Natural geometry of nonzero quaternions

Vladimir Trifonov
AMS
trifonov@member.ams.org

Abstract

It is shown that the group of nonzero quaternions carries a family of natural closed Friedmann-Lemaître-Robertson-Walker metrics.

Introduction.

The quaternion algebra $\mathbb{H}$ is one of the most important and well-studies objects in mathematics (e. g. [Wid02] and references therein) and physics (e. g. [Adl95] and references therein). It has a natural Hermitian form which induces a Euclidean scalar product on its additive vector space $S_{\mathbb{H}}$. There is also a family of natural indefinite scalar products of signature 2 on $S_{\mathbb{H}}$ [Tri95], induced by the structure tensor $\boldsymbol{H}$ of the quaternion algebra. This result came out of a study of relationship between natural metric properties of *unital* algebras and internal logic of topoi they generate. It was shown in [Tri95] that if the logic of a topos is bivalent Boolean then the generating algebra is isomorphic to the quaternion algebra with a family of natural scalar product of signature 2. Such scalar products can be defined on any linear algebra over a field $\mathbb{F}$. In this note we show that for a unital algebra these scalar products can be naturally extended over the Lie group of its invertible elements, producing a family of *principal metrics*. In particular, for the quaternion algebra, these metrics are closed Friedmann-Lemaître-Robertson-Walker.

Remark 0.1. Some of the notations and definitions are slightly nonstandard. We use the $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ device to denote tensor ranks; for example a one-form is a $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$-tensor. For clarity of the exposition we use $\square$ at the end of a *Proof*, and each *Remark* ends the sign appearing at the end of this line. $\diamond$

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Definition 0.1. An $\mathbb{F}$-*algebra*, A, is an ordered pair $(S_A, \boldsymbol{A})$, where S_A is a vector space over a field $\mathbb{F}$, and $\boldsymbol{A}$ is a $[{}^1_2]$-tensor on S_A, called the *structure tensor* of A. Each vector $\boldsymbol{a}$ of S_A is called an *element* of A, denoted $\boldsymbol{a} \in A$. The *dimensionality* of A is that of S_A.

Remark 0.2. This is an unconventional definition of a linear algebra over $\mathbb{F}$. Indeed, the tensor $\boldsymbol{A}$ induces a binary operation $S_A \times S_A \to S_A$, called the *multiplication* of A: to each pair of vectors $(\boldsymbol{a}, \boldsymbol{b})$ the tensor $\boldsymbol{A}$ associates a vector $\boldsymbol{ab} : S_A^* \to \mathbb{F}$, such that $(\boldsymbol{ab})(\tilde{\boldsymbol{\tau}}) = \boldsymbol{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}), \forall \tilde{\boldsymbol{\tau}} \in S_A^*$. An $\mathbb{F}$-algebra with an associative multiplication is called *associative.* An element $\boldsymbol{\imath}$, such that $\boldsymbol{a\imath} = \boldsymbol{\imath a} = \boldsymbol{a}, \forall \boldsymbol{a} \in A$ is called an *identity* of A. ◇

Definition 0.2. For an $\mathbb{F}$-algebra A and each nonzero one-form $\tilde{\boldsymbol{\tau}} \in S_A^*$, the tensor $\boldsymbol{A}[\tilde{\boldsymbol{\tau}}] := \tilde{\boldsymbol{\tau}} \lrcorner \boldsymbol{A}$ is called a *principal scalar product* on A, just in case it is symmetric, where $\lrcorner$ denotes contraction on the first index.

Definition 0.3. For each $\mathbb{F}$-algebra $A = (S_A, \boldsymbol{A})$, an $\mathbb{F}$-algebra $[A] = (S_A, [\boldsymbol{A}])$, with the structure tensor defined by

$$[\boldsymbol{A}](\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) := \boldsymbol{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) - \boldsymbol{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{b}, \boldsymbol{a}), \forall \tilde{\boldsymbol{\tau}} \in S_A^*, \boldsymbol{a}, \boldsymbol{b} \in S_A,$$

is called the *commutator* algebra of A.

