Spectral Analysis of Radial Dirac Operators in the Kerr-Newman Metric and its Applications to Time-periodic Solutions

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We investigate the existence of time-periodic solutions of the Dirac equation in the Kerr-Newman background metric. To this end, the solutions are expanded in a Fourier series with respect to the time variable t and the Chandrasekhar separation ansatz is applied so that the question of existence of a time-periodic solution is reduced to the solvability of a certain coupled system of ordinary differential equations. First, we prove the already known result that there are no time-periodic solutions in the non-extreme case. Then it is shown that in the extreme case for fixed black hole data there is a sequence of particle masses $(m_N)_{N\in\mathbb{N}}$ for which a time-periodic solution of the Dirac equation does exist. The period of the solution depends only on the data of the black hole described by the Kerr-Newman metric.

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I. INTRODUCTION

In this paper we consider the system consisting of a Kerr-Newman black hole and an electron. The Kerr-Newman metric describing the black hole is the most general electrovac solutions of Einstein's field equations. It describes a rotating, electrically charged, rotating black hole, see, e.g., [13] and [8]. We are interested in the stability of a system consisting of such a black hole and an electron. To this end, we have to consider the Dirac equation for the electron in the Kerr-Newman metric, see equations (1) and (2)–(6). The Dirac equation is a complicated system of partial differential equations in all four spacetime variables. The existence of a time-periodic solution of the Dirac equation would imply the stability of the system under consideration.

Since the black hole is rotating, the background metric is only axisymmetric, whereas in the non-rotating cases (the Schwarzschild and the Reissner-Nordstrøm geometries) the background metric is spherically symmetric. This loss of symmetry leads to a complicated coupling of the angular and the radial coordinates in the Dirac equation. Chandrasekhar [4] showed that, in spite of this complicated coupling, the Dirac equation can be separated into a system of ordinary differential equations, the so-called angular equation (13) and the radial equation (14). These differential equations have realisations as eigenvalue equations in appropriate Hilbert spaces. The eigenvalue parameter ω of the radial equation is interpreted as the energy of the electron. It should be emphasised that, due to the lack of spherical symmetry of the spacetime, the eigenvalues of the radial and the angular equation are not independent of each other. Hence to show the stability of the system under consideration, it does not suffice to find eigenvalues of the radial and the angular equation separately, but we need to show that the eigenvalues are compatible with each other.

In section II we present the separation ansatz for time-periodic solution of the Dirac equation in the Kerr-Newman background metric due to Chandrasekhar with mathematical rigour and we derive the radial and angular equation. In the next section we consider the radial equation. It has been shown by Belgiorno and Martellini [2] that the essential spectrum of the radial operator is all of the real axis. Schmid [10] proved that in the so-called non-extreme Kerr-Newman metric the radial equation has no eigenvalues. In the extreme Kerr-Newman metric there exists at most one eigenvalue. In this paper we use the Levinson theorem to obtain most of the results proved in [10]. In Theorem IV.5 we show that in the non-extreme Kerr-Newman metric the Dirac equation has no time-periodic solution that can be interpreted as a particle wave function. The case of the extreme Kerr-Newman metric is investigated in section V. It turns out that in this case there may be an eigenvalue of the radial equation. Schmid [10] proved a sufficient condition for the existence of a time-periodic solution of the Dirac equation. However, it is not easy to verify that for given

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particle data there are the black hole parameters such that this condition can be satisfied. This problem is discussed in section VI. We show that for fixed data of an extreme Kerr-Newman black hole there is a sequence of particle masses, such that the system consisting of the quantum particle and the black hole permits time-periodic solutions.

II. SEPARATION OF THE DIRAC EQUATION IN THE KERR-NEWMAN BACKGROUND METRIC

We consider the Dirac equation

$$(\widehat{\mathcal{R}}+\widehat{\mathcal{A}})\widehat{\Psi} \;=\; 0 \tag{1}$$

for a spin-$\frac{1}{2}$ particle with the mass $m \geq 0$ and the charge e in the Kerr-Newman geometry, where

$$\widehat{\mathcal{R}} \;:=\; \begin{pmatrix} imr & 0 & \sqrt{\Delta}\,\mathcal{D}_+ & 0 \\ 0 & -imr & 0 & \sqrt{\Delta}\,\mathcal{D}_- \\ \sqrt{\Delta}\,\mathcal{D}_- & 0 & -imr & 0 \\ 0 & \sqrt{\Delta}\,\mathcal{D}_+ & 0 & imr \end{pmatrix}, \tag{2}$$

$$\widehat{\mathcal{A}} \;:=\; \begin{pmatrix} -am\cos\theta & 0 & 0 & \mathcal{L}_+ \\ 0 & am\cos\theta & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & -am\cos\theta & 0 \\ -\mathcal{L}_- & 0 & 0 & am\cos\theta \end{pmatrix}, \tag{3}$$

$$\mathcal{D}_\pm \;:=\; \frac{\partial}{\partial r} \mp \frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi} - ieQr\right], \tag{4}$$

$$\mathcal{L}_\pm \;:=\; \frac{\partial}{\partial\theta} + \frac{\cot\theta}{2} \mp i\left[a\sin\theta\frac{\partial}{\partial t} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right], \tag{5}$$

$$\Delta(r) \;:=\; r^2 - 2Mr + a^2 + Q^2 \tag{6}$$

and $\widehat{\Psi}$ is the wave function of the spin-$\frac{1}{2}$ particle under consideration. If the so-called black hole condition

$$M^2 - a^2 - Q^2 \;\geq\; 0 \tag{7}$$

holds, then the Kerr-Newman geometry is interpreted as the spacetime geometry generated by a black hole with the mass $M \geq 0$, the electric charge Q and the angular moment J; if $M \neq 0$, then the so-called Kerr parameter $a = J/M$ is the angular momentum of the black hole per unit mass. The black hole condition (7) ensures that the function Δ can be written as the product

$$\Delta(r) \;=\; (r-r_+)(r-r_-) \tag{6'}$$

with

$$r_\pm \;=\; M \pm \sqrt{M^2-a^2-Q^2}. \tag{8}$$

The special case $M^2-a^2-Q^2=0$, that is the case where $r_+ = r_- = M$, is referred to as the extreme Kerr-Newman metric.

Let us recall that the Kerr-Newman metric is the most general electrovac solution of Einstein's field equations [8]. Special cases contained in the Kerr-Newman geometry are the Kerr geometry (if $Q=0$), the Reissner-Nordstrøm geometry (if $a=0$) and the Schwarzschild geometry (if $Q=0$ and $a=0$).

The existence of a solution $\widehat{\Psi}$ of (1) for $(r,\theta,\varphi,t) \in \widehat{\Omega} := (r_+,\infty)\times(0,\pi)\times(-\pi,\pi)\times(-\infty,\infty)$ such that for every fixed time t the function $\widehat{\Psi}(\cdot,\cdot,\cdot,t)$ lies in a suitable $\mathscr{L}^2$-space would imply the stability of the system consisting of the black hole and the spin-$\frac{1}{2}$ particle in its exterior.

