ON THE SPECTRAL PROPERTIES OF THE PERTURBED LANDAU HAMILTONIAN

GRIGORI ROZENBLUM AND GRIGORY TASHCHIYAN

Dedicated to our teacher, Professor Mikhail Solomyak on the occasion of his 75-th birthday

Abstract. The Landau Hamiltonian governing the behavior of a quantum particle in dimension 2 in a constant magnetic field is perturbed by a compactly supported magnetic field and a similar electric field. We describe how the spectral subspaces change and how the Landau levels split under this perturbation.

Contents

1. Introduction

The paper is devoted to the study of the spectrum of the Schrödinger and Pauli operators in the plane, with nonzero constant magnetic field perturbed by smooth compactly supported magnetic and electric fields.

The Landau Hamiltonian describing the motion of the quantum particle in two dimensions under the influence of the constant magnetic field is one of the classical models in quantum physics. It was introduced and studied in late 1920-s, see [10]. It turned out that the system possesses rather unusual spectral properties: the spectrum consists of discrete eigenvalues lying at the points of an arithmetic progression and all these eigenvalues have infinite multiplicity. These eigenvalues are traditionally called Landau levels (LL) and the corresponding spectral subspaces are called Landau subspaces.

A natural question arises, what happens with the spectrum of the Landau Hamiltonian under perturbations. If the perturbation is strong, say, it is a Coulomb electric field, it turns out more reasonable to consider the constant magnetic field as a perturbation, and it leads to Zeeman effect. On the other hand, one can consider weak perturbations of the Landau Hamiltonian by a fast decaying or even compactly supported electric or/and magnetic field. In this case it is natural to expect that the Landau levels will split, and the question consists in describing this splitting as well as the behavior of the spectral subspaces under the perturbation.

The case of the perturbation by an electric potential, moderately, power-like decaying at infinity, was first studied by Raikov in [12]. It was shown that each LL splits into an infinite sequence of eigenvalues, converging to the originating LL. The rate of convergence of these eigenvalues is also power-like and it is determined by the rate of decay of the perturbation at infinity, the asymptotics being described by a semi-classical formula. Advanced methods of micro-local analysis were used in [12].

The case of a fast decaying (or compactly supported) electric potential was dealt with much later, in [13], see also [11]. It turned out that the Landau levels still split, however in an unusual, not a semi-classical way. Most profoundly this can be seen for a compactly supported perturbation. The semi-classical considerations, formally applied, would lead to a finite number of eigenvalues splitting away from the LL. However, in fact, not a finite but an infinite number of eigenvalues split away, but they converge to the LL extraordinarily fast, super-exponentially. Moreover, the leading term in the asymptotics does not depend on the perturbation at all, and the next term (as it was found recently in [5]) is determined by the geometry of the support of the perturbation. This result can be also understood as justifying the first order Raleigh-Schrödinger perturbative correction, with properly chosen initial basis in the Landau subspace. Methods of micro-local analysis do not work here, and the results are obtained by a combinations of methods of complex analysis and variational considerations.

Considerably less is known about the perturbations of the Landau Hamiltonian by a weak magnetic field. The complication here has its roots in the fact that it is not the magnetic field itself but its potential that enters in the quantum Hamiltonian. Unless the total flux of the perturbing magnetic field vanishes, the perturbation of the (Schrödinger or Pauli) operator contains rather slowly decaying coefficients, so it is fairly strong even for a compactly supported perturbation of the field and it even may be not relatively compact if the perturbation goes to zero at infinity not sufficiently fast. Iwatsuka [9] proved that invariance of the essential spectrum still takes place, so Landau levels are the only possible limit points of eigenvalues in the gaps between them. After that, there were very few results in the spectral analysis of this problem (apart from the rather specific situation with the lowest LL). In the paper [4] the perturbation by a single Aharonov-Bohm solenoid was considered. For the standard self-adjoint extension, by means of explicitly solving the eigenvalue problem, it was established that the Landau levels are still intact, the eigenspaces change in a controlled way, and a finite number of eigenvalues split away from each LL. Hempel and Levendorski in [8] and Besch in [3] investigated the behavior of the eigenvalues in the gaps when a large parameter tending to infinity stands in front of the perturbation. The number of eigenvalues between the Landau levels grows quasi-classically, as a power of the parameter, but nothing was found out about the distribution of the eigenvalues for a fixed value of the parameter. However it was shown that one should not expect monotonicity of the eigenvalues in this parameter, even if the perturbing field has constant sign.

The aim of the present paper is to find out what happens with the spectrum and spectral subspaces of the Landau Hamiltonian under smooth compactly supported perturbations of the magnetic field, with additional perturbation by a compactly supported electric field (thus both the case of the magnetic Schrödinger and of the Pauli operators are taken care of). We show that the Landau levels split superexponentially, and under certain positivity conditions find the asymptotics of eigenvalues as they approach the LL. We find also a rather exact approximation to the spectral subspaces of the perturbed operator. It turns out that the spectral subspaces are perturbed fairly strongly. This effect is absent for the perturbation by the electric field only: we show that in the latter case the spectral subspaces change very weakly, in proper terms (the behavior of spectral subspaces was not investigated in [13], [11]).

Our approach is based upon the study of approximate creation and annihilation operators. The commutation relations for these operators (the deformed Heisenberg relations) enable us to construct approximate spectral subspaces. It is known that for the unperturbed operator the Landau subspaces are obtained by repeatedly applying the creation operator to the explicitly described infinite-dimensional space of zero modes of the Pauli operator. The Pauli operator with the perturbed magnetic field possesses an infinite-dimensional space of zero modes as well, moreover, this subspace can also be explicitly described. It turns out that the application of the same procedure for the perturbed operator, but using the approximate creation operator instead, produces a very good approximation for the spectral subspaces for the perturbed operator.

In the papers [13], [11], [5], the splitting of Landau levels under the electric perturbation was associated with the spectrum of certain

Toeplitz-type operators acting in the spectral subspaces. In our case, the study of the spectrum of the perturbed operator is based upon the analysis of Toeplitz-type operators in the space of zero modes, generalized in the sense that a differential operator replaces the multiplication by a function in the Toeplitz expression.

We prove the asymptotics of the eigenvalues split from the Landau level under a certain positivity condition imposed on the *effective* potential reflecting the combined influence of the electric and magnetic perturbations upon the spectrum. If this condition is not fulfilled, we can only prove the superexponential estimate from above. The obstacle here is the lack of knowledge of the spectrum of the Toeplitz-like operators with indefinite weight in the space of zero modes, which turns out to be a separate hard problem in complex analysis.

In Section 2 we define our main operators and establish a series of relations between them. Further, in Section 3, we construct the approximate spectral subspaces. In Section 4 we prove that they form a very good approximation for the actual spectral subspaces. As a by-product, we show that the spectral subspaces 'almost' do not change if just an electric perturbation is present. Section 5 is devoted to the analysis of the Toeplitz-like operators. In Sect. 6 we establish the estimates, and when possible, the asymptotics in the Landau levels splitting. The last section contains the proofs of several rather technical formulas and estimates.

Different results involve different smoothness conditions imposed on the perturbations. The upper estimate for the eigenvalue splitting is proved for C^2 compactly supported perturbations. On the other hand, the lower estimates granting the eigenvalue asymptotics, as well as the result on the approximate spectral subspaces are proved for infinitely differentiable perturbations. These latter smoothness conditions can be replaced by some finite smoothness, depending on the Landau level under consideration, but we do not go into these details.

The conditions of the perturbations to have compact support can be replaced by the requirements of controlled decay at infinity. In that case the results on the eigenvalue splitting and on the spectral subspaces approximation can also be obtained, with the eigenvalue estimates and asymptotics as well as the quality of spectral subspaces approximation depending on the rate of decay. These resulst will be published elsewhere.

The second author was partly supported by a grant from the Royal Swedish Academy of Sciences, he also thanks Chalmers University of Technology in Gothenburg for hospitality.

2. Magnetic Schrödinger and Pauli operators

We will denote the points in the plane $\mathbb{R}^2$ by $x = (x_1, x_2)$; it is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ by setting $z = x_1 + ix_2$. The Lebesgue measure will be denoted by dx and the derivatives by $\partial_k = \partial_{x_k}$; we set, as usual, $\bar{\partial} = (\partial_1 + i\partial_2)/2,\ \partial = (\partial_1 - i\partial_2)/2$.

The constant magnetic field will be denoted by $\mathbf{B}^\circ$; we fix the sign, $\mathbf{B}^\circ > 0$. With this magnetic field we associate the magnetic potential $\mathbf{A}^\circ(x) = (A_1^\circ, A_2^\circ) = \frac{\mathbf{B}^\circ}{2}(-x_2, x_1)$. Then the (unperturbed) magnetic Schrödinger operator in $L_2(\mathbb{R}^2)$ is defined as

$$\mathfrak{H}^\circ = -(\nabla - i\mathbf{A}^\circ)^2. \tag{2.1}$$

The Pauli operator describing the motion of a spin-$\frac{1}{2}$ particle acts in the space $L_2(\mathbb{R}^2)^2$ of two-component vector functions, it has the diagonal form, $\mathfrak{P}^\circ = \operatorname{diag}(\mathfrak{P}_+^\circ, \mathfrak{P}_-^\circ)$, where $\mathfrak{P}_\pm^\circ = \mathfrak{H}^\circ \pm \mathbf{B}^\circ$.

The spectrum of the Schrödinger operator was found by Landau in 1928 (see [10]) by separation of variables. However more conveniently the spectral structure of these operators is described by means of creation and annihilation operators introduced by Fock [6] and reflecting the algebraical structure of the problem (see, for example the detailed exposition in [7] or [2]). Let us introduce the complex magnetic potential $A^\circ = A_1^\circ + iA_2^\circ$ and set

$$\mathfrak{Q}^\circ = -2i\bar{\partial} - A^\circ,\ \overline{\mathfrak{Q}^\circ} = -2i\partial - \overline{A^\circ}. \tag{2.2}$$

These operators can be also expressed by means of the scalar potential of the magnetic field, the function

$$\Psi^\circ(z) = \frac{\mathbf{B}^\circ}{4}|z|^2 \tag{2.3}$$

solving the equation $\Delta\Psi^\circ = \mathbf{B}^\circ$:

$$\overline{\mathfrak{Q}^\circ} = -2ie^{\Psi^\circ}\partial e^{-\Psi^\circ},\ \mathfrak{Q}^\circ = -2ie^{-\Psi^\circ}\bar{\partial}e^{\Psi^\circ}.$$

The creation and annihilation operators $\overline{\mathfrak{Q}^\circ}, \mathfrak{Q}^\circ$ satisfy the following basic relation

$$[\mathfrak{Q}^\circ, \overline{\mathfrak{Q}^\circ}] = 2\mathbf{B}^\circ. \tag{2.4}$$

This relation, meaning that the operators $\mathfrak{Q}^\circ, \overline{\mathfrak{Q}^\circ}, 2\mathbf{B}^\circ$ generate a representation of the Heisenberg algebra, together with the formulas

$$\mathfrak{P}_+^\circ = \mathfrak{Q}^\circ\overline{\mathfrak{Q}^\circ}, \mathfrak{P}_-^\circ = \overline{\mathfrak{Q}^\circ}\mathfrak{Q}^\circ, \mathfrak{H}^\circ = \mathfrak{Q}^\circ\overline{\mathfrak{Q}^\circ} - \mathbf{B}^\circ = \overline{\mathfrak{Q}^\circ}\mathfrak{Q}^\circ + \mathbf{B}^\circ, \tag{2.5}$$

enable one to describe the spectrum completely, and the construction is presented in numerous physical and mathematical sources. Since $\overline{\mathfrak{Q}^\circ} = \mathfrak{Q}^{\circ *}$, the equation $\mathfrak{P}_-^\circ u = 0$ is equivalent to

$$\mathfrak{Q}^\circ u = e^{-\Psi^\circ}\bar{\partial}(e^{\Psi^\circ}u) = 0.$$

This means that $f = e^{\Psi^\circ}u$ is an entire analytical function in $\mathbb{C}$, such that after multiplication by $e^{-\Psi^\circ}$ it belongs to L_2. The space of such functions f is called Fock or Segal-Bargmann space $\mathcal{F}$ (see [1], [2], [7] for extensive discussion); when we need to specify the particular weight involved in the definition of $\mathcal{F}$ we will use the notation $\mathcal{F}_{\mathbf{B}^\circ}$. So, the null subspace of the operator $\mathfrak{P}_-^\circ$, i.e., its spectral subspace corresponding to the eigenvalue $\Lambda_0 = 0$, is $\mathcal{L}_0 = e^{-\Psi^\circ}\mathcal{F}$. After this, the commutation relations (2.4), (2.5) imply that $\Lambda_1 = 2\mathbf{B}^\circ$ is the lowest eigenvalue of the operator $\mathfrak{P}_+^\circ = \mathfrak{P}_-^\circ + 2\mathbf{B}^\circ$, with the same spectral subspace, and thus $\mathcal{L}_1 = \overline{\mathfrak{Q}^\circ}\mathcal{L}_0$ is the spectral subspace of $\mathfrak{P}_-^\circ$ with eigenvalue Λ_1 and so on. So, the spectrum of $\mathfrak{P}_-^\circ$ consists of eigenvalues, *Landau levels* $\Lambda_q = 2q\mathbf{B}^\circ$, $q = 0, 1, \dots$, and the spectra of $\mathfrak{H}^\circ$, $\mathfrak{P}_+^\circ$ consist, respectively, of $\Lambda_q + \mathbf{B}^\circ$ and $\Lambda_q + 2\mathbf{B}^\circ$. The operators $\overline{\mathfrak{Q}^\circ}, \mathfrak{Q}^\circ$ act between *Landau subspaces* $\mathcal{L}_q = \overline{\mathfrak{Q}^\circ}^q\mathcal{L}_0, q = 0, 1, \dots,$

$$\overline{\mathfrak{Q}^\circ} : \mathcal{L}_q \mapsto \mathcal{L}_{q+1},\ \mathfrak{Q}^\circ : \mathcal{L}_q \mapsto \mathcal{L}_{q-1},\ \mathfrak{Q}^\circ : \mathcal{L}_0 \mapsto 0, \tag{2.6}$$

and are, up to constant factors, isometries of Landau subspaces.

It is also convenient to introduce the spectral projection P_q° of $\mathfrak{P}_-^\circ$ corresponding to the eigenvalue $\Lambda_q = 2q\mathbf{B}^\circ$. The above reasoning can be expressed as

$$P_q^\circ = C_q^{-1}\overline{\mathfrak{Q}^\circ}^q P_0^\circ \mathfrak{Q}^{\circ q},\ C_q = q!(2\mathbf{B}^\circ)^q. \tag{2.7}$$

Now we introduce the perturbation $\mathbf{b} \in C_0^\infty(\mathbb{R}^2)$ of the magnetic field, and we set $\mathbf{B} = \mathbf{B}^\circ + \mathbf{b}$. Let ψ be a scalar potential for the field $\mathbf{b}$, a solution of the equation $\Delta\psi = \mathbf{b}$. Of course, ψ is defined up to a harmonic summand, the choice of ψ corresponds to the choice of gauge. The magnetic potential $\mathbf{a} = (a_1, a_2) = (-\partial_2\psi, \partial_1\psi)$, $\operatorname{curl}\mathbf{a} = \mathbf{b}$, is thus determined up to a gradient, and the complete scalar and vector magnetic potentials are

$$\Psi = \Psi^\circ + \psi,\ \mathbf{A} = \mathbf{A}^\circ + \mathbf{a}. \tag{2.8}$$

The crucial observation here is that although $\mathbf{b}$ has compact support, we should not expect the same, or even a rapid decay, from the magnetic potential $\mathbf{a}$, unless the total flux of $\mathbf{b}$, $\int \mathbf{b}(x)dx$ is zero. This can be seen in the easiest way by applying Green's formula, saying that the circulation of the vector field $\mathbf{a}$ along the circle with radius $\mathbf{R}$ equals the flux of $\mathbf{b} = \operatorname{curl}\mathbf{a}$ through the disk $|x| < \mathbf{R}$. Thus, if the total flux of $\mathbf{b}$ is nonzero and $\mathbf{R}$ is taken large enough, so that the support of $\mathbf{b}$ lies in the disk, the circulation of $\mathbf{a}$ is bounded away from zero, and thus $\mathbf{a}$ may not decay faster than $C\mathbf{R}^{-1}$, at least in some directions. This corresponds to a logarithmical growth rate of ψ.

We define the perturbed magnetic Schrödinger operator as in (2.1), with $\mathbf{A}^\circ$ replaced by $\mathbf{A}$:

$$\mathfrak{H} = -(\nabla - i\mathbf{A})^2,$$

and the components of Pauli operator as

$$\mathfrak{P}_\pm = \mathfrak{H} \pm \mathbf{B}.$$

It is easy to observe that the difference between $\mathfrak{H}$ and $\mathfrak{H}^\circ$ contains an operator of multiplication by $\mathbf{A}^\circ \cdot \mathbf{a}$, the latter function does not decay at infinity (again unless the total flux of $\mathbf{b}$ vanishes), and thus looks like being *not* a relatively compact perturbation of $\mathfrak{H}^\circ$. However, thanks to the special form of this term, the perturbation is still relatively compact if $\mathbf{a} \to 0$ at infinity, as was noticed by Besch [3].

