Fiber Hamiltonians in the non-relativistic quantum electrodynamics

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July 7, 2006

Abstract

A translation invariant Hamiltonian H in the nonrelativistic quantum electrodynamics is studied. This Hamiltonian is decomposed with respect to the total momentum $P_{\rm T}$:

$$H = \int_{\mathbb{R}^d}^{\oplus} H(P) dP,$$

where the self-adjoint fiber Hamiltonian $H(P)$ is defined for arbitrary values of coupling constants. It is discussed a relationship between rotation invariance of $H(P)$ and polarization vectors, and functional integral representations of n point Euclidean Green functions of $H(P)$ is given. From these, some applications concerning with degeneracy of ground states, ground state energy and expectation values of suitable observables with respect to ground states are given.

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1 Introduction and statements of results

In this paper we investigate self-adjoint operator $H(P)$ indexed by $P \in \mathbb{R}^d$ by means of a functional integral representation of $e^{-tH(P)}$. Operator $H(P)$ is derived from a translation invariant self-adjoint operator H acting in a Hilbert space $\mathcal{H}$ such that

$$[H, P_{\mathrm{T}j}] = 0, \quad j = 1, ..., d, \tag{1.1}$$

where $P_{\mathrm{T}} = (P_{\mathrm{T}1}, ..., P_{\mathrm{T}d})$ denotes the d-tuple of the total momentum operators with $\sigma(P_{\mathrm{T}j}) = \mathbb{R}$. Here $\sigma(T)$ denotes the spectrum of T. Hence $\mathcal{H}$ and H can be represented by constant fiber direct integrals:

$$\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{H}(P) dP, \quad H \cong \int_{\mathbb{R}^d}^{\oplus} H(P) dP. \tag{1.2}$$

In this paper we study the so-called Pauli-Fierz model in the nonrelativistic quantum electrodynamics, which describes an interaction between a quantum mechanical particle (electron) and a quantized radiation field. The Hamiltonian H of this system is defined as a self-adjoint operator minimally coupled to the quantized radiation field, which acts in $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\mathrm{b}}$, where $\mathcal{F}_{\mathrm{b}}$ is a boson Fock space. We impose an ultraviolet cutoff on H and work in the Coulomb gauge with $d-1$ polarization vectors. It is seen that H without external potentials is translation invariant. Namely H satisfies (1.1) for some total momentum operators, from which $\mathcal{H}$ and H can be decomposed such as (1.2). We shall show that $\mathcal{H}(P)$ is unitarily equivalent to $\mathcal{F}_{\mathrm{b}}$ and $H(P)$ is realized as a self-adjoint operator acting in $\mathcal{F}_{\mathrm{b}}$ for each $P \in \mathbb{R}^d$.

1.1 Statements of results

In this article we shall investigate (1) functional integral representations, (2) self-adjointness and essential self-adjointness, (3) ergodic properties of $e^{-tH(0)}$, (4) rotation invariance, (5) energy inequalities and (6) measures associated with ground states. (2),(3),(5) and (6) are studied through the functional integration (1).

(1) Functional integrations:
The polaron model $H_{\rm polaron}(P)$ is a typical example of fiber Hamiltonians, which is studied in [52, 51] by functional integrals. In [52, Appendix] the functional integral representation of $(\Omega, e^{-tH_{\rm polaron}(P)}\Omega)$ is shown, where Ω denotes a vacuum vector in a boson Fock space. Our motivation to construct (1.4) below comes from this. In [17, 20, 28] a functional integral representation of $(F, e^{-tH}G)_{\mathcal{H}}$ is given. It is the main achievement in this paper that a functional integral representation of $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}}$ is constructed for arbitrary total momentum $P \in \mathbb{R}^d$ and arbitrary values of coupling constants on a probability space $W \times Q_1$ equipped with the product measure, $db \otimes d\mu_1$, where db is a measure associated with the particle and $d\mu_1$ with the quantized radiation field. Although for example, it can be taken $C([0,\infty);\mathbb{R}^d)$ as W and the direct sum of the set of real Schwarz distributions, $\oplus^d \mathcal{S}'_{\rm real}(\mathbb{R}^{d+1})$, as Q_1, we do not specify them in this paper. See e.g., [17, 28] for a detail. Moreover a functional integral representation of an n point Euclidean Green function of the form