A. Time periodic solutions

In this paper we consider time-periodic solutions $\widehat{\Psi}$, that is, solutions such that

$$\widehat{\Psi}(r,\theta,\varphi,t) = \widehat{\Psi}(r,\theta,\varphi,t+\tfrac{2\pi}{\omega_0}) \qquad ((r,\theta,\varphi,t)\in\widehat{\Omega})$$

for some $\omega_0 > 0$. In this case, the solutions can be expanded in a Fourier series

$$\widehat{\Psi}(r,\theta,\varphi,t) \;=\; \sum_{\nu\in\mathbb{Z}} \exp(-i\nu\omega_0 t)\,\Psi^\nu(r,\theta,\varphi).$$

For physical reasons, each wave function Ψ^ν must be an element of the Hilbert space

$$\mathcal{H}_{r,\theta,\varphi} \;:=\; \mathscr{L}^2\Big((r_+,\infty)\times(0,\pi)\times(-\pi,\pi);\; \tfrac{r^2+a^2}{\Delta(r)}\,\sin\theta\; dr\,d\theta\,d\varphi\Big)^4 \tag{9}$$

with the inner product according to Finster et al. [6]

$$(\Psi,\Phi) \;=\; \int_{r_+}^{\infty}\int_0^{\pi}\int_{-\pi}^{\pi} \langle \Psi(r,\theta,\varphi)\,,\,\Phi(r,\theta,\varphi)\rangle_{\mathbb{C}^4}\,\frac{r^2+a^2}{\Delta(r)}\,\sin\theta\,dr\,d\theta\,d\varphi. \tag{10}$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{C}^4}$ is the usual scalar product on $\mathbb{C}^4$. In order to separate off also the φ-dependence of the solution Ψ^ν we use the complete system $\{\exp(i\kappa\varphi)\,:\,\kappa\in\mathbb{Z}+\frac{1}{2}\}$ in $\mathscr{L}^2((-\pi,\pi);\;d\varphi)$ to employ the ansatz

$$\Psi^\nu(r,\theta,\varphi) \;=\; \sum_{\kappa\in\mathbb{Z}+\frac{1}{2}} \exp(-i\kappa\varphi)\Psi^{\nu,\kappa}(r,\theta).$$

Thus, if

$$\widehat{\Psi}(r,\theta,\varphi,t) = \sum_{\nu\in\mathbb{Z}}\sum_{\kappa\in\mathbb{Z}+\frac{1}{2}} \exp(-i\omega_0\nu t)\exp(-i\kappa\varphi)\Psi^{\nu,\kappa}(r,\theta)$$

satisfies (1), then each $\Psi^{\nu,\kappa}$ is a solution of

$$(\mathcal{R}_{\nu,\kappa}+\mathcal{A}_{\nu,\kappa})\Psi^{\nu,\kappa} = 0 \qquad \text{on} \qquad (r_0,\infty)\times(0,\pi), \tag{11}$$

where $\mathcal{R}_{\nu,\kappa}$ and $\mathcal{A}_{\nu,\kappa}$ are obtained from (2) and (3) by replacing $\mathcal{D}_\pm$ and $\mathcal{L}_\pm$ by

$$\mathcal{D}_{\pm,\nu,\kappa} \;:=\; \frac{\partial}{\partial r} \pm \frac{i}{\Delta}\left[\omega_0\nu(r^2+a^2)+\kappa a + eQr\right],$$

$$\mathcal{L}_{\pm,\nu,\kappa} \;:=\; \frac{\partial}{\partial\theta} + \frac{\cot\theta}{2} \mp \left[a\omega_0\nu\sin\theta + \frac{\kappa}{\sin\theta}\right],$$

respectively (see the proof of Theorem IV.5).

B. Separation of the radial and the angular coordinate

To study the equation (11) we consider the formal differential expression

$$\mathfrak{A}_{\nu,\kappa} := \begin{pmatrix} -am\cos\theta & \mathcal{L}_{-,\nu,\kappa} \\ -\mathcal{L}_{+,\nu,\kappa} & am\cos\theta \end{pmatrix}$$

for $\theta\in(0,\pi)$. For any half integer κ the differential expression $\mathfrak{A}_{\nu,\kappa}$ has a unique self-adjoint realisation $A_{\nu,\kappa}$ in the Hilbert space $\mathscr{L}^2((0,\pi);\;\sin\theta\,d\theta)^2$ which has purely discrete spectrum $\sigma(A_{\nu,\kappa}) = \{\lambda_{\nu,\kappa,n}\,:\,n\in\mathbb{Z}\setminus\{0\}\}\subseteq\mathbb{R}$ where each eigenvalue has only simple multiplicity (see, e.g., Batic et al. [1], Winklmeier [16]). Hence there is a complete set of orthonormal eigenfunctions of $A_{\nu,\kappa}$

$$g^{\nu,\kappa,n} \;:=\; \begin{pmatrix} g_1^{\nu,\kappa,n} \\ g_2^{\nu,\kappa,n} \end{pmatrix} \;\in\; \mathscr{L}^2((0,\pi);\;\sin\theta\,d\theta)^2 \qquad (n\in\mathbb{Z}\setminus\{0\}) \tag{12}$$

with eigenvalues $\lambda_{\nu,\kappa,n}$. The family (12) allows us to make the ansatz

$$\Psi^{\nu,\kappa} \;=\; \sum_{n\in\mathbb{Z}\setminus\{0\}} \Psi^{\nu,\kappa,n}$$

with

$$\Psi^{\nu,\kappa,n}(r,\theta) \;=\; \begin{pmatrix} X_-^{\nu,\kappa,n}(r)\, g_2^{\nu,\kappa,n}(\theta) \\ X_+^{\nu,\kappa,n}(r)\, g_1^{\nu,\kappa,n}(\theta) \\ X_+^{\nu,\kappa,n}(r)\, g_2^{\nu,\kappa,n}(\theta) \\ X_-^{\nu,\kappa,n}(r)\, g_1^{\nu,\kappa,n}(\theta) \end{pmatrix} \qquad (r \in (r_+,\infty),\;\theta \in (0,\pi))$$

which leads to a separation of the angular coordinate θ and the radial coordinate r (cf. Chandrasekhar [4]): the angular function $g^{\nu,\kappa,n}$ satisfies the angular equation

$$A_{\nu,n} g^{\nu,\kappa,n} \;=\; \lambda_{\nu,\kappa,n} g^{\nu,\kappa,n} \tag{13}$$

with integrability condition $g^{\nu,\kappa,n} \in \mathscr{L}^2((0,\pi);\;\sin\theta\, d\theta)$ and the radial function $X^{\nu,\kappa,n} = {}^t(X_+^{\nu,\kappa,n},\, X_-^{\nu,\kappa,n})$ satisfies the radial equation

$$\begin{pmatrix} -imr - \lambda_{\nu,\kappa,n} & \sqrt{\Delta}\,\mathcal{D}_{-,\nu,\kappa} \\ \sqrt{\Delta}\,\mathcal{D}_{+,\nu,\kappa} & imr - \lambda_{\nu,\kappa,n} \end{pmatrix} \begin{pmatrix} X_+^{\nu,\kappa,n}(r) \\ X_-^{\nu,\kappa,n}(r) \end{pmatrix} \;=\; 0, \tag{14}$$

with the integrability condition arising from the inner product on $\mathcal{H}_{r,\theta,\varphi}$

$$\int_{r_+}^{\infty} \left(|X_+^{\nu,\kappa,n}(r)|^2 + |X_-^{\nu,\kappa,n}(r)|^2\right) \frac{r^2+a^2}{\Delta(r)}\, dr \;<\; \infty. \tag{15}$$

C. The radial equation

Let

$$W := \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ -1 & -1 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & -1/\sqrt{\Delta} \\ 1/\sqrt{\Delta} & 0 \end{pmatrix}, \quad \widetilde{f}^{\nu,\kappa,n}(r) \;:=\; \begin{pmatrix} \widetilde{f}_1^{\nu,\kappa,n}(r) \\ \widetilde{f}_2^{\nu,\kappa,n}(r) \end{pmatrix} \;:=\; W \begin{pmatrix} X_+^{\nu,\kappa,n}(r) \\ X_-^{\nu,\kappa,n}(r) \end{pmatrix}. \tag{16}$$

Then we obtain from (14)

$$\begin{aligned} 0 \;&=\; VW \begin{pmatrix} -imr - \lambda_{\nu,\kappa,n} & \sqrt{\Delta}\,\mathcal{D}_{-,\nu,\kappa} \\ \sqrt{\Delta}\,\mathcal{D}_{+,\nu,\kappa} & imr - \lambda_{\nu,\kappa,n} \end{pmatrix} W^{-1} \widetilde{f}^{\nu,\kappa,n}(r) \\ &= \begin{pmatrix} \dfrac{mr}{\sqrt{\Delta}} - \dfrac{a\kappa + eQr + \omega_0\nu(r^2+a^2)}{\Delta} & -\dfrac{d}{dr} + \dfrac{\lambda_{\nu,\kappa,n}}{\sqrt{\Delta}} \\ \dfrac{d}{dr} + \dfrac{\lambda_{\nu,\kappa,n}}{\sqrt{\Delta}} & -\dfrac{mr}{\sqrt{\Delta}} - \dfrac{a\kappa + eQr + \omega_0\nu(r^2+a^2)}{\Delta} \end{pmatrix} \widetilde{f}^{\nu,\kappa,n}(r). \end{aligned}$$

for $r \in (r_+,\infty)$ (see Winklmeier [16]).

