Symmetric Functional Model for Extensions of Hermitian Operators

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Abstract

This paper offers the functional model of a class of non-selfadjoint extensions of a Hermitian operator with equal deficiency indices. The explicit form of dilation of a dissipative extension is offered and the symmetric form of Sz.Nagy-Foiaş model as developed by B. Pavlov is constructed. A variant of functional model for a general non-selfadjoint non-dissipative extension is formulated. We illustrate the theory by two examples: singular perturbations of the Laplace operator in $L_2(\mathbb{R}^3)$ by a finite number of point interactions, and the Schrödinger operator on the half axis $(0, \infty)$ in the Weyl limit circle case at infinity.

Introduction

Functional model approach plays a prominent role in the study of non-selfadjoint and non-unitary operators on Hilbert space. The rich and comprehensive theory has been developed since pioneering works of M. Brodskiĭ, M. Livŝiç, B. Szökefalvi-Nagy, C. Foiaş, L. de Branges, and J. Rovnyak, see [N1], [N2] and references therein. The functional model techniques are based on the fundamental theorem of B. Szökefalvi-Nagy and C. Foiaş stating that each linear contraction T, $\|T\| \le 1$ on a separable Hilbert space H can be extended to an unitary operator U on a wider Hilbert space $\mathscr{H} \supset H$ such that $T^n = P_H U^n|_H$, $n \ge 0$, where P_H is the orthogonal projection from the space $\mathscr{H}$ onto its subspace H. Operator U is called **dilation** of the contraction T. An unitary operator U with such properties is not unique, but if the contraction T does not have reducing unitary parts (such operators are called completely non-unitary, or simple) and if U is **minimal** in the sense that the linear set $\{U^k H : k \in \mathbb{Z}\}$ is dense in the dilation space $\mathscr{H}$, then the unitary dilation U is unique up to an unitary equivalence. B. Szökefalvi-Nagy and C. Foiaş proved as well that the spectrum of the minimal unitary dilation of a simple contraction is absolutely continuous and coincides with the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. In the spectral representation of the unitary operator U, when U becomes a multiplication $f \mapsto k * f$, $k \in \mathbb{T}$ on some L_2 space of vector-functions f, the contraction $T = P_H U|_H$ takes the form of its functional model $T \cong P_H\, k * |_H$.

Originating in the specific problems of physics of the time, the initial research on functional model quickly shifted into the realm of "pure mathematics" and most of the model results are now commonly regarded as "abstract". One of the few exceptions is the scattering theory developed by P. Lax and R. Phillips [LP]. The theory was originally devised for the analysis of the scattering of electromagnetic and acoustic waves off compact obstacles. The research, however, not only resulted in important discoveries in the scattering theory, but deeply influenced the subsequent developments of the operator model techniques as well.

The connection between the Lax-Phillips approach and the Sz.-Nagy-Foiaş dilation theory is established by means of the Cayley transform that maps a bounded operator T such that $\mathcal{R}(T - I)$ is dense in H into a possibly unbounded operator $A := -i(T + I)(T - I)^{-1}$, $\mathcal{D}(A) := \mathcal{R}(T - I)$. If T is unitary, then A is selfadjoint, and when T is contractive, the imaginary part of the operator A (properly understood, if needed, in the sense of sesquilinear forms) is positive. The latter operators A are called **dissipative**. By definition, the selfadjoint dilation $\mathscr{A} = \mathscr{A}^*$ of a dissipative operator $A = -i(T + I)(T - I)^{-1}$ is the Cayley transform of the unitary dilation of T. Correspondingly, the dilation $\mathscr{A}$ is called **minimal** if the set $\{(\mathscr{A} - zI)^{-1} H : \operatorname{Im} z \ne 0\}$ is dense in $\mathscr{H}$.

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The main object of the Lax-Phillips scattering theory is a strongly continuous contractive group of operators on a Hilbert space. The generator of this group is a dissipative operator that describes the geometry of the scatterer. Its selfadjoint dilation is present in the problem statement from the very beginning, and as all other mathematical objects of the theory, allows a clear physical interpretation.

Another line of examples of the fruitful interplay between the functional model theory and mathematical physics originates in the works [P1], [P3], [P4], [P5] of B. Pavlov on dissipative Schrödinger operators with a complex potential on $L_2(\mathbb{R}^3)$ and with a dissipative boundary condition on $L_2(0, \infty)$. In comparison with the Lax-Phillips theory these studies are distinguished by the absence of the "natural" selfadjoint dilation known upfront. In both cases the selfadjoint dilations have to be "guessed" and explicitly assembled from the objects given in the initial problem statement. This approach eventually evolved into a recipe that not only allows to recover the selfadjoint dilation (see [K]), but also to build its spectral representation, obtaining the eigenfunction expansion of the original dissipative operator. The dilation and the model space used by B. Pavlov known as symmetric model are well suited for the study of differential operators, and as in the case of the Lax-Phillips theory, the objects emerging from the model considerations have clear physical meaning. (See [P6].) The technique of expansion by the dilation's eigenfunctions of absolutely continuous spectrum in order to pass to the spectral representation is well-known in the physical literature, where this otherwise formal procedure is properly rectified by the distribution theory. In application to the setting of a generic dissipative operator, this approach requires a certain adaptation of the rigged Hilbert spaces technique. (See [P6] for an example.)

The next step in the development was made by S. Naboko, who offered a "direct" method of passing on to the functional model representation for the dissipative operators with the relatively bounded imaginary part [Na], [Na1], [Na2]. The approach is based on the preceding works of B. Pavlov, but without resorting to the dilation's eigenfunctions of continuous spectrum, the spectral mapping is expressed in terms of boundary values of certain operator- and vector-valued functions analytic in the upper and lower half planes. In a sense, this is exactly what one should expect trying to justify the distributions by methods of the analytic functions theory [Br]. As an immediate benefit, this direct approach opened up the opportunity to include non-selfadjoint relatively bounded perturbations of a selfadjoint operator with the relative bound lesser than 1 in the model-based considerations. It turned out that for an operator of this class there exists a model space where the action of the operator can be expressed in a simple and precise form. The ability to abandon the dissipativity restriction imposed on the operator class suitable for the model-based study allowed S. Naboko to conduct the profound spectral analysis of additive perturbations of the selfadjoint operators, to develop the scattering theory for such perturbations, and to introduce valuable definitions of spectral subspaces of a non-selfadjoint non-dissipative operator. The idea of utilization of the functional model of a "close" operator for the study of the operator under consideration was adopted by N. Makarov and V. Vasyunin in [MV], who offered the analogue of S. Naboko's construction for an arbitrary bounded operator considered as a perturbation of an unitary. It comes quite naturally that the relationship between these two settings is established by the Cayley transform and we term this construction Naboko-Vasyunin model.

Although the question of model representation of a bounded operator became settled on the abstract level with the work [MV], the challenges with various applications to the physical problems remain to be addressed. (See [S] for valuable details on dissipative case.) Speaking of two basic examples of non-selfadjoint Schrödinger operators tracked back to the original works of B. Pavlov, it has to be noted that the example of the Schrödinger operator with a complex-valued potential can be studied from the more general point of view of relative bounded perturbations developed in [Na].[1] At the same time the second example, non-selfadjoint extensions of a Hermitian differential operator, mostly remains outside of the general theory since these operators could not be divided into a selfadjoint one, plus a relatively bounded additive perturbation. Consequently, in order to utilize the functional model approach for the study of extensions of Hermitian operators arising in the physical applications, one is left solely with the recipe of B. Pavlov. In other words, one has to "guess" the selfadjoint dilation and to prove the eigenfunction expansion theorem.

The present paper concerns the functional model construction for a wide class of extensions of Hermitian operators known in the literature as **almost solvable extensions**. Our approach is identical to that of S. Naboko and as such does not involve the eigenfunctions expansion at all. All considerations are carried out

[1] The functional model of additive perturbations has been applied to the spectral analysis of the transport operator in [NR], [KNR].

in the general setting of the model for non-dissipative non-selfadjoint operators. Although results obtained here are applicable to many interesting physical and mathematical problems, the limitations of almost solvable extensions theory hamper the study of the most interesting case of a multi-dimensional boundary value problem for the partial differential operators. (See Remark 1.5 for more details.) Dissipative extensions of Hermitian operators with finite deficiency indices are much easier to analyze. A few successful attempts that utilize B. Pavlov schema to examine operators of this class encountered in applications were published recently. In particular, Pavlov's approach to the model construction of dissipative extensions of Hermitian operators was followed by B. Allahverdiev in his works [A2], [A3], [A4], [A5], and some others, and by the group of authors [KaNR], [BN], [BKNR1], [BKNR2], where the theory of dissipative Schrödinger operator on finite interval was applied to the problems arising in the semiconductor physics. In comparison with these results, Section 2 below offers an abstract perspective on the selfadjoint dilation and its resolvent for a dissipative almost solvable extension, and more importantly, verifies correctness of many underlying arguments needed for the further development in the general situation. These abstract results are immediately applicable to any dissipative almost solvable extension, thereby relieve of the burden to prove them in each particular case. Since the eigenfunction expansion is not used in the model construction, all the objects are well defined and there is no need for special considerations with regard to formal procedures dealing with "generalized" vectors. Finally, the paper proposes a model of an almost solvable extension with no assumption of its dissipativity.

The paper is organized as follows. In Section 1 we briefly review some definitions and results pertinent to our study. The Section culminates with the calculation of the characteristic function of a non-selfadjoint almost solvable extension of a Hermitian operator expressed in terms of the extension's "parameter" and the Weyl function. (See the definitions below.) The relationship of these three objects is believed to be first obtained in the paper [P1] for a Hermitian operator with the deficiency indices $(1, 1)$, but seems to remain unnoticed. We take an opportunity and formulate this result in the more general setting of almost solvable extensions. In Section 2 we show how to build the functional model of a non-selfadjoint almost solvable extension of a Hermitian operator following the approach of [Na]. All the results are accompanied with the full proofs, starting from the exact form of dilation of a dissipative almost solvable extension and ending in the main model theorem for a general non-selfadjoint non-dissipative extension. In Section 3 the theory is illustrated by two examples of Hermitian operators with finite deficiency indices. We refrain from giving the model construction of non-selfadjoint extensions of these operators, because all such results are easily derived from the theory developed in Section 2.

We use symbol $\mathscr{B}(H_1, H_2)$ where H_1, H_2 are separable Hilbert spaces, for the Banach algebra of bounded operators, defined everywhere in H_1 with values in H_2. The notation $A : H_1 \to H_2$ is equivalent to $A \in \mathscr{B}(H_1, H_2)$. Also, $\mathscr{B}(H) := \mathscr{B}(H, H)$. The real axis, complex plane are denoted as $\mathbb{R}$, $\mathbb{C}$, respectively. Further, $\mathbb{C}_\pm := \{z \in \mathbb{C} \,:\, \pm \operatorname{Im} z > 0\}$, $\mathbb{R}_\pm := \{x \in \mathbb{R} \,:\, \pm\, x > 0\}$, where Im stands for the imaginary part of a complex number. The domain, range and kernel of a linear operator A are denoted as $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker(A)$; the symbol $\rho(A)$ is used for the resolvent set of A.

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1 Preliminaries

Let us recall a few basic facts about unbounded linear operators.

For a closed linear operator L with dense domain $\mathcal{D}(L)$ on a separable Hilbert space H a sesquilinear form $\Psi_L(\cdot, \cdot)$ defined on domain $\mathcal{D}(L) \times \mathcal{D}(L)$:

$$\Psi_L(f, g) = \frac{1}{i}[(Lf, g)_H - (f, Lg)_H], \quad f, g \in \mathcal{D}(L) \tag{1.1}$$

plays a role of the imaginary part of L in the sense that $2\operatorname{Im}(Lf, f) = \Psi_L(f, f)$, $f \in \mathcal{D}(L)$.

Definition 1.1. Operator L is called **dissipative** if

$$\operatorname{Im}(Lf, f) \geq 0, \quad f \in \mathcal{D}(L) \tag{1.2}$$

Definition 1.2. Operator L is called **maximal dissipative** if (1.2) holds and the resolvent $(L - zI)^{-1}$ exists for any $z \in \mathbb{C}_-$ as operator from $\mathscr{B}(H)$.

In what follows A denotes a closed and densely defined Hermitian operator on the separable Hilbert space H with equal deficiency indices $0 < n_+(A) = n_-(A) \leq \infty$. We will assume that A is simple, i.e. it has no reducing subspaces where it induces a self-adjoint operator. The adjoint operator A^* is closed and $A \subseteq A^*$ in a sense that $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $Ax = A^*x$ for $x \in \mathcal{D}(A)$.

1.1 Boundary triples and almost solvable extensions

An extension $\mathcal{A}$ of the operator A is called **proper**, if $A \subseteq \mathcal{A} \subseteq A^*$. The following definition, see [GG], [Bru], [Ko1], may be considered as an abstract version of the second Green's formula.

Definition 1.3. A triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings Γ_0, Γ_1 defined on the set $\mathcal{D}(A^*)$, is called a **boundary triple** for the operator A^* if the following conditions are satisfied:

1. The Green's formula is valid
$$(A^*f, g)_H - (f, A^*g)_H = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathcal{D}(A^*) \tag{1.3}$$

2. For any $Y_0, Y_1 \in \mathcal{H}$ there exist $f \in \mathcal{D}(A^*)$, such that $\Gamma_0 f = Y_0$, $\Gamma_1 f = Y_1$. In other words, the mapping $f \mapsto \Gamma_0 f \oplus \Gamma_1 f$, $f \in \mathcal{D}(A^*)$ into $\mathcal{H} \oplus \mathcal{H}$ is surjective.

The boundary triple can be constructed for any closed densely defined Hermitian operator with equal deficiency indices. Moreover, the space $\mathcal{H}$ can be chosen so that $\dim \mathcal{H} = n_+(A) = n_-(A)$. (See references above for further details.)

Definition 1.4. A proper extension $\mathcal{A}$ of the Hermitian operator A is called **almost solvable (a.s.)** if there exist a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* and an operator $B \in \mathscr{B}(\mathcal{H})$ such that
$$f \in \mathcal{D}(\mathcal{A}) \iff \Gamma_1 f = B\Gamma_0 f \tag{1.4}$$

Note that this definition implies inclusion $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(A^*)$ and in fact operator $\mathcal{A}$ is a restriction of A^* to the linear set $\{f \in \mathcal{D}(A^*) : \Gamma_1 f = B\Gamma_0 f\}$.

It can be shown (see [DM]) that if a proper extension $\mathcal{A}$ has regular points in both the upper and lower half planes, then this extension is almost solvable. In other words, there exist a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and an operator $B \in \mathscr{B}(\mathcal{H})$ such that $\mathcal{A} = A_B$. We will refer to the operator B as a "parameter" of the extension A_B.

Next Theorem summarizes some facts concerning a.s. extensions needed for the purpose of the paper.

Theorem 1.1. *Let A be a closed Hermitian operator with dense domain on a separable Hilbert space H with equal (finite or infinite) deficiency indices and $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triple for its adjoint A^*. Let $B \in \mathscr{B}(\mathcal{H})$ and A_B be the corresponding a.s. extension of A. Then*

1. $A \subset A_B \subset A^*$
2. $(A_B)^* \subset A^*, \quad (A_B)^* = A_{B^*}$
3. *A_B is maximal, i.e. $\rho(A_B) \neq \emptyset$*
4. *B is dissipative $\iff$ A_B is maximal dissipative*
5. $B = B^* \iff A_B = (A_B)^*$

Proof. The proof can be found in [GG], [DM]. Note that the last two assertions can easily be verified using equality
$$\Psi_{A_B}(f, g) = \frac{1}{i}[(A_B f, g) - (f, A_B g)] = \frac{1}{i}((B - B^*)\Gamma_0 f, \Gamma_0 g), \qquad f, g \in \mathcal{D}(A_B) \tag{1.5}$$
which directly follows from (1.3), (1.4). □

Remark 1.5. In many cases of operators associated with partial differential equations, the boundary triple constructed according to the results cited in Definition 1.3 could not be easily linked to the Green formula as traditionally understood in a sense of differential expressions. For example, let Ω be a smooth bounded domain in $\mathbb{R}^3$, and A be a minimal Hermitian operator in $L_2(\Omega)$ associated with the Laplace differential expression $-\Delta$ in Ω. Then A^* is defined on the set of functions $u \in L_2(\Omega)$ such that $\Delta u \in L_2(\Omega)$. The well known Green formula (see [AF], for example) suggest the "natural" definition of mappings Γ_0, Γ_1 as $\Gamma_0 : u \mapsto u|_{\partial\Omega}$, $\Gamma_1 : u \mapsto \frac{\partial u}{\partial n}|_{\partial\Omega}$, $u \in D(A^*)$ with the boundary space $\mathcal{H} = L_2(\partial\Omega)$. However, because there exist functions in $\mathcal{D}(A^*)$ that do not possess boundary values on $\partial\Omega$, operators Γ_0 and Γ_1 are not defined on the whole of $\mathcal{D}(A^*)$, and the theory of almost solvable extensions is inapplicable to this choice of triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

1.2 Weyl function

For a given boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for the operator A^* introduce an operator A_∞ as a restriction of A^* on the set of elements $y \in \mathcal{D}(A^*)$ satisfying condition $\Gamma_0 y = 0$:

$$A_\infty := A^*|_{\mathcal{D}(A_\infty)}, \quad \mathcal{D}(A_\infty) := \{y \in \mathcal{D}(A^*) \,:\, \Gamma_0 y = 0\} \tag{1.6}$$

Formally, the operator A_∞ is an almost solvable selfadjoint extension of A corresponding to the choice $B = \infty$. (See (1.4).) This justifies the notation. It turns out ([GG], [DM]), that the operator A_∞ is selfadjoint indeed. Further, for any $z \in \mathbb{C}_- \cup \mathbb{C}+$ the domain $\mathcal{D}(A^*)$ can be represented in the form of direct sum:

$$\mathcal{D}(A^*) = \mathcal{D}(A_\infty) \dot{+} \ker(A^* - zI) \tag{1.7}$$

according to the decomposition $f = y + h$ with $f \in \mathcal{D}(A^*)$, $y \in \mathcal{D}(A_\infty)$, and $h \in \ker(A^* - zI)$, where

$$y := (A_\infty - zI)^{-1}(A^* - zI)f, \quad h := f - y$$

Taking into account equality $\mathcal{D}(A_\infty) = \ker(\Gamma_0)$ and surjective property of Γ_0, it follows from the formula (1.7) that for each $e \in \mathcal{H}$ and $z \in \mathbb{C}_- \cup \mathbb{C}_+$ the equation $\Gamma_0 h = e$ has an unique solution that belongs to $\ker(A^* - zI)$. In other words, a restriction of operator Γ_0 on the set $\ker(A^* - zI)$ is invertible. Denote $\gamma(z)$ the corresponding inverse operator:

$$\gamma(z) = \left[\Gamma_0|_{\ker(A^* - zI)}\right]^{-1}, \quad z \in \mathbb{C}_- \cup \mathbb{C}_+. \tag{1.8}$$

By a simple computation we deduce from (1.3) with $f \in \mathcal{D}(A_\infty)$, $g \in \ker(A^* - zI)$ that

$$\gamma^*(\bar{z}) = \Gamma_1(A_\infty - zI)^{-1}, \quad z \in \mathbb{C}_- \cup \mathbb{C}_+. \tag{1.9}$$

Weyl function $M(\cdot)$ corresponding to the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is defined as an operator-function with values in $\mathscr{B}(\mathcal{H})$, such that for each $z \in \mathbb{C}_- \cup \mathbb{C}_+$, and $f_z \in \ker(A^* - zI)$

$$M(z)\Gamma_0 f_z = \Gamma_1 f_z \tag{1.10}$$

Another representation of $M(\cdot)$ easily follows from (1.8) and (1.10)

$$M(z) = \Gamma_1 \gamma(z), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{1.11}$$

Next Theorem sums up a few properties of the Weyl function.