$$(\Phi_0, \prod_{j=1}^n e^{-(s_j-s_{j-1})K} e^{-(t_j-t_{j-1})H(P_{j-1})}\Phi_j)_{\mathcal{F}_{\rm b}} \tag{1.3}$$

is also given. Here K denotes a second quantized operator and Φ_j, $j = 1, ..., n-1$, bounded multiplication operators. Since the interaction of the Pauli-Fierz Hamiltonian is introduced as a minimal coupling, we need a Hilbert space-valued stochastic integral to construct the functional integral representation of $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}}$. Actually we show that

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}} = \int_{W\times Q_1} \overline{\Psi_0}\Phi_t e^{-i\int_0^t \mathcal{A}_1 db(s)} e^{iP\cdot b(t)} db\otimes d\mu_1, \tag{1.4}$$

where the right-hand side above is in the Schrödinger representation instead of the Fock representation, $(b(s))_{0\le s}$ the d dimensional Brownian motion with respect to db and Ψ_0, Φ_t denotes some vectors. The integral $\int_0^t \mathcal{A}_1 db(s) = \sum_{\mu=1}^d \int_0^t \mathcal{A}_{\mu,s} db_\mu(s)$ denotes a Hilbert space-valued stochastic integral. See Section 3 for details. As far as we know it is the first time to give functional integral representations explicitly such as (1.4) of a fiber Hamiltonian minimally coupled to a quantized radiation field.

(2) Self-adjointness and essential self-adjointness:
In [30, 31], applying a functional integral representation, we established the self-adjointness of H for arbitrary values of coupling constants. The self-adjointness of $H(P)$ follows from that of H, which was done in [39]. In this paper as an application of the functional integral representation, we show the essential self-adjointness of a more singular operator $K(P)$, $P \in \mathbb{R}^d$, which is defined as $H(P)$ without the free field Hamiltonian $H_{\rm f}$, is shown in Theorem 2.3. The idea is to find an invariant domain by using (1.4).

(3) Ergodic properties and the uniqueness of ground states: The multiplicity of the ground state of the Pauli-Fierz Hamiltonian is estimated in e.g., [32, 35]. In [35] for a sufficiently small coupling constant, the uniqueness of the ground state of $H(P)$ is proved. We want to extend it for an arbitrary values of coupling constants. One merit of working in the Schrödinger representation is to define the positive cone:

$$\mathcal{K}_+ = \{\Psi \in \mathcal{H} | \Psi \geq 0\}, \quad \mathcal{K}_+^0 = \{\Psi \in \mathcal{H} | \Psi > 0\} \subset \mathcal{K}_+.$$

We say that bounded operator T is positivity preserving if and only if $T\mathcal{K}_+ \subset \mathcal{K}_+$ and positivity improving if and only if $T[\mathcal{K}_+ \setminus \{0\}] \subset \mathcal{K}_+^0$. We discuss some positivity properties of $e^{-tH(0)}$. One important translation invariant model is the so-called Nelson model $H_{\rm N}(P)$ [45] acting in a Hilbert space $\mathcal{H}_N$ with a fixed total momentum $P \in \mathbb{R}^d$. In a simpler way than (1.4) we can also give the functional integral representation of $e^{-tH_{\rm N}(P)}$. Since the interaction term of the Nelson model is linear, the integrand of the functional integral representation of $(\Psi, e^{-tH_{\rm N}(P)}\Phi)_{\mathcal{H}_N}$ is given by a Riemann integral instead of the stochastic integral as in (1.4) with the form

