Distant perturbation asymptotics in window-coupled waveguides. I. The non-threshold case

D. Borisov[a,b] and P. Exner[a,c]

a) Nuclear Physics Institute, Academy of Sciences, 25068 Řež near Prague, Czechia
b) Bashkir State Pedagogical University, October Revolution St. 3a, 450000 Ufa, Russia
c) Doppler Institute for Mathematical Physics and Applied Mathematics, Czech Technical University, Břehová 7, 11519 Prague, Czechia
borisovdi@yandex.ru, exner@ujf.cas.cz

We consider a pair of adjacent quantum waveguides, in general of different widths, coupled laterally by a pair of windows in the common boundary, not necessarily of the same length, at a fixed distance. The Hamiltonian is the respective Dirichlet Laplacian. We analyze the asymptotic behavior of the discrete spectrum as the window distance tends to infinity for the generic case, i.e. for eigenvalues of the corresponding one-window problems separated from the threshold.

1 Introduction

Quantum mechanics exhibits various effects which defy our intuition based on "classical" experience. A nice class of examples are bound states in hard-wall tubes induced solely by their geometric properties such as bends, protrusions, or "windows". Such systems are interesting not only *per se* but also from the practical point of view as models of various nanophysical devices, and in a reasonable approximation also of flat electromagnetic waveguides.

Among numerous questions such models pose an important one concerns behavior of the spectra in case of two distant perturbations. One can think of it as of an analogue of the exponential spectral shift for a pair of distant potential wells, despite the fact that the usual methods of the Schrödinger operator theory do not work here. The aim of the present paper is to study this problem in a model example of a pair of laterally coupled waveguides, or adjacent straight hard-wall strip in the plane, coupled by a pair of "windows" in the common boundary – we refer to [1], [2], [3] for a bibliography concerning such models.

In our recent paper [2] we dealt with the symmetric situation where the widths d_1, d_2 of the two channels were the same and so were the window widths a_1, a_2. The technique used in these papers employed substantially the fact that the problem can be decomposed into parts with a definite parity, which allows one to study a single-window problem with a perturbation which consists of an additional Dirichlet or Neumann boundary condition at a segment far from the window.

The approach based on symmetry works no longer if $a_1 \neq a_2$. The man aim of the present work is to demonstrate a different technique, suitable for the general case, which reduces the question to analysis of a boundary perturbation at the distant window. This technique follows the main ideas of [4], where the Dirichlet Laplacian in an n-dimensional tube with a pair of distant perturbations described by two arbitrary operators was studied. It was assumed in [4] that these operators are defined on functions from W_2^2 vanishing at the boundary, and this assumption was employed substantially. This is obviously not true in the problem we study, since the windows enlarge the domain of the Laplacian beyond the Sobolev space W_2^2. At the same time, the general approach of [4] works in our case with the appropriate modifications. Moreover, since we restrict ourselves to the two-dimensional case and specify the nature of the distant perturbations, we are able to obtain a more detailed result in comparison with the general case in [4].

In order not to make this study too technical we concentrate in this paper at the generic case when the "unperturbed" energy is an *isolated* eigenvalue of the one-window problem, leaving the computationally involved discussion of threshold resonances to a sequel. The problem will be properly formulated and the results stated in the next section; the rest of the paper is devoted to the proofs.

2 Statement of the problem and the results

Let $x = (x_1, x_2)$ be Cartesian coordinates in the plane, $\Pi^+ := \{x : 0 < x_2 < \pi\}$ and $\Pi^- := \{x : -d < x_2 < 0\}$. With the natural scaling properties in mind we may suppose without loss of generality that $d \leqslant \pi$. By $\gamma_\pm$ we denote two intervals $\gamma_\pm := \{x : |x_1 \mp l| < a_\pm,\ x_2 = 0\}$, from now on referred to as the *windows*. The numbers $a_\pm$ are assumed to be fixed throughout the paper while the distance $2l$ between the windows will be changing playing the role of a large parameter.

We set $\Pi := \Pi^+ \cup \Pi^- \cup \gamma_+ \cup \gamma_-$ (cf. Figure 1); the Hilbert space of our problem is $L_2(\Pi)$. We will employ the symbol H to denote Friedrichs extension of the negative Laplacian from the set $C_0^\infty(\Pi)$. We will use the symbols $\sigma_{\mathrm{ess}}(\cdot)$ and $\sigma_{\mathrm{disc}}(\cdot)$ to indicate the essential and discrete spectrum, respectively. As we have indicated in the introduction, this work is devoted to the study of the asymptotic behavior of isolated eigenvalues of H as $l \to +\infty$. In order to formulate the main results we have to introduce first some more notations.

Let Ω be an open set in $\mathbb{R}^2$ and $\gamma \subset \overline{\Omega}$. Throughout the paper $W_2^1(\Omega, \gamma)$

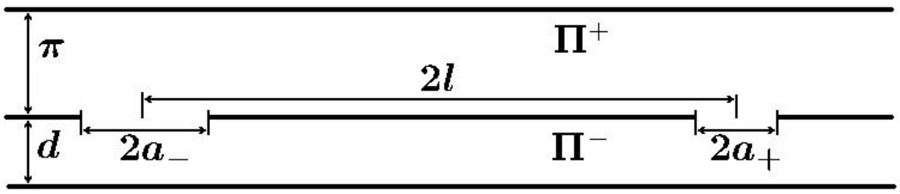

Figure 1: Window-coupled waveguides

will indicate the completion of the set of functions from $C^\infty(\overline{\Omega})$ having a compact support and vanishing in the vicinity of the set γ, taken with respect to the norm of the Sobolev space $W_2^1(\Omega)$.

We denote $\gamma_a := \{x : |x_1| < a,\ x_2 = 0\}$ so that $\Pi_a := \Pi^+ \cup \Pi^- \cup \gamma_a$ is the double waveguide with a single window centered at $x_1 = 0$, and $\Gamma_a := \partial\Pi_a$. Furthermore, we introduce the corresponding cut-off sets $\Pi_a^b := \Pi_a \cap \{x : |x_1| < b\}$ and $\Gamma_a^b := \Gamma_a \cap \{x : |x_1| < b\}$. Consider the negative Laplacian in $L_2(\Pi_a)$ and call $H(a)$ its Friedrichs extension in $L_2(\Pi_a)$ from the set $C_0^\infty(\Pi_a)$ on which it is symmetric; by $\lambda_m(a)$, $m = 1, 2, \ldots$, we denote the isolated eigenvalues of this operator arranged in the ascending order with the multiplicity taken into account.

The following results were demonstrated in [1], [3].

Proposition 2.1. *For any $a > 0$ the essential spectrum of $H(a)$ equals $[1, +\infty)$ while $\sigma_{\mathrm{disc}}(H(a))$ is non-empty consisting of finitely many simple eigenvalues. The eigenfunction associated with an eigenvalue $\lambda_m(a)$ has a definite parity: it is even or odd with respect to x_1 if m is odd or even, respectively. In the particular case $d = \pi$ the eigenfunctions are even in the variable x_2.*

The eigenfunctions associated with the eigenvalues $\lambda_n(a)$ will be denoted as $\psi_n(\cdot, a)$ and assumed to be normalized, i.e. to be unit vectors in $L_2(\Pi_a)$. It is easy to check that $\psi_n(\cdot, a) \in C^\infty(\Pi_a)$. We put $\sigma_* := \sigma_{\mathrm{disc}}\, H(a_+) \cup \sigma_{\mathrm{disc}}\, H(a_-)$; an element $\lambda_* \in \sigma_*$ of this set will be called *simple* if λ_* belongs to one of the sets $\sigma_{\mathrm{disc}}(H(a_\pm))$ only and *double* otherwise. Furthermore, we set $\boldsymbol{a} := (a_+, a_-)$.

With these preliminaries we can formulate the first main result of this paper.

Theorem 2.1. *For any $l > 0$, $a_\pm > 0$ the operator H has the essential spectrum equal to $[1, +\infty)$ and finitely many isolated eigenvalues. The number of the isolated eigenvalues of H is independent of the window distance provided $l \geqslant \max\{a_-, a_+\}$. In the limit $l \to +\infty$ each isolated eigenvalue of the operator H converges to one of the numbers from the set σ_* or to the threshold of $\sigma_{\mathrm{ess}}(H)$.*

By Ξ_a we indicate the set of all bounded domains $S \subset \Pi_a$ having smooth boundary and separated from the edges of γ_a by a positive distance; we stress that the case $\partial S \cap \partial\Pi_a \neq \emptyset$ is not excluded. For any λ such that $\operatorname{Re}\lambda \leqslant 1$ we denote

$\kappa_1^+ = \kappa_1^+(\lambda) := \sqrt{1-\lambda}$ and $\kappa_1^- = \kappa_1^-(\lambda) := \sqrt{\frac{\pi^2}{d^2} - \lambda}$, where the branch of the root is specified by the requirement that the functions are analytic in $\mathbb{S}_\delta$ and $\sqrt{1} = 1$.

The following statements will be proven in Section 3.

Proposition 2.2. *In the limit $x_1 \to \pm\infty$ the eigenfunction ψ_n of $H(a)$ behaves as*

$$\begin{aligned} &\psi_n(x,a) = (\pm 1)^{n+1} c(\lambda_n, a) \mathrm{e}^{-\kappa_1^+(\lambda_n)|x_1|} \sin x_2 + \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\sqrt{4-\lambda}|x_1|}), \quad && x_2 \in [0,\pi], \\ &\psi_n(x,a) = \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\kappa_1^-(\lambda)|x_1|}), && x_2 \in [-d, 0], \end{aligned} \tag{2.1}$$

if $d < \pi$, and

$$\psi_n(x,a) = (\pm 1)^{n+1} c(\lambda_n, a) \mathrm{e}^{-\kappa_1^+(\lambda_n)|x_1|} \sin |x_2| + \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\sqrt{4-\lambda}|x_1|}), \quad x_2 \in [-\pi, \pi], \tag{2.2}$$

in the case of equal-width channels, $d = \pi$. In these relations

$$c(\lambda_n, a) = \frac{1}{\pi\kappa_1^+(\lambda_n)} \int\limits_{\gamma_a} \psi_n(x,a) \mathrm{e}^{\kappa_1^+(\lambda_n) x_1} \, \mathrm{d}x_1 = \frac{(-1)^{n+1}}{\pi\kappa_1^+(\lambda_n)} \int\limits_{\gamma_a} \psi_n(x,a) \mathrm{e}^{-\kappa_1^+(\lambda_n) x_1} \, \mathrm{d}x_1, \tag{2.3}$$

and $c(\lambda_i, a) \neq 0$, $i = 1, 2$. The asymptotic relations (2.1), (2.2) give rise to valid formulæ when both their sides are differentiated.

Proposition 2.3. *For any $\lambda \in (-\infty, 1) \setminus \sigma_{\mathrm{disc}}(H(a))$ there exists a unique solution of the boundary value problem*

$$\begin{gathered} (\Delta + \lambda) U = 0, \quad x \in \Pi_a \setminus \gamma_a, \qquad U = 0, \quad x \in \partial\Pi_a, \\ \left.\frac{\partial U}{\partial x_2}\right|_{x_2 = +0} - \left.\frac{\partial U}{\partial x_2}\right|_{x_2 = -0} = \mathrm{e}^{\kappa_1^+(\lambda) x_1}, \quad x \in \gamma_a, \end{gathered} \tag{2.4}$$

belonging to $W_2^1(\Pi_a)$. For large values of $|x_1|$ this function is infinitely differentiable and in the limit $x_1 \to +\infty$ it behaves as

$$\begin{aligned} &U(x,\lambda,a) = c(\lambda,a) \mathrm{e}^{-\kappa_1^+(\lambda_n) x_1} \sin x_2 + \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\sqrt{4-\lambda} x_1}), \quad && x_2 \in [0,\pi], \\ &U(x,\lambda,a) = \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\kappa_1^-(\lambda) x_1}), && x_2 \in [-d,0], \end{aligned} \tag{2.5}$$

if $d < \pi$, and

$$U(x,\lambda,a) = c(\lambda,a) \mathrm{e}^{-\kappa_1^+(\lambda_n) x_1} \sin |x_2| + \mathcal{O}(\mathrm{e}^{-\operatorname{Re}\sqrt{4-\lambda} x_1}), \quad x_2 \in [-\pi,\pi], \tag{2.6}$$

in the case $d = \pi$, where the coefficient is given by

$$c(\lambda, a) = \frac{1}{\pi\kappa_1^+(\lambda)} \int\limits_{\gamma_a} U(x,\lambda,a) \mathrm{e}^{\kappa_1^+(\lambda) x_1} \, \mathrm{d}x_1. \tag{2.7}$$

This coefficient is negative for $\lambda < \lambda_1(a)$. The asymptotic relations (2.5), (2.6) give rise to valid formulæ when both their sides are differentiated.

For the double window $\gamma_+\cup\gamma_-$ we indicate by Ξ the set of all bounded domains $S\subset\Pi$ having smooth boundary and separated from the edges of $\gamma_\pm$ by a positive distance; the case $\partial S\cap\partial\Pi\neq\emptyset$ is again not excluded. For brevity we will introduce a two-valued symbol, $\tau:=1$ if $d<\pi$ and $\tau:=2$ if $d=\pi$.

Continuing the list of the main results we make the following claims.