Remark II.1. If we take into account only terms of first order in $1/r$ for large r, then the radial equation becomes

$$\begin{pmatrix} m - \dfrac{eQ}{r} + \omega_0\nu & -\dfrac{d}{dr} + \dfrac{\lambda_{\nu,n,\kappa}}{r} \\ \dfrac{d}{dr} + \dfrac{\lambda_{\nu,n,\kappa}}{r} & -m - \dfrac{eQ}{r} + \omega_0\nu \end{pmatrix} \widetilde{f}(r) = 0,$$

which is exactly the radial equation for the relativistic hydrogen atom, see, e.g., Bjorken and Drell [3].

Remark II.2. It is important to note that in the case $a \neq 0$ the eigenvalue parameter λ for the angular equation does depend on $\omega_0\nu$ whereas in the case $a = 0$ there is no such dependence. Hence, in the latter case, for a solution of the complete problem (14)–(13) first the the angular problem $(A_{\nu,\kappa} - \lambda_{\nu,\kappa})g = 0$ is solved for the eigenvalues $\lambda_{\nu,\kappa,n}$, and then the radial equation (14) can be attacked for fixed $\lambda_{\nu,\kappa,n}$. In the case $a = 0$, the eigenvalues can be calculated explicitly.

We introduce a new coordinate x such that

$$\frac{dx}{dr} = \frac{r^2+a^2}{\Delta(r)} \qquad (r > r_+), \tag{17}$$

that is,

$$x(r) = \begin{cases} r + \frac{r_+^2+a^2}{r_+-r_-}\log(r-r_+) - \frac{r_+^2+a^2}{r_+-r_-}\log(r-r_-) + x_0 & (r_+ \neq r_-), \\ r + 2r_+\log(r-r_+) - \frac{r_+^2+a^2}{r-r_+} + x_0 & (r_+ = r_-), \end{cases} \tag{18}$$

where x_0 is a constant of integration that can be set $x_0 = 0$. The correspondence $r > r_+$ and $x \in (-\infty, \infty)$ is a bijection. With the new coordinate x equation (14) becomes

$$\begin{aligned} H_{\nu,\kappa,n} f^{\nu,\kappa,n} &:= \begin{pmatrix} \frac{mr\sqrt{\Delta}}{r^2+a^2} - \frac{a\kappa + eQr}{r^2+a^2} & -\frac{d}{dx} + \frac{\lambda_{\nu,\kappa,n}\sqrt{\Delta}}{r^2+a^2} \\ \frac{d}{dx} + \frac{\lambda_{\nu,\kappa,n}\sqrt{\Delta}}{r^2+a^2} & -\frac{mr\sqrt{\Delta}}{r^2+a^2} - \frac{a\kappa + eQr}{r^2+a^2} \end{pmatrix} f^{\nu,\kappa,n} \\ &= \omega_0 \nu f^{\nu,\kappa,n}, \end{aligned} \tag{19}$$

where r has to be understood as $r(x)$ and $f^{\nu,\kappa,n}(x) = \widetilde{f}^{\nu,\kappa,n}(r(x))$, $f_j^{\nu,\kappa,n}(x) = \widetilde{f}_j^{\nu,\kappa,n}(r(x))$ for all $x \in (-\infty, \infty)$ and $j \in \{1, 2\}$. In view of (16) and (17) the integrability condition (15) becomes

$$\int_{-\infty}^{\infty} \| f^{\nu,\kappa,n}(x) \|_{\mathbb{C}^2}^2 \, dx = \int_{-\infty}^{\infty} \left[\left| f_1^{\nu,\kappa,n}(x) \right|^2 + \left| f_2^{\nu,\kappa,n}(x) \right|^2 \right] dx < \infty. \tag{20}$$

The operator $H_{\nu,\kappa,n}$ is formally symmetric in the Hilbert space $\mathcal{H}_x := \mathscr{L}^2((-\infty,\infty);\, dx)^2$, so it is natural to look for an operator theoretical realisation of $H_{\nu,\kappa,n}$ in the Hilbert space $\mathcal{H}_x$. The purpose of this note is to investigate the spectral properties of the self-adjoint operator $H_{\nu,\kappa,n}$ in $\mathcal{H}_x = \mathscr{L}^2((-\infty,\infty);\; dx)^2$ and to study the non-existence of non-trivial solutions satisfying (11). In section VI we will investigate the existence of so-called energy eigenvalues of $H_{\nu,\kappa,n}$ (cf. Schmid [10]).

Definition II.3. We call $\omega \in \mathbb{R}$ an *energy eigenvalue* of $H_{\nu,\kappa,n}$ if there are $\lambda_{\nu,\kappa,n}$ such that ω is an eigenvalue of $H_{\nu,\kappa,n}$ and $\lambda_{\nu,\kappa,n}$ is an eigenvalue of $A_{\nu,\kappa}$, that is, if equations (13) and (14) can be solved simultaneously with functions satisfying the corresponding integrability conditions.

III. THE OPERATOR H

In this and the following sections we consider the eigenvalue equation $(H_{\nu,\kappa,n} - \omega\nu_0) f^{\nu,\kappa,n} = 0$ from (19) on the Hilbert space $\mathcal{H}_x = \mathscr{L}^2((-\infty,\infty);\; dx)^2$.

If there is no ambiguity, we omit the indices ν, n and κ in the following for the sake of clarity; for instance, we write simply H instead of $H_{\nu,\kappa,n}$, λ instead of $\lambda_{\nu,\kappa,n}$ and ω instead of $\omega_0\nu$. We decompose the operator H into the sum

$$H = H_0 + V$$

where

$$H_0 = \begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}, \qquad \mathcal{D}(H_0) = \mathscr{C}_0^\infty(-\infty,\infty)^2,$$

$$V(x) = \begin{pmatrix} A(x) & B(x) \\ B(x) & C(x) \end{pmatrix},$$

$$A(x) = \frac{m\, r(x)\sqrt{\Delta(r(x))}}{r(x)^2+a^2} - \frac{a\kappa + eQr(x)}{r(x)^2+a^2},$$

$$
\begin{aligned}
B(x) &= \lambda \frac{\sqrt{\Delta(r(x))}}{r(x)^2 + a^2}, \\
C(x) &= -\frac{m\, r(x)\sqrt{\Delta(r(x))}}{r(x)^2 + a^2} - \frac{a\kappa + eQr(x)}{r(x)^2 + a^2}, \\
\Delta(x) &= (r(x) - r_+)(r(x) - r_-).
\end{aligned}
$$

The operator H_0 is symmetric and has a unique self-adjoint extension on the space $\mathscr{L}^2((-\infty,\infty);\ dx)^2$. Since $x \to -\infty$ is equivalent to $r(x) \to r_+$ and $x \to \infty$ is equivalent to $r(x) \to \infty$, we have

$$
\begin{aligned}
&\lim_{x\to-\infty} A(x) = -\frac{a\kappa + eQr_+}{r_+^2 + a^2} =: A_0, &&\lim_{x\to+\infty} A(x) = m, \\
&\lim_{x\to-\infty} C(x) = A_0, &&\lim_{x\to+\infty} C(x) = -m, \\
&\lim_{x\to\pm\infty} B(x) = 0
\end{aligned}
$$

which implies that the functions $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are bounded. Therefore, $H = H_0 + V$ has a unique self-adjoint extension which we again denote by H (see, e.g., Weidmann [15, Theorem 6.8]).

