On the inconsistency of the Bohm-Gadella theory with quantum mechanics

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Abstract. The Bohm-Gadella theory, sometimes referred to as the Time Asymmetric Quantum Theory of Scattering and Decay, is based on the Hardy axiom. The Hardy axiom asserts that the solutions of the Lippmann-Schwinger equation are functionals over spaces of Hardy functions. The preparation-registration arrow of time provides the physical justification for the Hardy axiom. In this paper, it is shown that the Hardy axiom is incorrect, because the solutions of the Lippmann-Schwinger equation do not act on spaces of Hardy functions. It is also shown that the derivation of the preparation-registration arrow of time is flawed. Thus, Hardy functions neither appear when we solve the Lippmann-Schwinger equation nor they should appear. It is also shown that the Bohm-Gadella theory does not rest on the same physical principles as quantum mechanics, and that it does not solve any problem that quantum mechanics cannot solve. The Bohm-Gadella theory must therefore be abandoned.

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1. Introduction

Recently, a new axiom for quantum mechanics, called the Hardy axiom, has been introduced [1]. This axiom is the bedrock of the Bohm-Gadella theory, which is meant to be a "*consistent and exact theory of quantum resonances and decay*" [2].

The Bohm-Gadella theory was initially formulated in the late 1970s [3–5], and it was a creative attempt to accommodate states for nonrelativistic resonances, the "Gamow vectors" [3–5]. A byproduct of the Bohm-Gadella theory was that the time evolution of the "Gamow vectors" is given by a semigroup rather than by a group, expressing time asymmetry at the microscopic level. The Bohm-Gadella theory was subsequently refined in [6–8]. With further improvements, it was summarized in [9]. The single most important mathematical piece of the Bohm-Gadella theory is the rigged Hilbert space of Hardy class.

In 1994, the Bohm-Gadella theory received a boost from the preparation-registration arrow of time [10, 11]. The preparation-registration arrow of time provides, by means of causality, the physical justification for the Hardy axiom and thus also for the Bohm-Gadella theory.

After several review articles, and after extending the theory to relativistic resonances [12], the assumptions of the Bohm-Gadella theory have been explicitly formulated as a new axiom for quantum mechanics, the Hardy axiom [1]. The state of the art of the Bohm-Gadella theory can be found in review [2].

If we denote the Lippmann-Schwinger bras by $\langle^{\pm}E|$, and the "in" and "out" wave functions by φ^+ and ψ^-, then the Hardy axiom states that $\langle^+E|\varphi^+\rangle \equiv \varphi^+(E)$ and $\langle^-E|\psi^-\rangle \equiv \psi^-(E)$ are Hardy functions. By using the Hardy axiom, the two major achievements of the Bohm-Gadella theory, the "Gamow vectors" and time asymmetry, are readily obtained.

Although its goals are worth pursuing, the Bohm-Gadella theory suffers from at least two major problems. First, the Bohm-Gadella theory is purely formal, that is, nobody has found a potential to which such a theory applies. The way the Bohm-Gadella theory works is by *presupposing* that the solutions of the Lippmann-Schwinger equation are distributions acting on two spaces of Hardy functions. In the Bohm-Gadella theory, the Lippmann-Schwinger equation is never solved, and one gets along by means of the Hardy-function assumption. Second, and more important, the content of the Hardy axiom is not a matter of assumption, but a matter of proof. The properties of the solutions of the Lippmann-Schwinger equation cannot be simply assumed; rather, one must solve the Lippmann-Schwinger equation and derive the properties of its solutions.

The present paper is devoted to a critical examination of the Bohm-Gadella theory. We will show that the solutions of the Lippmann-Schwinger equation do not comply with the Hardy axiom, thereby showing that the Bohm-Gadella theory is inconsistent with standard quantum mechanics. In order to show this, we will use the example of the spherical shell potential and the results of [13, 14], where that same potential was used to obtain the rigged Hilbert spaces that accommodate the Lippmann-Schwinger equation [13] and its analytic continuation [14].

As mentioned above, the justification for the Hardy axiom comes from causality, by way of the preparation-registration arrow of time. Since the preparation-registration arrow of time seems so compelling, one might still think that the Hardy axiom must be correct, and that there must be some subtlety missing in our analysis. To dispel any remaining doubts, we will show that the derivation of the preparation-registration arrow of time is actually flawed. Thus, neither Hardy functions appear when we solve the Lippmann-Schwinger equation, nor there is a physical reason why they should.

In addition, we will see that not only the Hardy axiom but also other aspects of the Bohm-Gadella theory are inconsistent with quantum mechanics. Furthermore, the two major achievements of the Bohm-Gadella theory, the resonance states and time asymmetry, can be achieved within standard quantum mechanics. Thus, the Bohm-Gadella theory is not only inconsistent with quantum mechanics, but also unnecessary.

In Sec. 2, we review the properties of Hardy functions. As well, we recall the precise statement of the Hardy axiom.

In Sec. 3, we show that the behavior of $\varphi^+(E)$ and $\psi^-(E)$ does not comply with the Hardy axiom. More specifically, we will show that whereas the Hardy axiom implies that the analytic continuation of $\varphi^+(E)$ (respectively $\psi^-(E)$) vanishes on the lower (respectively upper) infinite arc of the second sheet of the Riemann surface, the truly quantum mechanical $\varphi^+(E)$ and $\psi^-(E)$ blow up exponentially in that infinite arc.

In Sec. 4, we show that the derivation of the preparation-registration arrow of time is flawed.

In Sec. 5, we argue that the semiboundedness of the Hamiltonian seems to be the reason why the Hardy axiom does not apply in quantum mechanics.

In Secs. 6 and 7, we compare the most salient features of the Bohm-Gadella theory with standard quantum mechanics.