$$(\Psi, e^{-tH_{\rm N}(P)}\Phi)_{\mathcal{H}_N} = \int_{W \times Q_N} \overline{\Psi_0} \Phi_t e^{-\int_0^t \phi ds} e^{iP\cdot b(t)} db \otimes d\mu_N, \tag{1.5}$$

where $(Q_N, d\mu_N)$ is some probability space. In [25], by a positivity preserving argument and a hypercontractivity argument, the uniqueness of ground state of $H_{\rm N}(0)$ is proven. See also [50, 11]. Since $e^{-\int_0^t \phi ds} e^{iP\cdot b(t)}$ is strictly positive for $P = 0$ in the Schrödinger representation, it can be shown that, by (1.5), $e^{-tH_{\rm N}(0)}$ is positivity improving. From this it can be also concluded that the ground state of $H_{\rm N}(0)$ is unique due to the infinite dimensional version of the Perron Frobenius theorem.

Although we want to apply the Perron Frobenius theorem to the Pauli-Fierz model, we can not apply directly the positivity argument as for the Nelson model (1.5), since $e^{-i\int_0^t \mathcal{A}_1 db(s)}$ in (1.4) can not be only positive but also real. Let us consider the multiplication operator $T_t = e^{itx}$, $t \in \mathbb{R}$, in $L^2(\mathbb{R}_x)$. Although T_t is not positivity preserving operator, $\mathbb{F}T_t\mathbb{F}^{-1}$, where $\mathbb{F}$ denotes the Fourier transformation on $L^2(\mathbb{R})$, turns out to be a shift operator, i.e., $(f, \mathbb{F}T_t\mathbb{F}^{-1}g)_{L^2(\mathbb{R})} = (f, g(\cdot + t))_{L^2(\mathbb{R})} \geq 0$ for nonnegative functions f and g. Then $\mathbb{F}T_t\mathbb{F}^{-1}$ is a positivity preserving operator, but *not* a positivity improving operator:

$$\begin{array}{ccc} T_t = e^{itx} & \longrightarrow & \mathbb{F}T_t\mathbb{F}^{-1} \\ \text{multiplication} & & \text{shift} \end{array}$$

This idea was applied to H in [29]. In this paper we also do this for $H(0)$. We can show that $\vartheta e^{-i\int_0^t \mathcal{A}_1 db(s)}\vartheta^{-1}$ is positivity preserving for some unitary operator ϑ discussed in [27, 29], which corresponds to the Fourier transformation on $\mathcal{F}_{\rm b}$. Actually $\vartheta = \exp(i(\pi/2)N)$, where N denotes the number operator. Hence we can see that by the functional integral representation (1.4), $\vartheta e^{-tH(0)}\vartheta^{-1}$ is a positivity *improving* operator in Theorem 3.5, i.e.,

$$\vartheta e^{-tH(0)}\vartheta^{-1}[\mathcal{K}_+ \setminus \{0\}] \subset \mathcal{K}_+^0.$$

As a corollary, the uniqueness of the ground state of $H(0)$ is shown for arbitrary values of coupling constants if it exists.

(4) Rotation invariance and the degeneracy of ground states: Operator $H(P)$ has also a rotational symmetry. When the Hamiltonian includes a spin, a lower bound of the multiplicity, M, of ground states can be estimated by using this rotational symmetry. Let us add a spin to H which is denoted by H_σ. It can be shown that H_σ with suitable polarization vectors is rotation invariant around some unit vector $n \in \mathbb{R}^3$, which is inherited to operator $H_\sigma(P)$ with fixed total momentum P acting in $\mathbb{C}^2 \otimes \mathcal{F}_\mathrm{b}$. Then we shall see that $H_\sigma(P)$ is also decomposed with respect to the spectrum of the generator of the rotation around n, namely

$$H_\sigma(P) \cong H_\sigma(|P|n) = \bigoplus_{z\in\mathbb{Z}_{1/2}} H_\sigma(P,z), \quad \mathbb{C}^2 \otimes \mathcal{F}_\mathrm{b} = \bigoplus_{z\in\mathbb{Z}_{1/2}} \mathcal{F}_\mathrm{b}(z), \tag{1.6}$$

where $\cong$ denotes an unitary equivalence and $\mathbb{Z}_{1/2}$ the set of half integers. Although for a sufficiently small coupling constant, $M \geq 2$ is established in [35], applying the decomposition (1.6) and [48], we can see that $M \geq 2$ for arbitrary values of coupling constants. See Corollary 2.13.