Now let us look at the algebraic structure related with the perturbed operators. The perturbed creation and annihilation operators are defined similarly to (2.2):

$$\overline{\mathfrak{Q}} = -2i\partial - \overline{A},\ \mathfrak{Q} = -2i\bar{\partial} - A,$$

where A is the complex magnetic potential, $A = A^\circ + (a_1 + ia_2)$. The representation via the scalar potential takes the form

$$\overline{\mathfrak{Q}} = -2ie^{\Psi}\partial e^{-\Psi},\ \mathfrak{Q} = -2ie^{-\Psi}\bar{\partial}e^{\Psi}.$$

Most important for us are the commutation relations for $\overline{\mathfrak{Q}}, \mathfrak{Q}$:

$$[\mathfrak{Q}, \overline{\mathfrak{Q}}] = 2\mathbf{B} = 2\mathbf{B}^\circ + 2\mathbf{b}, \tag{2.9}$$

together with

$$\mathfrak{P}_+ = \mathfrak{Q}\overline{\mathfrak{Q}},\ \mathfrak{P}_- = \overline{\mathfrak{Q}}\mathfrak{Q},\ \mathfrak{P}_+ - \mathfrak{P}_- = 2\mathbf{B} = 2\mathbf{B}^\circ + 2\mathbf{b}, \tag{2.10}$$

$$\mathfrak{H} = \mathfrak{Q}\overline{\mathfrak{Q}} - \mathbf{B} = \overline{\mathfrak{Q}}\mathfrak{Q} + \mathbf{B}. \tag{2.11}$$

The relations (2.9)-(2.11) look similar to (2.4), (2.5), however the crucial difference is that the new ones contain variable functions on the right-hand side, do not represent the Heisenberg algebra any more and thus cannot be used for the exact spectral analysis. The spectra of Schrödinger and Pauli operators do not determine each other any more. The only information that one can obtain immediately, is the description of the lowest point of the spectrum of $\mathfrak{P}_-$. Since, again, $\overline{\mathfrak{Q}} = \mathfrak{Q}^*$, the equation $\mathfrak{P}_- u = 0$ is equivalent to $\mathfrak{Q}u = 0$, or $\bar{\partial}(\exp(\Psi)u) = 0$. So the function $f = u \exp \Psi$ is an entire analytical function such that $u = \exp(-\Psi)f \in L_2$. The space of entire functions with this property is, obviously, infinite-dimensional, it contains at least all polynomials of z, although it does not necessarily coincide with Fock space, since $\exp(-\psi)$ may have power growth (or decay). We denote this space by $\mathcal{F}^{\mathbf{b}}$.

The same reasoning, applied to $\mathfrak{P}_+$, shows that the null subspace of $\overline{\mathfrak{Q}}$, as well as the one of $\mathfrak{P}_+$ is zero: it consists of anti-analytical functions which belong to L_2 after the multiplication by the growing exponent $\exp(\Psi)$, and there are no such functions, except zero.

We denote the null-space of $\mathfrak{P}_-$, the space of *zero modes*, by $\mathcal{H}_0$. In [14] rather general sufficient conditions were found for infiniteness of dimension of $\mathcal{H}_0$; in particular they even do not require the magnetic field to be a weak perturbation of the constant one.

As it follows from the relative compactness of the perturbation, by Weyl's theorem, the essential spectrum of the perturbed operator $\mathfrak{P}_-$ consists of the same Landau levels Λ_q, and the eigenvalues in the gaps and below Λ_0 may only have Λ_q as their limit points. This latter fact was established much earlier by Iwatsuka [9], and the reasoning based upon the commutation relations remains valid even in the case of a slowly decaying magnetic perturbation, when the perturbation of the operator is not relatively compact any more. Since our considerations in the paper can be considered as an extension of the ones of Iwatsuka, we reproduce the reasoning from [9].

Theorem 2.1. *([9]) The essential spectrum of the operator* $\mathfrak{P}_-$ *consists of the points* Λ_q, $q = 0, 1, \ldots$. *The essential spectra of* $\mathfrak{H}$ *and* $\mathfrak{P}_+$ *consist, respectively, of the points* $\Lambda_q + \mathbf{B}^\circ$ *and* $\Lambda_q + 2\mathbf{B}^\circ$.

Proof. Note first that the operator of multiplication by $\mathbf{b}$ is relatively compact with respect to $\mathfrak{H}$ and $\mathfrak{P}_\pm$. Thus, by Weyl theorem, the operators $\mathfrak{P}_+$ and $\mathfrak{P}_- + 2\mathbf{B}^\circ = \mathfrak{P}_+ - 2\mathbf{b}$ have the same essential spectrum, i.e. the essential spectrum of $\mathfrak{P}_+$ is just the one of $\mathfrak{P}_-$, shifted by $2\mathbf{B}^\circ$. So, $\mathfrak{P}_+$ has no essential spectrum in $[0, 2\mathbf{B}^\circ)$. On the other hand, the nonzero spectra of $\mathfrak{P}_+$ and $\mathfrak{P}_-$ coincide, since these operators are just products of $\overline{\mathfrak{Q}}$ and $\mathfrak{Q}$ in the opposite order. Therefore $\mathfrak{P}_-$ has no essential spectrum in $(0, 2\mathbf{B}^\circ)$, and $2\mathbf{B}^\circ$ belongs to the essential spectrum. Again, this means that the interval $(2\mathbf{B}^\circ, 4\mathbf{B}^\circ)$ is free from the essential spectrum of $\mathfrak{P}_+$ and $4\mathbf{B}^\circ$ belongs to it. By repeating this reasoning, the complete description of the essential spectrum is obtained. $\square$

Now we add a perturbation by the electric potential. Let $V(x)$ be a real valued function in $C_0^\infty(\mathbb{R}^2)$. We introduce the operators

$$\mathfrak{H}(V) = \mathfrak{H} + V, \ \mathfrak{P}_\pm(V) = \mathfrak{P}_\pm + V. \tag{2.12}$$

Since the operator of multiplication by V is relatively compact with respect to $\mathfrak{H}, \mathfrak{P}_\pm$, the operators (2.12) have the same essential spectra as the respective unperturbed ones. In this paper we are going to study the behavior of the eigenvalues of $\mathfrak{H}(V), \mathfrak{P}_\pm(V)$ near these Landau levels.

3. Approximate spectral subspaces

In this section we construct the approximate spectral subspaces of the operators (2.12) under the condition $\mathbf{b} \in C_0^\infty$. To do this, we need some more properties of the space $\mathcal{H}_0$ of zero modes of $\mathfrak{P}_-$. We must keep in mind that the ambiguity in associating the vector potential with the magnetic field leads to the ambiguity in defining the magnetic operators themselves. The gauge transformations of the form $u \mapsto e^{i\Theta} u$ with real functions $\Theta(z)$ such that $e^{i\Theta}$ is smooth, realize the unitary equivalence of magnetic operators constructed by different choice of the potential.

As it was explained in Section 2, the space $\mathcal{H}_0$ consists of functions $u(z)$ having the form $u(z) = e^{-\Psi(z)} f(z)$ where $f(z)$ is an analytical entire function such that $e^{-\Psi} f \in L_2$, and $\Psi(z)$ is the potential in (2.8). Note first that the choice of Ψ does not affect $\mathcal{H}_0$, up to a gauge transformation. In fact, let Φ be a harmonic function in $\mathbb{R}^2$, so that $\Psi' = \Psi + \Phi$ is also a potential for the magnetic field $\mathbf{B} = \mathbf{B}^\circ + \mathbf{b}$,

$\Delta\Psi' = \mathbf{B}$. Denote by Θ the function harmonic conjugate to Φ, so that $F = \Phi + i\Theta$ is an entire analytical function. This gives us the representation

$$u(z) = e^{-i\Theta(z)}e^{-\Psi'(z)}(e^{F(z)}f(z)),$$

so the multiplication by $e^{i\Theta(z)}$ is the gauge transformation in question.

However the perturbation of the magnetic field leads to the perturbation of the space of zero modes that cannot be reduced to a gauge transformation. In fact, if there were a real function $\Theta(z)$ such that $e^{i\Theta}\mathcal{H}_0 = \mathcal{L}_0$, this would mean that the function $e^{-\psi+i\Theta}$ is analytical. However in this case the function $-\psi$, as the logarithm of the absolute value of an analytical function, must be harmonic, which it is not.

Nevertheless there is a close relation between $\mathcal{H}_0$ and $\mathcal{L}_0$, not reduced to a gauge transform. We fix the potential ψ, defining it as the logarithmic potential of the field b,

$$\psi(x) = (2\pi)^{-1}\int \mathbf{b}(y)\log|x-y|dy. \tag{3.1}$$

The behavior of the potential ψ at infinity is determined by σ, the total flux of the perturbing field divided by 2π,

$$\sigma = (2\pi)^{-1}\int \mathbf{b}(x)dx.$$

If $\sigma = 0$, ψ is a bounded function, as well as $e^{\pm\psi}$. So, the multiplication by the bounded function $m = e^{-\psi}$ maps $\mathcal{L}_0$ to $\mathcal{H}_0$. This mapping is, obviously, invertible.

Now suppose that $\sigma > 0$. Then the function $\psi(x)$ grows at infinity as $\sigma\log|x|$, $m(x) = e^{\psi(x)}$ grows as $|x|^\sigma$ and it is not bounded any more. The function $e^{-\psi(x)}$ is bounded. Thus the multiplication by m^{-1} is a bounded operator from $\mathcal{H}_0$ to $\mathcal{L}_0$. The inverse mapping is not bounded, but we show that, nevertheless, it is well defined on a certain subspace in $\mathcal{L}_0$. Denote by ν the smallest integer larger than σ. Fix ν arbitrary different points $z_1, \dots, z_\nu$ in $\mathbb{C}$. Denote by $p(z)$ the monic polynomial with simple zeros at the points $z_1, \dots, z_\nu$ and by $\tilde{\mathcal{L}}_0$ the space of functions in $\mathcal{L}_0$ having zeros at these points. Since the evaluation at a point is a bounded functional in the Fock space, $\tilde{\mathcal{L}}_0$ is a closed subspace in $\mathcal{L}_0$, with codimension ν. Now consider the space $\mathcal{L}_0^\nu = p(z)^{-1}\tilde{\mathcal{L}}_0$. By our construction, this space is still a part of $\mathcal{L}_0$ (however is not a closed subspace any more). Since at infinity we have $e^{\psi(z)} = O(p(z)^{-1})$, the operator of multiplication by $e^{\psi(z)}$ maps $\mathcal{L}_0^\nu$ to $\mathcal{H}_0$. In the case of a negative σ, the spaces $\mathcal{H}_0$ and $\mathcal{L}_0$ change places in the above reasoning.

The space $\mathcal{H}_0$ is closely related to the spaces $\mathfrak{F}^\rho$ introduced in [1], Section 3, consisting of entire analytical functions $f(z)$ that belong to L_2 after being multiplied by $\theta_\rho(z) = (1+|z|^2)^{-\rho/2}\exp(-(\mathbf{B}^\circ/4)|z|^2)$. Due to (3.1), the function $\xi(z) = \psi(z) - \frac{\sigma}{2}\log(1+|z|^2)$ is bounded, so we can write

$$e^{-\Psi(z)} = \theta_{-\sigma}(z)e^{-\xi(z)}.$$

Therefore our space $\mathcal{H}_0$ can be obtained from Bargmann's space $\theta_{-\sigma}(z)\mathfrak{F}^\sigma$ by multiplication by the bounded and boundedly invertible function $e^{-\xi(z)}$. This means that many properties of this Bargmann space are inherited by our space $\mathcal{H}_0$. What is important for us at the moment is the fact established in [1], Section 3.6, that the functions of the form $\theta_\sigma(z)z^n$ form a basis in $\theta_\sigma(z)\mathfrak{F}^\sigma$. Therefore this space possesses a dense set consisting of rapidly decaying functions (having the form $\theta_{-\sigma}(z)p(z)$, with polynomial p), and thus our null subspace $\mathcal{H}_0$ possesses a dense subspace of rapidly decaying functions as well.

Let $\delta_q = (\Lambda_q - \gamma, \Lambda_q + \gamma), q = 0, 1, 2, \dots,$ be intervals of the same size on the real axis, centered at the Landau levels $\Lambda_q = 2q\mathbf{B}^\circ$, and not containing the neighboring Landau levels $\Lambda_{q\pm1}$. We choose the size of δ_q in such way that neither of these intervals has the eigenvalues of $\mathfrak{P}_-$ at its endpoints. Moreover, since the lowest LL $\Lambda_0 = 0$ is an isolated point of the spectrum of $\mathfrak{P}_-$, we can choose the size of the intervals in such way that δ_0 contains only this point of spectrum. We denote by $\mathcal{H}_q$ the spectral subspace of $\mathfrak{P}_-$ corresponding to the interval δ_q and by P_q the corresponding spectral projection. Since, by Theorem 2.1, the spectrum of $\mathfrak{P}_-$ is discrete between Landau levels, the change of δ_q leads only to a finite-rank perturbation of P_q. As usual, the spectral projection P_q can be be expressed by means of the integration of the resolvent of $\mathfrak{P}_-$ along a closed contour Γ_q in the complex plane, not passing through the eigenvalues of $\mathfrak{P}_-$ and containing inside only those eigenvalues that lie in δ_q. Again, using the discreteness of the spectrum of $\mathfrak{P}_-$ between the Landau levels, we can choose these contours in such way that all they are obtained from Γ_0 by the shift along the real axis in the complex spectral plane, $\Gamma_q = \Gamma_0 + 2q\mathbf{B}^\circ$.

Now we are going to establish several properties of the subspaces $\mathcal{L}_q$, projections P_q and some related operators.

First, note the simple fact following directly from the spectral theorem.

Proposition 3.1. *For any $q = 0, 1, \dots,$ and any polynomial $p(\lambda)$ the operator $p(\mathfrak{P}_-)P_q$ is bounded, moreover the operator $(p(\mathfrak{P}_-) - p(\Lambda_q))P_q$ is compact.*

In fact, by the spectral theorem the nonzero spectrum of the operator $p(\mathfrak{P}_-)P_q$ consists of the points $p(\lambda_j)$ where λ_j are all points of spectrum of $\mathfrak{P}_-$ in δ_q and thus all $p(\lambda_j)$ live in a bounded interval. Moreover, $p(\lambda_j)$ may only have $p(\Lambda_q)$ as their limit point, by Theorem 2.1.

The following lemma will enable us later to establish a much stronger compactness property.

Lemma 3.2. *Let each of T_j, $j = 1, \dots, N$ be one of operators $\mathfrak{Q}$ or $\overline{\mathfrak{Q}}$. Then for some constants C, C', for any u in the domain of the operator*

$\mathfrak{P}_-^N$,

$$\|T_1T_2\dots T_Nu\|^2 \le C(\mathfrak{P}_-^N u, u) + C'\|u\|^2. \tag{3.2}$$

The proof of Lemma 3.2 can be found in Section 7.

Now we can establish the compactness property.

Proposition 3.3. *Let $T_1, \dots, T_N$ be a collection of operators, each being $\overline{\mathfrak{Q}}$ or $\mathfrak{Q}$, and let h_j, $j = 0, \dots N$ be functions with all derivatives bounded, $\mathcal{T} = h_0T_1h_1 \dots T_Nh_N$. Then for any q and for any polynomial $p(\lambda)$ the operators $\mathcal{T}(p(\mathfrak{P}_-) - p(\Lambda_q))P_q\mathcal{T}$ are compact.*

Proof. By commuting functions h_j and operators T_j (moving all functions to the left), we can transform the left operator $\mathcal{T}$ to the sum of terms having the form $\tilde{h}_\kappa\mathcal{T}_\kappa$ where $\tilde{h}_\kappa$ are bounded functions and $\mathcal{T}_\kappa$ is a product of no more than N operators $\overline{\mathfrak{Q}}, \mathfrak{Q}$. Similarly, in the copy of $\mathcal{T}$ that stands to the right from P_q, we move all functions to the utmost right positions, to get the representation of $\mathcal{T}$ as a sum of terms $\mathcal{T}_\varkappa\tilde{h}_\varkappa$, being the product of a bounded smooth function $\tilde{h}_\varkappa$ and no more than N operators $\overline{\mathfrak{Q}}, \mathfrak{Q}$.