Before turning to the main body of the paper, we would like to make three remarks. First, although we will focus on the spherical shell potential for zero angular momentum, our results are valid for any partial wave and for a large class of potentials that include, in particular, potentials of finite range. The reason why our results are valid for such a large class of potentials is that, ultimately, such results depend on whether one can analytically continue the Jost and scattering functions into the whole complex plane. Since such continuation is possible for potentials that fall off at infinity faster than any exponential [15], our results remain valid for a whole lot of potentials. What is more, when the tails of the potential fall off slower than exponentials, the analytic properties of the Jost and scattering functions are even in more disagreement with the Hardy axiom. Second, although the results of this paper render the rigged Hilbert space of Hardy class unsuitable for the Lippmann-Schwinger equation, the rigged Hilbert spaces of [13, 14] lie at the heart of the foundations of the Lippmann-Schwinger equation. And third, it is not that the math of the Bohm-Gadella theory is wrong, it is rather that the math of the Bohm-Gadella theory is inconsistent with quantum mechanics.

2. Brief review of Hardy functions

In this section, we list the definition and main properties of functions of Hardy class.

2.1. General properties of Hardy functions

A Hardy function $f(z)$ on the upper half of the complex plane $\mathbb{C}^+$ is a function satisfying the following conditions [16–20]:

(i) It is analytic on the open upper half-plane, i.e., on the set of complex numbers with positive imaginary part.

(ii) For any value of $y > 0$, the integral

$$\int_{-\infty}^{+\infty} \mathrm{d}E \, |f(E + \mathrm{i}y)|^2 \tag{2.1}$$

converges.

(iii) For all $y > 0$, these integrals are bounded by the same constant K,

$$\sup_{y>0} \int_{-\infty}^{+\infty} \mathrm{d}E \, |f(E + \mathrm{i}y)|^2 < K \, . \tag{2.2}$$

The set of Hardy functions on the upper half-plane, often referred to as Hardy functions from above, is a linear space that we denote by $\mathcal{H}^2_+$.

Similarly, Hardy functions on the lower half-plane $\mathbb{C}^-$ are analytic on the open lower half-plane, and for these functions the conditions (2.1) and (2.2) hold with $y < 0$. We denote the linear space of Hardy functions from below by $\mathcal{H}^2_-$.

Any Hardy function $f(z)$ has a unique boundary value $f(E)$ on the real axis:

$$\lim_{y \to 0} f(E \pm \mathrm{i}y) = f(E) \,, \quad f \in \mathcal{H}^2_\pm \,. \tag{2.3}$$

The boundary value $f(E)$ is square integrable, and its squared norm is also bounded by K:

$$\int_{-\infty}^{+\infty} \mathrm{d}E \, |f(E)|^2 < K \,. \tag{2.4}$$

Thus, a function in $\mathcal{H}^2_\pm$ uniquely determines a square integrable function on $\mathbb{R}$.

An important theorem, due to Titchmarsh [20], states that Hardy functions can be recovered from their boundary values on the real line. If $f(E)$ is the function representing the boundary values of a Hardy function $f(z)$ on $\mathbb{C}^\pm$, then

$$f(z) = \pm \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \mathrm{d}E \frac{f(E)}{E - z} \,, \tag{2.5}$$

where the signs $(+)$ and $(-)$ correspond to Hardy functions on the upper and lower half-planes, respectively. This one-to-one correspondence between the $\mathcal{H}^2_\pm$ functions and their boundary values on $\mathbb{R}$ allows the identification of $f(z)$ with $f(E)$ for $f \in \mathcal{H}^2_\pm$.

Another important theorem on Hardy functions is that by Paley and Wiener [16–19, 21]. The theorem asserts that if $f \in \mathcal{H}^2_+$, then the Fourier transform of f,

$$\widetilde{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}E \, \mathrm{e}^{-\mathrm{i}Et} f(E) \,, \tag{2.6}$$

has the property $\widetilde{f}(t) = 0$ for $t < 0$. Similarly, if $f \in \mathcal{H}^2_-$, then the Fourier transform of f has the property $\widetilde{f}(t) = 0$ for $t > 0$. The Paley-Wiener theorem is the major reason why Hardy functions have been widely used in causality problems in which the frequency (energy) runs through the entire real line [15, 22].

A theorem due to van Winter [23] establishes that a Hardy function can be recovered from just its boundary values on the positive real axis $\mathbb{R}^+$. Whether the recovered function is an element of $\mathcal{H}^2_+$ or $\mathcal{H}^2_-$ is determined by means of the Mellin transform. Thus, if we call $\mathcal{H}^2_+\big|_{\mathbb{R}^+}$ the space of boundary values on $\mathbb{R}^+$ of the functions in $\mathcal{H}^2_+$, and $\mathcal{H}^2_-\big|_{\mathbb{R}^+}$ the space of boundary values on $\mathbb{R}^+$ of the functions in $\mathcal{H}^2_-$, we have the following bijection:

$$\Theta \mathcal{H}^2_+ = \mathcal{H}^2_+\big|_{\mathbb{R}^+} \,, \tag{2.7a}$$

$$\Theta \mathcal{H}^2_- = \mathcal{H}^2_-\big|_{\mathbb{R}^+} \,, \tag{2.7b}$$

where the image of any $f_\pm(E) \in \mathcal{H}^2_\pm$ by Θ is a function which is equal to $f_\pm(E)$ for $E \in \mathbb{R}^+$ and is not defined for negative values of E.

The following are among the other interesting properties of Hardy functions [20]:

(i) Let us define the Hilbert transform for an $L^2(\mathbb{R})$ function f as

$$\mathfrak{H}f(E) = \frac{1}{\pi}\,\mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}x\,\frac{f(x)}{x-E}\,, \tag{2.8}$$

where $\mathcal{P}$ denotes the Cauchy principal value. The Hilbert transform is linear and its image also lies in $L^2(\mathbb{R})$. A square integrable complex function $f(E)$, with real part $u(E)$ and imaginary part $v(E)$, belongs to $\mathcal{H}^2_\pm$ if and only if

$$\mathfrak{H}u = \pm v \quad \text{and} \quad \mathfrak{H}v = \mp u\,. \tag{2.9}$$

In particular, a Hardy function cannot be either real or purely imaginary on the whole real line.