In what follows, a prime $'$ always denotes differentiation with respect to x.

Lemma III.1 (Asymptotic behaviour of V for $x \to -\infty$).

(i) *For $x \to -\infty$ the functions*

$$A(x) - A_0, \quad B(x), \quad C(x) - A_0$$

decay exponentially in the case $r_+ \neq r_-$, and they are of order $O(x^{-1})$ in the case $r_+ = r_-$. More precisely, in the latter case we have

$$
\begin{aligned}
A(x) - A_0 &= \frac{mM - \mu}{-x} + O\left(\frac{1}{x^2}\right) \qquad \text{as} \quad x \to -\infty, \\
A_0 - C(x) &= \frac{mM + \mu}{-x} + O\left(\frac{1}{x^2}\right) \qquad \text{as} \quad x \to -\infty, \\
B(x) &= \frac{\lambda}{x} + O\left(\frac{1}{x^2}\right) \qquad \text{as} \quad x \to -\infty,
\end{aligned}
$$

where

$$\mu := -\frac{2a\kappa M}{M^2 + a^2} - eQ\frac{M^2 - a^2}{M^2 + a^2}.$$

(ii) *The derivatives A', B' and C' are integrable with respect to x on $(-\infty, 0]$.*

(iii) *$A'(x) = O(x^{-2})$, $B'(x) = O(x^{-2})$ and $C'(x) = O(x^{-2})$ hold as $x \to +\infty$.*

Proof. We prove the assertions for the functions $B(x)$ and $A(x)$ only. The corresponding assertions for $C(x)$ can be obtained from those for $A(x)$ by substituting m by $-m$. We remark that (18) shows

$$r(x) - r_+ = O(\exp(\alpha x)) \qquad \text{as} \quad x \to -\infty \qquad (r_+ \neq r_-), \tag{21}$$

$$r(x) - r_+ = O(x^{-1}) \qquad \text{as} \quad x \to -\infty \qquad (r_+ = r_-) \tag{22}$$

for a positive constant α. To keep notation simple, let us write r instead of $r(x)$ in this proof.

(i) Note that

$$\frac{1}{r^2 + a^2} = \frac{1}{r_+^2 + a^2} - \frac{(r - r_+)(r + r_+)}{(r_+^2 + a^2)(r^2 + a^2)} = \frac{1}{r_+^2 + a^2} + O(r - r_+). \tag{23}$$

Hence it follows immediately that

$$B(x) \;=\; \lambda\frac{\sqrt{\Delta(r)}}{r^2+a^2} \;=\; \lambda\sqrt{\Delta(r)}\left(\frac{1}{r_+^2+a^2}+O(r-r_+)\right).$$

If we recall that $\Delta(r(x)) = (r(x)-r_+)(r(x)-r_-)$ and use the relations (21) and (22), we see that the assertion for B holds.

Now we prove the assertions for A. Simple calculations show

$$\begin{aligned}
A(x)-A_0 \;&=\; \frac{mr\sqrt{\Delta(r)}}{r^2+a^2}+\frac{(r^2+a^2)(a\kappa+eQr_+)-(r_+^2+a^2)(a\kappa+eQr)}{(r^2+a^2)(r_+^2+a^2)}\\
&=\; \frac{mr\sqrt{\Delta(r)}}{r^2+a^2}+\frac{(r-r_+)\left[(r-r_+)(a\kappa+eQr_+)+2a\kappa r_+ + eQ(r_+^2-a^2)\right]}{(r^2+a^2)(r_+^2+a^2)}\\
&=\; \frac{mr_+\sqrt{\Delta(r)}}{r^2+a^2}+\frac{r-r_+}{r^2+a^2}\left[m\sqrt{\Delta(r)}+\frac{(r-r_+)(a\kappa+eQr_+)+2a\kappa r_+ + eQ(r_+^2-a^2)}{r_+^2+a^2}\right].
\end{aligned}$$

Using the relations (23) and (21), we see that the assertion holds for the case $r_+ \neq r_-$. In the case $r_+ = r_-$ we have $\Delta(r) = (r-r_+)^2 = (r-M)^2$, hence we can continue the calculation above as follows

$$\begin{aligned}
A(x)-A_0 \;&=\; \frac{r-r_+}{r^2+a^2}\left[mr_+ + \frac{2a\kappa r_+ + eQ(r_+^2-a^2)}{r_+^2+a^2}+(r-r_+)\left(m+\frac{a\kappa+eQr_+}{r_+^2+a^2}\right)\right]\\
&=\; \frac{r-r_+}{r^2+a^2}\left[mr_+ - \mu + (r-r_+)\left(m+\frac{a\kappa+eQr_+}{r_+^2+a^2}\right)\right].
\end{aligned}$$

The assertion follows now from (23) and (22).

(ii) A simple calculation gives

$$\begin{aligned}
\frac{d}{dx}A(x) \;&=\; \frac{dr}{dx}\frac{d}{dr}A(x(r)) \;=\; \frac{\Delta(r)}{r^2+a^2}\frac{d}{dr}A(x(r))\\
&=\; \frac{\sqrt{\Delta(r)}}{r^2+a^2}\left\{m\left(\frac{\Delta(r)}{r^2+a^2}+\frac{r\Delta'(r)}{2(r^2+a^2)}-\frac{2r^2\Delta(r)}{(r^2+a^2)^2}\right)+\frac{2(a\kappa+eQr)r\sqrt{\Delta(r)}}{(r^2+a^2)^2}-\frac{eQ\sqrt{\Delta(r)}}{r^2+a^2}\right\} \qquad (24)\\
&=\; \begin{cases} O((r-r_+)^{1/2}) = O(\exp[(1/2)\alpha x]), & (r_+\neq r_-),\\ O((r-r+)^2) \;\; = O(x^{-2}), & (r_+ = r_-).\end{cases}
\end{aligned}$$

as $x\to-\infty$.

$$\begin{aligned}
\frac{d}{dx}B(x) \;&=\; \frac{dr}{dx}\frac{d}{dr}B(x(r)) \;=\; \frac{\Delta(r)}{r^2+a^2}\frac{d}{dr}B(x(r))\\
&=\; \lambda\frac{\sqrt{\Delta(r)}}{r^2+a^2}\left\{\frac{\Delta'(r)}{2(r^2+a^2)}-\frac{2r\Delta(r)}{(r^2+a^2)^2}\right\} \qquad (25)\\
&=\; \begin{cases} O((r-r_+)^{1/2}) = O(\exp[(1/2)\alpha x]), & (r_+\neq r_-),\\ O((r-r+)^2) \;\; = O(x^{-2}), & (r_+ = r_-).\end{cases}
\end{aligned}$$

as $x\to-\infty$.

(iii) Since

$$\frac{dr}{dx} = \frac{\Delta(r)}{r^2+a^2}\sim 1, \quad x\sim r \;\; \text{as} \;\; x\to+\infty,$$

and

$$2(r^2+a^2)\Delta(r)+r(r^2+a^2)\Delta'(r)-4r^2\Delta(r)=2Mr^3+2r^2(a^2-Q^2)-6a^2Mr+2a^2(a^2+Q^2)\sim r^3 \text{ as } r\to\infty$$

the assertion follows from (24). The proof of the assertion concerning $B'(x)$ follows directly from (25). $\square$

IV. ABSOLUTELY CONTINUOUS SPECTRUM

The following proposition has been shown by Belgiorno and Martellini [2].