For each of the terms arising in this way in the decomposition of $\mathcal{T}(p(\mathfrak{P}_-) - p(\Lambda_q))P_q\mathcal{T}$, we can write

$$\tilde{h}_\kappa\mathcal{T}_\kappa(p(\mathfrak{P}_-) - p(\Lambda_q))P_q\mathcal{T}_\varkappa\tilde{h}_\varkappa = [\tilde{h}_\kappa\mathcal{T}_\kappa(\mathfrak{P}_-^N + 1)^{-1}]\times \tag{3.3}$$
$$[(\mathfrak{P}_-^N + 1)(p(\mathfrak{P}_-) - p(\Lambda_q))P_q(\mathfrak{P}_-^N + 1)][(\mathfrak{P}_-^N + 1)^{-1}\mathcal{T}_\varkappa\tilde{h}_\varkappa].$$

In (3.3), the first factor in brackets is bounded by Lemma 3.2, and the middle factor is compact by Proposition 3.1. The last factor in brackets is also bounded, by Lemma 3.2 applied to the adjoint operator. $\square$

The result we have just proved indicates that, probably, the eigenvalues of the operator $\mathfrak{P}_-$, lying near Λ_q, must converge to Λ_q rather rapidly. This will be, in fact, established later on.

Now we are going to describe the main construction of the paper, the approximate spectral subspaces of the perturbed operator. Note, first of all, that it is sufficient to consider the operator $\mathfrak{P}_-(V)$. In fact, due to the commutation relations, we have

$$\mathfrak{H}(V) = \mathfrak{P}_-(V + \mathbf{b}) + \mathbf{B}^\circ, \mathfrak{P}_+(V) = \mathfrak{P}_-(V + 2\mathbf{b}) + 2\mathbf{B}^\circ,$$

and thus the operators differ from $\mathfrak{P}_-$ by a shift and by the electric type perturbations $V + \mathbf{b}$, $V + 2\mathbf{b}$. We start by finding approximate spectral subspaces of $\mathfrak{P}_-$. Adding V will be an easier task after this.

The subspaces approximating $\mathcal{H}_q$ will be defined in the following way:

$$\mathcal{G}_0 = \mathcal{H}_0,\ \mathcal{G}_q = \overline{\mathfrak{Q}}^q\mathcal{G}_0,\ q = 1, 2, \dots. \tag{3.4}$$

So we mimic the construction of the eigenspaces of the unperturbed Landau Hamiltonian, see (2.6). Our approximate spectral subspaces are obtained by applying the creation operators to the space of zero modes.

Of course, since we apply the unbounded operator $\overline{\mathfrak{Q}}$, we must show that all the time we do not leave the space L_2, and moreover, we have to prove that the subspaces $\mathcal{G}_q$ are closed. Both these properties, as well as some other results of the paper will be based upon the important Proposition 3.4 to follow. But first we need a definition.

Many times in the paper we will encounter operators that are products of several copies of $\mathfrak{Q}, \overline{\mathfrak{Q}}$, functions $\mathbf{B}, \mathbf{b}, U$ (for some smooth function U) and their derivatives. Let us assign the weight 2 to $\mathbf{B}^\circ, \mathbf{B}, \mathbf{b}, U$, the weight $l+2$ to the order l derivatives of $\mathbf{b}, U$ and the weight 1 to $\mathfrak{Q}$ and to $\overline{\mathfrak{Q}}$. For a product of operators and functions the weights of factors are added.

Proposition 3.4. *Let $q > 0$.*

(1) *There exists a function $\mathcal{Z}_q[\mathbf{b}] \in C_0^\infty$ depending only on q, $\mathbf{B}^\circ$, and $\mathbf{b}$ such that for any $u \in \mathcal{H}_0$,*

$$\|\overline{\mathfrak{Q}}^q u\|^2 = C_q\|u\|^2 + (\mathcal{Z}_q[\mathbf{b}]u, u),\ C_q = q!(2\mathbf{B}^\circ)^q. \tag{3.5}$$

The function $\mathcal{Z}_q[\mathbf{b}]$ is a polynomial of $\mathbf{b}$ and its derivatives up to the order $2q-2$ with coefficients depending on $\mathbf{B}^\circ$. All summands in $\mathcal{Z}_q[\mathbf{b}]$ have weight $2q$. The term with the highest derivative equals $2\Delta^q\mathbf{b}$; the term linear in $\mathbf{b}$ and not containing derivatives equals $C'_q\mathbf{B}^{\circ q-1}\mathbf{b}$, $C'_q = 2^q q! q$.

(2) *There exists a function $\mathcal{X}_q[\mathbf{b}, U] \in C_0^\infty$ depending only on q, $\mathbf{B}^\circ$, $\mathbf{b}$, and U such that for any $u \in \mathcal{H}_0$.*

$$(U\overline{\mathfrak{Q}}^q u, \overline{\mathfrak{Q}}^q u) = (\mathcal{X}_q[\mathbf{b}, U]u, u). \tag{3.6}$$

The function $\mathcal{X}_q[\mathbf{b}, U]$ is expressible as an order $2q$ linear differential operator acting on U, with coefficients depending polynomially on $\mathbf{b}$, its derivatives, and $\mathbf{B}^\circ$. All the terms in $\mathcal{X}_q[\mathbf{b}, U]$ have weight $2q+2$. The term with the highest derivative of U is $\Delta^q U$. The term with zero order derivative of U is $C_qU + \mathcal{Z}_q[\mathbf{b}]U$.

(3) *There exists a function $\mathcal{Y}_q[\mathbf{b}, U] \in C_0^\infty$ depending only on q, $\mathbf{B}^\circ$, $\mathbf{b}$, and U such that for any $u \in \mathcal{H}_0$.*

$$(U\overline{\mathfrak{Q}}^q u, \overline{\mathfrak{Q}}^{q+1} u) = (\mathcal{Y}_q[\mathbf{b}, U]u, u). \tag{3.7}$$

The function $\mathcal{Y}_q[\mathbf{b}, U]$ is expressible as an order $2q+1$ linear differential operator acting on U, with coefficients depending polynomially on $\mathbf{b}$, its derivatives, and $\mathbf{B}^\circ$. All the terms in $\mathcal{Y}_q[\mathbf{b}, U]$ have weight $2q+3$. The term with the highest derivative of U is $(-2i)\Delta^q\bar{\partial}U$. The term with zero order derivative of U is $C_qU + \mathcal{Z}_q[\mathbf{b}]U$.

Proof. First, in order to make the main mechanism of the proof more clear, we derive the relation (3.5) for $q = 1$ and $q = 2$. It is sufficient to prove (3.5) for $u = \exp(-\Psi)p(z)$ where $p(z)$ is a polynomial: such functions are dense in $\mathcal{H}_0$, and then we can pass to general u by the closure. In the process of our, rather lengthy, calculations we will repeatedly use the commutation relations (2.9) as well as

$$[\mathfrak{Q}, h] = -2i\bar{\partial}h, [\overline{\mathfrak{Q}}, h] = -2i\partial h, \text{for } h \in C^\infty. \tag{3.8}$$

So, let us consider the expression $\|\overline{\mathfrak{Q}}u\|^2$, $u \in \mathcal{H}_0$. We have

$$\begin{aligned} \|\overline{\mathfrak{Q}}u\|^2 &= (\overline{\mathfrak{Q}}u, \overline{\mathfrak{Q}}u) = (\mathfrak{Q}\overline{\mathfrak{Q}}u, u) \\ &= (2\mathbf{B}^\circ + 2\mathbf{b} + \overline{\mathfrak{Q}}\mathfrak{Q}u, u) = 2\mathbf{B}^\circ\|u\|^2 + (2\mathbf{b}u, u) \end{aligned} \tag{3.9}$$

(on the last step we used that $\mathfrak{Q}u = 0$). Thus we have $\mathcal{Z}_1[\mathbf{b}] = 2\mathbf{b}$.

The case $q = 2$ requires some more calculations. We start with

$$\|\overline{\mathfrak{Q}}^2u\|^2 = (\mathfrak{Q}^2\overline{\mathfrak{Q}}^2u, u).$$

Then we are going to commute $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$ so that only $\mathfrak{Q}$, and not $\overline{\mathfrak{Q}}$ would act on u, so we will move all copies of $\overline{\mathfrak{Q}}$ to the left and all copies of $\mathfrak{Q}$ to the right:

$$\begin{aligned} (\mathfrak{Q}^2\overline{\mathfrak{Q}}^2u, u) = (\mathfrak{Q}(2\mathbf{B}^\circ + 2\mathbf{b} + \overline{\mathfrak{Q}}\mathfrak{Q})\overline{\mathfrak{Q}}u, u) = 2\mathbf{B}^\circ\|\overline{\mathfrak{Q}}u\|^2 \\ +2(\mathfrak{Q}\mathbf{b}\overline{\mathfrak{Q}}u, u) + (\mathfrak{Q}\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}u, u). \end{aligned} \tag{3.10}$$

The first term in (3.10) has already the form we aim for, by (3.9). In the second term we commute $\mathfrak{Q}$ and $\mathbf{b}$ using (3.8):

$$2(\mathfrak{Q}\mathbf{b}\overline{\mathfrak{Q}}u, u) = 2(\mathbf{b}\mathfrak{Q}\overline{\mathfrak{Q}}u, u) + 2(-2i(\bar{\partial}\mathbf{b})\overline{\mathfrak{Q}}u, u). \tag{3.11}$$

In the first term in (3.11) we commute $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$, and in the second one commute $-2i(\bar{\partial}\mathbf{b})$ and $\overline{\mathfrak{Q}}$:

$$\begin{aligned} 2(\mathfrak{Q}\mathbf{b}\overline{\mathfrak{Q}}u, u) = 4(\mathbf{b}(\mathbf{B}^\circ + \mathbf{b})u, u)+ \\ 2(\mathbf{b}\overline{\mathfrak{Q}}\mathfrak{Q}u, u) + 2(\overline{\mathfrak{Q}}(-2i\bar{\partial}\mathbf{b})u, u) + (8(\partial\bar{\partial}\mathbf{b})u, u). \end{aligned}$$

Since $\mathfrak{Q}u = 0$, we arrive at

$$2(\mathfrak{Q}\mathbf{b}\overline{\mathfrak{Q}}u, u) = 4(\mathbf{b}(\mathbf{B}^\circ + \mathbf{b})u, u) + 2((\Delta\mathbf{b})u, u).$$

In the last term in (3.10) we commute the first pair $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$ and then the second one:

$$\begin{aligned} (\mathfrak{Q}\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}u, u) &= (\overline{\mathfrak{Q}}\mathfrak{Q}^2\overline{\mathfrak{Q}}u, u) + 2\mathbf{B}^\circ(\mathfrak{Q}\overline{\mathfrak{Q}}u, u) + 2(\mathbf{b}\mathfrak{Q}\overline{\mathfrak{Q}}u, u) \\ &= 2\mathbf{B}^\circ(\overline{\mathfrak{Q}}u, \overline{\mathfrak{Q}}u) + 4(\mathbf{b}(\mathbf{B}^\circ + \mathbf{b})u, u) + 2(\mathbf{b}\overline{\mathfrak{Q}}\mathfrak{Q}u, u). \end{aligned} \tag{3.12}$$

On the right-hand side of (3.12), the first term is again calculated by (3.9), the second term has the required form, and the last one vanishes. Now we collect all the transformed terms in (3.10) and obtain

$$\begin{aligned} \|\overline{\mathfrak{Q}}^2u\|^2 = 2\mathbf{B}^\circ(2\mathbf{B}^\circ\|u\|^2 + (2\mathbf{b}u, u)) + 4(\mathbf{b}(\mathbf{B}^\circ + \mathbf{b})u, u)+ \\ 2(\Delta\mathbf{b}u, u) + 2\mathbf{B}^\circ(2\mathbf{B}^\circ\|u\|^2 + (2\mathbf{b}u, u)) + 4(\mathbf{b}(\mathbf{B}^\circ + \mathbf{b})u, u) \\ = 2(2\mathbf{B}^\circ)^2\|u\|^2 + (\mathcal{Z}_2[\mathbf{b}]u, u). \end{aligned}$$

with

$$\mathcal{Z}_2[\mathbf{b}] = 8\mathbf{b}^2 + 12\mathbf{B}^\circ\mathbf{b} + 2\Delta\mathbf{b}.$$

Now we can formally describe how the procedure goes in the general case. In the expression $\mathfrak{Q}^q\overline{\mathfrak{Q}}^q$ we start commuting $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$, as well as these operators with the functions arising by the previous commutings. All the time we keep moving $\mathfrak{Q}$ to the right and $\overline{\mathfrak{Q}}$ to the left. As soon as $\mathfrak{Q}$ reaches the utmost right position or $\overline{\mathfrak{Q}}$ reaches the utmost left position, the corresponding term vanishes. We do this until no operators $\mathfrak{Q}, \overline{\mathfrak{Q}}$ are left, and only a function $C_q + \mathcal{Z}_q[\mathbf{b}]$ remains. To find C_q and describe the structure of $\mathcal{Z}_q[\mathbf{b}]$, we need only to do some bookkeeping of the terms arising when the operators and/or functions are commuted. The formulas (2.9), (3.8) show that the weight is preserved under commutation. So the constant term C_q has to be proportional to $\mathbf{B}^{\circ q}$, and it can arise only when the operators $\mathfrak{Q}, \overline{\mathfrak{Q}}$ are commuted, not the operators with functions. The number of such commutings needed to re-arrange all terms equals $q!$. The term linear in $\mathbf{b}$ and not containing derivatives results from the same procedure and has the form $C_q'\mathbf{B}^{\circ q-1}\mathbf{b}$, $C_q' = 2^n q!q$. The function $\mathcal{Z}_q[\mathbf{b}]$ arising as a result of moving all $\mathfrak{Q}$-s to the right and all $\overline{\mathfrak{Q}}$-s to the left will be a polynomial of $\mathbf{b}$ and its derivatives with weight $2q$. The highest derivative of $\mathbf{b}$ present in $\mathcal{Z}_q[\mathbf{b}]$ will thus be of order $2q-2$, and this term will be linear in $\mathbf{b}$. Such term arises when $2\mathbf{B}$ appears after the first commutation of $\overline{\mathfrak{Q}}, \mathfrak{Q}$ in $\mathfrak{Q}^q\overline{\mathfrak{Q}}^q$,

$$\mathfrak{Q}^q\overline{\mathfrak{Q}}^q = \mathfrak{Q}^{q-1}\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}^q + \mathfrak{Q}^{q-1}(2\mathbf{B}^\circ + 2\mathbf{b})\mathfrak{Q}^{q-1},$$

and then $q-1$ copies of $\mathfrak{Q}$ pass $2\mathbf{b}$ on their way to the right and $q-1$ copies of $\overline{\mathfrak{Q}}$ pass it on their way to the left, each such pair producing the Laplacian acting on $\mathbf{b}$.

The above reasoning takes care of (3.5). Parts 2) and 3) of the proposition are proved in the same way, with a similar bookkeeping in order to determine the structure of the resulting functions. We just demonstrate how Part 2) works for $q = 1$:

$$(\mathfrak{Q}U\overline{\mathfrak{Q}}u, u) = (\mathfrak{Q}\overline{\mathfrak{Q}}Uu, u) + (\mathfrak{Q}[U, \overline{\mathfrak{Q}}]u, u) = (\overline{\mathfrak{Q}}\mathfrak{Q}Uu, u)(= 0)+$$
$$(2\mathbf{B}Uu, u) + ([U, \overline{\mathfrak{Q}}]\mathfrak{Q}u, u)(= 0) + ([\mathfrak{Q}, [U, \overline{\mathfrak{Q}}]u, u) = (\mathcal{X}_1[\mathbf{b}, U]u, u);$$
$$\mathcal{X}_1[\mathbf{b}, U] = 2\mathbf{B}U + \Delta U. \qquad \square$$

When applying Proposition 3.4 and similar results, we need a certain compactness property. Such facts were used persistently in [13], [11], [5], but for the case of a constant magnetic field only.

Lemma 3.5. *Let $W(x)$ be a bounded function with compact support. Then for any differential operator $\mathfrak{L}$ with bounded coefficients the quadratic form*

$$\mathfrak{w}[u] = \int W(x)|\mathfrak{L}u|^2 dx$$

is compact in the space $\mathcal{H}_0$.

Remark 3.6. In section 5 we will find much more detailed information about such forms. However the proof we give here is also useful. We return to this approach later.

Proof. Let $u = e^{-\Psi} f$ be a function in $\mathcal{H}_0$, so $f(z)$ is an analytical function.

Take some $\mathbf{R}$ such that the support of the function W lies inside the circle $C_{\mathbf{R}}$ with radius $\mathbf{R}$ centered in the origin. For each $r \in (\mathbf{R}, 2\mathbf{R})$ we write the Cauchy representation for f:

$$f(z) = (2\pi i)^{-1} \int_{C_r} f(\zeta)(z-\zeta)^{-1} d\zeta. \tag{3.13}$$

Then for some fixed function $\xi(r) \in C_0^\infty(\mathbf{R}, 2\mathbf{R})$, $\int \xi(r)dr = 1$, we multiply (3.13) by $\xi(r)$ and integrate in r from $\mathbf{R}$ to $2\mathbf{R}$. This gives the integral representation of $f(x)$, $|x| < \mathbf{R}$, in the form $f(x) = \int_{\mathbf{R}<|y|<2\mathbf{R}} K(x,y) f(y) dy$, with smooth bounded kernel $K(x,y)$. After applying $\mathfrak{L}$ in x variable, we obtain a similar integral representation for $\mathfrak{L}u = \mathfrak{L}(e^{-\Psi} f)$:

$$\mathfrak{L}u(x) = \int_{\mathbf{R}<|y|<2\mathbf{R}} e^{-\Psi(x)} K^{\mathfrak{L}}(x,y) e^{\Psi(y)} u(y) dy = (\mathfrak{K}^{\mathfrak{L}} u)(x).$$

The integral operator $|W|^{1/2}\mathfrak{K}^{\mathfrak{L}}$ has a bounded kernel with compact support and therefore is compact in L_2, and thus the quadratic form $\mathfrak{w}[u]$ can be written as $\mathfrak{w}[u] = (\operatorname{sign} W |W|^{1/2}\mathfrak{K}^{\mathfrak{L}} u,\ |W|^{1/2}\mathfrak{K}^{\mathfrak{L}} u)$ and therefore it is compact. □

Now we are able to justify our construction of the spaces $\mathcal{G}_q$.