(ii) From *(i)*, we immediately see that $f(E) \in \mathcal{H}^2_\pm$ if and only if its complex conjugate $f^*(E) \in \mathcal{H}^2_\mp$.

(iii) A function f of $\mathcal{H}^2_+$ (respectively $\mathcal{H}^2_-$) vanishes in the infinite arc of the upper (respectively lower) half-plane:

$$\lim_{z\to\infty} f(z) = 0\,, \qquad f \in \mathcal{H}_\pm\,,\ z \in \mathbb{C}_\pm\,. \tag{2.10}$$

More precisely, Hardy functions behave for large values of z as $1/\sqrt{z}$ (cf. [19]).

(iv) The spaces $\mathcal{H}^2_+$ and $\mathcal{H}^2_-$ have a trivial intersection,

$$\mathcal{H}^2_+ \cap \mathcal{H}^2_- = \{0\}\,. \tag{2.11}$$

However, the spaces of functions which are restrictions of Hardy functions to the positive semiaxis $\mathbb{R}^+$, $\mathcal{H}^2_+\big|_{\mathbb{R}^+}$ and $\mathcal{H}^2_-\big|_{\mathbb{R}^+}$, have a nontrivial intersection [9],

$$\mathcal{H}^2_+\big|_{\mathbb{R}^+} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+} \neq \{0\}\,. \tag{2.12}$$

Moreover, the intersection $\mathcal{H}^2_+\big|_{\mathbb{R}^+} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$.

2.2. The rigged Hilbert spaces of Hardy class

Two pairs of rigged Hilbert spaces of Hardy class form the backbone of the Bohm-Gadella theory. The first pair is given by [9]

$$\mathcal{S}\cap\mathcal{H}^2_\pm \subset \mathcal{H}^2_\pm \subset (\mathcal{S}\cap\mathcal{H}^2_\pm)^\times, \tag{2.13}$$

where $\mathcal{S}$ denotes the Schwartz space on the real line. The second pair is constructed as follows. We mentioned earlier that Hardy functions are determined by their values on the positive real axis plus a specification which says if they are Hardy on the upper or the lower half-planes. Thus, we have defined the spaces $\Theta\mathcal{H}^2_+ = \mathcal{H}^2_+\big|_{\mathbb{R}^+}$ and $\Theta\mathcal{H}^2_- = \mathcal{H}^2_-\big|_{\mathbb{R}^+}$. Now consider

$$\mathcal{S}\cap\mathcal{H}^2_+\big|_{\mathbb{R}^+} = \Theta(\mathcal{S}\cap\mathcal{H}^2_+)\,, \tag{2.14a}$$

$$\mathcal{S}\cap\mathcal{H}^2_-\big|_{\mathbb{R}^+} = \Theta(\mathcal{S}\cap\mathcal{H}^2_-)\,. \tag{2.14b}$$

The spaces $\mathcal{S}\cap\mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$ are dense in $L^2(\mathbb{R}^+)$ and yield the second pair of rigged Hilbert spaces of Hardy class [9]:

$$\mathcal{S}\cap\mathcal{H}^2_\pm\big|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+) \subset (\mathcal{S}\cap\mathcal{H}^2_\pm\big|_{\mathbb{R}^+})^\times. \tag{2.15}$$

2.3. The Hardy axiom

If we denote the Lippmann-Schwinger bras by $\langle^{\pm}E|$, and if we denote the "in" and the "out" wave functions by φ^{+} and ψ^{-}, then the Hardy axiom states that

$$\langle^{+}E|\varphi^{+}\rangle = \varphi^{+}(E) \in \mathcal{S}\cap\mathcal{H}^2_{-}\big|_{\mathbb{R}^+}\,, \tag{2.16}$$

$$\langle^{-}E|\psi^{-}\rangle = \psi^{-}(E) \in \mathcal{S}\cap\mathcal{H}^2_{+}\big|_{\mathbb{R}^+}\,. \tag{2.17}$$

By (2.10), these assumptions imply, in particular, that the analytic continuation of $\varphi^{+}(E)$ and $\psi^{-}(E)$ tend to zero in, respectively, the lower and upper infinite arcs of the second sheet of the Riemann surface:

$$\lim_{z\to\infty}\varphi^{+}(z)=0\,,\quad z\in\mathbb{C}_{\rm II}^{-}\,,\qquad \text{if } \varphi^{+}(E) \text{ was a Hardy function,} \tag{2.18}$$

$$\lim_{z\to\infty}\psi^{-}(z)=0\,,\quad z\in\mathbb{C}_{\rm II}^{+}\,,\qquad \text{if } \psi^{-}(E) \text{ was a Hardy function,} \tag{2.19}$$

where $\mathbb{C}_{\rm II}^{\pm}$ denote the upper (+) and the lower (−) half-planes of the second sheet. As we will see in the next section, in quantum mechanics these limits are not zero, and therefore $\varphi^{+}(E)$ and $\psi^{-}(E)$ are not Hardy functions.

3. The asymptotics of the Lippmann-Schwinger equation vs. the asymptotics of the Bohm-Gadella theory

We are going to see that the limits (2.18) and (2.19) are in general not zero by way of a simple counterexample. We will focus on the limit (2.18), since the same arguments apply to the limit (2.19).

The simple counterexample we will use is the spherical shell potential:

$$V(\mathbf{x})\equiv V(r)=\begin{cases} 0 & 0<r<a \\ V_0 & a<r<b \\ 0 & b<r<\infty\,, \end{cases} \tag{3.1}$$

where V_0 is a positive constant that determines the strength of the potential, and a and b determine the positions in between which the potential is nonzero.