(5) Energy inequalities: As is seen in (1.4), P dependence on the integrand is just the exponent of the phase: $e^{iP\cdot b(t)}$. Trivial bound $|e^{iP\cdot b(t)}| \leq 1$ and $|\vartheta e^{-i\int_0^t \mathcal{A}_1 db(s)}\vartheta^{-1}\Psi| \leq \vartheta e^{-i\int_0^t \mathcal{A}_1 db(s)}\vartheta^{-1}|\Psi|$ are useful to estimate the ground state energy of $H(P)$ from below. Then it can be shown that $\inf\sigma(H(0)) \leq \inf\sigma(H(P))$ and $\inf\sigma(H(0)) \leq \inf\sigma(H)$. See Corollary 3.8.

(6) Measures associated with ground states: In [18, 19] spectral properties of the translation invariant model including the Nelson model and the Pauli-Fierz model are investigated, in which mainly the renormalized Nelson model with nonrelativistic or relativistic kinematic term is studied. See also [12, 21, 41, 42, 22, 54]. In [18], it is shown that a ground state of the fiber Hamiltonian of the Pauli-Fierz model exists for all values of coupling constants but $|P| < P_0$ with some P_0 for a massive case. In [13], it is extended to a massless case. Although the existence problem of ground states mentioned above is solved, it is not constructive. In [10] functional integrals are applied to study properties of ground state φ_g of the Nelson model, in which $(\varphi_\mathrm{g}, \mathcal{O}\varphi_\mathrm{g})_{\mathcal{H}_N}$ with suitable operator $\mathcal{O}$ is represented as

$$(\varphi_\mathrm{g}, \mathcal{O}\varphi_\mathrm{g})_{\mathcal{H}_N} = \int_{C(\mathbb{R};\mathbb{R}^d)} f_{\mathcal{O}}(q)d\mu_\infty(q) \tag{1.7}$$

with some function $f_{\mathcal{O}}$ and a probability measure $d\mu_\infty$ on $C(\mathbb{R};\mathbb{R}^d)$. This measure is constructed by taking an infinite time limit of the form (1.3). In this paper, we do not construct such a measure, since it is not easy to control the stochastic integral appeared in (1.4). Instead of this, as is studied in [23, Theorem 3.4.1], we construct a sequence of measures

$$\{e^{iP\cdot b(2t)}d\mu_{2t}\}_{t>0}$$

converging, in some sense, to $(\varphi_{\rm g}(P), \mathcal{O}\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}}$ with a ground state $\varphi_{\rm g}(P)$ of $H(P)$. Actually due to a double stochastic integral it has *informally* expressed as

$$d\mu_{2t} = \frac{1}{Z}\exp\left(-\frac{e^2}{4}\sum_{\alpha,\beta=1}^{d}\int_0^{2t}db_\alpha(s)\int_0^{2t}db_\beta(s')W_{\alpha\beta}(s-s', b(s)-b(s'))\right)db \tag{1.8}$$

and

$$W_{\alpha\beta}(t,x) = \int_{\mathbb{R}^d}(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2})\frac{|\hat{\varphi}(k)|}{\omega(k)}e^{-|s|\omega(k)}e^{-ikx}dk.$$

See Corollary 4.5 and Remark 4.6 for a detail. The properties of measure $d\mu_{2t}$ with $\int db_\alpha(s)\int db_\beta(s')$ replaced by $\int ds \int ds'$, which corresponds to the measure associated with the ground state of the Nelson model, is discussed in [7, 8, 40]. We shall discuss the existence of measures such as (1.8) on a continuous path space in [33].

