}\nabla_0 - ZeA(x_0)\big)^2 + \sum_{j=1}^{N} \frac{1}{2m_{\mathrm{e}}}\big\{\sigma_j \cdot \big(-\mathrm{i}\nabla_j + eA(x_j)\big)\big\}^2 \\ & + \sum_{1\le i<j\le N} \frac{e^2}{4\pi|x_i - x_j|} - \sum_{j=1}^{N} \frac{Ze^2}{4\pi|x_j - x_0|} + H_{\mathrm{f}}. \end{aligned} \tag{5}$$

Here ∇_j is the gradient w.r.t. x_j and σ_j are the Pauli spin matrices of the j-th electron. $A(x)$ is the quantized transverse vector potential and H_{f} the energy of the photons with dispersion relation $\omega(k) = |k|$, see (10), (11) for a precise definition. We use units such that the speed of light $c = 1$. An ultraviolet cutoff is always imposed. Otherwise H would not be properly defined. The infrared cutoff will be studied in detail.

As in the Schrödinger case, H commutes with the total momentum

$$P_{\mathrm{tot}} = P_{\mathrm{f}} + \sum_{j=0}^{N} \big(- \mathrm{i}\nabla_j \big) \tag{6}$$

with P_{f} the momentum of all photons. Hence, if $H(P)$ denotes the Hamiltonian at fixed total momentum P, as before one has the decomposition

$$H = \int_{\mathbb{R}^3}^{\oplus} H(P)\, \mathrm{d}P. \tag{7}$$

The problem is to understand under which conditions $H(P)$ has an eigenvalue at the bottom of its spectrum. Physically the corresponding eigenstate describes a stable atom dressed with a photon cloud and in motion with momentum P.

The case of a single charge, $N = 0$ in our notation, has been studied by J. Fröhlich in his ground-breaking thesis [11]. We borrow many of his insights. For a more current study of the low energy regime we refer to [6]. Very recently, the case of dressed atoms and ions, as governed by Hamiltonian (5), has been taken up by Amour, Grebert, and Guillot [2]. For $N \leq Z$ they succeed to prove that $H(P)$ has a ground state provided $|e|, |P|$, and the ultraviolet cutoff are sufficiently small. Our aim here is to completely avoid such smallness assumptions, by developing a strategy along the lines of [17]. There the author consider the hamiltonian $H_W = H + \sum_j W(x_j)$, i.e. they add a confining one-body potenital W, and prove the existence of absolute ground states provided a binding condition is satisfied. H_W does not conserve the total momentum and a decompostion as in (7) is not possible. If the ground state of H_W exists, then the atom is at rest. Thus, in some sense, the results in [17] cover the case when the total momentum vanishes.

The existence of a ground state for $H(P)$ will be established under four general assumptions. While their precise form will stated in due course, it should be helpful for the reader to understand their meaning in simple terms, first.

(i) $|P| < P_{\mathrm{c}}$ *(Cherenkov radiation).* If a single charge is accelerated to a speed above the speed of light it emits Cherenkov radiation and thereby slows down. Of course, physically, the electron has to move in a medium where light propagates with a speed less than c. Our point is only that the model Hamiltonian (5) knows about Cherenkov radiation. Mathematically Cherenkov radiation is reflected by the fact that there exists some P_{c} such that $H(P)$ has a ground state for $|P| < P_{\mathrm{c}}$, while $H(P)$ has no ground state for $|P| > P_{\mathrm{c}}$. It has been established already in [11] that $P_{\mathrm{c}} > (\sqrt{3} - 1)m_{\mathrm{n}}$ for $N = 0$, see also [35]. Even for $N = 0$, the converse statement, namely no ground state for P sufficiently large, is left as an open problem. To our knowledge, the only result in this direction is provided in

[3], where the case $N = 0$ is studied for small coupling to a scalar field.

(ii) *Energy inequality.* Let $E(P)$ be the bottom of the spectrum of $H(P)$. In our proof we need that

$$E(0) \leq E(P). \tag{8}$$

Physically such a property appears to be obvious. But even for a single charge with spin we have no method to establish (8). We are equally at loss to include Fermi statistics. On the other hand, in Section 7 we prove the inequality (8) for an arbitrary number of spinless charges satisfying Bose/Boltzmann statistics.

(iii) *Strictly positive binding energy.* Roughly speaking the binding condition states that energywise it is more favorable to assemble all electrons near the nucleus compared to having one or several electrons placed at infinity. The presence of the quantized radiation field complicates matter, but we will state a suitable binding condition which reduces to the known condition when the coupling to the field is ignored. Of course, to ensure the existence of a ground state then requires to establish the binding condition. We will prove it for a heavy nucleus and, in greater generality, for electrons without spin.

(iv) *Charge neutrality.* In H of (5) the charge e appears in the Coulomb potential and in the coupling to the quantized transverse vector potential $A(x)$. After all, both originate from the coupling to the Maxwell field. The particular splitting in (5) is due to quantizing in the Coulomb gauge. Mathematically it is often convenient to disregard such a link and to replace the Coulomb potential by a general pair potential. By neutrality we refer here to the charges entering in the coupling to the vector potential.

If $Z = N$, then the quantized radiation field sees a neutral charge. Thus, even for an atom in motion, the induced vector potential decays faster than $1/|x|$, which can indeed be accomodated in Fock space. If $Z \neq N$, then the quantized radiation field sees a non-zero charge. If the atom is at rest, $P = 0$, classically the transverse vector field vanishes and quantum mechanically $A(x)$ averaged in the ground state has a fast decay. On the other hand if, $P \neq 0$, then $A(x)$ decays as $1/|x|$, which cannot be accomodated in Fock space. The putative physical ground state has an infinite number of (virtual) photons. Therefore for $N \neq Z$ a ground state in Fock space can exist only at $P = 0$. Already for a single charge, such a property is a rather delicate phenomenon, see [6] for the best results available. The results in [2] and in our work are in agreement with such general reasoning. For a neutral assembly of charges no infrared cutoff is needed. However for a nonvanishing total charge we have to impose a suitable infrared cutoff.

Perhaps more than in other papers, one of our difficulties concerns the generality in which results are written out. As guiding principle we adopt that at least one physically accepted Hamiltonian should be covered. This requires to work in space dimension $d = 3$ and to have electrons with spin $1/2$. On the other hand the core of the mathematical argument may become hidden through over-explicit notation. For example, we will replace the Coulomb potential by a general pair potential from a class which includes the Coulomb potential, of course. The case of several spinless nuclei could be handled. If no statistics is included, our proof carries over without changes. To include Bose statistics requires extra efforts.

We provide a short outline of our paper. In Section 2 we define the Hamiltonian for charges coupled to the Maxwell field and state the main result, namely the existence of a ground state for $H(P)$ for P within a suitable range and under a strictly positive binding energy. In case of an atom with a heavy nucleus we provide explicit bounds on the range of P and for the validity of the binding condition.

The self-adjointness of $H(P)$ for arbitrary couplings and cutoffs is proven in Section 3. As an essential input we use the same property for the full Hamiltonian as established in [21] by the use of functional integral techniques.

For a single charge some general properties of $E(P) = \inf \operatorname{spec}(H(P))$ are demonstrated in [11]. In Section 4 we show how to extend them to an arbitrary number of charges, in fact in a slightly strengthened version by means of a variational technique.

In Section 5 we consider a non-zero photon mass by replacing in H_{f} the dispersion relation $\omega(k) = |k|$ by $\omega_m(k) = (k^2 + m^2)^{1/2}$, $m > 0$. We assume a strictly positive binding energy and combine the methods in [17] with the general properties of $E(P)$ from Section 4. This yields the existence of a ground state for a suitable range of P's. The remaining task is to remove the infrared cutoff, i.e., $m \to 0$, see Section 6. For a neutral system of charges the form factor $\hat{\varphi}(k)$ is allowed to have $\hat{\varphi}(0) = (2\pi)^{-3/2}$. For a non-neutral system the form factor has to vanish as $\hat{\varphi}(k) \simeq |k|$ for small k. Our method is based on pull-through which yields a bound on the number of soft photons and bounds on the derivative of the ground state wave function with respect to the momenta of the photons. In the appendices we collect some technical results.

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2 Definitions and main results

2.1 Fock space and second quantization

First we recall some basic facts. Let $\mathfrak{h}$ be a Hilbert space. The Fock space over $\mathfrak{h}$ is defined by

$$\mathfrak{F}(\mathfrak{h}) = \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathfrak{h},$$

where $\otimes_{\mathrm{s}}^{n}\mathfrak{h}$ means the n-fold symmetric tensor product of $\mathfrak{h}$ with the convention $\otimes_{\mathrm{s}}^{0}\mathfrak{h} = \mathbb{C}$. The vector $\Omega = 1 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathfrak{h})$ is called the Fock vacuum.

We denote by $a(f)$ the annihilation operator on $\mathfrak{F}(\mathfrak{h})$ with test vector $f \in \mathfrak{h}$ [31, Sec. X.7]. By definition, $a(f)$ is densely defined, closed, and antilinear in f.

The adjoint $a(f)^*$ is the adjoint of $a(f)$ and called the creation operator. Creation and annihilation operators satisfy the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g\rangle_{\mathfrak{h}} \mathbb{1}, \quad [a(f), a(g)] = 0 = [a(f)^*, a(g)^*]$$

on the finite particle subspace

$$\mathfrak{F}_0(\mathfrak{h}) = \bigcup_{m=1}^{\infty} \{\varphi = \varphi_0 \oplus \varphi_1 \oplus \cdots \in \mathfrak{F}(\mathfrak{h}) \,|\, \varphi_n = 0, \text{ for } n \geq m\},$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ denotes the inner product on $\mathfrak{h}$ and $\mathbb{1}$ denotes the identity operator. We introduce a further important subspace of $\mathfrak{F}(\mathfrak{h})$. Let $\mathfrak{s}$ be a subspace of $\mathfrak{h}$. We define

$$\mathfrak{F}_{\text{fin}}(\mathfrak{s}) = \text{Lin}\{a(f_1)^* \ldots a(f_n)^*\Omega, \ \Omega \,|\, f_1, \ldots, f_n \in \mathfrak{s}, \ n \in \mathbb{N}\},$$

where $\text{Lin}\{\cdots\}$ means the linear span of the set $\{\cdots\}$. If $\mathfrak{s}$ is dense in $\mathfrak{h}$, so is $\mathfrak{F}_{\text{fin}}(\mathfrak{s})$ in $\mathfrak{F}(\mathfrak{h})$.

Let b be a contraction operator from $\mathfrak{h}_1$ to $\mathfrak{h}_2$, i.e., $\|b\| \leq 1$. The linear operator $\Gamma(b) : \mathfrak{F}(\mathfrak{h}_1) \to \mathfrak{F}(\mathfrak{h}_2)$ is defined by

$$\Gamma(b) \restriction \otimes_{\text{s}}^n \mathfrak{h}_1 = \otimes^n b$$

with the convention $\otimes^0 b = \mathbb{1}$. It is well known that

$$\Gamma(b)a(f)^* = a(bf)^*\Gamma(b), \quad \Gamma(b)a(b^*f) = a(f)\Gamma(b).$$

For a densely defined closable operator c on $\mathfrak{h}$, $\text{d}\Gamma(c) : \mathfrak{F}(\mathfrak{h}) \to \mathfrak{F}(\mathfrak{h})$ is defined by

$$\text{d}\Gamma(c) \restriction \widehat{\otimes}_{\text{s}}^n \text{dom}(c) = \sum_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes \underset{j\,\text{th}}{c} \otimes \cdots \otimes \mathbb{1}$$

and

$$\text{d}\Gamma(c)\Omega = 0$$

where $\widehat{\otimes}$ means the algebraic tensor product and for any linear operator A, $\text{dom}(A)$ denotes the domain of A. Here in the j-th summand c is at the j-th entry. Clearly $\text{d}\Gamma(c)$ is closable and we denote its closure by the same symbol. As an example, the number operator N_{f} is given by $N_{\text{f}} = \text{d}\Gamma(\mathbb{1})$.

Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be Hilbert spaces. Then there exists an isometry $\mathsf{U} : \mathfrak{F}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \mathfrak{F}(\mathfrak{h}_1) \otimes \mathfrak{F}(\mathfrak{h}_2)$ such that

$$\mathsf{U}\Omega = \Omega \otimes \Omega,$$
$$\mathsf{U}a(h_1 \oplus h_2)\mathsf{U}^* = a(h_1) \otimes \mathbb{1} + \mathbb{1} \otimes a(h_2).$$

2.2 Definition of the Hamiltonian

We consider N electrons with mass $m_{\rm e}$ and charge $-e$, one nucleus with mass $m_{\rm n}$ and charge Ze, moving in 3-dimensional space and coupled to the quantized radiation field. The electrons are fermions with spin 1/2 and the nucleus is spinless. The Hilbert space of state vectors is

$$\mathcal{H}^{N+1} = L^2(\mathbb{R}^3) \otimes [A_N \otimes^N L^2(\mathbb{R}^3; \mathbb{C}^2)] \otimes \mathcal{F},$$

where $\mathcal{F}$ is the Fock space over $\oplus^2 L^2(\mathbb{R}^3)$,

$$\mathcal{F} = \mathfrak{F}(\oplus^2 L^2(\mathbb{R}^3))$$

and A_N denotes the antisymmetrizer. For $f \in L^2(\mathbb{R}^3)$, we define $a_r(f)^\#$, $r = 1, 2$, by

$$a_r(f)^\# = a\Big(\oplus_{j=1}^2 \delta_{rj} f\Big)^\#,$$

where $a^\#$ is either the creation or the annihilation operator on $\mathcal{F}$. It is convenient to use the notation $a_r(f) = \int_{\mathbb{R}^3} \overline{f}(k) a_r(k)\, {\rm d}k$.

The Hamiltonian of our system is the Pauli-Fierz Hamiltonian defined by

$$\begin{aligned} H_N =& \frac{1}{2m_{\rm n}}\Big(-{\rm i}\nabla_0 \otimes 1\!\!1 - ZeA(x_0)\Big)^2 + \sum_{j=1}^N \frac{1}{2m_{\rm e}}\Big\{\sigma_j \cdot \Big(-{\rm i}\nabla_j \otimes 1\!\!1 + eA(x_j)\Big)\Big\}^2 \\ & + V \otimes 1\!\!1 + 1\!\!1 \otimes H_{\rm f}. \end{aligned} \tag{9}$$

Here the quantized vector potential $A(x) = (A_1(x), A_2(x), A_3(x))$ is given by

$$A_\mu(x) = \sum_{r=1,2} \int_{\mathbb{R}^3} \frac{\chi_{\sigma,\kappa}(k)}{\sqrt{2(2\pi)^3 \omega(k)}} \Big\{ a_r(k)^* {\rm e}^{-{\rm i}k\cdot x} + a_r(k)\, {\rm e}^{{\rm i}k\cdot x} \Big\} e^r_\mu(k)\, {\rm d}k, \tag{10}$$

where the form factor $\chi_{\sigma,\kappa}$ $(0 \le \sigma < \kappa < \infty)$ is for simplicity choosen as $\chi_{\sigma,\kappa} = \chi_\kappa - \chi_\sigma$, χ_r with the indicator function of the ball of radius r. σ and κ is the infrared cutoff and ultraviolet cutoff, resp.. The polarization vector are denoted by $e^r = (e^r_1, e^r_2, e^r_3)$, $r = 1, 2$. Together with $k/|k|$ they form a basis, which for concreteness is taken as

$$e^1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \ e^2(k) = \frac{k}{|k|} \wedge e_1(k).$$

Then $e^r(k) \cdot e^s(k) = \delta_{rs}$ and $e^r(k) \cdot k = 0$ a.e.. $\sigma_j = (\sigma_{j1}, \sigma_{j2}, \sigma_{j3})$ denotes the spin matrix for the j-the particle. The Hamiltonian of the free photon field $H_{\rm f}$ is defined by

$$H_{\rm f} = {\rm d}\Gamma(\oplus^2 \omega), \quad \omega(k) = |k|. \tag{11}$$

We will prescribe the following conditions for V.

(V.1) V is a pair potential of the form

$$V(x_0,\dots,x_N)=\sum_{1\le i<j\le N}v(x_i-x_j)+\sum_{j=1}^{N}w(x_0-x_j)$$
$$=:\sum_{0\le i<j\le N}V_{ij}(x_i-x_j).$$

Each V_{ij} is infinitesimally small with respect to $-\Delta$ in the sense that there exists sufficiently small $\varepsilon>0$ and $b_\varepsilon>0$ such that

$$\|V_{ij}f\|\le\varepsilon\|-\Delta f\|+b_\varepsilon\|f\|,\quad f\in\mathrm{dom}(-\Delta),\tag{12}$$

where $-\Delta=-\sum_{j=0}^{N}\Delta_j$.

(V.2) v and w are in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Moreover $V_{ij}(x)\to0$ as $|x|\to\infty$.

As for the self-adjointness of H_N the following result is well-known.

Proposition 2.1 [23] *Assume (V.1). Then, for arbitrary Z, coupling e, mass $m_{\mathrm{e}},m_{\mathrm{n}}>0$ and cutoffs σ,κ with $0\le\sigma<\kappa<\infty$, H_N is self-adjoint on* $\mathrm{dom}(-\Delta\otimes\mathbb{1})\cap\mathrm{dom}(\mathbb{1}\otimes H_{\mathrm{f}})$ *and bounded from below. Moreover H_N is essentially self-adjoint on any core of $-\Delta\otimes\mathbb{1}+\mathbb{1}\otimes H_{\mathrm{f}}$.*

Remark 2.2 The proposition holds also for massive photons, i.e., for the dispersion relation $\omega_m(k)=\sqrt{k^2+m^2}$ instead of $\omega(k)=|k|$. The proof uses the functional integral representation for $m>0$ as established in [22] and is otherwise in essence identical to the one in [23].