Theorem 2.2. *Suppose that $\lambda_*\in\sigma_*$ is simple being an eigenvalue $\lambda_n(a_\pm)$ of the operator $H(a_\pm)$. Then there is a unique eigenvalue of the operator H converging to λ_* as $l\to+\infty$. This eigenvalue is simple and behaves asymptotically as follows,*

$$\lambda^\pm(l,\boldsymbol{a})=\lambda_*+\mu^\pm(l,\boldsymbol{a})\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}+\mathcal{O}(l^2\mathrm{e}^{-8\kappa_1^+(\lambda_*)l}+\mathrm{e}^{-2l(\kappa_1^+(\lambda_*)+\rho)}),\tag{2.8}$$

$$\mu^\pm(l,\boldsymbol{a}):=\tau\pi c(\lambda_*,a_\pm)c^2(\lambda_*,a_\mp)\kappa_1^+(\lambda_*),\tag{2.9}$$

where $\rho=\rho(\lambda):=\min\{\kappa_1^-(\lambda),\sqrt{4-\lambda}\}$ if $d<\pi$, and $\rho=\rho(\lambda):=\sqrt{4-\lambda}$ if $d=\pi$. The associated eigenfunction $\psi^\pm(x,l,\boldsymbol{a})$ satisfies the relation

$$\psi^\pm(x,l,\boldsymbol{a})=\psi_n(x_1\mp l,x_2,a_\pm)+\mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l})\tag{2.10}$$

in the norms of both the $W_2^1(\Pi)$ and $W_2^2(S)$ for each $S\in\Xi$.

Theorem 2.3. *Suppose that $\lambda_*\in\sigma_*$ is double and $\lambda_*=\lambda_n(a_-)=\lambda_m(a_+)$. Then there exist either two simple eigenvalues $\lambda^\pm(l,\boldsymbol{a})$ or one double eigenvalue $\lambda^-(l,\boldsymbol{a})=\lambda^+(l,\boldsymbol{a})$ of the operator H converging to λ_* as $l\to+\infty$. The asymptotic expansions of these eigenvalues read as follows,*

$$\lambda^\pm(l,\boldsymbol{a})=\lambda_*\pm|\mu(l,\boldsymbol{a})|\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}+\mathcal{O}(l\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}+\mathrm{e}^{-2\rho(\lambda_*)l}),\tag{2.11}$$

$$\mu(l,\boldsymbol{a})=(-1)^{m+1}\tau\pi\kappa_1^+(\lambda_*)c(\lambda_*,a_-)c(\lambda_*,a_+),\tag{2.12}$$

Theorem 2.4. *Suppose the hypothesis of Theorem 2.3 holds true. If $\mu(l,\boldsymbol{a})\neq0$, the eigenvalues $\lambda^+(l,\boldsymbol{a})$ $\lambda^-(l,\boldsymbol{a})$ do not coincide and are simple. The associated eigenfunctions $\psi^\pm(x,l,\boldsymbol{a})$ satisfy the relations*

$$\psi^\pm(x,l,\boldsymbol{a})=\psi_n(x_1+l,x_2,a_-)\mp\psi_m(x_1-l,x_2,a_+)\operatorname{sgn}\mu(l,\boldsymbol{a})+\mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l})\tag{2.13}$$

in the norms of $W_2^1(\Pi)$ and $W_2^2(S)$ for each $S\in\Xi$. If $\lambda^-(l,\boldsymbol{a})=\lambda^+(l,\boldsymbol{a})$ is a double eigenvalue, the associated eigenfunctions $\psi^\pm(x,l,\boldsymbol{a})$ satisfy the relations

$$\begin{aligned}\psi^+(x,l,\boldsymbol{a})&=\psi_n(x_1+l,x_2,a_-)+\mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}),\\ \psi^-(x,l,\boldsymbol{a})&=\psi_m(x_1-l,x_2,a_-)+\mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}),\end{aligned}\tag{2.14}$$

in the norm of $W_2^1(\Pi)$ and $W_2^2(S)$ for each $S\in\Xi$. Finally, if $\mu(l,\boldsymbol{a})=0$ and $\lambda^-(l,\boldsymbol{a})\neq\lambda^+(l,\boldsymbol{a})$, the eigenvalues $\lambda^\pm(l,\boldsymbol{a})$ are simple and the associated eigenfunctions satisfy the relations

$$\psi^\pm(x,l,\boldsymbol{a})=c_+^\pm\psi_n(x_1+l,x_2,a_-)+c_-^\pm\psi_m(x_1-l,x_2,a_+)+\mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}),\tag{2.15}$$

where the vectors $\boldsymbol{c}^\pm:=\begin{pmatrix}c_+^\pm\\ c_-^\pm\end{pmatrix}$ are nontrivial solutions to the system (4.20) with $\lambda=\lambda^\pm$ such that $\|\boldsymbol{c}^\pm\|_{\mathbb{R}^2}=1$.

The leading terms of the asymptotics (2.8), (2.11) are non-zero provided the corresponding coefficients $c(\lambda_*, a_\pm)$ are non-zero. We know from Propositions 2.2, 2.3 that this is true at least for $c(\lambda, a)$ as $\lambda \leqslant \lambda_1(a)$ or $\lambda = \lambda_2(a)$. For instance, if $\lambda_1(a_-) < \lambda_1(a_+)$, the eigenvalue of the operator H converging to $\lambda_1(a_-)$ has the asymptotic expansion (2.8), and the coefficient (2.9) of leading term is non-zero. Moreover, due to Proposition 2.3 this coefficient is negative. If $a_\pm$ are such that $\lambda_1(a_-) = \lambda_2(a_+)$, the eigenvalues of the operator H converging to $\lambda_* = \lambda_1(a_-) = \lambda_2(a_+)$ have the asymptotics expansions (2.11), and the coefficients of the leading terms are non-zero. By Theorem 2.4 the "perturbed" eigenvalues are simple and the associated eigenfunctions satisfy the identities (2.13) in this case. We also stress that in this case the leading terms of the asymptotic expansions (2.11) have the same modulus but different signs. This phenomenon is known in double-well problems with symmetric wells. It also occurs in the symmetric case, $a_- = a_+$ and $d_- = d_+$, as we have shown in [2].

We conjecture that the coefficient $c(\lambda, a)$ is non-zero for all values a and $\lambda < 1$. If it is true, this fact would imply that the leading terms in the asymptotics (2.8), (2.11) are non-zero. In turn, this fact together with Theorem 2.4 would imply that a double $\lambda_* \in \sigma_*$ splits into two simple "perturbed" eigenvalues and the formulæ (2.13) are valid for the associated eigenfunctions.

3 Analysis of the one-window problem

In this section we shall study the following boundary value problem,

$$\begin{gathered}(\Delta+\lambda)u = 0, \quad x \in \Pi_a \setminus \gamma_a, \qquad u = 0, \quad x \in \Gamma_a,\\ \frac{\partial u}{\partial x_2}\Big|_{x_2=+0} - \frac{\partial u}{\partial x_2}\Big|_{x_2=-0} = f, \quad x \in \gamma_a.\end{gathered} \tag{3.1}$$

The function f is assumed to be an element of $L_2(\gamma_a)$. A solution to this problem is understood in a generalized sense, more specifically, as a function belonging to $W_2^1(\Pi_a^b, \Gamma_a^b)$ for each $b > 0$ and satisfying the equation

$$-(\nabla u, \nabla \zeta)_{L_2(\Pi_a)} + \lambda (u, \zeta)_{L_2(\Pi_a)} - (f, \zeta)_{L_2(\gamma_a)} = 0 \tag{3.2}$$

for any $\zeta \in C_0^\infty(\Pi_a)$. By standard smoothness-improving results about solutions to elliptic boundary value problems, cf. [5, Ch. 4, §2], the said solution belongs to $C^\infty(\overline{\Pi^+} \cup \overline{\Pi^-} \setminus \overline{\gamma}_a)$. As we have said in the introduction we will deal in this paper with the non-threshold case only. Thus the parameter λ is supposed to belong to $\mathbb{S}_\delta$ for a fixed $\delta > 0$, where $\mathbb{S}_\delta$ is a set of all λ separated from the halfline $[1, +\infty)$ by a distance not less than δ.

We seek a solution to the problem (3.1) belonging to $L_2(\Pi_a)$. We fix $\beta > a$ and put $P := \{x : |x_1| < a + \beta,\ 0 < x_2 < d_0\}$. The number d_0 here is chosen so that $d_0 < d$, and the lowest eigenvalue of the negative Laplacian in P subject to

Dirichlet boundary condition on $\partial P \setminus \overline{\gamma}_a$ and to Neumann one on γ_a exceeds two. We consider the boundary value problem

$$(\Delta+\lambda)\widetilde{u}=0, \quad x\in P, \qquad \widetilde{u}=0, \quad x\in\partial P, \qquad \frac{\partial\widetilde{u}}{\partial x_2}=\frac{1}{2}f, \quad x\in\gamma_a, \tag{3.3}$$

which is again treated in the weak sense,

$$-(\nabla\widetilde{u},\nabla\zeta)_{L_2(P)}+\lambda(\widetilde{u},\zeta)_{L_2(P)}-\frac{1}{2}(f,\zeta)_{L_2(\gamma_a)}=0 \tag{3.4}$$

for each function $\zeta\in C^\infty(P)$ vanishing in a neighborhood of $\partial P\setminus\overline{\gamma}_a$. The problem (3.3) is uniquely solvable in the space $W_2^1(P,\partial P\setminus\gamma_a)$ and the solution belongs to $C^\infty(\overline{P}\setminus\overline{\gamma}_a)$ – see [7, Chap. II, §5, Rem. 5.1] and [5, Chap. IV, §2]).

Let $\chi_1=\chi_1(x)$ be an infinitely differentiable function, even w.r.t. the variable x_2, equal to one if $|x_1|<a+\beta/6$ and $|x_2|<d_0/6$, and vanishing for $|x_1|>a+\beta/3$ or $|x_2|>d_0/3$. We extend the function $\widetilde{u}$ in an even way for $x_2<0$ setting $\widetilde{u}(x):=\widetilde{u}(x_1,-x_2)$ as $x_2<0$ and denote $u_f(x):=\chi_1(x)\widetilde{u}(x)$.

Lemma 3.1. *The function u_f belongs to $W_2^1(\Pi_a,\Gamma_a)\cap C^\infty(\Pi_a)$ and satisfies the equation*

$$-(\nabla u_f,\nabla\zeta)_{L_2(\Pi_a)}+\lambda(u_f,\zeta)_{L_2(\Pi_a)}-(f,\zeta)_{L_2(\gamma_a)}=(F,\zeta)_{L_2(\Pi_a)} \tag{3.5}$$

for any $\zeta\in C_0^\infty(\Pi_a)$, where

$$F=T_1(\lambda,a)f:=2\nabla\widetilde{u}\cdot\nabla\chi_1+\widetilde{u}\Delta\chi_1\,.$$

The operator $T_1: L_2(\gamma_a)\to L_2(\{x:|x_1|<a+\beta/3,|x_2|<d_0/3\})$ is linear, bounded, and holomorphic in λ. The operator $T_2(\lambda,a)f:=u_f$ is linear, bounded, and holomorphic in λ as a map from $L_2(\gamma_a)$ into $W_2^1(\Pi_a,\Gamma_a)$, $W_2^2(S)$, and $W_2^2(\Pi^\pm\setminus\Pi_a^\beta)$, where $S\in\Xi_a$ is such that $S\subset\Pi^+$ or $S\subset\Pi^-$.

Proof. Let $\{\widetilde{u}^{(j)}\}$ be a sequence of functions from $C^\infty(\overline{P})$ vanishing in a neighborhood of $\partial P\setminus\overline{\gamma}_a$, which converges to $\widetilde{u}$ in $W_2^1(P)$. It is easy to see that the functions $u_f^{(j)}(x):=\chi_1(x)\widetilde{u}^{(j)}(x)$ belong to $W_2^1(\Pi_a,\Gamma_a)$, and that they converge to w_f in the norm of $W_2^1(\Pi_a)$ as $j\to\infty$, so $u_f\in W_2^1(\Pi_a,\Gamma_a)$. Next we observe that u_f belongs to $C^\infty(\Pi_a)$ as it follows from the fact that $\widetilde{u}\in C^\infty(\overline{P}^+\setminus\overline{\gamma}_a)$. Since the function u_f is even in the variable x_2, we find that for each $\zeta\in C_0^\infty(\Pi_a)$ the left-hand side of (3.5) equals twice the expression

$$-(\nabla u_f,\nabla\zeta^+)_{L_2(P)}+\lambda(u_f,\zeta^+)_{L_2(P)}-(f,\zeta^+)_{L_2(\gamma_a)}\,,$$

where $\zeta^+(x):=\zeta(x)+\zeta(x_1,-x_2)$. In view of (3.4) and the definition of χ_1 we get

$$\begin{aligned}&-(\nabla u_f,\nabla\zeta)_{L_2(\Pi_a)}+\lambda(u_f,\zeta)_{L_2(\Pi_a)}-(f,\zeta)_{L_2(\gamma_a)}\\ =&2\Bigg(-(\nabla\widetilde{u},\nabla(\chi_1\zeta^+))_{L_2(P)}+\lambda(u_f,\chi_1\zeta^+)_{L_2(P)}-\frac{1}{2}(f,\chi_1\zeta^+)_{L_2(\gamma_a)}\end{aligned}$$

$$
\begin{aligned}
&+ (\nabla\widetilde{u}, \zeta^+\nabla\chi_1)_{L_2(P)} - (\widetilde{u}\nabla\chi_1, \nabla\zeta^+)_{L_2(P)}\Big)\\
=&2\Big((\nabla\widetilde{u}, \zeta^+\nabla\chi_1)_{L_2(P)} - (\widetilde{u}\nabla\chi_1, \nabla\zeta^+)_{L_2(P)}\Big)\\
=&2\Big((\nabla\widetilde{u}, \zeta^+\nabla\chi_1)_{L_2(P)} + (\operatorname{div}\widetilde{u}\nabla\chi_1, \zeta^+)_{L_2(P)}\Big) = (F,\zeta)_{L_2(\Pi_a)}.
\end{aligned}
$$

The boundedness of the operator $T_1(\lambda, a)$ follows from the above mentioned theorems on improving smoothness of solutions to elliptic boundary value problems.

