LOWER BOUNDS ON THE LOWEST SPECTRAL GAP OF SINGULAR POTENTIAL HAMILTONIANS

SYLWIA KONDEJ AND IVAN VESELIĆ

ABSTRACT. We analyze Schrödinger operators whose potential is given by a singular interaction supported on a sub-manifold of the ambient space. Under the assumption that the operator has at least two eigenvalues below its essential spectrum we derive estimates on the lowest spectral gap. In the case where the sub-manifold is a finite curve in two dimensional Euclidean space the size of the gap depends only on the following parameters: the length, diameter and maximal curvature of the curve, a certain parameter measuring the injectivity of the curve embedding, and a compact sub-interval of the open, negative energy half-axis which contains the two lowest eigenvalues.

Dedicated to Krešimir Veselić on the occasion of his 65th birthday.

1. MODEL AND RESULTS

This paper studies quantum Hamiltonians with singular potentials, also called singular interactions. This kind of perturbations are particularly important for nanophysics because they model "leaky" nanostructures. More precisely, the Hamiltonians describe nonrelativistic quantum particles which are confined to a nanostructure, e.g. a thin semiconductor, with high probability, but still allowed to tunnel in and through the classically forbidden region.

An idealization of this situation is a d-dimensional quantum system with the potential supported by a finite collection of sub-manifolds $\Gamma \in \mathbb{R}^d$ whose geometry is determined by the semiconductor structure. In general the different manifolds in the collection Γ may have different dimensions. The corresponding Hamiltonian may be formally written as

$$(1) \qquad -\Delta - \alpha\delta(x - \Gamma),$$

where $\alpha > 0$ denotes the coupling constant.

Various results concerning the spectrum of Hamiltonians with singular pertubations were already obtained, for instance, in [BEKŠ94], and more recently in [EI01], [EY02], [EY03], [EK02], [EK03]. However almost nothing is known about gaps between successive eigenvalues. Some estimates for spectral gaps can be recovered from the results given in [EY02],[EK03] but only for the strong coupling constant case, i.e. $\alpha \to \infty$. The aim of this paper is to make progress in this field and obtain lower bounds for the first spectral gap $E_1 - E_0$, where E_1, E_0 are the two lowest eigenvalues.

Since singular potentials are a generalization of regular ones let us give a brief review of some facts known for the latter before formulating our results. It is well known that the double well potential with widely separated minima gives rise to eigenvalues which tend to be grouped in pairs. The classical result by Harrell (see [Har80]) shows that the magnitude of splitting is exponentially small with respect to the separation parameter, i.e. the distance between the wells. This leads naturally to the question whether for more general potentials V one can obtain bounds for eigenvalue splittings in terms of the geometry of V and a spectral parameter at or near the eigenvalues in question. The problem was studied by Kirsch and Simon in [KS85, KS87]. It was shown that for one dimensional Schrödinger operators $-\Delta + V$, where V is a smooth function supported on a set $[a, b]$, the eigenvalue gaps can be bounded in the following way

$$(2) \qquad\qquad E_n - E_{n-1} \geq \pi \lambda^2 e^{-\lambda(b-a)}, \quad n \in \mathbb{N}$$

where

$$\lambda = \max_{E \in \,]E_n, E_{n-1}[\,;x \in \,]a,b[} |E - V(x)|^{1/2},$$

cf. [KS85]. For the multi-dimensional case an exponential lower bound for the spectral gap $E_1 - E_0$ was found in [KS87].

Our main result can be considered as the analog for singular potentials of the results in [KS87]. Thus we return to the main topic of this paper and ask the question: can one find exponential lower bounds for the eigenvalues splittings for Schrödinger operators with singular interactions? We address this question for a two dimensional system with a potential supported by a finite curve (or more generally finitely many disconnected curves). The desired lower bound is expressed in terms of geometric properties of Γ. A crucial role is played by the diameter $2R$ of Γ.

- The main aim of this paper is to show the following lower bound (see Theorem 4.3)

$$E_1 - E_0 \geq \kappa_1^2 \mu_{\Gamma,\alpha}(\rho, \kappa_0)\mathrm{e}^{-C_0\rho}, \text{ with } \rho := \kappa_0 R$$

  where $\kappa_i = \sqrt{-E_i}$ and C_0 is a constant. The dependence of the function $\mu_{\Gamma,\alpha}$ on geometric features of Γ is given explicitly in equation (45).

To prove the above estimate we establish some auxiliary results which, in our opinion, are interesting in their own right. They, for example, concern the

- generalization to singular potentials of techniques developed in [DS84], [KS87] to estimate the first spectral gap,
- analysis of the behaviour of eigenfunctions: exponential decay, localization of maxima and nodal points,
- estimates for gradients of eigenfunction, in particular near the support of the singular potential.

The last mentioned point concerns a step in our strategy which is very different from the route taken in [KS87]. There, in fact, a gradient estimate of eigenfunctions is derived relying on the assumption that the potential is bounded — a situation quite opposite to ours.

The paper is organized as follows. In Section 2 we present some general facts about Hamiltonians with singular potentials. In Section 3 we adapt to the singular potential case an abstract formula for the first spectral gap which was derived in [DS84], [KS87] for regular potentials.

In Section 4 we specialize to the case where the support of the potential is a finite curve Γ in a two-dimensional Euclidean ambient space. In this situation we derive our most explicit lower bound on the first spectral gap in terms of geometric parameters of Γ. The proof of this result is contained in the three last sections.

Section 5 contains several estimates on the pointwise behaviour of eigenfunctions. In Section 6 we establish upper and lower bounds on gradients of eigenfunctions. Special attention and care are given to the behaviour near the support of the singular interaction. A technical estimate is deferred to Appendix A. Section 7 is devoted to the discussion of our results and of some open questions.

2. Generalized Schrödinger operators

We are interested in Hamiltonians with so called singular perturbations. In general, this kind of perturbation is localized on a set of Lebesgue measure zero. In this paper we consider more specifically operators with an interaction supported on an orientable, compact sub-manifold $\Gamma \subset \mathbb{R}^d$ of class C^2 and codimension one. The manifold Γ may, but need not, have a boundary.

The Hamiltonian with a potential perturbation supported on Γ can be formally written as

$$(3) \qquad -\Delta - \alpha\delta(x - \Gamma) \,,$$

where $\alpha > 0$ is a coupling constant.

To give (3) a mathematical meaning we have to construct the corresponding selfadjoint operator on $L^2 := L^2(\mathbb{R}^d)$. The scalar product and norm in L^2 will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Let us consider the Dirac measure σ_Γ in $\mathbb{R}^d$ with support on Γ, i.e. for any Borel set $G \subset \mathbb{R}^d$ we have

$$\sigma_\Gamma(G) := s_{d-1}(G \cap \Gamma) \,,$$

where s_{d-1} is the $d - 1$ dimensional surface measure on Γ. It follows from the theory of Sobolev spaces that the trace map

$$I_{\sigma_\Gamma} \colon W^{1,2} \to L^2(\sigma_\Gamma) \,, \quad \text{where} \quad W^{1,2} := W^{1,2}(\mathbb{R}^d) \,, \ L^2(\sigma_\Gamma) := L^2(\mathbb{R}^d, \sigma_\Gamma)$$

is a bounded operator. Using the trace map we construct the following sesquilinear form

$$(4) \qquad \mathcal{E}_{\alpha\sigma_\Gamma}(\psi, \phi) = \int_{\mathbb{R}^d} \nabla\psi(x)\nabla\overline{\phi}(x)\mathrm{d}x - \alpha \int_{\mathbb{R}^d} (I_{\sigma_\Gamma}\psi)(x)(I_{\sigma_\Gamma}\overline{\phi})(x)\mathrm{d}\sigma_\Gamma(x) \,,$$

for $\psi, \phi \in W^{1,2}$. From Theorem 4.1 in [BEKŠ94] we infer that the measure σ_Γ belongs to the generalized Kato class, which is a natural generalization of the notion of Kato class potentials. In particular, for such a measure and an arbitrary $a > 0$ there exists $b_a < \infty$ such that

$$\int_{\mathbb{R}^d} |(I_{\sigma_\Gamma}\psi)(x)|^2 \mathrm{d}\sigma_\Gamma(x) \leq a\|\nabla\psi\|^2 + b_a\|\psi\|^2 \,.$$

This, in turn, implies that the form $\mathcal{E}_{\alpha\sigma_\Gamma}$ is closed. Consequently there exists a unique selfadjoint operator $H_{\alpha\sigma_\Gamma}$ acting in L^2 associated to $\mathcal{E}_{\alpha\sigma_\Gamma}$. This operator $H_{\alpha\sigma_\Gamma}$ gives a precise meaning to the formal expression (3).

Remark 2.1. Using an argument from [BEKŠ94] we can define the operator $H_{\alpha\sigma_\Gamma}$ by appropriate selfadjoint boundary conditions on Γ. Denote by n: $\Gamma \to \mathbb{S}^d$ a global unit normal vectorfield on Γ. Let $D(\tilde{H}_{\alpha\sigma_\Gamma})$ denote the set of functions

$$\psi \in C(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \Gamma) \cap W^{2,2}(\mathbb{R}^d \setminus \Gamma)$$

which satisfy [1]

(5) $$\partial_\mathrm{n}^+ \psi(x) + \partial_\mathrm{n}^- \psi(x) = -\alpha\psi(x) \qquad \text{for } x \in \Gamma \,,$$

where

$$\partial_\mathrm{n}^+ \psi(x) := \lim_{\epsilon \searrow 0} \frac{\psi(x + \epsilon\mathrm{n}(x)) - \psi(x)}{\epsilon}$$

$$\partial_\mathrm{n}^- \psi(x) := \lim_{\epsilon \searrow 0} \frac{\psi(x - \epsilon\mathrm{n}(x)) - \psi(x)}{\epsilon} \,.$$

By Green's formula we have for $\psi, \phi \in D(\tilde{H}_{\alpha\sigma_\Gamma})$

$$-\int_{\mathbb{R}^2} (\Delta\psi(x)) \overline{\phi(x)} \, \mathrm{d}x = \mathcal{E}_{\alpha\sigma_\Gamma}(\psi, \phi) \,.$$

Using this equation we can conclude exactly as in Remark 4.1 of [BEKŠ94] that the closure of $-\Delta$ with domain $D(\tilde{H}_{\alpha\sigma_\Gamma})$ is the selfadjoint operator $H_{\alpha\sigma_\Gamma}$.

It can be immediately seen from formula (5) that the opposite choice of the orientation of the manifold Γ does not change the boundary condition. It is useful to note that the eigenfunctions of $H_{\alpha\sigma_\Gamma}$ belong to $C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \Gamma)$, cf.[EY03].

Remark 2.2. The manifold Γ may have several components. We will provide proofs for our results only in the case that Γ is connected. The modification for the general case consists basically in the introduction of a new index numbering the components. See also the Remark 4.2.