Theorem 1.2. *Let $M(\cdot)$ be the Weyl function (1.10), $z \in \mathbb{C}_- \cup \mathbb{C}_+$ and an operator $B \in \mathscr{B}(\mathcal{H})$ be a parameter of a.s. extension A_B of A. Following assertions hold:*

1. *$M(z)$ is analytic,*
2. *$Im\, M(z) \cdot Im\, z > 0$,*
3. *$[M(z)]^* = M(\bar{z})$,*
4. *$M(z) - M(\zeta) = (z - \zeta)\gamma^*(\bar{\zeta})\gamma(z), \quad z, \zeta \in \mathbb{C}_+ \cup \mathbb{C}_-$*

5. $z \in \rho(A_B) \iff (B - M(z))$ *is boundedly invertible in* $\mathcal{H}$

6. $(A_B - zI)^{-1} - (A_\infty - z)^{-1} = \gamma(z)(B - M(z))^{-1}\gamma^*(\bar{z}), \quad z \in \rho(A_B)$

Proof. The proof of the Theorem can be found in [DM]. □

It follows from the Theorem 1.2 that the Weyl function $M(\cdot)$ is a Herglotz function. It is analytic in the upper half plane, with positive imaginary part.

1.3 Characteristic function of an almost solvable extension

As before, let A be a simple densely defined Hermitian operator with equal deficiency indices and $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triple for A^*. Let $M(\cdot)$ be the Weyl function corresponding to that triple. According to the Theorem 1.1, for any $B \in \mathscr{B}(\mathcal{H})$ the extension A_B is selfadjoint if $B = B^*$. We shall assume that $B \neq B^*$ and calculate the characteristic function of the nonselfadjoint operator A_B. (See the definition below.) For simplicity sake we assume that the operator A_B is simple. In other words, A_B has no non-trivial selfadjoint parts. It turns out that there exists an elegant formula which ties together the characteristic function of A_B, Weyl function $M(\cdot)$ and the extension "parameter" B. In the particular case of one-dimensional Schrödinger operator on $\mathbb{R}_+$, this formula was obtained in [P1].

Let us recall the definition of characteristic function of a linear non-selfadjoint operator. In our narrative we follow the abstract approach developed by Štraus [St1].

For a closed linear operator L with dense domain $\mathcal{D}(L)$ introduce a linear set $\mathcal{G}(L)$:

$$\mathcal{G}(L) = \{g \in \mathcal{D}(L) : \Psi_L(f, g) = 0, \quad \forall f \in \mathcal{D}(L)\},$$

and a linear space $\mathfrak{L}(L)$ defined as closure of the quotient $\mathcal{D}(L)/\mathcal{G}(L)$ endowed with an inner product $[\xi, \eta]_\mathfrak{L} = \Psi_L(f, g)$, $\xi, \eta \in \mathfrak{L}(L)$, $f \in \xi$, $g \in \eta$, where $\Psi_L(f, g)$ is defined in (1.1). The inner product $[\cdot, \cdot]_\mathfrak{L}$ is symmetric and non-degenerate, but not necessarily positive. The non-degeneracy means the implication

$$[\xi, \eta]_\mathfrak{L} = 0, \forall \eta \in \mathfrak{L} \quad \Rightarrow \quad \xi = 0$$

Definition 1.6. A **boundary space** for the operator L is any linear space $\mathfrak{L}$ which is isomorphic to $\mathfrak{L}(L)$. A **boundary operator** for the operator L is the linear operator Γ with the domain $\mathcal{D}(L)$ and the range in the boundary space $\mathfrak{L}$ such that

$$[\Gamma f, \Gamma g]_\mathfrak{L} = \Psi_L(f, g), \quad f, g \in \mathcal{D}(L) \tag{1.12}$$

We shall assume that the operator L is non-selfadjoint and its resolvent set is non-empty: $\rho(L) \neq \emptyset$. Let $\mathfrak{L}$ endowed with an inner product $[\cdot, \cdot]$ be a boundary space for L with boundary operator Γ, and let $\mathfrak{L}'$ with an inner product $[\cdot, \cdot]'$ be a boundary space for $-L^*$ with boundary operator Γ' mapping $\mathcal{D}(L^*)$ onto $\mathfrak{L}'$.

Definition 1.7. A **characteristic function** of the operator L is an operator-valued function Θ_L defined on the set $\rho(L^*)$ whose values $\Theta_L(z)$ map $\mathfrak{L}$ into $\mathfrak{L}'$ according to the equality

$$\Theta_L(z)\Gamma f = \Gamma'(L^* - zI)^{-1}(L - zI)f, \quad f \in \mathcal{D}(L). \tag{1.13}$$

Since the right hand side of (1.13) is analytic with regard to $z \in \rho(L^*)$, the function Θ_L is analytic on $\rho(L^*)$.

Let us carry out the calculation of characteristic function of an a.s. extension A_B of the Hermitian operator A parameterized by the bounded operator $B \in \mathscr{B}(\mathcal{H})$.

Let $B = B_R + iB_I$ where $B_R = \frac{1}{2}(B + B^*)$ and $B_I = \frac{1}{2i}(B - B^*)$ be the real and the imaginary parts of operator B, and

$$E = clos\,\mathcal{R}(B_I), \quad \alpha = |2B_I|^{1/2}, \quad J = sign(B_I|_E) \tag{1.14}$$

Obviously, operators α and J commute as functions of the selfadjoint operator B_I. Note as well the involutional properties of the mapping J acting on the space E, namely, the equalities $J = J^* = J^{-1}$. If the operator B is dissipative (i.e. $B_I \geq 0$), then $J = I_E$ and $\alpha = (2B_I)^{1/2}$.

Using notation (1.14) the equality (1.5) can be rewritten in the form

$$\Psi_{A_B}(f,g) = 2(B_I\Gamma_0 f, \Gamma_0 g)_E = (J\alpha\Gamma_0 f, \alpha\Gamma_0 g)_E, \quad f,g \in \mathcal{D}(A_B)$$

where equality $2B_I|_E = \alpha J\alpha|_E$ holds due to the Spectral Theorem. According to the definition (1.12) we can choose the boundary space of the operator A_B to be the space E with the metric $[\cdot,\cdot] = (J\cdot,\cdot)_{\mathcal{H}} = (J\cdot,\cdot)_E$ and define the boundary operator Γ as the map

$$\Gamma : f \mapsto J\alpha\Gamma_0 f, \quad f \in \mathcal{D}(\Gamma), \quad \mathcal{D}(\Gamma) = \mathcal{D}(A_B) \tag{1.15}$$

Since $-A_B^* = -A_{B^*}$, see Theorem 1.1, we can repeat the arguments above and choose the boundary space of $-A_B^*$ to be the same Hilbert space E with the same metric $[\cdot,\cdot]' = [\cdot,\cdot] = (J\cdot,\cdot)_E$, and the boundary operator Γ' to be equal to the operator $\Gamma = J\alpha\Gamma_0$. Note that the metric $[\cdot,\cdot] = [\cdot,\cdot]'$ is positive if the operator B is dissipative.

Now we are ready to calculate the characteristic function of the operator A_B that corresponds to the chosen boundary spaces and operators. Let $z \in \rho(A_B^*)$ be a complex number and $f \in \mathcal{D}(A_B)$. Then from the equality $g_z = (A_B^* - zI)^{-1}(A_B - zI)f$ we obtain

$$A_B f - A_B^* g_z = z(f - g_z)$$

which due to inclusions $A_B \subset A^*$, $A_B^* \subset A^*$ shows that the vector $f - g_z$ belongs to the linear set $\ker(A^* - zI)$. By the Weyl function definition (1.10) the following equality holds for each $z \in \rho(A_B^*)$, $f \in \mathcal{D}(A_B)$

$$M(z)\Gamma_0(f - g_z) = \Gamma_1(f - g_z)$$

Since $f \in \mathcal{D}(A_B)$ and $g_z \in \mathcal{D}(A_B^*)$, the right hand side here can be rewritten in the form $B\Gamma_0 f - B^*\Gamma_0 g_z$, and after elementary regrouping we obtain

$$(M(z) - B)\Gamma_0 f = (M(z) - B^*)\Gamma_0 g_z$$

By virtue of Theorem 1.2 the operator $(M(z) - B^*)$ is boundedly invertible for $z \in \rho(A_B^*)$. Therefore,

$$\Gamma_0 g_z = (B^* - M(z))^{-1}(B - M(z))\Gamma_0 f$$

and due to (1.15),

$$\begin{aligned}
\Gamma' g_z &= J\alpha\Gamma_0 g_z = J\alpha(B^* - M(z))^{-1}(B - M(z))\Gamma_0 f \\
&= J\alpha(B^* - M(z))^{-1}\big[B^* - M(z) + (B - B^*)\big]\Gamma_0 f \\
&= J\alpha\big[I + 2i(B^* - M(z))^{-1}B_I\big]\Gamma_0 f = J\alpha\big[I + i(B^* - M(z))^{-1}\alpha J\alpha\big]\Gamma_0 f \\
&= \big[I_E + iJ\alpha(B^* - M(z))^{-1}\alpha\big]J\alpha\Gamma_0 f = \big[I_E + iJ\alpha(B^* - M(z))^{-1}\alpha\big]\Gamma f
\end{aligned}$$

so that finally for any $f \in \mathcal{D}(A_B)$ and $z \in \rho(A_B^*)$ the following equality holds

$$\Gamma'(A_B^* - zI)^{-1}(A_B - zI)f = \big[I_E + iJ\alpha(B^* - M(z))^{-1}\alpha\big]\Gamma f.$$

Now the comparison with the definition (1.13) yields that the characteristic function $\Theta_{A_B}(\cdot) : E \to E$ corresponding to the boundary operators and spaces chosen above is given by the formula

$$\Theta_{A_B}(z) = I_E + iJ\alpha(B^* - M(z))^{-1}\alpha|_E, \quad z \in \rho(A_B^*) \tag{1.16}$$

Similar calculations can be found in [Ko2].

A few remarks are in order. Following the schema followed above, it is easy to compute the characteristic function $\Theta_B(\cdot)$ of the operator B. Indeed, for $x, y \in \mathcal{H}$

$$\begin{aligned}
\Psi_B(x,y) &= \frac{1}{i}\big[(Bx,y)_{\mathcal{H}} - (x,By)_{\mathcal{H}}\big] = \frac{1}{i}\big((B - B^*)x, y\big)_{\mathcal{H}} = 2(B_I x, y)_{\mathcal{H}} = (J\alpha x, \alpha y)_E, \\
\Psi_{-B^*}(x,y) &= \Psi_B(x,y)
\end{aligned}$$

so that we can choose the space $E = clos\,\mathcal{R}(B_I)$ as a boundary space of the operators B and $-B^*$, see (1.14), and assume the boundary operators for B and $-B^*$ to be the mapping of the vector $x \in \mathcal{H}$ into $J\alpha x \in E$. Computations, similar to those conducted above, lead to the following expression for the characteristic function $\Theta_B(\cdot)$ of operator B:

$$\Theta_B(z) = I_E + iJ\alpha(B^* - zI)^{-1}\alpha|_E$$

Remark 1.8. Comparison with (1.16) shows that the characteristic function Θ_{A_B} of the extension A_B can be formally obtained by the substitution of zI in the expression for characteristic function Θ_B of the "parameter" operator B with the Weyl function $M(z)$ of the operator A. Or more formally,

$$\Theta_{A_B}(z) = \Theta_B(M(z)), \quad z \in \rho(B^*) \cap \rho(A_B^*).$$

This interesting formula can be traced back to the paper of B. Pavlov [P1].

Remark 1.9. Values of the characteristic operator function $\Theta_{A_B}(\cdot)$ in the upper half plane $\mathbb{C}_+$ are J-contractive operators in E, i.e. for $\varphi \in E$

$$\big(J\Theta_{A_B}(z)\varphi, \Theta_{A_B}(z)\varphi\big)_E \leq (J\varphi, \varphi)_E, \quad z \in \rho(A_B^*) \cap \mathbb{C}_+ \tag{1.17}$$

This result follows from the general contractive property of characteristic functions of linear operators obtained in [St1]. It is remarkable that the proof cited below does not require the knowledge of characteristic function itself. Its contractiveness follows directly from its definition.

Theorem 1.3. *[St1] Let $\mathfrak{L}$, $\mathfrak{L}'$, Γ, Γ' be the boundary spaces and boundary operators for the operators L and $-L^*$ respectively as described in Definition 1.6, $[\cdot,\cdot]$, $[\cdot,\cdot]'$ be the metrics in $\mathfrak{L}$, $\mathfrak{L}'$, and $\Theta_L(\cdot)$ be the characteristic function of L, see Definition 1.7. Then the following equality holds*

$$[\varphi, \varphi_1] - [\Theta_L(z)\varphi, \Theta_L(\zeta)\varphi_1]' = \frac{1}{i}(z - \bar{\zeta})(\Omega_z\varphi, \Omega_\zeta\varphi_1)_H \tag{1.18}$$

where $z, \zeta \in \rho(L^*)$, $\varphi, \varphi_1 \in \mathfrak{L}$, and operator Ω_z, $z \in \rho(L^*)$ is uniquely defined as the map $\Omega_z : \Gamma f \mapsto f - (L^* - zI)^{-1}(L - zI)f$, $f \in \mathcal{D}(L)$.

Proof. By polarization identity it is sufficient to show that (1.18) is valid for $z = \zeta$, $\varphi = \varphi_1$. Standard density arguments allow us to prove the statement of the Theorem only for the dense set of vectors $\{\varphi\}$ in $\mathfrak{L}$, for which there exist $f \in \mathcal{D}(L)$, such that $\varphi = \Gamma f$. Let φ be such a vector and $f \in \mathcal{D}(L)$ satisfies the condition $\Gamma f = \varphi$. Let g_z be the vector $(L^* - zI)^{-1}(L - zI)f$. Note that $g_z \in \mathcal{D}(L^*)$, $(Lf, g_z) = (f, L^*g_z)$, and $Lf - L^*g_z = z(f - g_z)$. Then

$$\begin{aligned}
&(z - \bar{z})(\Omega_z\varphi, \Omega_z\varphi)_H = (z - \bar{z})(\Omega_z\Gamma f, \Omega_z\Gamma f) = (z - \bar{z})(f - g_z, f - g_z) \\
&= \big(z(f - g_z), f - g_z\big) - \big(f - g_z, z(f - g_z)\big) = \big(Lf - L^*g_z, f - g_z\big) - \big(f - g_z, Lf - L^*g_z\big) \\
&= (Lf, f) + (L^*g_z, g_z) - (f, Lf) - (g_z, L^*g_z) = \big((Lf, f) - (f, Lf)\big) - \big((-L^*g_z, g_z) - (g_z, -L^*g_z)\big)
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{i}(z - \bar{z})(\Omega_z\varphi, \Omega_z\varphi)_H &= \Psi_L(f, f) - \Psi_{-L^*}(g_z, g_z) = [\Gamma f, \Gamma f] - [\Gamma' g_z, \Gamma' g_z]' \\
&= [\Gamma f, \Gamma f] - [\Theta_L(z)\Gamma f, \Theta_L(z)\Gamma f]' = [\varphi, \varphi] - [\Theta_L(z)\varphi, \Theta_L(z)\varphi]'.
\end{aligned}$$

The proof is complete. □

From (1.18) and the representation of the boundary spaces of operators A_B, $-A_B^*$ described above we conclude

$$(J\varphi, \varphi)_E - (J\Theta_{A_B}(z)\varphi, \Theta_{A_B}(z)\varphi)_E = (2\,\mathrm{Im}\,z)\|\Omega_z\varphi\|^2$$

which proves (1.17).

Remark 1.10. If the extension parameter B is a dissipative operator from $\mathscr{B}(\mathcal{H})$, then the corresponding extension A_B is a closed dissipative operator and its resolvent set $\rho(A_B)$ includes the lower half plane $\mathbb{C}_-$. The conjugate operator $(A_B)^*$ is an extension of A corresponding to the operator B^*, see the Theorem 1.1, so that the upper half plane $\mathbb{C}_+$ consists of the regular points of A_B^*. Since B is dissipative, the involution operator J defined in (1.14) is in fact the identity operator on E. Moreover, the metric of the boundary spaces $\mathfrak{L}$, $\mathfrak{L}'$ is positively defined, hence the space $\mathfrak{L}$ is a Hilbert space. It follows from (1.17) that in this case values of the characteristic function $\Theta_{A_B}(z)$, $z \in \mathbb{C}_+$ are contractive operators from $\mathscr{B}(E)$:

$$\|\Theta_{A_B}(z)\varphi\|_E \leq \|\varphi\|_E, \quad z \in \mathbb{C}_+,\ B \in \mathscr{B}(\mathcal{H}) \text{ is dissipative.} \tag{1.19}$$

Remark 1.11. Let $z \in \mathbb{C}_+$. According to Theorem 1.2, the imaginary part of the operator $M(z)$ is positive. In other words, values of the function $M(z)$ are dissipative operators in $\mathscr{B}(E)$. Assume $B = iI_{\mathcal{H}}$. This operator is dissipative, so is the corresponding extension A_{iI}. Since $B^* = -iI_{\mathcal{H}}$, $E = \mathcal{H}$, $J = I_{\mathcal{H}}$, $\alpha = (2|B_I|)^{1/2} = \sqrt{2}I_{\mathcal{H}}$, the characteristic function (1.16) can be written as

$$\Theta_{A_{iI}}(z) = I_{\mathcal{H}} + 2i\big(-iI_{\mathcal{H}} - M(z)\big)^{-1} = \big(M(z) - iI\big)\big(M(z) + iI\big)^{-1}.$$

We see that for $z \in \mathbb{C}_+$ the contractive function $\Theta_{iI}(z)$ and the Herglotz function $M(z)$ are related to each other via Cayley transform. In fact, operator function $\Theta_{A_{iI}}$ is a characteristic function of the Hermitian operator A as defined in [St2].

2 Functional model

In this Section a symmetric variant of model for a non-selfadjoint non-dissipative a.s extension of the Hermitian operator A is constructed.

Let $B \in \mathscr{B}(\mathcal{H})$ and A_B be the corresponding a.s. extension of A. Question of simultaneous simplicity of operators B and A_B was formulated in [S], and the author is unaware of any results which would shed light on the intricate relationship between selfadjoint parts of B and A_+. In the following it is always assumed that both B and A_B are simple operators. Further, by virtue of Theorem 1.1, A_B is maximal and the resolvent set of A_B is non-empty: $\rho(A_B) \neq \emptyset$. The conjugate operator $(A_B)^*$ is simple and maximal as well. It coincides with the extension of A parametrized by $B^* : (A_B)^* = A_{B^*}$. The characteristic function $\Theta_{A_B}(\cdot)$ is analytic on $\rho(A_B^*)$ with values in $\mathscr{B}(E)$ and J-contractive on $\rho(A_B^*) \cap \mathbb{C}_+$, see (1.16), (1.17) and (1.14) for the notation.

Assume that the operator $B = B_R + iB_I$, where $B_R := (1/2)(B + B^*)$, $B_I := (1/2i)(B - B^*)$, is not dissipative so that $J \neq I_E$. Along with operator $B = B_R + iJ\frac{\alpha^2}{2}$ consider a dissipative operator $B_+ := B_R + i|B_I| = B_R + i\frac{\alpha^2}{2}$ and let A_{B_+} be the corresponding a.s. extension of A. Then the operators B_+ and A_{B_+} are both dissipative, B_+ is bounded, and as shown in [Na], B_+ is simple. As mentioned above, these observations alone do not guarantee simplicity of A_{B_+}. Nevertheless, A_{B_+} is simple. This fact follows from the Theorem 2.1 below and explicit relationship between Cayley transformations of A_B and A_{B_+} found in [MV] in more general setting. Namely, it follows from [MV] that selfadjoint parts of A_B and A_{B_+} coincide. The same result can be obtained by methods developed in the system theory [Ar].

Finally, due to dissipativity the lower half plane $\mathbb{C}_-$ consists of the regular points of A_{B_+} and similarly, $\mathbb{C}_+ \subset \rho(A^*_{B_+})$.

According to Remarks 1.9 and 1.10, values of characteristic functions of two extensions A_B, A_{B_+} are J-contractive and contractive operators respectively in $\rho(A_B^*) \cap \mathbb{C}_+$. It turns out that these values are related via so called Potapov-Ginzburg transformation [AI]. This observation was first made in [Na] for additive perturbations of a selfadjoint operator and in [MV] for the general case. (Cf. [Ar] for an alternative, but equivalent approach.) We formulate this relationship in the special situation of almost solvable extensions of a Hermitian operator and sketch a simple proof based on findings of [Na].

