PURELY ABSOLUTELY CONTINUOUS SPECTRUM FOR SOME RANDOM JACOBI MATRICES

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Dedicated to Stanislav Molchanov on the occasion of his 65th birthday

Abstract. We consider random Jacobi matrices of the form

$$(J_\omega u)(n) = a_n(\omega)u(n+1) + b_n(\omega)u(n) + a_{n-1}(\omega)u(n-1)$$

on $\ell^2(\mathbb{N})$, where $a_n(\omega) = \tilde{a}_n + \alpha_n(\omega)$, $b_n(\omega) = \tilde{b}_n + \beta_n(\omega)$, $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ are sequences of bounded variation obeying $\tilde{a}_n \to 1$ and $\tilde{b}_n \to 0$, and $\{\alpha_n(\omega)\}$ and $\{\beta_n(\omega)\}$ are sequences of independent random variables on a probability space $(\Omega, dP(\omega))$ obeying

$$\sum_{n=1}^{\infty} \int_\Omega (\alpha_n^2(\omega) + \beta_n^2(\omega))\, dP(\omega) < \infty$$

and $\int_\Omega \alpha_n(\omega)\, dP(\omega) = \int_\Omega \beta_n(\omega)\, dP(\omega) = 0$ for each n. We further assume that there exists $C_0 > 0$ such that $1/C_0 < a_n(\omega) < C_0$ and $-C_0 < b_n(\omega) < C_0$ for every n and P a.e. ω. We prove that, for P a.e. ω, J_ω has purely absolutely continuous spectrum on $(-2, 2)$.

1. Introduction

In this paper we study self-adjoint Jacobi matrices of the form

$$J(\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where $a_j > 0$ and $b_j \in \mathbb{R}$. We consider only such matrices whose entries are bounded, so they define bounded self-adjoint operators on $\ell^2(\mathbb{N})$. We say that a sequence $\{c_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is *of bounded variation* if $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$. We denote by J_0 the free Laplacian on $\ell^2(\mathbb{N})$, which is the Jacobi matrix of the form (1.1) with $a_n = 1$ and $b_n = 0$. Our main result in this paper is the following:

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Theorem 1.1. *Let* $(\Omega, dP(\omega))$ *be a probability space and let* $J_\omega = J(\{a_n(\omega)\}_{n=1}^\infty, \{b_n(\omega)\}_{n=1}^\infty)$*, where* $a_n(\omega) = \tilde{a}_n + \alpha_n(\omega)$ *and* $b_n(\omega) = \tilde{b}_n + \beta_n(\omega)$*. Assume that*

(i) $\{\tilde{a}_n\}_{n=1}^\infty$ *and* $\{\tilde{b}_n\}_{n=1}^\infty$ *are real-valued sequences of bounded variation obeying* $\lim_{n\to\infty} \tilde{a}_n = 1$ *and* $\lim_{n\to\infty} \tilde{b}_n = 0$*.*

(ii) $\{\alpha_n(\omega)\}_{n=1}^\infty$ *and* $\{\beta_n(\omega)\}_{n=1}^\infty$ *are sequences of real-valued independent random variables on* $(\Omega, dP(\omega))$ *obeying*
$$\int_\Omega \alpha_n(\omega)\, dP(\omega) = \int_\Omega \beta_n(\omega)\, dP(\omega) = 0$$
for each n *and*
$$\sum_{n=1}^\infty \int_\Omega (\alpha_n^2(\omega) + \beta_n^2(\omega))\, dP(\omega) < \infty\,.$$

(iii) *There exists a constant* $C_0 > 0$ *such that* $1/C_0 < a_n(\omega) < C_0$ *and* $-C_0 < b_n(\omega) < C_0$ *for every* n *and* P *a.e.* ω*.*

Then, for P *a.e.* ω*,* J_ω *has purely absolutely continuous spectrum on* $(-2, 2)$ *with essential support* $(-2, 2)$*.*

Remarks. 1. Saying that the essential support of the absolutely continuous spectrum of J_ω is $(-2, 2)$ means that the absolutely continuous part of the spectral measure of J_ω is supported on $(-2, 2)$ and gives positive weight to any subset of $(-2, 2)$ that has positive Lebesgue measure.