Let P_{tot} be the total momentum operator, namely

$$P_{\mathrm{tot}}=-\mathrm{i}\sum_{j=0}^{N}\nabla_j\otimes\mathbb{1}+\mathbb{1}\otimes P_{\mathrm{f}},$$

where $P_{\mathrm{f}}=(P_{\mathrm{f},1},P_{\mathrm{f},2},P_{\mathrm{f},3})=(\mathrm{d}\Gamma(\oplus^2k_1),\mathrm{d}\Gamma(\oplus^2k_2),\mathrm{d}\Gamma(\oplus^2k_3))$ is the momentum operator of the electromagnetic field. Each component $P_{\mathrm{tot},j}$, $j=1,2,3$ of P_{tot} is essentially self-adjoint. We denote its closure by the same symbol $P_{\mathrm{tot},j}$. To obtain $H(P)$ in (7), the Hamiltonian at fixed total momentum P, formally we regard $P_{\mathrm{tot}}=P$ as a parameter and simply substitute in (9) as

$$-\mathrm{i}\nabla_0\otimes\mathbb{1}=P+\mathrm{i}\sum_{j=1}^{N}\nabla_j\otimes\mathbb{1}-\mathbb{1}\otimes P_{\mathrm{f}}.$$

In the resulting Hamiltonian we may then set $x_0=0$. To be more precise, let us define, for all $x_0\in\mathbb{R}^3$,

$$W(x_0)=\exp\{\mathrm{i}x_0\cdot(P_{\mathrm{tot}}+\mathrm{i}\nabla_0\otimes\mathbb{1})\}$$

acting on $\mathcal{H}^N$. Since $x_0 \to W(x_0)$ is strongly continuous, we can define the fiber direct integral operator

$$W = \int_{\mathbb{R}^3}^{\oplus} W(x_0)\,\mathrm{d}x_0$$

acting on $\mathcal{H}^{N+1} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}^N\,\mathrm{d}x_0$, where

$$\mathcal{H}^N = [A_N \otimes^N L^2(\mathbb{R}^3;\mathbb{C}^2)] \otimes \mathcal{F}.$$

Let $\mathcal{U}$ be the Fourier transformation with respect to the variable x_0, acting in $L^2(\mathbb{R}^3) \otimes [A_N \otimes^N L^2(\mathbb{R}^3;\mathbb{C}^2)]$,

$$(\mathcal{U}f)(P, x_1, \ldots, x_N) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}P\cdot x_0} f(x_0, x_1, \ldots, x_N)\,\mathrm{d}x_0.$$

The linear operator $U_F = \mathcal{U} \otimes \mathbb{1}$ is unitary on $\mathcal{H}^{N+1}$. Next we define a unitary operator on $\mathcal{H}^{N+1}$ by $U = U_F W$. The unitary operator U induces the identification of $\mathcal{H}^{N+1}$ with $\int_{\mathbb{R}^3}^{\oplus} \mathcal{H}^N\,\mathrm{d}P$, which is concretely given by

$$\begin{aligned}&(U\psi)^{(n)}(P, x_1, \ldots, x_N, k_1, \ldots, k_n)\\ =&(2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}(P-\sum_{j=1}^n k_j)\cdot x_0} \psi^{(n)}(x_0, x_1 - x_0, \ldots, x_N - x_0, k_1, \ldots, k_n)\,\mathrm{d}x_0\end{aligned}$$

for $\psi = \oplus_{n=0}^{\infty} \psi^{(n)} \in \mathcal{H}^{N+1}$. It is not hard to check that

$$U P_{\mathrm{tot},j} U^* = \int_{\mathbb{R}^3}^{\oplus} P_j\,\mathrm{d}P.$$

Hence the operator U provides the direct integral decomposition of $\mathcal{H}^{N+1}$ with respect to the value of the total momentum. It can be easily seen that $\mathrm{e}^{\mathrm{i}\lambda\cdot P_{\mathrm{tot}}} H_N \subseteq H_N\,\mathrm{e}^{\mathrm{i}\lambda\cdot P_{\mathrm{tot}}}$ for all $\lambda \in \mathbb{R}^3$, i.e., P_{tot} and H_N strongly commute. Thus UH_NU^* is a decomposable operator, i.e., UH_NU^* can be represented by the fiber direct integral

$$UH_NU^* = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(P)\,\mathrm{d}P. \tag{13}$$

Clearly $\mathcal{H}(P)$ is a self-adjoint operator for a.e. P acting in $\mathcal{H}^N$.

We introduce a dense subspace of $\mathcal{H}^N$ by

$$\mathcal{H}^N_{\mathrm{fin}} = [A_N \hat{\otimes}^N C_0^\infty(\mathbb{R}^3_x;\mathbb{C}^2)] \hat{\otimes}\, \mathfrak{F}_{\mathrm{fin}}(\oplus^2 C_0^\infty(\mathbb{R}^3)).$$

On $\mathcal{H}^N_{\mathrm{fin}}$ we can write down $\mathcal{H}(P)$ as follows,

$$\begin{aligned}\mathcal{H}(P) =& \sum_{j=1}^N \frac{1}{2m_{\mathrm{e}}} \Big\{\sigma_j \cdot \Big(-\mathrm{i}\nabla_j \otimes \mathbb{1} + eA(x_j)\Big)\Big\}^2\\ &+ \frac{1}{2m_{\mathrm{n}}} \Big(P + \mathrm{i}\sum_{j=1}^N \nabla_j \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}} - ZeA(0)\Big)^2\\ &+ \tilde{V} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}},\end{aligned} \tag{14}$$

where

$$\tilde{V}(x_1, \ldots, x_N) = \sum_{1 \le i < j \le N} v(x_i - x_j) + \sum_{j=1}^{N} w(x_j). \tag{15}$$

The symmetric operator $H(P)$ is now defined by the right hand side of (14). Clearly $H(P)$ is closable and we denote its closure by the same symbol. Note that, by (14),

$$\mathcal{H}(P) = H(P)$$

on the dense subspace $\mathcal{H}^N_{\mathrm{fin}}$.

2.3 Main results

Our first result concerns the self-adjointness of $H(P)$.

Theorem 2.3 *Assume (V.1). For arbitrary Z, coupling e, cutoffs σ, κ with $0 \le \sigma < \kappa < \infty$ and total momentum P, $H(P)$ is self-adjoint on $\cap_{j=1}^N \mathrm{dom}(-\Delta_j \otimes \mathbb{1}) \cap \mathrm{dom}(\mathbb{1} \otimes P_{\mathrm{f}}^2) \cap \mathrm{dom}(\mathbb{1} \otimes H_{\mathrm{f}})$ and essentially self-adjoint on any core of $-\sum_{j=1}^N \Delta_j \otimes \mathbb{1} + \mathbb{1} \otimes P_{\mathrm{f}}^2 + \mathbb{1} \otimes H_{\mathrm{f}}$. In particular, $H(P)$ is essentially self-adjoint on $\mathcal{H}^N_{\mathrm{fin}}$. Moreover*

$$U H_N U^* = \int_{\mathbb{R}^3}^{\oplus} H(P)\, \mathrm{d}P.$$

We introduce the energy inequality and the binding condition.

Let $H_m(P)$ be the Hamiltonian (14) with the photon dispersion relation $\omega_m(k) = \sqrt{k^2 + m^2}$ and $E_m(P)$ be the infinimum of the spectrum of $H_m(P)$, i.e.,

$$E_m(P) = \inf \mathrm{spec}(H_m(P))$$

with $\mathrm{spec}(A)$ denoting the spectrum of the linear operator A. The *energy inequality* reads

$$E_m(0) \le E_m(P) \tag{E.I.}$$

for any sufficiently small $m \ge 0$. As shorthand we set $H_0(P) = H(P)$ and let $E_0(P) = E(P)$.

Let $\mathcal{D}_R = \{\varphi \in \mathcal{H}^N_{\mathrm{fin}} \mid \varphi(x) = 0 \text{ for } |x| < R\}$ and introduce a threshold energy $\Sigma(P)$ by

$$\Sigma(P) = \lim_{R \to \infty} \Big(\inf_{\varphi \in \mathcal{D}_R, \|\varphi\| = 1} \langle \varphi, H(P) \varphi \rangle \Big). \tag{16}$$

The *binding condition* for our model is stated as

$$\Sigma(P) > E(P). \tag{B.C.}$$

In case of vanishing coupling to the Maxwell field the binding condition reduces to more standard versions based on cluster decomposition, as will be explained in Appendix D. We note that the binding condition depends on the parameter P. Let Λ be the set on which the binding condition is satisfied, i.e.,

$$\Lambda = \{P \in \mathbb{R}^3 \,|\, \Sigma(P) > E(P)\}.$$

First we treat neutral atoms. Mathematically the neutrality condition is expressed as

$$N = Z. \tag{N}$$

Theorem 2.4 *Assume (V.1), (V.2), (E.I.), (N), and the infrared cutoff $\sigma = 0$. If $P \in \Lambda$ and $|P| < m_{\rm n}$, then $H(P)$ has a ground state.*

The condition $P \in \Lambda$ is implicit. But it can be written more explicitly under stronger assumptions.

Let Π_N be the set of the subsets of $\{0, 1, 2, \ldots, N\}$. We denote by H_β the Hamiltonian of the form (9), but only refering to the particles in the set $\beta \in \Pi_N$, i.e.,

$$\begin{aligned} H_\beta =& \frac{1}{2m_{\rm n}}\Big(-\mathrm{i}\nabla_0 \otimes \mathbb{1} - ZeA(x_0)\Big)^2 + \sum_{j\in\beta\setminus\{0\}} \frac{1}{2m_{\rm e}}\Big\{\sigma_j \cdot \Big(-\mathrm{i}\nabla_j \otimes \mathbb{1} + eA(x_j)\Big)\Big\}^2 \\ &+ \sum_{i,j\in\beta, 0\le i<j\le N} V_{ij} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f}, \end{aligned}$$

if $0 \in \beta$, and, if $0 \notin \beta$,

$$\begin{aligned} H_\beta =& \sum_{j\in\beta} \frac{1}{2m_{\rm e}}\Big\{\sigma_j \cdot \Big(-\mathrm{i}\nabla_j \otimes \mathbb{1} + eA(x_j)\Big)\Big\}^2 \\ &+ \sum_{i,j\in\beta, 0\le i<j\le N} V_{ij} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f}. \end{aligned}$$

Let us introduce

$$E_\beta = \inf \mathrm{spec}(H_\beta)\ ,$$

and let $E_N = \inf \mathrm{spec}(H_N)$. (With our notation, $E_N = E_{\{0,1,\ldots,N\}}$.) The binding energy for the Hamiltonian H_N is defined by

$$E_{\rm bin} = \min\big\{E_\beta + E_{\bar\beta} \,|\, \beta \in \Pi_N \,\mathrm{and}\, \beta \ne \emptyset, \{0, 1, \ldots, N\}\big\} - E_N,$$

where $\bar\beta$ denotes the complement of β.

Theorem 2.5 *Assume (V.1), (V.2), (E.I.), (N), and the infrared cutoff $\sigma = 0$. If $E_{\rm bin} > 0$ and*

$$|P| < \min\big\{m_{\rm n}, \sqrt{2m_{\rm n}E_{\rm bin}}\big\},$$

then $H(P)$ has a ground state.

As explained before, for ions we need an infrared cutoff.

Theorem 2.6 *Assume (V.1), (V.2), (E.I.), and a non-neutral system, i.e., (N) does not hold. Suppose that* $\sigma > 0$.

(i) *If* $P \in \Lambda$ *and* $|P| < m_{\mathrm{n}}$, *then* $H(P)$ *has a ground state.*

(ii) *If* $E_{\mathrm{bin}} > 0$ *and* $|P| < \min\{m_{\mathrm{n}}, \sqrt{2m_{\mathrm{n}}E_{\mathrm{bin}}}\}$, *then* $H(P)$ *has a ground state.*

To establish the parameter values for which the binding condition holds is a difficult problem. Indeed, to prove $E_{\mathrm{bin}} > 0$ in case of a fixed nucleus is already very hard work [26]. Thus the reader might worry whether the binding condition can be satisfied at all. We will prove it for m_{n} sufficiently large.

In the limit $m_{\mathrm{n}} \to \infty$, H_N of (9) converges to H_N^∞ defined by

$$H_N^\infty = \sum_{j=1}^{N} \frac{1}{2m_{\mathrm{e}}} \Big\{ \sigma_j \cdot \Big(-\mathrm{i}\nabla_j \otimes \mathbb{1} + eA(x_j) \Big) \Big\}^2 + \tilde{V} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}},$$

where we have set $x_0 = 0$ and $\tilde{V}$ is defined in (15). Let $E_N^\infty = \inf \mathrm{spec}(H_N^\infty)$. With the cluster decomposition from above the binding energy for H_N^∞ is given by

$$E_{\mathrm{bin}}^\infty = \min \big\{ E_\beta^\infty + E_{\bar{\beta}}^\infty \,|\, \beta \subset \{1, \ldots, N\} \,\mathrm{and}\, \beta \neq \emptyset, \{1, \ldots, N\} \big\} - E_N^\infty.$$

In [26] conditions are provided under which $E_{\mathrm{bin}}^\infty > 0$.

Remark 2.7 In [26] $E_{\mathrm{bin}}^\infty > 0$ is proved for molecules and atoms with a smooth cutoff function $\hat{\varphi}$ instead of the sharp cutoff $\chi_{0,\kappa}$ used here. There is no difficulty in extending our main results to a smooth cutoff.

Proposition 2.8 *Assume (V.1), (V.2) and (E.I.). For sufficiently large* m_{n}, *the binding condition (B.C.) holds provided* $|P| < \sqrt{m_{\mathrm{n}} E_{\mathrm{bin}}^\infty}$.

Proof. See Appendix B. □

We remark that Proposition 2.8 is needed as an input for Theorem 2.6.

3 Proof of Theorem 2.3

Theorem 2.3 is proved in using the following strategy. Firstly we define a new Hamiltonian $\tilde{H}(P)$ which is self-adjoint and which coincides with $H(P)$ on a dense domain. Secondly we prove

$$U H_N U^* = \int_{\mathbb{R}^3}^{\oplus} \tilde{H}(P)\, \mathrm{d}P \tag{17}$$

and clarify the domain and the domain of essential self-adjointness of $\tilde{H}(P)$ by applying Proposition 2.1 and (17). Finally we show that this self-adjoint operator equals $H(P)$. Clearly, the essential point lies in the choice of $\tilde{H}(P)$. The reader might think that the simplest way to define a new Hamiltonian $\tilde{H}(P)$ is by just taking the Friedrichs extension $H_1(P)$ of $H(P)$. However, in this case, it seems difficult to establish the measurability of $H_1(P)$ in the sense that the map $P \to \langle \varphi, (H_1(P)+\mathrm{i})^{-1}\psi\rangle$ is measurable. On the other hand, the measurability of $\tilde{H}(P)$ is required to define $\int_{\mathbb{R}^3}^{\oplus} \tilde{H}(P)\,\mathrm{d}P$. Therefore we will adopt another construction for the Hamiltonian $\tilde{H}(P)$. We will see that the construction of the Hamiltonian $\tilde{H}(P)$ which will put to use in Section 7.

3.1 Definitions

Let

$$H_A = \frac{1}{2m_\mathrm{n}}\Big(-\mathrm{i}\nabla_1 \otimes \mathbb{1} - ZeA(-x_1)\Big)^2 + \frac{1}{2}\mathbb{1}\otimes H_\mathrm{f}.$$

By Proposition 2.1, H_A is self-adjoint on $\mathrm{dom}(-\Delta_1 \otimes \mathbb{1}) \cap \mathrm{dom}(\mathbb{1}\otimes H_\mathrm{f})$ for all e and cutoffs. For all $P \in \mathbb{R}^3$, let $\mathcal{V}(P)$ be a unitary operator defined by

$$\mathcal{V}(P) = \exp\Big\{\mathrm{i}x_1 \cdot \Big(P + \mathrm{i}\sum_{j=2}^{N} \nabla_j \otimes \mathbb{1} - \mathbb{1}\otimes P_\mathrm{f}\Big)\Big\}. \tag{18}$$

We introduce $K(P)$ by

$$K(P) = \mathcal{V}(P) H_A \mathcal{V}(P)^*, \tag{19}$$

then $K(P)$ is also self-adjoint for all e and $P \in \mathbb{R}^3$, and

$$K(P)\Psi = \frac{1}{2m_\mathrm{n}}\Big(P + \mathrm{i}\sum_{j=1}^{N}\nabla_j \otimes \mathbb{1} - \mathbb{1}\otimes P_\mathrm{f} - ZeA(0)\Big)^2\Psi + \frac{1}{2}\mathbb{1}\otimes H_\mathrm{f}\Psi$$

for $\Psi \in \mathcal{H}^N_{\mathrm{fin}}$.

Let

$$H_\mathrm{PF} = \sum_{j=1}^{N}\frac{1}{2m_\mathrm{e}}\Big\{\sigma_j\cdot\Big(-\mathrm{i}\nabla_j\otimes\mathbb{1} + eA(x_j)\Big)\Big\}^2 + \tilde{V}\otimes\mathbb{1} + \frac{1}{2}\mathbb{1}\otimes H_\mathrm{f}$$

acting in $\mathcal{H}^N$. By (V.1), $\tilde{V}$ is infinitesimally small with respect to $-\sum_{j=1}^N \Delta_j$. Hence, by Proposition 2.1, H_PF is self-adjoint on $\cap_{j=1}^N \mathrm{dom}(-\Delta_j\otimes\mathbb{1})\cap\mathrm{dom}(\mathbb{1}\otimes H_\mathrm{f})$, essentially self-adjoint on $\mathcal{H}^N_{\mathrm{fin}}$ for arbitrary coupling and cutoffs.

Now we define a densely defined symmetric form s_P as follows

$$\begin{aligned} Q(\mathsf{s}_P) &= \mathrm{dom}(|H_\mathrm{PF}|^{1/2}) \cap \mathrm{dom}(K(P)^{1/2}), \quad \text{(form domain)} \\ \mathsf{s}_P(\varphi,\psi) &= \langle |\hat{H}_\mathrm{PF}|^{1/2}\varphi, |\hat{H}_\mathrm{PF}|^{1/2}\psi\rangle + \langle K(P)^{1/2}\varphi, K(P)^{1/2}\psi\rangle \\ &\quad + \inf\mathrm{spec}(H_\mathrm{PF})\langle\varphi,\psi\rangle, \end{aligned}$$

for $\varphi, \psi \in Q(\mathsf{s}_P)$, where $\hat{A} = A - \inf \mathrm{spec}(A)$. s_P is closed and semibounded. Let $\tilde{H}(P)$ be the self-adjoint operator associated with s_P. Then $\tilde{H}(P)$ is a self-adjoint extension of $H_{\mathrm{PF}} + K(P)$ and the formula

$$\tilde{H}(P)\Psi = H(P)\Psi$$

holds for all $\Psi \in \mathcal{H}^N_{\mathrm{fin}}$.