We will work in the radial, position representation for zero angular momentum and take $\hbar^2/(2m)=1$. In this representation, the Hamiltonian H acts as

$$H=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+V(r)\,. \tag{3.2}$$

The Lippmann-Schwinger eigenfunctions are given by [13, 14, 24, 25]

$$\langle r|E^{\pm}\rangle \equiv \chi^{\pm}(r;E)=N(E)\,\frac{\chi(r;E)}{\mathcal{J}_{\pm}(E)}\,, \tag{3.3}$$

where $N(E)$ is a delta-normalization factor,

$$N(E)=\sqrt{\frac{1}{\pi}\frac{1}{\sqrt{E}}}\,, \tag{3.4}$$

$\chi(r;E)$ is the so-called regular solution,

$$\chi(r;E) = \begin{cases} \sin(\sqrt{E}\, r) & 0<r<a \\ \mathcal{J}_1(E)\mathrm{e}^{\mathrm{i}\sqrt{E-V_0}\, r} + \mathcal{J}_2(E)\mathrm{e}^{-\mathrm{i}\sqrt{E-V_0}\, r} & a<r<b \\ \mathcal{J}_3(E)\mathrm{e}^{\mathrm{i}\sqrt{E}\, r} + \mathcal{J}_4(E)\mathrm{e}^{-\mathrm{i}\sqrt{E}\, r} & b<r<\infty\,, \end{cases} \tag{3.5}$$

and $\mathcal{J}_\pm(E)$ are the Jost functions,

$$\mathcal{J}_+(E) = -2\mathrm{i}\mathcal{J}_4(E)\,, \tag{3.6a}$$

$$\mathcal{J}_-(E) = 2\mathrm{i}\mathcal{J}_3(E)\,. \tag{3.6b}$$

The explicit expressions for $\mathcal{J}_1$-$\mathcal{J}_4$ can be obtained by matching the values of $\chi(r;E)$ and of its derivative at the discontinuities of the potential.

If we denote the "in" wave function in the position representation by $\varphi^+(r)$, then the "in" wave function in the energy representation, $\varphi^+(E)$, is given by [13]

$$\varphi^+(E) = \int_0^\infty \mathrm{d}r\, \varphi^+(r)\overline{\chi^+(r;E)} = \int_0^\infty \mathrm{d}r\, \varphi^+(r)\chi^-(r;E)\,. \tag{3.7}$$

The analytic continuation of $\varphi^+(E)$ into the complex plane can be readily obtained by analytically continuing $\chi^-(r;E)$ [14]. The resulting analytic continuations are denoted by $\varphi^+(z)$ and $\chi^-(r;z)$:

$$\varphi^+(z) = \int_0^\infty \mathrm{d}r\, \varphi^+(r)\chi^-(r;z)\,. \tag{3.8}$$

In order to show that the limit (2.18) does not hold, we will compare that limit with the pace at which $\varphi^+(z)$ grows in the lower half-plane of the second sheet. From Eq. (3.8) and from the results of Ref. [14], we know that such a pace is determined by the falloff of $\varphi^+(r)$ as r tends to infinity and by the growth of $\chi^-(r;z)$, which we now obtain.

The growth of $\chi^-(r;z)$ is determined by the growths of the regular solution and of the inverse of $\mathcal{J}_-(z)$, as can be seen in Eq. (3.3). In its turn, the growth of the regular solution is given by the following estimate (see, for example, Eq. (12.6) in Ref. [24]):

$$|\chi(r;z)| \leq C\, \frac{|z|^{1/2}\, r}{1+|z|^{1/2}\, r}\, \mathrm{e}^{|\mathrm{Im}\sqrt{z}|r}\,, \quad z\in\mathbb{C}\,, \tag{3.9}$$

where C is a constant. In the lower half-plane of the second sheet, the growth of $1/\mathcal{J}_-(z)$ can be shown to be bounded by a constant [14]:

$$\left|\frac{1}{\mathcal{J}_-(z)}\right| \leq C\,, \quad z\in\mathbb{C}_{\mathrm{II}}^-\,. \tag{3.10}$$

The combination of the last two inequalities with Eq. (3.3) yields the growth of $\chi^-(r;z)$ in the lower half-plane of the second sheet:

$$\left|\chi^-(r;z)\right| \leq C\, \frac{|z|^{1/4}\, r}{1+|z|^{1/2}\, r}\, \mathrm{e}^{|\mathrm{Im}\sqrt{z}|r}, \quad z\in\mathbb{C}_{\mathrm{II}}^-\,. \tag{3.11}$$

Hence, when $z\in\mathbb{C}_{\mathrm{II}}^-$, $\chi^-(r;z)$ blows up *exponentially* as z tends to infinity.

By using the estimate (3.11) and the Gelfand-Shilov theory of M and Ω functions [26], one can obtain the growth of $\varphi^+(z)$ from the falloff of $\varphi^+(r)$, see [14]. The result is that if a and b are positive real numbers satisfying

$$\frac{1}{a}+\frac{1}{b}=1\,, \tag{3.12}$$

and if $\varphi^+(r)$ is an infinitely differentiable function whose tails fall off like $\mathrm{e}^{-r^a/a}$, then $\varphi^+(z)$ grows like $\mathrm{e}^{|\mathrm{Im}(\sqrt{z})|^b/b}$ in the infinite arc of the energy plane:

$$\text{If } \varphi^+(r)\sim \mathrm{e}^{-\frac{r^a}{a}} \text{ as } r\to\infty, \text{ then } \varphi^+(z)\sim \mathrm{e}^{\frac{|\mathrm{Im}(\sqrt{z})|^b}{b}} \text{ as } z\to\infty\,. \tag{3.13}$$

Thus, when in the position representation a function has tails, in the energy representation it blows up faster than any exponential, and therefore it cannot be a Hardy function, since the blowup (3.13) contradicts the limit (2.18) demanded by the Hardy axiom.