Proposition IV.1. $\sigma_{\text{ess}}(H)=\mathbb{R}$. $\square$

We point out that the proof of the proposition relies on the fact that

$$\lim_{x\to-\infty}[V(x)-A_0I_2]=0, \tag{26}$$

where I_2 is the 2×2 unit matrix. Lemma III.1 yields the following theorem.

Theorem IV.2. (i) *H has purely absolutely continuous spectrum in $\mathbb{R}\setminus\{A_0\}$.*

(ii) *H has purely absolutely continuous spectrum in $(-\infty,-m)$ and $(m,+\infty)$, that is, H is absolutely continuous in the complement of $[-m,m]\cap\{A_0\}$.*

Proof. (i) Lemma III.1 gives (26) and any component of V' is integrable at $-\infty$. Therefore we can prove the theorem in view of Weidmann [14], Schmidt [11] and also Thaller [12, Theorem 4.18].

(ii) The proof is the same as in (i) by using Lemma III.1 (iii).

$\square$

Remark IV.3. The above theorem has already been proven by Schmid with different means (see Schmid [10, Corollary 3.4]); in addition, he has shown that $\omega=m$ or $\omega=-m$ is not an eigenvalue of H ([10, Lemma 3.5]).

Theorem IV.4. *If $r_+\neq r_-$, then A_0 is not an eigenvalue of H.*

Proof. Let us assume that $U={}^t(u_1,u_2)\in\mathcal{H}_x$ satisfies

$$\begin{pmatrix} A & B-\frac{d}{dx} \\ B+\frac{d}{dx} & C\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=A_0\begin{pmatrix}u_1\\u_2\end{pmatrix},$$

that is,

$$U'=\begin{pmatrix}u_1'\\u_2'\end{pmatrix}=\begin{pmatrix}-B & A_0-C\\ A-A_0 & B\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}. \tag{27}$$

As seen in section III, $A(x)-A_0$, $B(x)$ and $C(x)-A_0$ decay exponentially as $x\to-\infty$. Therefore, the Levinson theorem (e.g. Eastham [5]) gives that there are two linearly independent solutions U^1, U^2 of (27) such that

$$U^1(x)=\left\{\begin{pmatrix}1\\0\end{pmatrix}+o(1)\right\} \quad\text{as}\quad x\to-\infty,$$
$$U^2(x)=\left\{\begin{pmatrix}0\\1\end{pmatrix}+o(1)\right\} \quad\text{as}\quad x\to-\infty.$$

Hence there are constants c_1 and c_2 such that $U(x)=c_1U^1(x)+c_2U^2(x)$, which is square integrable on $(-\infty,0)$ only if $c_1=c_2=0$. $\square$

As an important corollary we obtain the following theorem.

Theorem IV.5 (Non-existence of time-periodic solutions for $r_+ \neq r_-$). *Let $\widehat{\Psi} \in C^1((r_0,\infty)\times(0,\pi)\times[-\pi,\pi]\times\mathbb{R},\,\mathbb{C}^4)$ be a solution of* (1) *satisfying the periodicity conditions*

$$\begin{aligned}\widehat{\Psi}(r,\theta,\varphi,t) &= \widehat{\Psi}(r,\theta,\varphi,t+\tfrac{2\pi}{\omega_0}),\\ \widehat{\Psi}(r,\theta,\pi,t) &= \widehat{\Psi}(r,\theta,-\pi,t),\end{aligned}\qquad (r,\theta,\varphi,t)\in\widehat{\Omega}. \tag{28}$$

Furthermore, assume that for all $(\varphi,\,t)\in[-\pi,\,\pi]\times\mathbb{R}$ we have $\widehat{\Psi}(\,\cdot\,,\,\cdot\,,\,\varphi\,,\,t)\in C^0((r_0,\infty)\times(0,\pi);\;\mathcal{H}_{r,\theta})$ where

$$\mathcal{H}_{r,\theta} := \mathscr{L}^2\left((r_+,\infty)\times(0,\pi);\;\tfrac{r^2+a^2}{\Delta(r)}\sin\theta\,dr\,d\theta\right)^4$$

with the norm denoted by $\|\cdot\|_{\mathcal{H}_{r,\theta}}$ and the inner product denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}_{r,\theta}}$.

If $r_-\neq r_+$, then $\widehat{\Psi}\equiv 0$.

Proof. Let $\widehat{\Psi}$ be a time-periodic solution satisfying conditions of the theorem. Then $\widehat{\Psi}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,t)$ is an $\mathcal{H}_{r,\theta,\varphi}$-valued strongly continuous function with respect to t, since $\widehat{\Psi}(\,\cdot\,,\,\cdot\,,\varphi,t)$ is uniformly continuous in $\mathcal{H}_{r,\theta}$ with respect to $(\varphi,t)\in[-\pi,\pi]\times\mathbb{R}$. Therefore we can expand $\widehat{\Psi}(r,\theta,\varphi,t)$ as the Fourier series with respect to t

$$\widehat{\Psi}(r,\theta,\varphi,t) = \sum_{\nu\in\mathbb{Z}} \exp(-i\omega_0\nu t)\Psi^\nu(r,\theta,\varphi)$$

strongly in $\mathscr{L}^2([0,2\pi/\omega_0];\;\mathcal{H}_{r,\theta,\varphi}\,;\,dt)$, where

$$\Psi^\nu(r,\theta,\varphi) = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\exp(i\omega_0\nu t)\widehat{\Psi}(r,\theta,\varphi,t)\,dt, \qquad \sum_\nu \|\Psi^\nu\|^2_{\mathcal{H}_{r,\theta,\varphi}}<\infty.$$

Moreover, each $\Psi^\nu(r,\theta,\varphi)$ can be expanded as

$$\Psi^\nu(r,\theta,\varphi) = \sum_{\kappa\in\mathbb{Z}+(1/2)} \exp(-i\kappa\varphi)\Psi^{\nu,\kappa}(r,\theta)$$

strongly in $\mathscr{L}^2([-\pi,\pi]\,;\;\mathcal{H}_{r,\theta};\;d\varphi)$ with

$$\Psi^\nu(\theta,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(i\kappa\varphi)\Psi^\nu(r,\theta,\varphi)\,d\varphi, \qquad \sum_\kappa \|\Psi^{\nu,\kappa}\|^2_{\mathcal{H}_{r,\theta}}<\infty.$$

For every $\Phi\in\mathscr{C}_0^\infty((r_0,\infty)\times(0,\pi))^4$ of the form

$$\Phi(r,\theta) \;=\; \begin{pmatrix}\rho_-(r)\,\eta_2(\theta)\\ \rho_+(r)\,\eta_1(\theta)\\ \rho_+(r)\,\eta_2(\theta)\\ \rho_-(r)\,\eta_1(\theta)\end{pmatrix}$$

with $\rho_\pm(r)\in\mathscr{C}_0^\infty((r_+,\infty))$, $\eta_1(\theta)$, $\eta_2(\theta)\in\mathscr{C}_0^\infty((0,\pi))$ we obtain, by using (28),