Definition 2.3 (Resolvent of $H_{\alpha\sigma_\Gamma}$). Since we are interested in the discrete spectrum of $H_{\alpha\sigma_\Gamma}$ we restrict ourselves to values in the resolvent set with negative real part.

For $\kappa > 0$ denote by $R^\kappa := (-\Delta + \kappa^2)^{-1}$ the resolvent of the "free" Laplacian. It is an integral operator for whose kernel we write $G^k(x - x')$. Furthermore

---

[1] In [BEKŠ94] the notation $-\frac{\partial}{\partial n_-}$ is used for what we denote by ∂_n^-.

define $R^\kappa_{\sigma_\Gamma,\mathrm{d}x}$ as the integral operator with the same kernel but acting from L^2 to $L^2(\sigma_\Gamma)$. Let $R^\kappa_{\mathrm{d}x,\sigma_\Gamma}$ stand for its adjoint, i.e. $R^\kappa_{\mathrm{d}x,\sigma_\Gamma} f = G^\kappa * f\sigma_\Gamma$ and finally we introduce $R^\kappa_{\sigma_\Gamma,\sigma_\Gamma}$ defined by G^κ as an operator acting from $L^2(\sigma_\Gamma)$ to itself. In the following theorem we combine several results borrowed from [BEKŠ94] and [Pos04, Pos01].

Theorem 2.4. *(i) There is a $\kappa_0 > 0$ such that operator $I - \alpha R^\kappa_{\sigma_\Gamma,\sigma_\Gamma}$ in $L^2(\sigma_\Gamma)$ has a bounded inverse for any $\kappa \geq \kappa_0$.*
(ii) Assume that $I - \alpha R^\kappa_{\sigma_\Gamma,\sigma_\Gamma}$ is boundedly invertible. Then the operator

$$R^\kappa_{\alpha\sigma_\Gamma} = R^\kappa + \alpha R^\kappa_{\mathrm{d}x,\sigma_\Gamma}(I - \alpha R^\kappa_{\sigma_\Gamma,\sigma_\Gamma})^{-1} R^\kappa_{\sigma_\Gamma,\mathrm{d}x}$$

maps L^2 to L^2, $-\kappa^2 \in \rho(H_{\alpha\sigma_\Gamma})$ and $R^\kappa_{\alpha\sigma_\Gamma} = (H_{\alpha\sigma_\Gamma} + \kappa^2)^{-1}$.
(iii) Suppose $\kappa > 0$. The number $-\kappa^2$ is an eigenvalue of $H_{\alpha\sigma_\Gamma}$ iff $\ker(I - \alpha R^\kappa_{\sigma_\Gamma,\sigma_\Gamma}) \neq \{0\}$. Moreover,

$$\dim \ker(H_{\alpha\sigma_\Gamma} + \kappa^2) = \dim \ker(I - \alpha R^\kappa_{\sigma_\Gamma,\sigma_\Gamma}).$$

(iv) Assume $-\kappa^2$ is an eigenvalue of $H_{\alpha\sigma_\Gamma}$. Then for every $w_\kappa \in \ker(I - \alpha R^\kappa_{\sigma_\Gamma,\sigma_\Gamma})$ the function defined by

$$(6) \qquad\qquad \psi_\kappa := R^\kappa_{\mathrm{d}x,\sigma_\Gamma} w_\kappa .$$

is in $D(H_{\alpha\sigma_\Gamma})$ and satisfies $H_{\alpha\sigma_\Gamma}\psi_\kappa = -\kappa^2\psi_\kappa$.

Combining the statements (iii) and (iv) of the above theorem we get the equality

$$(7) \qquad\qquad \alpha I_{\sigma_\Gamma}\psi_\kappa = w_\kappa ,$$

which will be useful in the sequel, more precisely in equation (24).

Remark 2.5 (Some facts about the spectrum of $H_{\alpha\sigma_\Gamma}$)**.** Since the perturbation is supported on a compact set the essential spectrum of $H_{\alpha\sigma_\Gamma}$ is the same as for free Laplacian, i.e.

$$\sigma_{\mathrm{ess}}(H_{\alpha\sigma_\Gamma}) = [0, \infty[,$$

cf. [BEKŠ94]. From [BEKŠ94, EY03] we infer that $H_{\alpha\sigma_\Gamma}$ has nonempty discrete spectrum if $d = 2$ and α is positive. For $d \geq 3$ there is a critical value $\alpha_c > 0$ for the coupling constant such that the discrete spectrum of $H_{\alpha\sigma_\Gamma}$ is empty if and only if $\alpha \leq \alpha_c$. The discrete spectrum has been analyzed in various papers (see [EY02, EY03, EY04] and [Exn03]). It was shown, for example, that for a sub-manifold Γ without boundary we have the following asymptotics of the j-th eigenvalue of $H_{\alpha\sigma_\Gamma}$ in the strong coupling constant limit

$$(8) \qquad E_j(\alpha) = -\frac{\alpha^2}{4} + \mu_j + \mathcal{O}\left(\frac{\log \alpha}{\alpha}\right) \quad \text{as} \quad \alpha \to \infty,$$

where μ_j is the eigenvalue of an appropriate comparison operator. This operator is determined by geometric properties of Γ, i.e. its metric tensor. In the simplest case, when $\Gamma \subset \mathbb{R}^2$ is a closed curve of length L determined as the range of the arc length parameterization $[0, L] \ni s \mapsto \gamma(s) \in \mathbb{R}^2$, the comparison operator takes the form

$$-\frac{d^2}{ds^2} - \frac{\varkappa(s)^2}{4} \; : \; D\left(\frac{d^2}{ds^2}\right) \to L^2(0, L),$$

where $\frac{d^2}{ds^2}$ is the Laplace operator with periodic boundary conditions and $\varkappa :=$ $|\gamma''| \colon [0, L] \to \mathbb{R}$ is the curvature of Γ in the parameterization $\gamma \colon [0, L] \to \mathbb{R}^2$.

If the curve Γ is not closed analogous asymptotics as (8) can be proven. However now only upper and lower bounds on μ_j can be established, namely

$$\mu_j^N \leq \mu_j \leq \mu_j^D \,,$$

where μ_j^N, μ_j^D are eigenvalues corresponding to Neumann, respectively Dirichlet boundary conditions of a comparison operator.

3. The lowest spectral gap for singular perturbations

The aim of this section is to give general formulae for the first spectral gap of $H_{\alpha\sigma_\Gamma}$. Following the idea used for regular potentials (see for example ([DS84], [KS87]) we will introduce a unitary transformation defined by means of the ground state of $H_{\alpha\sigma_\Gamma}$.

Assume that there exists an eigenfunction ψ_0 of $H_{\alpha\sigma_\Gamma}$ which is positive almost everywhere. Such an eigenfunction is up to a scalar multiple uniquely defined, i.e. the corresponding eigenvalue is non-degenerate. We will show in Lemma 5.1 that, if $H_{\alpha\sigma_\Gamma}$ is a Hamiltonian in $\mathbb{R}^2$ with a singular potential supported on a curve, such a function ψ_0 exists and is the eigenfunction corresponding to the lowest eigenvalue of $H_{\alpha\sigma_\Gamma}$. Let us define the unitary transformation

$$U \colon L^2 \to L_{\psi_0}^2 := L^2(\mathbb{R}^d, \psi_0^2 \mathrm{d}x) \,, \quad Uf := \psi_0^{-1} f \,, \quad f \in L^2$$

and denote the eigenvalue corresponding to ψ_0 by E_0. Furthermore, consider the sesquilinear form

$$(9) \qquad \tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}(\psi, \phi) = \mathcal{E}_{\alpha\sigma_\Gamma}(U^{-1}\psi, U^{-1}\phi) - E_0(U^{-1}\psi, U^{-1}\phi) \,,$$

for $\psi, \phi \in D(\tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}) = W^{1,2}(\mathbb{R}^d, \psi_0^2 \mathrm{d}x)$.

Similarly as for regular potentials, after the unitary transformation the information about the singular potential is comprised in the weighted measure, i.e. we have

Theorem 3.1. *The form $\tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}$ admits the following representation*

$$(10) \qquad \tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}(\psi, \phi) = \int_{\mathbb{R}^d} (\nabla\psi)(\nabla\overline{\phi}) \, \psi_0^2 \mathrm{d}x \,, \quad \text{for} \quad \psi, \phi \in D(\tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}) \,.$$

Proof. To show the claim let us consider first the form $\mathcal{E}_{\alpha\sigma_\Gamma}(U^{-1}\psi, U^{-1}\phi)$ for $\psi, \phi \in C_0^\infty(\mathbb{R}^d)$. Using (4) we obtain by a straightforward calculation

$$\mathcal{E}_{\alpha\sigma_\Gamma}(U^{-1}\psi, U^{-1}\phi) = -\alpha \int_{\mathbb{R}^d} I_{\sigma_\Gamma}(\psi\overline{\phi}\psi_0^2) \mathrm{d}\sigma_\Gamma +$$

$$(11) \qquad \int_{\mathbb{R}^d} \left[(\nabla\psi)(\nabla\overline{\phi})\psi_0^2 + (\nabla\psi)\overline{\phi}\psi_0\nabla\psi_0 + \psi(\nabla\overline{\phi})\psi_0\nabla\psi_0 + \psi\overline{\phi}(\nabla\psi_0)^2 \right] \mathrm{d}x \,.$$

The last term in the above expression can be expanded by integrating by parts in the following way

$$(12) \qquad \int_{\mathbb{R}^d} \psi \overline{\phi} (\nabla \psi_0)^2 \mathrm{d}x = - \int_{\mathbb{R}^d} \nabla (\psi \overline{\phi} \nabla \psi_0) \psi_0 \mathrm{d}x$$
$$+ \int_{\mathbb{R}^d} I_{\sigma_\Gamma} (\psi \overline{\phi} \psi_0)(\partial_n^- \psi_0 + \partial_n^+ \psi_0) \mathrm{d}\sigma_\Gamma \, .$$

To deal with the last expression we expand by differentiation the term of the r.h.s. on (12) onto three components, use boundary conditions (5) and the fact that ψ_0 is the eigenfunction of the Laplacian with these boundary conditions. Finally, putting together (11) and (12) and inserting it to (9) we obtain the equivalence (10) on $C_0^\infty(\mathbb{R}^d)$. Extending it by continuity to $D(\tilde{\mathcal{E}}_{\alpha\sigma_\Gamma})$ we get the claim. $\qquad \square$

Since our aim is to estimate the spectral gap we assume that
(13)
the bottom of the spectrum of $H_{\alpha\sigma_\Gamma}$ consists of two isolated eigenvalues.