Lemma 3.1 *The mapping* $P \to (\tilde{H}(P) + \mathrm{i})^{-1}$ *is measurable, i.e., for all* $\varphi, \psi \in \mathcal{H}^N$*,* $P \to \langle \varphi, (\tilde{H}(P) + \mathrm{i})^{-1}\psi\rangle$ *is a measurable mapping.*

Proof. By Kato's strong Trotter product formula [30, Theorem S.21], we have

$$\mathrm{e}^{-t\tilde{H}(P)} = \text{s-}\lim_{n\to\infty}\left(\mathrm{e}^{-tH_{\mathrm{PF}}/n}\mathrm{e}^{-tK(P)/n}\right)^n. \tag{20}$$

Since $P \to \mathrm{e}^{-sK(P)} = \mathcal{V}(P)\mathrm{e}^{-sH_A}\mathcal{V}(P)^*$ is strongly continuous, $P \to \mathrm{e}^{-t\tilde{H}(P)}$ is measurable by (20). Therefore, we obtain the desired assertion. □

Thanks to the above lemma and [32, Theorem XIII.85], one can define a self-adjoint operator H' on $\mathcal{H}^N$ by

$$H' = \int_{\mathbb{R}^3}^{\oplus} \tilde{H}(P)\,\mathrm{d}P.$$

Proposition 3.2

$$UH_NU^* = \int_{\mathbb{R}^3}^{\oplus} \tilde{H}(P)\,\mathrm{d}P.$$

To prove this we need some preparations. Let

$$L = -\sum_{j=1}^N \Delta_j \otimes \mathbb{1} + \left(k \otimes \mathbb{1} + \mathrm{i}\sum_{j=1}^N \nabla_j \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}}\right)^2 + \mathbb{1} \otimes H_{\mathrm{f}}. \tag{21}$$

L is closable and we denote its closure by the same symbol.

Lemma 3.3 L *is essentially self-adjoint on*

$$\mathcal{V} = \left[A_N \hat{\otimes}^N C_0^\infty(\mathbb{R}^3_x; \mathbb{C}^2)\right] \hat{\otimes}\, C_0^\infty(\mathbb{R}^3_k)\, \hat{\otimes}\, \mathfrak{F}_{\mathrm{fin}}(\oplus^2 C_0^\infty(\mathbb{R}^3)). \tag{22}$$

and

$$L = U(-\Delta \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}})U^*. \tag{23}$$

Proof. Essential self-adjointness of L on $\mathcal{V}$ is proven by Nelson's commutator theorem [31, Theorem X.37] with a test operator $J = -\sum_{j=1}^N \Delta_j \otimes \mathbb{1} + k^2 \otimes \mathbb{1} + \mathbb{1} \otimes P_{\mathrm{f}}^2 + \mathbb{1} \otimes H_{\mathrm{f}} + \mathbb{1} \otimes \mathbb{1}$. We can confirm that (23) holds on $\mathcal{V}$. Since $\mathcal{V}$ is a core of L, we conclude (23) as an operator equality. □

Proof of Proposition 3.2

By Proposition 2.1 and the above lemma, UH_NU^* is essentially self-adjoint on $\mathcal{V}$. On $\mathcal{V}$ we can check that $UH_NU^* = H'$ which implies the proposition. □

3.2 Domain of self-adjointness for $H(P)$

We prove Theorem 2.3 by series of lemmata. The first lemma is a simple application of the closed graph theorem.

Lemma 3.4 *Let A and B be self-adjoint operators. Suppose that* $\mathrm{dom}(A) = \mathrm{dom}(B)$*. Then there exists $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1\|\varphi\|_A \leq \|\varphi\|_B \leq C_2\|\varphi\|_A,$$

where, for a linear operator T, $\|\varphi\|_T^2 = \|T\varphi\|^2 + \|\varphi\|^2$ for $\varphi \in \mathrm{dom}(T)$. In particular, A is essentially self-adjoint on any core of B and B is essentially self-adjoint on any core of A.

Proof. Let $\mathcal{D} = \mathrm{dom}(A) = \mathrm{dom}(B)$. Norm spaces $D_A = (\mathcal{D}, \|\cdot\|_A)$ and $D_B = (\mathcal{D}, \|\cdot\|_B)$ are both closed by the self-adjointness of A and B. Now let $i : D_A \to D_B$ defined by

$$i\varphi = \varphi, \quad \varphi \in D_A.$$

Then the graph of i is closed. Indeed let

$$\mathrm{gr}(i) = \{\varphi \oplus i\varphi \,|\, \varphi \in \mathcal{D}\} \subseteq D_A \oplus D_B$$

and let $\{\varphi_n \oplus i\varphi_n\}$ be a Cauchy sequence in $\mathrm{gr}(i)$. Then $\{\varphi_n\}$ is also Cauchy in D_A, D_B and the underlying Hilbert space. Thus there exists $\varphi = \lim_{n\to\infty} \varphi_n \in \mathcal{D}$, $\lim_{n\to\infty} A\varphi_n = A\varphi$ and $\lim_{n\to\infty} B\varphi_n = B\varphi$ by the closedness of self-adjoint operators. Therefore $\{\varphi_n \oplus i\varphi_n\}$ is a convergent sequence in $\mathrm{gr}(i)$. Applying the closed graph theorem, i is bounded and

$$\|\varphi\|_B \leq C\|\varphi\|_A$$

for some constant $C > 0$. From this B is essentially self-adjoint on any core of A. Interchanging the role of A and B, we also conclude the remaining assertion. □

Lemma 3.5 *Let A and B be positive decomposable operators on the Hilbert space $\int_M^\oplus \mathcal{X}\, \mathrm{d}\mu(m)$ with* $\mathrm{dom}(A) = \mathrm{dom}(B)$*. Then* $\mathrm{dom}(A(m)) = \mathrm{dom}(B(m))$*, furthermore the self-adjoint operator $A(m)$ is essentially self-adjoint on any core of $B(m)$, and the self-adjoint operator $B(m)$ is essentially self-adjoint on any core of* $\mathrm{dom}(A(m))$ *for μ-a.e. m.*

Proof. By Lemma 3.4, there is a constant $d > 0$ so that

$$\|A\varphi\| \leq d(\|B\varphi\| + \|\varphi\|), \quad \varphi \in \mathrm{dom}(A).$$

Hence $C := A(B + \mathbb{1})^{-1}$ is a bounded operator.

Since A and $(B+\mathbb{1})^{-1}$ are both decomposable, C is also decomposable. Therefore we can represent C as $C = \int_M^\oplus C(m)\, \mathrm{d}\mu(m)$. Moreover it is not hard to check that $C(m) = A(m)(B(m)+\mathbb{1})^{-1}$ for μ-a.e. m. (Note that $A(m)$ and $B(m)$ are both self-adjoint for μ-a.e..) Hence, $A(m)(B(m)+\mathbb{1})^{-1}$ is bounded and

$$\|A(m)(B(m)+\mathbb{1})^{-1}\| \leq \|C\|$$

for μ-a.e.. Thus $A(m)\,\mathrm{e}^{-tB(m)} = A(m)(B(m)+\mathbb{1})^{-1}(B(m)+\mathbb{1})\,\mathrm{e}^{-tB(m)}$ is bounded for all $t>0$. This means $\mathrm{e}^{-tB(m)}\mathrm{dom}(A(m)) \subseteq \mathrm{dom}(A(m))$ for all $t>0$. By applying [31, Theorem X.49], $B(m)$ is essentially self-adjoint on $\mathrm{dom}(A(m))$. Similarly $A(m)$ is essentially self-adjoint on $\mathrm{dom}(B(m))$ for μ-a.e.. Therefore $\mathrm{dom}(A(m)) = \mathrm{dom}(B(m))$ and we have the desired result by Lemma 3.4. □

Lemma 3.6 *Let*

$$L(P) = -\sum_{j=1}^{N}\Delta_j\otimes\mathbb{1} + \overline{\Big(P + \mathrm{i}\sum_{j=1}^{N}\nabla_j\otimes\mathbb{1} - \mathbb{1}\otimes P_{\mathrm{f}}\Big)^2} + \mathbb{1}\otimes H_{\mathrm{f}}.$$

acting in $\mathcal{H}^N$. *Then, for all* $P\in\mathbb{R}^3$, $L(P)$ *is self-adjoint on* $\cap_{j=1}^N \mathrm{dom}(-\Delta_j\otimes\mathbb{1})\cap\mathrm{dom}(\mathbb{1}\otimes P_{\mathrm{f}}^2)\cap\mathrm{dom}(\mathbb{1}\otimes H_{\mathrm{f}})$, *essentially self-ajoint on* $\mathcal{H}^N_{\mathrm{fin}}$. *Moreover*

$$L = \int_{\mathbb{R}^3}^{\oplus} L(P)\,\mathrm{d}P. \tag{24}$$

Proof. By the functional calculus, we confirm that $\mathrm{dom}(L(P)) = \cap_{j=1}^N\mathrm{dom}(\Delta_j\otimes\mathbb{1})\cap\mathrm{dom}(\mathbb{1}\otimes P_{\mathrm{f}}^2)\cap\mathrm{dom}(\mathbb{1}\otimes H_{\mathrm{f}})$. Thus by applying Lemma 3.4, $L(P)$ is essentially self-adjoint on $\mathcal{H}^N_{\mathrm{fin}}$. On the subspace $\mathcal{V}$ defined by (22) one can easily see (24). Thus we conclude (24) as an operator equality. □

Lemma 3.7 *Let* $\tilde{H}^{V=0}(P)$ *be the Hamiltonian* $\tilde{H}(P)$ *with* $V=0$. *Then, for all* $P\in\mathbb{R}^3$, *there is a finite constant* $C>0$ *independent of* P *such that*

$$\|\tilde{H}^{V=0}(P)\varphi\| \leq C\big(\|\tilde{H}(P)\varphi\| + \|\varphi\|\big), \quad \varphi\in\mathcal{H}^N_{\mathrm{fin}}.$$

Proof. Let $H_N^{V=0}$ be the Hamiltonian H_N with $V=0$. By Proposition 2.1, two self-adjoint operators $H_N^{V=0}$ and H_N have the same domain. Hence there is a constant $C>0$ such that

$$\|UH_N^{V=0}U^*\Psi\| \leq C\big(\|UH_NU^*\Psi\| + \|\Psi\|\big)$$

by Lemma 3.4. Let $M_n(P) = \{k\in\mathbb{R}^3\,|\,|k_j - P_j| \leq \frac{1}{2n},\ j=1,2,3\}$ for $P\in\mathbb{R}^3$. Taking $\Psi(k,x_1,\ldots,x_N) = \eta_n(k)\varphi(x_1,\ldots,x_N)$ with $\eta_n = n^{3/2}\chi_{M_n(P)}$ and $\varphi\in\mathcal{H}^N_{\mathrm{fin}}$, one has

$$\begin{aligned}&\Big(\int_{\mathbb{R}^3}|\eta_n(k)|^2\|\tilde{H}^{V=0}(k)\varphi\|^2\,\mathrm{d}k\Big)^{1/2}\\ \leq& C\Big(\int_{\mathbb{R}^3}|\eta_n(k)|^2\|\tilde{H}(k)\varphi\|^2\,\mathrm{d}k\Big)^{1/2} + C\|\varphi\|\end{aligned}$$

by Proposition 3.2. Noting that $k \to \tilde{H}^{V=0}(k)\varphi$ and $k \to \tilde{H}(k)\varphi$ are strongly continuous, we can conclude the assertion by taking $n \to \infty$. □

Proof of Theorem 2.3
By Proposition 2.1 and (23), UH_NU^* is self-adjoint on $\mathrm{dom}(L)$. By applying Lemma 3.5 and 3.6, $\tilde{H}(P)$ is self-adjoint on $\mathrm{dom}(L(P)) = \cap_{j=1}^N \mathrm{dom}(\Delta_j \otimes \mathbb{1}) \cap \mathrm{dom}(\mathbb{1} \otimes P_{\mathrm{f}}^2) \cap \mathrm{dom}(\mathbb{1} \otimes H_{\mathrm{f}})$ and essentially self-adjoint on $\mathcal{H}_{\mathrm{fin}}^N$ for $P \in \mathbb{R}^3 \backslash \mathcal{N}$ where $\mathcal{N}$ is a measure zero set.

Let $P_0 \in \mathcal{N}$. We introduce a linear operator $\delta_P \tilde{H}(P_0)$ by

$$\delta_P \tilde{H}(P_0) = \tilde{H}(P) - \tilde{H}(P_0).$$

For each $\Psi \in \mathcal{H}_{\mathrm{fin}}^N$ and $P \notin \mathcal{N}$,

$$\begin{aligned}\delta_P \tilde{H}(P_0)\Psi &= [\tilde{H}(P_0) - \tilde{H}(P)]\Psi \\ &= \frac{1}{2m_{\mathrm{n}}}[2(P-P_0)\cdot(X-P) + 3P^2 - P_0^2 - 2P_0 \cdot P]\Psi,\end{aligned}$$

where $X = \mathrm{i}\sum_{j=1}^N \nabla_j \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}} - ZeA(0)$. We prove that there is a constant C independent of P and $B_{P,P_0} > 0$ which is finite for all $P \notin \mathcal{N}$ such that

$$\|\delta_p \tilde{H}(P_0)\Phi\| \leq C|P - P_0|\big(\|\tilde{H}(P)\Phi\| + B_{P,P_0}\|\Phi\|\big) \tag{25}$$

for all $\Phi \in \mathrm{dom}(\tilde{H}(P))$. For $\Psi \in \mathcal{H}_{\mathrm{fin}}^N$ and $j = 1, 2, 3$,

$$\begin{aligned}\|(X_j - P_j)\Psi\| &\leq C_1\big(\|\tilde{H}^{V=0}(P)\Psi\| + \|\Psi\|\big) \\ &\leq C_2\big(\|\tilde{H}(P)\Psi\| + \|\Psi\|\big)\end{aligned}$$

by Lemma 3.7. Note that C_2 does not depend on P. From this, we obtain (25) for $\Phi \in \mathcal{H}_{\mathrm{fin}}^N$. Since $\mathcal{H}_{\mathrm{fin}}^N$ is a core of $\tilde{H}(P)$, we can extend the result to $\mathrm{dom}(\tilde{H}(P))$.

Since $\mathcal{N}$ has measure zero, there is a $P \in \mathbb{R}^3 \backslash \mathcal{N}$ such that $|P - P_0|C < 1$. Thus, by (25) and the Kato-Rellich theorem [31, Theorem X.12], $\tilde{H}(P_0) = \tilde{H}(P) + \delta_P \tilde{H}(P_0)$ is self-adjoint on $\mathrm{dom}(\tilde{H}(P)) = \mathrm{dom}(L(P))$ and essentially self-adjoint on any core of $\tilde{H}(P)$. Since, for all $P \in \mathbb{R}^3$, $\tilde{H}(P)$ is essentially self-adjoint on $\mathcal{H}_{\mathrm{fin}}^N$ and $H(P)\Psi = \tilde{H}(P)\Psi$ for $\Psi \in \mathcal{H}_{\mathrm{fin}}^N$, we have $H(P) = \tilde{H}(P)$ for all P. □

4 Properties of the ground state energy

Let $H_{N,m}$ be the Hamiltonian (9) with the photon dispersion relation $\omega_m(k) = \sqrt{k^2 + m^2}$ instead of $\omega(k) = |k|$. Note that Theorem 2.1 also holds for $H_{N,m}$ with arbitrary $m \geq 0$. Therefore $H_m(P)$ is self-adjoint on $\cap_{j=1}^N \mathrm{dom}(-\Delta_j \otimes \mathbb{1}) \cap \mathrm{dom}(\mathbb{1} \otimes P_{\mathrm{f}}^2) \cap \mathrm{dom}(\mathbb{1} \otimes H_{\mathrm{f},m})$, essentially self-adjoint on $\mathcal{H}_{\mathrm{fin}}^N$ for all e, m, Z, cutoffs and P under the assumptions (V. 1) and (V. 2). We denote the infinimum of the spectrum of $H_{N,m}$ by $E_{N,m}$. The purpose of this section is to prove a few simple properties of the function $E_m(P)$.

Theorem 4.1 *Assume (V.1), (V.2) and (E.I.). For all $m \geq 0$, Z, coupling e, and cutoffs $0 \leq \sigma < \kappa < \infty$, the following assertions hold.*

(i) *The function $f(P) = \frac{1}{2m_{\rm n}}P^2 - E_m(P)$ is a convex function. In particular, $f(P)$ and hence $E_m(P)$ are continuous in P.*

(ii) *For all $P \in \mathbb{R}^3$,*
$$E_m(P) - E_m(0) \leq \frac{1}{2m_{\rm n}}P^2.$$

(iii)
$$E_m(P-k) - E_m(P) \geq \begin{cases} -\frac{|k||P|}{m_{\rm n}} + \frac{k^2}{2m_{\rm n}} & \text{if } |k| \leq |P|, \\ -\frac{P^2}{2m_{\rm n}} & \text{if } |k| \geq |P| \end{cases}.$$

(iv) $E_m(0) = E_{N,m}$.