One could think of saving the limit (2.18) by imposing further restrictions on the falloff of $\varphi^+(r)$ at infinity, because such restrictions slow down the growth of $\varphi^+(z)$. The strongest restriction of this kind is to impose that $\varphi^+(r)$ be an infinitely differentiable function with compact support, $\varphi^+(r)\in C_0^\infty$. However, as we are going to see, functions of compact support do not comply with the limit (2.18) either.

Let us take $\varphi^+(r)\in C_0^\infty$ such that $\varphi^+(r)=0$ when $r>A$. Then,

$$\begin{aligned}
\left|\varphi^+(z)\right| &= \left|\int_0^\infty \mathrm{d}r\,\varphi^+(r)\chi^-(r;z)\right| && \text{by (3.8)} \\
&\leq \int_0^\infty \mathrm{d}r\, \left|\varphi^+(r)\right|\left|\chi^-(r;z)\right| \\
&= \int_0^A \mathrm{d}r\, \left|\varphi^+(r)\right|\left|\chi^-(r;z)\right| \\
&\leq C\,\frac{|z|^{1/4}A}{1+|z|^{1/2}A}\,\mathrm{e}^{|\mathrm{Im}\sqrt{z}|A}\int_0^A \mathrm{d}r\,\left|\varphi^+(r)\right| && \text{by (3.11)} \\
&\leq C\,\frac{|z|^{1/4}A}{1+|z|^{1/2}A}\,\mathrm{e}^{|\mathrm{Im}\sqrt{z}|A}\,;
\end{aligned} \tag{3.14}$$

that is, when $\varphi^+(r)\in C_0^\infty$, $\varphi^+(z)$ blows up *exponentially* in the infinite arc of $\mathbb{C}_{\mathrm{II}}^-$:

$$\text{If } |\varphi^+(r)|=0 \text{ when } r>A, \text{ then } \varphi^+(z)\sim \mathrm{e}^{A|\mathrm{Im}\sqrt{z}|} \text{ as } z\to\infty\,. \tag{3.15}$$

Quite a contrast to the limit (2.18).

Thus, no matter how small the tails of the wave functions in the position representation are, in the energy representation the wave functions blow up at least *exponentially* in the infinite arc of $\mathbb{C}_{\mathrm{II}}^-$, in contradiction to the limit (2.18)—the Hardy axiom is inconsistent with the Lippmann-Schwinger equation.

4. The preparation-registration arrow of time

The preparation-registration arrow of time provides the physical justification for the Hardy axiom. In this section, we will demonstrate that the derivation of the preparation-registration arrow of time is flawed, thereby demonstrating that the Hardy axiom lacks a physical basis.

The derivation of the preparation-registration arrow of time can be found, for example, in section 3 of Ref. [10] and in section III of Ref. [11], and it goes as follows. A scattering experiment consists of a preparation stage and a registration stage. In the preparation stage, a beam is prepared in an initial state $\varphi^{\rm in}$ before the interaction $V=H-H_0$ is turned on. The initial state $\varphi^{\rm in}$ evolves according to the free Hamiltonian H_0, $\varphi^{\rm in}(t)={\rm e}^{-{\rm i}H_0t}\varphi^{\rm in}$. When the beam reaches the interaction region, the free initial state $\varphi^{\rm in}$ turns into the exact "in" state φ^+, which evolves according to the total Hamiltonian H, $\varphi^+(t)={\rm e}^{-{\rm i}Ht}\varphi^+$. The beam then leaves the interaction region and ends up as the state $\varphi^{\rm out}$.

In the registration stage, the detector outside the interaction region does in general not detect $\varphi^{\rm out}$ but rather the overlap of $\varphi^{\rm out}$ with a final state $\psi^{\rm out}$. The state $\psi^{\rm out}$ evolves according to the free Hamiltonian and corresponds to an exact "out" state ψ^-. The "out" state ψ^- evolves according to the total Hamiltonian.

A well-known result of scattering theory is that the states $\varphi^{\rm in}$, φ^+ and $\psi^{\rm out}$, ψ^- satisfy

$$\varphi^{\rm in}(E)=\langle E|\varphi^{\rm in}\rangle=\langle^+E|\varphi^+\rangle=\varphi^+(E)\,, \tag{4.1}$$

$$\psi^{\rm out}(E)=\langle E|\psi^{\rm out}\rangle=\langle^-E|\psi^-\rangle=\psi^-(E)\,, \tag{4.2}$$

where $\langle E|$ are the eigenbras of H_0, and $\langle^\pm E|$ are the Lippmann-Schwinger bras.

One now takes $t=0$ as the time before which the preparation of $\varphi^{\rm in}$ is completed, and after which the registration of $\psi^{\rm out}$ begins. Then, as the mathematical statement for "no preparations for $t>0$," one takes

$$|\langle E|\varphi^{\rm in}(t)\rangle|=|\langle^+E|\varphi^+(t)\rangle|=0\,,\quad t>0\,, \tag{4.3}$$

for all energies, which implies

$$0=\int_{-\infty}^{+\infty}{\rm d}E\,\langle E|\varphi^{\rm in}(t)\rangle=\int_{-\infty}^{+\infty}{\rm d}E\,\langle^+E|\varphi^+(t)\rangle=\int_{-\infty}^{+\infty}{\rm d}E\,\langle^+E|{\rm e}^{-{\rm i}Ht}|\varphi^+\rangle\,, \tag{4.4}$$

or

$$0=\int_{-\infty}^{+\infty}{\rm d}E\,{\rm e}^{-{\rm i}Et}\varphi^+(E)\equiv\widetilde{\varphi}^+(t)\,,\quad \text{for } t>0\,. \tag{4.5}$$