1.2 Remarks and plan of the paper

Recently the spectral properties of a general version of the Pauli-Fierz model with a fixed total momentum is studied in [39] where the self-adjointness and energy inequalities are also shown. See also [43, 44] for some recent development for the massive Nelson model, and [1, 2, 3, 4, 48] for a relativistic model. The effective mass $m_{\rm eff}$ is defined by the inverse of the Hessian of the ground state energy $E(P)$ of a fiber Hamiltonian $H(P)$ at $P = 0$, i.e., $m_{\rm eff}^{-1} = \partial^2 E(P)/\partial|P|^2\lceil_{P=0}$. For the effective mass of the Pauli-Fierz model without infrared cutoff is studied in [14, 5], and its renormalization in e.g., [26, 34, 36, 38]. In this paper we do not discuss a relationship between the effective mass and functional integrals. See [9, 52] to this direction for the Nelson model. See [53] as a review of the recent development on this area.

This paper is organized as follows. In Section 2 we define the Pauli-Fierz Hamiltonian $H(P)$ for an arbitrary total momentum $P \in \mathbb{R}^d$ and an arbitrary coupling constant, and discuss a relationship between rotation invariance and polarization vectors. Moreover we introduce an operator $K(P)$ defined by $H(P)$ without the free Hamiltonian $H_{\rm f}$ of the field. In Section 3 we construct a functional integral representation of $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}}$ and show some applications including the diamagnetic inequality, the positivity improvingness of $\vartheta e^{-tH(0)}\vartheta^{-1}$ and the essential self-adjointness of $K(P)$. Section 4 is devoted to extending the functional integral representation to an n point Euclidean Green function and to giving applications.

2 The Pauli-Fierz Hamiltonian

2.1 Preliminaries and notations

Let us assume that an electron moves in the d dimensional space and is polarized to $d-1$ directions. Let $\mathcal{F}_{\rm b}$ be the Boson Fock space over $\mathcal{W} := \oplus^{d-1} L^2(\mathbb{R}^d)$, i.e.,

$$\mathcal{F}_{\rm b} := \bigoplus_{n=0}^{\infty} {\mathcal{F}_{\rm b}}^{(n)} := \bigoplus_{n=0}^{\infty} [\otimes_s^n \mathcal{W}],$$

where $\otimes_s^n \mathcal{W}$ denotes the n-fold symmetric tensor product of Hilbert space $\mathcal{W}$, i.e., $\otimes_s^n \mathcal{W} := S_n(\otimes^n W)$ with $\otimes_s^0 \mathcal{W} := \mathbb{C}$. Here S_n symmetrizes $\otimes^n W$, i.e.,

$$S_n(f_1 \otimes \cdots \otimes f_n) := \frac{1}{n!} \sum_{\sigma \in \wp_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},$$

where $\wp_n$ denotes the set of permutations of degree n. In this paper we denote the norm and the scalar product on a Hilbert space $\mathcal{K}$ by $\|f\|_{\mathcal{K}}$ and $(f,g)_{\mathcal{K}}$, respectively. The scalar product is linear in g and antiliner in f. Unless confusions arise we omit $\mathcal{K}$. $\mathcal{F}_{\rm b}$ can be identified with the set of ℓ_2-sequences $\{\Psi^{(n)}\}_{n=0}^{\infty}$ with $\Psi^{(n)} \in {\mathcal{F}_{\rm b}}^{(n)}$ such that $\sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2_{{\mathcal{F}_{\rm b}}^{(n)}} < \infty$ and $\mathcal{F}_{\rm b}$ is the Hilbert space endowed with the scalar product $(\Psi, \Phi)_{\mathcal{F}_{\rm b}} = \sum_{n=0}^{\infty} (\Psi^{(n)}, \Phi^{(n)})_{{\mathcal{F}_{\rm b}}^{(n)}}$. $\Omega = \{1, 0, 0, ...\} \in \mathcal{F}_{\rm b}$ is called as the Fock vacuum. The annihilation operator and the creation operator on $\mathcal{F}_{\rm b}$ are denoted by $a(f)$ and $a^*(f)$, $f \in W$, respectively, which are defined by