Let $E_1 = \inf_{\psi \perp \psi_0, \|\psi\|=1} \mathcal{E}_{\alpha\sigma_\Gamma}(\psi, \psi)$ denote the first excited eigenvalue and denote by ψ_1 a corresponding eigenfunction. It follows from (9) that

$$(14) \qquad E_1 - E_0 = \tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}[U\psi_1]/\|\psi_1\|^2 \, ,$$

where we use the abbreviation $\tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}[\phi] = \tilde{\mathcal{E}}_{\alpha\sigma_\Gamma}(\phi, \phi)$. If E_1 is degenerate the formula holds for *any* eigenfunction. Using Theorem 3.1, we have

Corollary 3.2. *The spectral gap between the two lowest eigenvalues of $H_{\alpha\sigma_\Gamma}$ is given by*

$$E_1 - E_0 = \int \left| \nabla \frac{\psi_1}{\psi_0} \right|^2 \psi_0^2 \mathrm{d}x / \|\psi_1\|^2 \, .$$

Let us note that notations ψ_i correspond to ψ_{κ_i} where $E_i = -\kappa_i^2$ from Theorem 2.4.

4. Estimates for the lowest spectral gap of Hamiltonians with interaction on a finite curve

The aim of this section is to derive explicit estimates for the lowest spectral gap of $H_{\alpha\sigma_\Gamma}$. We will use the general results obtained in Section 3, and apply them to a two dimensional system. More precisely, let $\Gamma \subset \mathbb{R}^2$ be a finite curve given as the range of the C^2-parameterization $[0, L] \ni s \mapsto \gamma(s) = (\gamma_1(s), \gamma_2(s)) \in \mathbb{R}^2$ without self-intersections. (Exception: If Γ is a closed curve the starting and end point of the curve coincide. In that case we also require that the first two derivatives of the parameterization γ coincide at the parameter values 0 and L.) We assume that Γ is parameterized by arc length.

Denote by m_Γ a constant satisfying $m_\Gamma < L$ if Γ is closed and $m_\Gamma \le L$ otherwise. By the C^2-differentiability assumption on γ, for each m_Γ, there exists a positive constant $M_\gamma := M_\gamma(m_\Gamma)$ such that

$$(15) \qquad M_\gamma(m_\Gamma)|s - s'| \le |\gamma(s) - \gamma(s')| \quad \text{for } s, s' \in \mathbb{R} \text{ with } |s - s'| \le m_\Gamma \, ,$$

where $|\gamma(s)| = \sqrt{\gamma_1(s)^2 + \gamma_2(s)^2}$. We choose $M_\gamma(m_\Gamma)$ to be the largest possible number satisfying inequality (15). Then $m_\Gamma \mapsto M_\gamma(m_\Gamma)$ is a non-increasing, continuous function.

The estimates we will derive depend on the geometry of the curve through its length L, the diameter of Γ, the maximum

$$K := \max_{s \in [0,L]} \varkappa(s)$$

of its curvature $\varkappa \colon [0, L] \to \mathbb{R}$, and the values $M_\gamma(L/2)$ and $M_{\tilde\gamma}(L/2)$, where $\tilde\gamma$ is defined in (29).

Remark 4.1. In fact our methods work also for C^1-curves which are piecewise C^2 regular. This means that there are finitely many values $0 =: s_1, \ldots, s_N := L$ such that $\gamma \colon]s_i, s_{i+1}[\to \mathbb{R}^2$ is of class C^2 for each $i \in \{1, \ldots, N-1\}$ and at each point $\gamma(s_i)$ the curvature of Γ jumps by an angle φ_i. In this case the proofs become somewhat more technical. The constants which under the global C^2-assumption depend only on the curvature are in the more general case additionally dependent on the angles $\varphi_2, \ldots, \varphi_{N-1}$.

Following the general discussion given in Section 2 we can construct the Hamiltonian $H_{\alpha\sigma_\Gamma}$ with a perturbation on Γ as the operator associated with the form $\mathcal{E}_{\alpha\sigma_\Gamma}$ given by (4). Furthermore, the operator $H_{\alpha\sigma_\Gamma}$ is associated to the boundary conditions (5).

To derive estimates for the lowest spectral gap we will work in an appropriate neighbourhood of Γ and to this aim we will introduce the following notation. For $\epsilon \geq 0$ let $\mathcal{C}_\epsilon$ be a convex hull of the set $\Gamma_\epsilon := \{x \in \mathbb{R}^2 \mid \operatorname{dist}(x, \Gamma) \leq \epsilon\}$. We denote

$$\mathcal{C} := \mathcal{C}_0, \quad R := \inf\{r > 0 \mid \exists x \in \mathbb{R}^2 : B_r(x) \supset \mathcal{C}_1\}.$$

Let $x_0 \in \mathbb{R}^2$ be such that $B_R := B_R(x_0) \supset \mathcal{C}_1$.

Remark 4.2. For a connected curve Γ we have clearly $R \leq 1 + \frac{L}{2}$. In the general situation, where Γ consists of several topological components this is no longer true, and R and L are completely independent parameters of our model.

We employ Corollary 3.2 and the Hölder inequality to obtain the lower bound for

$$(16) \qquad E_1 - E_0 \geq \frac{(\int_{B_R} |\nabla f| \mathrm{d}x)^2}{\|\psi_0\|^2 \|\psi_1\|^2} \inf_{x \in B_R} \psi_0(x)^4,$$

where $f := \psi_1/\psi_0$, cf. [KS87].

Set $\kappa_i := \sqrt{-E_i}$ for $i = 0, 1$. The main result of this section is contained in the following statement.

Theorem 4.3. *Suppose that assumption* (13) *is satisfied. Then the lowest spectral gap of $H_{\alpha\sigma_\Gamma}$ can be estimated as follows*

$$(17) \qquad E_1 - E_0 \geq \kappa_1^2 \, \mu_{\Gamma,\alpha}(\rho, \kappa_0) \, \mathrm{e}^{-C_0\rho}, \quad \rho := \kappa_0 R$$

where $\mu_{\Gamma,\alpha}(\cdot, \cdot)$ is a polynomial function and C_0 is an absolute constant.

The precise formula (45) for the function $\mu_{\Gamma,\alpha}$ is derived at the end of Section 6. For the proof of this theorem we need several lemmata estimating the behavior of all ingredients involved in the r.h.s. of (16). They are collected in the subsequent sections.

5. POINTWISE ESTIMATES ON THE EIGENFUNCTIONS

5.1. Lower bound for the ground state.
The first step is to obtain a lower bound for $\inf_{x \in B_R} \psi_0(x)$. The sought estimate is given in the following

Lemma 5.1. *(i) The ground state ψ_0 of $H_{\alpha\sigma_\Gamma}$ is a simple eigenfunction.*
(ii) The function ψ_0 is strictly positive on $\mathbb{R}^2$ and moreover we have

$$(18) \qquad \inf_{x \in B_R} \psi_0(x) \geq C_1 \kappa_0 \frac{\mathrm{e}^{-2\rho}}{1 + \sqrt{2\rho}} \|\psi_0\|, \quad \text{where} \quad \rho = \kappa_0 R,$$

and C_1 is a positive constant.

Remark 5.2. It is useful to note that the integral kernel of the inverse of the two dimensional Laplacian has the following representation

$$(19) \qquad G^{i\kappa}(x - x') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{ip(x-x')}}{p^2 + \kappa^2} \mathrm{d}p = \frac{1}{2\pi} K_0(\kappa|x - x'|),$$

where K_0 is the Macdonald function [AS72].

Proof of Lemma 5.1. (i) To prove the theorem we will use the representation $\psi_0 = R_{\mathrm{d}x,\sigma_\Gamma}^{\kappa_0} w_0$, cf. (6). Using the same argument as in [Exn05] we conclude that w_0 is a simple, positive eigenfunction of $R_{\sigma_\Gamma,\sigma_\Gamma}^{\kappa_0}$, (the argument from [Exn05] can be extended to curves which are not closed). This implies the simplicity of ψ_0.
(ii) Since the kernel of $R_{\mathrm{d}x,\sigma_\Gamma}^{\kappa_0}$ is a strictly positive function and w_0 is positive we have

$$(20) \qquad \psi_0(x) = (R_{\mathrm{d}x,\sigma_\Gamma}^{\kappa_0} w_0)(x) = \int_{\mathbb{R}^2} G^{\kappa_0}(x - x')w_0(x')\mathrm{d}\sigma_\Gamma(x') > 0.$$

Furthermore, using the representation $G^\kappa(\xi) = \frac{1}{2\pi} K_0(\kappa\xi)$ (see Remark 5.2) and the behavior of the Macdonald function K_0, cf. [AS72], one infers the existence of a constant $C_2 > 0$ such that

$$G^\kappa(\xi) > C_2 \frac{\mathrm{e}^{-\kappa\xi}}{1 + \sqrt{\kappa\xi}}.$$

Combining this with the positivity of w_0 and formula (20) we get

$$\inf_{x \in B_R} \psi_0(x) \geq \inf_{(x,x') \in B_R \times \Gamma} G^{\kappa_0}(x-x')\|w_0\|_{L^1(\sigma_\Gamma)} \geq C_2(1+\sqrt{2\rho})^{-1}\mathrm{e}^{-2\rho}\|w_0\|_{L^1(\sigma_\Gamma)}.$$

Moreover it follows from formula (23) in Section 5.3 that $2^{3/2}\pi\kappa_0\|\psi_0\| \leq \|w_0\|_{L^1(\sigma_\Gamma)}$ which completes the proof. $\qquad\square$

Let us note that the above result is analogous to the one obtained for regular potentials in [KS87]. However the method used there is mainly based on the Feynman–Kac formula which cannot be directly applied to singular potentials.

5.2. **Localizations of zeros and maxima of eigenfunctions.** To obtain an estimate on the gradient which is involved in (16) we need some informations on the behaviour of the functions ψ_0 and ψ_1. For this aim we will localize their zeros and maxima.

Let us recall that v is a *subsolution*, respectively *supersolution*, of the equation $(-\Delta - E)u = 0$ in an open set Ω, if $(-\Delta - E)v(x) \leq 0$, respectively $(-\Delta - E)v(x) \geq 0$, for all $x \in \Omega$. In the sequel we need the following fact, see e.g. Lemma 2.9 in [Agm85].

Lemma 5.3. *Let v be a subsolution of the equation $(-\Delta - E)u = 0$ in an open set Ω. Then $v_+ := \max\{v, 0\}$ is also a subsolution of the same equation in Ω.*

Our next task is to localize the maxima, minima and zeros of eigenfunctions of $H_{\alpha\sigma_\Gamma}$.

Proposition 5.4. *Let ψ be a real eigenfunction of $H_{\alpha\sigma_\Gamma}$ with negative eigenvalue E. Then all its maxima and minima lie on Γ. If ψ is not the ground state, at least one zero of ψ lies in $\mathcal{C}$.*

The analog of the proposition holds for proper potentials as well as singular ones in arbitrary space dimension.