2. Our proof actually shows something a bit stronger (see Remark 3 to Theorem 2.1 below), namely, that for P a.e. fixed ω, the purity of the absolutely continuous spectrum in $(-2, 2)$ will be stable under changing any finite number of entries in J_ω.

Theorem 1.1 is essentially an extension of a result of Kiselev-Last-Simon [7, Theorem 8.1], who obtained this theorem for the special case $\tilde{a}_n = 1$, $\tilde{b}_n = 0$, $\alpha_n(\omega) = 0$ (also see [4, 8, 14] for related earlier results). Part of the extension is in considering the non-trivial off-diagonal part, rather than just $a_n(\omega) = 1$ for all n. While this extension is fairly straight forward, it does add some technical complexity to the problem (partly due to the fact that one-step transfer matrices depend on pairs of neighboring a_n's and are thus not independent of each other). The more important extension in Theorem 1.1 is in adding the decaying perturbation of bounded variation $J(\{\tilde{a}_n - 1\}_{n=1}^\infty, \{\tilde{b}_n\}_{n=1}^\infty)$.

We note that the Jacobi matrix $J(\{\tilde{a}_n\}_{n=1}^\infty, \{\tilde{b}_n\}_{n=1}^\infty)$, where $\{\tilde{a}_n\}_{n=1}^\infty$ and $\{\tilde{b}_n\}_{n=1}^\infty$ are sequences of bounded variation obeying $\lim_{n\to\infty} \tilde{a}_n = 1$ and $\lim_{n\to\infty} \tilde{b}_n = 0$, is well known to have purely absolutely continuous spectrum on $(-2,2)$ with essential support $(-2,2)$. That is, adding the decaying perturbation of bounded variation $J(\{\tilde{a}_n-1\}_{n=1}^\infty,\{\tilde{b}_n\}_{n=1}^\infty)$ to the free Laplacian J_0 doesn't change its absolutely continuous spectrum. This fact is the discrete version of Weidmann's theorem [17] (see, e.g., [16] for a proof). One may thus be tempted to think that such a perturbation of bounded variation may never be important for absolutely continuous spectrum. However, one of us have recently constructed [11] an example of a Jacobi matrix $J(\{a_n\}_{n=1}^\infty,\{b_n\}_{n=1}^\infty)$ with $a_n=1$, $b_n=\tilde{b}_n+\hat{b}_n$, so that $\lim_{n\to\infty}\tilde{b}_n=\lim_{n\to\infty}\hat{b}_n=0$, $\{\tilde{b}_n\}_{n=1}^\infty$ is of bounded variation, $J(\{a_n\}_{n=1}^\infty,\{\hat{b}_n\}_{n=1}^\infty)$ (like $J(\{a_n\}_{n=1}^\infty,\{\tilde{b}_n\}_{n=1}^\infty)$) has purely absolutely continuous spectrum on $(-2,2)$ with essential support $(-2,2)$, but $J(\{a_n\}_{n=1}^\infty,\{b_n\}_{n=1}^\infty)$ has empty absolutely continuous spectrum. In particular, adding a decaying perturbation of bounded variation to a Jacobi matrix can fully "destroy" its absolutely continuous spectrum (even though such a perturbation does not change the absolutely continuous spectrum when added to the free Laplacian J_0).