In order to check that T_1 is holomorphic in the variable λ we just need to show that the mapping $f \mapsto \widetilde{u}$ is bounded and holomorphic as an operator family from $L_2(\gamma_a)$ into $W_2^1(P)$ and $W_2^2(S \cap P)$, where $S \in \Xi_a$ and $S \subset \overline{\Pi^+}$. To prove the last claim it is sufficient to reduce the boundary value problem to an operator equation in $W_2^1(P, \partial P \setminus \overline{\gamma}_a)$ in the standard way – see [5, Ch. II, §2] – and to apply then Proposition 4.5 of [6, Ch. XI, §4]. $\square$

We seek the solution to the problem (3.1) in the form $u = u_f + \widehat{u}$. As u_f is compactly supported, the function $\widehat{u}$ has to be an element of $L_2(\Pi_a)$. It follows from (3.2), (3.5) that the function $\widehat{u}$ must also obey the integral relation

$$
-(\nabla\widehat{u}, \nabla\zeta)_{L_2(\Pi_a)} + \lambda(\widehat{u}, \zeta)_{L_2(\Pi_a)} = -(F, \zeta)_{L_2(\Pi_a)} \tag{3.6}
$$

for any $\zeta \in C_0^\infty(\Pi_a)$. Thus $\widehat{u}$ has to solve the boundary value problem

$$
-(\Delta + \lambda)\widehat{u} = F, \quad x \in \Pi_a, \qquad u = 0, \quad x \in \Gamma_a, \tag{3.7}
$$

belonging to $L_2(\Pi_a)$ and $W_2^1(\Pi_a^b, \Gamma_a^b)$ for each $b > 0$. By Theorem 4.6.8 of [8, Ch. 4, §4.6] any solution of this problem belonging to $L_2(\Pi_a)$ is an element of the operator domain of $H(a)$. In this way the problem (3.7) can be cast into the form $(H(a) - \lambda)\widehat{u} = F$, which in turn gives $\widehat{u} = \big(H(a) - \lambda\big)^{-1} F$.

Let us next denote $T_3(\lambda, a) := T_2(\lambda, a) + (H(a) - \lambda)^{-1} T_1(\lambda, a)$. In order to analyze properties of this operator we need an additional notation and a lemma. For any numbers $b_1, b_2, b_3 \in \mathbb{R}$ we set $\Omega_\pm := \{x : \pm x_1 > b_1,\, b_2 < x_2 < b_3\}$ and $\omega_\pm := \{x : \pm x_1 > b_1\} \cap \partial\Omega_\pm$.

Lemma 3.2. *Let $v \in W_2^1(\Omega_\pm)$ be a solution to the problem*

$$
(\Delta + \lambda)v = 0, \quad x \in \Omega^\pm, \qquad v = 0, \quad x \in \omega^\pm,
$$

and $0 < b_2 - b_3 \leqslant \pi$, $\lambda \in \mathbb{S}_\delta$. Then the function v can be represented as

$$
v(x, \lambda) = \sum_{j=1}^{\infty} \alpha_j(\lambda) \exp\left(-\sqrt{\frac{\pi^2 j^2}{(b_3 - b_2)^2} - \lambda}\,(\pm x_1 - b_1)\right) \sin\frac{\pi j}{b_3 - b_2}(x_2 - b_2), \tag{3.8}
$$

where

$$\alpha_j(\lambda) := \frac{2}{b_3 - b_2} \int\limits_{b_2}^{b_3} v(a_1, x_2, \lambda) \sin \frac{\pi j}{b_3 - b_2}(x_2 - b_2)\, \mathrm{d}x_2. \tag{3.9}$$

The series (3.8) converges in the norms of $W_2^m(\{x : \pm x_1 > b_4, b_2 < x_2 < b_3\})$, $m \geqslant 0$, *for any* $b_4 > b_1$. *The coefficients* α_j *satisfy the condition*

$$\frac{\pi}{2} \sum_{j=1}^{\infty} |\alpha_j|^2 = \|v(\pm b_1, \cdot, \lambda)\|_{L_2(b_2, b_3)}. \tag{3.10}$$

This lemma is a particular case of Lemma 3.3 of [4] so we skip the proof.

Lemma 3.3. *The operator* $T_3(\lambda, a)$ *is bounded and meromorphic in* $\lambda \in \mathbb{S}_\delta$ *as a map from* $L_2(\gamma_a)$ *into* $W_2^1(\Pi_a, \Gamma_a)$ *and into* $W_2^2(S)$ *for each* $S \in \Xi_a$. *Its poles coincide with the eigenvalues of the operator* $H(a)$. *For any* λ *close to an eigenvalue* λ_n *of* $H(a)$ *the representation*

$$T_3(\lambda, a) = \frac{\psi_n}{\lambda - \lambda_n} T_4(a) + T_5(\lambda, a) \tag{3.11}$$

holds true. Here $T_4(a)f := (f, \psi_n)_{L_2(\gamma_a)}$ *and the operator* T_5 *is bounded and holomorphic in* $\lambda \in \mathbb{S}_\delta$ *as a map from* $L_2(\gamma_a)$ *into* $W_2^1(\Pi_a, \Gamma_a)$. *The operator* T_5 *is also bounded and holomorphic as a map into* $W_2^2(S)$ *for each* $S \in \Xi_a$.

Proof. In accordance with [9, Ch. 5, §3.5] the operator $(H(a) - \lambda)^{-1}$ is bounded and meromorphic in $L_2(\Pi_a)$, its poles coincide with the eigenvalues of $H(a)$ and for λ close to λ_n the representation

$$\big(H(a) - \lambda\big)^{-1} = -\frac{\psi_n}{\lambda - \lambda_n} (\cdot, \psi_n)_{L_2(\Pi_a)} + T_6(\lambda, a) \tag{3.12}$$

is valid, where the operator $T_6(\lambda, a)$ is bounded and holomorphic in λ in the vicinity of λ_n. The function $\check{u} := T_6(\lambda, a)F$ is a solution to the boundary value problem (3.7) with F replaced by $\check{F} := F - (F, \psi_n)_{L_2(\Pi_a)}\psi_n$; it means that

$$\|\nabla \check{u}\|^2_{L_2(\Pi_a)} - \lambda \|\check{u}\|^2_{L_2(\Pi_a)} = (\check{F}, \check{u})_{L_2(\Pi_a)}.$$

This relation together with (3.12) imply that the operator T_6 is bounded and holomorphic as a map into $W_2^1(\Pi_a)$ as well. Using again the smoothness-improving theorems mentioned above we conclude that the operator T_6 is also bounded and holomorphic in λ as a map into $W_2^2(S)$ for each $S \in \Xi_a$.

Since the function ψ_n is an element of $W_2^1(\Pi_a, \Gamma_a)$, the relation (3.5) is valid for $\zeta = \psi_n$. For any $f \in L_2(\gamma_a)$ the function $T_1(\lambda, a)f$ is compactly supported, hence we have

$$(T_1(\lambda_n, a)f, \psi_n)_{L_2(\Pi_a)} = -(\nabla u_f, \nabla \psi_n)_{L_2(\Pi_a)} + \lambda (u_f, \psi_n)_{L_2(\Pi_a)} - (f, \psi_n)_{L_2(\gamma_a)}.$$

According to Lemma 3.1, the function u_f belongs to $W_2^1(\Pi_a,\Gamma_a)$, which allows us to proceed with the calculations,

$$-(\nabla u_f,\nabla\psi_n)_{L_2(\Pi_a)}+\lambda(u_f,\psi_n)_{L_2(\Pi_a)}=0,$$
$$(T_1(\lambda_n,a)f,\psi_n)_{L_2(\Pi_a)}=-(f,\psi_n)_{L_2(\gamma_a)}=-T_4f.$$

Substituting the relation thus obtained together with (3.12) into the definition of the operator T_3 and taking into account Lemma 3.1, we arrive finally at the statement of the lemma. □

Let us next fix a number $\widetilde{a}>0$. For any $l\geqslant(a+\widetilde{a})$ we define operators $T_7^{\pm}(\lambda,l,a,\widetilde{a})$ which map an arbitrary $v\in W_2^1(\Pi_a^a)$ into the function

$$(T_7^{\pm}v)(x_1,\lambda,l):=\sum_{j=1}^{\infty}j\alpha_j^{\pm}\mathrm{e}^{\mp\kappa_j^+(\lambda)(x_1\mp a)}\mathrm{e}^{-2\kappa_j^+(\lambda)l}-\sum_{j=1}^{\infty}\frac{\pi j}{d}\beta_j^{\pm}\mathrm{e}^{\mp\kappa_j^-(\lambda)(x_1\mp a)}\mathrm{e}^{-2\kappa_j^-(\lambda)l},$$

$$\alpha_j^{\pm}=\frac{2}{\pi}\int\limits_0^{\pi}v(a,x_2)\sin jx_2\,\mathrm{d}x_2,\quad \beta_j^{\pm}=\frac{2}{d}\int\limits_{-d}^{0}v(a,x_2)\sin\frac{\pi j}{d}x_2\,\mathrm{d}x_2,$$

$$\kappa_j^+(\lambda):=\sqrt{j^2-\lambda},\quad \kappa_j^-(\lambda):=\sqrt{\frac{\pi^2j^2}{d^2}-\lambda},\quad j\geqslant 2. \tag{3.13}$$

The branch of the root in the definition of the functions κ_j is specified by the requirement that the functions are analytic in $\mathbb{S}_\delta$ and $\sqrt{1}=1$.

Lemma 3.4. *The operators $T_7^{\pm}:W_2^1(\Pi_a^a)\to L_2(\gamma_{\widetilde{a}})$ are well defined, bounded and holomorphic in $\lambda\in\mathbb{S}_\delta$. The estimates*

$$\left\|\frac{\partial^i T_7^{\pm}}{\partial\lambda^i}\right\|\leqslant Cl^i\mathrm{e}^{-(2l-a-\widetilde{a})\operatorname{Re}\kappa_1^+(\lambda)},\quad i=0,1,2,$$

hold true uniformly w.r.t. $\lambda\in\mathbb{S}_\delta$ and $l\geqslant(a+\widetilde{a})$.

Proof. We will prove the lemma for T_7^+ only, the argument for T_7^- is similar. The function u belongs to $W_2^1(\Pi_a^a)$, hence we have the estimate

$$\sum_{j=1}^{\infty}\left(|\alpha_j^{\pm}|^2+|\beta_j^{\pm}|^2\right)\leqslant C\|u\|^2_{W_2^1(\Pi_a)},$$

where the constant C is independent of $\lambda\in\mathbb{S}_\delta$ and $l\geqslant(a+\widetilde{a})$. Employing this inequality we infer that

$$\left\|\sum_{j=1}^{\infty}j\alpha_j^+\mathrm{e}^{-\kappa_j^+(\lambda)(\cdot-a)}\mathrm{e}^{-2\kappa_j^+(\lambda)l}\right\|_{L_2(\gamma_{\widetilde{a}})}\leqslant\sum_{j=1}^{\infty}j|\alpha_j|\mathrm{e}^{-2l\operatorname{Re}\kappa_j^+(\lambda)}\|\mathrm{e}^{-\kappa_j^+(\lambda)(\cdot-a)}\|_{L_2(-\widetilde{a},\widetilde{a})}$$
$$\leqslant C\sum_{j=1}^{\infty}\frac{j|\alpha_j^+|}{\sqrt{\operatorname{Re}\kappa_j^+(\lambda)}}\mathrm{e}^{-(2l-a-\widetilde{a})\operatorname{Re}\kappa_j^+(\lambda)}$$

$$
\begin{aligned}
&\leqslant C\left(\sum_{j=1}^{\infty}|\alpha_j|^2\right)^{1/2}\left(\sum_{j=1}^{\infty}\frac{j^2\mathrm{e}^{-2(2l-a-\widetilde{a})\operatorname{Re}\kappa_j^+(\lambda)}}{|\kappa_j^+(\lambda)|}\right)^{1/2}\\
&\leqslant C\|v\|_{W_2^1(\Pi_a^a)}\mathrm{e}^{-(2l-a-\widetilde{a})\operatorname{Re}\kappa_1^+(\lambda)}\left(\sum_{j=1}^{\infty}\frac{j^2\exp\left(-2(2l-a-\widetilde{a})\operatorname{Re}(\kappa_j^+(\lambda)-\kappa_1^+(\lambda))\right)}{\sqrt{|\operatorname{Im}\lambda|}}\right)^{1/2}\\
&\leqslant C\mathrm{e}^{-(2l-a-\widetilde{a})}\|v\|_{W_2^1(\Pi_a^a)},
\end{aligned}
$$

where C is independent of $\lambda\in\mathbb{S}_\delta$. In the same way one can prove that

$$
\left\|\sum_{j=1}^{\infty}\frac{\pi j}{d}\beta_j^+\mathrm{e}^{-\kappa_j^-(\lambda)(\cdot-a)}\mathrm{e}^{-2\kappa_j^-(\lambda)l}\right\|_{L_2(\gamma_{\widetilde{a}})}\leqslant C\mathrm{e}^{-(2l-a-\widetilde{a})\operatorname{Re}\kappa_1^-(\lambda)}\|v\|_{W_2^1(\Pi_a^a)},
$$