Theorem 2.1. *The characteristic functions $\Theta := \Theta_{A_B}$, $S := \Theta_{A_{B_+}}$ of two simple maximal a.s extensions A_B, A_{B_+} of the Hermitian operator A corresponding to the extension parameters $B, B_+ \in \mathscr{B}(\mathcal{H})$, where $B = B_R + iB_I$, $B_+ = B_R + i|B_I|$ are related to each other via following Potapov-Ginzburg transformation.*

$$\begin{aligned}
\Theta(z) &= \quad (X^- + X^+S(z)) \cdot (X^+ + X^-S(z))^{-1}, \\
\Theta(z) &= -(X^+ - S(z)X^-)^{-1} \cdot (X^- - S(z)X^+), \\
S(z) &= \quad (X^- + X^+\Theta(z)) \cdot (X^+ + X^-\Theta(z))^{-1}, \\
S(z) &= -(X^+ - \Theta(z)X^-)^{-1} \cdot (X^-\Theta(z)X^+)^{-1} \\
\Theta(\zeta) &= \quad (X^+ + X^-[S(\bar\zeta)]^*) \cdot (X^- + X^+[S(\bar\zeta)]^*)^{-1}, \\
\Theta(\zeta) &= -(X^- - [S(\bar\zeta)]^*X^+)^{-1} \cdot (X^+ - [S(\bar\zeta)]^*X^-), \\
[S(\bar\zeta)]^* &= \quad (X^+ + X^-\Theta(\zeta)) \cdot (X^- + X^+\Theta(\zeta))^{-1}, \\
[S(\bar\zeta)]^* &= -(X^- - \Theta(\zeta)X^+)^{-1} \cdot (X^+ - \Theta(\zeta)X^-)
\end{aligned} \tag{2.1}$$

Here $z \in \rho(A_B^*) \cap \mathbb{C}_+$, $\zeta \in \rho(A_B^*) \cap \mathbb{C}_-$ and $X^\pm := (I_E \pm J)/2$ are two complementary orthogonal projections in the space E.

Proof. The existence of Potapov-Ginzburg transformation S of a J-contractive operator Θ and formulae (2.1) can be found in the literature ([AI], [Ar]). On the other hand, it has been shown in the paper [Na] that the characteristic functions of two bounded operators $B = B_R + iB_I$ and $B_+ = B_R + i|B_I|$ are related via Potapov-Ginzburg transformation. Taking into account Remark 1.8 we arrive at the Theorem's assertion. $\square$

In what follows we will use the simplified notation Θ, S introduced in the Theorem 2.1 for the characteristic functions Θ_{A_B} and $\Theta_{A_{B_+}}$, respectively. Note that due to Remark 1.10, the analytic operator functions $S(z)$ and $S^*(\zeta) := [S(\bar\zeta)]^*$ are contractive if $z \in \mathbb{C}_+$, $\zeta \in \mathbb{C}_-$. Moreover, there exist non-tangential strong boundary values almost everywhere on the real axis: $S(k) := s-\lim_{\varepsilon\downarrow 0} S(k+i\varepsilon)$, $S^*(k) := s-\lim_{\varepsilon\downarrow 0} S^*(k-i\varepsilon)$, a.e. $k \in \mathbb{R}$. These boundary values are contractive and mutually conjugate operators for almost all $k \in \mathbb{R}$ ([NF]).

2.1 Symmetric form of Sz.Nagy-Foiaş model

The functional model of a dissipative operator can be derived from the B. Sz.-Nagy-C.Foiaş model for the contraction, whose Cayley transform it represents [NF]. An independent approach was given in the framework of acoustic scattering by P. Lax and R. Phillips [LP]. In our narrative we will use an equivalent model construction known as symmetric model as given by B. Pavlov in [P3], [P4] and and elaborated further in the paper [Na].

Let $\mathscr{A}$ be the minimal selfadjoint dilation of the simple dissipative operator A_{B_+}. In other words, the operator $\mathscr{A} = \mathscr{A}^*$ is defined on a wider space $\mathscr{H} \supset H$ such that (cf. [NF])

$$\begin{aligned} P_H(\mathscr{A} - zI)^{-1}|_H &= (A_{B_+} - zI)^{-1}, \quad z \in \mathbb{C}_- \\ P_H(\mathscr{A} - zI)^{-1}|_H &= (A^*_{B_+} - zI)^{-1}, \quad z \in \mathbb{C}_+ \end{aligned} \tag{2.2}$$

and $\mathscr{H} := span\{(\mathscr{A} - zI)^{-1}H \,:\, z \in \mathbb{C}_\pm\}$. Here $P_H : \mathscr{H} \to H$ is the orthogonal projection from the dilation space $\mathscr{H}$ onto H. The dilation $\mathscr{A}$ can be chosen in many ways. Following [P3], [P4], we will use the dilation space in the form of orthogonal sum $\mathscr{H} := D_- \oplus H \oplus D_+$, where $D_\pm := L_2(\mathbb{R}_\pm, E)$. The space H is naturally embedded into $\mathscr{H}$: $H \to 0 \oplus H \oplus 0$, whereas spaces $D_\pm$ are embedded into $L_2(E) = D_- \oplus D_+$. The dilation representation offered in the next Theorem is a straightforward generalization of B. Pavlov's construction [P5]. Its form was announced in [Pe] without a proof. (See [S], [K] for more general approach.)

Define a linear operator $\mathscr{A}$ by formula

$$\mathscr{A}\begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} = \begin{pmatrix} iv'_- \\ A^*u \\ iv'_+ \end{pmatrix}, \qquad \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} \in \mathcal{D}(\mathscr{A}), \tag{2.3}$$

where domain $\mathcal{D}(\mathscr{A})$ consists of vectors $(v_-, u, v_+) \in \mathscr{H}$, such that $v_\pm \in W_2^1(\mathbb{R}_\pm, E)$ and $u \in \mathcal{D}(A^*)$ satisfy two "boundary conditions":

$$\left.\begin{aligned} \Gamma_1 u - B_+\Gamma_0 u &= \alpha v_-(0) \\ \Gamma_1 u - B_+^*\Gamma_0 u &= \alpha v_+(0) \end{aligned}\right\} \tag{2.4}$$

Here boundary values $v_\pm(0) \in E$ are well defined according to imbedding theorems for spaces $W_2^1(\mathbb{R}_\pm, E)$.

Remark 2.1. There is a certain "geometrical" aspect of conditions (2.4). Indeed, the left hand side of relations (2.4) are vectors from $\mathcal{H}$, whereas vectors on the right hand side belong to the potentially "smaller" space $E \subset \mathcal{H}$. Since the vector $v_\pm(0) \in E$ can be chosen arbitrarily, it means that for $(v_-, u, v_+) \in \mathcal{D}(\mathscr{A})$.

$$\overline{\mathcal{R}\left(\Gamma_1 u - B_+\Gamma_0 u\right)} = \overline{\mathcal{R}\left(\Gamma_1 u - B_+^*\Gamma_0 u\right)} = E$$

Remark 2.2. By termwise substraction we obtain from (2.4):

$$(B_+ - B_+^*)\Gamma_0 u = i\alpha^2\Gamma_0 u = \alpha(v_+(0) - v_-(0)).$$

Standard arguments based on the functional calculus for bounded selfadjoint operator α combined with facts that $\mathcal{R}(\alpha)$ is dense in E and $v_\pm(0) \in E$ yields:

$$i\alpha\Gamma_0 u = v_+(0) - v_-(0), \quad (v_-, u, v_+) \in \mathcal{D}(\mathscr{A}). \tag{2.5}$$

Remark 2.3. Let $\mathscr{G}$ be a set of vectors $u \in \mathcal{D}(A^*)$ such that $(v_-, u, v_+) \in \mathcal{D}(\mathscr{A})$ with some $v_\pm \in D_\pm$. It is clear that $\mathscr{G}$ includes $\mathcal{D}(A_{B_+}) \cup \mathcal{D}(A^*_{B_+})$. Indeed, if for example $v_-(0) = 0$ in (2.4), then we conclude that $u \in \mathcal{D}(A_{B_+})$, whereas $v_+(0)$ can be chosen appropriately in order to satisfy the second condition (2.4). The same argument applied to the case $v_+(0) = 0$ shows that $\mathcal{D}(A^*_{B_+}) \subset \mathscr{G}$.

Now we can formulate main theorem concerning selfadjoint dilation of A_{B_+}. For notational convenience let us introduce following four operators

$$\begin{aligned} \mathrm{Y}_\pm &: y_\pm \mapsto i y_\pm', \quad \mathcal{D}(\mathrm{Y}_\pm) := W_2^1(\mathbb{R}_\pm, E) \\ \mathrm{Y}^0_\pm &: y_\pm \mapsto i y_\pm', \quad \mathcal{D}(\mathrm{Y}^0_\pm) := \overset{\circ}{W}{}_2^1(\mathbb{R}_\pm, E), \end{aligned}$$

where W_2^1, $\overset{\circ}{W}{}_2^1$ are usual Sobolev spaces [AF]. Direct computation shows that $(\mathrm{Y}_\pm)^* = (\mathrm{Y}^0_\pm)$ and $\rho(\mathrm{Y}_+) = \rho(\mathrm{Y}^0_-) = \mathbb{C}_+$, $\rho(\mathrm{Y}_-) = \rho(\mathrm{Y}^0_+) = \mathbb{C}_-$.

Theorem 2.2. *Operator $\mathscr{A}$ is a minimal selfadjoint dilation of the dissipative operator A_{B_+}. The resolvent of $\mathscr{A}$ is given by following formulae:*

$$(\mathscr{A} - zI)^{-1}\begin{pmatrix} h_- \\ h_0 \\ h_+ \end{pmatrix} = \begin{pmatrix} \psi_-(\xi) \\ (A_{B_+} - z)^{-1}h_0 - \gamma(z)(B_+ - M(z))^{-1}\alpha\psi_-(0) \\ (\mathrm{Y}^0_+ - z)^{-1}h_+ + e^{-iz\xi}\{i\alpha\Gamma_0(A_{B_+} - z)^{-1}h_0 + S^*(\bar z)\psi_-(0)\} \end{pmatrix}, \quad z \in \mathbb{C}_-$$

$$(\mathscr{A} - zI)^{-1}\begin{pmatrix} h_- \\ h_0 \\ h_+ \end{pmatrix} = \begin{pmatrix} (\mathrm{Y}^0_- - z)^{-1}h_- + e^{-iz\xi}\{-i\alpha\Gamma_0(A^*_{B_+} - z)^{-1}h_0 + S(z)\psi_+(0)\} \\ (A^*_{B_+} - z)^{-1}h_0 - \gamma(z)(B^*_+ - M(z))^{-1}\alpha\psi_+(0) \\ \psi_+(\xi) \end{pmatrix}, \quad z \in \mathbb{C}_+$$

where $(h_-, h_0, h_+) \in \mathscr{H}$, $\psi_\pm := (\mathrm{Y}_\pm - z)^{-1}h_\pm$, $z \in \mathbb{C}_\pm$.

Proof. Let $\mathcal{U} := (v_-, u, v_+) \in \mathcal{D}(\mathscr{A})$. Then

$$\begin{aligned} &(\mathscr{A}\mathcal{U}, \mathcal{U}) - (\mathcal{U}, \mathscr{A}\mathcal{U}) \\ &= (iv_-', v_-) + (A^*u, u) + (iv_+', v_+) - (v_-, iv_-') - (u, A^*u) - (v_+, iv_+') \\ &= i\int_{-\infty}^0 (v_-'\bar v_- + v_-\bar v_-')dk + i\int_0^{+\infty}(v_+'\bar v_+ + v_+\bar v_+')dk + (A^*u, u) - (u, A^*u) \\ &= i\|v_-(0)\|^2 - i\|v_+(0)\|^2 + (\Gamma_1 u, \Gamma_0 u) - (\Gamma_0 u, \Gamma_1 u). \end{aligned}$$

By substitution $\Gamma_1 u$ from (2.4) and (2.5) we obtain for two last summands

$$\begin{aligned} &(\Gamma_1 u, \Gamma_0 u) - (\Gamma_0 u, \Gamma_1 u) \\ &= (\alpha v_-(0) + B_+\Gamma_0 u, \Gamma_0 u) - (\Gamma_0 u, \alpha v_+(0) + B_+^*\Gamma_0 u) \\ &= (v_-(0), \alpha\Gamma_0 u) - (\alpha\Gamma_0 u, v_+(0)) \\ &= (v_-(0), (-i)[v_+(0) - v_-(0)]) - ((-i)[v_+(0) - v_-(0)], v_+(0)) \\ &= i(v_-(0), v_+(0)) - i\|v_-(0)\|^2 + i\|v_+(0)\|^2 - i(v_-(0), v_+(0)) \\ &= i\|v_+(0)\|^2 - i\|v_-(0)\|^2. \end{aligned}$$

Finally,

$$(\mathscr{A}\mathcal{U},\mathcal{U})-(\mathcal{U},\mathscr{A}\mathcal{U})=0,\quad \mathcal{U}\in\mathcal{D}(\mathscr{A}),$$

therefore $\mathscr{A}$ is Hermitian.

Further, it is easy to see on ground that $\|\psi_\pm(0)\|_E\le C\|\psi_\pm\|_{W_2^1(\mathbb{R}_\pm,E)}$ that operators defined by the right hand sides of formulae for resolvent of $\mathscr{A}$ in the Theorem's statement are bounded for corresponding $z\in\mathbb{C}_\pm$. If we show that they yield vectors that belong to the domain of operator $\mathscr{A}$ and they indeed describe inverse operators for $\mathscr{A}-zI$, it would mean that Hermitian operator $\mathscr{A}$ is closed and its deficiency indices equal zero. Hence $\mathscr{A}$ is selfadjoint.

Let $z\in\mathbb{C}_-$ be a complex number and $\mathcal{V}:=(\tilde v_-,\tilde u,\tilde v_+)$ be a vector from the right hand side of the corresponding resolvent equality under consideration. The first and third component of $\mathcal{V}$ obviously belong to the Sobolev spaces $W_2^1(\mathbb{R}_\pm,E)$. We need to verify first that $\mathcal{V}$ satisfies boundary conditions (2.4).

$$\begin{aligned}
&(\Gamma_1-B_+\Gamma_0)\tilde u\\
&=(\Gamma_1-B_+\Gamma_0)\big[(A_{B_+}-z)^{-1}h_0-\gamma(z)(B_+-M(z))^{-1}\alpha\psi_-(0)\big]\\
&=-(\Gamma_1-B_+\Gamma_0)\gamma(z)(B_+-M(z))^{-1}\alpha\psi_-(0)\\
&=-(M(z)-B_+)(B_+-M(z))^{-1}\alpha\psi_-(0)=\alpha\psi_-(0)=\alpha\tilde v_-(0)
\end{aligned}$$

where we used equalities $\Gamma_1\gamma(z)=M(z)$ and $\Gamma_0\gamma(z)=I_{\mathcal{H}}$, see (1.8), (1.11).
Further,

$$\begin{aligned}
&(\Gamma_1-B_+^*\Gamma_0)\tilde u\\
&=(\Gamma_1-B_+^*\Gamma_0)\big[(A_{B_+}-z)^{-1}h_0-\gamma(z)(B_+-M(z))^{-1}\alpha\psi_-(0)\big]\\
&=(\Gamma_1-B_+\Gamma_0)\tilde u+i\alpha^2\Gamma_0\big[(A_{B_+}-z)^{-1}h_0-\gamma(z)(B_+-M(z))^{-1}\alpha\psi_-(0)\big]\\
&=\alpha\psi_-(0)+i\alpha^2\Gamma_0(A_{B_+}-z)^{-1}h_0-i\alpha^2(B_+-M(z))^{-1}\alpha\psi_-(0)\\
&=i\alpha^2\Gamma_0(A_{B_+}-z)^{-1}h_0+\alpha\big[I-i\alpha(B_+-M(z))^{-1}\alpha\big]\psi_-(0)\\
&=\alpha\big[i\alpha\Gamma_0(A_{B_+}-z)^{-1}h_0+S^*(\bar z)\psi_-(0)\big]=\alpha\tilde v_+(0)
\end{aligned}$$

Thus, both conditions (2.4) are satisfied. Now consider $(\mathscr{A}-zI)\mathcal{V}$ for $z\in\mathbb{C}_-$. Since $\tilde v_-=(\Upsilon_--z)^{-1}h_-$ and $\tilde v_+=(\Upsilon_+^0-z)^{-1}h_++e^{-iz\xi}\tilde v_+(0)$ it is easy to see that $(\Upsilon_\pm-z)\tilde v_\pm=h_\pm$. Inclusions $A_{B_+}\subset A^*$ and $\mathcal{R}(\gamma(z))\subset\ker(A^*-zI)$ help to compute the middle component $(A^*-z)\tilde u$:

$$(A^*-z)\big[(A_{B_+}-z)^{-1}h_0-\gamma(z)(B_+-M(z))^{-1}\alpha\psi_-(0)\big]=h_0$$

Thus, $(\mathscr{A}-zI)(\mathscr{A}-zI)^{-1}=I$.

In order to check correctness of the equality $(\mathscr{A}-zI)^{-1}(\mathscr{A}-zI)=I$, let $\mathcal{U}:=(v_-,u,v_+)\in\mathcal{D}(\mathscr{A})$ and $z\in\mathbb{C}_-$ be a complex number. Then

$$\begin{aligned}
(\mathscr{A}-zI)^{-1}(\mathscr{A}-zI)U&=(\mathscr{A}-zI)^{-1}\begin{pmatrix}(\Upsilon_--zI)v_-\\(A^*-zI)u\\(\Upsilon_+-zI)v_+\end{pmatrix}\\
&=\begin{pmatrix}v_-(\xi)\\(A_{B_+}-z)^{-1}(A^*-z)u-\gamma(z)(B_+-M(z))^{-1}\alpha v_-(0)\\v_+^0(\xi)+e^{-iz\xi}\big\{i\alpha\Gamma_0(A_{B_+}-z)^{-1}(A^*-z)u+S^*(\bar z)v_-(0)\big\}\end{pmatrix}
\end{aligned}$$

where $v_+^0(\xi):=(\Upsilon_+^0-zI)^{-1}(\Upsilon_+-zI)v_+$.

We need to show first that the middle component here coincides with u. Note that vector $\Psi(z):=(A_{B_+}-zI)^{-1}(A^*-zI)u-u$ belongs to $\ker(A^*-zI)$, therefore the expression $[\gamma(z)]^{-1}\Psi(z)$ represents an element $\Gamma_0\Psi(z)$ from $\mathcal{H}$. Now we can rewrite the middle component as follows:

$$\begin{aligned}
&u+\gamma(z)\big[\Gamma_0\Psi(z)-(B_+-M(z))^{-1}\alpha v_-(0)\big]\\
&=u+\gamma(z)(B_+-M(z))^{-1}\big[(B_+-M(z))\Gamma_0\Psi(z)-\alpha v_-(0)\big]
\end{aligned}$$

By the definition (1.10) of Weyl function $M(\cdot)$ and the first of conditions (2.4), the expression in square brackets can be rewritten as

$$(B_+\Gamma_0 - \Gamma_1)\Psi(z) - (\Gamma_1 - B_+\Gamma_0)u = B_+\Gamma_0(\Psi(z) + u) - \Gamma_1(\Psi(z) + u).$$

The only thing left is the observation that $\Psi(z) + u$ belongs to the domain $\mathcal{D}(A_{B_+})$, hence this expression equals zero.

Because $v_+(\xi) = v_+^0(\xi) + e^{-iz\xi}v_+(0)$, in order to check correctness of the expression for the third component in the computations above we only need to show that

$$i\alpha\Gamma_0(A_{B_+} - z)^{-1}(A^* - z)u + S^*(\bar{z})v_-(0) = v_+(0)$$

Recalling that $S^*(\bar{z}) = I - i\alpha(B_+ - M(z))^{-1}\alpha$, $v_-(0) = v_+(0) - i\alpha\Gamma_0 u$ (see (1.16), (2.5)) and utilizing notation $\Psi(z)$ once again, we obtain

$$\begin{aligned}
&i\alpha\Gamma_0(A_{B_+} - z)^{-1}(A^* - z)u + S^*(\bar{z})v_-(0) \\
&= i\alpha\Gamma_0(\Psi(z) + u) + v_+(0) - i\alpha\Gamma_0 u - i\alpha(B_+ - M(z))^{-1}\alpha v_-(0) \\
&= v_+(0) + i\alpha\Gamma_0\Psi(z) - i\alpha(B_+ - M(z))^{-1}\alpha v_-(0) \\
&= v_+(0) + i\alpha(B_+ - M(z))^{-1}\big[(B_+ - M(z))\Gamma_0\Psi(z) - \alpha v_-(0)\big].
\end{aligned}$$

It was shown at the previous step that the expression in square brackets is equal to zero.