$$(a^*(f)\Psi)^{(n)} := \sqrt{n} S_n(f \otimes \Psi^{(n-1)})$$

with the domain

$$D(a^*(f)) := \{\{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm b} | \sum_{n=1}^{\infty} n \|S_n(f \otimes \Psi^{(n-1)})\|^2_{{\mathcal{F}_{\rm b}}^{(n)}} < \infty\},$$

and $a(f) = (a^*(\bar{f}))^*$. Since the creation operator and the annihilation operator are closable, we take their closed extension and denote them by the same symbols. Let $\mathcal{F}_{\rm b,fin}$ be the so-called finite particle subspace of $\mathcal{F}_{\rm b}$ defined by

$$\mathcal{F}_{\rm b,fin} := \{\{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm b} | \Psi^{(m)} = 0 \text{ for all } m \geq \exists M\}.$$

The annihilation operator and the creation operator leave $\mathcal{F}_{\rm b,fin}$ invariant and satisfy the canonical commutation relations on it:

$$[a(f), a^*(g)] = (\bar{f}, g)1, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0.$$

For $f = (f_1, ..., f_{d-1}) \in \oplus^{d-1} L^2(\mathbb{R}^d)$, we informally write $a^\sharp(f)$, where $a^\sharp$ stands a or a^*, as $a^\sharp(f) = \displaystyle\sum_{j=1}^{d-1} \int a^\sharp(k,j) f_j(k) dk$ with informal kernel $a^\sharp(k,j)$. Let T be a contraction operator on $L^2(\mathbb{R}^d)$. Then the contraction linear operator $\Gamma([T]_{d-1})$ on $\mathcal{F}_\mathrm{b}$ is defined by

$$\Gamma([T]_{d-1}) := \bigoplus_{n=0}^\infty \otimes^n [T]_{d-1},$$

where $[T]_\ell := \underbrace{T \oplus \cdots \oplus T}_{\ell}$. Unless confusions arise we write $\Gamma(T)$ for $\Gamma([T]_{d-1})$. For a self-adjoint operator h on $\mathcal{W}$, $\{\Gamma(e^{ith})\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{F}_\mathrm{b}$. Then by the Stone theorem [46], there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathcal{F}_\mathrm{b}$ such that

$$\Gamma(e^{ith}) = e^{itd\Gamma(h)}, \quad t \in \mathbb{R}.$$

$d\Gamma(h)$ is called as the second quantization of h. For a self-adjoint operator h in $L^2(\mathbb{R}^d)$, $d\Gamma([h]_{d-1})$ is simply denoted by $d\Gamma(h)$ unless confusion arises. The number operator is defined by $N := d\Gamma(1)$. Let

$$\omega(k) = |k|$$

be the multiplication operator on $L^2(\mathbb{R}^d)$. Define the free Hamiltonian H_f on $\mathcal{F}_\mathrm{b}$ by $H_\mathrm{f} := d\Gamma(\omega)$. The quantized radiation field $A_{\hat{\varphi}_\mu}(x)$, $x \in \mathbb{R}^d$, $\mu = 1, ..., d$, with a form factor φ is defined by

$$A_{\hat{\varphi}_\mu}(x) = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int e_\mu(k,j) \left(\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} a^*(k,j) e^{-ik\cdot x} + \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} a(k,j) e^{ik\cdot x} \right) dk,$$

which acts on $\mathcal{F}_\mathrm{b}$. Here $e(k,1), \cdots, e(k,d-1)$ denote generalized polarization vectors satisfying $k \cdot e(k,j) = 0$ and $e(k,i) \cdot e(k,j) = \delta_{ij} 1$, $i,j = 1, ..., d-1$, and $\hat{\varphi}$ is the Fourier transform of form factor φ given by $\hat{\varphi}(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \varphi(x) e^{-ik\cdot x} dx$. Note that

$$\sum_{j=1}^{d-1} e_\alpha(k,j) e_\beta(k,j) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} := \delta^\perp_{\alpha\beta}(k), \quad \alpha, \beta = 1, ..., d.$$

Throughout this paper we assume (A) below.