$$\begin{aligned}
0 \;&=\; \int_0^{2\pi/\omega_0} dt\int_{-\pi}^{\pi} d\varphi\left\langle(\widehat{\mathcal{R}}+\widehat{\mathcal{A}})\widehat{\Psi},\;\exp(-i\omega_0\nu t)\exp(-i\kappa\varphi)\Phi\right\rangle_{\mathcal{H}_{r,\theta}}\\
&=\; \int_0^{2\pi/\omega_0} dt\int_{-\pi}^{\pi} d\varphi\left\langle\widehat{\Psi},\;(\widehat{\mathcal{R}}+\widehat{\mathcal{A}})^*\exp(-i\omega_0\nu t)\exp(-i\kappa\varphi)\Phi\right\rangle_{\mathcal{H}_{r,\theta}}\\
&=\; \int_0^{2\pi/\omega_0} dt\int_{-\pi}^{\pi} d\varphi\left\langle\widehat{\Psi},\;\exp(-i\omega_0\nu t)\exp(-i\kappa\varphi)(\mathcal{R}_{\nu,\kappa}+\mathcal{A}_{\nu,\kappa})^*\Phi\right\rangle_{\mathcal{H}_{r,\theta}}\\
&=\; \int_0^{2\pi/\omega_0}\left\langle\widehat{\Psi},\;\exp(-i\omega_0\nu t)\exp(-i\kappa\varphi)(\mathcal{R}_{\nu,\kappa}+\mathcal{A}_{\nu,\kappa})^*\Phi\right\rangle_{\mathcal{H}_{r,\theta,\varphi}}dt\\
&=\; \frac{2\pi}{\omega_0}\int_{-\pi}^{\pi}\left\langle\Psi^\nu,\;\exp(-i\kappa\varphi)(\mathcal{R}^*_{\nu,\kappa}+\mathcal{A}^*_{\nu,\kappa})\Phi\right\rangle_{\mathcal{H}_{r,\theta}}d\varphi
\end{aligned}$$

$$= \frac{4\pi^2}{\omega_0} \left\langle \Psi^{\nu,\kappa},\ (\mathcal{R}^*_{\nu,\kappa} + A^*_{\nu,\kappa})\Phi \right\rangle_{\mathcal{H}_{r,\theta}} . \tag{29}$$

In the above calculation, the superscript * denotes the formal adjoint operator. As in (12), for fixed ν and κ let

$$g^{\nu,\kappa,n}(\theta) = \begin{pmatrix} g_1^{\nu,\kappa,n}(\theta) \\ g_2^{\nu,\kappa,n}(\theta) \end{pmatrix} \qquad (n \in \mathbb{Z} \setminus \{0\}),$$

be a complete family of orthonormal eigenfunctions of $A_{\nu,\kappa}$ with eigenvalues $\lambda_{\nu,\kappa,n}$, respectively. Then $\Psi^{\nu,\kappa}(r,\theta)$ can be expanded by means of $g^{\nu,\kappa,n}$ $(n \in \mathbb{Z} \setminus \{0\})$ as follows

$$\begin{aligned} \Psi^{\nu,\kappa}(r,\theta) &= \sum_{n\in\mathbb{Z}\setminus\{0\}} X_+^{\nu,\kappa,n}(r) \begin{pmatrix} 0 \\ g_1^{\nu,\kappa,n}(\theta) \\ g_2^{\nu,\kappa,n}(\theta) \\ 0 \end{pmatrix} + X_-^{\nu,\kappa,n}(r) \begin{pmatrix} g_2^{\nu,\kappa,n}(\theta) \\ 0 \\ 0 \\ g_1^{\nu,\kappa,n}(\theta) \end{pmatrix} \\ &= \sum_{n\in\mathbb{Z}\setminus\{0\}} \begin{pmatrix} X_-^{\nu,\kappa,n}(r)\, g_2^{\nu,\kappa,n}(\theta) \\ X_+^{\nu,\kappa,n}(r)\, g_1^{\nu,\kappa,n}(\theta) \\ X_+^{\nu,\kappa,n}(r)\, g_2^{\nu,\kappa,n}(\theta) \\ X_-^{\nu,\kappa,n}(r)\, g_1^{\nu,\kappa,n}(\theta) \end{pmatrix}, \end{aligned}$$

where the series converges in the strong sense in $\mathscr{L}^2((0,\pi);\ \sin\theta\, d\theta))^4$ and $X_\pm^{\nu,\kappa,n}(r)$ satisfies

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{r_+}^\infty (|X_+^{\nu,\kappa,n}(r)|^2 + |X_-^{\nu,\kappa,n}(r)|^2)\, \frac{r^2+a^2}{\Delta(r)}\, dr \,<\, \infty.$$

Since $\mathfrak{A}_{\nu,\kappa}$ on $\mathscr{C}_0^\infty((0,\pi))^2$ is essentially selfadjoint in $\mathscr{L}^2((0,\pi);\ \sin\theta\, d\theta)^2$, for any $g^{\nu,\kappa,n}(\theta)$ there exists a convergent sequence $\eta^\ell(\theta)$ $(\ell = 1,2,\cdots) \in \mathscr{C}_0^\infty((0,\pi))^2$ such that

$$A_{\nu,\kappa}\eta^\ell = A_{\nu,\kappa} \begin{pmatrix} \eta_1^\ell \\ \eta_2^\ell \end{pmatrix} \longrightarrow A_{\nu,\kappa} g^{\nu,\kappa,n} = \lambda_{\nu,\kappa,n} g^{\nu,\kappa,n}$$

in $\mathscr{L}^2((0,\pi);\ \sin\theta\, d\theta)^2$. Substituting Φ in (29) by

$$\Phi^\ell(r,\theta) = \begin{pmatrix} \rho_-(r)\, \eta_2^\ell(\theta) \\ \rho_+(r)\, \eta_1^\ell(\theta) \\ \rho_+(r)\, \eta_2^\ell(\theta) \\ \rho_-(r)\, \eta_1^\ell(\theta) \end{pmatrix}$$

and taking the limit $\ell \to \infty$, we have

$$\left\langle \sum_{m\in\mathbb{Z}\setminus\{0\}} \begin{pmatrix} X_-^{\nu,\kappa,m}(r)\, g_2^{\nu,\kappa,m}(\theta) \\ X_+^{\nu,\kappa,m}(r)\, g_1^{\nu,\kappa,m}(\theta) \\ X_+^{\nu,\kappa,m}(r)\, g_2^{\nu,\kappa,m}(\theta) \\ X_-^{\nu,\kappa,m}(r)\, g_1^{\nu,\kappa,m}(\theta) \end{pmatrix},\ (R^*_{\nu,\kappa} - \lambda_{\nu,\kappa,n}) \begin{pmatrix} \rho_-(r)\, g_2^{\nu,\kappa,n}(\theta) \\ \rho_+(r)\, g_1^{\nu,\kappa,n}(\theta) \\ \rho_+(r)\, g_2^{\nu,\kappa,n}(\theta) \\ \rho_-(r)\, g_2^{\nu,\kappa,n}(\theta) \end{pmatrix} \right\rangle_{\mathcal{H}_{r,\theta}} = 0,$$

which gives

$$0 = \int_{r_+}^\infty dr\, \frac{r^2+a^2}{\Delta(r)} \left\langle \begin{pmatrix} X_+^{\nu,\kappa,n} \\ X_-^{\nu,\kappa,n} \end{pmatrix}, \begin{pmatrix} imr - \lambda_{\nu,\kappa,n} & \mathcal{D}^*_{+,\nu,\kappa}\sqrt{\Delta} \\ \mathcal{D}^*_{-,\nu,\kappa}\sqrt{\Delta} & -imr - \lambda_{\nu,\kappa,n} \end{pmatrix} \begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix} \right\rangle_{\mathbb{C}^2},$$

which implies

$$\begin{pmatrix} -imr - \lambda_{\nu,\kappa,n} & \sqrt{\Delta}\, \mathcal{D}_{-,\nu,\kappa} \\ \sqrt{\Delta}\, \mathcal{D}_{+,\nu,\kappa} & imr - \lambda_{\nu,\kappa,n} \end{pmatrix} \begin{pmatrix} X_+^{\nu,\kappa,n}(r) \\ X_-^{\nu,\kappa,n}(r) \end{pmatrix} = 0.$$

If we set (cf. (16) and (20))

$$f^{\nu,\kappa,n}(x) = \begin{pmatrix} f_1^{\nu,\kappa,n}(r(x)) \\ f_2^{\nu,\kappa,n}(r(x)) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ -1 & -1 \end{pmatrix} \begin{pmatrix} X_+^{\nu,\kappa,n}(r(x)) \\ X_-^{\nu,\kappa,n}(r(x)) \end{pmatrix},$$

we have

$$H_{\nu,\kappa,n} f^{\nu,\kappa,n} = \omega_0 \nu f^{\nu,\kappa,n}$$

(see (19)). Then, Theorem IV.2 and Theorem IV.4 give $f^{\nu,\kappa,n} = 0$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $\kappa \in \mathbb{Z} + (1/2)$, which yields $\Psi^{\nu,\kappa}(r, \theta) = 0$. □

The non-existence of time-periodic solutions is shown by Finster et al. [6] by different means.