The resolvent formula in the case $z \in \mathbb{C}_+$ is verified analogously.

Finally, dilation equalities (2.2) are obvious for operators $(\mathscr{A} - zI)^{-1}$. Minimality of dilation $\mathscr{A}$ follows from the relation

$$span\{(\mathscr{A} - zI)^{-1}H \,:\, z \in \mathbb{C}_\pm\} = span\left\{\begin{pmatrix} e^{-iz_+\xi}\alpha\Gamma_0(A^*_{B_+} - z_+)^{-1}H \\ (A_{B_+} - z_-)^{-1}H + (A^*_{B_+} - z_+)^{-1}H \\ e^{-iz_-\xi}\alpha\Gamma_0(A_{B_+} - z_-)^{-1}H \end{pmatrix} \,:\, z_\pm \in \mathbb{C}_\pm\right\},$$

properties of exponents in $L_2(\mathbb{R}_\pm)$, and density of sets

$$\{\alpha\Gamma_0(A^*_{B_+} - z)^{-1}H \,:\, z \in \mathbb{C}_+\}, \quad \{\alpha\Gamma_0(A_{B_+} - z)^{-1}H \,:\, z \in \mathbb{C}_-\}$$

in E. This density is a simple consequence of the fact that E is a boundary space and $\alpha\Gamma_0$ is a boundary operator for A_{B_+}, $A^*_{B_+}$ as defined in Section 1.3.

The proof is complete. □

The spectral mapping that maps dilation $\mathscr{A}$ into the multiplication operator $f \mapsto k \cdot f$ on some L_2-space gives the model representation of the dissipative operator A_{B_+}:

$$\left.\begin{aligned} P_H(k - z)^{-1}|_H &\cong (A_{B_+} - zI)^{-1}, \quad z \in \mathbb{C}_-,\, k \in \mathbb{R} \\ P_H(k - z)^{-1}|_H &\cong (A^*_{B_+} - zI)^{-1}, \quad z \in \mathbb{C}_+,\, k \in \mathbb{R} \end{aligned}\right\} \tag{2.6}$$

Following [P3], [P4], [Na] we arrive at the model Hilbert space $\mathbf{H} = L_2\begin{pmatrix} I & S^* \\ S & I\end{pmatrix}$ by the factorization against elements with zero norm and subsequent completion of the linear set $\{\binom{\widetilde{g}}{g} \,:\, \widetilde{g}, g \in L_2(\mathbb{R}, E)\}$ of two-components E-valued vector functions with respect to the norm

$$\left\|\begin{pmatrix}\widetilde{g}\\ g\end{pmatrix}\right\|^2_{\mathbf{H}} := \int_{\mathbb{R}}\left\langle\begin{pmatrix} I & S^* \\ S & I\end{pmatrix}\begin{pmatrix}\widetilde{g}\\ g\end{pmatrix}, \begin{pmatrix}\widetilde{g}\\ g\end{pmatrix}\right\rangle_{E\oplus E} dk \tag{2.7}$$

Note that in general the completion operation makes it impossible to treat individual components $\widetilde{g}$, g of a vector $\binom{\widetilde{g}}{g} \in \mathbf{H}$ as regular L_2-functions. However, two equivalent forms of the $\mathbf{H}$-norm

$$\left\|\begin{pmatrix}\widetilde{g}\\ g\end{pmatrix}\right\|^2_{\mathbf{H}} = \|S\widetilde{g} + g\|^2_{L_2(E)} + \|\Delta_* g\|^2_{L_2(E)} = \|\widetilde{g} + S^* g\|^2_{L_2(E)} + \|\Delta\widetilde{g}\|^2_{L_2(E)},$$

where $\Delta := \sqrt{I - S^*S}$ and $\Delta_* := \sqrt{I - SS^*}$ show that for each $\binom{\widetilde{g}}{g} \in \mathbf{H}$ expressions $S\widetilde{g} + g$, $\widetilde{g} + S^*g$, $\Delta\widetilde{g}$, and $\Delta_* g$ are in fact usual square summable vector-functions from $L_2(E)$.

Subspaces in $\mathbf{H}$

$$\mathfrak{D}_+ := \begin{pmatrix} H_2^+(E) \\ 0 \end{pmatrix}, \quad \mathfrak{D}_- := \begin{pmatrix} 0 \\ H_2^-(E) \end{pmatrix}, \quad \mathfrak{H} := \mathbf{H} \ominus [\mathfrak{D}_+ \oplus \mathfrak{D}_-]$$

where $H_2^\pm(E)$ are Hardy classes of E-valued vector functions analytic in $\mathbb{C}_\pm$, are mutually orthogonal.[2] The subspace $\mathfrak{H}$ can be described explicitly:

$$\mathfrak{H} = \left\{ \begin{pmatrix} \widetilde{g} \\ g \end{pmatrix} \in \mathbf{H} \ : \ \widetilde{g} + S^*g \in H_2^-(E),\, S\widetilde{g} + g \in H_2^+(E) \right\}$$

Orthogonal projection $P_\mathfrak{H}$ from $\mathbf{H}$ onto $\mathfrak{H}$ is defined by the following formula

$$P_\mathfrak{H} \begin{pmatrix} \widetilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \widetilde{g} - P_+(\widetilde{g} + S^*g) \\ g - P_-(S\widetilde{g} + g) \end{pmatrix}, \quad \widetilde{g}, g \in L_2(E)$$

where $P_\pm$ are the orthogonal projections from $L_2(E)$ onto Hardy classes $H_2^\pm(E)$.

Following Lemma is a version of the corresponding result from [Na].

Lemma 2.4. *Let $u \in H$. Linear mappings*

$$u \mapsto \alpha\Gamma_0(A_{B_+}^* - z)^{-1}u, \quad u \mapsto \alpha\Gamma_0(A_{B_+} - \zeta)^{-1}u$$

are bounded operators from H into classes $H_2^+(E)$, $H_2^-(E)$, respectively, with the norms less then $\sqrt{2\pi}$, i.e. for $u \in H$ the following estimates hold

$$\|\alpha\Gamma_0(A_{B_+}^* - z)^{-1}u\|_{H_2^+(E)} \le \sqrt{2\pi}\|u\|,$$
$$\|\alpha\Gamma_0(A_{B_+} - \zeta)^{-1}u\|_{H_2^-(E)} \le \sqrt{2\pi}\|u\|$$

Proof. For a given vector $u \in H$ and $\zeta \in \mathbb{C}_-$, $\zeta = k - i\varepsilon$, $k \in \mathbb{R}$, $\varepsilon > 0$ denote $g_\zeta := (A_{B+} - \zeta)^{-1}u$. Then since $B_+ = B_R + i\frac{\alpha^2}{2}$ and $g_\zeta \in \mathcal{D}(A_{B_+})$, so that $B_+\Gamma_0 g_\zeta = \Gamma_1 g_\zeta$, we obtain

$$\begin{aligned}
& i\|\alpha\Gamma_0(A_{B_+} - \zeta)^{-1}u\|^2 = i\|\alpha\Gamma_0 g_\zeta\|^2 = i(\alpha^2\Gamma_0 g_\zeta, \Gamma_0 g_\zeta) \\
&= \left(i\frac{\alpha^2}{2}\Gamma_0 g_\zeta, \Gamma_0 g_\zeta\right) - \left(\Gamma_0 g_\zeta, i\frac{\alpha^2}{2}\Gamma_0 g_\zeta\right) \\
&= \left(\left(B_R + i\frac{\alpha^2}{2}\right)\Gamma_0 g_\zeta, \Gamma_0 g_\zeta\right) - \left(\Gamma_0 g_\zeta, \left(B_R + i\frac{\alpha^2}{2}\right)\Gamma_0 g_\zeta\right) \\
&= (B_+\Gamma_0 g_\zeta, \Gamma_0 g_\zeta) - (\Gamma_0 g_\zeta, B_+\Gamma_0 g_\zeta) = (\Gamma_1 g_\zeta, \Gamma_0 g_\zeta) - (\Gamma_0 g_\zeta, \Gamma_1 g_\zeta) \\
&= (A^* g_\zeta, g_\zeta) - (g_\zeta, A^* g_\zeta) = \big(A_{B_+} g_\zeta, g_\zeta\big) - \big(g_\zeta, A_{B_+} g_\zeta\big) \\
&= \big(A_{B_+}(A_{B_+} - \zeta)^{-1}u, (A_{B_+} - \zeta)^{-1}u\big) - \big((A_{B_+} - \zeta)^{-1}u, A_{B_+}(A_{B_+} - \zeta)^{-1}u\big) \\
&= \big(u, (A_{B_+} - \zeta)^{-1}u\big) - \big((A_{B_+} - \zeta)^{-1}u, u\big) + (\zeta - \bar{\zeta})\|g_\zeta\|^2
\end{aligned}$$

Here we used inclusion $A_{B_+} \subset A^*$ and Green formula (1.3). The remaining part of proof reproduces corresponding reasoning of paper [Na]. Let E_t, $t \in \mathbb{R}$ be the spectral measure of the selfadjoint dilation $\mathscr{A}$. Then

$$\begin{aligned}
& \frac{1}{2}\,\|\alpha\Gamma_0(A_{B_+} - \zeta)^{-1}u\|^2 = \frac{1}{2i}\left[\big(u, (\mathscr{A} - \zeta)^{-1}u\big) - \big((\mathscr{A} - \zeta)^{-1}u, u\big) + (\zeta - \bar{\zeta})\|g_\zeta\|^2\right] \\
&= \frac{1}{2i}\left([(\mathscr{A} - \bar{\zeta})^{-1} - (\mathscr{A} - \zeta)^{-1}]u, u\right) - \varepsilon\|g_\zeta\|^2 = \varepsilon\left\|(\mathscr{A} - \zeta)^{-1}u\right\|^2 - \varepsilon\|(A_{B+} - \zeta)^{-1}u\|^2 \\
&= \varepsilon\left\|(\mathscr{A} - k + i\varepsilon)^{-1}u\right\|^2 - \varepsilon\|(A_{B+} - k + i\varepsilon)^{-1}u\|^2 \\
&= \varepsilon\int_\mathbb{R} \frac{1}{(t-k)^2 + \varepsilon^2}\, d(E_t u, u) - \varepsilon\|(A_{B+} - k + i\varepsilon)^{-1}u\|^2
\end{aligned}$$

[2] Analytic functions from vector-valued Hardy classes $H_2^\pm(E)$ are equated with their boundary values existing almost everywhere on the real axis. These boundary values form two complementary orthogonal subspaces in $L_2(\mathbb{R}, E) = H_2^+(E) \oplus H_2^-(E)$. (See [RR] for details.)

By Fubini theorem,

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} \|\alpha\Gamma_0(A_{B_+}-k+i\varepsilon)^{-1}u\|^2dk = \\
&= \int_{\mathbb{R}}\left\{\varepsilon\int_{\mathbb{R}}\frac{1}{(t-k)^2+\varepsilon^2}d(E_tu,u)\right\}dk - \varepsilon\int_{\mathbb{R}}\|(A_{B+}-k+i\varepsilon)^{-1}u\|^2dk \\
&= \int_{\mathbb{R}}\left\{\varepsilon\int_{\mathbb{R}}\frac{1}{(t-k)^2+\varepsilon^2}dk\right\}d(E_tu,u) - \varepsilon\int_{\mathbb{R}}\|(A_{B+}-k+i\varepsilon)^{-1}u\|^2dk \\
&= \pi\int_{\mathbb{R}}d(E_tu,u) - \varepsilon\int_{\mathbb{R}}\|(A_{B+}-k+i\varepsilon)^{-1}u\|^2dk = \pi\|u\|^2 - \varepsilon\int_{\mathbb{R}}\|(A_{B+}-k+i\varepsilon)^{-1}u\|^2dk
\end{aligned}
$$

Hence,

$$
\|\alpha\Gamma_0(A_{B_+}-\zeta)^{-1}u\|^2_{H_2^-(E)} = \sup_{\varepsilon>0}\int_{\mathbb{R}}\|\alpha\Gamma_0(A_{B_+}-k+i\varepsilon)^{-1}u\|^2dk \le 2\pi\|u\|^2
$$

Another statement of the Lemma is proven analogously. □

It follows from the properties of Hardy classes $H_2^{\pm}$ that for each $u \in H$ there exist L_2-boundary values of the analytic vector-functions $\alpha\Gamma_0(A^*_{B_+}-zI)^{-1}u$ and $\alpha\Gamma_0(A_{B_+}-\zeta I)^{-1}u$ almost everywhere on the real axis. For these limits we will use the notation:

$$
\begin{aligned}
\alpha\Gamma_0(A^*_{B_+}-k-i0)^{-1}u &:= \lim_{\varepsilon\downarrow 0}\alpha\Gamma_0(A^*_{B_+}-(k+i\varepsilon))^{-1}u, \\
\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}u &:= \lim_{\varepsilon\downarrow 0}\alpha\Gamma_0(A_{B_+}-(k-i\varepsilon))^{-1}u, \\
&\quad u \in H \text{ and almost all } k \in \mathbb{R}.
\end{aligned} \tag{2.8}
$$

Note that the point set on the real axis where these limits exist depends on the vector $u \in H$. Moreover, the left hand side in (2.8) does not define any operator functions on the real axis $\mathbb{R}$. These expressions can only be understood as formal symbols for the limits that appear on the right hand side.

In accordance with [Na], introduce two linear mappings $\mathscr{F}_\pm : \mathscr{H} \to L_2(\mathbb{R}, E)$

$$
\begin{aligned}
\mathscr{F}_+ : (v_-, u, v_+) &\longmapsto -\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}u + S^*(k)\widehat{v}_-(k) + \widehat{v}_+(k) \\
\mathscr{F}_- : (v_-, u, v_+) &\longmapsto -\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A^*_{B_+}-k-i0)^{-1}u + \widehat{v}_-(k) + S(k)\widehat{v}_+(k)
\end{aligned}
$$

where $(v_-, u, v_+) \in \mathscr{H}$, and $\widehat{v}_\pm$ are the Fourier transforms of functions $v_\pm \in D_\pm$. By virtue of Paley-Wiener theorem, $\widehat{v}_\pm \in H_2^\pm(E)$, see [RR]. The distinguished role of mappings $\mathscr{F}_\pm$ is revealed in the next Theorem.

Theorem 2.3. *There exists an unique mapping Φ from the dilation space $\mathscr{H}$ onto the model space* $\mathbf{H}$ *with the properties:*

1. *Φ is an isometry.*
2. *$\widetilde{g} + S^*g = \mathscr{F}_+\mathfrak{h}$, $S\widetilde{g} + g = \mathscr{F}_-\mathfrak{h}$, where $\binom{\widetilde{g}}{g} = \Phi\mathfrak{h}$, $\mathfrak{h} \in \mathscr{H}$*
3. *For $z \notin \mathbb{R}$*
$$\Phi \circ (\mathscr{A} - zI)^{-1} = (k-z)^{-1} \circ \Phi,$$
where $\mathscr{A}$ is the minimal selfadjoint dilation of the operator A_{B_+}
4. *$\Phi H = \mathfrak{H}$, $\quad \Phi D_\pm = \mathfrak{D}_\pm$*

Property (3) means that Φ maps $\mathscr{A}$ into the multiplication operator on the space $\mathbf{H}$; therefore, the dissipative operator A_{B_+} is mapped into its model representation as required in (2.6).

The proof of Theorem is carried out at the end of this Section.

Computation of functions $\mathscr{F}_\pm h$, $h \in H$ can be further simplified. More precisely, there exists a formula which allows one to avoid the calculation of the resolvent of the dissipative operator A_{B_+}. To that end we recall the definition (1.6) of operator A_∞ given earlier. There exists a certain "resolvent identity" for A_∞ and A_{B_+}, which we will obtain next.

Let $\zeta \in \mathbb{C}_-$. Then the equation $(A_{B_+} - \zeta)\phi = h$ has an unique solution for each $h \in H$. We can represent this solution in the form of sum $\phi = f + g$, where $g := (A_\infty - \zeta)^{-1}h$ and $f \in \ker(A^* - \zeta)$. Obviously, $f = [(A_{B_+} - \zeta)^{-1} - (A_\infty - \zeta)^{-1}]h$. Since $\phi \in \mathcal{D}(A_{B_+})$ and $\Gamma_0 g = 0$, we have

$$0 = (\Gamma_1 - B_+\Gamma_0)\phi = \Gamma_1(f + g) - B_+\Gamma_0 f = M(\zeta)\Gamma_0 f + \Gamma_1 g - B_+\Gamma_0 f$$

Hence, $\Gamma_1 g = (B_+ - M(\zeta))\Gamma_0 f$ and since $0 \in \rho(B_+ - M(\zeta))$, we obtain

$$\Gamma_0 f = (B_+ - M(\zeta))^{-1}\Gamma_1 g.$$

The left hand side can be rewritten in the form

$$\Gamma_0 f = \Gamma_0(f + g) = \Gamma_0\phi = \Gamma_0(A_{B_+} - \zeta)^{-1}h$$

Now, by the definition of g,

$$\Gamma_0(A_{B_+} - \zeta I)^{-1}h = (B_+ - M(\zeta))^{-1}\Gamma_1(A_\infty - \zeta I)^{-1}h$$

Since vector $h \in H$ is arbitrary, it follows that

$$\Gamma_0(A_{B_+} - \zeta I)^{-1} = (B_+ - M(\zeta))^{-1}\Gamma_1(A_\infty - \zeta I)^{-1}, \quad \zeta \in \mathbb{C}_- \tag{2.9}$$

Similar computations yield the formula for the conjugate operator $A^*_{B_+}$:

$$\Gamma_0(A^*_{B_+} - zI)^{-1} = (B^*_+ - M(z))^{-1}\Gamma_1(A_\infty - zI)^{-1}, \quad z \in \mathbb{C}_+ \tag{2.10}$$

Substituting (2.9) and (2.10) into the definitions of functions $\mathscr{F}_\pm h$, $h \in H$ we arrive at the result ($h \in H$, $k \in \mathbb{R}$):

$$\begin{aligned} \mathscr{F}_+ h &= -\frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\downarrow 0} \alpha(B_+ - M(k - i\varepsilon))^{-1}\Gamma_1(A_\infty - (k - i\varepsilon))^{-1}h \\ \mathscr{F}_- h &= -\frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\downarrow 0} \alpha(B^*_+ - M(k + i\varepsilon))^{-1}\Gamma_1(A_\infty - (k + i\varepsilon))^{-1}h \end{aligned} \tag{2.11}$$

For each $h \in H$ these limits exist for almost any $k \in \mathbb{R}$ and represent two square integrable vector-functions.

The advantage of formulae (2.11) becomes apparent when, for example, the space $\mathcal{H}$ is finite dimensional. In this case all computations are reduced to the calculation of the resolvent of the selfadjoint operator A_∞ and the matrix inversion problem for the matrix-valued function $(B_+ - M(z))$, $z \in \mathbb{C}_-$[3].

Taking into account that $\Gamma_1(A_\infty - zI)^{-1} = \gamma^*(\bar{z})$, we obtain from (2.9) and (2.10) following relations. They will be used in the proof of Theorem 2.3.

$$\left.\begin{aligned} \Gamma_0(A_{B_+} - \zeta I)^{-1} &= (B_+ - M(\zeta))^{-1}\gamma^*(\bar{\zeta}), \quad \zeta \in \mathbb{C}_- \\ \Gamma_0(A^*_{B_+} - zI)^{-1} &= (B^*_+ - M(z))^{-1}\gamma^*(\bar{z}), \quad z \in \mathbb{C}_+ \end{aligned}\right\} \tag{2.12}$$

At last, for the sake of completeness, we formulate the theorem that describes the resolvent of operator A_{B_+} in the upper half plane. Its proof is based solely on the Hilbert resolvent identities and can be found in [AP]. It is curious to notice that in contrast with similar results of the next Section, the vectors on the right hand side of these formulae already belong to space H, making application of projection P_H redundant. In the notation below we customarily identify initial and model spaces and operators whose unitary equivalence is established by the isometry Φ in hope that it will not lead to confusion.