The last two estimates imply that the operator $T_7^+:W_2^1(\Pi_a^a)\to L_2(\gamma_{\widetilde{a}})$ is well defined and bounded. One can check easily that

$$
\begin{aligned}
\left(\frac{\partial T_7^+v}{\partial\lambda}\right)(x_1,\lambda,l):=&\sum_{j=1}^{\infty}\frac{j\alpha_j^+}{2\kappa_j^+(\lambda)}(x_1-a+2l)\mathrm{e}^{-\kappa_j^+(\lambda)(x_1-a)}\mathrm{e}^{-2\kappa_j^+(\lambda)l}\\
&-\sum_{j=1}^{\infty}\frac{\pi j\beta_j^+(x_1-a+2l)}{2\kappa_j^-(\lambda)d}\mathrm{e}^{-\kappa_j^-(\lambda)(x_1-a)}\mathrm{e}^{-2\kappa_j^-(\lambda)l}.
\end{aligned}
$$

Repeating the argument which yielded the estimate for T_7^+v we can establish that

$$
\left\|\frac{\partial T_7^\pm v}{\partial\lambda}\right\|_{L_2(\gamma_{\widetilde{a}})}\leqslant Cl\mathrm{e}^{-(2l-a-\widetilde{a})\operatorname{Re}\kappa_1^-(\lambda)}\|v\|_{W_2^1(\Pi_a^a)}
$$

with the constant C independent of $\lambda\in\mathbb{S}_\delta$ and $l\geqslant(a+\widetilde{a})$. Consequently, the operator $\frac{\partial T_7^+}{\partial\lambda}$ exists, it is bounded and the stated estimate for its norm holds true. The norm estimate for $\frac{\partial^2T_7^+}{\partial\lambda^2}$ is obtained in a similar way. $\square$

For any $l\geqslant(a+\widetilde{a})$ we define operators $T_8^\pm(\lambda,l,a,\widetilde{a})$ which map any $f\in L_2(\gamma_a)$ into the function

$$
\frac{\partial u}{\partial x_2}(x_1\pm 2l,+0,\lambda)-\frac{\partial u}{\partial x_2}(x_1\pm 2l,-0,\lambda),\quad x_1\in(-\widetilde{a},\widetilde{a}).
$$

Here u is a solution to the boundary value problem (3.1) belonging to $W_2^1(\Pi)$. Taking into account Lemma 3.3 together with the boundedness of the embedding $W_2^1(\Pi_a)$ into $L_2(\{x:|x_1\pm 2l|<\widetilde{a},x_2=0\})$, we conclude that the operators $T_8^\pm:L_2(\gamma_a)\to L_2(\gamma_{\widetilde{a}})$ are bounded and holomorphic in $\lambda\in\mathbb{S}_\delta$.

Lemma 3.5. *The poles of the operators $T_8^\pm$ coincide with the eigenvalues of the operator $H(a)$. For any compact set $K\subset\mathbb{S}_\delta$ separated from $\sigma_{\mathrm{disc}}(H(a))$ by a positive distance the estimates*

$$
\left\|\frac{\partial^iT_8^\pm}{\partial\lambda^i}\right\|\leqslant Cl^i\mathrm{e}^{-2l\operatorname{Re}\kappa_1^+(\lambda)},\quad i=0,1,\tag{3.14}
$$

hold true with C which is independent of $\lambda \in K$ and l. For any λ close to an eigenvalue λ_n of the operator $H(a)$ the representation

$$T_8^{\pm}(\lambda, l, a, \widetilde{a}) = \frac{\phi_n^{\pm}}{\lambda - \lambda_n} T_4(a) + T_9^{\pm}(\lambda, l, a, \widetilde{a}) \tag{3.15}$$

is valid, where

$$\phi_n^{\pm}(x_1, l, a) := \frac{\partial \psi_n}{\partial x_2}(x_1 \pm 2l, +0, a) - \frac{\partial \psi_n}{\partial x_2}(x_1 \pm 2l, -0, a), \quad x_1 \in (-\widetilde{a}, \widetilde{a}). \tag{3.16}$$

The operators $T_9^{\pm} : L_2(\gamma_a) \to L_2(\gamma_{\widetilde{a}})$ are bounded and holomorphic w.r.t. λ in the vicinity of λ_n and satisfy the estimates

$$\left\| \frac{\partial^i T_9^{\pm}}{\partial \lambda^i} \right\| \leqslant C l^{i+1} \mathrm{e}^{-2l \operatorname{Re} \kappa_1^{+}(\lambda)}, \quad i = 0, 1, \tag{3.17}$$

where the constant C is independent of λ and l.

Proof. Due to Lemma 3.2 we have

$$T_8^{\pm}(\lambda, l, a, \widetilde{a}) f = T_7^{\pm}(\lambda, l, a, \widetilde{a}) u. \tag{3.18}$$

Here u is a solution to the boundary value problem (3.1). Using this identity and the representation (3.11), we arrive at (3.15), where $T_9^{\pm}$ is bounded operator holomorphic in λ. Moreover,

$$\begin{aligned} T_9^{\pm}(\lambda, l, a, \widetilde{a}) =& \frac{T_7^{\pm}(\lambda, l, a, \widetilde{a}) - T_7^{\pm}(\lambda_n, l, a, \widetilde{a})}{\lambda - \lambda_n} T_4(a) + T_7^{\pm}(\lambda, l, a, \widetilde{a}) T_5^{\pm}(\lambda, a) \\ =& \left(\frac{1}{\lambda - \lambda_n} \int\limits_{\lambda_n}^{\lambda} \frac{\partial T_7^{\pm}}{\partial \lambda}(z, l, a, \widetilde{a}) \, \mathrm{d}z \right) T_4(a) + T_7^{\pm}(\lambda, l, a, \widetilde{a}) T_5^{\pm}(\lambda, a), \\ \frac{T_9^{\pm}}{\partial \lambda}(\lambda, l, a, \widetilde{a}) =& \left(\frac{1}{(\lambda - \lambda_n)^2} \int\limits_{\lambda_n}^{\lambda} \int\limits_{z_1}^{\lambda} \frac{\partial^2 T_7^{\pm}}{\partial \lambda^2}(z_2, l, a, \widetilde{a}) \, \mathrm{d}z_2 \, \mathrm{d}z_1 \right) T_4(a) \\ &+ \frac{\partial}{\partial \lambda} \left(T_7^{\pm}(\lambda, l, a, \widetilde{a}) T_5^{\pm}(\lambda, a) \right). \end{aligned}$$

Applying now Lemma 3.4 we obtain the estimates (3.17).

The operators $\frac{\partial^i T_3}{\partial \lambda^i}$, $i = 0, 1$, are bounded uniformly in $\lambda \in K$, thus in view of the relation (3.18) and Lemma 3.4 we arrive readily at the estimates (3.14). $\square$

Concluding this section we shall prove Propositions 2.2 and 2.3.

Proof of Proposition 2.2. Applying Lemma 3.2 to ψ_n with $b_1 = \pm a$, $b_2 = 0$, $b_3 = \pi$ and $b_2 = -d$, $b_3 = 0$, we obtain the formulæ (2.1), (2.2). The factor $(\pm 1)^{n+1}$ in these formulæ is due to the definite parity of ψ_n w.r.t. x_1. The formula (2.3) for $c_1^{(n)}$ follows from the chain of relations obtained by integration by parts,

$$
\begin{aligned}
0= & \int\limits_{\Pi^+} \mathrm{e}^{\pm\kappa_1^+(\lambda_n)x_1}\sin x_2(\Delta+\lambda_n)\psi_n(x,a)\,\mathrm{d}x \\
= & \int\limits_{\gamma_a} \mathrm{e}^{\pm\kappa_1^+(\lambda_n)x_1}\psi_n(x,a)\,\mathrm{d}x-(\pm 1)^{n+1}\pi\kappa_1^+(\lambda_n)c(\lambda_n,a).
\end{aligned}
$$

It remains to check the inequalities $c(\lambda_i,a)\neq 0$, $i=1,2$. The eigenfunction ψ_1 associated with the ground state can be chosen non-negative. Moreover, ψ_1 is not identically zero at γ_a, since otherwise it would be an eigenfunction of the negative Dirichlet Laplacian in Π^+ and would correspond to the eigenvalue $\lambda_1<1$. At the same time, the spectrum of the mentioned operator is the halfline $[1,+\infty)$. The described properties of ψ_1 and the formula (2.12) imply that $c(\lambda_1,a)\neq 0$.

According to Proposition 2.1 the eigenfunction ψ_2 is odd w.r.t. x_1. It allows us to modify the formula (2.3),

$$
\begin{aligned}
c(\lambda_2,a)= & \frac{1}{\pi\kappa_1^+(\lambda_2)}\int\limits_{\gamma_a}\psi_2(x,a)\sinh\kappa_1^+(\lambda_2)x_1\,\mathrm{d}x_1 \\
= & \frac{2}{\pi\kappa_1^+(\lambda_2)}\int\limits_{0}^{a}\psi_2(x_1,0,a)\sinh\kappa_1^+(\lambda_2)x_1\,\mathrm{d}x_1.
\end{aligned}
\tag{3.19}
$$

The eigenvalue λ_2 is the ground state of the negative Laplacian in $\Pi_a\cap\{x: x_1>0\}$ subject to Dirichlet boundary condition on $\partial\Pi_a\cap\{x: x_1>0\}$ and to Neumann one on $\{x: x_1=0, -d<x_2<\pi\}$. Hence $\psi_2(x_1,0,a)\geqslant 0$, and $\psi_2(x_1,0,a)\not\equiv 0$ as $x_1\in(0,a)$, and by (3.19) these inequalities imply that $c(\lambda_2,a)\neq 0$. $\square$

Proof of Proposition 2.3. The unique solvability of the problem (2.4) is ensured by Lemma 3.3. Moreover, we have $U=T_3(\lambda,a)\mathrm{e}^{\kappa_1^+(\lambda)x_1}$. The relations (2.5), (2.6) follow from Lemma 3.2, and the formula (2.7) is proved in the same way as (2.3).

Integrating by parts and employing the formula (2.7), we obtain a chain or identities,

$$
\begin{aligned}
0= & \int\limits_{\Pi} U(\Delta+\lambda)U\,\mathrm{d}x=\lambda\|U\|^2_{L_2(\Pi)}-\|\nabla U\|^2_{L_2(\Pi)}- \\
& -\int\limits_{\gamma_a} U\left(\frac{\partial U}{\partial x_2}\bigg|_{x_2=+0}-\frac{\partial U}{\partial x_2}\bigg|_{x_2=-0}\right)\mathrm{d}x_1 \\
= & \lambda\|U\|^2_{L_2(\Pi)}-\|\nabla U\|^2_{L_2(\Pi)}-\pi\kappa_1^+(\lambda)c(\lambda,a),
\end{aligned}
$$

which implies

$$
c(\lambda,a)=\frac{\lambda\|U\|^2_{L_2(\Pi)}-\|\nabla U\|^2_{L_2(\Pi)}}{\pi\kappa_1^+(\lambda)}. \tag{3.20}
$$

Since $U \in W_2^1(\Pi, \partial\Pi)$, the minimax principle yields the inequality

$$\|U\|^2_{L_2(\Pi)} \leqslant \frac{1}{\lambda_1(a)} \|\nabla U\|^2_{L_2(\Pi)}.$$

We substitute this inequality into the formula (3.20) and obtain

$$c(\lambda, a) \leqslant \frac{1}{\pi \kappa_1^+(\lambda)} \left(\frac{\lambda}{\lambda_1(a)} - 1 \right) \|\nabla U\|^2_{L_2(\Pi)} < 0,$$

if $\lambda < \lambda_1(a)$. □

4 Reduction of the perturbed problem

After this preliminary let us turn to our main problem; we are going to reformulate it as a suitable operator equation. Recall that we are looking for eigenvalues of the operator H, i.e. non-trivial $L_2(\Pi)$-solutions to the boundary value problem

$$-\Delta\psi = \lambda\psi, \quad x \in \Pi, \qquad \psi = 0, \quad x \in \partial\Pi. \tag{4.1}$$

We denote $Q^b := \{x : -b < x_1 < b, -d < x_2 < \pi\}$ and introduce the cut-off regions $\Pi^b := \Pi \cap Q^b$, $\Gamma^b := \partial\Pi \cap Q^b$. Solutions to the problem (4.1) can be identified with functions belonging to $W_2^1(\Pi^b, \Gamma^b)$ for any $b > 0$ such that

$$(\nabla\psi, \nabla\zeta)_{L_2(\Pi)} = \lambda(\psi, \zeta)_{L_2(\Pi)}, \tag{4.2}$$

holds for each $\zeta \in C_0^\infty(\Pi)$; it follows from the smoothness-improving theorem mentioned above that such a ψ belongs to $C^\infty(\Pi)$.

We assume that $\lambda \in \mathbb{S}_\delta$, with $\delta > 0$ is chosen in such a way that $\sigma_* \subset \mathbb{S}_\delta$. Let $f_\pm = f_\pm(\cdot, l) \in L_2(\gamma_{a_\pm})$ be an arbitrary pair of functions. Denote by $u_\pm$ the solutions of the problem (3.1) with $a = a_\pm$ and $f = f_\pm \in L_2(\gamma_{a_\pm})$ and assume that $u_\pm \in L_2(\Pi_{a_\pm})$. We will seek a solution to the problem (4.1) in the form

$$\psi(x, \lambda, l) = u_+(x_1 - l, x_2, \lambda, l) + u_-(x_1 + l, x_2, \lambda, l). \tag{4.3}$$

Suppose for a moment that the function ψ defined in this way solves the problem (4.1). In such a case the function ψ is infinitely differentiable at the points of the segments $\gamma_{a_\pm}$, and therefore

$$\frac{\partial\psi}{\partial x_2}(x_1, +0, \lambda, l) - \frac{\partial\psi}{\partial x_2}(x_1, -0, \lambda, l) = 0, \quad x \in \gamma_\pm.$$

Substituting from (4.3) into this identity, we obtain a pair of equations,

$$f_\pm(x_1) + \frac{\partial u_\mp}{\partial x_2}(x_1 \pm 2l, +0, \lambda, l) - \frac{\partial u_\mp}{\partial x_2}(x_1 \pm 2l, -0, \lambda, l) = 0, \quad x \in \gamma_{a_\pm}. \tag{4.4}$$

Denote $\boldsymbol{f} = (f_+, f_-) \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-})$. The following lemma states that the last equation is equivalent to the original problem (4.1).