As the mathematical statement for "no registrations for $t<0$," one takes

$$|\langle E|\psi^{\rm out}(t)\rangle|=|\langle^-E|\psi^-(t)\rangle|=0\,,\quad t<0\,, \tag{4.6}$$

for all energies, which implies

$$0=\int_{-\infty}^{+\infty}{\rm d}E\,\langle E|\psi^{\rm out}(t)\rangle=\int_{-\infty}^{+\infty}{\rm d}E\,\langle^-E|\psi^-(t)\rangle=\int_{-\infty}^{+\infty}{\rm d}E\,\langle^-E|{\rm e}^{-{\rm i}Ht}|\psi^-\rangle\,, \tag{4.7}$$

or

$$0=\int_{-\infty}^{+\infty}{\rm d}E\,{\rm e}^{-{\rm i}Et}\psi^-(E)\equiv\widetilde{\psi}^-(t)\,,\quad \text{for } t<0\,. \tag{4.8}$$

By the Paley-Wiener theorem, Eq. (4.5) implies that $\varphi^+(E) = \varphi^{\rm in}(E)$ is a Hardy function from below, and Eq. (4.8) implies that $\psi^-(E) = \psi^{\rm out}(E)$ is a Hardy function from above. Hence the claim that the Hardy axiom is rooted in causality principles.

We are now in a position to reveal the flaws of the preparation-registration arrow of time. For the sake of simplicity, we will focus on the "in" states, since the same arguments apply to the "out" states.

The first flaw lies in assumption (4.3). From such an assumption, it follows that

$$0 = \langle ^+E|\varphi^+(t)\rangle = {\rm e}^{-{\rm i}Et}\varphi^+(E) \tag{4.9}$$

for all energies. Hence,

$$0 = \varphi^+(E) \tag{4.10}$$

for all energies, which can happen only when φ^+ is identically 0. Thus, the preparation-registration arrow of time holds only in the meaningless case of the zero wave function.

Condition (4.5), on which condition the Hardy axiom rests, is weaker than the flawed assumption (4.3). One may then think of saving the Hardy axiom by assuming simply Eq. (4.5), which by the Paley-Wiener theorem is equivalent to assuming that $\varphi^+(E)$ is a Hardy function from below. This way of stating the Hardy axiom, which is the way used in Refs. [1,2], is tantamount to the causal condition "$\widetilde{\varphi}^+(t) = 0$ for $t > 0$." The problem is that this causal condition is dead-end, because the quantum mechanical time evolution of φ^+,

$${\rm e}^{-{\rm i}Ht}\varphi^+ = \int_0^\infty {\rm d}E\, {\rm e}^{-{\rm i}Et}|E^+\rangle\langle ^+E|\varphi^+\rangle \, , \tag{4.11}$$

is *different* from $\widetilde{\varphi}^+(t)$:

$$\varphi^+(t) = {\rm e}^{-{\rm i}Ht}\varphi^+ \ \neq \ \widetilde{\varphi}^+(t) \, . \tag{4.12}$$

It is clear from this equation that any statement on the the causal behavior of $\widetilde{\varphi}^+(t)$ will in general not apply to the true quantum mechanical evolved state $\varphi^+(t)$ (even if φ^+ were a Hardy function). Therefore, there is no reason why the solutions of the Lippmann-Schwinger equation should be related to Hardy functions.

One could have foreseen that the causality principles of the Hardy axiom are inconsistent with quantum mechanics by recalling that in standard scattering theory, a beam is prepared at $t = -\infty$, hits the target, and outgoes to infinity at $t = +\infty$, and therefore its time evolution lasts from $t = -\infty$ till $t = +\infty$.

Three remarks are in order here. (1) Because ${\rm e}^{-{\rm i}Ht}$ is unitary, ${\rm e}^{-{\rm i}Ht}\varphi^+ = 0$ if and only if $\varphi^+ \equiv 0$. Thus, there is little hope of having $\varphi^+(t) = 0$ for any $t > 0$, since such condition forces φ^+ to vanish. (2) If the argument behind the preparation-registration arrow of time was physically sound, it would apply to any quantum mechanical energy distribution, not just to those associated with the solutions of the Lippmann-Schwinger equation. Also, it would apply to any system, be it quantum or classical. (3) The entire real line of energies has been used in Eq. (4.5), whereas the quantum mechanical time evolution (4.11) uses the physical spectrum $[0, \infty)$.

5. The spectrum of the Hamiltonian is bounded from below

That the Paley-Wiener theorem makes use of the entire real line of energies through the Fourier transform (2.6) seems to indicate that Hardy functions fit systems where energy (frequency) integrations are taken over the whole real line, rather than quantum mechanical systems, the majority of which have spectra bounded from below. This important point is known to some authors. For example, Nussenzveig [15] and van Kampen [22] utilize $\mathcal{H}^2_\pm$ functions in causality problems of classical electromagnetic scattering, where the frequency integrals are taken over the entire real line. However, both Nussenzveig [15] and van Kampen [27] avoid the use of $\mathcal{H}^2_\pm$ functions in non-relativistic scattering, where the spectrum is bounded from below. In this section, we are going to delve into why Hardy functions are not suitable for systems whose spectrum is bounded from below.

We start by recalling a basic result of the spectral theory of linear operators on a Hilbert space. Let A denote a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $\mathrm{Sp}(A)$ denote its spectrum. For simplicity, let us assume that the spectrum of A is purely continuous. Associated with each $\lambda \in \mathrm{Sp}(A)$, there is a ket $|\lambda\rangle$ that is a (generalized) eigenvector of A:

$$A|\lambda\rangle = \lambda|\lambda\rangle \, . \tag{5.1}$$

In practical applications, Eq. (5.1) is solved in the position representation:

$$\langle x|A|\lambda\rangle = \lambda\langle x|\lambda\rangle \, . \tag{5.2}$$

This equation has in general solutions for any complex λ, even though the spectrum of a self-adjoint operator is real. The λ's of $\mathrm{Sp}(A)$ are distinguished from the rest of real and complex numbers by the fact that their corresponding eigenfunctions $\langle x|\lambda\rangle$ are polynomially bounded [28]. The solutions of Eq. (5.2) for $\lambda \in \mathrm{Sp}(A)$ can then be used to express the norm of a sufficiently smooth function f and the expectation of A in f:

$$(f,f) = \int_{\mathrm{Sp}(A)} \mathrm{d}\lambda \, \langle f|\lambda\rangle\langle\lambda|f\rangle = \int_{\mathrm{Sp}(A)} \mathrm{d}\lambda \, |f(\lambda)|^2 < \infty \, , \tag{5.3}$$

$$(f,Af) = \int_{\mathrm{Sp}(A)} \mathrm{d}\lambda \, \lambda\langle f|\lambda\rangle\langle\lambda|f\rangle = \int_{\mathrm{Sp}(A)} \mathrm{d}\lambda \, \lambda|f(\lambda)|^2 < \infty \, . \tag{5.4}$$

It is important to note that in these integrals, λ runs over the physical spectrum.