Proof. Let ψ be any eigenfunction of the operator $H_{\alpha\sigma_\Gamma}$ to the eigenvalue $E < 0$. Then for any $\epsilon > 0$, on the complement of Γ_ϵ we have $-\Delta\psi = E\psi$. For $\psi_+ := \max\{\psi, 0\}, \psi_- := \max\{-\psi, 0\}$ we have again

$$(21) \qquad -\Delta\psi_+ \leq E\psi_+ \quad -\Delta\psi_- \leq E\psi_- \quad \text{on } \mathbb{R}^2 \setminus \Gamma_\epsilon \,.$$

By the strong maximum principle, ψ_+, ψ_- assume their maxima inside Γ_ϵ, unless they are constant, cf. for instance Thm. 2.2 in [GT83]. The latter can only occur if we consider a bounded region with boundary conditions which are different from Dirichlet ones. Thus the minima and maxima of ψ are contained in $\bigcap_{\epsilon>0}\Gamma_\epsilon = \Gamma$. If ψ is not the ground state, at least one of its zeros is contained in $\mathcal{C}$ since ψ is real. $\qquad\square$

5.3. **Relation between norms.**

Lemma 5.5. *Let ψ be an eigenfunction of $H_{\alpha\sigma_\Gamma}$ to $E = -\kappa^2$. We have*

$$(22) \qquad \|\psi\| \leq \frac{L\alpha}{2^{3/2}\pi\,\kappa}\|\psi\|_\infty \,,$$

where L is the length of Γ.

Proof. To prove the claim we will use the representation

$$\psi(x) \equiv \psi_\kappa(x) = (R^\kappa_{\mathrm{d}x,\sigma_\Gamma}w_\kappa)(x) = \int_{\mathbb{R}^2} G^\kappa(x - x')w_\kappa(x')\mathrm{d}\sigma_\Gamma(x') \,,$$

cf. (20) and the Fourier transform of G^κ given by (19). A straightforward calculation yields

$$(23) \qquad \|\psi\|^2 \leq \frac{1}{(2\pi)^3}\int_{\mathbb{R}^2}\frac{1}{(p^2 + \kappa^2)^2}\mathrm{d}p\|w\|^2_{L^1(\sigma_\Gamma)} = \frac{1}{2^3\pi^2}\kappa^{-2}\|w\|^2_{L^1(\sigma_\Gamma)} \,.$$

Combining relation (7) with the fact that the maxima and minima of ψ lie on Γ we obtain

$$(24) \qquad\qquad \|w\|_{L^1(\sigma_\Gamma)} \leq L\|w\|_\infty = \alpha L\|\psi\|_\infty \,.$$

Applying (24) to (23) we get the desired inequality. $\square$

Lemma 5.6. *Let ψ be an eigenfunction of $H_{\alpha\sigma_\Gamma}$ with the corresponding eigenvalue $-\kappa^2$. There exists a positive absolute constant η_0 such that*

$$(25) \qquad\qquad \|\psi\| \geq \frac{\kappa R}{(c_5^\Gamma \alpha + 1)^2(\kappa^2 + 1)} \mathrm{e}^{-\eta_0\kappa R}\|\psi\|_\infty \,,$$

where $c_5^\Gamma := \max\{2c_4^\Gamma, c_3^\Gamma + c_4^\Gamma \log(\max\{1, L\})\}$ and c_3^Γ and c_4^Γ are taken from Corollary 6.7.

Proof. In the sequel we will use the following fact: If the curve Γ is parameterized by arc length, then at each point $x \in \Gamma$ the vector tangential to Γ and the unit normal vector form an orthogonal basis.

Denote by v a point in $\mathbb{R}^2$ where ψ assumes its maximal value, i.e. $\psi(v) = \|\psi\|_\infty$. We know from Lemma 5.4 that $v \in \Gamma$. Denote by A the square centered at v with sidelength $2b$ whose sides are parallel to the tangential, respectively the normal vector of the curve Γ at the point v. Let x be an arbitrary point in A. To estimate $|\psi(x) - \psi(v)|$ we would like to apply the fundamental theorem of calculus to the gradient of ψ and an auxiliary curve connecting x and v. The natural choice would be a line segment joining the two points, however this segment might be tangential to Γ. In this geometric situation we do not have good control of $\nabla\psi$.

For this reason we consider the following picewise linear curve connecting v and x: join x by a linear segment parallel to the normal vector of Γ at the point v to the boundary of A, do the same for v. Along this boundary edge of A joint the two line segments by a third line segment. This curve has at most length $4b$

Now we complete the proof of the lemma along the lines of the proof of Proposition 6.10 using the upper bound on the gradient obtained in Proposition 6.8. In the present situation the argument is actually somewhat simpler than in the proof of Proposition 6.10. One has to choose the parameter $b = \rho\left((c_5^\Gamma \alpha + 1)^2(\kappa^2 + 1)\right)^{-1} \mathrm{e}^{-\eta_0\rho}$ with an appropriate positive constant η_0, cf.(42), and then establish analogues of the inequalities (42) and (43).

This argument implies that

$$|\psi(x) - \psi(v)| \leq \frac{1}{2}\|\psi\|_\infty$$

for all $x \in A$ if the sidelength of A obeys

$$2b = 2\frac{\kappa R}{(c_5^\Gamma \alpha + 1)^2(\kappa^2 + 1)}\mathrm{e}^{-\eta_0\rho} \,.$$

Finally, by a straightforward calculation we get

$$\|\psi\|^2 \geq \int_A \psi(x)^2 \mathrm{d}x \geq b^2\|\psi\|_\infty^2 \,,$$

which completes the desired result. □

5.4. **Exponential decay of eigenfunctions.** The following result on the exponential decay of eigenfunctions from [Agm85] will be useful in the sequel.

Lemma 5.7. *Let $\tilde{R} > R$ and ψ be an eigenfunction of $H_{\alpha\sigma_\Gamma}$ corresponding to the eigenvalue $E = -\kappa^2$, where $\kappa > 0$. Set $\phi(x) = \phi(|x|) = \sqrt{\tilde{R}/|x|} \; e^{-\kappa(|x|-\tilde{R})}$. Then the following estimate holds*

$$|\psi(x)| \leq \|\psi\|_\infty \phi(x) \;\; \text{for} \;\; x \in D^c := \mathbb{R}^d \setminus D, \; D := \{x \mid |x| < \tilde{R}\} \,.$$

Proof. ϕ is a supersolution and $|\psi|$ is by Lemma 5.3 a subsolution of the equation $(-\Delta - E)u = 0$ in D^c. Thus for any constant $C_4 > 0$ the function

(26) $$F = (C_4\phi - |\psi|)_-$$

is a supersolution. We can choose the constant $C_4 \geq \sup_{|x|=\tilde{R}} |\psi(x)|$ such that F vanishes identically on ∂D. The maximum principle implies that the supremum of F on the closed set D^c is assumed at its boundary. Therefore

$$F \equiv 0 \text{ on } D^c \,.$$

This implies the statement of the lemma. □

6. ESTIMATES ON THE GRADIENT OF EIGENFUNCTIONS

The aim of this section is to derive a lower bound for the expression $\int_{B_R} |\nabla f| \mathrm{d}x$ involved in (16) where $f := \psi_1/\psi_0$.

The strategy here is the following. First we will derive upper bounds for $\nabla\psi_0$, $\nabla\psi_1$. Combining this with the inequality

(27) $$|\nabla f| \leq \frac{|\psi_1| \; |\nabla\psi_0|}{\psi_0^2} + \frac{|\nabla\psi_1|}{\psi_0}$$

and using Lemma 5.1 we will get an upper bound for ∇f. The estimate on the gradient gives a quantitive upper bound for the variation of the function f and is used in Proposition 6.10 to provide a lower bound for $\int_{B_R} |\nabla f| \mathrm{d}x$.

6.1. **Preliminary estimates on certain integrals.** To derive upper and lower bounds on gradients the following geometric notions and generalized distance functions will prove useful.

Definition 6.1. Let S be a line segment of the length $2b$ intersecting the curve Γ at the mid point. We call $y \in \mathbb{R}^2 \setminus \Gamma$ an *S-admissible point*, if the following hold:
there is a unit vector e parallel to S (up to orientation) such that

$$\Theta_S := \{y - te \mid t \in [0, \infty[\}$$

intersects Γ. Denote $t_1 := \min\{t \in [0, \infty[\mid y - te \in \Gamma\}$ and let $s_y \in [0, L]$ be the parameter value such that

$$\gamma(s_y) = y - t_1 e \quad (\in \Gamma \cap \Theta_S) \,.$$

Denote by θ the angle at which Θ_S and Γ intersect at $\gamma(s_y)$, more precisely

$$\cos\theta := \langle e, t(s_y) \rangle$$

where $t(s_y) := \gamma'(s_y) \in \mathbb{R}^2$ is the (unit) tangential vector to the curve Γ at the point $\gamma(s_y)$. Assume that the angle θ is neither zero nor π, i.e.

$$|\cos \theta| < 1 \,.$$

Denote by $d_S(y) = t_1$ the distance between y and $\gamma(s_y)$, which is also the distance form y to Γ along Θ_S. Thus for any $y \in S$ we have $d_S(y) \leq b$.

Remark 6.2. In our application in the proof of Proposition 6.10. we will only need to consider line segments which intersect Γ at an angle which is at least $\pi/6$, i.e. we have $|\cos \theta| \leq \sqrt{3}/2$. Therefore we assume this bound in the sequel.

In the following we assume that S is a line segment intersecting Γ and $y \in \mathbb{R}^2 \setminus \Gamma$ an S-admissible point with vector e in the sense of Definition 6.1.

Lemma 6.3. *Define the function $\phi \colon [0, L] \to \mathbb{R}$ as the angle $\phi(s)$ between the vector e and the vector $\gamma(s) - \gamma(s_y)$, more precisely*

$$g(s) := \cos \phi(s) := \langle e, \tilde{t}(s) \rangle \,,$$

where $\tilde{t}(s) := \tau(s)/|\tau(s)|$ is the normalization of the vector $\tau(s) := \gamma(s) - \gamma(s_y)$. Then we have

$$(28) \qquad |g(s) - g(s_y)| \leq \frac{2K}{M_\gamma(m_\Gamma)} |s - s_y| \quad \text{for} \quad |s - s_y| \leq m_\Gamma \,.$$

Proof. In the proof we will denote by s' a generic value in the interval $[s_y, s]$ and it may change from estimate to estimate.