Theorem 1.1 is connected with a recent result of Breuer-Last [1] which says, roughly speaking, that absolutely continuous spectrum of Jacobi matrices which is associated with bounded generalized eigenfunctions is stable under square-summable random perturbations of the form $J(\{\alpha_n(\omega)\}_{n=1}^\infty,\{\beta_n(\omega)\}_{n=1}^\infty)$. In particular, their result imply that, with probability one, J_ω of Theorem 1.1 has absolutely continuous spectrum with essential support $(-2,2)$, but it doesn't exclude by itself the possibility of embedded singular spectrum in $(-2,2)$. If the probability distributions of the random variables $\{\alpha_n(\omega)\}_{n=1}^\infty$ and $\{\beta_n(\omega)\}_{n=1}^\infty$ happen to be absolutely continuous with respect to the Lebesgue measure (in fact, it suffices for only some of the distributions to be absolutely continuous, for example, absolute continuity of the distribution of $\beta_1(\omega)$ or of the distributions of any consecutive pair $\beta_n(\omega)$, $\beta_{n+1}(\omega)$ would suffice), then one can use spectral averaging (see, e.g., [15, Theorem I.8]) to conclude almost sure purely absolutely continuous spectrum in $(-2,2)$, namely, to recover Theorem 1.1 from the general result of [1]. The main ingredient of Theorem 1.1 which doesn't follow from the result of [1] is the purity of the absolutely continuous spectrum in $(-2,2)$ even in cases where all of the probability distributions of the random variables $\{\alpha_n(\omega)\}_{n=1}^\infty$ and $\{\beta_n(\omega)\}_{n=1}^\infty$ are singular with respect to the Lebesgue measure. This ingredient is connected with a key technical difference between the analysis of the current paper and [1]. Here we develop, following the original approach of [7], estimates which are uniform in energy over subintervals of $(-2,2)$ and which yield bounds on certain integrated (over energies) quantities. Such an approach cannot possibly work in the more general context considered by [1], where the obtained estimates are per individual energy and uniformity over energy ranges cannot hold. The current paper and [1] can thus be viewed as complementary to each other.

We note that much of the original Kiselev-Last-Simon result [7, Theorem 8.1] discussed above can be seen as a special case of the celebrated deterministic result of Deift-Killip [3], which has been considerably strengthened by Killip-Simon [6]. The Killip-Simon theorem says, among other things, that any Jacobi matrix J for which $J - J_0$ is Hilbert-Schmidt has absolutely continuous spectrum with essential support $(-2, 2)$ (this doesn't exclude singular spectrum embedded in $(-2, 2)$). Thus, by spectral averaging, one can recover [7, Theorem 8.1] for the case of absolutely continuous probability distributions. While various variants and extensions of the Killip-Simon result exist (see, e.g., [9, 10, 13, 18] and references therein), it seems that the sum-rule techniques underlying all these deterministic results cannot handle a general decaying perturbation of bounded variation, which may have an arbitrarily slow decay rate.

Stanislav Molchanov has been among the most important contributors to the theory of random Schrödinger operators ever since his seminal paper with Goldsheid and Pastur [5], which gave the first proof of Anderson localization for such an operator. It is a pleasure to dedicate this paper to him on the occasion of his 65th birthday. This research was supported in part by The Israel Science Foundation (Grant No. 188/02) and in part by Grant No. 2002068 from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

2. Proof of Theorem 1.1

Our proof of Theorem 1.1 relies on controlling the asymptotic growth of the norms of the 2×2 transfer matrices associated with the problem. These are defined by

$$\mathcal{T}_{n,m}(E, \omega) = \mathcal{T}_n(E, \omega)\mathcal{T}_{n-1}(E, \omega) \ldots \mathcal{T}_m(E, \omega) ,$$

where

$$\mathcal{T}_j(E, \omega) = \begin{pmatrix} (E - b_j(\omega))/a_j(\omega) & -a_{j-1}(\omega)/a_j(\omega) \\ 1 & 0 \end{pmatrix} .$$

We use the following result of [12]:

Theorem 2.1. *Suppose there is some $m \in \mathbb{N}$ so that*

$$\liminf_{n\to\infty} \int_a^b \|\mathcal{T}_{n,m}(E, \omega)\|^p \, dE < \infty$$

for some $p > 2$. Then (a, b) is in the essential support of the absolutely continuous spectrum of J_ω and the spectrum of J_ω is purely absolutely continuous on (a, b).

Remarks. 1. This is essentially Theorem 1.3 of [12]. While [12] only discusses Jacobi matrices with $a_n = 1$, the result easily extends to our more general context.