The eigenfunctions of the Hamiltonian (3.2) are given by Eq. (3.3). As Eq. (3.9) shows, the eigenfunctions (3.3) are polynomially bounded only when $E \geq 0$, and therefore the spectrum of H is the positive real line. Thus, when we apply the last two equations to H and φ^+, we obtain

$$(\varphi^+,\varphi^+) = \int_0^\infty \mathrm{d}E \, \langle\varphi^+|E^+\rangle\langle^+E|\varphi^+\rangle = \int_0^\infty \mathrm{d}E \, |\varphi^+(E)|^2 < \infty \, , \tag{5.5}$$

$$(\varphi^+,H\varphi^+) = \int_0^\infty \mathrm{d}E \, E\langle\varphi^+|E^+\rangle\langle^+E|\varphi^+\rangle = \int_0^\infty \mathrm{d}E \, E|\varphi^+(E)|^2 < \infty \, . \tag{5.6}$$

Note that in these two integrals, the energy ranges from 0 to ∞.

We now recall that the Hardy axiom implies that the following integrals over the entire real line are finite:

$$(\varphi^+, \varphi^+)_{\text{Hardy}} = \int_{-\infty}^{+\infty} \mathrm{d}E \, \langle \varphi^+ | E^+ \rangle \langle ^+E | \varphi^+ \rangle = \int_{-\infty}^{+\infty} \mathrm{d}E \, |\varphi^+(E)|^2 < K < \infty \,, \tag{5.7}$$

$$(\varphi^+, H\varphi^+)_{\text{Hardy}} = \int_{-\infty}^{+\infty} \mathrm{d}E \, E \langle \varphi^+ | E^+ \rangle \langle ^+E | \varphi^+ \rangle = \int_{-\infty}^{+\infty} \mathrm{d}E \, E |\varphi^+(E)|^2 < \infty \,. \tag{5.8}$$

From Eqs. (5.5)-(5.8) it follows that

$$(\varphi^+, \varphi^+)_{\text{Hardy}} = (\varphi^+, \varphi^+) + \int_{-\infty}^{0} \mathrm{d}E \, \langle \varphi^+ | E^+ \rangle \langle ^+E | \varphi^+ \rangle \,, \tag{5.9}$$

$$(\varphi^+, H\varphi^+)_{\text{Hardy}} = (\varphi^+, H\varphi^+) + \int_{-\infty}^{0} \mathrm{d}E \, E \langle \varphi^+ | E^+ \rangle \langle ^+E | \varphi^+ \rangle \,. \tag{5.10}$$

Thus, if the Hardy axiom held, the integrals over the negative real line would be finite. As we are going to see, this is not so.

When $\varphi^+(r) \in C_0^\infty$ and $E \in (-\infty, 0)$, the estimate (3.14) becomes

$$\left|\varphi^+(E)\right| \leq C \, \frac{|E|^{1/4} A}{1 + |E|^{1/2} A} \, \mathrm{e}^{|E|^{1/2} A} \,, \quad E < 0 \,, \tag{5.11}$$

and therefore $\varphi^+(E)$ blows up exponentially when E tends to $-\infty$,

$$\varphi^+(E) \sim \mathrm{e}^{|E|^{1/2} A} \,, \quad E \to -\infty \,. \tag{5.12}$$

In addition, as shown in Sec. 3, when $\varphi^+(r)$ has non-vanishing tails, the growth of $\varphi^+(z)$ outpaces (5.12). Thus, the following integrals do not converge for very many reasonable wave functions:

$$\int_{-\infty}^{0} \mathrm{d}E \, \langle \varphi^+ | E^+ \rangle \langle ^+E | \varphi^+ \rangle = \int_{-\infty}^{0} \mathrm{d}E \, |\varphi^+(E)|^2 = \infty \,, \tag{5.13}$$

$$\int_{-\infty}^{0} \mathrm{d}E \, E \langle \varphi^+ | E^+ \rangle \langle ^+E | \varphi^+ \rangle = \int_{-\infty}^{0} \mathrm{d}E \, E |\varphi^+(E)|^2 = \infty \,, \tag{5.14}$$

and

$$(\varphi^+, \varphi^+)_{\text{Hardy}} = \infty \,, \tag{5.15}$$

$$(\varphi^+, H\varphi^+)_{\text{Hardy}} = \infty \,, \tag{5.16}$$

in contradiction with the Bohm-Gadella assumptions (5.7) and (5.8).

Note that if the spectrum of H were the entire real line, the Bohm-Gadella assumptions (5.7)-(5.8) would hold, because in such case the integrals in Eqs. (5.3)-(5.4) would be taken over the full real line. Hence, the Hardy axiom might be suitable for systems whose spectrum is the entire real line, although it is definitely not suitable for systems whose spectrum is bounded from below. Nevertheless, even if the Hardy axiom were allowed for Hamiltonians whose spectrum is the whole real line, such Hamiltonians are very rear, and they cannot be used in scattering theory, because the (continuous) spectrum of a scattering Hamiltonian coincides with the spectrum of the kinetic energy operator, which is $[0, \infty)$.