First we calculate and derive a bound for g'. A calculation using the product rule gives

$$\frac{d}{ds}\left(\frac{\tau(s)}{|\tau(s)|}\right) = \frac{\tau'(s)\langle \tau(s), \tau(s) \rangle - \tau \langle \tau(s), \tau'(s) \rangle}{(\langle \tau(s), \tau(s) \rangle)^{3/2}} \,.$$

This can be expressed more geometrically by means of the orthogonal projection $P(s) \colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ onto the line orthogonal to the vector $\tau(s)$. The formula is $P(s) = I - \frac{|\tau\rangle\langle\tau|}{\langle \tau(s), \tau(s) \rangle} = |\tilde{n}(s)\rangle\langle\tilde{n}(s)|$, where we use the Dirac brac-ket notation, $\tilde{n}(s)$ denotes a unit vector perpendicular to $\tilde{t}(s)$ and I is the identity operator. Then we have

$$\frac{d}{ds}(\tilde{t}(s)) = \frac{P(s)\tau'(s)}{|\tau(s)|} \,.$$

Now, since $\tau(s) = \gamma(s) - \gamma(s_y)$ we obtain $\tau'(s) = \gamma'(s)$ and consequently

$$g'(s) = \frac{\langle e, \tilde{n}(s)\rangle\langle\tilde{n}(s), \gamma'(s)\rangle}{|\tau(s)|} \,.$$

For s close to s_y we expect $\langle \tilde{n}(s), \gamma'(s) \rangle \approx \langle n(s_y), \gamma'(s_y) \rangle = 0$, where $n(s)$ is a unit normal vector to the curve at the point $\gamma(s)$. Let us make this more precise.

From Taylor's formula for the curve $\gamma \colon [0, L] \to \mathbb{R}^2$ it follows

$$\tau(s) = \gamma(s) - \gamma(s_y) = (s - s_y)t(s_y) + \frac{(s - s_y)^2}{2}\gamma''(s'), \text{ for } s' \in [s_y, s]$$

and therefore we get

$$\left|\frac{\tau(s)}{s-s_y}-t(s_y)\right|\leq\frac{K}{2}|s-s_y|\,.$$

This gives the following estimates

$$\left||\tau(s)|-|s-s_y|\right|\leq|\tau(s)-(s-s_y)t(s_y)|\leq\frac{(s-s_y)^2}{2}|\gamma''(s')|\leq\frac{K\,(s-s_y)^2}{2}$$

and

$$\left|\frac{\tau(s)}{s-s_y}-\frac{\tau(s)}{|\tau(s)|}\right|\leq|\tau(s)|\left|\frac{|\tau(s)|-(s-s_y)}{(s-s_y)\tau(s)}\right|\leq\frac{K\,|s-s_y|}{2}\,,$$

which, in turn, imply

$$|\tilde{t}(s)-t(s_y)|\leq\left|\frac{\tau(s)}{|\tau(s)|}-\frac{\tau(s)}{s-s_y}\right|+\left|\frac{\tau(s)}{s-s_y}-t(s_y)\right|\leq K|s-s_y|\,.$$

An easy calculation shows that $|\tilde{n}(s)-n(s_y)|=|\tilde{t}(s)-t(s_y)|$. Combining this we the above inequalities we can estimate the sought expression for $|s-s_y|\leq m_\Gamma$

$$|g'(s)|=\frac{1}{|\tau(s)|}\,|\langle e,\tilde{n}(s)\rangle\langle\tilde{n}(s),\gamma'(s)\rangle|\leq\frac{1}{M_\gamma(m_\Gamma)|s-s_y|}\,|\langle\tilde{n}(s),\gamma'(s)\rangle|$$

$$\leq\frac{1}{M_\gamma(m_\Gamma)|s-s_y|}\left(|\langle n(s_y),\gamma'(s)\rangle|+|n(s_y)-\tilde{n}(s)|\,|\gamma'(s)|\right).$$

Using the formula $\gamma'(s)=\gamma'(s_y)+(s-s_y)\gamma''(s')$ for $s'\in[s_y,s]$ we arrive at

$$|g'(s)|\leq\frac{2K}{M_\gamma(m_\Gamma)}\,.$$

Since g is continuously differentiable, by Taylor's formula there is a number $s'\in[s_y,s]$ such that

$$g(s)=g(s_y)+(s-s_y)g'(s')\,.$$

This finally implies

$$|g(s)-g(s_y)|\leq|s-s_y|\|g'\|_\infty\leq\frac{2K}{M_\gamma(m_\Gamma)}|s-s_y|\,.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 6.4. *Let S be a line segment intersecting Γ and $y\in\mathbb{R}^2\setminus\Gamma$ an S-admissible point with vector e in the sense of Definition 6.1. Set $\delta_0=\delta_0(\theta,K,M_\gamma(L/2),L)=\min\{\frac{L}{2},\frac{M_\gamma(L/2)}{2K}\frac{1-|\cos\theta|}{2}\}$ and $\tau\geq\frac{1}{2}(1-|\cos\theta|)>0$. Then we have for all $s\in[s_y-\delta_0,s_y+\delta_0]\cap[0,L]$*

$$|y-\gamma(s)|^2\geq\tau\,\left(d_S(y)^2+|\gamma(s_y)-\gamma(s)|^2\right).$$

Proof. Since

$$\tilde{t}(s)=\frac{\tau(s)}{|\tau(s)|}=\frac{\gamma(s)-\gamma(s_y)}{|s-s_y|}\left|\frac{s-s_y}{\gamma(s)-\gamma(s_y)}\right|,$$

the equalities $\lim_{s\to s_y}\tilde{t}(s)=\tau(s_y)\,|\tau(s_y)|^{-1}=t(s_y)$ and $\lim_{s\to s_y}g(s)=\lim_{s\to s_y}\langle e,\tilde{t}(s)\rangle=\langle e,t(s_y)\rangle=\cos\theta$ hold. For δ_0 as in the statement of the lemma we have by (28)

$$|g(s)-\cos\theta|\leq\frac{1}{2}(1-|\cos\theta|)\quad\text{for all}\quad|s-s_y|\leq\delta_0\,,$$

which implies $|g(s)| \leq \frac{1}{2}(1 + |\cos\theta|) < 1$. Now the cosine formula gives us

$$|y - \gamma(s)|^2 = d_S(y)^2 + |\gamma(s_y) - \gamma(s)|^2 - 2d_S(y)|\gamma(s_y) - \gamma(s)|g(s) \,.$$

Set now $\tilde{\tau} := 1 - |g(s)|$. By the definition of δ_0 we have $\tilde{\tau} \geq \tau$, which is positive since $|\cos\theta| < 1$ by Definition 6.1. Therefore the binomial formula implies

$$|y - \gamma(s)|^2 \geq \tilde{\tau}\left(d_S(y)^2 + |\gamma(s_y) - \gamma(s)|^2\right) \,.$$

$\square$

Remark 6.5. In the following we will need a lower bound for $|(\gamma(s) - \gamma(s_y)(s - s_y)^{-1}|$ uniform with respect to $s \in [0, L]$. Such a lower bound does not exists if Γ is a closed curve parameterized by γ in such a way that $s_y = 0$. In this case define a new parameterization $\tilde{\gamma} \colon [0, L] \to \mathbb{R}^2$ by

$$(29) \qquad \tilde{\gamma}(s) := \begin{cases} \gamma(s + \frac{L}{2}) \text{ for } s \in [0, \frac{L}{2}[\\ \gamma(s - \frac{L}{2}) \text{ for } s \in [\frac{L}{2}, L]. \end{cases}$$

It is easily seen that any arc segment $\hat{\Gamma} \subset \Gamma$ of length $L/2$ or less which contains the point $\gamma(0)$ in its interior (relative to the set Γ) cannot contain the point $\gamma(L/2) = \tilde{\gamma}(0)$. This shows that

$$(30) \quad \inf_{\gamma_1} M_{\gamma_1}(m_\Gamma) = \min\{M_\gamma(m_\Gamma), M_{\tilde{\gamma}}(m_\Gamma)\} =: \tilde{M}(m_\Gamma) \quad \text{for } m_\Gamma \leq L/2 \,,$$

where γ_1 runs over all arc length parameterizations of Γ. Thus $M := \tilde{M}(L/2)$ can be used for all parameterizations as an lower bound in (15) for intervals no longer then $L/2$.

Choose now a parameterization γ_1 of the curve such that $s_y = L/2$. This implies that for any $s \in [0, L]$ we have $|s - s_y| \leq L/2$ and thus $|\gamma_1(s) - \gamma_1(s_y)| \geq M_{\gamma_1}(L/2)|s - s_y| \geq M|s - s_y|$.

In the following we will again write γ instead of γ_1, but the subsequent estimates are not affected by this change since we have the universal bound (30). For the same reason we will also suppress the dependence on the parameterization of some constants c_i^Γ which depend on the value $M_\gamma(L/2)$, since it can be bounded independently of the chosen parameterization γ using M.

In the next lemma we will use the abbreviation $\mathcal{A}_{\delta_0} := [s_y - \delta_0, s_y + \delta_0] \cap [0, L]$, $A_{\delta_0} := \gamma(\mathcal{A}_{\delta_0})$.

Lemma 6.6. *(i) Let $y \in B_R \setminus \Gamma$ be an S-admissible point. For M as in Remark 6.5 and $|y - \gamma(s_y)| \leq b_1 := M\delta_0/2$ we have*

$$(31) \qquad \frac{1}{2\pi}\int_{A_{\delta_0}} \frac{d\sigma_\Gamma(x)}{|y - x|} \leq c_1^\Gamma - c_2^\Gamma \log d_S(y) \,,$$

where the constant c_1^Γ depends only on θ, K, M, L and c_2^Γ only on M.
(ii) Moreover we have

$$(32) \qquad \frac{1}{2\pi}\int_{\Gamma \setminus A_{\delta_0}} \frac{d\sigma_\Gamma(x)}{|y - x|} \leq \frac{2}{\pi M}(\log L + |\log \delta_0|) \,.$$

If Γ is not a closed curve, we can replace M by $M_\gamma(L)$ by the monotonicity of the function $M_\gamma(\cdot)$.

Proof. We first prove statement (i). Since $|s_y - s| \leq \delta_0 \leq L/2$ we get $|\gamma(s_y) - \gamma(s)| \geq M|s_y - s|$. Using now Lemma 6.4 we obtain

$$|y - \gamma(s)|^2 \geq \tau \left(d_S(y)^2 + M^2|s_y - s|^2 \right) .$$

Applying the above inequality we can estimate the integral

$$\frac{1}{2\pi} \int_{A_{\delta_0}} \frac{d\sigma_\Gamma(x)}{|y - x|} = \frac{1}{2\pi} \int_{\mathcal{A}_{\delta_0}} \frac{ds}{|y - \gamma(s)|} \leq \frac{1}{\pi M \sqrt{\tau}} \int_0^{\delta_0} \frac{d\tilde{s}}{\sqrt{(d_S(y)/M)^2 + \tilde{s}^2}}$$

$$= \frac{1}{\pi M \sqrt{\tau}} \left(-\log(d_S(y)/M) + \log \left(\delta_0 + \sqrt{(d_S(y)/M)^2 + \delta_0^2} \right) \right)$$

$$\leq \frac{1}{\pi M \sqrt{\tau}} \left(-\log d_S(y) + \log M + \log \left(\left(1 + \frac{\sqrt{5}}{2} \right) \delta_0 \right) \right) ,$$

where in the last estimate we use the inequality $d_S(y) \leq b_1$. Using the explicite formula for δ_0 one can check that the dependence of the constants c_1^Γ, c_2^Γ is precisely as stated in the lemma.