Definition 0.4. A finite dimensional associative $\mathbb{R}$-algebra with an identity is called a *unital* algebra.

Lemma 0.1. *The set $\mathcal{A}$ of all invertible elements of a unital algebra A is a Lie group with respect to the multiplication of A whose Lie algebra of $\mathcal{A}$ is the commutator algebra $[A]$.*

Proof. See, for example, [Pos82] for a proof of this simple lemma. □

Remark 0.3. For each basis $(\boldsymbol{e}_j)$ on the vector space S_A of a unital algebra, there is a natural basis field on $\mathcal{A}$, namely the basis $(\hat{\boldsymbol{e}}_j)$ of left invariant vector fields generated by $(\boldsymbol{e}_j)$, associating to each point $\boldsymbol{a} \in A$ a basis $(\hat{\boldsymbol{e}}_j)(\boldsymbol{a})$ on the tangent space $T_{\boldsymbol{a}}\mathcal{A}$. We call $(\hat{\boldsymbol{e}}_j)$ a *proper frame generated by* $(\boldsymbol{e}_j)$. The value of $(\hat{\boldsymbol{e}}_j)$ at $\boldsymbol{a}$ is referred to as a *proper basis* (at $\boldsymbol{a}$) generated by $(\boldsymbol{e}_j)$, and denoted $(\hat{\boldsymbol{e}}_j)(\boldsymbol{a})$. In particular, $(\hat{\boldsymbol{e}}_j)(\boldsymbol{\imath})$, the proper basis at the identity generated by $(\boldsymbol{e}_j)$ coincides with $(\boldsymbol{e}_j)$. ◇

Definition 0.5. For a unital algebra A, let $(\hat{\boldsymbol{e}}_j)$ be a proper frame on $\mathcal{A}$, generated by a basis $(\boldsymbol{e}_j)$ on S_A. The *structure field* of the Lie group $\mathcal{A}$ is a tensor field $\boldsymbol{\mathcal{A}}$ on $\mathcal{A}$, assigning to each point $\boldsymbol{a} \in \mathcal{A}$ a $[\frac{1}{2}]$-tensor $\boldsymbol{\mathcal{A}}(\boldsymbol{a})$ on $T_a\mathcal{A}$, with components $\mathcal{A}^i_{jk}(\boldsymbol{a})$ in the basis $(\hat{\boldsymbol{e}}_j)(\boldsymbol{a})$, defined by

$$\mathcal{A}^i_{jk}(\boldsymbol{a}) := A^i_{jk}, \quad \forall \boldsymbol{a} \in \mathcal{A},$$

where A^i_{jk} are the components of the structure tensor $\boldsymbol{A}$ in the basis $(\boldsymbol{e}_j)$.

Definition 0.6. For a unital algebra A and each $\boldsymbol{a} \in \mathcal{A}$, an $\mathbb{F}$-algebra $\mathcal{A}(\boldsymbol{a}) = (T_{\boldsymbol{a}}\mathcal{A}, \boldsymbol{\mathcal{A}}(\boldsymbol{a}))$ is called the *tangent algebra* of the Lie group $\mathcal{A}$ at $\boldsymbol{a}$.

Remark 0.4. It is easy to see that for each $\boldsymbol{a} \in \mathcal{A}$, the tangent algebra $\mathcal{A}(\boldsymbol{a})$ is isomorphic to A; in particular, each $A(\boldsymbol{a})$ is unital. $\diamond$

Definition 0.7. For a unital algebra A and a twice differentiable real function $\mathfrak{T}$ on the Lie group $\mathcal{A}$, a *principal metric on* $\mathcal{A}$ is a $[\frac{0}{2}]$-tensor field $\boldsymbol{\mathfrak{T}} := d\mathfrak{T} \lrcorner \boldsymbol{\mathcal{A}}$, where $d\mathfrak{T}$ is the gradient of $\mathfrak{T}$, such that $\boldsymbol{\mathfrak{T}}(\boldsymbol{a})$ is a principal scalar product on $\mathcal{A}(\boldsymbol{a}), \forall \boldsymbol{a} \in \mathcal{A}$.