Lemma 4.1. *To any solution $\boldsymbol{f} \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-})$ of (4.4) and functions $u_\pm$ solving (3.1) for $a = a_\pm$, $f = f_\pm$ there exists a unique $L_2(\Pi)$-solution of (4.1) given by (4.3). Reversely, for any solution ψ of (4.1) there are unique $\boldsymbol{f} \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-})$ solving (4.4) and unique functions $u_\pm \in L_2(\Pi_{a_\pm})$ satisfying (3.1) with $a = a_\pm$, $f = f_\pm$ such that ψ is given by (4.3). This equivalence holds for any $\lambda \in \mathbb{S}_\delta$ and $l \geqslant \max\{a_-, a_+\} + 1$.*

Proof. Suppose that $\boldsymbol{f} \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-})$ is a solution to the equations (4.4), where the functions $u_\pm \in L_2(\Pi_{a_\pm})$ solve the problem (3.1) $a = a_\pm$ and $f = f_\pm$. We define ψ in accordance with (4.3). The functions $u_\pm$ are elements of $L_2(\Pi)$, hence the same is true for ψ. Moreover, the function ψ belongs obviously to $W_2^1(\Pi^b, \Gamma^b)$ for each $b > 0$ and vanishes on Γ.

Let us check that the function ψ satisfies the equation (4.2). To this purpose, we indicate by $\chi_2 = \chi_2(x_1)$ an infinitely differentiable cut-off function being equal to one if $|x_1 + l| < \max\{a_+, a_-\} + 1/2$ and vanishing if $|x_1 + l| > \max\{a_+, a_-\} + 1$. For any $\zeta \in C_0^\infty(\Pi)$ we have

$$
\begin{aligned}
&\big(\nabla u_+(x_1 - l, x_2, \lambda, l), \nabla\zeta\big)_{L_2(\Pi)} - \lambda\big(u_+(x_1 - l, x_2, \lambda, l), \zeta\big)_{L_2(\Pi)} \\
&= \big(\nabla u_+(x_1 - l, x_2, \lambda, l), \nabla(\zeta\chi_2)\big)_{L_2(\Pi)} - \lambda\big(u_+(x_1 - l, x_2, \lambda, l), \zeta\chi_2)\big)_{L_2(\Pi)} \\
&+ \big(\nabla u_+(x_1 - l, x_2, \lambda, l), \nabla(\zeta(1 - \chi_2))\big)_{L_2(\Pi)} - \lambda\big(u_+(x_1 - l, x_2, \lambda, l), \zeta(1 - \chi_2)\big)_{L_2(\Pi)}.
\end{aligned}
\tag{4.5}
$$

Since $u_+(\cdot)$ is an element of $C^\infty(\Pi_{a_+} \setminus \overline{Q}^{a_+})$, we can integrate by parts,

$$
\begin{aligned}
&\big(\nabla u_+(x_1 - l, x_2, \lambda, l), \nabla\zeta\chi_2)\big)_{L_2(\Pi)} - \lambda\big(u_+(x_1 - l, x_2, \lambda, l), \zeta\chi_2\big)_{L_2(\Pi)} \\
&= -\int\limits_{\gamma_-} \left(\frac{\partial u_+}{\partial x_2}(x_1 - l, +0, \lambda, l) - \frac{\partial u_+}{\partial x_2}(x_1 - l, -0, \lambda, l)\right)\zeta \,\mathrm{d}x_1 - \\
&- \int\limits_{\Pi} \zeta\chi_2(\Delta + \lambda)u_+(x_1 - l, x_2, \lambda, l)\,\mathrm{d}x = \int\limits_{\gamma_-} f_-(x_1 + l)\zeta\,\mathrm{d}x_1.
\end{aligned}
$$

We have employed here the equation satisfied by u_+ as well as the equation (4.4) for f_+. Since $\zeta(x_1 + l, x_2)(1 - \chi_2(x_1 + l)) \in C_0^\infty(\Pi_{a_+})$, we can use the identity (3.2) to infer that

$$
\begin{aligned}
\big(\nabla u_+(x_1 - l, x_2, \lambda, l), \nabla\zeta(1 - \chi_2)\big)_{L_2(\Pi)} - \big(u_+(x_1 - l, x_2, \lambda, l), \zeta(1 - \chi_2)\big)_{L_2(\Pi)} \\
= -\int\limits_{\gamma_+} f_+(x_1 - l)\zeta\,\mathrm{d}x_1.
\end{aligned}
$$

We substitute now the last two relations into (4.5) and arrive at the identity

$$
\begin{aligned}
&\big(\nabla u_+(x_1 - l, x_2, \lambda, l), \nabla\zeta\big)_{L_2(\Pi)} - \lambda\big(u_+(x_1 - l, x_2, \lambda, l), \zeta\big)_{L_2(\Pi)} \\
&= (f_-(x_1 + l), \zeta)_{L_2(\gamma_-)} - (f_+(x_1 - l), \zeta)_{L_2(\gamma_+)}.
\end{aligned}
$$

In the same way one can check that

$$
\begin{aligned}
&\big(\nabla u_-(x_1+l,x_2,\lambda,l),\nabla\zeta\big)_{L_2(\Pi)}-\lambda\big(u_-(x_1+l,x_2,\lambda,l),\zeta\big)_{L_2(\Pi)}\\
&=(f_+(x_1-l),\zeta)_{L_2(\gamma_+)}-(f_-(x_1+l),\zeta)_{L_2(\gamma_-)};
\end{aligned}
$$

summing the last two relations we arrive at the relation (4.2) for the function ψ.

Let ψ be a solution to the problem (4.1) belonging to $L_2(\Pi)$. By smoothness-improving theorems the function ψ belongs to $C^\infty(\{x:-1\leqslant x_1\leqslant 1, 0\leqslant x_2\leqslant\pi\})$ and to $C^\infty(\{x:-1\leqslant x_1\leqslant 1, -d\leqslant x_2\leqslant 0\})$. This allows us to define the numbers

$$
\begin{aligned}
\alpha_j^\pm=\alpha_j^\pm(\lambda,l)&:=\frac{2}{\pi}\int\limits_0^\pi\left(\psi(0,x_2,\lambda,l)\pm\frac{1}{\kappa_j^+}\frac{\partial\psi}{\partial x_1}(0,x_2,\lambda,l)\right)\sin jx_2\,\mathrm{d}x_2,\\
\beta_j^\pm=\beta_j^\pm(\lambda,l)&:=\frac{2}{d}\int\limits_0^\pi\left(\psi(0,x_2,\lambda,l)\pm\frac{1}{\kappa_j^-}\frac{\partial\psi}{\partial x_1}(0,x_2,\lambda,l)\right)\sin\frac{\pi j}{d}x_2\,\mathrm{d}x_2.
\end{aligned}
$$

Using these numbers, we introduce the functions $u_\pm$ in the following way:

$$
\begin{aligned}
u_\pm(x_1\mp l,x_2,\lambda,l)&:=\sum_{j=1}^\infty\alpha_j^\pm(\lambda,l)\mathrm{e}^{\pm\kappa_j^+x_1}\sin jx_2, &&\pm x_1\leqslant 0,\quad x_2\in(0,\pi),\\
u_\pm(x_1\mp l,x_2,\lambda,l)&:=\sum_{j=1}^\infty\beta_j^+(\lambda,l)\mathrm{e}^{\pm\kappa_j^-x_1}\sin\frac{\pi j}{d}x_2, &&\pm x_1\leqslant 0,\quad x_2\in(-d,0),\\
u_\pm(x_1\mp l,x_2,\lambda,l)&:=\psi(x,\lambda,l)-u_\pm(x_1\pm l,x_2,\lambda), &&\pm x_1>0,\quad x_2\in(-d,\pi).
\end{aligned}
$$

Proceeding in the same way as in the proof of Lemma 4.1 in [4], we check that the functions $u_\pm$ are well defined and

$$
u_\pm\in W_2^1(\Pi_{a_\pm},\Gamma_{a_\pm})\cap W_2^2(S),\quad S\in\Xi_a, \tag{4.6}
$$

$$
(\Delta+\lambda)u_\pm(x_1\mp l,x_2,\lambda,l)=0,\quad x\in\Pi\setminus\{x:x_1=0\}. \tag{4.7}
$$

The relation (4.3) follows from the definition of the functions $u_\pm$. Now we set

$$
f_\pm(x_1,l):=-\frac{\partial u_\mp}{\partial x_2}(x_1\pm 2l,+0,\lambda,l)+\frac{\partial u_\mp}{\partial x_2}(x_1\pm 2l,-0,\lambda,l),\quad x_1\in(-a_\pm,a_\pm); \tag{4.8}
$$

in view of (4.6) we can conclude that $f_\pm\in L_2(\gamma_{a_\pm})$. We also note that the definition of $u_\pm$ and the smoothness of ψ at $\gamma_\pm$ imply

$$
f_\pm(x_1,l)=\frac{\partial u_\pm}{\partial x_2}(x_1,+0,\lambda,l)-\frac{\partial u_\pm}{\partial x_2}(x_1,-0,\lambda,l),\quad x_1\in(-a_\pm,a_\pm). \tag{4.9}
$$

Let us check the integral equation (3.2) for the function $u_+=u_+(x,\lambda)$. Taking into account (4.7), (4.9), and integrating by parts, we get

$$
-(\nabla u_+,\nabla\zeta)_{L_2(\Pi_a)}+\lambda(u_+,\zeta)_{L_2(\Pi_a)}
$$

$$= \left(\frac{\partial u_+}{\partial x_2}(x_1, +0, \lambda) - \frac{\partial u_+}{\partial x_2}(x_1, -0, \lambda), \zeta \right)_{L_2(\gamma_{a_+})} = (f_+, \zeta)_{L_2(\gamma_{a_+})}.$$

for any $\zeta \in C_0^\infty(\Pi_{a_+})$. In the same way one can check that

$$-(\nabla u_-, \nabla \zeta)_{L_2(\Pi_a)} + \lambda (u_-, \zeta)_{L_2(\Pi_a)} = (f_-, \zeta)_{L_2(\gamma_{a_-})}$$

for any $\zeta \in C_0^\infty(\Pi_{a_-})$, thus $u_\pm$ are solutions to the problem (3.1) with $a = a_\pm$ and $f = f_\pm$. The equations (4.4) follow from (4.8). $\square$

Suppose that $\lambda \in \mathbb{S}_\delta \setminus \sigma_*$. In that case the functions $u_\pm$ introduced above can be represented as $u_\pm = T_3(\lambda, a_\pm) f_\pm$, thus the equations (4.4) become

$$\boldsymbol{f} + T_8(\lambda, l, \boldsymbol{a}) \boldsymbol{f} = 0, \tag{4.10}$$

where the operator $T_8 : L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-}) \to L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-})$ is defined by

$$T_8(\lambda, l, \boldsymbol{a}) \boldsymbol{f} := \big(T_8^+(\lambda, l, a_-, a_+) f_-, T_8^-(\lambda, l, a_+, a_-) f_+ \big).$$

Now we are ready to demonstrate the first one of our main results.

Proof of Theorem 2.1. If $a_\pm = 0$ the essential spectrum of the operator H is obviously $[1, +\infty)$, and an elementary argument using Dirichlet-Neumann bracketing [10, Ch. XIII, §15] and the minimax principle [10, Ch. XIII, §1] shows that the threshold of the essential spectrum of H is one, i.e. $\sigma_{\text{ess}}(H) \subseteq [1, +\infty)$. The opposite inclusion can be shown easily; one needs to employ Weyl's criterion (see, for instance, proof of Lemma 2.1 in [4]).

The operator H being self-adjoint, its isolated eigenvalues are real, and in view of the above observation they are smaller than one; we arrange them conventionally in the ascending order counting multiplicity. Next we use bracketing again in a way analogous to [1]: we add Neumann boundaries at segments corresponding to x_1 at the endpoints of $\gamma_\pm$ and $x_2 \in (-d, \pi)$. In this way we get an operator estimating H from below, and since only the window parts contribute to the spectrum below one we infer by minimax that H has finitely many eigenvalues for any $l > 0$ and their number has a bound independent of l.

Let $K \subset \mathbb{S}_\delta$ be any compact set separated from σ_* by a positive distance. By the estimates (3.14) the operator T_8 has a norm being strictly less than one for $\lambda \in K$ and l large enough. For such λ and l the equation (4.10) has thus a trivial solution only, and in view of Lemma 4.1 this implies that the operator H has no eigenvalues in the set K if l is large enough. This means that each eigenvalue of the operator H has to converge to one of the numbers from the set σ_* or to the threshold of the essential spectrum. $\square$

The eigenvalues H, i.e. those λ for which the problem (4.1) has a nontrivial $L_2(\Pi)$-solution, coincide in view of Lemma 4.1 with the values of λ for which the equation (4.4) has a nontrivial solution. In the case considered here we deal only with the eigenvalues of H which converge to a value $\lambda_* \in \sigma_*$ separated from the threshold, in other words, being smaller than one.