4.1 Proof of (i)

The functional
$$\langle \Psi, [H_m(P) - \frac{1}{2m_{\rm n}}P^2]\Psi\rangle\ ,$$
is linear in P and hence $E_m(P) - \frac{1}{2m_{\rm n}}P^2$, being the infimum of this expression over all normalized vectors Ψ, is a concave function of P. Thus $f(P)$ is convex. □

4.2 Proof of (ii)

Let T be the time reversal operator which is defined by complex conjugating the wave function, reversing all photon momenta, multiplying by $(-1)^{\mathbb{1}\otimes N_2}$ where $N_2 := \mathrm{d}\Gamma(0 \oplus \mathbb{1})$ is the number operator of photons in the 2 polarization state and multiplying the spinor by $\Pi_{j=1}^{N}\sigma_{j2}$ with $\sigma_j = (\sigma_{j1}, \sigma_{j2}, \sigma_{j3})$ $j = 1, \ldots, N$. Clearly $TP_{\rm f}T = -P_{\rm f}$, $TA(x_j)T = -A(x_j)$ and $TB(x_j)T = -B(x_j)$. Moreover , $T\sigma_j T = -\sigma_j$. Hence $H_m(P)$ and $H_m(-P)$ are (antiunitarily) equivalent and therefore $E_m(-P) = E_m(P)$. From this the function f introduced in (i) satisfies $f(-P) = f(P)$. Since f is convex by (i),
$$-E_m(0) = f(0) = f\big(\frac{1}{2}P - \frac{1}{2}P\big) \leq \frac{1}{2}f(P) + \frac{1}{2}f(-P) = f(P).$$

Thus we conclude (ii). □

4.3 Proof of (iii)

Property (iii) is a direct consequence of the following general proposition:

Proposition 4.2 *Let $F(P)$ be a function that satisfies the following conditions:*

(a) $F(0) \le F(P)$,

(b) $F(P) \le \dfrac{P^2}{2} + F(0)$,

(c) $g(P) = \dfrac{P^2}{2} - F(P)$ *is a convex function.*

Then

$$F(P-k) - F(P) \ge \begin{cases} -|k||P| + \frac{k^2}{2} & \text{if } |k| \le |P|, \\ -\frac{P^2}{2} & \text{if } |k| \ge |P| \end{cases} .$$

Proof. See Appendix A. □

4.4 Proof of (iv)

The inequality $E_{N,m} \le E_m(0)$ is a consequence of the fact that $E_{N,m}$ is given by a less restrictive minimization problem than $E_m(0)$. To prove the converse we simply note that due to the direct integral representation of $H_{N,m}$ in terms of $H_m(P)$ we get that

$$\langle \Psi, H_{N,m}\Psi \rangle \ge \int_{\mathbb{R}^3} |f(P)|^2 E_m(P) dP \tag{26}$$

for some function $f(P)$ with $\int_{\mathbb{R}^3} |f(P)|^2 dP = 1$. Since, by assumption $E_m(0) \le E_m(P)$, the claim is proved. □

5 Existence of the ground state for massive photons

In this section, we concentrate on the existence of a ground state with massive photons, $m > 0$. Throughout this section, we assume (V.1), (V.2), (E.I.) and $m > 0$.

Let $\Sigma_m(P)$ be the threshold energy $\Sigma(P)$ in the case of massive photons. Likewise let Λ_m be the set of P's satisfying the binding condition for the massive case.

Theorem 5.1 *Assume that $\Lambda_m \neq \emptyset$. Then, for $P \in \Lambda_m$ and $|P| < m_{\mathrm{n}}$, $H_m(P)$ has a ground state.*

We will prove this theorem by series of propositions and lemmata. The basic idea of our proof is taken from [17]. The easiest case $N = 1$ will be worked and explicitely. It is not hard to extend this proof to general N.

First we prove the following.

Proposition 5.2 *Let* $\Delta_m(P)=\inf_k[E_m(P-k)-E_m(P)+\omega_m(k)]$ *and let* $\delta_m(P)=\min\{\Delta_m(P),\Sigma_m(P)-E_m(P)\}$. *For all* $P\in\mathbb{R}^3$,

$$\inf \mathrm{ess.\,spec}(H_m(P))\geq E_m(P)+\delta_m(P).$$

We first need some prepreations. Let j_1 and j_2 be two smooth localization functions so that $j_1^2+j_2^2=1$ and j_1 is supported in a ball of radius L. We introduce linear operators $\tilde{j}_1$ and $\tilde{j}_2$ on $\oplus^2L^2(\mathbb{R}^3)$ by

$$\tilde{j}_i(f_1\oplus f_2)=j_i(-\mathrm{i}\nabla_k)f_1\oplus j_i(-\mathrm{i}\nabla_k)f_2,\quad i=1,2.$$

Now we define $j:\oplus^2L^2(\mathbb{R}^3)\to[\oplus^2L^2(\mathbb{R}^3)]\oplus[\oplus^2L^2(\mathbb{R}^3)]$ by $jf=\tilde{j}_1f\oplus\tilde{j}_2f$ for each $f\in\oplus^2L^2(\mathbb{R}^3)$. Note that $j^*j=\mathbb{1}$.

Let U be the isometry from $\mathfrak{F}([\oplus^2L^2(\mathbb{R}^3)]\oplus[\oplus^2L^2(\mathbb{R}^3)])$ to $\mathcal{F}\otimes\mathcal{F}$ given in Section 2.1 and set

$$\check{\Gamma}(j)=\mathsf{U}\Gamma(j):\mathcal{F}\to\mathcal{F}\otimes\mathcal{F}.$$

From the definition it follows that

$$\check{\Gamma}(j)a_r(f)^{\#}=[a_r(j_1(-\mathrm{i}\nabla_k)f)^{\#}\otimes\mathbb{1}+\mathbb{1}\otimes a_r(j_2(-\mathrm{i}\nabla_k)f)^{\#}]\check{\Gamma}(j).$$

Since j is an isometry, so is $\check{\Gamma}(j)$. We remark that, for a multiplication operator h on $L^2(\mathbb{R}^3)$,

$$\begin{aligned}&\big\|\big\{\mathrm{d}\Gamma(\oplus^2h)-\check{\Gamma}(j)^*[\mathrm{d}\Gamma(\oplus^2h)\otimes\mathbb{1}+\mathbb{1}\otimes\mathrm{d}\Gamma(\oplus^2h)]\check{\Gamma}(j)\big\}\Psi\big\|\\ \leq&\big(\big\|[j_1(-\mathrm{i}\nabla_k),h]\big\|+\big\|[j_1(-\mathrm{i}\nabla_k),h]\big\|\big)\|N_{\mathrm{f}}\Psi\|\end{aligned}\tag{27}$$

holds by the definition (or see, e.g., [10, Section 2]).

Let $H_m^{\otimes}(P)$ be a self-adjoint operator on $\mathcal{H}^N\otimes\mathcal{F}$ $(N=1)$ assoiated with the form sum

$$\begin{aligned}&\frac{1}{2m_{\mathrm{e}}}\Big\{\sigma\cdot\Big(p\otimes\mathbb{1}\ +e\mathbb{1}\otimes A(x_1)\Big)\Big\}^2\otimes\mathbb{1}\\ &+\frac{1}{2m_{\mathrm{n}}}\Big(P-p\otimes\mathbb{1}\otimes\mathbb{1}-\mathbb{1}\otimes P_{\mathrm{f}}\otimes\mathbb{1}-\mathbb{1}\otimes\mathbb{1}\otimes P_{\mathrm{f}}-Ze\mathbb{1}\otimes A(0)\otimes\mathbb{1}\Big)^2\\ &+\tilde{V}\otimes\mathbb{1}\otimes\mathbb{1}+\mathbb{1}\otimes H_{\mathrm{f},m}\otimes\mathbb{1}+\mathbb{1}\otimes\mathbb{1}\otimes H_{\mathrm{f},m},\end{aligned}\tag{28}$$

where $p=-\mathrm{i}\nabla_{x_1}$. Note that $H_m^{\otimes}(P)$ can be written as

$$H_{1,m}\otimes\mathbb{1}+\mathbb{1}\otimes H_{\mathrm{f},m}+J^{\otimes}(P),$$

where $J^{\otimes}(P)$ is defined by the second term in (28).

Lemma 5.3 (i) *For* $\varphi\in\mathcal{H}^N_{\mathrm{fin}}$,

$$\langle\varphi,H_m(P)\varphi\rangle=\langle\check{\Gamma}(j)\varphi,H_m^{\otimes}(P)\check{\Gamma}(j)\varphi\rangle+o_L(\varphi)$$

where $o_L(\varphi)$ is the error term which satisfies

$$|o_L(\varphi)| \leq \tilde{o}(L^0)(\|H_m(P)\varphi\|^2 + \|\varphi\|^2).$$

Here $\tilde{o}(L^0)$ is a function of L does not depend on φ and vanishes as $L \to \infty$.
(ii) *For $\varphi \in \mathcal{H}_{\mathrm{fin}}^N \hat{\otimes} \mathfrak{F}_{\mathrm{fin}}(\oplus^2 C_0^\infty(\mathbb{R}^3))$,*

$$\langle \varphi, H_m^{\otimes}(P)\varphi\rangle \geq \langle \varphi, [E_m(P) + (\mathbb{1} - P_\Omega)\Delta_m(P)]\varphi\rangle,$$

where P_Ω is the orthogonal projection onto $\mathcal{H}^N \otimes \Omega$.

Proof. (i) In [17, Lemma A.1] the following assertion has already been proven,

$$\langle \varphi, H_{1,m}\varphi\rangle = \langle \varphi, \check{\Gamma}(j)^*[H_{1,m} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f},m}]\check{\Gamma}(j)\varphi\rangle + o_L(\varphi).$$

So it suffices to prove

$$\langle \varphi, J(P)\varphi\rangle = \langle \varphi, \check{\Gamma}(j)^* J^{\otimes}(P)\check{\Gamma}(j)\varphi\rangle + o_L(\varphi),$$

where

$$J(P) = \frac{1}{2m_{\mathrm{n}}}\Big(P - p \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}} - Ze\mathbb{1} \otimes A(0)\Big)^2.$$

Let $X = P - p \otimes \mathbb{1} - ZeA(0)$. We can easily check

$$J(P) - \check{\Gamma}(j)^* J^{\otimes}(P)\check{\Gamma}(j) = (X - \mathbb{1} \otimes P_{\mathrm{f}})Q + Q(X - \mathbb{1} \otimes P_{\mathrm{f}}) - Q^2,$$

where

$$Q = X - \mathbb{1} \otimes P_{\mathrm{f}} - \check{\Gamma}(j)^*\Big(X \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}} \otimes \mathbb{1} - \mathbb{1} \otimes \mathbb{1} \otimes P_{\mathrm{f}}\Big)\check{\Gamma}(j).$$

Therefore it is enough to show $\|Q\Psi\| = o_L(\Psi)$ for $\Psi \in \mathcal{H}_{\mathrm{fin}}^N$. On the one hand, in [17, Lemma A. 1], it is already proven that

$$\Big\|\Big(X - \check{\Gamma}(j)^* X \otimes \mathbb{1}\check{\Gamma}(j)\Big)\Psi\Big\| = o_L(\Psi).$$

On the other hand, by Theorem 2.3, we have

$$\|\mathbb{1} \otimes N_{\mathrm{f}}\Psi\| \leq C\Big(\|H_m(P)\Psi\| + \|\Psi\|\Big)$$

for some $C > 0$ (which depends on m) and therefore, by (27),

$$\begin{aligned}
&\Big\|\Big[\mathbb{1} \otimes P_{\mathrm{f}i} - \check{\Gamma}(j)^*\Big(\mathbb{1} \otimes P_{\mathrm{f}i} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes P_{\mathrm{f}i}\Big)\check{\Gamma}(j)\Big]\Psi\Big\| \\
\leq&\big(\big\|[j_1(-\mathrm{i}\nabla_k), k_i]\big\| + \big\|[j_2(-\mathrm{i}\nabla_k), k_i]\big\|\big)\,\|\mathbb{1} \otimes N_{\mathrm{f}}\Psi\| \\
\leq&\frac{\text{const.}}{L}\Big(\|H_m(P)\Psi\| + \|\Psi\|\Big), \quad i = 1, 2, 3,
\end{aligned}$$

where we use the fact $\|[j_l(-\mathrm{i}\nabla_k), k_i]\| \le \mathrm{const.}/L$ $(l = 1, 2)$. Hence we have the desired assertion.

(ii) Before we start the proof, we need some preparations. Let S_n be the permutation group of degree n. For $k^{(n)} = (k_1^{(n)}, \dots, k_n^{(n)}) \in \mathbb{R}^{3n}$, $k_j^{(n)} \in \mathbb{R}^3$ and $\sigma \in \mathsf{S}_n$, set $k_\sigma^{(n)} = (k_{\sigma(1)}^{(n)}, \dots, k_{\sigma(n)}^{(n)})$. We introduce a closed subspace $L^2_{\mathrm{sym}}(\mathbb{R}^{3\alpha_1} \times \mathbb{R}^{3\alpha_2})$ of $L^2(\mathbb{R}^{3\alpha_1} \times \mathbb{R}^{3\alpha_2})$, consists of functions satisfying

$$\psi\big(k_{1,\sigma_1}^{(\alpha_1)}; k_{2,\sigma_2}^{(\alpha_2)}\big) = \psi\big(k_1^{(\alpha_1)}; k_2^{(\alpha_2)}\big)$$

for any $\sigma_j \in \mathsf{S}_{\alpha_j}$, $j = 1, 2$. Let h be a multiplication operator on $L^2(\mathbb{R}^3)$ by the function $h(k)$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, we define a linear operator $h^{(\alpha)}$ on $L^2_{\mathrm{sym}}(\mathbb{R}^{3\alpha_1} \times \mathbb{R}^{3\alpha_2})$ by

$$\big(h^{(\alpha)}\psi\big)\big(k_1^{(\alpha_1)}; k_2^{(\alpha_2)}\big) = \sum_{r=1,2} \sum_{l=1}^{\alpha_r} h\big(k_{rl}^{(\alpha_r)}\big)\psi\big(k_1^{(\alpha_1)}; k_2^{(\alpha_2)}\big),$$

where $k_{rl}^{(\alpha_r)}$ is the l-th component of $k_r^{(\alpha_r)} = (k_{r1}^{(\alpha_r)}, \dots, k_{r\alpha_r}^{(\alpha_r)})$. It is well-known that there is a natural identification such that

$$\mathcal{F} = \bigoplus_{\alpha_1,\alpha_2=0}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^{3\alpha_1} \times \mathbb{R}^{3\alpha_2}), \quad \mathrm{d}\Gamma(\oplus^2 h) = \bigoplus_{\alpha_1,\alpha_2=0}^{\infty} h^{(\alpha)}.$$

Note that the Hilbert space $\mathcal{H}^N \otimes \mathcal{F}$ has the following direct sum decomposition:

$$\mathcal{H}^N \otimes \mathcal{F} = \bigoplus_{\alpha \in \mathbb{N}_0^2} \mathcal{H}^N \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3\alpha_1} \times \mathbb{R}^{3\alpha_2}).$$

The restriction of $H_m^{\otimes}(P)$ to the subspace $\mathcal{H}^N \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3\alpha_1} \times \mathbb{R}^{3\alpha_2})$, $\alpha \ne (0,0)$, is given by

$$\begin{aligned}(H_m^{\otimes}\Psi)(k_1^{(\alpha_1)}; k_2^{(\alpha_2)}) =& H_m\Big(P - \sum_{r=1,2}\sum_{l=1}^{\alpha_r} k_{rl}^{(\alpha_r)}\Big)\Psi(k_1^{(\alpha_1)}; k_2^{(\alpha_2)}) \\ &+ \sum_{r=1,2}\sum_{l=1}^{\alpha_r} \omega_m(k_{rl}^{(\alpha_r)})\Psi(k_1^{(\alpha_1)}; k_2^{(\alpha_2)})\end{aligned}$$

for $\Psi \in \mathcal{H}^N \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3\alpha_1} \times \mathbb{R}^{3\alpha_2})$. Thus

$$\begin{aligned}\langle \Psi, H_m^{\otimes}(P)\Psi\rangle \ge& \int \Big[E_m\Big(P - \sum_{r=1,2}\sum_{l=1}^{\alpha_r} k_{rl}^{(\alpha_r)}\Big) + \sum_{r=1,2}\sum_{l=1}^{\alpha_r} \omega_m(k_{rl}^{(\alpha_r)})\Big] \\ &\times \big\|\Psi(k_1^{(\alpha_1)}; k_2^{(\alpha_2)})\big\|^2_{L^2(\mathbb{R}^3)\otimes\mathcal{F}} \mathrm{d}k_1^{(\alpha_1)}\mathrm{d}k_2^{(\alpha_2)} \\ \ge& \big(\Delta_m(P) + E_m(P)\big)\|\Psi\|^2, \end{aligned} \tag{29}$$

where we use the fact $\omega_m(k_1) + \omega_m(k_2) \geq \omega_m(k_1 + k_2)$. On the other hand, on "0-particle space" $\mathcal{H}^N \otimes \Omega$, we have

$$\langle \varphi \otimes \Omega, H_m^{\otimes}(P)\varphi \otimes \Omega\rangle \geq E_m(P)\|\varphi \otimes \Omega\|^2. \tag{30}$$

Combining (29) and (30) we obtain (ii). □

Let ϕ and $\bar{\phi}$ be nonnegative C^∞ functions with $\phi^2 + \bar{\phi}^2 = 1$, ϕ identically 1 on the unit ball, and vanishing outside the ball of radius 2. Let $\phi_R(x) = \phi(x/R)$. For any $\Psi \in \mathcal{H}_{\text{fin}}^N$,

$$\begin{aligned}\langle \Psi, H_m(P)\Psi\rangle =& \langle \phi_R\Psi, H_m(P)\phi_R\Psi\rangle + \langle \bar{\phi}_R\Psi, H_m(P)\bar{\phi}_R\Psi\rangle \\ & - C\langle \Psi, (\nabla\phi_R)^2\Psi\rangle - C\langle \Psi, (\nabla\bar{\phi}_R)^2\Psi\rangle \end{aligned} \tag{31}$$

with $C = 1/2m_{\text{n}} + 1/2m_{\text{e}}$. The last two term vanish, if we take $R \to \infty$.