Theorem 2.4. *For* $\binom{\widetilde{g}}{g} \in H$

$$\begin{aligned} (A_{B_+} - zI)^{-1}\begin{pmatrix}\widetilde{g}\\ g\end{pmatrix} &= (k - z)^{-1}\begin{pmatrix}\widetilde{g} - [S(z)]^{-1}(S\widetilde{g} + g)(z)\\ g\end{pmatrix}, \quad z \in \mathbb{C}_+ \\ (A^*_{B_+} - \zeta I)^{-1}\begin{pmatrix}\widetilde{g}\\ g\end{pmatrix} &= (k - \zeta)^{-1}\begin{pmatrix}\widetilde{g}\\ g - [S^*(\bar{\zeta})]^{-1}(\widetilde{g} + S^*g)(\zeta)\end{pmatrix}, \quad \zeta \in \mathbb{C}_- \end{aligned}$$

Here $(S\widetilde{g} + g)(z)$ *and* $(\widetilde{g} + S^*g)(\zeta)$ *are values of the analytical continuation of the functions* $S\widetilde{g} + g \in H^+_2(E)$ *and* $\widetilde{g} + S^*g \in H^-_2(E)$ *into complex points* $z \in \mathbb{C}_+$, $\zeta \in \mathbb{C}_-$, *respectively.*

[3] Recall that $(B^*_+ - M(\bar{z}))^{-1} = \left[(B_+ - M(z))^{-1}\right]^*$, $z \in \mathbb{C}_-$

The remaining part of this Section outlines principal steps of the proof of Theorem 2.3 in the form of a few Propositions.

Introduce a linear set in $\mathscr{H}$ by the formula

$$\mathscr{W} := \left\{ \sum_{j=1}^{n} a_j(\mathscr{A} - \zeta_j I)^{-1} v_- + \sum_{s=1}^{m} b_s(\mathscr{A} - z_s I)^{-1} v_+, \quad v_\pm \in D_\pm \right\}, \tag{2.13}$$

where $\zeta_j \in \mathbb{C}_-$, $z_s \in \mathbb{C}_+$, $a_j, b_s \in \mathbb{C}$, $j = 1, 2, \ldots n < \infty$, $s = 1, 2, \ldots m < \infty$.

Proposition 2.5. *Set $\mathscr{W}$ is dense in the dilation space $\mathscr{H}$.*

This Proposition is equivalent to the completeness of incoming and outgoing waves of Lax-Phillips theory [LP], or completeness of incoming and outgoing eigenfunctions of continuous spectra of the dilation [P4].

Proof. Since $\mathrm{s} - \lim_{t\to\infty} \pm it(\mathscr{A} \pm itI)^{-1} = I_{\mathscr{H}}$, the inclusion $D_+ \oplus D_- \subset \overline{\mathscr{W}}$ is obvious. Hence, $\mathscr{W}^\perp \subset H$. Further, $(\mathscr{A} - zI)^{-1}\mathscr{W} \subset \mathscr{W}$ and $\mathscr{A}$ is selfadjoint. It follows that $\overline{\mathscr{W}}$ and $\mathscr{W}^\perp$ are invariant subspaces of $\mathscr{A}$. Noticing that A_{B_+} is simple and $\mathscr{A}|_{\mathscr{W}^\perp} = A_{B_+}|_{\mathscr{W}^\perp}$ since $\mathscr{A}$ is the dilation of A_{B_+}, we conclude that $\mathscr{W}^\perp = \{0\}$. ☐

Introduce a linear set $\mathcal{W}$ as projection of $\mathscr{W}$ onto H. According to Theorem 2.2,

$$\mathcal{W} = \left\{ \sum_{j=1}^{n} a_j \gamma(\zeta_j)(B_+ - M(\zeta_j))^{-1}\alpha\psi_j + \sum_{s=1}^{m} b_s \gamma(z_s)(B_+^* - M(z_s))^{-1}\alpha\phi_s \right\},$$

where $\zeta_j \in \mathbb{C}_-$, $z_s \in \mathbb{C}_+$, $\psi_j, \phi_s \in E$, $a_j, b_s \in \mathbb{C}$, $j = 1, 2, \ldots n < \infty$, $s = 1, 2, \ldots m < \infty$.

Corollary 2.6. *The set $\mathcal{W}$ is dense in H.*

Following example of [Na], we define the spectral mapping $\Phi : \mathscr{H} \to \mathbf{H}$ initially on the dense set $(D_-, \mathcal{W}, D_+)$ in $\mathscr{H}$. Let $\mathscr{V} := (v_-, v_0, v_+) \in (D_-, \mathcal{W}, D_+)$, where

$$v_0 := \sum_j a_j \gamma(\zeta_j)(B_+ - M(\zeta_j))^{-1}\alpha\psi_j + \sum_s b_s \gamma(z_s)(B_+^* - M(z_s))^{-1}\alpha\phi_s \tag{2.14}$$

in the notation introduced earlier. Let us define the mapping Φ as follows

$$\Phi : \begin{pmatrix} v_- \\ v_0 \\ v_+ \end{pmatrix} \mapsto \begin{pmatrix} \widehat{v}_+ + \frac{i}{\sqrt{2\pi}}\left[\sum_j \frac{a_j}{k-\zeta_j} S^*(\bar{\zeta}_j)\psi_j + \sum_s \frac{b_s}{k-z_s}\phi_s\right] \\ \widehat{v}_- - \frac{i}{\sqrt{2\pi}}\left[\sum_j \frac{a_j}{k-\zeta_j}\psi_j + \sum_s \frac{b_s}{k-z_s} S(z_s)\phi_s\right] \end{pmatrix} \tag{2.15}$$

Here $\widehat{v}_\pm$ are Fourier transforms of functions $v_\pm \in L_2(\mathbb{R}_\pm, E)$. Our task is to prove that so defined map Φ possesses all the properties stated in Theorem 2.3.

First of all, observe that the mapping satisfying conditions (1) and (2) is unique. It follows directly from the definition of the norm in $\mathbf{H}$. (See (2.7).) Secondly, equalities $\Phi D_\pm = \mathfrak{D}_\pm$ for mapping (2.15) hold true by virtue of Paley-Wiener theorem. Moreover, since Fourier transform $v_\pm \mapsto \widehat{v}_\pm$ is isometric, restrictions $\Phi|_{D_\pm}$ are isometries onto $\mathfrak{D}_\pm$.

Proposition 2.7. *In notation of Corollary 2.6*

$$\Phi(0, \mathcal{W}, 0) \subset \mathfrak{H}$$

Proof. We need to show that vectors on the right hand side of (2.15) where $v_\pm = 0$ are orthogonal to $\mathfrak{D}_\pm$. Due to linearity and linear independence, it is sufficient to show that for each $j = 1, 2, \ldots n$ and $s = 1, 2, \ldots m$ vectors

$$\frac{1}{k - \zeta_j}\begin{pmatrix} S^*(\bar{\zeta}_j)\psi_j \\ -\psi_j \end{pmatrix}, \quad \frac{1}{k - z_s}\begin{pmatrix} \phi_s \\ -S(z_s)\phi_j \end{pmatrix}$$

are orthogonal to $(H_2^+(E), H_2^-(E))$ in $\mathbf{H}$. Let $h_\pm \in H_2^\pm(E)$ be two vector functions, so that $(h_+, h_-) \in (H_2^+(E), H_2^-(E))$. Then omitting index j, we have for $\zeta \in \mathbb{C}_-$

$$\begin{aligned}
&\left(\frac{1}{k-\zeta}\begin{pmatrix} S^*(\bar\zeta)\psi \\ -\psi \end{pmatrix}, \begin{pmatrix} h_+ \\ h_- \end{pmatrix}\right)_{\mathbf{H}} \\
&= \big((k-\zeta)^{-1} S^*(\bar\zeta)\psi, (h_+ + S^* h_-)\big)_{L^2(E)} - \big((k-\zeta)^{-1}\psi, (Sh_+ + h_-)\big)_{L^2(E)} \\
&= \big((k-\zeta)^{-1} S^*(\bar\zeta)\psi, h_+\big)_{L^2(E)} - \big((k-\zeta)^{-1}\psi, Sh_+\big)_{L^2(E)} \\
&= -\left(\frac{S^*(k) - S^*(\bar\zeta)}{k-\zeta}\psi, h_+\right)_{L^2(E)} = 0.
\end{aligned}$$

Similarly, for $z \in \mathbb{C}_+$

$$\left(\frac{1}{k-z}\begin{pmatrix} \phi \\ -S(z)\phi \end{pmatrix}, \begin{pmatrix} h_+ \\ h_- \end{pmatrix}\right)_{\mathbf{H}} = \left(\frac{S(k)-S(z)}{k-z}\phi, h_-\right)_{L^2(E)} = 0.$$

Here we used inclusions $(k-\zeta)^{-1} \in H_2^+$, $(k-z)^{-1} \in H_2^-$ and analytical continuation of bounded operator functions S and S^* to the upper and lower half planes correspondingly. The proof is complete. □

Later it will be shown that Φ maps the space H on the whole $\mathfrak{H}$ isometrically, therefore $\Phi(0, \mathcal{W}, 0)$ is dense in $\mathfrak{H}$.

Proposition 2.8. *Almost everywhere on the real axis*

$$\begin{pmatrix} I & S^* \\ S & I \end{pmatrix} \Phi\mathscr{V} = \begin{pmatrix} \mathscr{F}_+\mathscr{V} \\ \mathscr{F}_-\mathscr{V} \end{pmatrix}$$

where $\mathscr{V} = (v_-, v_0, v_+) \in (D_-, \mathcal{W}, D_+)$.

Proof. The statement is obviously true if $\mathscr{V}$ belongs to the set $D_- \oplus 0 \oplus D_+$. We only need to consider the case $\mathscr{V} = (0, v_0, 0)$ with $v_0 \in \mathcal{W}$, see (2.14). Arguments of linearity and independence of terms in (2.14) show that it is sufficient to verify the statement only when each sum consists of just one element. Using definitions of $\mathscr{F}_\pm$ we reduce the claim to the following equalities where indices are omitted for convenience:

$$\begin{aligned}
&i\alpha\Gamma_0(A_{B_+} - k + i0)^{-1}\big[a\gamma(\zeta)(B_+ - M(\zeta))^{-1}\alpha\psi + b\gamma(z)(B_+^* - M(z))^{-1}\alpha\phi\big] \\
&\qquad = \frac{a}{k-\zeta}[S^*(\bar\zeta) - S^*(k)]\psi + \frac{b}{k-z}[I - S^*(k)S(z)]\phi \\
&i\alpha\Gamma_0(A^*_{B_+} - k - i0)^{-1}\big[a\gamma(\zeta)(B_+ - M(\zeta))^{-1}\alpha\psi + b\gamma(z)(B_+^* - M(z))^{-1}\alpha\phi\big] \\
&\qquad = -\frac{a}{k-\zeta}[I - S(k)S^*(\bar\zeta)]\psi + \frac{b}{k-z}[S(k) - S(z)]\phi
\end{aligned}$$

Regrouping terms we come to four relations to be proven for almost all $k \in \mathbb{R}$:

$$\begin{aligned}
-\frac{S^*(k) - S^*(\bar\zeta)}{k-\zeta}\,\psi &= i\alpha\Gamma_0(A_{B_+} - k + i0)^{-1}\gamma(\zeta)(B_+ - M(\zeta))^{-1}\alpha\psi \\
\frac{I - S^*(k)S(z)}{k-z}\,\phi &= i\alpha\Gamma_0(A_{B_+} - k + i0)^{-1}\gamma(z)(B_+^* - M(z))^{-1}\alpha\phi \\
-\frac{I - S(k)S^*(\bar\zeta)}{k-\zeta}\,\psi &= i\alpha\Gamma_0(A^*_{B_+} - k - i0)^{-1}\gamma(\zeta)(B_+ - M(\zeta))^{-1}\alpha\psi \\
\frac{S(k) - S(z)}{k-z}\,\phi &= i\alpha\Gamma_0(A^*_{B_+} - k - i0)^{-1}\gamma(z)(B_+^* - M(z))^{-1}\alpha\phi
\end{aligned} \tag{2.16}$$

Let $\lambda = k - i\varepsilon$, $k \in \mathbb{R}$, $\varepsilon > 0$. Then, since $S^*(\bar\lambda) = I - i\alpha(B_+ - M(\lambda))^{-1}\alpha$ and $M(\lambda) - M(\zeta) = (\lambda-\zeta)\gamma^*(\bar\lambda)\gamma(\zeta)$ (see (1.16) and Theorem 1.2):

$$\begin{aligned}
&S^*(\bar\lambda) - S^*(\bar\zeta) = -i\alpha(B_+ - M(\lambda))^{-1}\alpha + i\alpha(B_+ - M(\zeta))^{-1}\alpha \\
&= i\alpha(B_+ - M(\lambda))^{-1}\big[-(B_+ - M(\zeta)) + (B_+ - M(\lambda))\big](B_+ - M(\zeta))^{-1}\alpha \\
&= -i\alpha(B_+ - M(\lambda))^{-1}\big[M(\lambda) - M(\zeta)\big](B_+ - M(\zeta))^{-1}\alpha \\
&= -i(\lambda-\zeta)\alpha(B_+ - M(\lambda))^{-1}\gamma^*(\bar\lambda)\gamma(\zeta)(B_+ - M(\zeta))^{-1}\alpha
\end{aligned}$$

Now the first relation of (2.12) yields:

$$-\frac{S^*(\bar\lambda)-S^*(\bar\zeta)}{\lambda-\zeta}\psi = i\alpha\Gamma_0(A_{B_+}-\lambda I)^{-1}\gamma(\zeta)(B_+-M(\zeta))^{-1}\alpha\psi$$

In accordance with the limiting procedure (2.8), we obtain the first formula in (2.16) as $\varepsilon\downarrow 0$. Similarly,

$$\begin{aligned} I-S^*(\bar\lambda)S(z) &= i\alpha(B_+-M(\lambda))^{-1}\alpha - i\alpha(B_+^*-M(z))^{-1}\alpha \\ &\qquad + i^2\alpha(B_+-M(\lambda))^{-1}\alpha^2(B_+^*-M(z))^{-1}\alpha \\ &= i\alpha(B_+-M(\lambda))^{-1}\big[(B_+^*-M(z))-(B_+-M(\lambda))+i\alpha^2\big](B_+^*-M(z))^{-1}\alpha \\ &= i\alpha(B_+-M(\lambda))^{-1}\big[M(\lambda)-M(z)\big](B_+^*-M(z))^{-1}\alpha \end{aligned}$$

The last expression was calculated at the previous step. The same line of reasoning applied to this case proves correctness of the second formula in (2.16) for almost all $k\in\mathbb{R}$.

Two last relations in (2.16) are verified analogously. Finally, the statement of the Proposition is valid on the whole space $\mathscr{H}$ due to uniqueness of mapping satisfying conditions (1), (2) of Theorem 2.3. The proof is complete. □

Proposition 2.9. *Operator Φ defined in (2.15) is an isometry from the dilation space $\mathscr{H}$ to the model space* $\mathbf{H}$.

Due to this Proposition the mapping (2.15) is uniquely extended to the isometry from the whole space $\mathscr{H}$ into $\mathbf{H}$. In what follows we will use the same symbol Φ for this extension.

Proof. It is sufficient to prove that restriction of Φ to the space H is an isometry. To that end compute norm of the vector $\Phi(0,v_0,0)$ in $\mathbf{H}$. Denote $\mathscr{V}=(0,v_0,0)$, where v_0 is defined in (2.14). Then, slightly abusing the notation, we have

$$\begin{aligned} \|\Phi\mathscr{V}\|^2_{\mathbf{H}} &= \left(\begin{pmatrix} I & S^*\\ S & I\end{pmatrix}\Phi\mathscr{V},\Phi\mathscr{V}\right)_{L_2\oplus L_2} = \left(\begin{pmatrix}\mathscr{F}_+\mathscr{V}\\ \mathscr{F}_-\mathscr{V}\end{pmatrix},\Phi\mathscr{V}\right)_{L_2\oplus L_2} \\ &= \left(\begin{pmatrix}\mathscr{F}_+v_0\\ \mathscr{F}_-v_0\end{pmatrix},\begin{pmatrix}\frac{i}{\sqrt{2\pi}}\big[\sum_j\frac{a_j}{k-\zeta_j}S^*(\bar\zeta_j)\psi_j+\sum_s\frac{b_s}{k-z_s}\phi_s\big]\\ -\frac{i}{\sqrt{2\pi}}\big[\sum_j\frac{a_j}{k-\zeta_j}\psi_j+\sum_s\frac{b_s}{k-z_s}S(z_s)\phi_s\big]\end{pmatrix}\right)_{L_2\oplus L_2} \end{aligned}$$

Since $\mathscr{F}_+v_0\in H_2^-(E)$, $\mathscr{F}_-v_0\in H_2^+(E)$, $(k-\zeta_j)^{-1}\in H_2^+$, and $(k-z_s)^{-1}\in H_2^-$, we obtain by the residue method that

$$\begin{aligned} \|\Phi\mathscr{V}\|^2_{\mathbf{H}} &= \frac{i}{\sqrt{2\pi}}\Big[\sum_s\bar b_s\left(\mathscr{F}_+v_0,(k-z_s)^{-1}\phi_s\right)_{L_2(E)}-\sum_j\bar a_j\left(\mathscr{F}_-v_0,(k-\zeta_j)^{-1}\psi_j\right)_{L_2(E)}\Big] \\ &= \frac{i}{\sqrt{2\pi}}\Big[2\pi i\sum_s\bar b_s((\mathscr{F}_+v_0)(\bar z_s),\phi_s)_E+2\pi i\sum_j\bar a_j((\mathscr{F}_-v_0)(\bar\zeta_j),\psi_j)_E\Big] \\ &= \sum_s\bar b_s(\alpha\Gamma_0(A_{B_+}-\bar z_s)^{-1}v_0,\phi_s)+\sum_j\bar a_j(\alpha\Gamma_0(A^*_{B_+}-\bar\zeta_j)^{-1}v_0,\psi_j) \end{aligned}$$

It follows from (2.12) that

$$\begin{aligned} \|\Phi\mathscr{V}\|^2_{\mathbf{H}} &= \sum_s\bar b_s(\alpha(B_+-M(\bar z_s))^{-1}\gamma^*(z_s)v_0,\phi_s)+\sum_j\bar a_j(\alpha(B_+^*-M(\bar\zeta_j))^{-1}\gamma^*(\zeta_j)v_0,\psi_j) \\ &= \Big(v_0,\sum_j a_j\gamma(\zeta_j)(B_+-M(\zeta_j))^{-1}\alpha\psi_j+\sum_s b_s\gamma(z_s)(B_+^*-M(z_s))^{-1}\alpha\phi_s\Big) \\ &= \|v_0\|^2. \end{aligned}$$

Thus, Φ is an isometry from $\mathscr{H}$ to $\mathbf{H}$. □

Proposition 2.10.