(ii) First, let us note that for all $s \in [0, L] \setminus \mathcal{A}_{\delta_0}$ and $|y - \gamma(s_y)| \leq b_1$ we have

$$|y - \gamma(s)| \geq |\gamma(s_y) - \gamma(s)| - |y - \gamma(s_y)| \geq M|s_y - s| - \frac{M\delta_0}{2} \geq \frac{M|s_y - s|}{2} .$$

Consequently we get the following inequality

$$\frac{1}{2\pi} \int_{\Gamma \setminus A_{\delta_0}} \frac{d\sigma_\Gamma(x)}{|y - \gamma(s)|} \leq \frac{1}{\pi M} \int_{[0,L] \setminus \mathcal{A}_{\delta_0}} \frac{ds}{|s_y - s|}$$

$$\leq \frac{2}{\pi M} \int_{\delta_0}^L \frac{ds}{s} \leq \frac{2}{\pi M} \left(\log L + |\log \delta_0| \right) ,$$

which completes the proof. $\qquad\square$

Corollary 6.7. *Let $y \in \mathbb{R}^2 \setminus \Gamma$ be an S-admissible point in the sense of Definition 6.1. Assume that $d_S(y) \leq \min\{1, b_1\}$ and $|\cos\theta| \leq \sqrt{3}/2$. Then there exist constants $c_3^\Gamma = c_3^\Gamma(M, K)$ and $c_4^\Gamma = c_4^\Gamma(M)$ such that*

$$(33) \qquad \frac{1}{2\pi} \int_\Gamma \frac{d\sigma_\Gamma(x)}{|y - x|} \leq c_3^\Gamma + c_4^\Gamma \left(\log L + |\log d_S(y)| \right) .$$

6.2. **Upper bound on the gradient.** In the sequel we will be interested in the behavior of the gradient of an eigenfunction in some neighbourhood of Γ. To this end we consider a line segment S intersecting Γ and an S-admissible point $y \in \mathbb{R}^2 \setminus \Gamma$ as in Definition 6.1. We assume $b \leq \min\{1, b_1\}$ and $|\cos\theta| \leq \sqrt{3}/2$.

Proposition 6.8. *Let ψ be an eigenfunction of $H_{\alpha\sigma_\Gamma}$ corresponding to the negative eigenvalue $E = -\kappa^2$. Let S be a line segment intersecting Γ and $y \in \mathbb{R}^2 \setminus \Gamma$ an S-admissible point. Then we have with the notation from Definition 6.1 and Lemma 6.6 (ii): if $b \leq \min\{1, b_1\}$, then*

$$(34) \qquad |\nabla\psi(y)| \leq \left[\mathcal{S}_{\kappa,\Gamma} + \alpha c_4^\Gamma |\log d_S(y)| \right] \|\psi\|_\infty ,$$

where

$$\mathcal{S}_{\kappa,\Gamma} := 2\kappa^2(R + 1) + C_5(\kappa R)^{1/2} e^{\kappa R} + \alpha \left(c_3^\Gamma + c_4^\Gamma \log(\max\{1, L\}) \right) ,$$

c_3^Γ *depends only on* M, K *and* c_4^Γ *depends only on* M.

Remark 6.9. For the reader's convenience let us write down the estimate (34) in the case that S is perpendicular on Γ, $\theta = \pi/2$ and $d_S(y) = \text{dist}(y, \Gamma)$. Then we have

$$|\nabla \psi(y)| \leq \left[\mathcal{S}_{\kappa, \Gamma} + \alpha c_4^\Gamma |\log \text{dist}(y, \Gamma)| \right] \|\psi\|_\infty \,.$$

Proof. The fundamental solution of the Laplace equation in two dimensions is given by

$$\mathcal{G}(x, y) = \mathcal{G}(|x - y|) = \frac{\log |x - y|}{2\pi} \,.$$

Let Ω be a domain in $\mathbb{R}^2$ and $u \in C^2(\bar{\Omega})$. Then the Green's representation formula

$$u(y) = \int_{\partial \Omega} \left(u(x) \frac{\partial \mathcal{G}(x, y)}{\partial \nu_x} - \mathcal{G}(x, y) \frac{\partial u(x)}{\partial \nu_x} \right) \mathrm{d}\sigma_\Gamma(x) + \int_\Omega \mathcal{G}(x, y) \Delta u(x) \mathrm{d}x \text{ for } y \in \Omega$$

holds. Here $\mathrm{d}s(x)$ denotes the surface element and $\frac{\partial}{\partial \nu_x}$ the outer normal derivative at x. Let Ω be a bounded domain with positive distance to the curve Γ. Consequently, for the eigenfunction ψ satisfying $\Delta \psi = \kappa^2 \psi$ on Ω the Green's representation formula implies

$$\psi(y) = \kappa^2 \int_\Omega \mathcal{G}(x, y) \psi(x) \mathrm{d}x + \int_{\partial \Omega} \left(\psi(x) \frac{\partial \mathcal{G}(x, y)}{\partial \nu_x} - \mathcal{G}(x, y) \frac{\partial \psi(x)}{\partial \nu_x} \right) \mathrm{d}\sigma_\Gamma(x) \,.$$

Now choose a monotone increasing sequence $\Omega_n, n \in \mathbb{N}$ of domains as above such that $\bigcup_n \Omega_n = \Gamma^c$, where Γ^c stands for the complement of Γ. Then we have for any $y \in \Gamma^c$

$$(35) \quad \psi(y) =$$

$$= \lim_{n \to \infty} \left(\kappa^2 \int_{\Omega_n} \mathcal{G}(x, y) \psi(x) \mathrm{d}x + \int_{\partial \Omega_n} (\psi(x) \frac{\partial \mathcal{G}(x, y)}{\partial \nu_x} - \mathcal{G}(x, y) \frac{\partial \psi(x)}{\partial \nu_x}) \mathrm{d}\sigma_\Gamma(x) \right)$$

$$= \kappa^2 \int_{\mathbb{R}^2} \mathcal{G}(x, y) \psi(x) \mathrm{d}x - \alpha \int_\Gamma \mathcal{G}(x, y) \psi(x) \mathrm{d}\sigma_\Gamma(x) \,.$$

Here we have used several facts. Firstly, given $y \in \Gamma^c$ the functions $\partial_{\nu_x} \mathcal{G}(\cdot, y)$, $\mathcal{G}(\cdot, y)$ and $\psi(\cdot)$ are continuous. Secondly, the part of the boundary $\partial \Omega_n$ which tends to infinity has a vanishing contribution to the integral in the limit $n \to \infty$. The remainder of the boundary $\partial \Omega_n$ tends for $n \to \infty$ to two copies of Γ with opposite orientation, i.e. opposite outward normal derivative, and formula (5) holds. In view of the exponential decay established in Lemma 5.7 the first term in the last line of (35) is finite. Now, taking the gradient of ψ and using the chain rules we obtain

$$(36) \qquad \nabla \psi(y) = \kappa^2 \int_{\mathbb{R}^2} \nabla_y \mathcal{G}(x, y) \psi(x) \mathrm{d}x - \alpha \int_\Gamma \nabla_y \mathcal{G}(x, y) \psi(x) \mathrm{d}\sigma_\Gamma(x) \,.$$

To deal with the singularity $\nabla_y \mathcal{G}(x, y) = \frac{1}{2\pi |x-y|}$ we split the integral over $\mathbb{R}^2$ in two regions, the ball $B_{R+1} = B_{R+1}(x_0)$ and its complement B_{R+1}^c. Employing

Lemma 5.7 we get

$$(37) \quad |\nabla \psi(y)| \le \kappa^2 \left(\int_{B_{R+1}} \frac{1}{2\pi|x-y|} \mathrm{d}x + \int_{B_{R+1}^c} \frac{1}{2\pi|x-y|} \phi(x)\mathrm{d}x \right) \|\psi\|_\infty$$
$$+ \int_\Gamma \frac{\alpha}{2\pi|x-y|} \mathrm{d}\sigma_\Gamma(x)\|\psi\|_\infty \,.$$

The first integral can be estimated by

$$\int_{B_{R+1}} \frac{1}{2\pi|x-y|} \mathrm{d}x \le \int_{B_{2(R+1)}(y)} \frac{1}{2\pi|x-y|} \mathrm{d}x = 2R+2 \,.$$

Using again Lemma 5.7 and the fact that $R \ge 1$ we can estimate the second integral

$$\int_{B_{R+1}^c} \frac{1}{2\pi|x-y|} \sqrt{\frac{(R+1)}{|x|}} \mathrm{e}^{-\kappa(|x|-(R+1))}\mathrm{d}x \le C_5 \kappa^{-3/2}\sqrt{R}\mathrm{e}^{\kappa R} \,,$$

where $C_5 = 2\int_0^\infty \sqrt{x}\mathrm{e}^{-x}\mathrm{d}x$. By Corollary 6.7 there exist constants c_3^Γ and c_4^Γ such that the estimate

$$(38) \quad \int_\Gamma \frac{\alpha}{2\pi|x-y|} \mathrm{d}\sigma_\Gamma(x) \le \alpha \left(c_3^\Gamma + c_4^\Gamma(\log L + |\log d_S(y)|)\right)$$

is valid. Combining the above estimates we obtain the claim. $\qquad\square$

6.3. **Lower bound on the gradient.** In Proposition 5.4 we have localized the zeros, minima and maxima of eigenfunctions of $H_{\alpha\sigma_\Gamma}$. Choose two points $v_0, v_1 \in \Gamma$ such that

$$\psi_1(v_0) = \inf_{x\in\mathbb{R}^2} \psi_1(x) < 0 \quad \text{and} \quad \psi_1(v_1) = \sup_{x\in\mathbb{R}^2} \psi_1(x) > 0 \,,$$

Taking appropriate scalar multiples we may assume the normalization $\psi_1(v_1) = \|\psi_1\|_\infty = 1$ and $\|\psi_0\|_\infty = 1$. This means that $f(v_0) < 0$ and moreover

$$f(v_1) = \frac{\|\psi_1\|_\infty}{\psi_0(v_1)} \ge \frac{\|\psi_1\|_\infty}{\|\psi_0\|_\infty} = 1 \,.$$

The following lemma states a lower bound on $\int_{B_R} |\nabla f(x)|\mathrm{d}x$.