Proposition 3.7. *The sets $\mathcal{G}_q$ defined in* (3.4) *are closed subspaces in L_2.*

Proof. The fact that $\mathcal{G}_q \subset L_2$ follows directly from Proposition 3.4. Next, the relation (3.5) can be written as

$$(P_0 \mathfrak{Q}^q \overline{\mathfrak{Q}}^q u, u) = C_q(u,u) + (P_0 \mathcal{Z}_q[\mathbf{b}]u, u);\ u = \Psi_0 u \in \mathcal{H}_0 \tag{3.14}$$

(recall that P_0 is the orthogonal projection onto $\mathcal{H}_0$, $P_0 u = u$). In the second term in (3.14), by Lemma 3.5, the operator $P_0 \mathcal{Z}_q[\mathbf{b}]$ is compact in $\mathcal{H}_0$, and therefore we can understand (3.14) as showing that the operator $C_q^{-1} P_0 \mathfrak{Q}^q$ is a left parametrix for $\overline{\mathfrak{Q}}^q : \mathcal{H}_0 \to L_2$. This implies that the range of $\overline{\mathfrak{Q}}^q$ is closed. □

The null space of $\overline{\mathfrak{Q}}$ and therefore of $\overline{\mathfrak{Q}}^q$ is zero. Consider the operator $\overline{\mathfrak{Q}}^q$ as acting from $\mathcal{G}_0 = \mathcal{H}_0$ to $\mathcal{G}_q$. This is a bounded invertible operator, therefore the inverse, that we denote by $\overline{\mathfrak{Q}}^{-q}$, is a bounded operator from $\mathcal{G}_q$ to $\mathcal{G}_0$. This inverse operator is a compact perturbation of $\mathfrak{Q}^q$.

We will use Proposition 3.4 and Lemma 3.5 many times throughout the paper. An immediate useful corollary of the latter lemma is the following.

Corollary 3.8. *Under the conditions of Proposition 3.4, for any $\epsilon > 0$ there exists a subspace $\mathscr{M} \subset \mathcal{H}_0$ such that*

$$\|\overline{\mathfrak{Q}}^q v\|^2 \geq (1-\epsilon)C_q\|v\|^2,\ v \in \mathscr{M},$$

and the codimension of $\mathscr{M}$ in $\mathcal{H}_0$ is finite.

4. Approximate spectral projections

The aim of this section is to prove that the subspaces $\mathcal{G}_q$ are very good approximations to the actual spectral subspaces $\mathcal{H}_q$ of the operator $\mathfrak{P}_-$, as well as to the spectral subspaces of $\mathfrak{P}_-(V)$. Closeness of subspaces will be measured by closeness of orthogonal projections onto them. Recall that the projection onto $\mathcal{H}_q$ is denoted by P_q. Let Q_q be the orthogonal projection onto $\mathcal{G}_q$.

Theorem 4.1. *The projections P_q and Q_q are* close*: for any N, and any collection of the operators T_j, $j = 1, \dots, N$, each of T_j being $\mathfrak{Q}$ or $\overline{\mathfrak{Q}}$, the operator $\mathcal{T}(P_q - Q_q)\mathcal{T}$, is compact, $\mathcal{T} = T_1T_2\dots T_N$.*

In justifying the theorem, we need two technical lemmas which will be proved in Section 7.

Both lemmas concern the operator properties of products of many copies of the resolvents of $\mathfrak{P}_+$ and $\mathfrak{P}_-$, the creation and annihilation operators $\overline{\mathfrak{Q}}$ and $\mathfrak{Q}$, functions h_j with all derivatives bounded, and, possibly, the spectral projection P_q onto $\mathcal{H}_q$. In such product, we assign order 1 to $\overline{\mathfrak{Q}}$ and $\mathfrak{Q}$, order -2 to the resolvent, order 0 to functions and projections. The order of the product is defined as the sum of orders of factors.

Lemma 4.2. *Let the operator $\mathfrak{A}$ be the product of creation, annihilation operators, resolvents, and functions h_j, have negative order, and let at least one of the functions h_j have compact support. Then $\mathfrak{A}$ is compact.*

Lemma 4.3. *Let the operator $\mathfrak{A}$ be the product of creation, annihilation operators, resolvents, functions and a spectral projection P_q. Then $\mathfrak{A}$ is bounded. If, additionally, at least one of the functions has compact support then $\mathfrak{A}$ is compact.*

The proof of the theorem goes in the following way. We construct an intermediate operator $\mathfrak{S}_q$ with range in $\mathcal{G}_q$ and prove that $\mathfrak{S}_q$ is close both to P_q and Q_q.

For $q > 0$ we define the operator $\mathfrak{S}_q$ by the relation

$$\mathfrak{S}_q = C_q^{-1}\overline{\mathfrak{Q}}^q P_0 \mathfrak{Q}^q,\ C_q = q!(2\mathbf{B}^\circ)^q. \tag{4.1}$$

So, our expression for the approximate spectral projection is just a natural modification of the exact formula (2.7) for the unperturbed operator. Equivalently, the operator $\mathfrak{S}_q$ can be described by the formula

$$\mathfrak{S}_q = C_q^{-1}\mathfrak{G}\mathfrak{G}^*, \qquad \mathfrak{G} = \overline{\mathfrak{Q}}^q P_0;$$

in Proposition 3.4 this operator is shown to be bounded.

The proof of Theorem 4.1 will consist of two parts, showing that $\mathfrak{S}_q$ is close to P_q and showing that it is close to Q_q.

Proposition 4.4. *The operator $\mathfrak{S}_q$ is close to the projection P_q.*

Proof. We give the detailed proof for $q = 1$ first, and and then explain the natural changes one should make for larger q.

We denote by $R_-(\zeta)$ the resolvent of the operator $\mathfrak{P}_-$; we also need the resolvent $R_+(\zeta)$ of the operator $\mathfrak{P}_+$. The projection P_0 can be expressed via Riesz integral

$$P_0 = (2\pi i)^{-1} \int_{\Gamma_0} R_-(\zeta) d\zeta,$$

where Γ_0 is the closed curve, encircling the zero point Λ_0, defined, together with curves Γ_q, in Sect.3. We are going to transform the expression for the resolvent $R_-(\zeta)$ using the commutation relations (2.9), (2.10), (2.11), and (3.8). After this, the crucial observation is that the integral of $R_\pm^k$, $k \geq 2$ along a closed contour vanishes. This enables us to dispose of terms that are not compact before integration.

We start by writing

$$R_-(\zeta) = (\mathfrak{P}_- - \zeta)^{-1} = (\mathfrak{P}_+ - 2\mathbf{B}^\circ - 2\mathbf{b} - \zeta)^{-1}.$$

Next we use the resolvent equation to get

$$\begin{gathered} R_-(\zeta) = (\mathfrak{P}_+ - 2\mathbf{B}^\circ - \zeta)^{-1} - \\ (\mathfrak{P}_- - \zeta)^{-1}(2\mathbf{b})(\mathfrak{P}_+ - 2\mathbf{B}^\circ - \zeta)^{-1} = R_+(2\mathbf{B}^\circ + \zeta) - H(\zeta) \end{gathered} \tag{4.2}$$

We multiply (4.2) by $\overline{\mathfrak{Q}}$ from the left and by $\mathfrak{Q}$ from the right, as (4.1) requires. For the first term we use that

$$\begin{gathered} \overline{\mathfrak{Q}} R_+(2\mathbf{B}^\circ + \zeta)\mathfrak{Q} = \overline{\mathfrak{Q}}(\mathfrak{Q}\overline{\mathfrak{Q}} - 2\mathbf{B}^\circ - \zeta)^{-1}\mathfrak{Q} = \\ \overline{\mathfrak{Q}}\mathfrak{Q}(\overline{\mathfrak{Q}}\mathfrak{Q} - 2\mathbf{B}^\circ - \zeta)^{-1} = \mathfrak{P}_- R_-(2\mathbf{B}^\circ + \zeta). \end{gathered} \tag{4.3}$$

Integration gives

$$\int_{\Gamma_0} \overline{\mathfrak{Q}}(\mathfrak{P}_- - 2\mathbf{B}^\circ - \zeta)^{-1}\mathfrak{Q} d\zeta = \mathfrak{P}_- \int_{\Gamma_1} (\mathfrak{P}_- - \zeta)^{-1} d\zeta = 2\pi i \mathfrak{P}_- P_1.$$

By Theorem 2.1, the point $\Lambda_1 = 2\mathbf{B}^\circ$ is an isolated point of the essential spectrum of $\mathfrak{P}_-$, therefore the nonzero eigenvalues of $\mathfrak{P}_- P_1$ may converge only to Λ_1, and thus $(\mathfrak{P}_- - 2\mathbf{B}^\circ)P_1 = \mathfrak{P}_- P_1 - 2\mathbf{B}^\circ P_1$ is compact. Moreover, by Proposition 3.3, this compactness is preserved after multiplication by any product of creation and annihilation operators.

Now we consider the second term, $H(\zeta)$ in (4.2). We are going to show that for any operator of the type $\mathcal{T}$ the integral of $\mathcal{T}H(\zeta)\mathcal{T}$ along Γ_0 is compact. First we apply the resolvent formula (4.2) k times to the first factor in $H(\zeta)$, with k large enough. This operation will produce terms of order $-4, -6, \dots$, containing factors $R_-(\zeta)$ and $\mathbf{b}$, and the remainder term of order $-2k$ containing these factors with, additionally, one factor $R_+(2\mathbf{B}^\circ+\zeta)$. This last remainder term, for k sufficiently large, is compact by Lemma 4.2, even after the multiplication by $\mathcal{T}$ from the left and from the right.

The leading terms in $H(\zeta)$, having orders -4, -6,…, will be transformed by repeatedly commuting $\mathbf{b}$ and $R_-(\zeta)$ and then the resulting commutants again with $R_-(\zeta)$ and so on. Under commuting $R_-(\zeta)$ with a function, with $\mathfrak{Q}$, or with $\overline{\mathfrak{Q}}$, this factor $R_-(\zeta)$ moves to the left or to the right, and one more product in the sum composing $\mathcal{T}H(\zeta)\mathcal{T}$ arises, of order lower by 1. In this commuting procedure we aim for collecting the factors $R_-(\zeta)$ together. As soon as a term with all $R_-(\zeta)$ standing together arises, we leave it alone and do not transform any more. After sufficiently many commutations, we arrive at a collection of terms in $\mathcal{T}H(\zeta)\mathcal{T}$ of negative order, which are compact by Lemma 4.2. The terms of nonnegative order, the ones to which this Lemma cannot be applied, will have the form $G_1R_-(\zeta)^kG_2$ with $k>1$ and some operators G_1, G_2. These terms vanish after integration over Γ_0.

Now we demonstrate in detail how this procedure works. Let $\mathcal{T}=1$. For the case $q=1$, which we are considering at the moment, it suffices to apply (4.2) twice and write

$$\begin{gathered}H(\zeta)=2R_-(\zeta)\mathbf{b}R_+(2\mathbf{B}^\circ+\zeta)=2R_-(\zeta)\mathbf{b}R_-(\zeta)\\ -4R_-(\zeta)\mathbf{b}R_-(\zeta)\mathbf{b}R_+(2\mathbf{B}^\circ+\zeta).\end{gathered}\tag{4.4}$$

We multiply (4.4) by $\overline{\mathfrak{Q}}$ and $\mathfrak{Q}$, as (4.1) requires, and then the last term takes the form $\overline{\mathfrak{Q}}R_-(\zeta)\mathbf{b}R_-(\zeta)\mathbf{b}R_+(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}$. This is an operator of order -2, containing as a factor the compactly supported function $\mathbf{b}$, so it is compact for any $\zeta\in\Gamma_1$ by Lemma 4.2. As for the first term in (4.4), we handle it in the following way (we suppress ζ as long as it does not cause a misunderstanding)

$$\begin{gathered}R_-\mathbf{b}R_-=\mathbf{b}R_-^2-R_-[\mathfrak{P}_-,\mathbf{b}]R_-^2=\\ \mathbf{b}R_-^2-[\mathfrak{P}_-,\mathbf{b}]R_-^3+R_-[\mathfrak{P}_-,[\mathfrak{P}_-,\mathbf{b}]]R_-^3\end{gathered}\tag{4.5}$$

(we used here that $[R_-,L]=-R_-[\mathfrak{P}_-,L]R_-$ for any operator L.) Again we multiply (4.5) by $\overline{\mathfrak{Q}}$ and $\mathfrak{Q}$. In the first and the second terms on the right-hand side all entries of the resolvent are collected together, so these terms vanish after the integration; the last one has order -4 and is compact by Lemma 4.2. If the operator $\mathcal{T}$ is present, we continue expanding (4.4) sufficiently long:

$$\begin{gathered}H(\zeta)=2R_-(\zeta)\mathbf{b}R_+(2\mathbf{B}^\circ+\zeta)=2R_-\mathbf{b}R_-\\ -4R_-\mathbf{b}R_-\mathbf{b}R_+(2\mathbf{B}^\circ+\zeta)\dots\\ =\sum_{j=1}^{k}(-2R_-\mathbf{b})^jR_-+(-2R_-\mathbf{b})^{k+1}R_+(2\mathbf{B}^\circ+\zeta).\end{gathered}\tag{4.6}$$

We choose k large enough, so that the last term in (4.6) will be of negative order even after the multiplication by $\mathcal{T}$. In all remaining terms in (4.6) we commute R_- with other operators, moving all factors R_- say, to the right. For example, the term $R_-\mathbf{b}R_-\mathbf{b}R_-$ will be handled in the following way

$$R_-\mathbf{b}R_-\mathbf{b}R_- = \mathbf{b}R_-^2\mathbf{b}R_- + R_-[\mathbf{b},\mathfrak{P}_-]R_-^2\mathbf{b}R_-$$
$$= \mathbf{b}R_-\mathbf{b}R_-^2 + \mathbf{b}R_-^2[\mathbf{b},\mathfrak{P}_-]R_-\mathbf{b}R_- + R_-[\mathbf{b},\mathfrak{P}_-]R_-^2\mathbf{b}R_-.$$

The second and third terms in the last line are operators of order -7, and they stay compact after multiplication by four operators $\mathfrak{Q}$, $\overline{\mathfrak{Q}}$, which gives compactness of corresponding terms in $\mathcal{T}\overline{\mathfrak{Q}}H(z)\mathfrak{Q}\mathcal{T}$ for an operator $\mathcal{T}$ containing just one factor. The first term in the last line has order -6, so we have to make one more commutation:

$$\mathbf{b}R_-\mathbf{b}R_-^2 = \mathbf{b}^2R_-^3 + \mathbf{b}R_-[\mathbf{b},\mathfrak{P}_-]R_-^3. \tag{4.7}$$

Now the first term in (4.7) vanishes after integration, and the second one has order -7. If the operator $\mathcal{T}$ contains more terms, we have to continue commuting, moving the factors R_- to the right. If there is a separately standing R_- in some term, we can move it to the right, and the order of the term with commutant decreases by one. We cannot do this and, correspondingly, decrease the order, only if all R_- stand together, but such terms vanish after the integration. This reasoning establishes our property for $\mathfrak{S}_1$.