$$\Phi\circ(\mathscr{A}-zI)^{-1}=(k-z)^{-1}\circ\Phi,\quad z\notin\mathbb{R}$$

Proof. The statement is a consequence of Proposition 2.9, property (2) of the Theorem 2.3, which is proven in Proposition 2.8, and equalities

$$\mathscr{F}_\pm \circ (\mathscr{A} - zI)^{-1} = (k-z)^{-1} \circ \mathscr{F}_\pm, \quad z \notin \mathbb{R}$$

to be established. For $(h_-, h_0, h_+) \in \mathscr{H}$ and $z \in \mathbb{C}_+$ denote as (h'_-, h'_0, h'_+) the vector $(\mathscr{A} - zI)^{-1}(h_-, h_0, h_+)$. Since

$$h_\pm = \Big(i\frac{d}{d\xi} - z\Big) h'_\pm,$$

by exercising integration by parts, we obtain for Fourier transforms $\widehat{h}'_\pm$, $\widehat{h}_\pm$:

$$(k-z)\widehat{h}'_\pm = \widehat{h}_\pm \pm \frac{i}{\sqrt{2\pi}} h'_\pm(0).$$

Then, according to the definition of $\mathscr{F}_-$ and Theorem 2.2,

$$\begin{aligned}
&\mathscr{F}_-(h'_-, h'_0, h'_+)\\
&= -\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A^*_{B_+} - k - i0)^{-1}h'_0 + \widehat{h}'_-(k) + S(k)\widehat{h}'_+(k)\\
&= -\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A^*_{B_+} - k - i0)^{-1}[(A^*_{B_+} - z)^{-1}h_0 - \gamma(z)(B^*_+ - M(z))^{-1}\alpha h'_+(0)]\\
&\qquad\qquad + \frac{1}{k-z}\big[(\widehat{h}_- + S\widehat{h}_+) + \frac{i}{\sqrt{2\pi}}(Sh'_+(0) - h'_-(0))\big]\\
&= \frac{1}{k-z}\mathscr{F}_-(h_-, h_0, h_+) + \frac{1}{k-z}\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A^*_{B_+} - z)^{-1}h_0\\
&\qquad + \frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A^*_{B_+} - k - i0)^{-1}\gamma(z)(B^*_+ - M(z))^{-1}\alpha h'_+(0)\\
&\qquad + \frac{1}{k-z}\frac{i}{\sqrt{2\pi}}\big[h'_+(0) - h'_-(0) + i\alpha(B^*_+ - M(k+i0))^{-1}\alpha h'_+(0)\big]
\end{aligned}$$

We need to show that the sum of last three terms is equal to zero. To that end we consider the sum of the first and the third summands at the non-real point $\lambda = k + i\varepsilon$, $k \in \mathbb{R}$, $\varepsilon > 0$, $\lambda \neq z$. Substitute $h'_+(0) - h'_-(0) = i\alpha\Gamma_0 h'_0$ and $h'_0 = (A^*_{B_+} - zI)^{-1}h_0 - \gamma(z)(B^*_+ - M(z))^{-1}\alpha h'_+(0)$ and conduct necessary computations.

$$\begin{aligned}
&\frac{1}{\lambda - z}\frac{1}{\sqrt{2\pi}}\Big[\alpha\Gamma_0(A^*_{B_+} - z)^{-1}h_0 - \alpha\Gamma_0 h'_0 - \alpha(B^*_+ - M(\lambda))^{-1}\alpha h'_+(0)\Big]\\
&= \frac{1}{\lambda - z}\frac{1}{\sqrt{2\pi}}\big[\alpha\Gamma_0\gamma(z)(B^*_+ - M(z))^{-1}\alpha h'_+(0) - \alpha(B^*_+ - M(\lambda))^{-1}\alpha h'_+(0)\big]\\
&= -\frac{1}{\lambda - z}\frac{1}{\sqrt{2\pi}}\alpha(B^*_+ - M(\lambda))^{-1}\left[M(\lambda) - M(z)\right](B^*_+ - M(z))^{-1}\alpha h'_+(0)\\
&= -\frac{1}{\sqrt{2\pi}}\alpha(B^*_+ - M(\lambda))^{-1}\gamma^*(\bar\lambda)\gamma(z)(B^*_+ - M(z))^{-1}\alpha h'_+(0)\\
&= -\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A^*_{B_+} - \lambda I)^{-1}\gamma(z)(B^*_+ - M(z))^{-1}\alpha h'_+(0),
\end{aligned}$$

where at the last step we employed relation (2.12). According to Lemma 2.4, this vector function is analytic in the upper half plane $\lambda \in \mathbb{C}_+$. More precisely, it belongs to the Hardy class $H^+_2(E)$. The only thing left is to observe that its boundary values as $\varepsilon \downarrow 0$ annihilate the second term in the expression for $\mathscr{F}_-(h'_-, h'_0, h'_+)$ above.

Now we turn to the lengthier computation of $\mathscr{F}_+(\mathscr{A}-zI)^{-1}(h_-,h_0,h_+)$.

$$\begin{aligned}
&\mathscr{F}_+(\mathscr{A}-zI)^{-1}(h_-,h_0,h_+)=\mathscr{F}_+(h'_-,h'_0,h'_+)\\
&=-\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}h'_0+S^*(k)\widehat{h}'_-(k)+\widehat{h}'_+(k)\\
&=-\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}[(A^*_{B_+}-z)^{-1}h_0-\gamma(z)(B^*_+-M(z))^{-1}\alpha h'_+(0)]\\
&\qquad\qquad+\frac{1}{k-z}\big[(S^*\widehat{h}_-+\widehat{h}_+)+\frac{i}{\sqrt{2\pi}}(h'_+(0)-S^*h'_-(0))\big]
\end{aligned}$$

Let $\lambda=k-i\varepsilon$, $k\in\mathbb{R}$, $\varepsilon>0$ be a number in the lower half plane. Let us compute vectors $h'_+(0)-S^*(\bar\lambda)h'_-(0)$ and $\alpha\Gamma_0(A_{B_+}-\lambda)^{-1}(A^*_{B_+}-z)^{-1}h_0$. Using Theorem 2.2, we have

$$\begin{aligned}
&h'_+(0)-S^*(\bar\lambda)h'_-(0)\\
&=h'_+(0)-S^*(\bar\lambda)\big[-i\alpha\Gamma_0(A^*_{B_+}-z)^{-1}h_0+S(z)h'_+(0)\big]\\
&=\big(I-S^*(\bar\lambda)S(z)\big)h'_+(0)+iS^*(\bar\lambda)\alpha\Gamma_0(A^*_{B_+}-z)^{-1}h_0\\
&=i(\lambda-z)\alpha\Gamma_0(A_{B_+}-\lambda)^{-1}\gamma(z)(B^*_+-M(z))^{-1}\alpha h'_+(0)+iS^*(\bar\lambda)\alpha\Gamma_0(A^*_{B_+}-z)^{-1}h_0
\end{aligned}$$

where at the last step we make use of computations for $I-S^*(\bar\lambda)S(z)$ conducted in the proof of Proposition 2.8. Note that almost everywhere on the real axis there exist boundary values of both sides of this formula as $\varepsilon\downarrow 0$.
With the help of Theorem 1.2 and relations (2.12) we obtain

$$\begin{aligned}
&\alpha\Gamma_0(A_{B_+}-\lambda)^{-1}(A^*_{B_+}-z)^{-1}h_0\\
&=\alpha(B_+-M(\lambda))^{-1}\Gamma_1(A_\infty-\lambda)^{-1}\big[(A_\infty-z)^{-1}+\gamma(z)(B^*_+-M(z))^{-1}\gamma^*(\bar z)\big]h_0\\
&=(\lambda-z)^{-1}\alpha(B_+-M(\lambda))^{-1}\Gamma_1\big[(A_\infty-\lambda)^{-1}-(A_\infty-z)^{-1}\big]h_0\\
&\qquad+\alpha(B_+-M(\lambda))^{-1}\gamma^*(\bar\lambda)\gamma(z)(B^*_+-M(z))^{-1}\gamma^*(\bar z)h_0\\
&=(\lambda-z)^{-1}\alpha\Gamma_0(A_{B_+}-\lambda)^{-1}h_0\\
&\qquad+(\lambda-z)^{-1}\alpha(B_+-M(\lambda))^{-1}\big[-(B^*_+-M(z))+(M(\lambda)-M(z))\big](B^*_+-M(z))^{-1}\gamma^*(\bar z)h_0\\
&=(\lambda-z)^{-1}\alpha\Gamma_0(A_{B_+}-\lambda)^{-1}h_0-(\lambda-z)^{-1}\alpha(B_+-M(\lambda))^{-1}\big[B_+-M(\lambda)-i\alpha^2\big](B^*_+-M(z))^{-1}\gamma^*(\bar z)h_0\\
&=(\lambda-z)^{-1}\alpha\Gamma_0(A_{B_+}-\lambda)^{-1}h_0-(\lambda-z)^{-1}\big[I-i\alpha(B_+-M(\lambda))^{-1}\alpha\big]\alpha(B^*_+-M(z))^{-1}\gamma^*(\bar z)h_0\\
&=(\lambda-z)^{-1}\alpha\Gamma_0(A_{B_+}-\lambda)^{-1}h_0-(\lambda-z)^{-1}S^*(\bar\lambda)\alpha\Gamma_0(A^*_{B_+}-z)^{-1}h_0
\end{aligned}$$

Again, both sides of this relation have boundary values almost everywhere on the real axis, since they both belong to the Hardy class $H_2^-(E)$. Passing $\varepsilon\downarrow 0$, substitute obtained results to the calculations of $\mathscr{F}_+(h'_-,h'_0,h'_+)$ started above.

$$\begin{aligned}
&\mathscr{F}_+(\mathscr{A}-zI)^{-1}(h_-,h_0,h_+)\\
&=-\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}(A^*_{B_+}-z)^{-1}h_0+\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}\gamma(z)(B^*_+-M(z))^{-1}\alpha h'_+(0)\\
&\quad+\frac{1}{k-z}(S^*\widehat{h}_-+\widehat{h}_+)+\frac{1}{k-z}\frac{i}{\sqrt{2\pi}}\times\\
&\qquad\times\Big[i(k-z)\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}\gamma(z)(B^*_+-M(z))^{-1}\alpha h'_+(0)+iS^*(k)\alpha\Gamma_0(A^*_{B_+}-z)^{-1}h_0\Big]\\
&=\frac{1}{k-z}\Big[-\frac{1}{\sqrt{2\pi}}\alpha\Gamma_0(A_{B_+}-k+i0)^{-1}h_0+S^*\widehat{h}_-+\widehat{h}_+\Big]\\
&\qquad\qquad+\frac{1}{k-z}\frac{1}{\sqrt{2\pi}}\Big[S^*(k)\alpha\Gamma_0(A^*_{B_+}-z)^{-1}h_0-S^*(k)\alpha\Gamma_0(A^*_{B_+}-z)^{-1}h_0\Big]\\
&=(k-z)^{-1}\mathscr{F}_+(h_-,h_0,h_+).
\end{aligned}$$

The desired equality is established. Finally, the case $z\in\mathbb{C}_-$ can be considered analogously. The proof is complete. □

Proposition 2.11. *The isometrical operator Φ maps $\mathscr{H}$ onto $\mathbf{H}$.*

Proof. As specified above, we use the same symbol Φ for the closure of the mapping defined in (2.15). We only need to show that $\mathcal{R}(\Phi)$ coincides with the whole space $\mathbf{H}$. It is already known that Φ maps $D_-\oplus 0\oplus D_+$ onto $\mathfrak{D}_+\oplus\mathfrak{D}_-$ isometrically and the linear set $\vee_{\lambda\notin\mathbb{R}}(\mathscr{A}-\lambda I)^{-1}(D_-\oplus 0\oplus D_+)$ is dense in $\mathscr{H}$. Owing to Proposition 2.10, this set is mapped by the isometry Φ into the set $\vee_{\lambda\notin\mathbb{R}}(k-\lambda)^{-1}(\mathfrak{D}_+\oplus\mathfrak{D}_-)$, which is dense in $(L_2(E),L_2(E))$. By the definition of norm in $\mathbf{H}$, this set is dense in $\mathbf{H}$. The range of an isometry is a closed subspace, and that observation completes the proof. $\square$

2.2 Naboko-Vasyunin model of non-selfadjoint non-dissipative operator

In the paper [Na] S. Naboko proposed a solution to the problem of the functional model construction for a non-selfadjoint non-dissipative operator. His method was revisited later in the work [MV] where it was taken as a foundation for the functional model of an arbitrary bounded operator. The key idea of this approach is to use the Sz.Nagy-Foiaş model of a dissipative operator that is "close" in a certain sense to the initial operator and to describe the properties of the latter in this model space. It turned out that such dissipative operator can be pointed to in a very natural, but not obvious, way. Namely, one arrives at that operator by replacing the imaginary part of the initial non-dissipative operator with its absolute value. In other words, the "close" dissipative operator for $A+iV$, where $A=A^*$ and $V=V^*$ is A-bounded operator with the relative bound lesser than 1, is the operator $A+i|V|$. Similar results are obtained in [MV] for a bounded operator considered as an additive perturbation of an unitary one.

The theory developed in [Na] becomes inapplicable in the general situation of an unbounded non-dissipative operator, since it could not be represented as a sum of its real and imaginary parts with the imaginary part relatively bounded. The Makarov-Vasyunin schema [MV] still holds its value in this case and could be employed for the model construction, provided that one works with the Cayley transform of the initial unbounded operator. However, in applications to problems arising in physics, the computational complexity and inherited inconvenience of Cayley transforms makes this method less attractive than the direct approach of [Na].

Almost solvable extensions of a Hermitian operator are an example when the functional model can be constructed by the method of paper [Na] without resorting to the Cayley transform. In this section we will use notation introduced earlier and explain how to obtain the formulae for the resolvent $(A_B-zI)^{-1}$ acting on the Sz.Nagy-Foiaş model space of the "close" dissipative operator A_{B_+}. Essentially, all the computations are based on some relationships between the resolvents $(A_B-zI)^{-1}$ and $(A_{B_+}-zI)^{-1}$, quite similar to the identities between the resolvents of operators A_∞ and A_{B_+} obtained in the previous Section.

Let $\zeta\in\rho(A_B)\cap\mathbb{C}_-$, $\phi\in\mathcal{D}(A_B)$ and $(A_B-\zeta I)\phi=h$. We will represent ϕ as a sum of two vectors $\phi=f+g$, where $f\in\ker(A^*-\zeta I)$ and $g=(A_{B_+}-\zeta I)^{-1}h$. Noting that $\Gamma_1\phi=B\Gamma_0\phi$ and $\Gamma_1 g=B_+\Gamma_0 g$ we obtain:

$$\begin{aligned}0&=\Gamma_1\phi-B\Gamma_0\phi=(\Gamma_1-B\Gamma_0)(f+g)\\&=\Gamma_1 f-B\Gamma_0 f+\Gamma_1 g-B\Gamma_0 g=M(\zeta)\Gamma_0 f-B\Gamma_0 f+B_+\Gamma_0 g-B\Gamma_0 g\\&=(M(\zeta)-B)\,\Gamma_0 f+(B_+-B)\Gamma_0 g\end{aligned}$$

Therefore, $\Gamma_0 f=(B-M(\zeta))^{-1}(B_+-B)\Gamma_0 g$, so that for $\Gamma_0\phi=\Gamma_0 f+\Gamma_0 g$ we have

$$\Gamma_0\phi=\left[I+(B-M(\zeta))^{-1}(B_+-B)\right]\Gamma_0 g$$

Now we apply the operator α to both sides of this equation and recall that

$$\phi=(A_B-\zeta I)^{-1}h,\quad g=(A_{B_+}-\zeta I)^{-1}h,\quad B_+-B=i\alpha X^-\alpha$$

where $X^-=(I_E-J)/2$. Thus for each $h\in H$:

$$\alpha\Gamma_0(A_B-\zeta I)^{-1}h=\left[I+i\alpha\,(B-M(\zeta))^{-1}\alpha X^-\right]\alpha\Gamma_0(A_{B_+}-\zeta I)^{-1}h$$

Similar computations with the operators B and B_+ interchanged yield equality

$$\alpha\Gamma_0(A_{B_+}-\zeta I)^{-1}h=\left[I-i\alpha\,(B_+-M(\zeta))^{-1}\alpha X^-\right]\alpha\Gamma_0(A_B-\zeta I)^{-1}h$$

Introduce an analytic operator-function $\Theta_-(\zeta)$, $\zeta \in \mathbb{C}_-$

$$\left.\begin{aligned} \Theta_-(\zeta) &:= I - i\alpha\left(B_+ - M(\zeta)\right)^{-1}\alpha X^- \\ &= X^+ + S^*(\bar\zeta)X^-, \qquad \zeta\in\mathbb{C}_- \end{aligned}\right\} \tag{2.17}$$

where $X^+ = (I_E + J)/2$ and $S(\cdot)$ is the characteristic function of the operator A_{B_+} as defined in the Theorem 2.1. The second equality (2.17) can be easily verified with the help of representation (1.16) for the characteristic function of an a.s. extension and the identity $X^+ + X^- = I_E$. Indeed, from (1.16) we obtain

$$\begin{aligned} X^+ + S^*(\bar\zeta)X^- &= X^+ + [I_E - i\alpha(B_+ - M(\zeta))^{-1}\alpha]X^- \\ &= X^+ + X^- - i\alpha(B_+ - M(\zeta))^{-1}\alpha X^- = \Theta_-(\zeta). \end{aligned}$$

The preceding formulae now can be rewritten in the form of operator equalities:

$$\left.\begin{aligned} \alpha\Gamma_0(A_{B_+} - \zeta I)^{-1} &= \Theta_-(\zeta)\,\alpha\Gamma_0(A_B - \zeta I)^{-1}, \quad \zeta\in\mathbb{C}_- \\ \alpha\Gamma_0(A_B - \zeta I)^{-1} &= \Theta_-^{-1}(\zeta)\,\alpha\Gamma_0(A_{B_+} - \zeta I)^{-1}, \quad \zeta\in\mathbb{C}_-\cap\rho(A_B) \end{aligned}\right\} \tag{2.18}$$

The inverse function $\Theta_-^{-1}(\cdot) = [\Theta_-(\cdot)]^{-1}$ has the form similar to (2.17):

$$\begin{aligned} \Theta_-^{-1}(\zeta) &= I + i\alpha\left(B - M(\zeta)\right)^{-1}\alpha X^- \\ &= X^+ + \Theta^*(\bar\zeta)X^-, \qquad \zeta\in\mathbb{C}_-\cap\rho(A_B) \end{aligned} \tag{2.19}$$

where Θ is the characteristic function of A_B.