Proposition 6.10. *There exists a positive constant β_0 such that*

$$\int_{B_R} |\nabla f(y)|\mathrm{d}y \ge \frac{(\kappa_0\rho)^2\rho}{(\alpha c_5^\Gamma + 1)^6(\kappa_0^2+1)^4\zeta(\kappa_0)} \mathrm{e}^{-\beta_0\rho} \,,$$

where

$$\zeta(\kappa_0) := \begin{cases} 1 & \text{for} \quad \kappa_0 \ge \frac{1}{2} \\ -\frac{\log\kappa_0}{\log 2} & \text{for} \quad \kappa_0 < \frac{1}{2} \end{cases} \,, \qquad \rho = \kappa_0 R \,,$$

$c_5^\Gamma = \max\{2c_4^\Gamma, c_3^\Gamma + c_4^\Gamma \log(\max\{1, L\})\}$ *and c_3^Γ and c_4^Γ are taken from Corollary 6.7.*

Proof. As was already mentioned, in order to estimate ∇f we will relay on Proposition 6.8 which gives upper bounds on $\nabla \psi_0$ and $\nabla \psi_1$. Recall that $E_0 = -\kappa_0^2$, $E_1 = -\kappa_1^2$ are eigenvalues corresponding to ψ_0, respectively ψ_1. First let us note that since $\mathcal{S}_{\kappa,\Gamma}$ is an increasing function of κ and $\kappa_0 > \kappa_1$ the inequality (34) implies

$$(39) \qquad |\nabla \psi_i(y)| \le \left[\mathcal{S} + \alpha c_4^\Gamma |\log d_S(y)| \right] \|\psi_i\|_\infty, \quad \text{for } i = 0, 1,$$

where we abbreviate

$$\mathcal{S} := \mathcal{S}_{\kappa_0,\Gamma} = 2\kappa^2(R+1) + C_5 \rho^{1/2} e^\rho + \alpha \left(c_3^\Gamma + c_4^\Gamma \log(\max\{1, L\}) \right).$$

To make use of the inequality (27) we need some estimates for ψ_0^{-1} which, in fact, can be directly derived from Lemma 5.1, i.e. we have

$$(40) \qquad \sup_{y \in B_R} \psi_0^{-1}(y) \le C_1^{-1} \kappa_0^{-1} (1 + \sqrt{2\rho}) e^{2\rho} \|\psi_0\|^{-1}.$$

Combining this with the statement of Lemma 5.6 and using our normalization $\|\psi_0\|_\infty = \|\psi_1\|_\infty = 1$ we get

$$\sup_{y \in B_R} \psi_0^{-1}(y) \le TD,$$

where

$$T := C_1^{-1} \frac{(c_5^\Gamma \alpha + 1)^2 (\kappa_0^2 + 1)}{\kappa_0 \rho}, \quad D := (1 + \sqrt{2\rho}) e^{(\eta_0 + 2)\rho},$$

and $c_5^\Gamma = \max\{2c_4^\Gamma, c_3^\Gamma + c_4^\Gamma(\log \max\{1, L\})\}$. Applying the above inequalities to (27) and using again our normalization we have

$$(41) \qquad |\nabla f(y)| \le (\mathcal{S} + \alpha c_4^\Gamma |\log d_S(y)|) TD(TD + 1).$$

Now choose two parallel line segments S_0 and S_1, which are not tangential to Γ, of length $2b$ and such that $S_i \cap \Gamma = v_i$ is the midpoint of S_i for $i = 0, 1$. Thus any $y \in S_i$ is a S_i-admissible point and the expression $d_{S_i}(y)$ is well defined.

We can suppose without loss of generality, that the line passing through v_0 and v_1 is the y_1-coordinate axis, $v_0 = (0, 0)$ and $\mathcal{L} = \text{dist}(v_0, v_1)$, in other words $v_1 = (\mathcal{L}, 0)$. Furthermore, denote by $\theta_\mathcal{L}$ the angle between the y_1-axis and the segment S_0. Let us note that it is always possible to choose the segments S_i in such a way that the smallest angle formed with the tangential vectors of Γ at the points v_i are at least $\pi/6$ and simultanously $\theta_\mathcal{L}$ is also at least $\pi/6$.

Our first task is to estimate the behavior of f near v_0. With the parameterization assumed above any $y \in S_0$ thus has coordinates $y = (y_1, y_2)$ where $y_2 = y_1 \tan \theta_\mathcal{L} = d_{S_0}(y) \sin \theta_\mathcal{L}$. For such y we obtain using the fundamental theorem of calculus and inequality (41)

$$(42) \quad |f(y) - f(v_0)| = \left| \int_0^{d_{S_0}(y)} \nabla f(\tau \cos \theta_\mathcal{L}, \tau \sin \theta_\mathcal{L}) \cdot \begin{pmatrix} \cos \theta_\mathcal{L} \\ \sin \theta_\mathcal{L} \end{pmatrix} d\tau \right|$$

$$\le \int_0^{d_{S_0}(y)} |\nabla f(\tau \cos \theta_\mathcal{L}, \tau \sin \theta_\mathcal{L})| d\tau \le \xi(b, \kappa_0, \rho),$$

where $\xi(b, \kappa_0, \rho) := b(\mathcal{S} + \alpha c_4^\Gamma(|\log b| + 1))TD(TD+1)$. Choosing b small enough we can make $\xi(b, \kappa_0, \rho)$ arbitrarily small. More precisely, by Lemma A.1 and Corollary A.2 we know that there exists a positive constant β_0 such that

$$(43) \qquad \xi(b, \kappa_0, \rho) \leq \frac{1}{4} \quad \text{for} \quad b = \frac{(\kappa_0\rho)^2\rho}{(\alpha c_5^\Gamma + 1)^6(\kappa_0^2 + 1)^4\zeta(\kappa_0)}e^{-\beta_0\rho},$$

where the function ζ is defined in the statement of the proposition.

Finally, using (42) we get

$$f(y) \leq \frac{1}{4} \quad \text{on} \quad S_0.$$

Similarly, for S_1, respectively b small enough we obtain $f(y) \geq 3/4$ for all $y \in S_1$. Using these inequalities we estimate the integral of the gradient of f on the strip $\mathcal{T} := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = \tau\cos\theta_\mathcal{L} + l, y_2 = \tau\sin\theta_\mathcal{L}, \tau \in [-b, b], l \in [0, \mathcal{L}]\}$

$$\int_\mathcal{T} |\nabla f(y)| \mathrm{d}y \geq \int_\mathcal{T} |\partial_{y_1} f(y)| \mathrm{d}y$$

$$\geq |\sin\theta_\mathcal{L} \int_{-b}^{b} (f(\tau\cos\theta_\mathcal{L} + \mathcal{L}, \tau\sin\theta_\mathcal{L}) - f(\tau\cos\theta_\mathcal{L}, \tau\sin\theta_\mathcal{L}))\, \mathrm{d}\tau| \geq |\sin\theta_\mathcal{L}|b,$$

where we again employ the fundamental theorem of calculus. Using the fact that $|\sin\theta_\mathcal{L}| \geq 1/2$ we get

$$(44) \qquad \int_{B_R} |\nabla f(y)| \mathrm{d}y \geq \frac{(\kappa_0\rho)^2\rho}{2(\alpha c_5^\Gamma + 1)^6(\kappa_0^2 + 1)^4\zeta(\kappa_0)}e^{-\beta_0\rho}.$$

$\square$

Proof of Theorem 4.3. Inserting the inequalities given in Lemmata 5.1, 5.5, 5.6 and 6.10 into the estimate (16) yields

$$E_0 - E_1 \geq \kappa_1^2\, \mu_{\Gamma,\alpha}(\rho, \kappa_0)\, e^{-(8+2\eta_0+2\beta_0)\rho},$$

where

$$(45) \qquad \mu_{\Gamma,\alpha}(\rho, \kappa_0) := C_7 \frac{(\kappa_0\rho)^8}{(L\alpha)^2(1 + \sqrt{2\rho})^4(\kappa_0^2 + 1)^{10}(c_5^\Gamma\alpha + 1)^{16}\xi(\kappa_0)^2},$$

where C_7 is an absolute constant, $c_5^\Gamma := \max\{2c_4^\Gamma, c_3^\Gamma + c_4^\Gamma\log(\max\{1, L\})\}$ and c_3^Γ and c_4^Γ are taken from Corollary 6.7. This proves the claim. $\square$

7. Closing remarks and open questions

Dependence on the second eigenvalue. Apart from the parameter κ_0 corresponding to the ground state energy, κ_1 is also involved in the lower bound for the first spectral gap

$$(46) \qquad E_1 - E_0 \geq \kappa_1^2\mu_{\Gamma,\alpha}(\rho, \kappa_0)e^{-C_0\rho}, \quad \text{with } \rho := \kappa_0 R.$$

In fact, the appearance of κ_1 here is natural since we assumed explicitly the existence of the second, isolated eigenvalue. For effective estimates of the spectral gap we would need lower bounds on κ_1^2. The following observation is helpful in many situations. Suppose that for a Hamiltonian $H_{\alpha\sigma_\Gamma}$ the value $C_{\alpha,\gamma}$ is a lower bound for κ_1^2. Then $C_{\alpha,\gamma}$ is also the corresponding lower bound for all

$H_{\tilde{\alpha},\sigma_{\tilde{\Gamma}}}$, where $\tilde{\alpha} \geq \alpha$ and $\tilde{\Gamma} \supset \Gamma$. The above statement is a direct consequence of the form sum representation of the Hamiltonian and the min-max theorem. In general the question whether a second eigenvalue exists is quite involving and will be discussed elsewhere.

Strong coupling constant case. There is one case where the function counting the number of eigenvalues is known. This is the situation where the singular interaction is very strong, more precisely $\alpha \to \infty$, cf. Remark 2.5. In particular, if Γ consists of one closed curve, the asymptotic behaviour of the eigenvalues implies the following estimate on the first spectral gap

$$E_1 - E_0 = \mu_1 - \mu_0 + \mathcal{O}\left(\frac{\log \alpha}{\alpha}\right) \quad \text{for} \quad \alpha \to \infty \,,$$

where μ_0, μ_1 are the two lowest eigenvalues of an appropriate comparison operator. This operator is defined as the negative Laplacian with periodic boundary conditions on $[0, L]$ plus a regular potential. Therefore, the estimate on the first spectral gap for $H_{\alpha \sigma_\Gamma}$ can be expressed by the gap for a Schrödinger operator with an ordinary potential. Denote by ϕ_0 the ground state of the comparison operator and

$$a := \left(\frac{\max_{x \in [0,L]} \phi_0(x)}{\min_{x \in [0,L]} \phi_0(x)}\right)^2 \,.$$

Theorem 1.4 in [KS87] implies

$$a^{-1}\left(\frac{2\pi}{L}\right)^2 \leq \mu_1 - \mu_0 \leq a\left(\frac{2\pi}{L}\right)^2 \,.$$

Note that the quotient a is independent of scaling by L. Since we are considering a single closed curve, $L/2 < R$ and thus $\mu_1 - \mu_0 \geq a^{-1}\left(\frac{\pi}{R}\right)^2$. Hence in the considered situation the lowest spectral gap decreases only polynomially in $1/R$, rather than exponentially as estimated in Theorem 4.3.