Let

$$\Sigma_{m,R}(P) = \inf_{\varphi \in \mathcal{D}_R,\, \|\varphi\|=1} \langle \varphi, H_m(P)\varphi\rangle.$$

Lemma 5.4 *For all $\Psi \in \operatorname{dom}(H_m(P))$ we have*

$$\begin{aligned}&\langle \Psi, H_m(P)\Psi\rangle \\ \geq& (E_m(P) + \delta_{m,R}(P))\|\Psi\|^2 - \Delta_m(P)\|\phi_R\Gamma(\tilde{j}_1)\Psi\|^2 + o(1)\|\Psi\|^2_{H_m(P)},\end{aligned} \tag{32}$$

where $\delta_{m,R}(P) = \min\{\Delta_m(P), \Sigma_{m,R}(P) - E_m(P)\}$, $o(1)$ is the error term vanishing uniformly in Ψ as both $L, R \to \infty$ and $\|\Psi\|^2_{H_m(P)} := \|H_m(P)\Psi\|^2 + \|\Psi\|^2$.

Proof. Clearly

$$\langle \bar{\phi}_R\Psi, H_m(P)\bar{\phi}_R\Psi\rangle \geq \Sigma_{m,R}(P)\|\bar{\phi}_R\Psi\|^2.$$

Thus, noting $\|P_\Omega\check{\Gamma}(j)\Phi\| = \|\Gamma(\tilde{j}_1)\Phi\|$, we obtain (32) by Lemma 5.3 and (31) for $\Psi \in \mathcal{H}_{\text{fin}}^N$. Since $\mathcal{H}_{\text{fin}}^N$ is a core of $H_m(P)$, this inequality extends to $\operatorname{dom}(H_m(P))$. □

Proof of Proposition 5.2

For any $\lambda \in \operatorname{ess.spec}(H_m(P))$, there is a sequence $\{\Psi_n\}$ such that $\|\Psi_n\| = 1$, $\text{w-}\lim\limits_{n\to\infty} \Psi_n = 0$, and $\lim_{n\to\infty} \|(H_m(P) - \lambda)\Psi_n\| = 0$. For any $n \in \mathbb{N}$,

$$\langle \Psi_n, H_m(P)\Psi_n\rangle \geq E_m(P) + \delta_{m,R}(P) - \Delta_m(P)\|\phi_R\Gamma(\tilde{j}_1)\Psi_n\|^2 + o(1)\|\Psi_n\|^2_{H_m(P)}$$

by Lemma 5.4. First, take $n \to \infty$. Notice that

$$\|\phi_R\Gamma(\tilde{j}_1)\Psi_n\|^2 = \langle \phi_R^2\Gamma(\tilde{j}_1^2)\Psi_n, (\mathbb{1}+p^2\otimes\mathbb{1}+\mathbb{1}\otimes H_{\text{f},m})^{-1/2}(\mathbb{1}+p^2\otimes\mathbb{1}+\mathbb{1}\otimes H_{\text{f},m})^{1/2}\Psi_n\rangle.$$

Since $(\mathbb{1} + p^2 \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{f},m})^{-1/2}\phi_R\Gamma(\tilde{j}_1)$ is compact on every finite particle space and $\langle \Psi_n, \mathbb{1}\otimes N_{\text{f}}\Psi_n\rangle$ is uniformly bounded on account of the positive photon mass, we have $\|\phi_R\Gamma(\tilde{j}_1)\Psi_n\| \to 0$ as $n \to \infty$ and $\lambda \geq E_m(P) + \delta_{m,R}(P) + o(1)(\lambda^2 + 1)$. Taking $R \to \infty$ and $L \to \infty$, we obtain $\lambda \geq E_m(P) + \delta_m(P)$. This means

$$\inf \operatorname{ess.spec}(H_m(P)) \geq E_m(P) + \delta_m(P). \quad \square$$

Proposition 5.5 *For* $|P| < m_{\mathrm{n}}$*,* $\Delta_m(P) > 0$*.*

Proof. Let $\Delta_m(P : k) = E_m(P-k) - E_m(P) + \omega_m(k)$. Note that $\Delta_m(P) \geq \min\{\inf_{|k|\leq|P|} \Delta_m(P : k), \inf_{|k|\geq|P|} \Delta_m(P : k)\}$. Thus, it sufficies to show that $\inf_{|k|\leq|P|} \Delta_m(P : k) > 0$ and $\inf_{|k|\geq|P|} \Delta_m(P : k) > 0$. Applying Theorem 4.1 (iii), we obtain

$$\inf_{|k|\leq|P|} \Delta_m(P : k) \geq \inf_{|k|\leq|P|} \left\{\omega_m(k) - \frac{|k||P|}{m_{\mathrm{n}}}\right\} > 0$$

whenever $|P| < m_{\mathrm{n}}$. Moreover, again by Theorem 4.1 (iii),

$$\begin{aligned}\inf_{|k|\geq|P|} \Delta_m(P : k) &\geq \inf_{|k|\geq|P|} \Big\{ -\frac{P^2}{2m_{\mathrm{n}}} + \omega_m(k)\Big\} \\ &= -\frac{P^2}{2m_{\mathrm{n}}} + \sqrt{P^2+m^2} > 0\end{aligned}$$

whenever $|P| < \sqrt{2m_{\mathrm{n}}\Big(m_{\mathrm{n}} + \sqrt{m_{\mathrm{n}}^2+m^2}\Big)}$. $\square$

Proof of Theorem 5.1
By Proposition 5.5, $\delta_m(P) > 0$ for $P \in \Lambda_m$. Thus, by Proposition 5.2, one has $\inf \operatorname{ess.spec}(H_m(P)) - E_m(P) \geq \delta_m(P) > 0$, which implies Theorem 5.1. $\square$

6 Proof of Theorems 2.4, 2.5, and 2.6

6.1 Exponential decay

By the following lemma we can reduce the binding condition with massive photons to the one with massless ones.

Lemma 6.1 (i) $E_m(P) \to E(P)$ *as* $m \to 0$*.*

(ii) $\Sigma_m(P)$ *is a convergent sequence and* $\lim_{m\to 0} \Sigma_m(P) \geq \Sigma(P)$*.*

(iii) *Suppose that* $P \in \Lambda$*. Then there exists* $\mathsf{m} > 0$ *such that, for all* $\mathsf{m} > m \geq 0$*,* $P \in \Lambda_m$*.*

Proof. (i) For $m_1 \geq m_2$, $H_{m_1}(P) \geq H_{m_2}(P)$. Thus $\{E_m(P)\}$ is monotonically decreasing and $\lim_{m\to 0} E_m(P)$ exists. Clearly $E(P) \leq \lim_{m\to 0} E_m(P)$.

We will prove $E(P) \geq \lim_{m\to 0} E_m(P)$. For arbitrary $\varepsilon > 0$, there is $\varphi \in \mathcal{H}_{\mathrm{fin}}^N$ such that $\|\varphi\| = 1$ and

$$\langle\varphi, H(P)\varphi\rangle \leq E(P) + \varepsilon.$$

Noting $H_m(P) \leq H(P) + m\mathbb{1} \otimes N_{\mathrm{f}}$, we have

$$\begin{aligned}E_m(P) &\leq \langle\varphi, H_m(P)\varphi\rangle \\ &\leq \langle\varphi, [H(P) + m\mathbb{1}\otimes N_{\mathrm{f}}]\varphi\rangle \\ &\leq E(P) + \varepsilon + m\|\mathbb{1}\otimes N_{\mathrm{f}}^{1/2}\varphi\|^2.\end{aligned}$$

Taking the limit $m \to 0$ we obtain

$$\lim_{m\to 0} E_m(P) \le E(P) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{m\to 0} E_m(P) \le E(P)$ follows.

(ii) For $m_1 \ge m_2$, we can easily see that $\Sigma_{m_1}(P) \ge \Sigma_{m_2}(P)$. Accordingly, $\{\Sigma_m(P)\}$ is monotically decreasing and has a finite limit $\tilde{\Sigma}(P) := \lim_{m\to 0} \Sigma_m(P)$. Note that, for all $m > 0$, $\Sigma_m(P) \ge \Sigma(P)$. Thus we have $\tilde{\Sigma}(P) \ge \Sigma(P)$.

(iii) Let $P \in \Lambda$. Then $\alpha = \Sigma(P) - E(P) > 0$. For all $\varepsilon > 0$ so that $\alpha - 2\varepsilon > 0$, there is a $\mathsf{m} > 0$ so that, for all m with $\mathsf{m} - m < \varepsilon$, $|\tilde{\Sigma}(P) - \Sigma_m(P)| < \varepsilon$ and $|E(P) - E_m(P)| < \varepsilon$. Then

$$\begin{aligned}\Sigma_m(P) - E_m(P) &= \Sigma(P) - E(P) + \big\{(\Sigma_m(P) - \tilde{\Sigma}(P)) + (\tilde{\Sigma}(P) - \Sigma(P)) \\ &\quad + (E(P) - E_m(P))\big\} \\ &\ge \alpha - 2\varepsilon > 0.\end{aligned}$$

This means $P \in \Lambda_m$ if $m < \mathsf{m}$. $\Box$

Lemma 6.2 *Let β be a real numbers and $\alpha = \sum_{j=1}^N (1/2m_\mathrm{e}) + N/m_\mathrm{n}$. For $P \in \Lambda$ suppose that $E(P) + \alpha\beta^2 < \Sigma(P)$. For each $P \in \Lambda$ and $|P| < m_\mathrm{n}$, let $\Psi_{P,m}$ be a normalized ground state for $H_m(P)$. Then, for $m > 0$ sufficiently small and R sufficiently large,*

$$\left\|\mathrm{e}^{\beta|x|}\Psi_{P,m}\right\|^2 \le C_\beta\, \mathrm{e}^{4\beta R}\Big(\frac{1}{\Sigma(P) - E(P) - \alpha\beta^2 + o(1)} + 1\Big),$$

where C_β is a positive constant depends on β but independent of R, m, and $o(1)$ is the error term vanishing as $m \to 0$ and $R \to \infty$.

Remark 6.3 Existence of $\Psi_{P,m}$ is guaranteed by Theorem 5.1 and Lemma 6.1 for small m.

Proof. Note first that each $G \in C^\infty(\mathbb{R}^{3N})$ with $G, |\nabla_j G| \in L^\infty(\mathbb{R}^{3N})$,

$$[[H_m(P), G], G] = -\sum_{j=1}^N \frac{1}{m_\mathrm{e}}\big(\nabla_j G\big)^2 - \frac{1}{m_\mathrm{n}}\Big(\sum_{j=1}^N \nabla_j G\Big)^2. \tag{33}$$

Take $G(x) = \chi(x/R)\mathrm{e}^{f(x)}$ where $f(x) = \beta|x|/(1+\varepsilon|x|)$ and $0 \le \chi \le 1$ is a smooth function that is identically 1 outside the ball radius 2, and 0 inside the ball radius 1. With a slight modification of [17, Proof of Lemma 6.2], we get

$$\begin{aligned}&\Big\langle G\Psi_{P,m}, \Big\{H_m(P) - E_m(P) - \sum_{j=1}^N \frac{1}{2m_\mathrm{e}}|\nabla_j f|^2 - \frac{1}{2m_\mathrm{n}}\Big(\sum_{j=1}^N \nabla_j f\Big)^2\Big\} G\Psi_{P,m}\Big\rangle \\ &\le C_\beta\, \mathrm{e}^{4\beta R}\end{aligned} \tag{34}$$

by (33). Using the facts $\Sigma_{m,R}(P) \geq \Sigma_{0,R}(P)$ for any m, $|\nabla_j f| \leq \beta$, and Lemma 6.1 (i), we obtain

$$\begin{aligned}\text{LHS of (34)} &\geq (\Sigma_{m,R}(P) - E_m(P) - \alpha\beta^2)\|G\Psi_{P,m}\|^2 \\ &\geq \big\{\Sigma(P) - E(P) - \alpha\beta^2 + (\Sigma_{0,R}(P) - \Sigma(P)) \\ &\quad + (E(P) - E_m(P))\big\}\|G\Psi_{p,m}\|^2 \\ &= \big(\Sigma(P) - E(P) - \alpha\beta^2 + o(1)\big)\|G\Psi_{P,m}\|^2.\end{aligned}$$

Therefore the assertion follows by taking $\varepsilon \to 0$. □

6.2 A photon number bound and photon derivative bound

Let

$$\begin{aligned}\mathfrak{P}_j &= -\mathrm{i}\nabla_j \otimes \mathbb{1} + eA(x_j), \quad j = 1, \ldots, N, \\ \mathfrak{P}_0 &= P + \mathrm{i}\sum_{j=1}^{N} \nabla_j \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}} - ZeA(0).\end{aligned}$$

For later use we first prove the following.

Lemma 6.4 *Assume (V.1), (V.2) and (E.I.). Suppose that* $|P| < m_{\mathrm{n}}$. *Let* $\Delta_m(P:k) := E_m(P-k) - E_m(P) + \omega_m(k)$. *Then the following assertion hold for any* $m \geq 0$, *coupling* e, *and cutoffs* σ, κ.

(i) $\Delta_m(P:k) \geq (1 - |P|/m_{\mathrm{n}})|k|$. *Thus* $H_m(P-k) - E_m(P) + \omega_m(k)$ *has the bounded inverse, denoted by* $\mathcal{R}_{P,m}(k)$, *for* $k \neq 0$.

(ii) $\|\mathcal{R}_{P,m}(k)\| \leq C/|k|$, *where* C *is a positive constant independent of* m *and* k.

(iii) $\|\mathfrak{P}_{j,l}\mathcal{R}_{P,m}(k)\| \leq C(1 + 1/|k|)$, $j = 0, 1, \ldots, N$, $l = 1, 2, 3$.

(iv) $\|[H_m(P) - E_m(P)]\mathcal{R}_{P,m}(k)\|\chi_{0,\kappa}(k) \leq C(1+\kappa)\chi_{0,\kappa}(k)$.

Proof. (i) If $|k| \leq |P|$, the claim follows by Theorem 4.1 (iii). Suppose that $|k| > |P|$. Then, since $|P||k|/m_{\mathrm{n}} > P^2/2m_{\mathrm{n}}$, we have

$$\begin{aligned}\Delta_m(P:k) &\geq E_m(P-k) - E_m(P) + |k| \\ &\geq |k| - \frac{P^2}{2m_{\mathrm{n}}} \geq |k| - \frac{|P||k|}{m_{\mathrm{n}}} = \Big(1 - \frac{|P|}{m_{\mathrm{n}}}\Big)|k|\end{aligned}$$

by Theorem 4.1. (ii) immediately follows from (i).

(iii) This is a direct consequence of Lemma C.1 and (i).

(iv) Note that

$$[H_m(P) - E_m(P)]\mathcal{R}_{P,m}(k) = \mathbb{1} + [H_m(P) - H_m(P-k) - \omega_m(k)]\mathcal{R}_{P,m}(k).$$

For all $\Psi \in \mathcal{H}_{\mathrm{fin}}^N$,

$$[H_m(P) - H_m(P-k) - \omega_m(k)]\Psi = [2k \cdot (k - \mathfrak{P}_0) - (k^2 + \omega_m(k))]\Psi.$$

Therefore

$$\big\|\big(H_m(P) - H_m(P-k) - \omega_m(k)\big)\Psi\big\| \le 2|k| \sum_{j=1}^{3} \|(\mathfrak{P}_{0,j} - k_j)\Psi\| + (k^2 + \omega_m(k))\|\Psi\|.$$

Since there is a constant C independent of P, m and k such that

$$\|(\mathfrak{P}_{0,j} - k_j)\Psi\| \le C\big(\|H_m(P-k)\Psi\| + \|\Psi\|\big), \quad j = 1, 2, 3,$$

by Lemma C.1, one has

$$\begin{aligned}
&\big\|[H_m(P) - H_m(P-k) - \omega_m(k)]\mathcal{R}_{P,m}(k)\Psi\big\| \\
&\le C\Big[|k|\Big(\|H_m(P-k)\mathcal{R}_{P,m}(k)\Psi\| + \|\mathcal{R}_{P,m}(k)\Psi\|\Big) \\
&\quad + (k^2 + \omega_m(k))\|\mathcal{R}_{P,m}(k)\Psi\|\Big].
\end{aligned}$$

Notice that

$$H_m(P-k)\mathcal{R}_{P,m}(k)\Psi = \Big\{\mathbb{1} + \mathcal{R}_{P,m}(k)[E_m(P) - \omega_m(k)]\Big\}\Psi.$$

Thus, considering $\Delta_m(P:k) \ge (1 - |P|/m_{\mathrm{n}})|k|$ by (i),

$$\begin{aligned}
\|H_m(P-k)\mathcal{R}_{P,m}(k)\| &\le 1 + \Delta_m(P:k)^{-1}|E_m(P) - \omega_m(k)| \\
&\le C\,(1 + |k|^{-1})
\end{aligned}$$

for $|k| \le |P|$. As for $\omega_m(k)\|\mathcal{R}_{P,m}(k)\|$ ($|k| \le |P|$), we have to be more careful. By Theorem 4.1 (iii),

$$\|\mathcal{R}_{P,m}(k)\| \le \Delta_m(P:k)^{-1} \le [\omega_m(k) - |k||P|/m_{\mathrm{n}}]^{-1}$$

and hence

$$\begin{aligned}
\omega_m(k)\|\mathcal{R}_{P,m}(k)\| &\le \omega_m(k)\big[\omega_m(k) - |k||P|/m_{\mathrm{n}}\big]^{-1} \\
&= 1 + \frac{|k||P|/m_{\mathrm{n}}}{\omega_m(k) - |k||P|/m_{\mathrm{n}}} \\
&\le 1 + \frac{|P|}{m_{\mathrm{n}} - |P|} < \infty.
\end{aligned}$$

Combining these results, one concludes that

$$\big\|[H_m(P) - E_m(P)]\mathcal{R}_{P,m}(k)\big\|\chi_{0,\kappa}(k) \le C\,\kappa\chi_{0,\kappa}(k)$$

for $|k| \le |P|$.

Similarly, we have, for $|k| > |P|$,

$$
\begin{aligned}
|k| \, \|H_m(P-k)\mathcal{R}_{P,m}(k)\| &\leq C\,|k|, \\
|k| \, \|\mathcal{R}_{P,m}(k)\| &\leq C, \\
\omega_m(k)\|\mathcal{R}_{P,m}(k)\| &\leq C.
\end{aligned}
$$

Hence the assertion follows. □

Proposition 6.5 (photon number bound) *Assume (V.1), (V.2), (E.I.) and (N). Suppose that $\sigma = 0$. Then*

$$
\|a_r(k)\Psi_{P,m}\| \leq C_\kappa \frac{\chi_{0,\kappa}(k)}{|k|^{1/2}},
$$

where C_κ is a positive constant independent of k and m, but depends on κ .