Now we explain how one handles the operators $\mathfrak{S}_q$, $q>1$. It is convenient to use the induction in q. So, supposing that for all $q<p$ the compactness is proved, we start transforming the expression (4.1) for $q=p$. First we extract the principal term. We write, using (4.2),

$$\overline{\mathfrak{Q}}^pR_-(\zeta)\mathfrak{Q}^p = \overline{\mathfrak{Q}}^p(R_+(2\mathbf{B}^\circ+\zeta) - H(\zeta))\mathfrak{Q}^p. \tag{4.8}$$

The term $H(\zeta)$ produces a compact contribution, in the same way as for $p=1$. For the leading term in (4.8) we use (4.3):

$$\overline{\mathfrak{Q}}^pR_+(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}^p = \overline{\mathfrak{Q}}^{p-1}\mathfrak{P}_-R_-(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}^{p-1}.$$

We commute $\overline{\mathfrak{Q}}^{p-1}$ and $\mathfrak{P}_-$ in the last expression to get

$$\overline{\mathfrak{Q}}^pR_+(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}^p = \mathfrak{P}_-\overline{\mathfrak{Q}}^{p-1}R_-(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}^{p-1} + [\overline{\mathfrak{Q}}^{p-1},\mathfrak{P}_-]R_-(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}^{p-1}. \tag{4.9}$$

The first term in (4.9) has the form, similar to the one we have started with in (4.8), just with p replaced by $p-1$, and the inductive assumption can be applied. The commutator $[\overline{\mathfrak{Q}}^{q-1},\mathfrak{P}_-]$ in the second term in (4.9) is the sum of $p-1$ expressions of the form

$$\overline{\mathfrak{Q}}^j[\overline{\mathfrak{Q}},\mathfrak{P}_-]\overline{\mathfrak{Q}}^{q-2-j} = 2\mathbf{B}^\circ\overline{\mathfrak{Q}}^{p-1} + 2\overline{\mathfrak{Q}}^{j+1}\mathbf{b}\overline{\mathfrak{Q}}^{p-2-j},\ j=0,\dots,p-2. \tag{4.10}$$

This gives additionally $p-1$ terms of the form $2\mathbf{B}^\circ\overline{\mathfrak{Q}}^{p-1}R_-(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}^{p-1}$ in our integrand. So we have transformed $\overline{\mathfrak{Q}}^p R_+(2\mathbf{B}^\circ+\zeta)\overline{\mathfrak{Q}}^p$ to $(2(p-1)\mathbf{B}^\circ+\mathfrak{P}_-)\overline{\mathfrak{Q}}^{p-1}R_-(2\mathbf{B}^\circ+\zeta)\mathfrak{Q}^{p-1}$ plus remainder terms.

After p such steps we arrive at the expression

$$\mathfrak{M}R_-(2p\mathbf{B}^\circ+\zeta);\ \mathfrak{M}=\prod_{j=1}^{p}(2(p-j)\mathbf{B}^\circ+\mathfrak{P}_-),$$

which, after the change of variables $\zeta'=2p\mathbf{B}^\circ+\zeta$ and integration along Γ_p, gives $2\pi i\mathfrak{M}P_p$. The latter operator is a compact perturbation of $2\pi iC_pP_p$, by Proposition 3.1, and, as usual, this perturbation remains compact after multiplication by $\mathcal{T}$.

The remainder terms arising in the expressions similar to (4.8), are taken care of by means of Lemma 4.2, as in the case of $q=1$: they contain at least two resolvents separated by a function and the commutation procedure succeeds. Another kind of remainders are the ones obtained in the relations similar to (4.10), substituted into (4.9). They contain only one resolvent. However after the integration such terms take the form $\mathcal{T}'P\mathcal{T}''$ where P is one of projections P_q, and $\mathcal{T}',\mathcal{T}''$ are products of creation, annihilation operators and functions, moreover at least one of these functions has compact support. By Lemma 4.3, such operators are compact. □

So we have proved that the operator $\mathfrak{S}_q$ is close to the projection P_q.

Proposition 4.5. *The operator $\mathfrak{S}_q$ is close to the projection Q_q onto the approximate spectral subspace $\mathcal{G}_q$.*

Proof. First of all, since the operator $\mathfrak{S}_q$ is a self-adjoint operator close to *some* projection, it must have the essential spectrum as any projection has, this means, consisting only of the points 0 and 1. Moreover, since the range of $\mathfrak{S}_q$ lies in $\mathcal{G}_q$, the operator $\mathfrak{S}_q$ has $\mathcal{G}_q$ as an invariant subspace and it annuls the orthogonal complement to $\mathcal{G}_q$. So we may consider $\mathfrak{S}_q$ in $\mathcal{G}_q$. We are going to show first that zero is not a limit point of nonzero eigenvalues of $\mathfrak{S}_q$; this will imply that zero is an isolated point of the spectrum of $\mathfrak{S}_q$ and the range of $\mathfrak{S}_q$ is closed in $\mathcal{G}_q$. To show this, we consider the self-adjoint operator $C_q^{-1}\mathfrak{G}^*\mathfrak{G}:\mathcal{H}_0\to\mathcal{H}_0$, $\mathfrak{G}=\overline{\mathfrak{Q}}^qP_0$. By Proposition 3.4 and Lemma 3.5, this operator is bounded, and, moreover,

$$C_q^{-1}\mathfrak{G}^*\mathfrak{G}=P_0(1+\mathfrak{K})P_0 \tag{4.11}$$

with a compact operator $\mathfrak{K}$. This implies that the point 0 is an isolated point of the spectrum of $C_q^{-1}\mathfrak{G}^*\mathfrak{G}:\mathcal{H}_0\to\mathcal{H}_0$, and therefore 0 is an isolated point of the spectrum of $\mathfrak{S}_q=C_q^{-1}\mathfrak{G}\mathfrak{G}^*$. So the range of $\mathfrak{S}_q$ is closed. Further on, it follows from (4.11) that $\mathfrak{G}^*=P_0\mathfrak{Q}^q:\mathcal{G}_q\to\mathcal{H}_0$ has range of finite codimension in $\mathcal{H}_0$, and since $\overline{\mathfrak{Q}}^q$ maps $\mathcal{H}_0$ onto $\mathcal{G}_q$ (by the definition of the latter space), the range of $\mathfrak{S}_q$ has finite codimension in $\mathcal{G}_q$. These properties, taken together, imply that $\mathfrak{S}_q$ is a compact perturbation of the projection Q_q acting in the range of Q_q, $\mathfrak{S}_q = Q_q(1+\mathfrak{K}_1)Q_q$, with a compact operator $\mathfrak{K}_1$. Since the range of $\mathfrak{S}_q$ has finite co-dimension in $\mathcal{G}_q$, this relation can also be written as $\mathfrak{S}_q - Q_q = \mathfrak{S}_q\mathfrak{K}_2\mathfrak{S}_q + \mathfrak{F}$ with a compact operator $\mathfrak{K}_2$ and $\mathfrak{F}$ having finite rank. By Lemma 4.3, the operator $\mathfrak{S}_q$ stays bounded after multiplication by any collection of creation and annihilation operators, and this establishes closeness of $\mathfrak{S}_q$ and Q_q. □

Propositions 4.4 and 4.5, taken together, prove Theorem 4.1.

Now we add a perturbation by a compactly supported smooth electric potential $V(x)$. We are going to show that under such perturbation the spectral subspaces 'almost' do not change. This observation justifies the usual physical intuition hinting that in the first order of perturbation theory only the eigenvalues change, while the spectral subspaces change only in the second order. Note that by our construction of the approximate spectral subspaces we have shown that this physical intuition is wrong in the case of a magnetic perturbation.

Note, first of all, that the perturbation of $\mathfrak{P}_-$ by V is relatively compact, therefore, again, the spectrum of the operator $\mathfrak{P}_-(V) = \mathfrak{P}_- + V$ between Landau levels is discrete. We can change the contours Γ_q a little, so that they do not pass through the eigenvalues of $\mathfrak{P}_-(V)$. We denote by $\mathcal{H}_q^V$ the spectral subspaces of $\mathfrak{P}_-(V)$ corresponding to the spectrum inside Γ_q, by P_q^V the corresponding spectral projections, and by $R_-^V(\zeta)$ the resolvent of $\mathfrak{P}_-(V)$.

Proposition 4.6. *The projections P_q^V and Q_q are close in the sense used in Theorem 4.1.*

Proof. We will prove that the projection P_q^V is close to P_q, then the result will follow from Theorem 4.1.

We use the representation of projections by means of resolvents,

$$P_q^V = (2\pi i)^{-1}\int_{\Gamma_q} R_-^V(\zeta)d\zeta.$$

The similar expression for P_q and the resolvent formula give

$$P_q^V - P_q = -(2\pi i)^{-1}\int_{\Gamma_q} R_-(\zeta)VR_-^V(\zeta)d\zeta = \tag{4.12}$$

$$(2\pi i)^{-1}\sum_{k=1}^{2N-1}\int_{\Gamma_q}(-R_-(\zeta)V)^k R_-(\zeta)d\zeta$$

$$+(2\pi i)^{-1}\int_{\Gamma_q}(-R_-(\zeta)V)^N R_-^V(\zeta)(-VR_-(\zeta))^N d\zeta.$$

The last term in (4.12) is compact and stays compact after the multiplication by the creation and annihilation operators, as soon as N is large enough, by Lemma 4.2. With the leading terms in (4.12), we can perform the same procedure as when proving Proposition 4.4. We commute the resolvent with V and with the terms arising by commutation and so on, aiming to collect the resolvents together all the time. As a results we arrive at a number of terms of sufficiently negative order, thus compact before the integration, and terms with all resolvents collected together, and thus vanishing after the integration. $\square$

5. Spectrum of Toeplitz-type operators

We move on to the study of the splitting of Landau levels of our operators. Similar to [13], [11], [5], the properties of the eigenvalues of the perturbed operators depend on the properties of the spectrum of certain Toeplitz-type operators.

Usually, by Toeplitz operator one understands an operator of the form $\mathfrak{T}^W = PWP$ where P is the orthogonal projection onto some subspace $\mathcal{G}$ in L_2 and W is the operator of multiplication by some function. Alternatively, if we consider the Toeplitz operator in $\mathcal{G}$ the operator can be written as $\mathfrak{T}^W = PW$.

In [13], [11] such operators were considered, with $\mathcal{G}$ being one of Landau subspaces and the electric potential V acting as W. Later, in [5], Toeplitz type operators at the lowest LL were considered, with some differential operators acting as W. We will study this latter kind of Toeplitz type operators, with the space $\mathcal{H}_0$ of zero modes of the perturbed Pauli operator acting as $\mathcal{G}$.

We recall that the projection onto the space $\mathcal{H}_0 = \mathcal{G}_0$ of zero modes of $\mathfrak{P}_-$ is denoted by P_0. We consider linear differential operators $\mathfrak{L}$ of order l with bounded coefficients, $\mathfrak{L}u = \sum_{\alpha+\beta\le l} p_{\alpha\beta}\partial_1^\alpha\partial_2^\beta u$. We suppose that all coefficients $p_{\alpha\beta}$ have compact support in the disk $|x| \le \mathbf{R}$ and $|p_{\alpha\beta}| \le \mathbf{M}$. With any such operator $\mathcal{L}$ we associate the Toeplitz-type operator in $\mathcal{H}_0$:

$$\mathfrak{T}^{\mathfrak{L}} v = P_0\mathfrak{L}^*\mathfrak{L}v,\ v \in \mathcal{H}_0.$$

This operator corresponds to the quadratic form

$$\mathfrak{t}^{\mathfrak{L}}[u] = \int |\mathfrak{L}u|^2 dx,\ u \in \mathcal{H}_0.$$

We are going to study the spectrum of $\mathfrak{T}^{\mathfrak{L}}$. We denote by $\lambda_n^\pm = \lambda_n^\pm(\mathfrak{L})$ positive, resp., negative eigenvalues of the operator $\mathfrak{T}^{\mathfrak{L}}$. The distribution functions $n_\pm(\lambda)$, $\lambda > 0$, are defined as $n_\pm(\lambda) = n_\pm(\lambda;\mathfrak{L}) = \#\{n:\ \pm\lambda_n^\pm(\mathfrak{L}) > \lambda\}$. The singular numbers $s_n(\mathfrak{L})$ of the operator $\mathfrak{T}^{\mathfrak{L}}$ are just the absolute values of $\lambda_n^\pm$ ordered non-increasingly, and their distribution function equals $n(\lambda;\mathfrak{L}) = n_+(\lambda;\mathfrak{L}) + n_-(\lambda,\mathfrak{L})$. We denote by $\Xi(\lambda)$ the function $\frac{1}{2}\frac{|\ln\lambda|}{\ln|\ln\lambda|}$.

Proposition 5.1. *For an operator $\mathfrak{L}$ with bounded compactly supported coefficients,*

$$\limsup_{\lambda\to 0} n_\pm(\lambda;\mathfrak{L})\Xi(\lambda)^{-1} \le 1. \tag{5.1}$$

The estimate (5.1) means that the eigenvalues of $\mathfrak{T}^{\mathfrak{L}}$ converge to zero extremely rapidly, super-exponentially:

$$\limsup_{n\to\infty}(n!\lambda_n^{\pm}(\mathfrak{L}))^{\frac{1}{n}} < \infty. \tag{5.2}$$

The estimates (5.1), (5.2) do not exclude the possibility that the operator $\mathfrak{T}^{\mathfrak{L}}$ has only a finite number of positive and/or negative eigenvalues, counting multiplicity.

Proof. We start by proving a special version of the proposition, dealing with the case when the operator $\mathfrak{L}$ is just the multiplication by a bounded compactly supported function W. The corresponding Toeplitz operator will be denoted by $\mathfrak{T}^W$

We apply the classical variational approach. Consider the quadratic form $\mathfrak{t}^W[v]$ of the operator $\mathfrak{T}^W$:

$$\mathfrak{t}^W[v] = \int W(x)|v(x)|^2 dx,\ v \in \mathcal{H}_0.$$

The variational description of eigenvalues (Glazman lemma) implies

$$n_+(\lambda; W) = \min\operatorname{codim}\{\mathscr{L} \subset \mathcal{H}_0 : \mathfrak{t}^W[v] \le \lambda\|v\|^2,\ v \in \mathscr{L}\}.$$

We recall the description of the space $\mathcal{H}_0$ to get

$$n_+(\lambda; W) = \min\operatorname{codim}\{\mathscr{M} \subset \mathcal{F}^{\mathbf{b}} : \mathfrak{t}^W[e^{-\Psi}f] \le \lambda\|e^{-\Psi}f\|^2, f \in \mathscr{M}\}, \tag{5.3}$$

where $\mathcal{F}^{\mathbf{b}}$ is the space of entire analytical functions, defined in Sect.2.

Now we are going to estimate the quantity in (5.3) from above by means of changing the forms and the spaces. First, if we replace $\|e^{-\Psi}f\|^2$ in (5.3) by something smaller, then there will be fewer subspaces $\mathscr{M}$ where the inequality holds and therefore $\min\operatorname{codim}$ may become only larger. So we can replace $\|e^{-\Psi}f\|^2$ by $C\|e^{-(1+\epsilon)\Psi^\circ}f\|^2$ where Ψ° is the potential defined in (2.3) and C is some constant, since, due to our estimates of Ψ, discussed in Sect.3, $e^{-\Psi} \ge Ce^{-(1+\epsilon)\Psi^\circ}$. After this, if we replace the enveloping space $\mathcal{F}^{\mathbf{b}}$ in (5.3) by a larger one, $\mathcal{F}_{(1+\epsilon)\mathbf{B}^\circ}$, then the codimension of subspaces may only grow. Therefore, we arrive at

$$n_+(\lambda, W) \le \tag{5.4}$$
$$\min\operatorname{codim}\{\mathscr{M} \subset \mathcal{F}_{(1+\epsilon)\mathbf{B}^\circ} : \mathfrak{t}^W[e^{-\Psi}f] \le \lambda\|Ce^{-(1+\epsilon)\Psi^\circ}f\|^2, f \in \mathscr{M}\}.$$

Now we notice that $\mathfrak{t}^W[e^{-\Psi}f] = \int We^{-2\Psi}|f|^2dx$, and $We^{-2\Psi}$ is again a bounded function with compact support. Therefore we can apply estimates for the right-hand side in (5.4) established in [13], Proposition 3.2 or [11], Lemma 6.1, which lead to the inequality (5.1).

The estimate we have just established can even be improved, if we use the results of [5] to majorize the right-hand side in (5.4):

$$\limsup_{n\to\infty}(n!\lambda_n^{\pm}(W))^{\frac{1}{n}} \le \frac{\mathbf{B}^\circ}{2}\operatorname{Cap}(\operatorname{supp} W),$$

where Cap (supp W) is the logarithmical capacity of the support of W.

Now we pass to the general case. Let all coefficients $p_{\alpha\beta}$ of the operator $\mathfrak{L}$ have support in the disk with radius $\mathbf{R}$ centered at 0 and $|p_{\alpha\beta}| \le \mathbf{M}$.

Denote by χ the characteristic function of the disk $\mathbf{D}$ of radius $2\mathbf{R}$. By Cauchy integral formula, applied as in Lemma 3.5, each term of the form $p_{\alpha\beta}\partial_1^\alpha\partial_2^\beta u$, $u \in \mathcal{H}_0$ can be expressed as $T_{\alpha\beta}(\chi u)$, with a bounded operator $T_{\alpha\beta}$. The upper bound for the norm of this operator depends only on the order of derivatives $\alpha+\beta$, on the size of the coefficient $p_{\alpha\beta}$ and on the radius $\mathbf{R}$. Since the quadratic form $\mathfrak{t}^{\mathfrak{L}}$ of the operator $\mathfrak{T}^{\mathfrak{L}}$ has the form $\mathfrak{t}^{\mathfrak{L}}[u] = \| \sum p_{\alpha\beta}\partial_1^\alpha\partial_1^\beta u\|^2 = \| \sum p_{\alpha\beta}\partial_1^\alpha\partial_1^\beta \chi u\|^2$, we have

$$|\mathfrak{t}^{\mathfrak{L}}[u]| \le \varkappa^{-1}\|\chi u\|^2 = \varkappa^{-1}\mathfrak{t}^{\chi}[u],\ u \in \mathcal{H}_0 \tag{5.5}$$

for a certain $\varkappa$. Thus the singular numbers of the operator $\mathfrak{T}^{\mathfrak{L}}$ are majorated by the ones of the operator $\mathfrak{T}^W$ with $W = \chi$, and for such operators the required estimate is already proved. □

It is more convenient for our needs to formulate Proposition 5.1, directly in the variational form, declaring the existence of subspaces with prescribed properties.