1 Quaternion algebra.

Definition 1.1. A four dimensional $\mathbb{R}$-algebra, $\mathbb{H} = (S_{\mathbb{H}}, \boldsymbol{H})$, is called a *quaternion algebra* (with *quaternions* as its elements), if there is a basis on $S_{\mathbb{H}}$, in which the components of the structure tensor $\boldsymbol{H}$ are given by the entries of the following matrices,

$$H^0_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ H^1_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$H^2_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ H^3_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

We refer to such a basis as *canonical*.

Remark 1.1. The vectors of the canonical basis are denoted $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. A quaternion algebra is unital, with the first vector of the canonical basis, $\mathbf{1}$, as its identity. Since $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is a basis on a real vector space, any quaternion $\boldsymbol{a}$ can be presented as $a^0\mathbf{1}+a^1\boldsymbol{i}+a^2\boldsymbol{j}+a^3\boldsymbol{k}$, $a^\beta \in \mathbb{R}$. A quaternion $\bar{\boldsymbol{a}} = a^0\mathbf{1}-a^1\boldsymbol{i}-a^2\boldsymbol{j}-a^3\boldsymbol{k}$ is called *conjugate* to $\boldsymbol{a}$. We refer to a^0 and $a^p\boldsymbol{i}_p$ as the *real* and *imaginary part* of $\boldsymbol{a}$, respectively. Quaternions of the form $a^0\mathbf{1}$ are in one-to-one correspondence with real numbers, which is often denoted, with certain notational abuse, as $\mathbb{R} \subset \mathbb{H}$. $\diamond$

Remark 1.2. A linear transformation $S_{\mathbb{H}} \to S_{\mathbb{H}}$ with the following components in the canonical basis,

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{B} \end{pmatrix}, \mathbf{B} \in SO(3),$$

takes $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ to a basis $(\boldsymbol{i}_\beta)$ in which the components (1) of the structure tensor will *not* change, and neither will the multiplicative behavior of vectors of $(\boldsymbol{i}_\beta)$. Thus, we have a class of canonical bases parameterized by elements of $SO(3)$. $\diamond$

Theorem 1.1. *Every principal scalar product on* $\mathbb{H}$ *is of signature* 2.

Proof. For the quaternion algebra the components of the structure tensor $\boldsymbol{H}$ in a canonical basis are given by (1). A one-form $\tilde{\boldsymbol{\tau}}$ on $S_{\mathbb{H}}$ with components $\tilde{\tau}_\beta$ in (the dual of) a canonical basis $(\boldsymbol{i}_\beta)$ contracts with the structure tensor into a $\left[{0 \atop 2}\right]$-tensor on $S_{\mathbb{H}}$ with the following components in the basis $(\boldsymbol{i}_\beta)$:

$$\begin{pmatrix} \tilde{\tau}_0 & \tilde{\tau}_1 & \tilde{\tau}_2 & \tilde{\tau}_3 \\ \tilde{\tau}_1 & -\tilde{\tau}_0 & \tilde{\tau}_3 & -\tilde{\tau}_2 \\ \tilde{\tau}_2 & -\tilde{\tau}_3 & -\tilde{\tau}_0 & \tilde{\tau}_1 \\ \tilde{\tau}_3 & \tilde{\tau}_2 & -\tilde{\tau}_1 & -\tilde{\tau}_0 \end{pmatrix}.$$