Our aim is to solve the equation (4.4) and to obtain in this way an equation for the aforementioned values of λ. Consider a $\lambda_* \in \sigma_*$; if $\lambda_* = \lambda_n$ is an eigenvalue of the operator $H(a_+)$ we set

$$\boldsymbol{\phi}_*^+(\cdot,l) := \big(0, \phi_n^-(\cdot,l,a_+)\big) \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-}), \quad T_4^+\boldsymbol{f} := (f_+,\psi_n)_{L_2(\gamma_{a_+})},$$

where ϕ_n^- is determined by ψ_n in accordance with (3.16) and ψ_n is an eigenfunction associated with λ_n, in the opposite case we put

$$\boldsymbol{\phi}_*^+(\cdot,l) := (0,0) \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-}), \quad T_4^+\boldsymbol{f} := 0.$$

Analogously, if $\lambda_* = \lambda_n$ is an eigenvalue of $H(a_-)$ we set

$$\boldsymbol{\phi}_*^-(\cdot,l) := \big(\phi_n^+(\cdot,l,a_-), 0\big) \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-}), \quad T_4^-\boldsymbol{f} := (f_-,\psi_n)_{L_2(\gamma_{a_-})},$$

where ϕ_n^+ corresponds to ψ_n according to (3.16) and ψ_n is an eigenfunction associated with λ_n, otherwise

$$\boldsymbol{\phi}_*^-(\cdot,l) := (0,0) \in L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-}), \quad T_4^-\boldsymbol{f} := 0.$$

Given a number $\lambda_* \in \sigma_*$, we consider the equation (4.4) for λ in the vicinity of λ_*. Assume first that $\lambda \neq \lambda_*$, in which case the equation (4.4) is equivalent to (4.10). In view of Lemma 3.5 the operator T_8 is bounded and meromorphic as a function of $\lambda \in \mathbb{S}_\delta$, and the numbers $\lambda_* \in \sigma_*$ are poles of T_8. For any λ close to λ_* the operator T_8 can be thus represented as

$$T_8(\lambda,l,\boldsymbol{a}) = \boldsymbol{\phi}_*^+(\cdot,l)\frac{T_4^+}{\lambda-\lambda_*} + \boldsymbol{\phi}_*^-(\cdot,l)\frac{T_4^-}{\lambda-\lambda_*} + T_9(\lambda,l,\boldsymbol{a}), \tag{4.11}$$

where the operator T_9 acts as

$$T_9(\lambda,l,\boldsymbol{a})\boldsymbol{f} := \big(T_8^+(\lambda,l,a_-,a_+)f_-, T_9^-(\lambda,l,a_+,a_-)f_+\big)$$

if $\lambda_* \in \sigma_{\mathrm{disc}}(H(a_+)) \setminus \sigma_{\mathrm{disc}}(H(a_-))$,

$$T_9(\lambda,l,\boldsymbol{a})\boldsymbol{f} := \big(T_9^+(\lambda,l,a_-,a_+)f_-, T_8^-(\lambda,l,a_+,a_-)f_+\big)$$

if $\lambda_* \in \sigma_{\mathrm{disc}}(H(a_-)) \setminus \sigma_{\mathrm{disc}}(H(a_+))$, and finally,

$$T_9(\lambda,l,\boldsymbol{a})\boldsymbol{f} := \big(T_9^+(\lambda,l,a_-,a_+)f_-, T_9^-(\lambda,l,a_+,a_-)f_+\big)$$

if $\lambda_* \in \sigma_{\mathrm{disc}}(H(a_-)) \cap \sigma_{\mathrm{disc}}(H(a_+))$. The operator T_9 on $L_2(\gamma_{a_+}) \oplus L_2(\gamma_{a_-})$ is bounded and holomorphic w.r.t. λ in the vicinity of λ_*, and the estimate

$$\left\|\frac{\partial^i T_9}{\partial\lambda^i}\right\| \leqslant Cl^{i+1}\mathrm{e}^{-2l\,\mathrm{Re}\,\kappa_1^+(\lambda)}, \quad i=0,1, \tag{4.12}$$

holds true with a constant C which is independent on λ and l.

We substitute the representation (4.11) into (4.10) to obtain

$$\boldsymbol{f}+\frac{T_4^+\boldsymbol{f}}{\lambda-\lambda_*}\boldsymbol{\phi}_*^+ +\frac{T_4^-\boldsymbol{f}}{\lambda-\lambda_*}\boldsymbol{\phi}_*^- +T_9\boldsymbol{f}=0.$$

Since the norm of T_9 is small for large l due to (4.12), the operator $(\mathrm{I}+T_9)^{-1}$ is well defined being bounded in $L_2(\gamma_{a_+})\oplus L_2(\gamma_{a_-})$. We apply this operator to the last equation arriving at

$$\boldsymbol{f}+\frac{T_4^+\boldsymbol{f}}{\lambda-\lambda_*}\boldsymbol{\Phi}_*^+ +\frac{T_4^-\boldsymbol{f}}{\lambda-\lambda_*}\boldsymbol{\Phi}_*^- =0, \tag{4.13}$$

where $\boldsymbol{\Phi}_*^\pm(\cdot,\lambda,l)=(\mathrm{I}+T_9(\lambda,l,\boldsymbol{a}))^{-1}\boldsymbol{\phi}_*^\pm(\cdot,l)$. The last equation implies that

$$\boldsymbol{f}=c_+\boldsymbol{\Phi}_*^+ +c_-\boldsymbol{\Phi}_*^- \tag{4.14}$$

for some numbers $c_\pm$. We substitute from here into (4.13) obtaining

$$\boldsymbol{\Phi}_*^+\left(c_+\left(1+\frac{A_{11}}{\lambda-\lambda_*}\right)+c_-\frac{A_{12}}{\lambda-\lambda_*}\right)+\boldsymbol{\Phi}_*^-\left(c_+\frac{A_{21}}{\lambda-\lambda_*}+c_-\left(1+\frac{A_{22}}{\lambda-\lambda_*}\right)\right)=0, \tag{4.15}$$

where the quantities $A_{ij}=A_{ij}(\lambda,l)$ are defined by

$$\begin{aligned} A_{11}(\lambda,l)&:=T_4^+\Phi_*^+(\cdot,\lambda,l), \quad & A_{12}(\lambda,l)&:=T_4^+\Phi_*^-(\cdot,\lambda,l),\\ A_{21}(\lambda,l)&:=T_4^-\Phi_*^+(\cdot,\lambda,l), \quad & A_{22}(\lambda,l)&:=T_4^-\Phi_*^-(\cdot,\lambda,l). \end{aligned}$$

The definition of $\boldsymbol{\Phi}_*^\pm$ together with the estimate (4.12) imply for l large enough

$$\boldsymbol{\Phi}_*^\pm=\boldsymbol{\phi}_*^\pm+\mathcal{O}\big(\mathrm{e}^{-2l\operatorname{Re}\kappa_1^+(\lambda)}\|\boldsymbol{\phi}_*^\pm\|\big). \tag{4.16}$$

If $\boldsymbol{\phi}_*^+\neq 0$, and $\boldsymbol{\phi}_*^-=0$, in particular, we have

$$\boldsymbol{\Phi}_*^+\neq 0,\quad \boldsymbol{\Phi}_*^-=0,\quad A_{12}=A_{22}=0, \tag{4.17}$$

and in this case the equation (4.15) holds if and only if

$$c_+\left(1+\frac{A_{11}}{\lambda-\lambda_*}\right)=0.$$

If $\boldsymbol{f}$ corresponds to an eigenfunction ψ of the problem (4.1) by (4.3), the number c_+ is non-zero. Indeed, in the opposite case (4.14) and (4.17) would imply that $\boldsymbol{f}=0$, which by Lemma 4.1 results in $\psi=0$. Consequently, the equation (4.10) has in this case a nontrivial solution if and only if

$$\lambda-\lambda_*+A_{11}(\lambda,l)=0. \tag{4.18}$$

If λ is a root of this equation, the corresponding nontrivial solution of (4.10) can be expressed as (4.14) with $c_+ \neq 0$ and $c_- = 0$.

In the case $\boldsymbol{\phi}_*^+ = 0$ and $\boldsymbol{\phi}_*^- \neq 0$ similar arguments lead us to the conclusion that the equation (4.10) has a nontrivial solution if and only if

$$\lambda - \lambda_* + A_{22}(\lambda, l) = 0, \tag{4.19}$$

and the corresponding non-trivial solution can be written as (4.14) with the coefficients $c_+ = 0$ and $c_- \neq 0$.

Finally, if both the functions $\boldsymbol{\phi}_*^\pm$ are non-zero, they are linearly independent by definition and the same is true for the functions $\boldsymbol{\Phi}_*^\pm$. Hence the equation (4.10) holds if and only if

$$\left((\lambda - \lambda_*)\mathrm{E} + \mathrm{A}(\lambda, l)\right) \boldsymbol{c} = 0, \tag{4.20}$$

where E is the unit matrix, and

$$\mathrm{A}(\lambda, l) := \begin{pmatrix} A_{11}(\lambda, l) & A_{12}(\lambda, l) \\ A_{21}(\lambda, l) & A_{22}(\lambda, l) \end{pmatrix}, \quad \boldsymbol{c} := \begin{pmatrix} c_+ \\ c_- \end{pmatrix}.$$

The column $\boldsymbol{c}$ is non-zero, since otherwise (4.14) and (4.17) would imply $\boldsymbol{f} = 0$, thus the system (4.20) of linear equations has a nontrivial solution if and only if

$$\det\left((\lambda - \lambda_*)\mathrm{E} + \mathrm{A}(\lambda, l)\right) = 0, \tag{4.21}$$

which can be rewritten as

$$(\lambda - \lambda_*)^2 + (\lambda - \lambda_*) \operatorname{tr} \mathrm{A}(\lambda, l) + \det \mathrm{A}(\lambda, l) = 0\,; \tag{4.22}$$

the corresponding non-trivial solution of the equation (4.10) is given by (4.14), where $\left(\begin{smallmatrix} c_+ \\ c_- \end{smallmatrix}\right)$ is a nontrivial solution of (4.20).

Assume now that $\lambda = \lambda_*$. Let λ_* coincide with an eigenvalue λ_n of the operator $H(a_+)$ being not at the same time an eigenvalue of $H(a_-)$. In this case we again can claim that $u_- = T_3(\lambda_*, a_-)f_-$, on the other hand, the boundary value problem for u_+ with $\lambda = \lambda_*$ is solvable if and only if

$$0 = \int\limits_{\gamma_{a_+}} f_+ \psi_n(x, a_+)\,\mathrm{d}x = T_4^+ \boldsymbol{f}. \tag{4.23}$$

This follows from Lemma 3.3. The function u_+ is given by $u_+ = T_5(\lambda_*, a_+)f_+ - c_+\psi_+$, where c_+ is a constant. We can substitute now the described $u_\pm$ into (4.4) and obtain

$$\begin{aligned} \boldsymbol{f} + T_9(\lambda, l, \boldsymbol{a})\boldsymbol{f} &= c_+ \boldsymbol{\phi}_*^+, \\ \boldsymbol{f} &= c_+ \boldsymbol{\Phi}_*^+. \end{aligned} \tag{4.24}$$

This function will generate a solution to the problem (4.1) if and only if (4.18) holds true. Substituting (4.24) into (4.23), we arrive at the equation (4.18) with

$\lambda=\lambda_*$. If $c_+\neq 0$ holds in (4.24) we see that the formula (4.23) coincides with (4.14) with $c_-=0$. Consequently, in the case $\lambda_*\in\sigma(H(a_-))\setminus\sigma(H(a_+))$ the equation (4.18) determines all the values of λ in the vicinity of λ_* for which the equation (4.4) has a nontrivial solution; these nontrivial solutions are given by (4.14) with $c_+\neq 0$ and $c_-=0$.

In the same way one can check that the equation (4.19) determines the sought values of λ in the case when λ_* is an eigenvalue of the operator $H(a_-)$ and not of $H(a_+)$. The corresponding nontrivial solutions of (4.4) have $c_+=0$ and $c_-\neq 0$.

Finally, if $\lambda_*\in\sigma_*$ is double and $\lambda=\lambda_*$, the solvability conditions of the boundary value problems for u are $T_4^\pm \boldsymbol{f}=0$. If this holds true, the functions $u_\pm$ are given by $u_\pm=T_5(\lambda,a_\pm)-c_\pm\psi_\pm(\cdot)$, where $c_\pm$ are constants and $\psi_\pm$ are the eigenfunctions of $H(a_\pm)$ associated with λ_*. The equation (4.4) becomes

$$\boldsymbol{f}+T_9(\lambda_*,l,\boldsymbol{a})\boldsymbol{f}=c_+\boldsymbol{\phi}_*^+ + c_-\boldsymbol{\phi}_*^-,$$

which yields the relation (4.14). The solvability conditions $T_4^\pm\boldsymbol{f}=0$ are nothing else than the system of linear equations (4.20). In this way (4.14), (4.20), and (4.21) describe the sought values of λ in the vicinity of λ_* and the corresponding nontrivial solutions of (4.4).

5 Proofs of Theorems 2.2-2.4

Now we are going to demonstrate the remaining part of our claims.