Now we turn to the similar, but lengthier, computations for the resolvents of the operators A_B and A_{B_+} in the upper half plane. For $z\in\mathbb{C}_+\cap\rho(A_B^*)$ and $h\in H$ we represent the vector $\phi\in\mathcal{D}(A_{B_+}^*)$ such that $(A_{B_+}^* - zI)\phi = h$ in the form $\phi = f + g$, where $f\in\ker(A^* - zI)$ and $g = (A_B^* - zI)^{-1}h$. Then

$$0 = (\Gamma_1 - B_+^*\Gamma_0)\phi = \left(M(z) - B_+^*\right)\Gamma_0 f + (B^* - B_+^*)\Gamma_0 g$$

Therefore,

$$\Gamma_0 f = \left(B_+^* - M(z)\right)^{-1}(B^* - B_+^*)\Gamma_0 g = i\left(B_+^* - M(z)\right)^{-1}\alpha X^-\alpha\Gamma_0 g$$

and

$$\Gamma_0\phi = \Gamma_0 f + \Gamma_0 g = \left[I + i\left(B_+^* - M(z)\right)^{-1}\alpha X^-\alpha\right]\Gamma_0 g$$

After substitution of $\phi = (A_{B_+}^* - zI)^{-1}h$ and $g = (A_B^* - zI)^{-1}h$ we obtain

$$\alpha\Gamma_0(A_{B_+}^* - zI)^{-1}h = \left[I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^-\right]\alpha\Gamma_0(A_B^* - zI)^{-1}h$$

Since this identity is valid for each $h\in H$, in particular, for $h\in\mathcal{R}(A_B - zI)$ it follows that on the domain $\mathcal{D}(A_B)$

$$\begin{aligned} &\alpha\Gamma_0(A_{B_+}^* - zI)^{-1}(A_B - zI) \\ &\quad = \left[I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^-\right]\times J\cdot J\alpha\Gamma_0(A_B^* - zI)^{-1}(A_B - zI) \end{aligned}$$

Noting that $J\alpha\Gamma_0(A_B^* - zI)^{-1}(A_B - zI)f = \Theta(z)J\alpha\Gamma_0 f$ for any $f\in\mathcal{D}(A_B)$ by the definition of characteristic function (see calculations preceding (1.16)), we arrive at the formulae

$$\alpha\Gamma_0(A_{B_+}^* - zI)^{-1}(A_B - zI) = \left[I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^-\right]\times J\Theta(z)J\alpha\Gamma_0$$

and, if $z\in\mathbb{C}_+\cap\rho(A_B)$

$$\alpha\Gamma_0(A_{B_+}^* - zI)^{-1} = \left[I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^-\right]J\Theta(z)J\alpha\Gamma_0(A_B - zI)^{-1}$$

Denote Θ_+ the operator function from the right hand side and compute it.

$$\begin{aligned}\Theta_+(z) &= \left[I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^-\right]J\Theta(z)J\\
&= \left[I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^-\right]J\left[I + iJ\alpha\left(B^* - M(z)\right)^{-1}\alpha\right]J\\
&= I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^- + i\alpha\left(B^* - M(z)\right)^{-1}\alpha J\\
&\qquad + (i)^2\alpha\left(B_+^* - M(z)\right)^{-1}\alpha X^-\alpha\left(B^* - M(z)\right)^{-1}\alpha J\\
&= I + i\alpha\left(B_+^* - M(z)\right)^{-1}\left[-X^-\left(B^* - M(z)\right) + \left(B_+^* - M(z)\right) + i\alpha X^-\alpha\right]\left(B^* - M(z)\right)^{-1}\alpha J\\
&= I + 2i\alpha\left(B_+^* - M(z)\right)^{-1}\left[-X^-\left(B^* - M(z)\right) + B^* - M(z)\right]\left(B^* - M(z)\right)^{-1}\alpha J\\
&= I + i\alpha\left(B_+^* - M(z)\right)^{-1}X^+\alpha J = X^- + \left[I + i\alpha\left(B_+^* - M(z)\right)^{-1}\alpha\right]X^+\\
&= X^- + S(z)X^+\end{aligned}$$

Therefore,

$$\left.\begin{aligned}\alpha\Gamma_0(A_{B_+}^* - zI)^{-1}(A_B - zI) = \Theta_+(z)\alpha\Gamma_0, \qquad z\in\mathbb{C}_+\\
\text{where } \Theta_+(z) = I + i\alpha\bigl(B_+^* - M(z)\bigr)^{-1}\alpha X^+ = X^- + S(z)X^+\end{aligned}\right\} \tag{2.20}$$

Values of operator-function $\Theta_+(z)$ are invertible operators if $z\in\mathbb{C}_+\cap\rho(A_B)$; simple computations show that

$$\Theta_+^{-1}(z) = I - i\alpha\left(B - M(z)\right)^{-1}\alpha X^+ = X^- + \Theta^*(\bar z)X^+, \tag{2.21}$$

Finally we obtain the counterpart for (2.18):

$$\left.\begin{aligned}\alpha\Gamma_0(A_{B_+}^* - zI)^{-1} = \Theta_+(z)\alpha\Gamma_0(A_B - zI)^{-1}, \quad z\in\mathbb{C}_+\\
\alpha\Gamma_0(A_B - zI)^{-1} = \Theta_+^{-1}(z)\alpha\Gamma_0(A_{B_+}^* - zI)^{-1}, \quad z\in\mathbb{C}_+\cap\rho(A_B)\end{aligned}\right\} \tag{2.22}$$

Now we can compute how the spectral mappings $\mathscr{F}_\pm$ translate the resolvent of the operator A_B into the "model" terms. For $\lambda_0\in\mathbb{C}_-\cap\rho(A_B)$, $\zeta\in\mathbb{C}_-$ and $h\in H$ with the assistance of (2.18) we have

$$\begin{aligned}&\alpha\Gamma_0(A_{B_+} - \zeta I)^{-1}(A_B - \lambda_0 I)^{-1}h\\
&\quad= \Theta_-(\zeta)\alpha\Gamma_0(A_B - \zeta I)^{-1}(A_B - \lambda_0 I)^{-1}h\\
&\quad= (\zeta - \lambda_0)^{-1}\Theta_-(\zeta)\alpha\Gamma_0[(A_B - \zeta I)^{-1} - (A_B - \lambda_0 I)^{-1}]h\\
&\quad= (\zeta - \lambda_0)^{-1}\left[\alpha\Gamma_0(A_{B_+} - \zeta I)^{-1} - \Theta_-(\zeta)\alpha\Gamma_0(A_B - \lambda_0 I)^{-1}\right]h\\
&\quad= (\zeta - \lambda_0)^{-1}\left[\alpha\Gamma_0(A_{B_+} - \zeta I)^{-1} - \Theta_-(\zeta)\Theta_-^{-1}(\lambda_0)\alpha\Gamma_0(A_{B_+} - \lambda_0 I)^{-1}\right]h\end{aligned}$$

Assume $\zeta = k - i\varepsilon$, $k\in\mathbb{R}$, $\varepsilon > 0$. We obtain the expression for $\mathscr{F}_+(A_B - \lambda_0 I)^{-1}h$ when $\varepsilon\to 0$. (See definitions of $\mathscr{F}_\pm$ after the Lemma 2.4.) Taking into account assertion (2) of the Theorem 2.3 and noting that boundary values $\Theta_-(k - i0)$ of the bounded analytic operator-function Θ_- exist in the strong operator topology almost everywhere on the real axis (see (2.17)), we deduce from the formula above that for $(\widetilde g, g) = \Phi h$, $k\in\mathbb{R}$:

$$\begin{aligned}&\left[\mathscr{F}_+(A_B - \lambda_0 I)^{-1}h\right](k)\\
&\quad= (k - \lambda_0)^{-1}\left[(\widetilde g + S^*g)(k - i0) - \Theta_-(k - i0)\Theta_-^{-1}(\lambda_0)(\widetilde g + S^*g)(\lambda_0)\right].\end{aligned}$$

The model representations of functions $\mathscr{F}_-(A_B - \lambda_0 I)^{-1}h$ and $\mathscr{F}_\pm(A_B - \mu_0 I)^{-1}h$, where $\mu_0\in\mathbb{C}_+\cap\rho(A_B)$ are computed quite similarly and below we sum up all these formulae:

$$\begin{aligned}\mathscr{F}_+(A_B - \lambda_0 I)^{-1}h &= \frac{1}{k - \lambda_0}\left[(\widetilde g + S^*g)(k) - \Theta_-(k)\Theta_-^{-1}(\lambda_0)(\widetilde g + S^*g)(\lambda_0)\right]\\
\mathscr{F}_-(A_B - \lambda_0 I)^{-1}h &= \frac{1}{k - \lambda_0}\left[(\,S\widetilde g + g\,)(k) - \Theta_+(k)\Theta_-^{-1}(\lambda_0)(\widetilde g + S^*g)(\lambda_0)\right]\\
\mathscr{F}_+(A_B - \mu_0 I)^{-1}h &= \frac{1}{k - \mu_0}\left[(\widetilde g + S^*g)(k) - \Theta_-(k)\Theta_+^{-1}(\mu_0)(\,S\widetilde g + g\,)(\mu_0)\right]\\
\mathscr{F}_-(A_B - \mu_0 I)^{-1}h &= \frac{1}{k - \mu_0}\left[(\,S\widetilde g + g\,)(k) - \Theta_+(k)\Theta_+^{-1}(\mu_0)(\,S\widetilde g + g\,)(\mu_0)\right]\end{aligned}$$

where $h \in H$, $(\widetilde{g}, g) = \Phi h$, $\lambda_0 \in \mathbb{C}_- \cap \rho(A_B)$, $\mu_0 \in \mathbb{C}_+ \cap \rho(A_B)$, and for almost all $k \in \mathbb{R}$ there exist strong limits $\Theta_\pm(k) := s - \lim_{\varepsilon \downarrow 0} \Theta_\pm(k \pm i\varepsilon)$.

The main theorem describes the action of operator A_B in the model space $\mathbf{H} = L_2\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}$ of dissipative operator A_{B_+}. As before, for the notational convenience we use the same symbols for objects whose unitary equivalence is established by the isometry Φ.

Theorem 2.5. *For $\lambda_0 \in \mathbb{C}_- \cap \rho(A_B)$, $\mu_0 \in \mathbb{C}_+ \cap \rho(A_B)$, $(\widetilde{g}, g) \in H$*

$$(A_B - \lambda_0 I)^{-1}\begin{pmatrix} \widetilde{g} \\ g \end{pmatrix} = P_H (k - \lambda_0)^{-1} \begin{pmatrix} \widetilde{g} \\ g - X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0) \end{pmatrix}$$

$$(A_B - \mu_0 I)^{-1}\begin{pmatrix} \widetilde{g} \\ g \end{pmatrix} = P_H (k - \mu_0)^{-1} \begin{pmatrix} \widetilde{g} - X^+ \Theta_+^{-1}(\mu_0)(S\widetilde{g} + g)(\mu_0) \\ g \end{pmatrix}$$

Here P_H is the orthogonal projection from $\mathbf{H}$ onto H.

Proof. The proof is identical to the proof of the corresponding result of [Na]. For the most part it is based on the identities for $\mathscr{F}_\pm(A_B - \lambda_0 I)^{-1}$, $\mathscr{F}_\pm(A_B - \mu_0 I)^{-1}$ obtained earlier.

Let us verify the Theorem's assertion for $\lambda_0 \in \mathbb{C}_- \cap \rho(A_B)$. The case of the resolvent in the upper half plane is considered analogously. According to Theorem 2.3 we only need to show that functions $(S\widetilde{g}' + g')$ and $(\widetilde{g}' + S^* g')$ where $(\widetilde{g}', g')$ is the vector on the right hand side of the corresponding formula satisfy following conditions

$$\mathscr{F}_+(A_B - \lambda_0 I)^{-1} h = (\widetilde{g}' + S^* g')$$
$$\mathscr{F}_-(A_B - \lambda_0 I)^{-1} h = (S\widetilde{g}' + g')$$

with $\Phi h = (\widetilde{g}, g)$. Since

$$\begin{aligned}
\begin{pmatrix} \widetilde{g}' \\ g' \end{pmatrix} &= P_H (k - \lambda_0)^{-1} \begin{pmatrix} \widetilde{g} \\ g - X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\widetilde{g}}{k-\lambda_0} - P_+ \frac{1}{k-\lambda_0}\left[\widetilde{g} + S^* g - S^* X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\right] \\ \frac{g - X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)}{k-\lambda_0} - P_- \frac{1}{k-\lambda_0}\left[S\widetilde{g} + g - X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\right] \end{pmatrix} \\
&= \frac{1}{k-\lambda_0} \begin{pmatrix} \widetilde{g} - (\widetilde{g} + S^* g)(\lambda_0) + S^*(\lambda_0) X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0) \\ g - X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0) \end{pmatrix},
\end{aligned}$$

we have with the help of (2.17) and (2.20)

$$\begin{aligned}
&\widetilde{g}' + S^* g' \\
&= \frac{1}{k-\lambda_0}\left[(\widetilde{g} + S^* g) - (\widetilde{g} + S^* g)(\lambda_0) + (S^*(\bar{\lambda}_0) - S^*) X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\right] \\
&= \frac{1}{k-\lambda_0}\Big[(\widetilde{g} + S^* g) - \big(\Theta_-(\lambda_0) - (S^*(\bar{\lambda}_0) - S^*) X^-\big) \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\Big] \\
&= \frac{1}{k-\lambda_0}\Big[(\widetilde{g} + S^* g) - \Theta_-(k) \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\Big] = \mathscr{F}_+(A_B - \lambda_0 I)^{-1} u
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{S g}' + g' \\
&= \frac{1}{k-\lambda_0}\left[(S\widetilde{g} + g) - S(\widetilde{g} + S^* g)(\lambda_0) - (I - SS^*(\bar{\lambda}_0)) X^- \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\right] \\
&= \frac{1}{k-\lambda_0}\left[(S\widetilde{g} + g) - \big(S\Theta_-(\lambda_0) + X^- - SS^*(\bar{\lambda}_0) X^-\big) \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\right] \\
&= \frac{1}{k-\lambda_0}\left[(S\widetilde{g} + g) - \big(SX^+ + X^-\big) \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\right] \\
&= \frac{1}{k-\lambda_0}\left[(S\widetilde{g} + g) - \Theta_+(k) \Theta_-^{-1}(\lambda_0)(\widetilde{g} + S^* g)(\lambda_0)\right] = \mathscr{F}_+(A_B - \lambda_0 I)^{-1} u.
\end{aligned}$$

The proof is complete. ∎

Remark 2.12. Operators $X^-\Theta_-^{-1}(\lambda_0)$, $X^+\Theta_+^{-1}(\mu_-)$ in the Theorem 2.5 can be replaced with $X^-\Theta^*(\bar\lambda_0)X^-$ and $X^+\Theta^*(\bar\mu_0)X^+$, respectively. For the proof see (2.19), (2.21) and identities $X^-X^+ = X^+X^- = 0$.

Remark 2.13. All assertions of the Theorem 2.5 remain valid if the operator J is formally substituted by $-J$ or $\pm I_E$. Compare with [Na] for details. Following theorem is a consequence of this observation obtained from the Theorem 2.5 by the substitution $J \to -J$. Note that its claim can be verified independently by passing on to adjoint operators in the formulae of Theorem 2.5.

Theorem 2.6. *For* $\lambda_0 \in \mathbb{C}_- \cap \rho(A_B^*)$, $\mu_0 \in \mathbb{C}_+ \cap \rho(A_B^*)$, $(\widetilde{g}, g) \in H$

$$(A_B^* - \lambda_0 I)^{-1}\begin{pmatrix}\widetilde{g}\\ g\end{pmatrix} = P_H(k-\lambda_0)^{-1}\begin{pmatrix}\widetilde{g}\\ g - X^+\Theta(\lambda_0)X^+(\widetilde{g} + S^*g)(\lambda_0)\end{pmatrix}$$
$$(A_B^* - \mu_0 I)^{-1}\begin{pmatrix}\widetilde{g}\\ g\end{pmatrix} = P_H(k-\mu_0)^{-1}\begin{pmatrix}\widetilde{g} - X^-\Theta(\mu_0)X^-(\,S\widetilde{g} + g\,)(\mu_0)\\ g\end{pmatrix}$$

Assuming $J = I_E$ in the statement of the Theorem 2.5, we arrive at the Sz.Nagy-Foiaş model of dissipative operator A_{B_+}, see (2.6) and Theorem 2.4.

Remark 2.14. It is unknown whether the operator A_∞ can be efficiently represented in the model space $\mathbf{H} = L_2\begin{pmatrix} I & S^* \\ S & I\end{pmatrix}$. The computations, analogous to the carried out above, fail to yield "resolvent identities" that could be used for the desired model representation of the operator A_∞.

At this point we close our discussion of the functional model of the operator A_B and turn to the illustrations of the developed theory.

3 Examples

In this section we offer two examples of calculation of Weyl function.

The first example is a Hermitian operator that models the finite set of δ-interactions of quantum mechanics ([BF]). A recently published preprint [BMN] offers a description of the boundary triple of this operator in the case of a single δ-interaction. It does not touch upon more general situation; however, a generalization to the case considered below is quite evident. The paper [BMN] is not concerned with any questions related to the functional model of nonselfadjoint extensions.

The second example is the Hermitian operator generated by the differential expression $l[y] = -y'' + q(z)y$ in $L^2(0,\infty)$ with a real-valued potential $q(x)$ such that the Weyl limit circle case at infinity is observed. Explicit construction of the selfadjoint dilation of a dissipative extension of this operator and subsequent spectral analysis in terms of its characteristic function are carried out in the paper [A1] in complete accordance with B. Pavlov's schema.

In this Section we content ourselves with description of convenient boundary triples and computation of corresponding Weyl functions. The construction of the functional models is not given here, since the model perspective on any a.s. non-selfadjoint extension of these operators can be easily derived from the exposition of Section 2.

3.1 Point interactions in $\mathbb{R}^3$

Let $\{x_s\}_{s=1}^n$ $(n < \infty)$ be the finite set of distinct points in $\mathbb{R}^3$.We define a Hermitian operator A as a closure of the restriction of Laplace operator $-\Delta$ on $H = L_2(\mathbb{R}^3)$ to the set of smooth functions vanishing in the neighborhood of $\cup_s \mathrm{x}_s$. It is known ([BF], [P2]), that

$$\mathcal{D}(A) = \left\{u \in W_2^2(\mathbb{R}^3) \,:\, u(x_s) = 0, \quad s = 1, 2, ...n\right\}$$

The deficiency indices $n_\pm(A)$ are equal to (n,n). The domain of conjugate operator A^* is described in the following Theorem borrowed from [P2].

Theorem 3.1. *The domain $\mathcal{D}(A^*)$ of conjugate operator A^* consists of the functions $u \in L_2(\mathbb{R}^3) \cap W_2^2(\mathbb{R}^3 \setminus \cup_s x_s)$ with the following asymptotic expansion in the neighborhood of $\{x_s\}_{s=1}^n$*

$$u(x) \sim u_-^s/|x - x_s| + u_0^s + O(|x - x_s|^{1/2}), \quad x \to x_s, \quad s = 1, 2, ...n.$$

For given vectors $u, v \in \mathcal{D}(A^*)$ the analogue of the second Green's formula holds:

$$(A^* u, v)_H - (u, A^* v)_H = \sum_{s=1}^{n} \left(u_0^s \bar{v}_-^s - u_-^s \bar{v}_0^s \right)$$

It is easy to show that the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for the operator A^* can be chosen in the form ($u \in \mathcal{D}(A^*)$):

$$\mathcal{H} = \mathbb{C}^n, \quad \Gamma_0 u = (u_-^1, u_-^2, \dots u_-^n)^T, \quad \Gamma_1 u = (u_0^1, u_0^2, \dots u_0^n)^T$$

In order to compute Weyl function corresponding to this boundary triple let us fix a complex number $z \in \mathbb{C}_- \cup \mathbb{C}_+$ and let y_z be a vector from $\ker(A^* - zI)$, so that $y_z \in \mathcal{D}(A^*)$ and $-\Delta y_z = z y_z$. Note that vector y_z is uniquely represented in the form of linear combination

$$y_z(x) = \sum_{s=1}^{n} C_s \frac{\exp(ik|x - x_s|)}{|x - x_s|},$$

where $k = \sqrt{z}$, $\operatorname{Im} z > 0$, and $\{C_s\}_{s=1}^n$ are some constants. Noting that in the neighborhoods of points $\{x_s\}_{s=1}^n$ asymptotically

$$\frac{\exp(ik|x - x_s|)}{|x - x_s|} \sim \frac{1}{|x - x_s|} + ik + O(|x - x_s|), \qquad \text{as } x \to x_s$$

and obviously

$$\lim_{x \to x_j} \frac{\exp(ik|x - x_s|)}{|x - x_s|} = \frac{\exp(ik|x_j - x_s|)}{|x_j - x_s|}, \quad j \neq s,$$

we easily compute both vectors $\Gamma_0 y_z$, $\Gamma_1 y_z$.

$$\begin{aligned} \Gamma_0 y_z &= (C_1, C_2, \dots C_n)^T \\ \Gamma_1 y_z &= \Bigg(ik \cdot C_1 + \sum_{s=2}^{n} C_s \frac{\exp(ik|x_1 - x_s|)}{|x_1 - x_s|}, \dots \\ &\quad \dots ik \cdot C_j + \sum_{s \neq j}^{n} C_s \frac{\exp(ik|x_j - x_s|)}{|x_j - x_s|}, \dots \\ &\quad \dots ik \cdot C_n + \sum_{s=1}^{n-1} C_s \frac{\exp(ik|x_{n-1} - x_s|)}{|x_{n-1} - x_s|} \Bigg)^T \end{aligned}$$

Comparison of these formulae with the definition $\Gamma_1 y_z = M(z) \Gamma_0 y_z$ of Weyl function yields its explicit form. It is a $(n \times n)$-matrix function $M(z) = \|M_{sj}(z)\|_1^n$ with elements

$$M_{sj}(z) = \begin{cases} ik, & s = j \\ \langle s, j \rangle, & s \neq j \end{cases}$$

where $k = \sqrt{z}$, $k \in \mathbb{C}_+$ and

$$\langle s, j \rangle := \frac{\exp(ik|x_s - x_j|)}{|x_s - x_j|}, \quad s \neq j, \quad s, j = 1, 2, \dots n$$

Note that the selfadjoint operator A_∞ defined as a restriction of A^* to the set $\{y \in \mathcal{D}(A^*) : \Gamma_0 y = 0\}$ is the Laplace operator $-\Delta$ in $L_2(\mathbb{R}^3)$ with the domain $\mathcal{D}(A_\infty) = W_2^2(\mathbb{R}^3)$. At the same time it is the Friedrichs extension of operator A. The special role of extension A_∞ with regard to the functional model construction was pointed out in Section 2.