2. As noted in [12], this theorem is an extension of an idea of Carmona [2].

3. While the fact that (a, b) is in the essential support of the absolutely continuous spectrum isn't explicitly stated in [12, Theorem 1.3], this easily follows from spectral averaging and the fact that the $\liminf_{n\to\infty} \int_a^b \|\mathcal{T}_{n,m}(E, \omega)\|^p \, dE < \infty$ condition is invariant to changing any finite number of entries in the Jacobi matrix.

To prove Theorem 1.1 we fix an $\varepsilon > 0$ and use Theorem 2.1 on the interval $\Delta = (-2 + \varepsilon, 2 - \varepsilon)$. Since ε is arbitrary, Theorem 1.1 would follow.

One of the technical difficulties we need to overcome is the fact that each of the matrices $\mathcal{T}_n(E, \omega)$ depends on both $\alpha_n(\omega)$ and $\alpha_{n-1}(\omega)$. Thus, $\mathcal{T}_n(E, \omega)$'s for different n's are not independent. To overcome this, we introduce the matrices

$$\mathcal{A}_n(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & a_{n-1}(\omega) \end{pmatrix}$$

and define

$$\mathcal{P}_n(E, \omega) = \mathcal{A}_{n+1}(\omega)\mathcal{T}_n(E, \omega)\mathcal{A}_n^{-1}(\omega) = \begin{pmatrix} \frac{E - b_n(\omega)}{a_n(\omega)} & -1/a_n(\omega) \\ a_n(\omega) & 0 \end{pmatrix}.$$

Now,

$$\mathcal{P}_{n,m}(E, \omega) \equiv \mathcal{P}_n \mathcal{P}_{n-1} \ldots \mathcal{P}_m = \mathcal{A}_{n+1}(\omega)\mathcal{T}_{n,m}(E, \omega)\mathcal{A}_m^{-1}(\omega) ,$$

and since $\|\mathcal{A}_n\| \leq C_0$, $\|\mathcal{A}_n^{-1}\| \leq C_0$, it suffice to prove that for a.e. $\omega \in \Omega$,

$$\liminf_{n\to\infty} \int_\Delta \|\mathcal{P}_{n,m}(E, \omega)\|^4 \, dE < \infty .$$

We choose an m so that for every $n \geq m$, $\sqrt{\frac{1-\varepsilon/3}{1+\varepsilon/3}} < \tilde{a}_n$ and $|\tilde{b}_n| < \varepsilon/3$. Then for every $E \in \Delta$, $|E - \tilde{b}_n| < 2(1 - \varepsilon/3)$, hence $\left|\frac{E-\tilde{b}_n}{\tilde{a}_n}\right| < 2\sqrt{1 - \frac{\varepsilon^2}{9}}$, and we can define $k_n \in (0, \pi)$ by

$$2 \cos k_n = (E - \tilde{b}_n)/\tilde{a}_n .$$

Since $|\cos k_n| < \sqrt{1-\frac{\varepsilon^2}{9}}$, we have for every $n \geq m$ and $E \in \Delta$,

$$\sin k_n > \varepsilon/3. \tag{2.1}$$

We want to separate the deterministic and the random parts of $\mathcal{P}_n$. Since

$$\frac{E-\tilde{b}_n}{\tilde{a}_n} - \frac{E-b_n(\omega)}{a_n(\omega)} = \frac{(E-\tilde{b}_n)(a_n(\omega)-\tilde{a}_n) + \tilde{a}_n(b_n(\omega)-\tilde{b}_n)}{\tilde{a}_n a_n(\omega)}$$

$$= \frac{\alpha_n(\omega)}{a_n(\omega)} 2\cos k_n + \frac{\beta_n(\omega)}{a_n(\omega)},$$

we have

$$\frac{E-b_n(\omega)}{a_n(\omega)} = \frac{\tilde{a}_n}{a_n(\omega)}\left(2\cos k_n - \frac{\beta_n(\omega)}{\tilde{a}_n}\right)$$