Proof of Theorem 2.2. We will give the proof for the case $\lambda_*=\lambda_n(a_-)$, the argument for $\lambda_*=\lambda_n(a_+)$ is similar. In accordance with the results of the previous section the eigenvalue $\lambda^-(a,l)$, if it exists, it must be a root of the equation (4.19). Let us prove first that there is a unique root which converges to λ_* as $l\to+\infty$. Proposition 2.2 implies that the relation

$$\phi_n^+(x_1,l,a_-)=\tau c(\lambda_*,a_-)\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}\mathrm{e}^{-\kappa_1^+(\lambda_*)x_1}+\mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l}), \tag{5.1}$$

holds in the norm of $L_2(\gamma_{a_+})$, hence by the definition of $\boldsymbol{\phi}_*^-$ we have

$$\boldsymbol{\phi}_*^-(x_1,l)=\tau c(\lambda_*,a_-)\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}(\mathrm{e}^{-\kappa_1^+(\lambda_*)x_1},0)+\mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l}). \tag{5.2}$$

This formula in combination with the estimate (4.12) lead to the relation

$$A_{22}(\lambda,l)=\mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}). \tag{5.3}$$

Since T_9 is holomorphic w.r.t. λ and has a small norm for large l, we infer that the left-hand side of the last equation is holomorphic in λ. For a small δ take the circle of those λ such that $|\lambda-\lambda_*|=\delta$. In view of (5.3) the function A_{22} satisfies the estimate $|A_{22}|<\delta$ if l is large enough and $|\lambda-\lambda_*|=\delta$; by Rouché theorem it implies that the function $\lambda\mapsto\lambda-\lambda_*+A_{22}(\lambda,l)$ has the same number of zeros in the disk $\{\lambda : |\lambda-\lambda_*|<\delta\}$ as the function $\lambda\mapsto\lambda-\lambda_*$ does. The number δ is arbitrary, so we can conclude that there is a unique root of the equation (4.19) converging to λ_* as $l\to+\infty$. As a consequence, there exists a unique eigenvalue of the operator H converging to λ_* as $l\to+\infty$; we will denote this eigenvalue as $\lambda^-(l,\boldsymbol{a})$. The estimate (5.3) implies at the same time that

$$\lambda^-(l,\boldsymbol{a})-\lambda_*=\mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}). \tag{5.4}$$

Let us derive the asymptotic expansion (2.8) for the eigenvalue $\lambda^-(l,\boldsymbol{a})$. In order to do it, we will need to know the asymptotic behavior for A_{22} in a way more precise than (5.3). For the sake of brevity we will write shortly λ instead of $\lambda^-(l,\boldsymbol{a})$. The relations (5.2) together with the estimates (4.12), (5.4) imply that

$$\begin{aligned}A_{22}(\lambda,l)&=T_4^-(\mathrm{I}+T_9(\lambda,l,\boldsymbol{a}))^{-1}\boldsymbol{\phi}_*^-(\cdot,l)\\&=T_4^-\boldsymbol{\phi}_*^-(\cdot,l)-T_4^-T_9(\lambda,l,\boldsymbol{a})\boldsymbol{\phi}_*^-(\cdot,l)+\mathcal{O}(\|T_9\|^2\|\boldsymbol{\phi}_*^-\|)\\&=-T_4^-T_9(\lambda_*,l,\boldsymbol{a})\boldsymbol{\phi}_*^-(\cdot,l)+\mathcal{O}\left(|\lambda-\lambda_*|\left\|\frac{\partial T_9}{\partial\lambda}\right\|\|\boldsymbol{\phi}_*^-\|+\|T_9\|^2\|\boldsymbol{\phi}_*^-\|\right)\\&=-T_4^-T_8^-(\lambda,l,\boldsymbol{a})\phi_n^+(\cdot,l,a_-)+\mathcal{O}\left(|\lambda-\lambda_*|l^2\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}+l\mathrm{e}^{-6\kappa_1^+(\lambda_*)l}\right).\end{aligned} \tag{5.5}$$

Taking into account the estimate (3.14) for $\|T_8^-\|$ and the relation (5.1), we can proceed with the calculations obtaining

$$\begin{aligned}A_{22}(\lambda,l)=&-\tau c(\lambda,a_-)\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}\big(T_8^-(\lambda_*,l,a_+,a_-)\mathrm{e}^{-\kappa_1^+(\lambda_*)x_1},\psi_*\big)_{L_2(\gamma_{a_-})}\\&+\mathcal{O}\left(|\lambda-\lambda_*|l^2\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}+\mathrm{e}^{-2l(\kappa_1^+(\lambda_*)+\rho(\lambda_*))}\right),\end{aligned} \tag{5.6}$$

where we have denoted $\psi_*(x)=\psi_n(x,a_-)$. In view of the relation (3.18) the function $T_8^-(\lambda_*,l,a_+,a_-)\mathrm{e}^{\kappa_1^+(\lambda_*)x_1}$ coincides with $T_7^-(\lambda,l,a_+,a_-)u$, where u is the solution to the problem (3.1) with $a=a_+$, $\lambda=\lambda_*$, and $f=\mathrm{e}^{-\kappa_1^+(\lambda_*)x_1}$. It is clear that $u(x)=U(-x_1,x_2,\lambda_*,a_+)$, and in view of (2.3), (2.5), (2.6) we obtain

$$\begin{aligned}\big(T_8^-(\lambda_*,l,&a_+,a_-)\mathrm{e}^{-\kappa_1^+(\lambda_*)x_1},\psi_*\big)_{L_2(\gamma_{a_-})}\\&=c(\lambda_*,a_+)\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}\big(\mathrm{e}^{\kappa_1^+(\lambda_*)x_1},\psi_*\big)_{L_2(\gamma_{a_-})}+\mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l})\\&=\pi c(\lambda_*,a_+)c(\lambda_*,a_-)\kappa_1^+(\lambda_*)\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}+\mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l}).\end{aligned}$$

Substituting these identities into (5.6), we finally arrive at the following formula,

$$A_{22}(\lambda,l)=\mu^-(l,\boldsymbol{a})\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}+\mathcal{O}\left(|\lambda-\lambda_*|l^2\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}+\mathrm{e}^{-2l(\kappa_1^+(\lambda_*)+\rho(\lambda_*))}\right),$$

where $\mu^-(l,\boldsymbol{a})$ is defined by (2.9). It allows us to rewrite the equation (4.19) as

$$(\lambda-\lambda_*)\big(1+\mathcal{O}(l^2\mathrm{e}^{-4\kappa_1^+(\lambda_*)l})\big)=\mu^-(l,\boldsymbol{a})\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}+\mathcal{O}\left(\mathrm{e}^{-2l(\kappa_1^+(\lambda_*)+\rho(\lambda_*))}\right);$$

expressing $(\lambda-\lambda_*)$ from here we get the asymptotic expansion (2.8) and the formula (2.9).

Next we have to prove the asymptotic expansion for the eigenfunction associated with λ^-. The nontrivial solution of the equation (4.4) is given by (4.14) with $c_+ = 0$ and $c_- = 1$, i.e. as $\boldsymbol{f} = \boldsymbol{\Phi}_*^-$. We substitute it into the relation $u_+ = T_3(\lambda^+, a_+)f_+$ and take into account (5.2), (4.16); this yields

$$u_+ = T_3(\lambda^+, a_+)f_+ = \mathcal{O}(\|\boldsymbol{\phi}_*^-\|) = \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}),$$

which holds true in $W_2^1(\Pi_{a_+})$ and in $W_2^2(S)$ for each $S \in \Xi_{a_+}$. If $\lambda^- \neq \lambda_*$, we obtain similarly with the help of Lemma 3.3

$$u_- = T_3(\lambda^-, a_-)f_- = \frac{\psi_*}{\lambda^- - \lambda_*}T_4^- \boldsymbol{\Phi}_*^- + T_5^-(\lambda^-, a_-)f_- = \frac{A_{22}(\lambda^-, l)\psi_*}{\lambda^- - \lambda_*} + \mathcal{O}(\|\boldsymbol{\phi}_*^-\|). \tag{5.7}$$

Due to the equation (4.19) it follows that

$$u_- = -\psi_* + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l})$$

holds in $W_2^1(\Pi_a)$ and $W_2^2(S)$ for each $S \in \Xi_{a_-}$. If $\lambda^- = \lambda_*$, the last relation holds again; in order to prove it, it is sufficient to employ the identity

$$u_- = T_5(\lambda_*, a_-)f_- - c_-\psi_* = T_5(\lambda_*, a_-)f_- - \psi_*.$$

The relations obtained in this way together with (4.3) lead to (2.10). □

Proof of Theorem 2.3. The general lines of the proof are similar to those of the previous one. According to the results of the previous section the eigenvalues of H converging to λ_* are roots of the equation (4.22). First we will check that the function at the left-hand side of this equation has either two simple zeroes or one second-order zero converging to λ_* as $l \to +\infty$.

To this aim, we need to estimate the functions A_{ij}. Proposition 2.2 implies

$$\phi_m^-(x_1, l, a_+) = (-1)^m \tau c(\lambda_*, a_+)\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}\mathrm{e}^{-\kappa_1^+(\lambda_*)x_1} + \mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l}), \tag{5.8}$$

This formula together with (5.1) allow us to conclude that

$$A_{ij}(\lambda, l) = \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}), \tag{5.9}$$

hence for any small δ we have the inequality

$$|(\lambda - \lambda_*)\operatorname{tr} A(\lambda, l) + \det A(\lambda, l)| < \delta^2 \quad \text{as} \quad |\lambda - \lambda_*| = \delta,$$

if l is large enough. Since the functions A_{ij} are holomorphic, by Rouché theorem this inequality implies that the function $\lambda \mapsto D(\lambda, l) := \det\big((\lambda - \lambda_*)\mathrm{E} + \mathrm{A}(\lambda, l)\big)$ has the same number of zeroes (with the order taken into account) as the function $\lambda \mapsto (\lambda - \lambda_*)^2$ does. The last function has λ_* as a second-order zero, of course, so it follows that the function $D(\cdot,l)$ has either two simple zeroes or a second-order zero, converging to λ_* as $l\to+\infty$. In what follows we denote these roots as $\lambda^\pm$, the case of the second-order zero corresponds to the equality $\lambda^+=\lambda^-$.

As it was established in the previous section, the nontrivial solutions of the equation (4.4) associated with the roots of (4.22) are given by (4.14) with the coefficients $c_\pm$ solving the system of linear equations (4.20). If the numbers $\lambda^\pm$ solve (4.21), the system (4.20) has at least one nontrivial solution corresponding to λ^+ and λ^-.

Suppose that $\lambda^+\neq\lambda^-$. Then $\lambda^\pm$ are simple zeroes of the function $D(\cdot,l)$, and in view of the above discussion the system (4.20) has exactly one non-trivial solution for $\lambda=\lambda^+$ and $\lambda=\lambda^-$. Hence in the case $\lambda^+\neq\lambda^-$ the operator H has exactly two simple eigenvalues converging to λ_* as $l\to+\infty$.

Let us check that if the system (4.20) has two linear independent solutions referring to $\lambda=\lambda^\pm$ it follows that $\lambda^\pm$ is a second-order zero of the function $D(\cdot,l)$. Indeed, two linear independent solutions exist if and only if

$$A_{11}(\lambda^\pm,l)=A_{22}(\lambda^\pm,l)=\lambda_*-\lambda^\pm,\quad A_{12}(\lambda^\pm,l)=A_{21}(\lambda^\pm,l)=0. \tag{5.10}$$

The derivative of $D(\lambda,l)$ with respect to λ equals

$$\begin{aligned}\frac{\partial D}{\partial\lambda}(\lambda,l)=&2(\lambda-\lambda_*)+(A_{11}(\lambda,l)+A_{22}(\lambda,l))\\&+(\lambda-\lambda_*)\left(\frac{\partial A_{11}}{\partial\lambda}(\lambda,l)+\frac{\partial A_{22}}{\partial\lambda}(\lambda,l)\right)\\&+A_{11}(\lambda,l)\frac{\partial A_{22}}{\partial\lambda}(\lambda,l)-A_{12}(\lambda,l)\frac{\partial A_{21}}{\partial\lambda}(\lambda,l)\\&+A_{22}(\lambda,l)\frac{\partial A_{11}}{\partial\lambda}(\lambda,l)-A_{21}(\lambda,l)\frac{\partial A_{12}}{\partial\lambda}(\lambda,l).\end{aligned}$$

Substituting from (5.10) into this expression, we see that

$$\frac{\partial D}{\partial\lambda}(\lambda,l)=0\quad\text{as}\quad\lambda=\lambda^\pm,$$

thus $\lambda^\pm$ is a second-order zero.