Proposition 5.2. *There exists a constant $\varkappa$ depending only on $l, \mathbf{R}, \mathbf{M}$, such that for any $\epsilon > 0$ and for any $\lambda > 0$, sufficiently small, there exists a subspace $\mathscr{M}(\lambda) \in \mathcal{H}^0$ such that the codimension of $\mathscr{M}(\lambda)$ is no greater than $(1+\epsilon)\Xi(\lambda)$, and for any order l differential operator $\mathfrak{L}$ with coefficients $p_{\alpha\beta}$ supported in the disk with radius $\mathbf{R}$ and satisfying $|p_{\alpha\beta}| \le \mathbf{M}$*

$$\mathfrak{t}^{\mathfrak{L}}[u] \le \varkappa\lambda\|u\|^2, u \in \mathscr{M}(\lambda). \tag{5.6}$$

The important feature in this formulation is that the subspace $\mathscr{M}(\lambda)$ of controlled codimension can be chosen in such way that it services simultaneously all Toeplitz-like operators generated by differential operators $\mathfrak{L}$ of fixed order and with coefficients subject to the above restrictions.

Proof. By (5.5), as soon as the chosen subspace $\mathscr{M}(\lambda)$ services the Toeplitz operator $\mathfrak{T}^\chi$, the inequality (5.6) holds for all operators $\mathfrak{L}$ subject to our conditions. □

We formulate the eigenvalue estimate from below for operators $\mathfrak{T}^W$ only, under the condition that the function W is non-negative. This latter condition can be somewhat relaxed, however some kind of positivity requirements still remain. In fact, starting from [13], [11], the positivity condition for lower spectral estimates remains a serious obstacle in this field.

Proposition 5.3. *Let the function W be non-negative and be greater than some $c > 0$ on an open set. Then*

$$\liminf_{\lambda \to 0} n_+(\lambda; W)\Xi(\lambda)^{-1} \geq 1.$$

Proof. We use the variational description of $n_\pm(\lambda; W)$:

$$n_+(\lambda; W) = \max \dim\{\mathscr{L} \subset \mathcal{H}_0 : \mathfrak{t}_0[v] < \lambda \|v\|^2,\ v \in \mathscr{L}, v \neq 0\}. \quad (5.7)$$

Again, using the description of the space $\mathcal{H}_0$ of zero modes, we can rewrite (5.7) as

$$n_+(\lambda; W) = \max \dim\{\mathscr{M} \subset \mathcal{F}^{\mathbf{b}} : \mathfrak{t}_0[e^{-\Psi} f] > \lambda \|e^{-\Psi} f\|^2, f \in \mathscr{M}, f \neq 0\} \quad (5.8)$$

By replacing $\|e^{-\Psi} f\|^2$ by some larger quantity, we narrow the choice of subspaces $\mathscr{M}$ where the inequality in (5.8) holds. So, we replace $\|e^{-\Psi} f\|^2$ by $\|Ce^{-(1-\epsilon)\Psi^\circ} f\|^2$ with an arbitrary $\epsilon \in (0, 1)$ and some C. This leads to decreasing of the right-hand side of (5.8). A further decreasing is obtained by narrowing the enveloping space: we replace $\mathcal{F}^{\mathbf{b}}$ by $\mathcal{F}_{(1-\epsilon)\mathbf{B}^\circ}$. This transforms the right-hand side in (5.8) to the form, where the estimate from below is already proved, again in [13], Proposition 3.2 or [11], Lemma 6.1. □

We will need the uniform variational version of the latter estimate.

Proposition 5.4. *Let Ω be an open set, $c > 0$. Then for any $\epsilon > 0$ and for any $\lambda > 0$, small enough, depending on ϵ there exists a subspace $\mathscr{L} = \mathscr{L}(\lambda) \subset \mathcal{H}_0$ such that $(Wv, v) > \lambda \|v\|^2$, $v \in \mathscr{L} \setminus \{0\}$ for all functions $W \geq 0$ satisfying $W \geq c$ on Ω and* $\dim \mathscr{L}(\lambda) \geq (1 - \epsilon)\Xi(\lambda)$.

The result follows from the estimate $(Wv, v) \geq c(\chi_\Omega v, v)$.

6. Perturbed eigenvalues

The passage from the spectral estimates for Toeplitz-type operators to the ones for the Pauli operator has been performed in a different way in [12], [13] and in [11]. A short and neat reasoning in [12], [13] based upon the proper version of the Birman-Schwinger principle requires positivity of the perturbing operator which fails to be positive in our case. Therefore we follow the approach of [11], Sect. 9, using variational considerations, however certain new features appear.

The particular form of the variational principle we use, derived immediately from the spectral theorem, is the following. For a self-adjoint operator $\mathfrak{A}$ and an interval (s, r) in the real axis we denote by $N(s, r; \mathfrak{A})$ the total multiplicity of the spectrum of $\mathfrak{A}$ in (s, r); when this quantity is finite, it is the number of eigenvalues of $\mathfrak{A}$ in (s, r), counting multiplicities.

Proposition 6.1. *Let $\mathfrak{A}$ be a self-adjoint operator with domain $\mathfrak{D}(\mathfrak{A})$. Then, for $s < r$,*

$$N(s, r; \mathfrak{A})$$
$$= \max \dim \left\{ \mathscr{L} \subset \mathfrak{D}(\mathfrak{A}) : \|(\mathfrak{A} - \mu)u\|^2 < \tau^2 \|u\|^2, u \in \mathscr{L} \backslash \{0\} \right\} \quad (6.1)$$
$$= \min \operatorname{codim} \left\{ \mathscr{L} \subset \mathfrak{D}(\mathfrak{A}) : \|(\mathfrak{A} - \mu)u\|^2 \geq \tau^2 \|u\|^2, u \in \mathscr{L} \right\}, \quad (6.2)$$

where $\mu = (r + s)/2$ and $\tau = (r - s)/2$.

When using Proposition 6.1, one looks for a subspace $\mathscr{L}$ where the inequality in (6.1) is fulfilled. The dimension of this subspace gives an estimate from below for $N(s, r; \mathfrak{A})$. On the other hand, having found a subspace $\mathscr{L}$ where the inequality in (6.2) is satisfied, we can be sure that the codimension of this subspace estimates $N(s, r; \mathfrak{A})$ from above.

Considering perturbations of the Landau Hamiltonian by general electric and magnetic fields, we should accept the possibility that the eigenvalues may split away from the Landau level both up and down. Therefore we need rather advanced notations. We fix a Landau level $\Lambda = \Lambda_q = 2q\mathbf{B}^\circ$ and set $s_\pm = \Lambda \pm \mathbf{B}^\circ$. For given $\lambda \in (0, \mathbf{B}^\circ/4)$ we define $\mu_\pm = (\Lambda \pm \lambda + s_\pm)/2$, $\tau = \frac{1}{2}\mathbf{B}^\circ - \frac{\lambda}{2}$. Thus, $\mu_\pm$ is the midpoint of the interval between $\Lambda \pm \lambda$ and $s_\pm$, τ is the half-length of this interval. For the perturbation given by an operator $\mathfrak{V}$ we denote by $N_+(\lambda; q, \mathbf{b}, \mathfrak{V})$ the eigenvalue counting function $N(\Lambda_q + \lambda, s_+; \mathfrak{P}_- + \mathfrak{V})$ where $\mathfrak{P}_-$ is the Pauli operator with magnetic field $\mathbf{B} = \mathbf{B}^\circ + \mathbf{b}$; similarly, $N_-(\lambda; q, \mathbf{b}, \mathfrak{V})$ is defined as $N(s_-, \Lambda_q - \lambda; \mathfrak{P}_- + \mathfrak{V})$. Thus, N_+, N_- count the eigenvalues converging to Λ from above, resp., from below.

6.1. **Upper estimate.** We start with the upper estimate. It declares that the eigenvalues split away from the Landau level at least superexponentially rapidly. This estimate is rather robust: is does not require much smoothness of the perturbing magnetic field or of the electric potential. It is convenient to keep all the previous notations in the paper for a smooth case, and use the superscript $\sharp$ for the non-smooth versions.

So, let $\mathbf{b}^\sharp \in C_0^2(\mathbb{R}^2)$ be a magnetic field with compact support, $\psi^\sharp \in C^4$ be its scalar potential, $\Delta\psi^\sharp = \mathbf{b}^\sharp$, and $\mathbf{a}^\sharp \in C^3$ its vector potential. The Pauli operator associated to the field $\mathbf{B}^\circ + \mathbf{b}^\sharp$ will be denoted by $\mathfrak{P}_-^\sharp$.

We consider the smoothened magnetic field, $\mathbf{b} = \omega * \mathbf{b}^\sharp$, where $\omega \in C_0^\infty$ is a smooth function with compact support, $\int \omega dx = 1$, and $\omega(x)$ depends only on $|x|$. Thus the field $\mathbf{b}$ is smooth and compactly supported. Of course, $\psi = \omega * \psi^\sharp$ is a potential for $\mathbf{b}$, $\Delta\psi = \mathbf{b}$. Moreover, since $\psi^\sharp$ is a harmonic function outside the support of $\mathbf{b}^\sharp$, by the mean value property, ψ coincides with $\psi^\sharp$ outside some compact. Correspondingly the vector potential $\mathbf{a}$ of the field $\mathbf{b}$ coincides with $\mathbf{a}^\sharp$ outside a compact.

As a result, the Pauli operator $\mathfrak{P}_-^\sharp$ can be expressed as

$$\mathfrak{P}_-^\sharp = \mathfrak{P}_- + \mathfrak{M},$$

where $\mathfrak{P}_-$ is the Pauli operator with smooth magnetic field $\mathbf{B} = \mathbf{B}^\circ + \mathbf{b}$ and $\mathfrak{M}$ is a first order differential operator with C^2-coefficients having compact support.

Further on, we consider the operator $\mathfrak{P}_-^\sharp + V$ as a perturbation of $\mathfrak{P}_-$,

$$\mathfrak{P}_-^\sharp + V = \mathfrak{P}_- + \mathfrak{V},\ \mathfrak{V} = \mathfrak{M} + V,\ V \in C_0^2(\mathbb{R}^2).$$

Now we can formulate the main upper estimate.

Theorem 6.2. *If the perturbations* $\mathbf{b}^\sharp$ *and* V *belong to* $C_0^2(\mathbb{R}^2)$ *then*

$$\limsup_{\lambda \to 0+} N_\pm(\lambda, \mathfrak{P}_-^\sharp + V)\Xi(\lambda)^{-1} \leq 1. \tag{6.3}$$

Proof. We will prove the 'minus' version of the Theorem; the 'plus' version differs just by non-essential details. Keeping in mind Proposition 6.1, we are going to construct a subspace $\mathscr{L}$ in the domain of the operator $\mathfrak{P}_-$ such that the inequality

$$\|(\mathfrak{P}_- + \mathfrak{V} - \mu)u\|^2 \geq \tau^2\|u\|^2$$

holds for all $u \in \mathscr{L}$, and estimate its codimension.

We represent an arbitrary function u as $u = u_1 + u_2$ where $u_1 \in \mathcal{G} = \mathcal{G}_q$ belongs to the approximate spectral subspace constructed in Section 3 and u_2 belongs to the subspace $\hat{\mathcal{G}}$ orthogonal to $\mathcal{G}$, $u_1 = Q_q u$, $u_2 = \hat{Q}_q u$, $\hat{Q}_q = 1 - Q_q$.

Then the inequality we aim for takes the form

$$\|(\mathfrak{P}_- + \mathfrak{V} - \mu)(u_1 + u_2)\|^2 - \tau^2\|u_1\|^2 - \tau^2\|u_2\|^2 \geq 0. \tag{6.4}$$

The left-hand side in (6.4) can be written as

$$[\|(\mathfrak{P}_- + \mathfrak{V} - \mu)u_1\|^2 - \tau^2\|u_1\|^2] + [\|(\mathfrak{P}_- + \mathfrak{V} - \mu)u_2\|^2 - \tau^2\|u_2\|^2] \\ +2\Re((\mathfrak{P}_- + \mathfrak{V} - \mu)u_1, (\mathfrak{P}_- + \mathfrak{V} - \mu)u_2). \tag{6.5}$$

Following the pattern of the proof of Proposition 9.2 in [11], we are going to find subspaces of controlled codimension in $\mathcal{G}$ and in $\hat{\mathcal{G}}$, where the first and the second terms in (6.5) are positive, with some margin, while the third term in (6.5) is majorized by this margin.

The first term. We start with the first term. Since $u_1 \in \mathcal{G}$, by our construction, u_1 can be written in a unique way as $u_1 = \overline{\mathfrak{Q}}^q v_1$, $v_1 \in \mathcal{H}_0$, and thus the term takes the form

$$\|(\mathfrak{P}_- + \mathfrak{V} - \mu)u_1\|^2 - \tau^2\|u_1\|^2 = \|(\mathfrak{P}_- + \mathfrak{V} - \mu)\overline{\mathfrak{Q}}^q v_1\|^2 - \tau^2\|\overline{\mathfrak{Q}}^q v_1\|^2 \tag{6.6}$$

$$= \|(\mathfrak{P}_- - \mu)\overline{\mathfrak{Q}}^q v_1\|^2 - \tau^2\|\overline{\mathfrak{Q}}^q v_1\|^2 + 2\Re((\mathfrak{P}_- - \mu)\overline{\mathfrak{Q}}^q v_1, \mathfrak{V}\overline{\mathfrak{Q}}^q v_1) \\ +(\mathfrak{V}\overline{\mathfrak{Q}}^q v_1, \mathfrak{V}\overline{\mathfrak{Q}}^q v_1).$$

The first two terms in (6.6) can be transformed by means of Proposition 3.4; they produce $C_q((\Lambda-\mu)^2-\tau^2)\|v_1\|^2$ as well as $C_q(W_1 v_1, v_1)$ with some compactly supported function W_1. The last term is non-negative and can be ignored when estimating the whole expression from below. As for the remaining, next to last term in (6.6), we have

$$2\Re((\mathfrak{P}_- - \mu)\overline{\mathfrak{Q}}^q v_1, \mathfrak{V}\overline{\mathfrak{Q}}^q v_1) \geq -(\|\chi\mathfrak{L}_1 v_1\|^2 + \|\mathfrak{L}_2 v_1\|^2),$$

where $\chi\mathfrak{L}_1, \mathfrak{L}_2$ are order $q+2$ and order $q+1$ differential operators with bounded compactly supported coefficients, $\mathfrak{L}_1 = (\mathfrak{P}_- - \mu)\overline{\mathfrak{Q}}^q$, $\mathfrak{L}_2 = \mathfrak{V}\overline{\mathfrak{Q}}^q$, and χ is a characteristic function of an open set containing the support of coefficients of $\mathfrak{V}$, so that $\chi\mathfrak{V} = \mathfrak{V}$.

As a result, we obtain

$$\begin{aligned} \|(\mathfrak{P}_- + \mathfrak{V} - \mu)u_1\|^2 - \tau^2\|u_1\|^2 \geq C_q((\Lambda-\mu)^2 - \tau^2)\|v_1\|^2 \\ +(Wv_1, v_1) - \|\chi\mathfrak{L}_1 v_1\|^2 - \|\mathfrak{L}_2 v_1\|^2. \end{aligned} \tag{6.7}$$

The expression $((\Lambda-\mu)^2-\tau^2)$ equals $(\Lambda-\mu-\tau)(\Lambda-\mu+\tau) = \lambda(\Lambda-s)$. Now we are going to chose our first subspace. We fix $\epsilon \in (0, \frac{1}{3})$. By Proposition 5.2, there exists a subspace $\mathscr{M}_1$ in $\mathcal{H}_0$ such that $|(Wv_1, v_1)|, \|\chi\mathfrak{L}_1 v_1\|^2, \|\mathfrak{L}_2 v_1\|^2$ are all no greater than $\frac{1}{10}\lambda(\Lambda-s)\|v_1\|^2$ and the codimension of $\mathscr{M}_1$ in $\mathcal{H}_0$ is less than $(1+\epsilon)\Xi(\frac{1}{10}(\Lambda-s)\lambda)$ for λ small enough. So, for $v_1 \in \mathscr{M}_1$ the right-hand side of (6.7) is no less than $\frac{3}{4}C_q(\Lambda-s)\lambda\|v_1\|^2$. This means that on the subspace $\mathscr{L}_1 = \overline{\mathfrak{Q}}^q\mathscr{M}_1$, which has the same codimension in $\mathcal{G}$ as $\mathscr{M}_1$ has in $\mathcal{H}_0$, the left hand side of (6.7) is no less than $\frac{3}{4}C_q(\Lambda-s)\lambda\|v_1\|^2$. The operator $\overline{\mathfrak{Q}}^q : \mathcal{H}_0 \to \mathcal{G}$ is boundedly invertible, so we can majorize $\|v_1\|$ by $||u_1||$, as in Corollary 3.8, therefore, for some constant c_1, not depending on ϵ, λ,

$$\|(\mathfrak{P}_- + \mathfrak{V} - \mu)u_1\|^2 - \tau^2\|u_1\| \geq c_1\lambda\|u_1\|^2, \ u_1 \in \mathscr{L}_1.$$

The subspace $\mathscr{L}_1$ has codimension no greater than $(1+\epsilon)\Xi(\frac{1}{10}(\Lambda-s)\lambda)$ in $\mathcal{G}$.