and thus

$$\mathcal{P}_n(E,\omega) = \begin{pmatrix} \frac{E-b_n(\omega)}{a_n(\omega)} & -1/a_n(\omega) \\ a_n(\omega) & 0 \end{pmatrix}$$

$$= \frac{\tilde{a}_n}{a_n(\omega)} \begin{pmatrix} 2\cos k_n - \frac{\beta_n(\omega)}{\tilde{a}_n} & -1/\tilde{a}_n \\ a_n^2(\omega)/\tilde{a}_n & 0 \end{pmatrix}$$

$$= \frac{\tilde{a}_n}{a_n(\omega)} \left[\begin{pmatrix} 2\cos k_n & -1/\tilde{a}_n \\ \tilde{a}_n & 0 \end{pmatrix} + \frac{1}{\tilde{a}_n} \begin{pmatrix} -\beta_n(\omega) & 0 \\ a_n^2(\omega) - \tilde{a}_n^2 & 0 \end{pmatrix} \right].$$

To control the growth of transfer matrices, [7] uses the EFGP transform, which is connected with the following equality:

$$\begin{pmatrix} 0 & \sin k \\ 1 & -\cos k \end{pmatrix} \begin{pmatrix} 2\cos k & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix} \begin{pmatrix} 0 & \sin k \\ 1 & -\cos k \end{pmatrix}.$$

We use the following modification:

$$\begin{pmatrix} 0 & \sin k \\ \tilde{a} & -\cos k \end{pmatrix} \begin{pmatrix} 2\cos k & -1/\tilde{a} \\ \tilde{a} & 0 \end{pmatrix} = \begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix} \begin{pmatrix} 0 & \sin k \\ \tilde{a} & -\cos k \end{pmatrix}.$$

Denote

$$\mathcal{C}_n(E) = \begin{pmatrix} 0 & \sin k_n \\ \tilde{a}_n & -\cos k_n \end{pmatrix}, \quad \text{so} \quad \mathcal{C}_n(E)^{-1} = \begin{pmatrix} \cot k_n/\tilde{a}_n & 1/\tilde{a}_n \\ 1/\sin k_n & 0 \end{pmatrix},$$

$$\mathcal{B}_n(E) = \begin{pmatrix} \cos k_n & \sin k_n \\ -\sin k_n & \cos k_n \end{pmatrix}, \quad \mathcal{D}_n(E) = \begin{pmatrix} 2\cos k_n & -1/\tilde{a}_n \\ \tilde{a}_n & 0 \end{pmatrix}.$$

Based on the previous computations, we have

$$\mathcal{P}_n(E,\omega) = \mathcal{D}_n(E) + \mathcal{Q}_n(E,\omega)\,, \tag{2.2}$$

where

$$\mathcal{D}_n(E) = \mathcal{C}_n(E)^{-1}\mathcal{B}_n(E)\mathcal{C}_n(E) \tag{2.3}$$

and

$$\mathcal{Q}_n(E,\omega) = \frac{1}{a_n(\omega)}\left[-\alpha_n(\omega)\mathcal{D}_n(E) + \begin{pmatrix} -\beta_n(\omega) & 0 \\ a_n^2(\omega) - \tilde{a}_n^2 & 0 \end{pmatrix}\right]. \tag{2.4}$$

We define $\mathcal{D}_{i,j}(E) \equiv \mathcal{D}_i(E)\mathcal{D}_{i-1}(E)\dots\mathcal{D}_j(E)$ for $i \geq j$. These matrices define the deterministic part of $\mathcal{P}_{n,m}(E,\omega)$ and are uniformly bounded in the following sense:

Lemma 2.2. *There exists a constant C_Δ, such that for every $E \in \Delta$ and $n \geq m$, $\|\mathcal{D}_{n,m}(E)\| \leq C_\Delta$ and $\|\mathcal{D}_{n,m}(E)^{-1}\| \leq C_\Delta$.*