Proof. From the pull-through formula for $a_r(k)$ one concludes

$$
\begin{aligned}
a_r(k)H_m(P)\Psi_{P,m} = &[H_m(P-k)+\omega_m(k)]a_r(k)\Psi_{P,m} \\
&- \sum_{j=1}^{N} \frac{1}{m_{\mathrm{e}}} \mathfrak{P}_j \cdot \mathfrak{K}_{j,r}^{(m)}(x_j,k)\Psi_{P.m} \\
&- \frac{1}{m_{\mathrm{n}}} \mathfrak{P}_0 \cdot \mathfrak{K}_{0,r}^{(m)}(0,k)\Psi_{P,m} \\
&- \sum_{j=1}^{N} \frac{\mathrm{i}\sigma_j}{2m_{\mathrm{e}}} \cdot k \wedge \mathfrak{K}_{j,r}^{(m)}(x_j,k)\Psi_{P,m} - \frac{\mathrm{i}\sigma_0}{2m_{\mathrm{n}}} \cdot k \wedge \mathfrak{K}_{0,r}^{(m)}(0,k)\Psi_{P,m},
\end{aligned}
$$

where

$$
\mathfrak{K}_{j,r}^{(m)}(k,x) := e_j \frac{\chi_{0,\kappa}(k)e^r(k)}{\sqrt{2(2\pi)^3\omega_m(k)}} \mathrm{e}^{-\mathrm{i}k\cdot x}, \quad j = 0,1,\ldots,N, \; r = 1,2,3
$$

with $e_0 = Ze$ and $e_j = -e$ for $j = 1,\ldots,N$. (Note that in the above we use $k \cdot e^r(k) = 0$.) Thus it follows that

$$
\begin{aligned}
&\big[H_m(P-k) - E_m(P) + \omega_m(k)\big] a_\lambda(k)\Psi_{P,m} \\
=&\sum_{j=0}^{N} \frac{1}{m_j} \mathfrak{P}_j \cdot \mathfrak{K}_{j,r}^{(m)}(0,k)\Psi_{P,m} + \sum_{j=1}^{N} \frac{1}{m_j} \mathfrak{P}_j \cdot \delta\mathfrak{K}_{j,r}^{(m)}(x_j,k)\Psi_{P,m} \\
&+ \sum_{j=1}^{N} \frac{\mathrm{i}\sigma_j}{2m_{\mathrm{e}}} \cdot k \wedge \mathfrak{K}_{j,r}^{(m)}(x_j,k)\Psi_{P,m} + \frac{\mathrm{i}\sigma_0}{2m_{\mathrm{n}}} \cdot k \wedge \mathfrak{K}_{0,r}^{(m)}(0,k)\Psi_{P,m},
\end{aligned}
$$

where $\delta\mathfrak{K}_{j,r}^{(m)}(x,k) := \mathfrak{K}_{j,r}^{(m)}(x,k) - \mathfrak{K}_{j,r}^{(m)}(0,k)$ and $m_0 = m_{\mathrm{n}}, \; m_j = m_{\mathrm{e}}\,(j = 1,\ldots,N)$.

By Lemma 6.4 (i), $H_m(P-k)-E_m(P)+\omega_m(k)$ has the bounded inverse $\mathcal{R}_{P,m}(k)$ for any $m\geq 0$. We also note that

$$\frac{1}{m_{\rm e}}\mathfrak{P}_j = {\rm i}[H_m(P),x_j]+\frac{1}{m_{\rm n}}\mathfrak{P}_0,\quad j=1,\ldots,N.$$

Accordingly we have

$$\begin{aligned}
&a_r(k)\Psi_{P,m}\\
=&{\rm i}\mathcal{R}_{P,m}(k)[H_m(P)-E_m(P)]\sum_{j=1}^N \mathfrak{K}_{j,r}^{(m)}(0,k)\cdot x_j\Psi_{P,m}\\
&+\frac{1}{m_{\rm n}}\sum_{j=0}^N \mathcal{R}_{P,m}(k)\mathfrak{P}_0\cdot\mathfrak{K}_{j,r}^{(m)}(0,k)\Psi_{P,m}\\
&+\sum_{j=1}^N\frac{1}{m_{\rm e}}\mathcal{R}_{P,m}(k)\mathfrak{P}_j\cdot\delta\mathfrak{K}_{j,r}^{(m)}(x_j,k)\Psi_{P,m}\\
&+\sum_{j=1}\mathcal{R}_{P,m}(k)\frac{{\rm i}\sigma_j}{2m_{\rm e}}\cdot k\wedge\mathfrak{K}_{j,r}^{(m)}(x_j,k)\Psi_{P,m}+\mathcal{R}_{P,m}(k)\frac{{\rm i}\sigma_0}{2m_{\rm n}}\cdot k\wedge\mathfrak{K}_{0,r}^{(m)}(0,k)\Psi_{P,m}\\
=:&I_1(k)+I_2(k)+I_3(k)+I_4(k).
\end{aligned}\tag{35}$$

By the neutrality condition (N), $I_2=0$ and by Lemma 6.2 and 6.4, one concludes that

$$\|I_1(k)\|,\ \|I_4(k)\|\leq\frac{C}{|k|^{1/2}}\chi_{0,\kappa}(k).$$

As for $I_3(k)$, noting $|\delta\mathfrak{K}_{j,r}^{(m)}(x_j,k)|\leq\{2(2\pi)^3\}^{-1/2}|e_j||k|^{1/2}|x_j|\chi_{0,\kappa}(k)$, we have

$$\|I_3(k)\|\leq\frac{C_\kappa}{|k|^{1/2}}\chi_{0,\kappa}(k)$$

by Lemma 6.4. □

Lemma 6.6 *Assume (V.1), (V.2) and (E.I.). For all $m>0$ and $|P|<m_{\rm n}$,*

$$\nabla_k\mathcal{R}_{P,m}(k)=\mathcal{R}_{P,m}(k)\Big[\frac{1}{m_{\rm n}}(\mathfrak{P}_0-k)-\frac{k}{\omega_m(k)}\Big]\mathcal{R}_{P,m}(k)\tag{36}$$

in the operator norm topology.

Proof. By the second resolvent formula, we have

$$\begin{aligned}
&\mathcal{R}_{P,m}(k+h)-\mathcal{R}_{P,m}(k)\\
=&\mathcal{R}_{P,m}(k+h)[H_m(P-k)-H_m(P-k-h)+\omega_m(k)-\omega_m(k+h)]\mathcal{R}_{P,m}(k)\\
=&\mathcal{R}_{P,m}(k+h)\Big[\frac{1}{m_{\rm n}}h\cdot(\mathfrak{P}_0-k)-h\cdot\nabla_k\omega_m(k)+O(h^2)\Big]\mathcal{R}_{P,m}(k).
\end{aligned}$$

Thus passing through the limiting argument, the assertion (36) follows. □

Proposition 6.7 (photon derivative bound) *Assume (V.1), (V.2), (E.I.), (N) and $\sigma = 0$. Suppose that $P \in \Lambda$ and $|P| < m_{\rm n}$. Then, for $|k| < \kappa$ and $(k_1, k_2) \neq 0$,*

$$\|\nabla_k a_r(k)\Psi_{P,m}\| \leq \frac{C_\kappa}{|k|^{1/2}\sqrt{k_1^2+k_2^2}}, \tag{37}$$

where C_κ is a positive constant independent of k, m.

Proof. By (35) we obtain

$$\begin{aligned}
&\nabla_k a_r(k)\Psi_{P,m}\\
=&\mathrm{i}\big[\nabla_k \mathcal{R}_{P,m}(k)\big][H_m(P)-E_m(P)]\sum_{j=1}^N \mathfrak{K}_{j,r}^{(m)}(0,k)\cdot x_j\Psi_{P,m} &(38)\\
&+\mathrm{i}\mathcal{R}_{P,m}(k)[H_m(P)-E_m(P)]\sum_{j=1}^N \nabla_k\big[\mathfrak{K}_{j,r}^{(m)}(0,k)\cdot x_j\big]\Psi_{P,m} &(39)\\
&+\sum_{j=1}^N \frac{1}{m_{\rm e}}\nabla_k\big[\mathcal{R}_{P,m}(k)\big]\mathfrak{P}_j\cdot\delta\mathfrak{K}_{j,r}^{(m)}(x_j,k)\Psi_{P,m} &(40)\\
&+\sum_{j=1}^N \frac{1}{m_{\rm e}}\mathcal{R}_{P,m}(k)\nabla_k\big[\mathfrak{P}_j\cdot\delta\mathfrak{K}_{j,r}^{(m)}(x_j,k)\big]\Psi_{P,m}. &(41)\\
&+\nabla_k I_4(k). &(42)
\end{aligned}$$

Applying Lemma 6.2, 6.4 and (36), we estimate the norms of (38) and (39) to obtain

$$\|(38)\|,\quad \|(40)\| \leq \frac{C_\kappa}{|k|^{3/2}}.$$

Considering the fact $|\nabla_k e_r(k)| \leq C/\sqrt{k_1^2+k_2^2}$ $(k_1,k_2)\neq(0,0)$, we also estimate (39) and (41) with results

$$\|(39)\|,\quad \|(41)\| \leq \frac{C_\kappa}{|k|^{1/2}\sqrt{k_1^2+k_2^2}}.$$

Similarly we can estimate $\|\nabla_k I_4(k)\|$. This implies the assertion in (37). □

6.3 Proof of Theorem 2.4

This proof is a slight modification of [17, Theorem 2.1] and we only provide on the outline, for details, see [17]. For $P \in \Lambda$ and $|P| < m_{\rm n}$, $H_m(P)$ has a normalized ground state $\Psi_{P,m}$ whenever m is sufficiently small by Theorem 5.1 and Lemma 6.1. Take $m_1 > m_2 > \cdots$ tending to 0 and denote Ψ_{P,m_j} by $\Psi_{P,j}$. The sequence $\{\Psi_{P,j}\}$ is a minimizing sequence for $H(P)$. Indeed

$$E_{m_j}(P) = \langle\Psi_{P,j}, H_{m_j}(P)\Psi_{P,j}\rangle \geq \langle\Psi_{P,j}, H_0(P)\Psi_{P,j}\rangle \geq E(P),$$

Thus $\langle \Psi_{P,j}, H(P)\Psi_{P,j}\rangle \to E(P)$ as $j \to \infty$ by Lemma 6.1. Since $\|\Psi_{P,j}\| = 1$, there is a subsequence $\{\Psi_{P,j'}\}$ of $\{\Psi_{P,j}\}$ which has a weak limit Ψ_P. Because

$$0 \le \langle \Psi_P, (H(P) - E(P))\Psi_P\rangle \le \liminf_{j\to\infty} \langle \Psi_{P,j'}, (H(P) - E(P))\Psi_{P,j'}\rangle = 0,$$

it suffices to prove that $\|\Psi_P\| = 1$. (This means the strong convergence of $\{\Psi_{P,j'}\}$.) Note that, by Proposition 6.5,

$$\langle \Psi_{P,j'}, \mathbb{1} \otimes N_{\mathrm{f}}\Psi_{P,j'}\rangle \le C < \infty,$$

where C is a positive constant independent of j'. Hence it sufficies to show the L^2-convergence of each n-photon component $\Psi^{(n)}_{P,j'}$, where we write $\Psi_{P,j'} = \oplus_{n=0}^{\infty}\Psi^{(n)}_{P,j'}$. From the exponential decay, it follows that, for each $R > 0$,

$$\begin{aligned}\|\tilde{\chi}_R\Psi_{P,j'}\| &= \|\tilde{\chi}_R\,\mathrm{e}^{-\beta|x|}\,\mathrm{e}^{\beta|x|}\Psi_{P,j'}\|\\ &\le C\,\mathrm{e}^{-\beta R},\end{aligned}$$

where $\tilde{\chi}_R := 1 - \chi_R$. Accordingly it suffices to show the L^2-convergence in the domain $|x| < R$. By Proposition 6.5, $\Psi^{(n)}_{P,j'}(x_1, \dots, x_N, k_1, \dots, k_n) = 0$ if $|k_i| > \kappa$ for some i. By putting these facts together, it suffices to show L^2-convergence for $\Psi^{(n)}_{P,j'}$ restricted to the bounded domain

$$\Omega_R := \{(x, k_1, \dots, k_n) \,|\, |x| < R, |k_i| < \kappa, i = 1, \dots, n\} \subset \mathbb{R}^{3(N+n)}.$$

By Proposition 6.7, $\{\Psi^{(n)}_{P,j'}\}_{j'}$ is a bounded sequence in $W^{1,p}(\Omega_R)$ for each $p < 2$ and $R > 0$. (It is not hard to check that

$$\|\nabla_{k_i}\Psi^{(n)}_{P,j'}\|^p_{L^p(\Omega_R)} \le C\int_{|k|<\kappa} \mathrm{d}k\|\nabla_k a_r(k)\Psi_{P,j'}\|^p \le Const < \infty$$

and $\|\nabla_x\Psi^{(n)}_{P,j'}\|^p_{L^p(\Omega_R)} \le Const < \infty$.) From the weak convergence of $\{\Psi^{(n)}_{P,j'}\}$ in $L^2(\Omega_R)$, $\Psi^{(n)}_{P,j'}$ weakly converges to $\Psi^{(n)}_P$ in $W^{1,p}(\Omega_R)$. Now we can apply the Rellich-Kondrachov theorem [25, Theorem 8.9]. Then $\{\Psi^{(n)}_{P,j'}\}$ converges strongly to $\{\Psi^{(n)}_P\}$ in $L^q(\Omega_R)$ of $1 \le q \le 3p(N+n)/3(N+n) - p$. If we choose p as $2 > p > 6(N+n)/[2+3(N+n)]$, we obtain the strong convergence of $\{\Psi_{P,j'}\}$ in $L^2(\Omega_R)$. □

6.4 Proof of Theorem 2.5 and 2.6

Let

$$\Sigma^{(N)} = \lim_{R\to\infty}\left(\inf_{\varphi\in\tilde{\mathcal{D}}_R, \|\varphi\|=1}\langle\varphi, H_N\varphi\rangle\right)$$

with

$$\tilde{\mathcal{D}}_R = \{\varphi \in \mathcal{H}^{N+1}_{\mathrm{fin}} \,|\, \varphi(x) = 0, \text{ if } |x| < R\}.$$

Lemma 6.8 (i) *For all* $P \in \mathbb{R}^3$,

$$\Sigma^{(N)} \leq \Sigma(P).$$

(ii) $\Sigma^{(N)} = \min\left\{E_\beta + E_{\bar\beta} \,|\, \beta \in \Pi_N \text{ and } \beta \neq \emptyset, \{0,1,\ldots,N\}\right\}$.

Proof. (i) Assume that there is a $P_0 \in \mathbb{R}^3$ such that $\Sigma^{(N)} > \Sigma(P_0)$ and set $\gamma := \Sigma^{(N)} - \Sigma(P_0) > 0$. There exists $R_0 > 0$ so that, for all $R > R_0$, $\gamma_R := \Sigma_R^{(N)} - \Sigma_R(P_0) > 0$. Here $\Sigma_R^{(N)}$ and $\Sigma_R(P)$ stands for $\inf_{\varphi\in\tilde{\mathcal{D}}_R, \|\varphi\|=1}\langle\varphi, H_N\varphi\rangle$ and $\inf_{\varphi\in\mathcal{D}_R, \|\varphi\|=1}\langle\varphi, H(P)\varphi\rangle$ respectively. (Note that $\lim_{R\to\infty}\gamma_R = \gamma$.) Take R as $R > R_0$. This R is kept fixed in the following. There is a $\varphi \in \mathcal{D}_R$, $\|\varphi\| = 1$ so that

$$\langle\varphi, H(P_0)\varphi\rangle \leq \Sigma_R^{(N)} - \gamma_R/2.$$

Since $\langle\varphi, H(P)\varphi\rangle$ is continuous in P, there is a $\delta > 0$ such that, for all P with $|P - P_0| \leq \delta$,

$$\langle\varphi, H(P)\varphi\rangle \leq \Sigma_R^{(N)} - \gamma_R/4.$$

Pick $f \in C^\infty(\mathbb{R}^3)$ as $\mathrm{supp} f \subseteq \{P \in \mathbb{R}^3 \,|\, |P - P_0| \leq \delta\}$, $\|f\| = 1$ and define $\varphi_f := f \times \varphi \in \mathcal{H}^N$. Then we have

$$\langle\varphi_f, UH_NU^*\varphi_f\rangle \leq \Sigma_R^{(N)} - \gamma_R/4.$$

Notice that $U^*\varphi_f(x) = 0$ if $|x| < R/2N$. Since $\mathcal{H}_{\mathrm{fin}}^{N+1}$ is a core of H_N, there is a sequence $\{\varphi_n\}$ in $\mathcal{H}_{\mathrm{fin}}^{N+1}$ so that $\|\varphi_n\| = 1$, $\varphi_n \to U^*\varphi_f$ and $H_N\varphi_n \to H_NU^*\varphi_f$ as $n \to \infty$. Let j and $\bar{j}$ be C^∞ functions with $j^2 + \bar{j}^2 = 1$, j identically 1 on the unit ball and vanishing outside the ball of radius 2. Set $j_R(x) = j(4Nx/R)$ and $\bar{j}_R(x) = \bar{j}(4Nx/R)$. Then one gets

$$\langle\varphi_n, H_N\varphi_n\rangle = \langle j_R\varphi_n, H_Nj_R\varphi_n\rangle + \langle\bar{j}_R\varphi_n, H_N\bar{j}_R\varphi_n\rangle + o_R(\varphi_n)$$

by the IMS localization formula. For all $\varepsilon > 0$, there is a n' such that, for all $n > n'$,

$$|\langle\varphi_n, H_N\varphi_n\rangle - \langle\varphi_f, UH_NU^*\varphi_f\rangle| < \varepsilon.$$