The only way to make this symmetric is to put $\tilde{\tau}_1 = -\tilde{\tau}_1$, $\tilde{\tau}_2 = -\tilde{\tau}_2$, $\tilde{\tau}_3 = -\tilde{\tau}_3$, which yields $\tilde{\tau}_1 = \tilde{\tau}_2 = \tilde{\tau}_3 = 0$:

$$A[\tilde{\boldsymbol{\tau}}]_{\alpha\beta} = \begin{pmatrix} \tilde{\tau}_0 & 0 & 0 & 0 \\ 0 & -\tilde{\tau}_0 & 0 & 0 \\ 0 & 0 & -\tilde{\tau}_0 & 0 \\ 0 & 0 & 0 & -\tilde{\tau}_0 \end{pmatrix}. \tag{2}$$

. $\square$

2 Natural structures on $\mathcal{H}$.

There is a class of canonical bases on $S_{\mathbb{H}}$ (see Remark 1.2) whose members differ from one another by a rotation in the hyperplane of pure imaginary quaternions. Each canonical basis $(\boldsymbol{i}_\beta)$ induces a *canonical* coordinate system $(w,\ x,\ y,\ z)$ on $S_{\mathbb{H}}$, considered as a (linear) manifold, and therefore also on its submanifold $\mathcal{H}$ of nonzero quaternions: a quaternion $\boldsymbol{a} = a^\beta \boldsymbol{i}_\beta$ is assigned coordinates $(w = a^0,\ x = a^1,\ y = a^2,\ z = a^3)$. This coordinate system covers both $S_{\mathbb{H}}$ and $\mathcal{H}$ with a single patch. Since $\mathbf{0} \notin \mathcal{H}$, at least one of the coordinates is always nonzero for any point $\boldsymbol{a} \in \mathcal{H}$. For a differentiable function $R : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ there is a system of natural spherical coordinates $(\eta,\ \chi,\ \theta,\ \phi)$ on $\mathcal{H}$, related to the canonical coordinates by

$$w = R(\eta)\cos(\chi), \quad x = R(\eta)\sin(\chi)\sin(\theta)\cos(\phi),$$
$$y = R(\eta)\sin(\chi)\sin(\theta)\sin(\phi), \quad z = R(\eta)\sin(\chi)\cos(\theta).$$

Each canonical basis $(\boldsymbol{i}_\beta)$ can be considered a basis on the vector space of the Lie algebra of $\mathcal{H}$, i. e., the tangent space $T_{\mathbf{1}}\mathcal{H} \cong S_{\mathbb{H}}$ to $\mathcal{H}$ at the point (1, 0, 0, 0), the identity of the group $\mathcal{H}$. There are several natural basis fields on $\mathcal{H}$ induced by each basis $(\boldsymbol{i}_\beta)$. First of all, there are two coordinate basis fields, the *canonical frame*, $(\partial_w,\ \partial_x,\ \partial_y,\ \partial_z)$ and the corresponding *spherical frame* $(\partial_\eta,\ \partial_\chi,\ \partial_\theta,\ \partial_\phi)$. We also have a noncoordinate basis field, the *proper frame* $(\hat{\boldsymbol{i}}_\beta)$, of left invariant vector fields on $\mathcal{H}$, induced by the canonical basis. For each frame $(\boldsymbol{f}_\beta)$ on $\mathcal{H}$, its value at $\boldsymbol{a}$, i. e., a basis on $T_{\boldsymbol{a}}\mathcal{H}$, is denoted $(\boldsymbol{f}_\beta)(\boldsymbol{a})$. A left invariant vector field $\hat{\boldsymbol{u}}$ on $\mathcal{H}$, generated by a vector $\boldsymbol{u} \in S_{\mathbb{H}}$ with components (u^β) in a canonical basis, associates to each point $\boldsymbol{a} \in \mathcal{H}$ with coordinates $(w,\ x,\ y,\ z)$ a vector $\hat{\boldsymbol{u}}(\boldsymbol{a}) \in T_{\boldsymbol{a}}\mathcal{H}$ with the components $\hat{u}^\beta(\boldsymbol{a}) = (\boldsymbol{a}\boldsymbol{u})^\beta$ in the basis $(\partial_w,\ \partial_x,\ \partial_y,\ \partial_z)(\boldsymbol{a})$ on $T_{\boldsymbol{a}}\mathcal{H}$:

$$\hat{u}^0(\boldsymbol{a}) = wu^0 - xu^1 - yu^2 - zu^3, \quad \hat{u}^1(\boldsymbol{a}) = wu^1 + xu^0 + yu^3 - zu^2,$$
$$\hat{u}^2(\boldsymbol{a}) = wu^2 - xu^3 + yu^0 + zu^1, \quad \hat{u}^3(\boldsymbol{a}) = wu^3 + xu^2 - yu^1 + zu^0. \quad (3)$$

The system (3) contains sufficient information to compute transformation between the frames. For example, the transformation between the spherical and proper frames is given by

$$\begin{pmatrix} R/\dot{R} & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \frac{\cos\chi\cos\theta\cos\phi+\sin\chi\sin\phi}{\sin\chi} & \frac{\cos\chi\cos\theta\sin\phi+\sin\chi\cos\phi}{\sin\chi} & \frac{\cos\chi\sin\theta}{\sin\chi} \\ 0 & \frac{\sin\chi\cos\theta\cos\phi-\cos\chi\sin\phi}{\sin\chi\sin\theta} & \frac{\sin\chi\cos\theta\sin\phi+\cos\chi\cos\phi}{\sin\chi\sin\theta} & -1 \end{pmatrix},$$

where $\dot{R} := \frac{dR}{d\eta} : \mathbb{R} \to \mathbb{R} \setminus \{0\}$.

Definition 2.1. A Lorentzian metric on a four dimensional manifold is called *closed FLRW* (Friedmann-Lemaître-Robertson-Walker) if there is a coordinate system $(\eta, \chi, \theta, \phi)$, such that in the corresponding coordinate frame the components of the metric are given by the entries of the following matrix:

$$\pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathsf{a}^2 & 0 & 0 \\ 0 & 0 & -\mathsf{a}^2 \sin^2(\chi) & 0 \\ 0 & 0 & 0 & -\mathsf{a}^2 \sin^2(\chi) \sin^2(\theta) \end{pmatrix},$$

where $\mathsf{a} : \mathbb{R} \to \mathbb{R}$, referred to as the *scale factor*, is a function of η only.

3 Principal metrics on $\mathcal{H}$.

Theorem 3.1. *Every principal metric of* $\mathbb{H}$ *is closed FLRW.*

Proof. Let $\tilde{\boldsymbol{\tau}}$ and $(\boldsymbol{i}_\beta)$ be a one-form and a canonical basis on $S_{\mathbb{H}}$, respectively. For each point $\boldsymbol{a} \in \mathcal{H}$ the $\mathbb{R}$-algebra $\mathcal{H}(\boldsymbol{a}) := (T_{\boldsymbol{a}}\mathcal{H}, \boldsymbol{\mathcal{H}}(\boldsymbol{a}))$ is the tangent algebra, at $\boldsymbol{a}$, of the Lie group $\mathcal{H}$. For each $\boldsymbol{a} \in \mathcal{H}$ the components of the structure tensor $\boldsymbol{\mathcal{H}}(\boldsymbol{a})$ and a principal scalar product, $\boldsymbol{\mathcal{H}}[\tilde{\boldsymbol{\tau}}]$ of $\mathcal{H}(\boldsymbol{a})$ in the basis $(\hat{\boldsymbol{\imath}}_\beta)(\boldsymbol{a})$ are given by (1) and (2), respectively. Therefore, the components of a principal metric, $\boldsymbol{\mathcal{T}}$, in the proper frame $(\hat{\boldsymbol{\imath}}_\beta)$ must have the form