Proof. Since the determinant of $\mathcal{D}_{n,m}(E)$ is 1, we have $\|\mathcal{D}_{n,m}(E)\| = \|\mathcal{D}_{n,m}(E)^{-1}\|$ for every E. It is thus sufficient to consider $\|\mathcal{D}_{n,m}(E)\|$. From (2.3), we have $\mathcal{D}_{n,m} = \mathcal{C}_n^{-1}\mathcal{B}_n(\mathcal{C}_n\mathcal{C}_{n-1}^{-1})\mathcal{B}_{n-1}\dots\mathcal{C}_m$, and thus, since $\|\mathcal{B}_i\| = 1$, we get

$$\|\mathcal{D}_{n,m}\| \leq \left\|\mathcal{C}_n^{-1}\right\| \left(\prod_{j=m+1}^{n} \left\|\mathcal{C}_j\mathcal{C}_{j-1}^{-1}\right\|\right) \|\mathcal{C}_m\|\,.$$

Since

$$\left\|\mathcal{C}_j\mathcal{C}_{j-1}^{-1}\right\| = \left\|(\mathcal{C}_{j-1} + \mathcal{C}_j - \mathcal{C}_{j-1})\mathcal{C}_{j-1}^{-1}\right\| \leq 1 + \|\mathcal{C}_j - \mathcal{C}_{j-1}\| \left\|\mathcal{C}_{j-1}^{-1}\right\|\,,$$

$$|\cos k_j - \cos k_{j-1}| = \frac{1}{2}\left|(E - \tilde{b}_j)/\tilde{a}_j - (E - \tilde{b}_{j-1})/\tilde{a}_{j-1}\right|\,,$$

and

$$|\sin k_j - \sin k_{j-1}| = \frac{|\cos k_j + \cos k_{j-1}|}{|\sin k_j + \sin k_{j-1}|}|\cos k_j - \cos k_{j-1}|\,,$$

we see that for some constant C, uniformly on Δ,

$$\left\|\mathcal{C}_j\mathcal{C}_{j-1}^{-1}\right\| \leq 1 + C(|\tilde{a}_j - \tilde{a}_{j-1}| + |\tilde{b}_j - \tilde{b}_{j-1}|).$$

Using the fact that $1 + x \leq e^x$ for $x \geq 0$, we thus conclude that

$$\|\mathcal{D}_{n,m}(E)\| \leq \left\|\mathcal{C}_n^{-1}(E)\right\| \|\mathcal{C}_m(E)\| \exp\left(C\sum_{n=1}^{\infty}(|\tilde{a}_{n+1} - \tilde{a}_n| + |\tilde{b}_{n+1} - \tilde{b}_n|)\right)\,,$$

which is uniformly bounded on Δ. This proves the lemma. ∎

Proof of Theorem 1.1. If $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis in $\mathbb{R}^2$, then for any 2×2 matrix $\mathcal{M}$,

$$\|\mathcal{M}\| \leq \|\mathcal{M}\mathbf{e}_1\| + \|\mathcal{M}\mathbf{e}_2\| \,.$$

Thus, if we show that there exists a constant C, such that for every $\mathbf{e} \in \mathbb{R}^2$ with $\|\mathbf{e}\| = 1$ and every $n \geq m$,

$$\int_\Omega \int_\Delta \|\mathcal{P}_{n,m}(E,\omega)\mathbf{e}\|^4 \, dE \, dP(\omega) \leq C \,, \tag{2.5}$$

then it would follow that

$$\int_\Omega \int_\Delta \|\mathcal{P}_{n,m}(E,\omega)\|^4 \, dE \, dP(\omega) < 16C \,,$$

and, by Fatou's lemma, we will have

$$\int_\Omega \liminf_{n\to\infty} \int_\Delta \|\mathcal{P}_{n,m}(E,\omega)\|^4 \, dE \, dP(\omega) < 16C \,.$$

This would mean that, for P a.e. ω,

$$\liminf_{n\to\infty} \int_\Delta \|\mathcal{P}_{n,m}(E,\omega)\|^4 \, dE < \infty \,,$$

and, as discussed above, this would prove the theorem.