It is more complicated to check existence of a double eigenvalue of the operator H if $\lambda^+=\lambda^-=:\widetilde{\lambda}$. It is equivalent to the fact that for $\lambda=\widetilde{\lambda}$ the system (4.20) has two linear independent solutions, and this in turn is equivalent to the relations (5.10). Let us prove that they hold. Consider the boundary value problem

$$\begin{gathered}(\Delta+\lambda)u=0,\quad x\in\Pi_a\setminus(\gamma_+\cup\gamma_-),\qquad u=0,\quad x\in\partial\Pi,\\ \left.\frac{\partial u}{\partial x_2}\right|_{x_2=+0}-\left.\frac{\partial u}{\partial x_2}\right|_{x_2=-0}=-g_\pm,\quad x\in\gamma_\pm.\end{gathered} \tag{5.11}$$

Here $g_\pm\in L_2(\gamma_\pm)$ are arbitrary functions, and the parameter λ is supposed to range in a small neighborhood of λ_* without coinciding with λ_* and $\widetilde{\lambda}$. This problem is uniquely solvable provided we seek a $L_2(\Pi)$-solution to (5.11). In a complete analogy with the proof of Lemma 4.1 one can check easily that the problem (5.11) is equivalent to the equation

$$\boldsymbol{f} + T_8(\lambda, l, \boldsymbol{a})\boldsymbol{f} = \boldsymbol{g}, \tag{5.12}$$

where $\boldsymbol{g} = (g_+, g_-) \in L_2(\gamma_{a_-}) \oplus L_2(\gamma_{a_+})$, while the solution u of (5.11) is given by

$$u(x, \lambda, l) = u_+(x_1 - l, x_2, \lambda, l) + u_-(x_1 + l, x_2, \lambda, l), \quad u_\pm := T_3(\lambda, a_\pm)f_\pm.$$

We can solve the equation (5.12) in the same way as the equation (4.10), obtaining as a result that

$$\boldsymbol{f} + \frac{T_4^+\boldsymbol{f}}{\lambda - \lambda_*}\boldsymbol{\Phi}_*^+ + \frac{T_4^-\boldsymbol{f}}{\lambda - \lambda_*}\boldsymbol{\Phi}_*^- = \boldsymbol{G}, \quad \boldsymbol{G} := \big(\mathrm{I} + T_9(\lambda, l, \boldsymbol{a})\big)^{-1}\boldsymbol{g}. \tag{5.13}$$

Hence the function $\boldsymbol{f}$ is of the form

$$\boldsymbol{f} = C_+\boldsymbol{\Phi}_*^+ + C_-\boldsymbol{\Phi}_*^- + \boldsymbol{G}, \tag{5.14}$$

where $C_\pm = C_\pm(\lambda, l)$ are constants to be found. Denoting $\boldsymbol{C} := \left(\begin{smallmatrix} C_+ \\ C_- \end{smallmatrix}\right)$ and substituting (5.14) into (5.13), we obtain an equation for $\boldsymbol{C}$,

$$\big((\lambda - \lambda_*)\mathrm{E} + \mathrm{A}(\lambda, l)\big)\boldsymbol{C} = \boldsymbol{h}, \quad \boldsymbol{h} := \begin{pmatrix} -T_4^+\boldsymbol{G} \\ -T_4^-\boldsymbol{G} \end{pmatrix}. \tag{5.15}$$

The solution of this system is given by Cramer's formula,

$$\begin{aligned} C_+(\lambda, l) &= \frac{A_{12}(\lambda, l)T_4^-\boldsymbol{G} - \big(\lambda - \lambda_* + A_{22}(\lambda, l)\big)T_4^+\boldsymbol{G}}{D(\lambda, l)}, \\ C_-(\lambda, l) &= \frac{A_{21}(\lambda, l)T_4^+\boldsymbol{G} - \big(\lambda - \lambda_* + A_{11}(\lambda, l)\big)T_4^-\boldsymbol{G}}{D(\lambda, l)}. \end{aligned} \tag{5.16}$$

Using now (5.15) and Lemma 3.3, we infer that

$$\begin{aligned} u_+(\cdot, \lambda, l) = &- C_+(\lambda, l)\psi_m(\cdot, a_+) + C_+(\lambda, l)T_5(\lambda, a_+)\Phi_{*,+}^+ \\ &+ C_-(\lambda, l)T_5(\lambda, a_+)\Phi_{*,+}^- + T_5(\lambda, a_+)G_+, \\ u_-(\cdot, \lambda, l) = &- C_-(\lambda, l)\psi_n(\cdot, a_-) + C_-(\lambda, l)T_5(\lambda, a_-)\Phi_{*,-}^+ \\ &+ C_-(\lambda, l)T_5(\lambda, a_-)\Phi_{*,-}^- + T_5(\lambda, a_-)G_-, \end{aligned} \tag{5.17}$$

where $\Phi_{*,\pm}^\pm$ and $G_\pm$ are the components of the vectors $\boldsymbol{\Phi}_*^\pm$ and $\boldsymbol{G}$,

$$\boldsymbol{\Phi}_*^\pm = (\Phi_{*,+}^\pm, \Phi_{*,-}^\pm), \quad \boldsymbol{G} = (G_+, G_-).$$

Since the number $\widetilde{\lambda}$ is a second-order zero of $D(\cdot, l)$, we conclude from (5.16) that the coefficients $C_\pm$ have, in general, a second-order pole at $\widetilde{\lambda}$, and the same is true for $u_\pm$. Taking into account (5.17), we conclude that the solution of (5.11) can be represented as

$$u(x,\lambda,l)=u_{-1}^{+}(x,\lambda,l)C_{+}(\lambda,l)+u_{-1}^{-}(x,\lambda,l)C_{-}(\lambda,l)+\mathcal{O}(1),\quad \lambda\to\widetilde{\lambda}. \tag{5.18}$$

In a complete analogy with the proof of Lemma 3.3 one can check easily that the solution of the problem (5.11) has a simple pole at $\widetilde{\lambda}$. Hence the function $u_{-1}^{+}(x,\lambda,l)C_{+}(\lambda,l)+u_{-1}^{-}(x,\lambda,l)C_{-}(\lambda,l)$ has a simple pole at $\widetilde{\lambda}$. For x from a neighborhood of γ_+ this function satisfies due to (5.16), (5.17) the relation

$$\begin{aligned}&D(\lambda,l)\left(u_{-1}^{+}(x,\lambda,l)C_{+}(\lambda,l)+u_{-1}^{-}(x,\lambda,l)C_{-}(\lambda,l)\right)\\&=\Big(\big(\lambda-\lambda_*+A_{22}(\lambda,l)\big)T_4^{+}\boldsymbol{G}-A_{12}(\lambda,l)T_4^{-}\boldsymbol{G}\Big)\psi_m^{+}(x_1-l,x_2)+\mathcal{O}(\mathrm{e}^{-2\kappa_1^{+}(\lambda_*)l}).\end{aligned}$$

Since $\widetilde{\lambda}$ is by assumption a second-order zero of $D(\cdot,l)$, the obtained identity yields that

$$\widetilde{\lambda}-\lambda_*+A_{22}(\widetilde{\lambda},l)=A_{12}(\widetilde{\lambda},l)=0.$$

Observing the behavior of the function u for x in the vicinity of γ_-, one can prove in the same way that

$$\widetilde{\lambda}-\lambda_*+A_{11}(\widetilde{\lambda},l)=A_{21}(\widetilde{\lambda},l)=0.$$

This completes the check of the relations (5.10) for $\lambda^+=\lambda^-$ showing that in this case the operator H has a double eigenvalue converging to λ_* as $l\to+\infty$.

We proceed to calculation of the asymptotic expansions for the root(s) of the equation (4.22). Substituting the estimates (5.9) into (4.22) we obtain

$$\lambda-\lambda_*=o(\mathrm{e}^{-\kappa_1^{+}(\lambda_*)l}). \tag{5.19}$$

This relation in combination with (5.1), (5.5) and the estimate (3.14) imply that

$$A_{22}(\lambda,l)=\mathcal{O}(l\mathrm{e}^{-4\kappa_1^{+}(\lambda_*)l}). \tag{5.20}$$

It is easy to establish an expression for A_{11} similar to (5.5), which together with (5.8) and (5.19) yield

$$A_{11}(\lambda,l)=\mathcal{O}(l\mathrm{e}^{-4\kappa_1^{+}(\lambda_*)l}). \tag{5.21}$$

Proceeding in the same way as in (5.5) we obtain a chain of relations,

$$\begin{aligned}A_{12}(\lambda,l)&=T_4^{+}(\mathrm{I}+T_9(\lambda,l,\boldsymbol{a}))^{-1}\boldsymbol{\phi}_*^{-}(\cdot,l)=T_4^{+}\boldsymbol{\phi}_*^{-}(\cdot,l)+\mathcal{O}(\|T_9\|\|\boldsymbol{\phi}_*^{-}\|)\\&=(\phi_n^{+}(\cdot,l,a_-),\psi_m(\cdot,a_+))_{L_2(\gamma_{a_+})}+\mathcal{O}(l\mathrm{e}^{-4\kappa_1^{+}(\lambda_*)l}).\end{aligned}$$

Due to (5.1) and (2.3) we have

$$(\phi_n^{+}(\cdot,l,a_-),\psi_m(\cdot,a_+))_{L_2(\gamma_{a_+})}$$

$$
\begin{aligned}
&= c(\lambda_*, a_-)\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}\big(\mathrm{e}^{-\kappa_1^+(\lambda_*)x_1}, \psi_m(\cdot, a_+)\big)_{L_2(\gamma_{a_+})} + \mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l}) \\
&= \mu(l, \boldsymbol{a})\mathrm{e}^{-2\kappa_1^+(\lambda_*)l} + \mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l}),
\end{aligned}
$$

where $\mu(l, \boldsymbol{a})$ is given by (2.12). Consequently,

$$
A_{12}(\lambda, l) = \mu(l, \boldsymbol{a})\mathrm{e}^{-2\kappa_1^+(\lambda_*)l} + \mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l} + l\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}), \tag{5.22}
$$

and in the same way one can show that

$$
A_{21}(\lambda, l) = \mu(l, \boldsymbol{a})\mathrm{e}^{-2\kappa_1^+(\lambda_*)l} + \mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l} + l\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}). \tag{5.23}
$$

The equation (4.22) is equivalent to the following pair of the equations,

$$
\lambda - \lambda_* = \frac{-\operatorname{tr} A(\lambda, l) \pm \sqrt{(A_{11}(\lambda, l) - A_{22}(\lambda, l))^2 + 4A_{12}(\lambda, l)A_{21}(\lambda, l)}}{2}. \tag{5.24}
$$

If $c(\lambda_*, a_-)c(\lambda_*, a_+) = 0$, these equations together with (5.20)–(5.23) imply that

$$
\lambda - \lambda_* = \mathcal{O}(\mathrm{e}^{-2\rho(\lambda_*)l} + l\mathrm{e}^{-4\kappa_1^+(\lambda_*)l}).
$$

which proves the asymptotic expansion (2.11) in the case $\mu(l, \boldsymbol{a}) = 0$.

Suppose on the contrary that $c(\lambda_*, a_-)c(\lambda_*, a_+) \neq 0$. In this case the function $(A_{11} - A_{22})^2 + 4A_{12}A_{21}$ is non-zero as $\lambda = \lambda_*$, and therefore its square root is holomorphic w.r.t. λ. Using this fact and the relations (5.20)–(5.23), one can show easily in analogy with the similar argument for the equation (4.22) that each of the equations (5.24) has a unique root converging to λ_* as $l \to +\infty$. Hence one of the roots of (4.22) satisfies the first of the equations (5.24), while the other satisfies the other one. Substituting now from (5.20)–(5.23) into (5.24), we arrive immediately at the asymptotics (2.11), (2.12) in the case $\mu(l, \boldsymbol{a}) \neq 0$. $\square$

Proof of Theorem 2.4. Let $\boldsymbol{c}$ be a nontrivial solution to the system (4.20), where λ is λ^+ or λ^-. Without loss of generality we may assume that $\|\boldsymbol{c}\|_{\mathbb{R}^2} = 1$. Modifying (4.14), we choose the corresponding nontrivial solution of the equation (4.4) as $\boldsymbol{f} = -c_+\boldsymbol{\Phi}_*^+ - c_-\boldsymbol{\Phi}_*^-$. In analogy with (5.7) we then obtain

$$
\begin{aligned}
u_- &= \frac{\psi_n(\cdot, a_-)}{\lambda - \lambda_*}T_4^-\boldsymbol{f} + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}) \\
&= -\frac{c_+A_{21}(\lambda, l) + c_-A_{22}(\lambda, l)}{\lambda - \lambda_*}\psi_n(\cdot, a_-) + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}),
\end{aligned}
$$

which holds true in $W_2^1(\Pi_{a_-})$ and $W_2^2(S)$ for each $S \in \Xi_{a_-}$. Employing now the system (4.20) we can write

$$
c_+A_{21}(\lambda, l) + c_-A_{22}(\lambda, l) = -c_-(\lambda - \lambda_*),
$$

hence

$$
u_- = c_-\psi_n(\cdot, a_-) + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}),
$$

and in the same way one can prove that

$$u_+ = c_+\psi_m(\cdot, a_+) + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}).$$

in the norm of $W_2^1(\Pi_{a_-})$ and $W_2^2(S)$ for each $S \in \Xi_{a_-}$. The last two relations prove the sought formulæ (2.15).

Suppose that $\lambda^+ = \lambda^-$, then (4.20) has two nontrivial solutions, which means that $(\lambda - \lambda_*)\mathrm{E} + \mathrm{A}(\lambda, l) = 0$; we can choose these solutions as $(c_+, c_-) = (-1, 0)$ and $(c_+, c_-) = (0, -1)$. Substituting these values into (2.15), we arrive at (2.14).

Suppose that $\mu(l, \boldsymbol{a}) \neq 0$. In view of (2.11) it implies that $\lambda^+(\lambda, a) \neq \lambda^-(\lambda, a)$, i.e. that $\lambda^\pm(\lambda, a)$ are simple eigenvalues. In this case the relations (2.11) and (5.21), (5.22) yield

$$\begin{aligned}
&\lambda^\pm - \lambda_* + A_{11}(\lambda^\pm, l) = \pm|\mu(l, \boldsymbol{a})|\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}(1 + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l})) \neq 0, \\
&A_{12}(\lambda^\pm, l) = \mu(l, \boldsymbol{a})\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}(1 + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l})) \neq 0.
\end{aligned} \tag{5.25}$$

Since the matrix $(\lambda^\pm - \lambda_*)\mathrm{E} + \mathrm{A}(\lambda^\pm, l)$ has rank one, we can choose nontrivial solutions of (4.20) as

$$\begin{aligned}
c_+^\pm &:= \pm\frac{\sqrt{2}(\lambda^\pm - \lambda_* + A_{11}(\lambda^\pm, l))}{\sqrt{(\lambda^\pm - \lambda_* + A_{11}(\lambda^\pm, l)^2 + A_{12}^2(\lambda^\pm, l)}}, \\
c_-^\pm &:= \pm\frac{\sqrt{2}A_{12}(\lambda^\pm, l)}{\sqrt{(\lambda^\pm - \lambda_* + A_{11}(\lambda^\pm, l)^2 + A_{12}^2(\lambda^\pm, l)}}.
\end{aligned}$$

In view of to (5.25) we then have

$$c_+^\pm = 1 + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}), \quad c_-^\pm = \mp \operatorname{sgn}\mu(l, \boldsymbol{a}) + \mathcal{O}(\mathrm{e}^{-2\kappa_1^+(\lambda_*)l}).$$

Substituting from here into (2.15) we arrive immediately at (2.13). □

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