$$\begin{pmatrix} \tilde{\tau} & 0 & 0 & 0 \\ 0 & -\tilde{\tau} & 0 & 0 \\ 0 & 0 & -\tilde{\tau} & 0 \\ 0 & 0 & 0 & -\tilde{\tau} \end{pmatrix}, \tag{4}$$

for some function $\tilde{\tau} : \mathcal{H} \to \mathbb{R} \setminus \{0\}$. In other words, any principal metric on $\mathcal{H}$ is obtained by contraction of a one-form field $\tilde{\boldsymbol{\tau}}$, whose components in $(\hat{\boldsymbol{\imath}}_\beta)$ are $(\tilde{\tau}, 0, 0, 0)$, with the structure field $\boldsymbol{\mathcal{H}}$. This one-form is exact, i. e., there exists a twice differentiable function $\mathcal{T}$, such that $d\mathcal{T} = \tilde{\boldsymbol{\tau}}$. In the spherical frame with $R(\eta) = \exp(\eta)$ the components of $\tilde{\boldsymbol{\tau}}$ are also $(\tilde{\tau},\ 0,\ 0,\ 0)$, and,

$$d\mathcal{T}_0 = \frac{\partial \mathcal{T}}{\partial \eta} = \tilde{\tau}, \quad d\mathcal{T}_1 = \frac{\partial \mathcal{T}}{\partial \chi} = d\mathcal{T}_2 = \frac{\partial \mathcal{T}}{\partial \theta} = d\mathcal{T}_3 = \frac{\partial \mathcal{T}}{\partial \phi} = 0. \tag{5}$$

It follows from (5) that both $\mathcal{T}$ and $\tilde{\tau}$ depend on η only. Since $\frac{\partial \mathcal{T}}{\partial \eta}$ is differentiable, $\tilde{\tau}$ must be at least continuous. Therefore $\tilde{\tau}$ cannot change sign, because $\tilde{\tau}(\eta) \neq 0, \forall \eta \in \mathbb{R}$. Computing the components of the principal metric $\mathfrak{T}$ in the spherical frame we get

$$\mathcal{T}_{\alpha\beta} = \begin{pmatrix} \tau(\eta)(\frac{\dot{R}}{R})^2 & 0 & 0 & 0 \\ 0 & -\tau(\eta) & 0 & 0 \\ 0 & 0 & -\tau(\eta)\sin^2(\chi) & 0 \\ 0 & 0 & 0 & -\tau(\eta)\sin^2(\chi)\sin^2(\theta) \end{pmatrix}.$$

If $\tau(\eta) > 0$, we take $R(\eta)$ such that $\tau(\eta)(\frac{\dot{R}}{R})^2 = 1$, which yields

$$R(\eta) = exp \int \frac{d\eta}{\pm\sqrt{\tau(\eta)}} \quad . \tag{6}$$

In other words, with $R(\eta)$ satisfying (6), the components of the principal metric in the spherical frame are

$$\mathcal{T}_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathsf{a}^2 & 0 & 0 \\ 0 & 0 & -\mathsf{a}^2\sin^2(\chi) & 0 \\ 0 & 0 & 0 & -\mathsf{a}^2\sin^2(\chi)\sin^2(\theta) \end{pmatrix},$$

where the scale factor $\mathsf{a}(\eta) := \sqrt{\tau(\eta)}$.

If $\tau(\eta) < 0$, similar considerations show that the metric is also closed FLRW with the scale factor $\mathsf{a}(\eta) := \sqrt{-\tau(\eta)}$. $\square$

Corollary 3.1. $\mathcal{T}$ *is a monotonous function of* η *for each principal metric* $\mathfrak{T}$ *of* $\mathcal{H}$.

Thus the natural geometry of the group of nonzero quaternions $\mathcal{H}$ is defined by a family of closed Friedmann-Lemaître-Robertson-Walker metrics with monotonous scale factors.

References

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