To prove the estimate (2.5), define $\mathbf{v}_l(E,\omega) = \mathcal{D}_{n,l}(E)\mathcal{P}_{l-1,m}(E,\omega)\mathbf{e}$. Using equation (2.2), we get

$$\begin{aligned}
\mathbf{v}_{l+1} &= \mathcal{D}_{n,l+1}(\mathcal{D}_l + \mathcal{Q}_l)\mathcal{P}_{l-1,m}\mathbf{e} \\
&= (\mathcal{I} + \mathcal{D}_{n,l+1}\mathcal{Q}_l\mathcal{D}_{n,l}^{-1})\mathbf{v}_l \\
&= (\mathcal{I} + \mathcal{R}_l)\mathbf{v}_l \,,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_l &\equiv \mathcal{D}_{n,l+1}\mathcal{Q}_l\mathcal{D}_{n,l}^{-1} \\
&= \frac{-\alpha_l}{a_l}\mathcal{I} + \frac{-\beta_l}{a_l}\mathcal{D}_{n,l+1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathcal{D}_{n,l}^{-1} + \frac{a_l^2 - \tilde{a}_l^2}{a_l}\mathcal{D}_{n,l+1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\mathcal{D}_{n,l}^{-1} \,.
\end{aligned}$$

We want to estimate the average of

$$\|\mathbf{v}_{l+1}\|^4 = \langle (\mathcal{I}+\mathcal{R}_l)\mathbf{v}_l, (\mathcal{I}+\mathcal{R}_l)\mathbf{v}_l \rangle^2 = (\|\mathbf{v}_l\|^2 + 2\langle \mathbf{v}_l, \mathcal{R}_l\mathbf{v}_l \rangle + \|\mathcal{R}_l\mathbf{v}_l\|^2)^2$$

$$= \|\mathbf{v}_l\|^4 + \|\mathcal{R}_l\mathbf{v}_l\|^4 + 2\|\mathbf{v}_l\|^2 \|\mathcal{R}_l\mathbf{v}_l\|^2 + 4\|\mathbf{v}_l\|^2 \langle \mathbf{v}_l, \mathcal{R}_l\mathbf{v}_l \rangle$$

$$+ 4\|\mathcal{R}_l\mathbf{v}_l\|^2 \langle \mathbf{v}_l, \mathcal{R}_l\mathbf{v}_l \rangle + 4\langle \mathbf{v}_l, \mathcal{R}_l\mathbf{v}_l \rangle^2 .$$

Since $\mathbf{v}_l(\omega)$ is independent of $\alpha_l(\omega)$ and $\beta_l(\omega)$, we have

$$\int_\Omega \|\mathbf{v}_l\|^2 \langle \mathbf{v}_l, \mathcal{R}_l\mathbf{v}_l \rangle \, dP(\omega) = \int_\Omega \frac{-\alpha_l}{a_l}\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^4 \, dP(\omega)$$

$$+ \int_\Omega \frac{-\beta_l}{a_l}\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^2 \left\langle \mathbf{v}_l, \mathcal{D}_{n,l+1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{D}_{n,l}^{-1}\mathbf{v}_l \right\rangle dP(\omega)$$

$$+ \int_\Omega \frac{a_l^2 - \tilde{a}_l^2}{a_l}\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^2 \left\langle \mathbf{v}_l, \mathcal{D}_{n,l+1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathcal{D}_{n,l}^{-1}\mathbf{v}_l \right\rangle dP(\omega) .$$

Since $1/C_0 < a_l(\omega) < C_0$, we see that

$$\frac{\alpha_l(\omega)}{a_l(\omega)} = \frac{\alpha_l(\omega)}{\tilde{a}_l} + O(\alpha_l^2(\omega)) ,$$

$$\frac{\beta_l(\omega)}{a_l(\omega)} = \frac{\beta_l(\omega)}{\tilde{a}_l} + O(\beta_l(\omega)\alpha_l(\omega)) ,$$

and

$$\frac{a_l^2(\omega) - \tilde{a}_l^2}{a_l(\omega)} = 2\alpha_l(\omega) + O(\alpha_l^2(\omega)) .$$