Singular perturbation on an infinite curve. It is very natural to pose the question whether the results obtained in the present paper can be extended to Hamiltonians with a singular potential supported on an infinite curve. If the curve is asymptotically straight in an appropriate sense then the essential spectrum is the same as in the case of a straight line. If the curve is non-straight, the existence of at least one isolated eigenvalue was shown in [EI01] and the function counting the number of eigenvalues for the strong coupling constant case was derived in [EY02].

The estimate obtained in the main Theorem 3.1 should hold for infinite curves as well. However, to obtain this result, one has to analyse the behavior of certain eigenfunctions and prove their exponential decay, a problem which is not encountered in the case of a finite curve. We postpone this question to a subsequent publication.

Related estimates about eigenvalue splittings for certain infinite quantum waveguides have been derived in [BE04].

Appendix A. Proof of inequality (43)

In this appendix we complete a technical estimate which is needed in the proof of Proposition 6.10. More precisely, we prove here the inequality (43) on the function $\xi(b, \kappa_0, \rho)$.

For the reader's convenience let us recall some notation introduced in the proof of Proposition 6.10. Define

$$\xi(b, \kappa_0, \rho) := b(\mathcal{S} + \alpha c_4^\Gamma(|\log b| + 1))TD(TD + 1)\,,$$

where

$$T := C_1^{-1} \frac{(c_5^\Gamma \alpha + 1)^2(\kappa_0^2 + 1)}{\kappa_0 \rho}\,,\quad D := (1 + \sqrt{2\rho})\mathrm{e}^{(\eta_0 + 2)\rho}$$

$\rho = \kappa_0 R$ and $\mathcal{S} := \kappa_0^2(2R + 2) + C_5\rho^{1/2}\mathrm{e}^\rho + \alpha(c_3^\Gamma + c_4^\Gamma \log(\max\{1, L\}))$. Let us introduce a one parameter family of functions defined by

$$b_\beta(\rho, \kappa_0) = \frac{(\kappa_0\rho)^2\rho}{(\alpha c_5^\Gamma + 1)^6(\kappa_0^2 + 1)^4\zeta(\kappa_0)}\mathrm{e}^{-\beta\rho}\,,\quad \zeta(\kappa_0) := \left\{ \begin{array}{ll} 1 & \text{for } \kappa_0 \geq \frac{1}{2} \\ -\frac{\log \kappa_0}{\log 2} & \text{for } \kappa_0 < \frac{1}{2} \end{array} \right.\,.$$

Lemma A.1. *There exists an absolute constant C_9 such that for $\beta > 2\eta_0 + 5$ we have*

$$\xi(b_\beta, \kappa_0, \rho) \leq \frac{C_9}{\beta - 2\eta_0 - 5}\,,$$

uniformly in ρ and κ_0.

Proof. In the following proof we will use the fact that the terms in the enumerator of T as well as $\zeta(\kappa_0)$ are larger or equal 1. Using the formula for T we obtain

$$\begin{aligned}
\xi(b, \kappa_0, \rho) &\leq bT^2(\mathcal{S} + \alpha c_4^\Gamma(|\log b| + 1))D(D + \kappa_0\rho) \\
&\leq bT^2(\mathcal{S} + 2\alpha c_4^\Gamma|\log b|)D(D + \kappa_0\rho)\,,
\end{aligned} \tag{47}$$

where in the last inequality we assume that $b < e^{-1}$. Furthermore applying the explicit form for $\mathcal{S}$ we get by a straightforward calculation that the right hand side of (47) is bounded from above by

$$\begin{aligned}
(48)\quad & bT^2(\kappa_0^2 + 1)^2(\alpha c_5^\Gamma + 1)^2\zeta(\kappa_0) \\
& \left(3 + 2\rho + C_5\rho^{1/2}\mathrm{e}^\rho + \frac{|\log b|}{(\kappa_0^2 + 1)(\alpha c_5^\Gamma + 1)\zeta(\kappa_0)}\right)D(D + \rho)\,.
\end{aligned}$$

Employing the definition of b_β and inserting this in the expression (48) we get that (48) is smaller or equal to

$$(49)\quad C_1^{-2}\rho\mathrm{e}^{-\beta\rho}\left(3 + 2\rho + C_5\rho^{1/2}\mathrm{e}^\rho + \frac{|\log b_\beta|}{(\kappa_0^2 + 1)(\alpha c_5^\Gamma + 1)\zeta(\kappa_0)}\right)D(D + \rho)\,.$$

Let us estimate now the logarithmic term in (49). Using again the formula for b_β and properties of the logarithmic function we get

$$\frac{|\log b_\beta|}{(\kappa_0^2 + 1)(\alpha c_5^\Gamma + 1)\zeta(\kappa_0)}$$

$$\leq \frac{|\log \zeta(\kappa_0)|}{\zeta(\kappa_0)} + 2\frac{|\log \kappa_0|}{\zeta(\kappa_0)(\kappa_0^2 + 1)} + 3|\log \rho| + \beta\rho + \frac{\log\left((\kappa_0^2 + 1)^4(\alpha c_5^\Gamma + 1)^6\right)}{(\kappa_0^2 + 1)(\alpha c_5^\Gamma + 1)}$$

$$\leq C_8 + 3|\log \rho| + \beta\rho\,.$$

In the last inequality we estimated the first, second and last term by a constant C_8. Inserting this to (49) we obtain that (49) is bounded from above by

$$(50) \qquad C_1^2 \rho \mathrm{e}^{-\beta\rho}(C_8 + 2\rho + C_5\rho^{1/2}\mathrm{e}^\rho + 3|\log \rho| + \beta\rho)D(D + \rho)\,.$$

Employing now the explicit form for D we estimate

$$\text{expression } (49) \leq \Xi_\beta(\rho)\,,$$

where

$$\Xi_\beta(\rho) := C_1^2 \rho \mathrm{e}^{(-\beta + 2\eta_0 + 5)\rho}(C_8 + 2\rho + C_5\rho^{1/2} + 3|\log \rho| + \beta\rho)(1 + \sqrt{2\rho})(1 + \sqrt{2\rho} + \rho)\,.$$

Now we estimate the maximum of the functions $\rho \mapsto \Xi_\beta(\rho)$ and conclude that there exists a positive constant C_9 such that

$$\Xi_\beta(\rho) \leq \frac{C_9}{\beta - 2\eta_0 - 5}\,,$$

for $\beta > 2\eta_0 + 5$. This proves the desired claim. $\qquad \square$

Corollary A.2. *There exists constant β_0 such that for any $\beta \geq \beta_0$ we have*

$$\xi(b_\beta, \kappa_0, \rho) \leq \frac{1}{4}\,,$$

for all values of κ_0, ρ.

Of course, β_0 should be chosen in such a way that $b_{\beta_0} \leq b_1$, where b_1 is defined in Lemma 6.6. This is always possible because

$$b_\beta(\rho, \kappa_0) \leq \rho^3 \mathrm{e}^{-\beta\rho} \leq 9\mathrm{e}^{-3}\beta^{-3}\,.$$

References

[Agm85]   Sh. Agmon. Bounds on exponential decay of eigenfunctions of Schrödinger operators. In *Schrödinger operators (Como, 1984)*, volume 1159 of *Lecture Notes in Math.*, pages 1–38. Springer, Berlin, 1985.

[AS72]   M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, New York, 1972).

[BE04]   D. Borisov and P. Exner. Exponential splitting of bound states in a waveguide with a pair of distant windows. *J. Phys. A*, 37(10):3411–3428, 2004.

[BEKŠ94] J. F. Brasche, P. Exner, Yu. A. Kuperin, and P. Šeba. Schrödinger operators with singular interactions. *J. Math. Anal. Appl.*, 184(1):112–139, 1994.

[DS84]   E. B. Davies and B. Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.*, 59(2):335–395, 1984.

[EI01]   P. Exner and T. Ichinose. Geometrically induced spectrum in curved leaky wires. *J. Phys. A*, 34(7):1439–1450, 2001.

[EK02]   P. Exner and S. Kondej. Curvature-induced bound states for a δ interaction supported by a curve in $\mathbb{R}^3$. *Ann. Henri Poincaré*, 3(5):967–981, 2002.

[EK03]   P. Exner and S. Kondej. Bound states due to a strong δ interaction supported by a curved surface. *J. Phys. A*, 36(2):443–457, 2003.

[Exn03] P. Exner. Spectral properties of Schrödinger operators with a strongly attractive δ interaction supported by a surface. In *Waves in periodic and random media (South Hadley, MA, 2002)*, volume 339 of *Contemp. Math.*, pages 25–36. Amer. Math. Soc., Providence, RI, 2003.

[Exn05] P. Exner. An isoperimetric problem for leaky loops and related mean-chord inequalities. *J. Math. Phys.*, 46(6):062105, 2005. http://arxiv.org/abs/math-ph/0501066.

[EY02]   P. Exner and K. Yoshitomi. Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop. *J. Geom. Phys.*, 41(4):344–358, 2002.

[EY03]   P. Exner and K. Yoshitomi. Eigenvalue asymptotics for the Schrödinger operator with a δ-interaction on a punctured surface. *Lett. Math. Phys.*, 65(1):19–26, 2003.

[EY04]   P. Exner and K. Yoshitomi. Erratum: "Eigenvalue asymptotics for the Schrödinger operator with a δ-interaction on a punctured surface" [Lett. Math. Phys. **65** (2003), no. 1, 19–26]. *Lett. Math. Phys.*, 67(1):81–82, 2004.

[GT83]   D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Springer, Berlin, 1983.

[Har80] E. M. Harrell. Double wells. *Comm. Math. Phys.*, 75(3):239–261, 1980.

[KS85]   W. Kirsch and B. Simon. Universal lower bounds of eigenvalue splittings for one dimensional Schrödinger operators. *Commun. Math. Phys.*, 97:453–460, 1985.

[KS87]   W. Kirsch and B. Simon. Comparison theorems for the gap of Schrödinger operators. *J. Funct. Anal.*, 75:396–410, 1987.

[Pos01] A. Posilicano. A Kreĭn-like formula for singular perturbations of self-adjoint operators and applications. *J. Funct. Anal.*, 183(1):109–147, 2001.

[Pos04] A. Posilicano. Boundary triples and Weyl functions for singular perturbations of self-adjoint operators. *Methods Funct. Anal. Topology*, 10(2):57–63, 2004.

(S. Kondej) Institute of Physics, University of Zielona Gora, ul. Prof. Z. Szafrana 4a, Zielona Gora, Poland
  E-mail address: skondej@proton.if.uz.zgora.pl

(I. Veselić) Fakultät für Mathematik, 09107 TU Chemnitz & Emmy-Noether-Programme of the DFG, Germany
  URL: www.tu-chemnitz.de/mathematik/schroedinger/members.php