The second term. In the next term in (6.5), $\|(\mathfrak{P}_- - \mu + \mathfrak{V})u_2\|^2 - \tau^2\|u_2\|^2$, we can write $u_2 = (1-Q_q)u_2 = (1-P_q)u_2 + (P_q - Q_q)u_2$ where P_q is the spectral projection of $\mathfrak{P}_-$ corresponding to the neighborhood $\delta_q = (\Lambda-\gamma, \Lambda+\gamma)$ of $\Lambda = \Lambda_q$, defined in Section 3. We denote by $\tilde{P}$ the spectral projection of $\mathfrak{P}_-$ corresponding to the set $(s - \mathbf{B}^\circ/2, \Lambda - \gamma] \cup [\Lambda+\gamma, \Lambda+\mathbf{B}^\circ)$. Since the spectrum of $\mathfrak{P}_-$ is discrete in $(\Lambda - 2\mathbf{B}^\circ, \Lambda) \cap (\Lambda, \Lambda+2\mathbf{B}^\circ)$, the projection $\tilde{P}$ has finite rank, and thus u_2 can be expressed as

$$u_2 = (1 - P_q - \tilde{P})u_2 + (P_q - Q_q + \tilde{P})u_2 = g + h. \tag{6.8}$$

Constructed as above, the function g is lying in the spectral subspace of $\mathfrak{P}_-$ corresponding to the exterior of the interval $(\Lambda - 3\mathbf{B}^\circ/2, \Lambda+\mathbf{B}^\circ)$. So, by means of the spectral theorem, we can estimate $\|(\mathfrak{P}_- - \mu)g\|$

from below as $\|(\mathfrak{P}_- - \mu)g\| \geq (\tau + \mathbf{B}^\circ/2)\|g\|$. Therefore

$$\|(\mathfrak{P}_- - \mu)g\|^2 - \tau^2\|g\|^2 \geq c_2\|g\|^2, \tag{6.9}$$

as well as

$$\|(\mathfrak{P}_- - \mu)g\|^2 - \tau^2\|g\|^2 \geq c_3\|(\mathfrak{P}_- - \mu)g\|^2 \tag{6.10}$$

with some constants c_2, c_3 not depending on λ.

Now let us consider $h = (P_q - Q_q + \tilde{P})u_2$. As we have proved in Section 4, the operator $P_q - Q_q$ is compact and remains compact after multiplication by any combination of creation and annihilation operators. This property remains intact after adding the finite rank operator $\tilde{P}$. So, we can write (6.8) as

$$u_2 = g + \mathfrak{K}u_2 \tag{6.11}$$

with a compact operator $\mathfrak{K}$. Therefore, on a subspace of finite codimension, (6.11) can be rewritten as

$$u_2 = g + \mathfrak{K}'g$$

with a compact operator $\mathfrak{K}'$ which, again, remains compact after multiplication by $\mathfrak{Q}, \overline{\mathfrak{Q}}$. As a result, for ϵ fixed, we can find a subspace of finite codimension in $\mathcal{G}' \cap \mathrm{Dom}\,(\mathfrak{P}_-)$ such that

$$\frac{\|u_2\|}{\|g\|} \in (1-\epsilon, 1+\epsilon), \ \frac{\|(\mathfrak{P}_- - \mu)u_2\|}{\|(\mathfrak{P}_- - \mu)g\|} \in (1-\epsilon, 1+\epsilon).$$

Therefore the inequalities of the form (6.9), (6.10) can be written for u_2 as well, just with a slight worsening of constants, the value of which is not important:

$$\|(\mathfrak{P}_- - \mu)u_2\|^2 - \tau^2\|u_2\|^2 \geq c_4\|u_2\|^2,$$

as well as

$$\|(\mathfrak{P}_- - \mu)u_2\|^2 - \tau^2\|u_2\|^2 \geq c_5\|(\mathfrak{P}_- - \mu)u_2\|^2. \tag{6.12}$$

Now we can consider the complete second term in (6.5). This expression can be written as

$$\|(\mathfrak{P}_- + \mathfrak{V} - \mu)u_2\|^2 - \tau^2\|u_2\|^2 = \|(\mathfrak{P}_- - \mu)u_2\|^2 - \tau^2\|u_2\|^2 \tag{6.13}$$
$$+\|\mathfrak{V}u_2\|^2 + 2\Re((\mathfrak{P}_- - \mu)u_2, \mathfrak{V}u_2), \ u_2 \in \hat{\mathcal{G}}.$$

The form $\|\mathfrak{V}u_2\|^2$ is compact with respect to $\|(\mathfrak{P}_- - \mu)u_2\|^2$, therefore on a subspace of finite codimension we have $\|\mathfrak{V}u_2\|^2 < \frac{c_5}{4}\|(\mathfrak{P}_- - \mu)u_2\|^2$. Further on, by Cauchy inequality,

$$|2\Re((\mathfrak{P}_- - \mu)u_2, \mathfrak{V}u_2)| \leq \frac{c_5}{4}\|(\mathfrak{P}_- - \mu)u_2\|^2 + \frac{4}{c_5}\|\mathfrak{V}u_2\|^2. \tag{6.14}$$

Again, using the same relative compactness, we can restrict ourselves to a subspace of finite codimension so that the last term in (6.14) is no greater than $\frac{c_5}{4}\|(\mathfrak{P}_- - \mu)u_2\|^2$. After substituting these estimates into (6.13) and recalling (6.12), we obtain

$$\|(\mathfrak{P}_- + \mathfrak{V} - \mu)u_2\|^2 - \tau^2\|u_2\|^2 \geq \frac{3c_5}{4}\|(\mathfrak{P}_- - \mu)u_2\|^2 \geq c_6\|u_2\|^2 \quad (6.15)$$

on a subspace $\mathscr{L}_2$ of finite codimension in $\mathcal{G}' \cap \mathrm{Dom}\,(\mathfrak{P}_-)$. Note that the subspaces here do not depend on the value of λ.

The third term. We are going to show now that the third term in (6.5) is majorized by the sum of two first ones. We can write it in the form

$$2\Re((\mathfrak{P}_-+\mathfrak{V}-\mu)u_1, (\mathfrak{P}_-+\mathfrak{V}-\mu)u_2) = 2\Re((\mathfrak{P}_-+\mathfrak{V}-\mu)^2u_1, u_2). \quad (6.16)$$

The function $u_1 \in \mathcal{G}_q$ can be written as $u_1 = \overline{\mathfrak{Q}}^q v_1$ for some $v_1 \in \mathcal{H}_0$. So we have

$$\begin{aligned}(\mathfrak{P}_- + \mathfrak{V} - \mu)^2 u_1 = (\overline{\mathfrak{Q}}\mathfrak{Q} - \mu)^2\overline{\mathfrak{Q}}^q v_1 + (\mathfrak{P}_- - \mu)\mathfrak{V}u_1 + \quad (6.17)\\ \mathfrak{V}(\mathfrak{P}_- - \mu)u_1 + V^2u_1 = (\overline{\mathfrak{Q}}\mathfrak{Q} - \mu)^2\overline{\mathfrak{Q}}^q v_1 + G.\end{aligned}$$

We start with the first term in (6.17).

In the expression $\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}^q v_1$ we perform our usual commuting procedure moving $\mathfrak{Q}$ to the right all the time. As soon as $\mathfrak{Q}$ reaches the utmost right position, the corresponding term vanishes. Thus we arrive at the representation

$$\begin{aligned}\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}^q v = 2\sum \overline{\mathfrak{Q}}^j(\mathbf{B}^\circ + \mathbf{b})\overline{\mathfrak{Q}}^{q-j}v_1 \quad (6.18)\\ = 2q\mathbf{B}^\circ\overline{\mathfrak{Q}}^q v_1 + \sum_{j=1}^{q}\overline{\mathfrak{Q}}^j\mathbf{b}\overline{\mathfrak{Q}}^{q-j}v = \Lambda u_1 + F_1.\end{aligned}$$

In a similar way, we transform the expression $\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}\mathfrak{Q}u_1 = \overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}^q v_1$: we move both copies of the operator $\mathfrak{Q}$ to the utmost right position, where they vanish, being applied to $v_1 \in \mathcal{H}_0$. What is left has the form

$$\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}\mathfrak{Q}\overline{\mathfrak{Q}}^q v_1 = \Lambda^2 u_1 + F_2. \quad (6.19)$$

The functions F_1, F_2 in (6.18), (6.19) are compositions of several, no more than q, operators $\overline{\mathfrak{Q}}$ and at least one function with compact support, applied to v_1. The term $G = ((\mathfrak{P}_- - \mu)\mathfrak{V} + \mathfrak{V}(\mathfrak{P}_- - \mu) + \mathfrak{V}^2)\overline{\mathfrak{Q}}^q v_1$ in (6.17) involves the differential operator of order $q+2$ with bounded compactly supported coefficients, applied to v_1. So we have

$$(\mathfrak{P}_- + V - \mu)^2 u_1 = (\Lambda - \mu)^2 u_1 - 2\mu F_1 + F_2 + G.$$

We substitute this expression into (6.16) and use the fact that u_1 and u_2 are orthogonal. This leads us to

$$\begin{aligned}2|\Re((\mathfrak{P}_- + V - \mu)^2 u_1, u_2)| = 2|\Re((-2\mu F_1 + F_2 + G), u_2)|\\ \leq M\| - 2\mu F_1 + F_2 + G\|^2 + M^{-1}\|u_2\|^2,\end{aligned}$$

where $M > 0$ can be chosen arbitrarily. We fix M larger than $16c_5^{-1}$, where c_5 is the constant in (6.15). After this, we take ϵ smaller than

$\frac{c_5}{16M}$. By Proposition 5.2 we can choose the subspace $\mathscr{M}_1 \subset \mathcal{H}_0$, constructed when considering the first term in (6.4), in such way that the codimension of $\mathscr{M}_1$ in $\mathcal{H}_0$ is no greater than $(1+\epsilon)\Xi(\varkappa\lambda)$ for some κ and that $\|-2\mu F_1 + F_2 + G\|^2$ is no greater than $\frac{c_1}{2}\lambda\|v_1\|^2$. With all parameters chosen in this way, for $u_1 \in \mathscr{L}_1 = \overline{\mathfrak{Q}}^q\mathscr{M}_1$ and $u_2 \in \mathscr{L}_2$ the third term in (6.5) is majorated by the sum of the first two terms, and thus the whole expression in (6.5) is non-negative for $u = u_1 + u_2 \in \mathscr{L}_1 \oplus \mathscr{L}_2$. The subspace $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2$ has codimension no greater than $(1+\epsilon)\Xi(\kappa\lambda) + N(\epsilon)$. As a result, due to ϵ being arbitrary and to the fact that $\Xi(\kappa\lambda) \equiv \Xi(\lambda), \lambda \to 0$, we have

$$\limsup_{\lambda\to 0} N_-(\lambda; q, \mathbf{b}, \mathfrak{V})\Xi(\lambda)^{-1} = \limsup_{\lambda\to 0} N_-(\lambda; q, \mathbf{b}, \mathfrak{V})\Xi(\varkappa\lambda)^{-1} \le (1+\epsilon)$$

Finally, we let $\epsilon \to 0$, and obtain (6.3). □

6.2. **Lower estimate.** Now we establish an estimate for $N_\pm(\lambda)$ from below. The conditions for this estimate to hold are expressed in the terms of an effective potential constructed for each LL from the magnetic field $\mathbf{b}$ and the electric potential V. The complicated form of this effective potential reflects the complicated character of the influence of the magnetic perturbation onto the behavior of eigenvalues: we remind again that the latter dependence is not monotone. Our construction requires considerable smoothness of $\mathbf{b}$ and V; in order to avoid extra technicalities, we suppose here that both are infinitely smooth.

For a fixed LL $\Lambda = \Lambda_q$, with the same $s_\pm, \mu_\pm, \tau$ as before, we define the effective potentials as

$$W_\pm = W_\pm[\mathbf{b}, V, \lambda] = -(\Lambda \pm \lambda + 2\mathbf{B}^\circ)(s + 2\mathbf{B}^\circ)\mathcal{Z}_q[\mathbf{b}] - \mathcal{Z}_{q+2}[\mathbf{b}] \quad (6.20)$$
$$+2(\mu_\pm + 3\mathbf{B}^\circ)\mathcal{Z}_{q+1}[\mathbf{b}] - \mathcal{X}_q[\mathbf{b}, 4(2\mathbf{B} - \mathbf{b} + \mu_\pm)\mathbf{b} - (4\mathbf{B} - 2\mu_\pm + V)V]$$
$$-2\mathcal{X}_{q+1}[\mathbf{b}, V - 3\mathbf{b}] - 4\Im\mathcal{Y}_q[\mathbf{b}, \partial V - 2\partial b],$$

where the expressions $\mathcal{Z}, \mathcal{X}, \mathcal{Y}$ are described in Proposition 3.4.

Theorem 6.3. *Suppose that for λ close to zero the potential $W_\pm$ is non-negative and, moreover, there exists an open set Ω where $W_\pm \ge c > 0$ for c independent on λ. Then*

$$\liminf_{\lambda\to 0} N_\pm(\lambda, q, \mathbf{b}, V)\Xi(\lambda)^{-1} \ge 1 \quad (6.21)$$

Proof. We consider the eigenvalues below Λ and prove the estimate (6.21) with 'minus' sign. The other case is proved in an almost identical way.

We are going to construct the subspace $\mathscr{L}$ in $\mathcal{G} = \mathcal{G}_q$, having dimension asymptotically greater than $(1-\epsilon)\Xi(\lambda)$ such that the inequality

$$\|(\mathfrak{P}_- + V - \mu)u\|^2 < \tau^2\|u\|^2 \quad (6.22)$$

is satisfied for all $u \in \mathscr{L} \setminus \{0\}$. To do this, we set $u = \overline{\mathfrak{Q}}^q v$, $v \in \mathcal{H}_0$, and thus reduce both parts of (6.22) to quadratic forms on the lowest Landau level using Proposition 3.4 and then apply Proposition 5.3.