(A) *Form factor $\hat{\varphi}$ satisfies that $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$ and $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$.*

$A_{\hat{\varphi}_\mu}(x)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{b,fin}}$, and its unique self-adjoint extension is denoted by the same symbol. The Hilbert space $\mathcal{H}$ of state vectors for the total system under consideration is given by the tensor product Hilbert space:

$$\mathcal{H} := L^2(\mathbb{R}^d_x) \otimes \mathcal{F}_\mathrm{b}.$$

Under the identification of $\mathcal{H}$ with the set of $\mathcal{F}_\mathrm{b}$-valued L^2-functions on $\mathbb{R}^d$, i.e.,

$$\mathcal{H} \cong \int^\oplus_{\mathbb{R}^d} \mathcal{F}_\mathrm{b} dx, \tag{2.1}$$

we define the self-adjoint operator $A_{\hat{\varphi}}$ on $\mathcal{H}$ by

$$A_{\hat{\varphi}_\mu} := \int_{\mathbb{R}^d}^{\oplus} A_{\hat{\varphi}_\mu}(x)dx, \quad \mu = 1, ..., d,$$

i.e., $(A_{\hat{\varphi}_\mu}F)(x) = A_{\hat{\varphi}_\mu}(x)F(x)$ and

$$D(A_{\hat{\varphi}_\mu}) := \{F \in \mathcal{H} | F(x) \in D(A_{\hat{\varphi}_\mu}(x)) \text{ a.e. } x \in \mathbb{R}^d \text{ and } \int_{\mathbb{R}^d} \|A_{\hat{\varphi}_\mu}(x)F(x)\|^2_{\mathcal{F}_{\rm b}} dx < \infty\}.$$

We set $A_{\hat{\varphi}} = (A_{\hat{\varphi}_1}, ..., A_{\hat{\varphi}_d})$. From the fact that $k \cdot e(k, j) = 0$ we have

$$\nabla \cdot A_{\hat{\varphi}} = \sum_{\mu=1}^{d} [\nabla_\mu, A_{\hat{\varphi}_\mu}] = 0$$

as an operator. We use the identification (2.1) without notices. We define the decoupled self-adjoint Hamiltonian H_0 by

$$H_0 := (-\frac{1}{2}\Delta + V) \otimes 1 + 1 \otimes H_{\rm f}, \quad D(H_0) := D(-\Delta \otimes 1) \cap D(1 \otimes H_{\rm f}).$$

The total Hamiltonian H, the so-called Pauli-Fierz Hamiltonian, is described by the minimal coupling, $-i\nabla_\mu \otimes 1 \to -i\nabla_\mu \otimes 1 - eA_{\hat{\varphi}_\mu}$, to H_0. Then

$$H := \frac{1}{2}(-i\nabla \otimes 1 - eA_{\hat{\varphi}})^2 + V \otimes 1 + 1 \otimes H_{\rm f},$$

where $e \in \mathbb{R}$ is a coupling constant. As a mathematical interest we introduce another operator K by

$$K := \frac{1}{2}(-i\nabla \otimes 1 - eA_{\hat{\varphi}})^2 + V \otimes 1.$$

It is well known that

$$\|a^\sharp(f)\phi\| \le \|f/\sqrt{\omega}\| \|H_{\rm f}^{1/2}\Psi\| + \|f\|\|\Psi\|, \quad \Psi \in D(H_{\rm f}^{1/2}). \tag{2.2}$$

Assumption $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$ implies that H is symmetric, and $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$ that $(-i\nabla \otimes 1)A_{\hat{\varphi}} + A_{\hat{\varphi}}(-i\nabla \otimes 1)$ and ${A_{\hat{\varphi}}}^2$ are relatively bounded with respect to $-\Delta \otimes 1 + 1 \otimes H_{\rm f}$. The proposition below is established in [30, 31].