Thus, for all $n > n'$,

$$\langle j_R\varphi_n, H_Nj_R\varphi_n\rangle + \langle\bar{j}_R\varphi_n, H_N\bar{j}_R\varphi_n\rangle + o_R(\varphi_n) - \varepsilon \leq \Sigma_R^{(N)} - \gamma_R/4.$$

Since $\bar{j}_R\varphi_n/\|\bar{j}_R\varphi_n\| \in \tilde{\mathcal{D}}_{R/2N}$, we have

$$\langle j_R\varphi_n, H_Nj_R\varphi_n\rangle + \Sigma_{R/2N}^{(N)}\|\bar{j}_R\varphi_n\|^2 + o_R(\varphi_n) - \varepsilon \leq \Sigma_R^{(N)} - \gamma_R/4.$$

We will discuss the limit $n \to \infty$. Note that

$$\begin{aligned}\langle j_R\varphi_n, H_Nj_R\varphi_n\rangle &= \langle j_R^2\varphi_n, H_N\varphi_n\rangle + o_R(\varphi_n)\\ &\to \langle j_R^2U^*\varphi_f, H_NU^*\varphi_f\rangle + o_R(U^*\varphi_f) \quad (n\to\infty).\end{aligned}$$

Here we use the fact $\lim_{n\to\infty} o_R(\varphi_n) = o_R(U^*\varphi_f)$ because

$$|o_R(\varphi_n)| \leq \tilde{o}(R^0)(\|H_N\varphi_n\|^2 + \|\varphi_n\|^2).$$

By the fact $j_R U^*\varphi_f = 0$, we conclude that $\lim_{n\to\infty}\langle j_R\varphi_n, H_N j_R\varphi_n\rangle = o_R(U^*\varphi_f)$. Taking the limit $n \to \infty$, we get

$$\Sigma^{(N)}_{R/2N}\|\bar{j}_R U^*\varphi_f\|^2 + o_R(U^*\varphi_f) - \varepsilon \leq \Sigma^{(N)}_R - \gamma_R/4.$$

Since ε is arbitrary and $\bar{j}_R U^*\varphi_f = U^*\varphi_f$, we get

$$\Sigma^{(N)}_{R/2N} + o_R(U^*\varphi_f) \leq \Sigma^{(N)}_R - \gamma_R/4.$$

Therefore, taking the limit $R \to \infty$, we conclude that

$$\Sigma^{(N)} \leq \Sigma^{(N)} - \gamma/4.$$

This is a contradiction. Proof of (ii) is a slight modification of the one of [16, Theorem 3]. □

Proof of Theorem 2.5

Note that $\{P \in \mathbb{R}^3 \,|\, E(P) \leq \Sigma^{(N)}\} \subseteq \Lambda$ by the above lemma. By the property $E(P) \leq E(0) + P^2/2m_{\mathrm{n}}$ (Theorem 4.1), one also has $\{P \in \mathbb{R}^3 \,|\, E(0) + P^2/2m_{\mathrm{n}} \leq \Sigma^{(N)}\} \subseteq \Lambda$. Considering the facts $E(0) = E_N$ (Theorem 4.1) and Lemma 6.8 (ii), we obtain $\{P \in \mathbb{R}^3 \,|\, |P| < \sqrt{2m_{\mathrm{n}}E_{\mathrm{bin}}}\} \subseteq \Lambda$. Now Theorem 2.5 follows from Theorem 2.4. □

Proof of Theorem 2.6

Basic idea of the proof is almost same as Theorem 2.4 and 2.5. Since the system is not neutral, the term $I_2(k)$ in (35) does not vanish. We can calculate the contribution of $I_2(k)$ as $|I_2(k)| \leq \mathrm{const.}|k|^{-3/2}\chi_{0,\kappa}(k)$ in the photon number bound and $|\nabla_k I_2(k)| \leq \mathrm{const.}|k|^{-3/2} \times (k_1^2 + k_2^2)^{-1/2}$ for $|k| < \kappa$ in the photon derivative bound by Lemma 6.4. If we take the infrared cutoff σ as $\sigma > 0$, these singularities at origin $k = 0$ do not influence our proof of Theorem 2.4 and 2.5, and the same arguments still hold. □

7 Spinless electrons, Boltzmann statistics

In this section we consider an arbitrary collection of charges with no symmetry condition on the wave function imposed. The Hamiltonian is given by

$$H_N = \sum_{j=0}^{N} \frac{1}{2m_j}\Big(-\mathrm{i}\nabla_j \otimes \mathbb{1} - e_j A(x_j)\Big)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}. \tag{43}$$

H_N acts on $[\otimes^{N+1} L^2(\mathbb{R}^3)] \otimes \mathcal{F}$. We require $m_j > 0$, while e_j is arbitrary, $j = 0, \ldots, N$. Note that the neutrality condition (N) can then be rewritten as

$$\sum_{j=0}^{N} e_j = 0. \tag{N'}$$

Moreover, because we do not consider any statistics of the particles, our assumptions for potential are generalized as follows:

(V'.1) V is a pair potential of the form

$$V(x_0, \ldots, x_N) = \sum_{0 \le i < j \le N} V_{ij}(x_i - x_j)$$

and each V_{ij} is infinitesimally small with respect to $-\Delta$,

(V'.2) each V_{ij} is in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $V_{ij}(x) \to 0$ as $|x| \to \infty$.

Following the argument in Section 2.2, H_N admits the decomposition

$$H_N = \int_{\mathbb{R}^3}^{\oplus} H(P)\, \mathrm{d}P.$$

We estabilish the energy inequality (E.I.).

Proposition 7.1 *Assume (V'.1). Then, the energy inequality (E.I.) holds for arbitrary photon mass m, couplings $e_1, \ldots, e_N$, and cutoffs σ, κ.*

Proof. See next subsection. □

Using this proposition we infer the following assertions.

Theorem 7.2 *Assume (V'.1), (V'.2), and (N'). Suppose that the infrared cutoff $\sigma = 0$ holds. If $P \in \Lambda$ and $|P| < m_0$, then $H(P)$ has a ground state.*

Theorem 7.3 *Assume (V'.1), (V'.2), and (N'). Suppose that the infrared cutoff $\sigma = 0$ holds. Moreover, suppose $E_{\mathrm{bin}} > 0$, and $|P| < \min\left\{m_0, \sqrt{2m_0 E_{\mathrm{bin}}}\right\}$. Then $H(P)$ has a ground state.*

Theorem 7.4 *Assume (V'.1), (V'.2) and that the system is not neutral in the sense that (N') does not hold. Suppose that $\sigma > 0$. Then $H(P)$ has a ground state for $P \in \Lambda$ and $|P| < m_0$. Moreover if $E_{\mathrm{bin}} > 0$, then $H(P)$ has a ground state for $|P| < \min\{m_0, \sqrt{2m_0 E_{\mathrm{bin}}}\}$.*

Let h_N be the Hamiltonian H_N ignoring the quantized radiation field, i.e.,

$$h_N = -\sum_{j=0}^{N} \frac{\Delta_j}{2m_j} + V.$$

For h_N one can define an binding energy $\mathsf{e}_{\mathrm{bin}}$ in correspondence to E_{bin}, see Appendix D.

Proposition 7.5 *For all σ, κ with $0 \leq \sigma < \kappa < \infty$, one has*

$$E_{\rm bin} \geq \mathsf{e}_{\rm bin} - \alpha(\kappa^2 - \sigma^2)$$

with $\alpha = \pi \sum_{j=0}^{N} (e_j^2/16\pi^2 m_j)$. Thus if $\mathsf{e}_{\rm bin} > 0$ and $\kappa^2 - \sigma^2 < \mathsf{e}_{\rm bin}/\alpha$, then $H(P)$ has a ground state for $|P| < \min\{m_0, \sqrt{2m_0 E_{\rm bin}}\}$.

Proof. Let h_β be the Hamiltonian H_β omitting the quantized radiation field and $E(h_\beta) = \inf \mathrm{spec}(h_\beta)$ (see Appendix D for details). By the diamagnetic inequality (see, e.g. [20]), one concludes

$$E_\beta \geq E(h_\beta)$$

for all $\beta \in \Pi_N$. On the other hand, for $f \in \mathrm{dom}(-\Delta)$ with $\|f\| = 1$,

$$E_N \leq \langle f \otimes \Omega, H_N f \otimes \Omega \rangle = \Big\langle f, \Big[-\sum_{j=0}^{N} \frac{1}{2m_j} \Delta_j + V + \alpha(\kappa^2 - \sigma^2) \Big] f \Big\rangle,$$

which implies

$$E_N \leq E(h_N) + \alpha(\kappa^2 - \sigma^2),$$

where $E(h_N) = \inf \mathrm{spec}(h_N)$. Combining both results yields the assertion. □

Example We consider the hydrogen atom, i.e., $N = 1$ and $V_{01}(x_0 - x_1) = -\, e^2/4\pi|x_0 - x_1|$ $(e_0 = -e,\ e_1 = e)$. The system is neutral and we allow $\sigma = 0$. By Proposition 7.5, one concludes that $E_{\rm bin} > 0$ if

$$\frac{\mu e^4}{32\pi^2} - \frac{e^2 \kappa^2}{16\pi^2 \mu} > 0 \tag{44}$$

with $1/\mu = 1/m_0 + 1/m_1$, because $\mathsf{e}_{\rm bin} = E(h_{\{0\}}) + E(h_{\{1\}}) - E(h_1) = -E(h_1) = \mu e^4/32\pi^2$. This rough estimate provides us with the following imformation.

(1) In case of hydrogen in nature $e^2/4\pi \simeq 1/137$ and the ultraviolet cutoff κ must satisfy

$$\kappa < \sqrt{\frac{2\pi}{137}} \mu.$$

(2) If we regard e as the coupling parameter, $E_{\rm bin} > 0$ provided

$$\frac{\sqrt{2}\kappa}{\mu} < e.$$

The stronger the coupling e, the larger the admissible ultraviolet cutoff κ.

Remark 7.6 In [14] the binding condition $\mathsf{e}_{\rm bin} > 0$ has been proven for the hydrogen molecule H_2 with spin 0 nuclei. The antisymmetry of the electronic part of the wave function can be absorbed into a spin singlet state. Using this result, Theorem 7.3 implies the existence of the ground state for a hydrogen-like molecule coupled to the radiation field, provided κ is not too large. Thus if κ is not too large, also the hydrogen molecule coupled to the radiation field has a ground state.

7.1 Proof of Proposition 7.1

We will treat the case $m = 0$ for the notational convenience. All arguments hold for $m > 0$, also. Let $\mathcal{W} = \oplus^3 L^2(\mathbb{R}^3)$ and q be the bilinear form defined by

$$q(f,g) = \frac{1}{2}\sum_{\mu,\nu=1}^{3}\int_{\mathbb{R}^3} d_{\mu\nu}(k)\hat{f}_\mu(k)\hat{g}_\nu(k)\,\mathrm{d}k, \quad f,g \in \mathcal{W},$$

where $d_{\mu\nu}(k) = \sum_{r=1,2} e^r_\mu(k)e^r_\nu(k) = \delta_{\mu\nu} - k_\mu k_\nu/|k|^2$. Let (Q,μ) be the probability measure space for the mean zero Gaussian random variables $\{\phi(f) \mid f \in \mathcal{W}\}$ with covariance given by

$$\int_Q \phi(f)\phi(g)\,\mathrm{d}\mu(\phi) = \frac{1}{2}q(f,g).$$

The photon Fock space $\mathcal{F}$ can be naturally identified with $L^2(Q, \mathrm{d}\mu)$ [23]. This representation is called the Schrödinger representation. Under this identification $\mathcal{H}^M \cong L^2(\mathbb{R}^{3M} \times Q, \mathrm{d}x \otimes \mathrm{d}\mu)$ for arbitrary $M \in \mathbb{N}$. The unitary operator from $\mathcal{H}^M$ to $L^2(\mathbb{R}^{3M} \times Q, \mathrm{d}x \otimes \mathrm{d}\mu)$ corresponding to this natural identification is denoted by $\tilde{S}_M$.

Let $(\mathcal{X}, \nu)$ be a σ-finite measure space. $f \in L^2(\mathcal{X}, \nu)$ is called positive if f is nonnegative a.e. and is not the zero function. A bounded operator A is positivity preserving if $\langle f_1, Af_2\rangle \geq 0$ for all positive f_1 and $f_2 \in L^2(\mathcal{X}, \mathrm{d}\nu)$. If A is positivity preserving,

$$|Af| \leq A|f| \quad \text{a.e.} \tag{45}$$

for any $f \in L^2(\mathcal{X}, \mathrm{d}\nu)$[18, **?**]. One advantage of the Schrödinger representation is the following fact: the operator $\mathsf{S}_{M+1}\,\mathrm{e}^{-tH_M}\mathsf{S}^*_{M+1}$ is a positivity preserving operator in $L^2(\mathbb{R}^{3(M+1)} \times Q, \mathrm{d}x \otimes \mathrm{d}\mu)$, where $\mathsf{S}_{M+1} = \tilde{S}_{M+1}\,\exp\{\mathrm{i}\frac{\pi}{2}\mathbb{1} \otimes N_\mathrm{f}\}$ [21].

From now on, we fix $N \in \mathbb{N}$ arbitrarily and denote $\mathsf{S} = \mathsf{S}_N$ for notational simplicity.

Lemma 7.7 *Let $\mathcal{V}(P)$ and $K(P)$ be the operators defined by (18) and (19) respectively.*

(i) $\mathsf{S}\mathcal{V}(0)\mathsf{S}^*$ *is positivity preserving.*

(ii) $\mathsf{S}\,\mathrm{e}^{-sK(0)}\mathsf{S}^*$ *is positivity preserving for all $s > 0$.*

Proof. (i) Since $\exp\{\mathrm{i}x_1 \cdot \nabla_j \otimes \mathbb{1}\}$ and $\exp\{\mathrm{i}x_1 \cdot \mathbb{1} \otimes P_\mathrm{f}\}$ are translations, the result follows.

(ii) Note that $\mathsf{S}\mathcal{V}(0)\mathsf{S}^*, \mathsf{S}\mathcal{V}(0)^*\mathsf{S}^*$ and $\mathsf{S}\,\mathrm{e}^{-sH_A}\mathsf{S}^*$ are positivity preserving. Thus $\mathsf{S}\,\mathrm{e}^{-sK(0)}\mathsf{S}^*$ is also positivity preserving by the fact $\mathrm{e}^{-sK(0)} = \mathcal{V}(0)\,\mathrm{e}^{-sH_A}\mathcal{V}(0)^*$. □

Lemma 7.8 *For all $P \in \mathbb{R}^3$, the following holds.*

(i) $|\mathsf{S}\,\mathcal{V}(P)\mathsf{S}^*F| \leq \mathsf{S}\,\mathcal{V}(0)\mathsf{S}^*|F|$ *a.e..*

(ii) $|\mathsf{S}\,\mathrm{e}^{-sK(P)}\mathsf{S}^*F| \leq \mathsf{S}\,\mathrm{e}^{-sK(0)}\mathsf{S}^*|F|$ *a.e.*.

Proof. (i) For a.e. x and ϕ,

$$\begin{aligned}|(\mathsf{S}\,\mathcal{V}(P)\mathsf{S}^*F)(x,\phi)| &= |\mathrm{e}^{\mathrm{i}x_1\cdot P}(\mathsf{S}\,\mathcal{V}(0)\mathsf{S}^*F)(x,\phi)| \\ &\leq |(\mathsf{S}\,\mathcal{V}(0)\mathsf{S}^*F)(x,\phi)| \\ &\leq (\mathsf{S}\,\mathcal{V}(0)\mathsf{S}^*|F|)(x,\phi)\end{aligned}$$

by Lemma 7.7.

(ii) By (i), Lemma 7.7, and the fact that $\mathsf{S}\,\mathrm{e}^{-sH_A}\mathsf{S}^*$ is positivity preserving,

$$\begin{aligned}|\mathsf{S}\,\mathrm{e}^{-sK(P)}\mathsf{S}^*F| &= |\mathsf{S}\,\mathcal{V}(P)\mathsf{S}^*\mathsf{S}\,\mathrm{e}^{-sH_A}\mathsf{S}^*\mathsf{S}\,\mathcal{V}(P)^*\mathsf{S}^*F| \\ &\leq \mathsf{S}\,\mathcal{V}(0)\mathsf{S}^*|\mathsf{S}\,\mathrm{e}^{-sH_A}\mathsf{S}^*\mathsf{S}\,\mathcal{V}(P)^*\mathsf{S}^*F| \\ &\leq (\mathsf{S}\mathcal{V}(0)\mathsf{S}^*)(\mathsf{S}\mathrm{e}^{-sH_A}\mathsf{S}^*)|\mathsf{S}\mathcal{V}(P)^*\mathsf{S}^*F| \\ &\leq (\mathsf{S}\mathcal{V}(0)\mathsf{S}^*)(\mathsf{S}\mathrm{e}^{-sH_A}\mathsf{S}^*)(\mathsf{S}\mathcal{V}(0)^*\mathsf{S}^*)|F| \\ &= \mathsf{S}\,\mathrm{e}^{-sK(0)}\mathsf{S}^*|F|\end{aligned}$$

for a.e.. □

Proposition 7.9 *For all $t > 0$ and $P \in \mathbb{R}^3$,*

$$|\mathsf{S}\,\mathrm{e}^{-tH(P)}\mathsf{S}^*F| \leq \mathsf{S}\,\mathrm{e}^{-tH(0)}\mathsf{S}^*|F| \quad \textit{a.e.}.$$

Proof. Let $A_n(P) = (\mathrm{e}^{-tH_{\mathrm{PF}}/n}\,\mathrm{e}^{-tK(P)/n})^n$ for all $n \in \mathbb{N}$. By Kato's strong product formula [30, Theorem S.21], $\text{s-}\lim_{n\to\infty} A_n(P) = \mathrm{e}^{-tH(P)}$. For all $n \in \mathbb{N}$,

$$|\mathsf{S}\,A_n(P)\mathsf{S}^*F| \leq \mathsf{S}\,A_n(0)\mathsf{S}^*|F|$$

by Lemma 7.8 and the fact that $\mathsf{S}\,\mathrm{e}^{-sH_{\mathrm{PF}}}\mathsf{S}^*$ is positivity preserving. Taking the limit $n \to \infty$, we get the desired result. □

Proof of Proposition 4.1.
By Proposition 7.9 we get

$$\langle F, \mathsf{S}\,\mathrm{e}^{-tH(P)}\mathsf{S}^*F\rangle \leq \langle |F|, \mathsf{S}\,\mathrm{e}^{-tH(0)}\mathsf{S}^*|F|\rangle.$$

for $F \in L^2(\mathbb{R}^{3N} \times Q, \mathrm{d}x \otimes \mathrm{d}\mu)$. From this we immediately obtain the desired result. □

A Proof of Proposition 4.2

We start with a lemma about convex functions. Denote by $\mathcal{C}$ the set of convex functions $g : \mathbb{R}^n \to \mathbb{R}$ that satisfy $0 \leq g(x) \leq |x|^2/2$.