V. THE CASE $r_+ = r_-$

In the previous section we have seen that there are no eigenvalues of (1) in the case $r_+ \neq r_-$. In this section we discuss whether A_0 is an eigenvalue of H in the case $r_+ = r_-$. Recall that in this case the function Δ has only one zero and that $r_+ = r_- = M$.

Theorem V.1. *If*

$$\lambda^2 + m^2 M^2 - \mu^2 \leq \frac{1}{4},$$

then A_0 is not an eigenvalue of H.

Proof. Let us assume that $U = {}^t(u_1, u_2) \in \mathcal{H}_x$ satisfies (27), that is,

$$U' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -B & A_0 - C \\ A - A_0 & B \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Lemma III.1 yields for $x \to -\infty$

$$U'(x) = -\frac{1}{x} \begin{pmatrix} \lambda & mM + \mu \\ mM - \mu & -\lambda \end{pmatrix} U(x) + O\left(x^{-2}\right) U(x).$$

If we introduce $s = \log(-x)$, we have

$$\frac{d}{ds} U(x(s)) = \begin{pmatrix} \lambda & mM + \mu \\ mM - \mu & -\lambda \end{pmatrix} U(x(s)) + O\left(\exp(-s)\right) U(x(s)) \qquad \text{as} \quad s \to +\infty \tag{30}$$

The matrix

$$S := \begin{pmatrix} \lambda & mM + \mu \\ mM - \mu & -\lambda \end{pmatrix}$$

can be diagonalised by a non-singular matrix T as

$$T^{-1} S T = \begin{pmatrix} \sqrt{\lambda^2 + m^2 M^2 - \mu^2} & 0 \\ 0 & -\sqrt{\lambda^2 + m^2 M^2 - \mu^2} \end{pmatrix},$$

if $\lambda^2 + m^2 M^2 - \mu^2 \neq 0$. If $\lambda^2 + m^2 M^2 - \mu^2 = 0$, its Jordan canonical form is

$$T^{-1} S T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

According to Theorem 1.8.1 and Theorem 1.10.1 in Eastham [5], we have two linearly independent solutions $U^1(s)$, $U^2(s)$ of (30) such that if $\lambda^2 + m^2 M^2 - \mu^2 \neq 0$,

$$\begin{aligned} U^1(s) &= \{v_1 + o(1)\} \exp\left(\sqrt{\lambda^2 + m^2 M^2 - \mu^2}\, s\right) &&= \{v_1 + o(1)\} (-x)^{\sqrt{\lambda^2 + m^2 M^2 - \mu^2}}, \\ U^2(s) &= \{v_2 + o(1)\} \exp\left(-\sqrt{\lambda^2 + m^2 M^2 - \mu^2}\, s\right) &&= \{v_2 + o(1)\} (-x)^{-\sqrt{\lambda^2 + m^2 M^2 - \mu^2}} \end{aligned}$$

as $x \to -\infty$ and, if $\lambda^2 + m^2M^2 - \mu^2 = 0$,

$$U^1(s) \;=\; v_1 + o(1),$$

$$U^2(s) \;=\; sv_1 + v_2 + o(s) \;=\; v_1 \log(-x) + v_2 + o(\log(-x))$$

as $x \to -\infty$, where $T = (v_1, v_2)$.

Therefore a necessary condition for the existence of an $\mathscr{L}^2(-\infty, 0)$-eigenfunction is

$$\sqrt{\lambda^2 + m^2M^2 - \mu^2} \in (1/2, \infty), \qquad \text{i.e.} \quad \lambda^2 + m^2M^2 - \mu^2 > 1/4. \qquad \square$$

Remark V.2. The above discussions show that for ω to be an eigenvalue of H the following conditions are necessary:

$$r_+ = r_- = M \qquad \text{(Theorem IV.4)}, \tag{31}$$

$$\omega = A_0 = -\frac{a\kappa + eQr_+}{r_+^2 + a^2} \qquad \text{(Theorem IV.2 (i))}, \tag{32}$$

$$\omega^2 < m^2 \qquad \text{(Theorem IV.2 (ii)), Remark IV.3)}, \tag{33}$$

$$\lambda^2 + m^2M^2 - \mu^2 > \frac{1}{4} \qquad \text{(Theorem V.1)}. \tag{34}$$

However, the solvability of the above system is not yet sufficient for the existence of an energy eigenvalue.

Schmid [10] showed that if in addition either

$$\beta - \sigma\lambda = 0, \qquad \alpha + \eta = 0 \tag{35.0}$$

or

$$1 + N + \alpha + \eta = 0 \qquad \text{for some non-negative integer } N, \tag{35.N}$$

holds, then the solvability of the system (32)–(35.N) is sufficient for the existence of an eigenvalue ω of H.

In the above formulae, we used

$$\sigma := \operatorname{sign}\omega, \qquad \eta := \sqrt{\lambda^2 + M^2m^2 - \mu^2}, \qquad \mu = 2M\omega + eQ = -\frac{2a\kappa M}{M^2 + a^2} - eQ\,\frac{M^2 - a^2}{M^2 + a^2},$$

$$\alpha := \frac{Mm^2 - \omega\mu}{\sqrt{m^2 - \omega^2}}, \qquad \beta := \frac{(M|\omega| - \sigma\mu)m}{\sqrt{m^2 - \omega^2}}.$$

Note that the variable η is denoted by κ in [10].

VI. ENERGY EIGENVALUES IN THE CASE $r_+ = r_-$

If ω is an energy eigenvalue of (1), then there must be an $N \in \mathbb{N}_0$ such that ω satisfies the complicated system of conditions (32)–(34) and (35.N). It is not clear that for given data of the black hole and the particle there are tuples $(\nu\omega_0,\, \kappa,\, \lambda_{\nu,\kappa,n})$ that solve the system (32)–(35.N). Schmid [10] has shown that in the so-called Kerr case (i.e. if $Q = 0$) for fixed data of the spin-$\frac{1}{2}$ particle there exist two sequences $(a_N^{\pm})_{N\in\mathbb{N}}$ such that for $a = a_N$ the value $\omega_N = -\frac{\eta}{2a_N^{\pm}}$ is an energy eigenvalue.

Here we fix the black hole data M, Q and a and vary the mass of the fermion to obtain the existence of energy eigenvalues in the case $r_- = r_+$.

Theorem VI.1. *Fix $M > 0$, a, Q, $e \in \mathbb{R}$, $\nu \in \mathbb{Z}$ and $\kappa \in \mathbb{Z} + \frac{1}{2}$. Let $\omega := -\frac{\kappa a + eQM}{a^2 + M^2}$. Take $\lambda = \lambda_{\nu,\kappa,n}$ for any sufficiently large $|n|$. If $\omega(eQ + M\omega) \le 0$ then there are no energy eigenvalues of $H_{\nu,\kappa,n}$. If $\omega(eQ + M\omega) > 0$ then there is a sequence $(m_N)_{N\in\mathbb{N}} \subseteq (|\omega|, \infty)$ such that if $m \in \{m_N : N \in \mathbb{N}\}$, then ω is an energy eigenvalue of $H_{\nu,\kappa,n}$. For N_0 large enough, the sequence $(m_N)_{N\ge N_0}$ is monotonously decreasing and converges to $|\omega|$.*

Before we proof the theorem, let us emphasise that $\lambda = \lambda_{\nu,\kappa,n}$ does depend also on m. Therefore, we denote it by $\lambda = \lambda_{\nu,\kappa,n}(m)$. In order to check the condition (34) we prepare the following lemma.