The advocates of the Bohm-Gadella theory may argue that in their theory, they integrate over the physical spectrum, negative-energy integrals such as those in (5.13) and (5.14) appear only by analytic continuation, and such negative-energy integrals are actually taken over the negative energy real line of the second sheet. The problem is that, although there is nothing wrong with analytically continuing from the physical spectrum into the negative real line of the second sheet, the resulting integrals are not convergent, as shown above. Assuming that they are convergent is in contradiction with the semiboundedness of the Hamiltonian.

6. The Bohm-Gadella theory vs. standard quantum mechanics

The Bohm-Gadella theory deviates from standard quantum mechanics in many essential aspects. In this section, we point out the most important ones.

In the Bohm-Gadella theory, the time evolution of the Lippmann-Schwinger bras $\langle^{\pm}E|$ and kets $|E^{\pm}\rangle$ is given by semigroups [29], whereas in standard quantum mechanics it is given by a group [13]. In standard quantum mechanics, only the time evolution of the analytic continuation of $\langle^{\pm}E|$ and $|E^{\pm}\rangle$ is given by a semigroup [14].

In the Bohm-Gadella theory, $\langle^{+}E|$ and $|E^{-}\rangle$ are analytically continued from the upper rim of the cut into the lower half-plane of the second sheet, and $\langle^{-}E|$ and $|E^{+}\rangle$ are continued from the lower rim of the cut into the upper half-plane of the second sheet [2]. By contrast, in standard quantum mechanics, $\langle^{\pm}E|$ and $|E^{\pm}\rangle$ are all continued from the upper rim of the cut into the whole complex plane [14].

In the Bohm-Gadella theory, the Lippmann-Schwinger bras and kets act on two different rigged Hilbert spaces [9]. In standard quantum mechanics, the Lippmann-Schwinger bras and kets act, in the position representation, on one and the same rigged Hilbert space [13].

In contrast to scattering theory, which seeks asymptotic completeness, the Bohm-Gadella theory forgoes asymptotic completeness:

> "[$\cdots$] *we change just one axiom (Hilbert space and/or asymptotic completeness) to a new axiom which distinguishes between (in-)states and (out)observables using Hardy spaces.*" (See the abstract and page 2324 of [2].)

The Bohm-Gadella theory still uses the Møller operators and the S matrix, although it is not explained how it is possible to have them without asymptotic completeness.

In the Bohm-Gadella theory,

> "[$\cdots$] *the Hamiltonian operator and boundary conditions alone do not specify a resonance state.*" (See [2], page 2332.)

By contrast, in standard resonance theory, the Gamow states are obtained by solving the Schrödinger equation subject to purely outgoing boundary conditions [30].

In the Bohm-Gadella theory, the Lippmann-Schwinger equation is never solved, and the properties of its solutions are assumed and promoted to an axiom. This (Hardy) axiom becomes the cornerstone of the theory:

> "*The unified theory of resonances and decay requires this new axiom.*" (See [2], page 2333.)

By contrast, as shown in the present paper, the Hardy axiom is not only unnecessary but also inconsistent with the standard quantum mechanical theory of resonances and decay.

On top of the above problems, the Bohm-Gadella theory lacks a simple quantum mechanical scattering system to which it applies.

7. The "Gamow vectors" and time asymmetry

The Bohm-Gadella theory was devised to accommodate resonance states in a well-defined manner. Time asymmetry was a byproduct. Since resonance states and time asymmetry are two important aspects of quantum mechanics, and since the Bohm-Gadella theory has turned out to be flawed, one may wonder if those two aspects can be described within ordinary quantum mechanics. Fortunately, they can.

Time asymmetry is perfectly accounted for by means of (advanced and retarded) propagators, see for example [31] and also [14].

The resonance states originally introduced by Gamow do not need the Hardy axiom to be well defined. Moreover, as will be shown in a forthcoming paper, the resonance states of the Bohm-Gadella theory, which are defined as

$$|z_{\mathrm{R}}^{-}\rangle \equiv \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}E\, \frac{|E^{-}\rangle}{E-z_{\mathrm{R}}}\,, \tag{7.1}$$

where z_{R} is the resonance energy, are not the same as Gamow's original state. In addition, the "Gamow vectors" (7.1) make use of the whole real line of energies. Thus, they do not make physical sense, because they assign a physical meaning to the decay of a resonance state $|z_{\mathrm{R}}^{-}\rangle$ into a scattering state $|E^{-}\rangle$ of negative energy,

$$\langle^{-}E|z_{\mathrm{R}}^{-}\rangle = \frac{\mathrm{i}}{2\pi}\frac{1}{E-z_{\mathrm{R}}}\,, \tag{7.2}$$

where E belongs to the second sheet for $E<0$. By contrast, quantum mechanical decay only occurs into energies of the physical spectrum. Once again, the negative energies are used inappropriately.

8. Conclusions

A thorough examination of the Bohm-Gadella theory has revealed its inconsistency with standard quantum mechanics:

- Contrary to the Hardy axiom, the wave functions $\varphi^{+}(E)$ and $\psi^{-}(E)$ are not Hardy functions.

- The derivation of the preparation-registration arrow of time is flawed.
- The resonance states of the Bohm-Gadella theory are not the same as Gamow's original states.
- That $\varphi^+(E)$ and $\psi^-(E)$ are not Hardy functions seems to stem from the semiboundedness of the Hamiltonian, because Hardy functions seem more adequate to describe causality in systems where energy (frequency) integrations are taken over the whole real line.
- The Bohm-Gadella theory does not rest on the same principles as quantum mechanics.
- The major achievements of the Bohm-Gadella theory, namely time asymmetry and the rigorous construction of resonance states, can be achieved rigorously within standard quantum mechanics.

The Bohm-Gadella theory must therefore be abandoned. Not abandoning the Bohm-Gadella theory would force us "*to bend the rules of standard quantum mechanics*" (see the abstract of [32]), which is unnecessary, because standard quantum mechanics is capable of describing scattering, decay and time asymmetry in a consistent manner.

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