3.2 Schrödinger operator in the Weyl limit circle case

The second example is the Hermitian operator A defined as a closure in the Hilbert space $H = L_2(\mathbb{R}_+)$ of the minimal operator generated by differential expression

$$l[y] = -y'' + q(x)y \tag{3.1}$$

on domain $C_0^\infty(\mathbb{R}_+)$. We assume the potential $q(x)$ to be a real-valued continuous function such that for the expression (3.1) the Weyl limit circle case at infinity is observed. The deficiency indices of A are equal to $(2, 2)$ and both solutions of equation $l[y] = \lambda y$ are functions from $L_2(\mathbb{R}_+)$ for any $\lambda \in \mathbb{C}$, see [T], [CL]. The conjugate operator A^* is generated by the same differential expression (3.1) on the class of absolutely continuous functions y from $L_2(\mathbb{R}_+)$ whose derivatives are locally absolutely continuous and $l[y]$ is square integrable.

Let $v_1(x)$, $v_2(x)$, $x \in \mathbb{R}$ be two linearly independent solutions of the equation $l[y] = 0$ satisfying conditions at $x = 0$:

$$v_1(0) = 1, \quad v_1'(0) = 0, \quad v_2(0) = 0, \quad v_2'(0) = 1,$$

For our purposes we will use the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for the operator A^* described in [A1]. The space $\mathcal{H}$ is two-dimensional: $\mathcal{H} = \mathbb{C}^2$, and the mappings $\Gamma_0, \Gamma_1 : \mathcal{D}(A^*) \to \mathbb{C}^2$ are defined as

$$\Gamma_0 y = \begin{pmatrix} y'(0) \\ \mathcal{W}[y, v_2]\big|_\infty \end{pmatrix}, \quad \Gamma_1 y = \begin{pmatrix} -y(0) \\ \mathcal{W}[y, v_1]\big|_\infty \end{pmatrix}, \quad y \in \mathcal{D}(A^*) \tag{3.2}$$

where $\mathcal{W}[f, g] := fg' - f'g$ is the Wronsky determinant of two functions f, g from $\mathcal{D}(A^*)$.

In order to compute the corresponding Weyl function $M(\cdot)$ let us fix a complex number $\lambda \in \mathbb{C}_+$ and let ψ_λ, $\phi_\lambda(x)$ be the solutions of the equation $l[y] = \lambda y$ satisfying

$$\psi_\lambda(0) = 1, \quad \psi_\lambda'(0) = 0, \quad \phi_\lambda(0) = 0, \quad \phi_\lambda'(0) = 1. \tag{3.3}$$

Both functions ϕ_λ, ψ_λ are square integrable on the real half axis $\mathbb{R}_+$, their Wronsky determinant is independent on $x \in \mathbb{R}_+$ and is equal to one: $\mathcal{W}[\psi_\lambda, \phi_\lambda] = 1$. The functions ψ_λ, ϕ_λ are linearly independent vectors in $L_2(\mathbb{R}_+)$ and any solution y_λ of the equation $(A^* - \lambda I)y_\lambda = 0$ is their linear combination $y_\lambda = C_1\psi_\lambda + C_2\phi_\lambda$ with some constants $C_1, C_2 \in \mathbb{C}$. According to (3.2),

$$\begin{aligned}
\Gamma_0 y_\lambda &= \begin{pmatrix} y_\lambda'(0) \\ \mathcal{W}[y_\lambda, v_2]\big|_\infty \end{pmatrix} = \begin{pmatrix} C_2 \\ C_1 \cdot \mathcal{W}[\psi_\lambda, v_2]\big|_\infty + C_2 \cdot \mathcal{W}[\phi_\lambda, v_2]\big|_\infty \end{pmatrix} \\
\Gamma_1 y_\lambda &= \begin{pmatrix} -y_\lambda(0) \\ \mathcal{W}[y_\lambda, v_1]\big|_\infty \end{pmatrix} = \begin{pmatrix} -C_1 \\ C_1 \cdot \mathcal{W}[\psi_\lambda, v_1]\big|_\infty + C_2 \cdot \mathcal{W}[\phi_\lambda, v_1]\big|_\infty \end{pmatrix}
\end{aligned}$$

Let $M(\lambda) = \|m_{ij}(\lambda)\| = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix}$ be the Weyl function being sought. Since $\Gamma_1 y_\lambda = M(\lambda)\Gamma_0 y_\lambda$ by the definition, the equalities

$$\begin{aligned}
&-C_1 = m_{11}(\lambda) \cdot C_2 + m_{12}(\lambda) \cdot \Big\{ C_1 \cdot \mathcal{W}[\psi_\lambda, v_2]\big|_\infty + C_2 \cdot \mathcal{W}[\phi_\lambda, v_2]\big|_\infty \Big\} \\
&C_1 \cdot \mathcal{W}[\psi_\lambda, v_1]\big|_\infty + C_2 \cdot \mathcal{W}[\phi_\lambda, v_1]\big|_\infty = \\
&\qquad = m_{21}(\lambda) \cdot C_2 + m_{22}(\lambda) \cdot \Big\{ C_1 \cdot \mathcal{W}[\psi_\lambda, v_2]\big|_\infty + C_2 \cdot \mathcal{W}[\phi_\lambda, v_2]\big|_\infty \Big\}
\end{aligned}$$

should be valid for any $C_1, C_2 \in \mathbb{C}$. The solution of this linear system is easy to compute:

$$\begin{aligned}
m_{11}(\lambda) &= \big(\mathcal{W}[\phi_\lambda, v_2]/\mathcal{W}[\psi_\lambda, v_2]\big)\big|_\infty \\
m_{12}(\lambda) &= (-1)/\mathcal{W}[\psi_\lambda, v_2]\big|_\infty \\
m_{21}(\lambda) &= \mathcal{W}[\psi_\lambda, v_1]\big|_\infty - \big(\mathcal{W}[\phi_\lambda, v_1]/\mathcal{W}[\phi_\lambda, v_2]\big)\big|_\infty \cdot \mathcal{W}[\psi_\lambda, v_2]\big|_\infty \\
m_{22}(\lambda) &= \big(\mathcal{W}[\phi_\lambda, v_1]/\mathcal{W}[\phi_\lambda, v_2]\big)\big|_\infty
\end{aligned}$$

Expression for $m_{21}(\lambda)$ above can be further simplified

$$\begin{aligned}
m_{21}(\lambda) = {} & \mathcal{W}[\phi_\lambda, v_2]^{-1} \cdot \big(\mathcal{W}[\phi_\lambda, v_1] \cdot \mathcal{W}[\psi_\lambda, v_2] - \mathcal{W}[\psi_\lambda, v_1] \cdot \mathcal{W}[\phi_\lambda, v_2]\big)\big|_\infty \\
= {} & \big(\mathcal{W}[\phi_\lambda, v_2]|_\infty\big)^{-1} \cdot \lim_{b\to\infty} \big((\phi_\lambda v_1' - \phi_\lambda' v_1)(\psi_\lambda v_2' - \psi_\lambda' v_2) - \\
& \qquad\qquad (\psi_\lambda v_1' - \psi_\lambda' v_1)(\phi_\lambda v_2' - \phi_\lambda' v_2)\big)\big|_b \\
= {} & \big(\mathcal{W}[\phi_\lambda, v_2]|_\infty\big)^{-1} \cdot \lim_{b\to\infty} \big(\phi_\lambda \psi_\lambda' (v_1 v_2' - v_1' v_2) - \phi_\lambda' \psi_\lambda (v_1 v_2' - v_1' v_2)\big)\big|_b \\
= {} & \big(\mathcal{W}[\phi_\lambda, v_2]|_\infty\big)^{-1} \cdot \lim_{b\to\infty} \mathcal{W}[\phi_\lambda, \psi_\lambda]|_b \cdot \mathcal{W}[v_1, v_2]|_b \\
= {} & -\big(\mathcal{W}[\phi_\lambda, v_2]|_\infty\big)^{-1}
\end{aligned}$$

Finally, for the Weyl function we obtain the formula

$$M(\lambda) = \big\{\mathcal{W}[\psi_\lambda, v_2]\big|_\infty\big\}^{-1} \begin{pmatrix} \mathcal{W}[\phi_\lambda, v_2]\big|_\infty & -1 \\ -1 & \mathcal{W}[\psi_\lambda, v_1]\big|_\infty \end{pmatrix}, \quad \lambda \in \mathbb{C}_+ \tag{3.4}$$

There exists another representation of the Weyl function (3.4) derived from the work of M.G.Krein [Kr]. Introduce following functions

$$\left.\begin{aligned}
D_0(x,\lambda) &= -\lambda \int_0^x \phi_\lambda(s) v_2(s) ds \\
D_1(x,\lambda) &= 1 + \lambda \int_0^x \phi_\lambda(s) v_1(s) ds \\
E_0(x,\lambda) &= 1 - \lambda \int_0^x \psi_\lambda(s) v_2(s) ds \\
E_1(x,\lambda) &= \lambda \int_0^x \psi_\lambda(s) v_1(s) ds
\end{aligned}\right\} \tag{3.5}$$

Noticing that Cauchy function of the differential operator $-\frac{d^2}{dx^2} + q(x)$ coincides with $v_1(x)v_2(s) - v_1(s)v_2(x)$, after a short computation we conclude that

$$\begin{aligned}
\mathcal{W}[\psi_\lambda, v_2] &= E_0(x,\lambda) \qquad & \mathcal{W}[\phi_\lambda, v_2] &= D_0(x,\lambda) \\
\mathcal{W}[\psi_\lambda, v_1] &= -E_1(x,\lambda) \qquad & \mathcal{W}[\phi_\lambda, v_1] &= -D_1(x,\lambda)
\end{aligned}$$

Consequently, the Weyl function (3.4) can be rewritten in the form

$$M(\lambda) = \big(E_0(\lambda)\big)^{-1} \begin{pmatrix} D_0(\lambda) & -1 \\ -1 & -E_1(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+$$

where

$$D_0(\lambda) := \lim_{x\to+\infty} D_0(x,\lambda), \qquad E_j(\lambda) := \lim_{x\to+\infty} E_j(x,\lambda), \quad j = 0, 1.$$

These limits exist due to the square integrability of the functions ψ_λ, ϕ_λ, v_1, v_2, when $\lambda \in \mathbb{C}_+$, see (3.5). Moreover, these limits are entire functions of the variable $\lambda \in \mathbb{C}$.

The selfadjoint operator A_∞ is generated by the expression (3.1) and boundary condition $\Gamma_0 y = \binom{y'(0)}{\mathcal{W}[y,v_2]|_\infty} = 0$. It is well known that the spectrum of the operator A_∞ consists of pure eigenvalues with the multiplicity equal to one. By the definition (3.3) the solution ψ_λ satisfies $\Gamma_0 \psi_\lambda = 0$ if the Wronsky determinant $\mathcal{W}[\psi_\lambda, v_2] = E_0(x,\lambda)$ tends to zero as $x \to \infty$. It means that the zeroes of the entire function $E_0(\lambda)$ in the "denominator" of the Weyl function are the eigenvalues of the operator A_∞ with the corresponding eigenvectors ψ_λ.

References

[AF] R.A. Adams, J.F. Fournier, *Sobolev Spaces*, Elsevier Science, Oxford, UK , 2003.

[AP] V.M. Adamyan, B.S. Pavlov, *Trace formula for dissipative operators* (Russian) Vestnik Leningrad. Univ. Mat. Mekh. Astronom., **2** (1979), 5–9.

[A1] B.P. Allahverdiev, *On dilation theory and spectral analysis of dissipative Schrödinger operators in Weyl's limit-circle case* (Russian), Izv. Akad. Nauk. SSSR Ser. Mat. **54** (1990), 242–257; English transl.: Math. USSR Izv. **36** (1991), 247–262.

[A2] B.P. Allahverdiev, *Spectral analysis of nonselfadjoint Schrodinger operators with a matrix potential*, J. Math. Anal. Appl. **303** (2005), 208-219.

[A3] B.P. Allahverdiev, *A nonself-adjoint singular Sturm-Liouville problem with a spectral parameter in the boundary condition*, Math. Nachr. **278, 7-8** (2005), 743-755.

[A4] B.P. Allahverdiev, *Spectral analysis of dissipative Dirac operators with general boundary conditions*, J. Math. Anal. Appl. **283** (2003), 287–303.

[A5] B.P. Allahverdiev, *Dilation and Functional Model of Dissipative Operator Generated by an Infinite Jacobi Matrix*, Math. and Comp. Modelling **38, 3** (2003), 989–1001.

[Ar] D.Z Arov, *Passive linear stationary dynamic systems.* (Russian) Sibirsk. Mat. Zh. **20, 2** (1979), 211–228.; English Transl.: Siberian Mathematical Journal, 20 (2), 149–162.

[AI] T.Ya. Azizov, I.S. Iokhvidov, *Linear Operators in Spaces With an Indefinite Metric*, John Wiley & Sons Inc., 1989.

[BN] M. Baro, H. Neidhardt *Dissipative Schrödinger-type operators as a model for generation and recombination*, J. Math. Phys. **44, 6** (2003), 2373–2401.

[BKNR1] M. Baro, H.-Chr. Kaiser, H. Neidhardt, J. Rehberg, *Dissipative Schrödinger–Poisson systems*, J. Math. Phys. **45, 1** (2004), 21–43.

[BKNR2] M. Baro, H.-Chr. Kaiser, H. Neidhardt, J. Rehberg, *A Quantum Transmitting Schrödinger–Poission System*, Rev. Math. Phys. **16, 3** (2004), 281–330.

[BMN] J. Behrndt, M. Malamud, H. Neidhardt, *Scattering matrices and Weyl functions*, Preprint (2006).

[BF] F.A. Berezin, L.D. Faddeev, *Remark on the Schrödinger equation with singular potential.* (Russian) Dokl. Akad. Nauk SSSR **137** (1961), 1011–1014.

[Br] H.-J. Bremermann, *Distributions, Complex Variables and Fourier Transforms*, Addison-Wesley, Reading, Massachusetts, 1965.

[Bru] V.M. Bruk, *A certain class of boundary value problems with a spectral parameter in the boundary condition.* (Russian) Mat. Sb. (N.S.) **100 (142)**, **2** (1976), 210–216.

[CL] E.A Coddington, N. Levinson, *Theory of ordinary differential equations*, New-York-Toronto-London: McGraw-Hill, 1955.

[DM] V.A. Derkach and M.M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps,* J. Funct. Anal. **95** (1991), 1-95.

[GG] V.I. Gorbachuk and M.L. Gorbachuk, *Boundary value problems for operator differential equations.* Translated and revised from the 1984 Russian original. Mathematics and its Applications (Soviet Series), 48. Kluwer Academic Publishers Group, Dordrecht, 1991.

[KaNR] H.-Ch. Kaiser, H. Neidhardt, J. Rehberg, On 1-*dimensional dissipative Schrödinger–type operators, their dilations, and eigenfunction expansions*. Math. Nachr. **252** (2003), 51–69.

[Ko1] A.N.Kočubeĭ, *On extension of symmetric operators and symmetric binary relations* (Russian) Mat. Zametki **17** (1975), 41-48.

[Ko2] A.N. Kočubeĭ, *Characteristic functions of symmetric operators and their extensions.* (Russian) Izv. Akad. Nauk Armyan. SSR Ser. Mat. **15, 3** (1980), 219–232.

[Kr] M.G. Kreĭn, *On the indeterminate case of the Sturm-Liouville boundary problem in the interval* $(0, \infty)$. (Russian) Izvestiya Akad. Nauk SSSR. Ser. Mat. **16** (1952). 293–324.

[K] Yu. Kudryashov, *Symmetric and selfadjoint dilations of dissipative operators.* (Russian) Teor. Funktsiĭ, Funktsional. Anal. i Prilozhen. **37** (1982), 51–54.

[KNR] Yu.A. Kuperin, S.N. Naboko, R.V. Romanov, *Spectral analysis of transport operator: Functional model approach*, Indiana Univ. Math. J. **51, 6** (2002), 1389-1426.

[LP] P. Lax, R. Phillips, *Scattering theory.* Pure and Applied Mathematics, Vol. 26 Academic Press, New York-London, 1967.

[MV] N.G. Makarov, V.I. Vasyunin, *A model for noncontractions and stability of the continuous spectrum.* Lect.Notes in Math., **864** (1981), 365–412.

[Na] S.N. Naboko, *Functional model of perturbation theory and its applications to scattering theory.* (Russian) Boundary value problems of mathematical physics, 10. Trudy Mat. Inst. Steklov. **147** (1980), 86–114.

[Na1] S.N. Naboko, *Absolutely continuous spectrum of a nondissipative operator, and a functional model. I.* (Russian) Investigations on linear operators and the theory of functions, VII. Zap. Naučn. Sem. Leningrad. Otdel Mat. Inst. Steklov. (LOMI) **65** (1976), 90–102. English transl.: J. Sov. Math. **16, 3** (1981)

[Na2] S.N. Naboko, *Absolutely continuous spectrum of a nondissipative operator, and a functional model. II.* (Russian) Investigations on linear operators and the theory of functions, VIII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **73** (1977). English transl.: J. Math. Sci. **34, 6** (1986), 2090 - 2101.

[NR] S.N. Naboko, R.V. Romanov, *Spectral singularities, Szökefalvi-Nagy-Foias functional model and the spectral analysis of the Boltzmann operator.* Recent advances in operator theory and related topics (Szeged, 1999), 473–490, Oper. Theory Adv. Appl., **127**, Birkhäuser, Basel, 2001.

[NF] B. Szökefalvi-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York, 1970.

[N1] N.K. Nikolskiĭ, *Treatise on the Shift Operator*, Springer-Verlag, Berlin etc., 1986.

[N2] N.K. Nikolskiĭ, *Operators, Functions, and Systems: An Easy reading, v.2: Model Operators and Systems*, AMS, 2002.

[P1] B.S. Pavlov, *On a non-selfadjoint Schrödinger operator.* (Russian) Probl. Math. Phys., No. 1, Spectral Theory and Wave Processes (1966) pp. 102–132 Izdat. Leningrad. Univ., Leningrad.

[P2] B.S. Pavlov, *A model of zero-radius potential with internal structure.* (Russian) Teoret. Mat. Fiz. **59, 3** (1984), 345–353. English transl.: Theoretical and Mathematical Physics, **59, 3**, pp. 544–550, Springer, New York.

[P3] B.S. Pavlov, *On conditions for separation of the spectral components of a dissipative operator*, Izv. Akad. Nauk SSSR, Ser. Matem., **39** (1975), 123–148 (Russian); English transl.: Math USSR Izvestiya, **9** (1976), 113–137.

[P4] B.S. Pavlov, *Selfadjoint dilation of a dissipative Schrödinger operator, and expansion in its eigenfunction.* (Russian) Mat. Sb. (N.S.) **102 (144), 4** (1977), 511–536.

[P5] B.S. Pavlov, *Dilation theory and spectral analysis of nonselfadjoint differential operators*, Proc. 7th Winter School, Drobobych 1974, 3-69 (1976) (Russian); English transl: Transl. II. Ser., Am. Math. Soc. **115** (1981), 103–142.

[P6] B.S. Pavlov, *A Remark on Spectral Meaning of the Symmetric Functional Model*, in Operator Theory: Advances and Applications, Vol. **154**, 163–177, Birkhäuser Verlag, Basel/Schiwtzerland, 2004.

[Pe] A.M. Petrov, *Spectral projectors of some classes of nonselfadjoint operators.* (Russian) Dynamical systems, No. **7** (Russian), 109–114, 160, "Vishcha Shkola", Kiev, 1988.; English transl. in J. Math. Sci. 65 (1993), no. 2, 1575–1578.

[RR] M. Rosenblum, J. Rovnyak, *Hardy calsses and operator theory*, Oxford University Press, 1985.

[S] B.S. Solomyak, *A functional model for dissipative operators. A coordinate-free approach.* (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **178** (1989), Issled. Linein. Oper. Teorii Funktsii. 18, 57–91, 184–185; Translation in J. Soviet Math. **61** (1992), no. **2**, 1981–2002.

[St1] A.V. Štraus, *Characteristic functions of linear operators.* (Russian) Izv. Akad. Nauk. SSSR, Ser. Mat., **24, 1** (1960), 43–74; English transl. in AMS Transl. **(2) 40** (1964), 1–37.

[St2] A.V. Štraus, *On Extensions and characteristic function of a symmetric operator.* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 186–207; English transl. in Math. USSR - Izvestija, **2** (1968), 181–204.

[T] E.S. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I.* Second Edition Clarendon Press, Oxford 1962.