Thus, since $\alpha_l(\omega)$ and $\beta_l(\omega)$ are of zero mean, Lemma 2.2 implies that for some constant C, uniformly on Δ and in l,

$$\left| \int_\Omega \|\mathbf{v}_l\|^2 \langle \mathbf{v}_l, \mathcal{R}_l\mathbf{v}_l \rangle \, dP(\omega) \right| \le C \int_\Omega (\alpha_l^2 + \beta_l^2)\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^4 \, dP(\omega).$$

In a similar way, one can see that for a sufficiently large constant C,

$$\int_\Omega \langle \mathbf{v}_l, \mathcal{R}_l\mathbf{v}_l \rangle^2 \, dP(\omega) \le C \int_\Omega (\alpha_l^2 + \beta_l^2)\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^4 \, dP(\omega),$$

$$\int_\Omega \|\mathbf{v}_l\|^2 \|\mathcal{R}_l\mathbf{v}_l\|^2 \, dP(\omega) \le C \int_\Omega (\alpha_l^2 + \beta_l^2)\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^4 \, dP(\omega),$$

$$\int_\Omega \|\mathcal{R}_l \mathbf{v}_l\|^4 \, dP(\omega) \leq C \int_\Omega (\alpha_l^2 + \beta_l^2)\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^4 \, dP(\omega)$$

and

$$\left| \int_\Omega \|\mathcal{R}_l \mathbf{v}_l\|^2 \langle \mathbf{v}_l, \mathcal{R}_l \mathbf{v}_l \rangle \, dP(\omega) \right| \leq C \int_\Omega (\alpha_l^2 + \beta_l^2)\, dP(\omega) \int_\Omega \|\mathbf{v}_l\|^4 \, dP(\omega).$$

Thus, we obtain that for some constant C, uniformly on Δ and in l,

$$\int_\Omega \|\mathbf{v}_{l+1}\|^4 \, dP(\omega) \leq \left(1 + C \int_\Omega (\alpha_l^2 + \beta_l^2)\, dP(\omega)\right) \int_\Omega \|\mathbf{v}_l\|^4 \, dP(\omega).$$

Since

$$\sum_{n=1}^\infty \int_\Omega (\alpha_n^2(\omega) + \beta_n^2(\omega))\, dP(\omega) < \infty\,,$$

the inequality $1 + x \leq e^x$ for $x > 0$ implies that

$$\prod_{l=m}^n \left(1 + C \int_\Omega (\alpha_l^2(\omega) + \beta_l^2(\omega))\, dP(\omega)\right)$$

is uniformly (in m and n) bounded from above by some constant $\tilde{C}$, and so, uniformly on Δ and in n,

$$\int_\Omega \|\mathbf{v}_{n+1}(E,\omega)\|^4 \, dP(\omega) \leq \tilde{C} \int_\Omega \|\mathbf{v}_m(E,\omega)\|^4 \, dP(\omega)\,,$$

which implies

$$\int_\Delta \int_\Omega \|\mathbf{v}_{n+1}(E,\omega)\|^4 \, dP(\omega)\, dE \leq \tilde{C} \int_\Delta \int_\Omega \|\mathbf{v}_m(E,\omega)\|^4 \, dP(\omega)\, dE.$$

Thus, by Fubini's theorem, we see that

$$\int_\Omega \int_\Delta \|\mathbf{v}_{n+1}(E,\omega)\|^4 \, dE\, dP(\omega) \leq \tilde{C} \int_\Omega \int_\Delta \|\mathbf{v}_m(E,\omega)\|^4 \, dE\, dP(\omega).$$

Since $\mathbf{v}_{n+1}(E,\omega) = \mathcal{P}_{n,m}(E,\omega)\mathbf{e}$ and $\mathbf{v}_m(E,\omega) = \mathcal{D}_{n,m}(E)\mathbf{e}$, Lemma 2.2 implies that (2.5) holds. From this Theorem 1.1 follows. $\square$

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