Lemma VI.2. *Fix $\nu \in \mathbb{Z}$ and $\kappa \in \mathbb{Z} + \frac{1}{2}$. If $|n|$ is sufficiently large, then the inequality (34) holds for any $m > 0$, $M > 0$ and $\mu \in \mathbb{R}$, that is,*

$$\lambda_{\nu,\kappa,n}^2 + m^2M^2 - \mu^2 \;>\; \frac{1}{4}$$

Proof. It follows from standard perturbation theory (applied to the angular operator with m as perturbation parameter, see Winklmeier [16], Kato [9]) that

$$\left|\frac{d}{dm}\lambda_{\nu,\kappa,n}(m)\right| \;\leq\; \left\|\begin{pmatrix}-a\cos\theta & 0\\ 0 & a\cos\theta\end{pmatrix}\right\| \;=\; |a|,$$

hence, for $m > |\omega|$,

$$|\lambda_{\nu,\kappa,n}(|\omega|)| - |a|(m-|\omega|) \;\leq\; |\lambda_{\nu,\kappa,n}(m)| \;\leq\; |\lambda_{\nu,\kappa,n}(|\omega|)| + |a|(m-|\omega|) \tag{36}$$

Let $\widetilde{m} = M^{-1}\sqrt{\mu^2+1/4}$. Since the sequence $(\lambda_{\nu,\kappa,n})_n$ is monotonously increasing and unbounded from below and from above, there is an integer n_0 such that

$$|\lambda_{\nu,\kappa,n}(|\omega|)| \;>\; |a|(\widetilde{m}-|\omega|) + \sqrt{\mu^2+1/4}$$

for all $|n| \geq n_0$. If $m \in \big(|\omega|,\, \widetilde{m}\big]$, we have

$$\begin{aligned}|\lambda_{\nu,\kappa,n}(m)| \;&\geq\; |\lambda_{\nu,\kappa,n}(|\omega|)| - |a|(m-|\omega|)\\ &\geq\; |\lambda_{\nu,\kappa,n}(|\omega|)| - |a|(\widetilde{m}-|\omega|) > \sqrt{\mu^2+1/4}\end{aligned}$$

which implies

$$\sqrt{\lambda_{\nu,\kappa,n}(m)^2 + M^2m^2 - \mu^2} \;\geq\; \sqrt{\mu^2+1/4+m^2M^2-\mu^2} > 1/2.$$

If $m > \widetilde{m} = M^{-1}\sqrt{\mu^2+1/4}$, then we have

$$\sqrt{\lambda_{\nu,\kappa,n}(m)^2 + M^2m^2 - \mu^2} \;>\; \sqrt{M^2\widetilde{m}^2-\mu^2} \;=\; 1/2.$$

$\square$

Now we shall prove Theorem VI.1.

Proof of Theorem VI.1. By definition, ω satisfies condition (32). As seen in Lemma VI.2, there is an $n_0 \in \mathbb{N}$ such that condition (34) is satisfied for all $|n| \geq n_0$. From now on, let us assume that condition (34) Next we consider the conditions (35.0) and (35.N). To this end we compute

$$\begin{aligned}\alpha+\eta \;&=\; \frac{Mm^2-\omega\mu}{\sqrt{m^2-\omega^2}} + \sqrt{\lambda_{\kappa,n}^2 + M^2m^2-\mu^2}\\ &=\; -\frac{\omega(eQ+M\omega)}{\sqrt{m^2-\omega^2}} + M\sqrt{m^2-\omega^2} + \sqrt{\lambda_{\kappa,n}^2+M^2m^2-\mu^2}.\end{aligned}$$

If $\omega(eQ+M\omega) \leq 0$ then $\alpha+\eta > \frac{1}{2}$ and condition (35.N) cannot be satisfied for any $N \in \mathbb{N}_0$. Assume now that $\omega(eQ+M\omega) > 0$. Then the function

$$\begin{aligned}A : (|\omega|,\,\infty) \;&\longrightarrow\; \mathbb{R},\\ m \;&\mapsto\; -\frac{\omega(eQ+M\omega)}{\sqrt{m^2-\omega^2}} + M\sqrt{m^2-\omega^2} + \sqrt{\lambda_{\nu,\kappa,n}(m)^2+M^2m^2-\mu^2}\end{aligned}$$

is continuous, satisfies $\lim_{m\searrow|\omega|} A(m) = -\infty$, $\lim_{m\to\infty} A(m) = \infty$ in view of (36). Hence for every $N \in \mathbb{N}$ there is at least one $m_N \in (|\omega|, \infty)$ such that $A(m_N) = -1-N$ and therefore satisfies condition (35.N). Since the function A is monotonously increasing in an interval $(|\omega|, |\omega|+\delta)$ for a sufficiently small $\delta > 0$, it follows that for N large enough there is only one m_N satisfying $A(m_N) = -1-N$ and that the sequence $(m_N)_N$ is decreasing. $\square$

Remark VI.3. The proof shows that for fixed m only a finite number of $\lambda_{\nu,\kappa,n}$ is allowed in order to satisfy condition (35.N). The closer m is to $|\omega|$, the more (and the larger) values for $\lambda_{\nu,\kappa,n}$ are allowed.

Remark VI.4. The condition $\omega(eQ + M\omega) > 0$ is satisfied if the ratio eQ/κ is sufficiently small since

$$\begin{aligned}
\omega(eQ + M\omega) &= -\frac{a}{(a^2 + M^2)^2}(\kappa a + eQM)(eQa - \kappa M) \\
&= \frac{a}{(a^2 + M^2)^2}[aM\kappa^2 + eQ(M^2 - a^2)\kappa - aMe^2Q^2] \\
&= \frac{a^2 M}{(a^2 + M^2)^2}\left[\left(\kappa - \frac{eQa}{2M} + \frac{eQM}{2a}\right)^2 - \frac{e^2Q^2}{4}\left(\left(\frac{a}{M} - \frac{M}{a}\right)^2 + 4\right)\right] \\
&= -\frac{a^2\kappa^2 M}{4(a^2 + M^2)^2}\left[\left(\frac{2eQ}{\kappa} + \left(\frac{a}{M} - \frac{M}{a}\right)\right)^2 - \left(4 + \left(\frac{a}{M} - \frac{M}{a}\right)^2\right)\right].
\end{aligned}$$

Remark VI.5. Let ω be an energy eigenvalue, $m \in \{m_N \,:\, N \in \mathbb{N}\}$ (see Theorem VI.1), and $f^{\nu,\kappa,n}$ the eigenfunctions of $H_{\nu,\kappa,n}$. If we set

$$\begin{pmatrix} X_+^{\nu,\kappa,n}(r) \\ X_-^{\nu,\kappa,n}(r) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ -i & -1 \end{pmatrix} \begin{pmatrix} f_1^{\nu,\kappa,n}(x) \\ f_2^{\nu,\kappa,n}(x) \end{pmatrix}$$

then

$$\widehat{\Psi}(r, \theta, \phi, t) = \exp(-i\omega t)\,\exp(-i\kappa\varphi) \begin{pmatrix} X_-^{\nu,\kappa,n}(r)\, g_2^{\nu,\kappa,n}(\theta) \\ X_+^{\nu,\kappa,n}(r)\, g_1^{\nu,\kappa,n}(\theta) \\ X_+^{\nu,\kappa,n}(r)\, g_2^{\nu,\kappa,n}(\theta) \\ X_-^{\nu,\kappa,n}(r)\, g_1^{\nu,\kappa,n}(\theta) \end{pmatrix}$$

is a time-periodic solution of (1).

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