Lemma A.1 *Fix any two points P and Q in $\mathbb{R}^n$. Then the function*

$$\Delta(P,Q) := \sup\{g(P) - g(Q) \,:\, g \in \mathcal{C}\}$$

equals

$$\Delta(P,Q) = \begin{cases} Q \cdot (P-Q) + |P-Q||Q|, & \text{if } |P-Q| \leq |Q|, \\ P^2/2, & \text{if } |P-Q| \geq |Q|. \end{cases}$$

Moreover the maximizer is given by

$$g(x) = \begin{cases} Q \cdot (x-Q) + |x-Q||Q|, & \text{if } |x-Q| \leq |Q|, \\ |x|^2/2, & \text{if } |x-Q| \geq |Q|. \end{cases}$$

Proof. First we set $g(Q) = A$ where $A > 0$ is an arbitrary number less than $Q^2/2$. Next we consider all the rays starting at (Q, A) that are tangent to the surface $z = x^2/2$ ($x \in \mathbb{R}^n$). Such a ray is given in parametrized form by

$$x(t) = Q + te, \quad z(t) = A + tE,$$

where e is a unit vector in $\mathbb{R}^n$ and E is a real number. As we said, this ray has to touch the surface at the point $(Q + t_0 e, A + t_0 E)$ which means that

$$A + t_0 E = (Q + t_0 e)^2/2$$

together with the tangency condition $(e, E) \perp (Q + t_0 e, -1)$. From this one sees that

$$t_0^2 = Q^2 - 2A > 0$$

and

$$E = Q \cdot e + t_0.$$

Thus, for every direction e there are two touching points

$$x_0 = Q \pm e\sqrt{Q^2 - 2A}, \quad z_0 = Q^2 - A \pm Q \cdot e\sqrt{Q^2 - 2A}.$$

Note that the x components of the touching points sit on a sphere in $\mathbb{R}^n$ given by the equation $(x - Q)^2 = Q^2 - 2A$.

The point about these touching segments is the following. Every function $g \in \mathcal{C}$ with $g(Q) = A$ must have its graph below this segment, in other words

$$g(Q + te) \leq A + tE$$

for all t with $t^2 \leq Q^2 - 2A$. Thus, if P is inside the sphere, i.e.,

$$(P - Q)^2 \leq Q^2 - 2A,$$

we have that $P = Q + te$ and hence

$$g(P) \leq A + tE = A + t(Q \cdot e + t_0) = A + Q \cdot (P - Q) + |P - Q|\sqrt{Q^2 - 2A},$$

noting that t and t_0 need to have the same sign. Thus

$$g(P) - g(Q) \leq Q \cdot (P - Q) + |P - Q|\sqrt{Q^2 - 2A}.$$

Next we consider the case P is outside the sphere. Clearly in this case the largest value for $g(P)$ is $P^2/2$ and hence in this case

$$g(P) - g(Q) \leq P^2/2 - A.$$

Thus we have that

$$g(P) - g(Q) \leq \begin{cases} Q \cdot (P - Q) + |P - Q|\sqrt{Q^2 - 2A}, & \text{if } (P - Q)^2 \leq Q^2 - 2A, \\ P^2/2 - A, & \text{if } (P - Q)^2 \geq Q^2 - 2A. \end{cases}$$

Note that for $(P - Q)^2 = Q^2 - 2A$ we find that

$$Q \cdot (P - Q) + |P - Q|\sqrt{Q^2 - 2A} = P^2/2 - A.$$

Next we claim that

$$g(P) - g(Q) \leq \begin{cases} Q \cdot (P - Q) + |P - Q||Q|, & \text{if } (P - Q)^2 \leq Q^2, \\ P^2/2, & \text{if } (P - Q)^2 \geq Q^2. \end{cases}$$

This is obvious on the set of all Qs with $(P - Q)^2 \leq Q^2 - 2A$ and for all those that satisfy $(p - Q)^2 \geq Q^2$. Thus, it remains to show that for all Qs that satisfy $Q^2 - 2A \leq (P - Q)^2 \leq Q^2$,

$$P^2/2 - A \leq Q \cdot (P - Q) + |P - Q||Q|,$$

which is the same as

$$(|Q| - |P - Q|)^2/2 \leq A.$$

Since $|P - Q| \leq |Q|$ it suffices to show that

$$|Q| - |P - Q| \leq \sqrt{2A}$$

or

$$|P - Q| \geq |Q| - \sqrt{2A}.$$

Since, by assumption $|P - Q| \geq \sqrt{Q^2 - 2A}$ this follows once we show that

$$\sqrt{Q^2 - 2A} \geq |Q| - \sqrt{2A}.$$

Squaring both sides yields

$$Q^2 - 2A \geq Q^2 - 2|Q|\sqrt{2A} + 2A$$

or equivalently

$$|Q| \geq \sqrt{2A}$$

which follows from the fact that $A \leq Q^2/2$. □

Proof of Proposition 4.2

Write $F(P)$ as

$$F(P) = \frac{P^2}{2} + F(0) - h(P)$$

where $h(P)$ is convex. From (b) in Proposition 4.2 we get $h(P) \geq 0$ and from (a) we learn that $h(P) \leq P^2/2$. Hence

$$\begin{aligned} F(P-k) - F(P) &= \frac{(P-k)^2}{2} - \frac{P^2}{2} - [h(P-k) - h(P)] \\ &= -k \cdot P + \frac{k^2}{2} - [h(P-k) - h(P)]. \end{aligned}$$

Using the lemma above we get

$$h(P-k) - h(P) \leq \begin{cases} -P \cdot k + |k||P|, & \text{if } |k| \leq |P|, \\ (P-k)^2/2, & \text{if } |k| \geq |P| \end{cases}$$

and hence

$$\begin{aligned} F(P-k) - F(P) &\geq -k \cdot P + \frac{k^2}{2} - \begin{cases} -P \cdot k + |k||P|, & \text{if } |k| \leq |P|, \\ (P-k)^2/2, & \text{if } |k| \geq |P| \end{cases} \\ &= \begin{cases} -|k||P| + \frac{k^2}{2}, & \text{if } |k| \leq |P|, \\ -\frac{P^2}{2}, & \text{if } |k| \geq |P|. \end{cases} \end{aligned}$$

This proves the proposition. □

B Proof of Proposition 2.8

In order to clarify the dependence of m_{n}, we denote our Hamiltonian by $H(P; m_{\mathrm{n}})$ instead of $H(P)$. Also we denote the bottom of spectrum of $H(P; m_{\mathrm{n}})$ by $E(P; m_{\mathrm{n}})$.

Lemma B.1 (i) *$E(P; m_{\mathrm{n}}) \to E_N^\infty$ as $m_{\mathrm{n}} \to \infty$.*

(ii) *$\Sigma(P; m_{\mathrm{n}}) \geq \Sigma_N^\infty$ for all m_{n} and P, where $\Sigma(P; m_{\mathrm{n}})$ and Σ_N^∞ denote the threshold energy correspoinding to $H(P; m_{\mathrm{n}})$ and H_N^∞ which are similarly defined by (16).*

Proof. (i) For all $m_{\mathrm{n}} > 0$, $H(P; m_{\mathrm{n}})$ and H_N^∞ are both essentially self-adjoint on $\mathcal{H}_{\mathrm{fin}}^N$ by Proposition 2.1 and Theorem 2.3. Moreover, for all $\varphi \in \mathcal{H}_{\mathrm{fin}}^N$, $H(P; m_{\mathrm{n}})\varphi \to H_N^\infty \varphi$ as $m_{\mathrm{n}} \to \infty$. Therefore $H(P; m_{\mathrm{n}}) \to H_N^\infty$ in the strongly resolvent sense by [30, Theorem VIII.25] which implies the desired result by [30, Theorem VIII.24].

(ii) This follows from the operator inequality $H(P; m_{\rm n}) \geq H_N^\infty$. □

Proof of Proposition 2.8
Note first that, by [16],

$$\Sigma_N^\infty = \min\big\{E_\beta^\infty + E_{\bar\beta}^\infty \,|\, \beta \subset \{1,\dots,N\} \,\text{and}\, \beta \neq \emptyset, \{1,\dots,N\}\big\}.$$

By Lemma B.1 (i), there is a $m_{\rm n} > 0$ such that

$$E(0; m_{\rm n}) - E_N^\infty < \frac{E_{\rm bin}^\infty}{2}.$$

(Remark, here, that $E(0; m_{\rm n}) \geq E_N^\infty$.) Hence, by Lemma B.1 (ii) and Theorem 4.1 (ii),

$$\begin{aligned}
\Sigma(P; m_{\rm n}) - E(P; m_{\rm n}) &\geq \Sigma_N^\infty - E(0; m_{\rm n}) - \frac{P^2}{2m_{\rm n}} \\
&\geq \Sigma_N^\infty - E_N^\infty - \frac{P^2}{2m_{\rm n}} - \frac{E_{\rm bin}^\infty}{2} \\
&= \frac{E_{\rm bin}^\infty}{2} - \frac{P^2}{2m_{\rm n}}.
\end{aligned}$$

Thus if $|P| < \sqrt{m_{\rm n} E_{\rm bin}^\infty}$, the binding condition (B.C.) follows. □

C A uniform estimate for $\mathfrak{P}_j$

Let

$$\begin{aligned}
\mathfrak{P}_j &= -\mathrm{i}\nabla_j \otimes \mathbb{1} + eA(x_j), \quad j = 1,\dots,N, \\
\mathfrak{P}_0 &= P + \mathrm{i}\sum_{j=1}^N \nabla_j \otimes \mathbb{1} - \mathbb{1} \otimes P_{\rm f} - ZeA(0).
\end{aligned}$$

Lemma C.1 *For each $j = 0,1,\dots,N$, $l = 1,2,3$ and $\varphi \in \mathrm{dom}(H_m(P))$, there is a constant C independent of m and P such that*

$$\|\mathfrak{P}_{j,l}\varphi\| \leq C\big(\|H_m(P)\varphi\| + \|\varphi\|\big).$$

Proof. Throughout this proof, we use the symbol $H^{V=0}(P)$ which means the Hamiltonian (14) with $V = 0$. By Lemma 3.7 (and the fact $\tilde H(P) = H(P)$ and $\tilde H^{V=0}(P) = H^{V=0}(P)$), there is a constant $C_1 > 0$ independent of m and P so that

$$\langle\varphi, H^{V=0}(P)\varphi\rangle \leq C_1\big(\langle\varphi, H(P)\varphi\rangle + \|\varphi\|^2\big)$$

for $\varphi \in \mathcal{H}_{\rm fin}^N$. Since $H(P) \leq H_m(P)$, we have

$$\begin{aligned}
\|\mathfrak{P}_{j,l}\varphi\|^2 &\leq \langle\varphi, H^{V=0}(P)\varphi\rangle \\
&\leq C_1\big(\langle\varphi, H(P)\varphi\rangle + \|\varphi\|^2\big) \\
&\leq C_1\big(\langle\varphi, H_m(P)\varphi\rangle + \|\varphi\|^2\big) \\
&\leq 2C_1\big(\|H_m(P)\varphi\|^2 + \|\varphi\|^2\big).
\end{aligned}$$

Since $\mathcal{H}_{\rm fin}^N$ is a core of $H_m(P)$, the lemma follows. □

D Binding condition for the Schrödinger atom

We consider the $(N+1)$-particle Schrödinger operator acting in $L^2(\mathbb{R}^{3(N+1)})$ given by (1), to repeat,

$$h_N = -\sum_{j=0}^{N} \frac{1}{2m_j}\Delta_j + \sum_{0\le i<j\le N} V_{ij}(x_i - x_j), \tag{46}$$

where $m_0 = m_\mathrm{n}$ and $m_j = m_\mathrm{e}$ for $j = 1, \ldots, N$. The purpose of this appendix is to prove that in this case our binding condition (B.C.) reduces to the conventional binding condition.

Let R be the center of mass

$$R = \frac{1}{m_\mathrm{tot}} \sum_{j=0}^{N} m_j x_j$$

and define $\mathcal{X} = \{x \in \mathbb{R}^{3(N+1)} \mid R = 0\}$. In the $3N$-dimensional vector space $\mathcal{X}$, we use atomic coordinates $y_i = x_i - x_0,\ i = 1, \ldots, N$. Since we can identify $\mathcal{X}$ as $\mathbb{R}^{3N}$ under atomic coordinates, we obtain the following identification

$$L^2(\mathbb{R}^{3(N+1)}) = L^2(\mathcal{X}^c) \otimes L^2(\mathcal{X}) = L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^{3N}).$$

Moreover, our Hamiltonian can be expressed as

$$h_N = -\frac{1}{2m_\mathrm{tot}}\Delta_R \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{h}, \tag{47}$$

where

$$\tilde{h} = -\sum_{j=1}^{N} \frac{1}{2\mu_j}\Delta_{y_j} + \sum_{i<j} \frac{1}{m_0}\nabla_{y_i}\cdot\nabla_{y_j} + \tilde{V}, \quad \frac{1}{\mu_j} := \frac{1}{m_0} + \frac{1}{m_j}, \tag{48}$$

$$\tilde{V}(y_1, \ldots, y_N) = \sum_{j=1}^{N} V_{0j}(y_j) + \sum_{1\le i<j\le N} V_{ij}(y_i - y_j).$$

Remark that the total momentum $P_\mathrm{tot} = \sum_{j=0}^{N}(-\mathrm{i}\nabla_j)$ is represented by $P_\mathrm{tot} = -\mathrm{i}\nabla_R$ in our coordinates. Let $\mathcal{F}$ be the Fourier transformation with respect to R. Clearly $\mathcal{F}$ is unitary and $\mathcal{F}P_\mathrm{tot}\mathcal{F}^* = k$ (as multiplication operator). Thus $\mathcal{F}$ yields a spectral representaiton of P_tot. Furtheremore, by (47), one obtains

$$\mathcal{F}h_N\mathcal{F}^* = \frac{k^2}{2m_\mathrm{tot}} \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{h}.$$

Thus we are lead to the following fibre direct integral represetation of $\mathcal{F}h_N\mathcal{F}^*$,

$$\mathcal{F}h_N\mathcal{F}^* = \int_{\mathbb{R}^3}^{\oplus} h(P)\,\mathrm{d}P, \quad h(P) = \frac{P^2}{2m_\mathrm{tot}} + \tilde{h}.$$

For a self-adjoint operator A on $L^2(\mathbb{R}^{3d})$, we introduce

$$\Sigma(A) = \lim_{R\to\infty}\Big(\inf_{\varphi\in\mathcal{D}_{d,R},\,\|\varphi\|=1}\langle\varphi, A\varphi\rangle\Big),$$

where $\mathcal{D}_{d,R} = \{\varphi \in C_0^\infty(\mathbb{R}^{3d}) \,|\, \varphi(x) = 0, \text{ if } |x| < R\}$. For a self-adjoint operator A bounded from below, $E(A)$ stands for $\inf \operatorname{spec}(A)$.

Proposition D.1 *For all P,*

$$\begin{aligned}\Sigma(h(P)) - E(h(P)) &= \inf \operatorname{ess.spec}(h(P)) - E(h(P))\\ &= \inf \operatorname{ess.spec}(\tilde{h}) - E(\tilde{h}),\end{aligned}$$

where $\operatorname{ess.spec}(A)$ *means the essential spectrum of the linear operator A. Thus, if $\Sigma(h(P)) - E(h(P)) > 0$ for some P, then $h(P)$ has a ground state for all P.*

Proof. Clearly

$$\Sigma(h(P)) = \frac{P^2}{2m_{\text{tot}}} + \Sigma(\tilde{h}), \quad E(h(P)) = \frac{P^2}{2m_{\text{tot}}} + E(\tilde{h}).$$

We also note that $\Sigma(\tilde{h}) = \inf \operatorname{ess.spec}(\tilde{h})$ by [28]. Hence

$$\begin{aligned}\Sigma(h(P)) - E(h(P)) &= \inf \operatorname{ess.spec}(\tilde{h}) - E(\tilde{h})\\ &= \inf \operatorname{ess.spec}(h(P)) - E(h(P)). \quad \square\end{aligned}$$

Let Π_N be the set of the subsets of $\{0, 1, 2, \ldots, N\}$. We denote by h_β the Hamiltonian of the form (46), but only for the particles in the set β,

$$h_\beta = -\sum_{j\in\beta}\frac{1}{2m_j}\Delta_j + \sum_{i,j\in\beta,0\le i<j\le N} V_{ij}.$$

The binding energy e_{bin} for h_N is defined by

$$\mathsf{e}_{\text{bin}} = \min\{E(h_\beta) + E(h_{\bar{\beta}}) \,|\, \beta \in \Pi_N,\ \beta \ne \emptyset, \{0, 1, \ldots, N\}\} - E(h_N)$$

Proposition D.2 *For all P,*

$$\mathsf{e}_{\text{bin}} = \Sigma(h(P)) - E(h(P)).$$

Thus, if $\mathsf{e}_{\text{bin}} > 0$, $h(P)$ has a ground state for all P.

Proof. By the HVZ-theorem [9, Theorem 3.7], we have

$$\mathsf{e}_{\text{bin}} = \inf \operatorname{ess.spec}(\tilde{h}) - E(\tilde{h})$$

and the assertion follows from Proposition D.1. $\square$

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