So, we have

$$\|(\mathfrak{P}_- + V - \mu)u\|^2 - \tau^2\|u\|^2 = (\mathfrak{P}_- u, \mathfrak{P}_- u) + 2\Re(Vu, \mathfrak{P}_- u) + \quad (6.23)$$
$$(\mu^2 - \tau^2)\|u\|^2 + (Vu, Vu) - 2\mu(Vu, u) - 2\mu(\mathfrak{P}_- u, u).$$

We substitute $u = \overline{\mathfrak{Q}}^q v,\ v \in \mathcal{H}_0$ into (6.23) and consider each term. First, by (3.5), we have

$$(\mu^2 - \tau^2)\|u\|^2 = (\mu^2 - \tau^2)C_q\|v\|^2 + (\mu^2 - \tau^2)(\mathcal{Z}_q[\mathbf{b}]v, v). \quad (6.24)$$

By (3.6) applied for $U = V$ and then for $U = V^2$, we obtain

$$-2\mu(Vu, u) = -2\mu(\mathcal{X}_q[\mathbf{b}, V]v, v);\ (Vu, Vu) = (\mathcal{X}_q[\mathbf{b}, V^2]v, v). \quad (6.25)$$

For the last term in (6.23), we have

$$-2\mu(\mathfrak{P}_- u, u) = -2\mu(\overline{\mathfrak{Q}}\mathfrak{Q}u, u) = -2\mu(\mathfrak{Q}\overline{\mathfrak{Q}}u, u) + 4\mu\mathbf{B}^\circ(u, u) \quad (6.26)$$
$$+4\mu(\mathbf{b}u, u) = -2\mu C_{q+1}\|v\|^2$$
$$-2\mu(\mathcal{Z}_{q+1}[\mathbf{b}]v, v) + 4\mu\mathbf{B}^\circ C_q\|v\|^2 + 4\mu\mathbf{B}^\circ(\mathcal{Z}_q[\mathbf{b}]v, v) + 4\mu(\mathcal{X}_q[\mathbf{b}, \mathbf{b}]v, v).$$

Further on, commuting $\mathfrak{Q}, \overline{\mathfrak{Q}}$, we get

$$2\Re(Vu, \mathfrak{P}_- u) = 2\Re(Vu, \overline{\mathfrak{Q}}\mathfrak{Q}u) = 2\Re(\overline{\mathfrak{Q}}Vu, \overline{\mathfrak{Q}}u) - 4(\mathbf{B}Vu, u) \quad (6.27)$$
$$= 2(V\overline{\mathfrak{Q}}u, \overline{\mathfrak{Q}}u) + 2\Re([\overline{\mathfrak{Q}}, V]u, \overline{\mathfrak{Q}}u) - 4(\mathbf{B}Vu, u).$$

To the first and the third terms in (6.27), we apply (3.6), and to the second term we apply (3.7) (with $U = -2i\partial V = [\overline{\mathfrak{Q}}, V]$), obtaining

$$2\Re(Vu, \mathfrak{P}_- u) \quad (6.28)$$
$$= 2(\mathcal{X}_{q+1}[\mathbf{b}, V]v, v) - 4(\mathcal{X}_q[\mathbf{b}, \mathbf{B}V]v, v) + 4\Im(\mathcal{Y}_q[\mathbf{b}, \partial V]v, v).$$

Finally, for the term $(\mathfrak{P}_- u, \mathfrak{P}_- u)$ we have

$$\|\mathfrak{P}_- u\|^2 = ((\mathfrak{Q}\overline{\mathfrak{Q}} - 2\mathbf{B})u, (\mathfrak{Q}\overline{\mathfrak{Q}} - 2\mathbf{B})u) = (\mathfrak{P}_-\overline{\mathfrak{Q}}u, \overline{\mathfrak{Q}}u) \quad (6.29)$$
$$+4\mathbf{B}^{\circ 2}(u, u) + 4((2\mathbf{B}^\circ + \mathbf{b})\mathbf{b}u, u) - 4\mathbf{B}^\circ\|\overline{\mathfrak{Q}}u\|^2 - 4\Re(\overline{\mathfrak{Q}}\mathbf{b}u, \overline{\mathfrak{Q}}u).$$

All these terms are of the form already considered in (6.26), (6.27),(6.28). Thus we obtain

$$(\mathfrak{P}_- u, \mathfrak{P}_- u) = C_{q+2}\|v\|^2 + (\mathcal{Z}_{q+2}[\mathbf{b}]v, v) - 2\mathbf{B}^\circ C_{q+1}\|v\|^2 \quad (6.30)$$
$$-2\mathbf{B}^\circ(\mathcal{Z}_{q+1}[\mathbf{b}]v, v) - 2(\mathcal{X}_{q+1}[\mathbf{b}, \mathbf{b}]v, v) + 4\mathbf{B}^{\circ 2}(C_q\|v\|^2 + (\mathcal{Z}_q[\mathbf{b}]v, v))$$
$$-4\mathbf{B}^\circ(C_{q+1}\|v\|^2 + (\mathcal{Z}_{q+1}[\mathbf{b}]v, v)) + 4(\mathcal{X}_q[\mathbf{b}, (2\mathbf{B}^\circ + \mathbf{b})\mathbf{b}]v, v)$$
$$-4(\mathcal{X}_{q+1}[\mathbf{b}, \mathbf{b}]v, v) - 8\Im(\mathcal{Y}_q[\mathbf{b}, \partial\mathbf{b}]v, v).$$

Now we collect the expressions (6.24), (6.25), (6.26), (6.28) (6.30) to get

$$\begin{gathered}\|(\mathfrak{P}_-+V-\mu)u\|^2-\tau^2\|u\|^2=(\mu^2-\tau^2)(C_q\|v\|^2+(\mathscr{Z}_q[\mathbf{b}]v,v)) \quad (6.31)\\ -2\mu(\mathscr{X}_q[\mathbf{b},V]v,v)+(\mathscr{X}_q[\mathbf{b},V^2]v,v)-2\mu C_{q+1}\|v\|^2-2\mu(\mathscr{Z}_{q+1}[\mathbf{b}]v,v)\\ +4\mu\mathbf{B}^\circ C_q\|v\|^2+4\mu\mathbf{B}^\circ(\mathscr{Z}_q[\mathbf{b}]v,v)+4\mu(\mathscr{X}_q[\mathbf{b},\mathbf{b}]v,v)+2(\mathscr{X}_{q+1}[\mathbf{b},V]v,v)\\ -4(\mathscr{X}_q[\mathbf{b},\mathbf{b}V]v,v)+4\Im(\mathscr{Y}_q[\mathbf{b},\partial V]v,v)+C_{q+2}\|v\|^2+(\mathscr{Z}_{q+2}[\mathbf{b}]v,v)\\ -2\mathbf{B}^\circ C_{q+1}\|v\|^2-2\mathbf{B}^\circ(\mathscr{Z}_{q+1}[\mathbf{b}]v,v)-2(\mathscr{X}_{q+1}[\mathbf{b},\mathbf{b}]v,v)+4\mathbf{B}^{\circ 2}C_q\|v\|^2\\ +4\mathbf{B}^{\circ 2}(\mathscr{Z}_q[\mathbf{b}]v,v)-4\mathbf{B}^\circ C_{q+1}\|v\|^2-4\mathbf{B}^\circ(\mathscr{Z}_{q+1}[\mathbf{b}]v,v)\\ +4(\mathscr{X}_q[\mathbf{b},(2\mathbf{B}^\circ+\mathbf{b})\mathbf{b}]v,v)-4(\mathscr{X}_{q+1}[\mathbf{b},\mathbf{B}]v,v)-8\Im(\mathscr{Y}_q[\mathbf{b},\partial\mathbf{b}]v,v).\end{gathered}$$

The terms with $\|v\|^2$ have the coefficient

$$\begin{gathered}(\mu^2-\tau^2)C_q-2\mu C_{q+1}+4\mu\mathbf{B}^\circ C_q+C_{q+2}-2\mathbf{B}^\circ C_{q+1}+4\mathbf{B}^{\circ 2}C_q\\ -4\mathbf{B}^\circ C_{q+1}=((\mu^2-\tau^2)-2\mu\Lambda+\Lambda^2)C_q=\lambda(\Lambda-s)C_q.\end{gathered}$$

The sum of all remaining terms in (6.31) equals exactly $(-W_-v,v)$, where W_- is given by (6.20). Now, by Proposition 5.4 it is possible to find a subspace $\mathscr{M}(\lambda)\subset\mathcal{H}_0$ such that $(Wv,v)>\lambda(L-s)C_q\|v\|^2$ for $v\in\mathcal{H}_0$ and dimension of $\mathscr{M}(\lambda)$ is greater than $\Xi(\lambda(L-s)C_q)(1-o(\lambda))$. This is the subspace we need. □

If the conditions of Theorem 6.3 are satisfied then the conditions of Theorem 6.2 are satisfied as well and we arrive at the asymptotic formula

$$\lim_{\lambda\to 0}N_\pm(\lambda,q,\mathbf{b},V)\Xi(\lambda)^{-1}=1 \quad (6.32)$$

7. Proofs of technical lemmas

Proof of Lemma 3.2. As usual, we may prove the inequality for functions u in the Schwartz space first, and then extend it to all functions for which the right-hand side of (3.2) is finite, by continuity.

We will prove Lemma by a double induction. For $N=1$ the statement is obvious: $||\mathfrak{Q}u||^2=(\mathfrak{P}_-u,u)$, $||\overline{\mathfrak{Q}}u||^2=(\mathfrak{P}_+u,u)=(\mathfrak{P}_-u,u)+2(\mathbf{B}u,u)$. Now, suppose that for some N_0 we have established (3.2) for all $N<N_0$ and for all combinations of T_j. The (exterior) induction step will consist of proving (3.2) for $N=N_0$.

To do this, we are going to show first that one can replace $\mathfrak{P}_-$ by $\mathfrak{P}_+$ on the right-hand side of (3.2) for all $N<N_0$. This will follow as soon as we prove the inequality

$$(\mathfrak{P}_-^Nu,u)\le C(\mathfrak{P}_+^Nu,u)+C'(u,u),\ N\le N_0 \quad (7.1)$$

(note that the case $N=N_0$ *is* included here.) This fact we again prove by induction (interior) in N. The statement is obvious for $N=1$.

Suppose that we have proved it for all $N < N_1 \leq N_0$ and we will check it for $N = N_1$. In fact, by (2.10),

$$\mathfrak{P}_-^{N_1} = \mathfrak{P}_+^{N_1} + \mathfrak{R}, \tag{7.2}$$

where $\mathfrak{R}$ is a sum of many terms $\mathfrak{R}_\kappa$. Each $\mathfrak{R}_\kappa$ is the product of less then N_1 factors $\mathfrak{P}_-$ and some factors $2\mathbf{B}$. In other words, this means that $\mathfrak{R}_\kappa$ is a product of no more than $N_1 - 1$ operators $\mathfrak{Q}$, no more than $N_1 - 1$ operators $\overline{\mathfrak{Q}}$ and several functions, with all derivatives bounded. Now we transform each term $\mathfrak{R}_\kappa$ in the following way. We move half of the factors $\mathfrak{Q}, \overline{\mathfrak{Q}}$ in $\mathfrak{R}_\kappa$ to the left and the other half of these operators to the right, commuting them with functions and using the relations (3.8).

So, many terms will be produced, having the form

$$T_1 \dots T_k F T_{k+1} \dots T_{k+m}, \tag{7.3}$$

where $k, m < N_1$, each T_j is $\mathfrak{Q}$ or $\overline{\mathfrak{Q}}$ and F is a bounded function. Now we can estimate each term in (7.3):

$$\begin{aligned} |(T_1 \dots T_k F T_{k+1} \dots T_{k+m} u, u)| &= |(F T_{k+1} \dots T_{k+m} u, T_k^* \dots T_1^* u)| \\ &\leq C \|F T_{k+1} \dots T_{k+m} u\| \, \|T_k^* \dots T_1^* u\|. \end{aligned} \tag{7.4}$$

The right-hand side in (7.4) can be estimated by the interior and the exterior inductive assumptions, since there are less than N_1 and less than N_0 factors T_j involved. Summing all these estimates over all terms generated by $\mathfrak{R}$, we arrive at

$$(\mathfrak{R}u, u) \leq C(\mathfrak{P}_+^{N_1 - 1} u, u) + C'(u, u),$$

which, together with (7.2) leads to (7.1) for $N = N_1$. Changing places of $\mathfrak{P}_+$ and $\mathfrak{P}_-$, we obtain the dual inequality

$$(\mathfrak{P}_+^N u, u) \leq C(\mathfrak{P}_-^N u, u) + C'(u, u), \ N \leq N_0. \tag{7.5}$$

Now, the inductive step in the exterior induction goes rather easily. For $N = N_0$, let us have N operators $T_1, \dots, T_N$. Suppose that $T_N = \mathfrak{Q}$. Then we can write $||T_1 \dots T_N u||^2$ as $||T_1 \dots T_{N-1} v||^2$ with $v = \mathfrak{Q}u$. By the inductive assumption and (7.1), we have $||T_1 \dots T_{N-1} v||^2 \leq C(\mathfrak{P}_+^{N-1} v, v) + C'(v, v)$, the first term here equals $C(\mathfrak{P}_-^N u, u)$ so we arrive at (3.2). In the case of $T_N = \overline{\mathfrak{Q}}$, we write the left-hand side in (3.2) as $||T_1 \dots T_{N-1} v||^2$ with $v = \overline{\mathfrak{Q}}u$. By the inductive assumption, this quantity is majorized by $(\mathfrak{P}_-^{N-1} v, v) + (v, v)$ which equals $(\mathfrak{P}_+^N u, u) + (\mathfrak{P}_+ u, u)$. Applying (7.5), we again obtain the estimate (3.2). □

Proof of Lemma 4.2. If $\mathfrak{A}$ satisfies the conditions of the Lemma and we commute some entry of the resolvent with a neighboring function or the operators $\mathfrak{Q}$ or $\overline{\mathfrak{Q}}$ then all resulting operators will again satisfy these conditions. So we perform such commutings by the following rule. First, move one copy of the resolvent to the utmost right position.

Then if there are three or more operators $\mathfrak{Q}$, $\overline{\mathfrak{Q}}$ in a row in front of some resolvent, commute the utmost right of these operator with resolvent. After performing this operation as long as it is needed, we arrive at the situation where in each summand no more than two operators $\mathfrak{Q}, \overline{\mathfrak{Q}}$ stand in front of each copy of the resolvent. Each such group is a bounded operator, by Lemma 3.2. At least one copy of the resolvent has no more than one operator $\mathfrak{Q}, \overline{\mathfrak{Q}}$ in front of it, because the order of the whole product is negative. After this, we move the function h that has compact support to the position in front of this particular copy of the resolvent. In this process, the structure of all products remains the same, moreover, when we commute h with resolvent, a compact term arises. Finally, we arrive at the term $hTR_\pm$ or $hR_\pm$, where T is $\mathfrak{Q}$ or $\overline{\mathfrak{Q}}$. This operator is, obviously, compact. All other factors in the product are bounded, therefore the whole product is compact. □

Proof of Lemma 4.3. Boundedness of the operator in question is derived from Lemma 3.2 in the same way as Proposition 3.3 is proved. By means of commuting, preserving the overall structure of all arising terms, we arrive at a sum of operators of the form

$$\tilde{h}_\kappa \mathcal{T}_\kappa P_q \mathcal{T}_\varkappa \tilde{h}_\varkappa = \tag{7.6}$$
$$[\tilde{h}_\kappa \mathcal{T}_\kappa (\mathfrak{P}_- + 1)^{-N}][(\mathfrak{P}_- + 1)^N P_q (\mathfrak{P}_- + 1)^N][(\mathfrak{P}_- + 1)^{-N} \mathcal{T}_\varkappa \tilde{h}_\varkappa],$$

where $\mathcal{T}_\varkappa, \mathcal{T}_\kappa$ are products of no more than N operators $\mathfrak{Q}, \overline{\mathfrak{Q}}$. The middle term in (7.6) is bounded by Proposition 3.1, and the extreme terms are bounded by Lemma 3.2. If, say, the function $\tilde{h}_\kappa$ is compactly supported, we can take N larger than the number of terms in $\mathcal{T}_\kappa$, and then the operator $\tilde{h}_\kappa \mathcal{T}_\kappa (\mathfrak{P}_- + 1)^{-N}$ has negative order and is compact by Lemma 4.2. □

References

[1] Bargmann, V. On a Hilbert space of analytic functions and an associated integral transform. Part II. A family of related function spaces. Application to distribution theory. Comm. Pure Appl. Math. 20, 1–101 (1967)

[2] Berezin, F., Shubin, M.: The Schrödinger Equation, Kluwer Academic Publishers, Dordrecht, 1991.

[3] Besch, A.: Eigenvalues in spectral gaps of the two-dimensional Pauli operator. J. Math. Phys. 41, 7918-7931 (2000)

[4] Exner, P. Stovicek, P., Vytras, P.: Generalized boundary conditions for the Aharonov-Bohm effect combined with a homogeneous magnetic field. J. Math. Phys. 43, 2151–2168 (2002)

[5] Filonov, N., Pushnitski, A.: Spectral asymptotics of Pauli operators and orthogonal polynomials in complex domains. Comm. Math. Phys. 264, 759 – 772 (2006)

[6] Fock, V.: Konfigurationsraum und zweite Quantelung, Zeit. für Fys. 75, 622–647 (1932)

[7] Folland G.: Harmonic Analysis in Phase Space, Ann.Math. Studies, 122, Princeton Univ. Press, Princeton, 1989.

[8] Hempel, R. Levendorski, S.: On eigenvalues in gaps for perturbed magnetic Schrödinger operators. J. Math. Phys. 39, 63–78 (1998)

[9] Iwatsuka, A.: The essential spectrum of two-dimensional Schrödinger operators with perturbed constant magnetic fields. J. Math. Kyoto Univ. 23, 475–480 (1983)

[10] Landau, L., Lifshitz, E.: Quantum Mechanics (Non-relativistic theory). 3rd edition, Pergamon Press, Oxford, 1977.

[11] Melgaard, M., Rozenblum, G.: Eigenvalue asymptotics for weakly perturbed Dirac and Schrödinger operators with constant magnetic fields of full rank. Comm. Partial Differential Equations 28, 697–736 (2003)

[12] Raikov, G.: Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips. Comm. Partial Differential Equations 15, 407–434 (1990)

[13] Raikov, G. Warzel, S.: Quasi-classical versus non-classical spectral asymptotics for magnetic Schroödinger operators with decreasing electric potentials. Rev. Math. Phys. 14, 1051–1072 (2002)

[14] Rozenblum, G., Shirokov, N.: Infiniteness of zero modes for the Pauli operator with singular magnetic field. Jour. of Functional Analysis 233, 135-172 (2006)

(G. Rozenblum) Department of Mathematics, Chalmers University of Technology, and University of Gothenburg, Chalmers Tvärgatan,3 , S-412 96 Gothenburg, Sweden

E-mail address: `grigori@math.chalmers.se`

(G. Tashchiyan) Department of Mathematics, St. Petersburg University for Telecommunications, St.Petersburg, 198504, Russia

E-mail address: `grigori.tashchiyan@gmail.com`