Proposition 2.1 *Assume that V is relatively bounded with respect to $-\Delta$ with a relative bound strictly smaller than one. Then*

(1) H is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of self-adjoint operator $-(1/2)\Delta \otimes 1 + 1 \otimes H_{\rm f}$, and bounded from below,

(2) K is essentially self-adjoint on $C^\infty(\Delta \otimes 1) \cap C^\infty(1 \otimes N)$ and bounded from below, where $C^\infty(T) := \cap_{n=1}^\infty D(T^n)$.

We denote the self-adjoint extension of $K\lceil_{C^\infty(\Delta\otimes 1)\cap C^\infty(1\otimes N)}$ by the same symbol K. Throughout this paper we assume that Vsatisfies the same assumptions in Proposition 2.1.

2.2 Translation invariance

In this subsection we set $V = 0$. Define the field momentum by $P_{\mathrm{f}\mu} := d\Gamma(k_\mu)$, $\mu = 1, ..., d$, and the total momentum

$$P_{\mathrm{T}\mu} := \overline{-i\nabla_\mu \otimes 1 + 1 \otimes P_{\mathrm{f}\mu}}, \quad \mu = 1, ..., d,$$

and set $P_\mathrm{f} := (P_{\mathrm{f}1}, ..., P_{\mathrm{f}d})$, $P_\mathrm{T} := (P_{\mathrm{T}1}, ..., P_{\mathrm{T}d})$, where $\overline{X}$ denotes the closure of closable operator X. It is seen that H is translation invariant [30, (5.23)], i.e.,

$$e^{isP_{\mathrm{T}\mu}} H e^{-isP_{\mathrm{T}\mu}} = H, \quad s \in \mathbb{R}, \quad \mu = 1, ..., d.$$

We shall decompose H on $\sigma(P_{\mathrm{T}\mu}) = \mathbb{R}$. Operator $H(P)$, $P \in \mathbb{R}^d$, acting in $\mathcal{F}_\mathrm{b}$ is defined by

$$H(P) := \frac{1}{2}(P - P_\mathrm{f} - eA_{\hat{\varphi}}(0))^2 + H_\mathrm{f}, \quad D(H(P)) := D(H_\mathrm{f}) \cap D({P_\mathrm{f}}^2).$$

Note that $H(P)$ is a well defined symmetric operator on $D(H_\mathrm{f}) \cap D({P_\mathrm{f}}^2)$ by assumption **(A)**. For a sufficiently small e, the self-adjointness of $H(P)$ is easily shown by using (2.2) and the Kato-Rellich theorem. In order to show the self-adjointness of $H(P)$ for an arbitrary $e \in \mathbb{R}$, we need to make a detour.

Theorem 2.2 *$H(P)$ is self-adjoint and*

$$\int_{\mathbb{R}^d}^{\oplus} H(P) dP \cong H. \tag{2.3}$$

Although it is not a physically reasonable model, it is of interest to study the essential self-adjointness of another operator $K(P)$ defined by

$$K(P) := \frac{1}{2}(P - P_\mathrm{f} - eA_{\hat{\varphi}}(0))^2, \quad D(K(P)) := D({P_\mathrm{f}}^2) \cap D(H_\mathrm{f}).$$

It is not clear that $K(P)$ is self-adjoint even for a sufficiently small e by the lack of H_f. The quadratic form $\tilde{Q}_P(\Psi, \Phi)$ is given by

$$\tilde{Q}_P(\Psi, \Phi) := \frac{1}{2}\sum_{\mu=1}^{d} ((P - P_\mathrm{f} - eA_{\hat{\varphi}}(0))_\mu \Psi, (P - P_\mathrm{f} - eA_{\hat{\varphi}}(0))_\mu \Phi)_{\mathcal{F}_\mathrm{b}},$$

$$D(\tilde{Q}_P) := \cap_{\mu=1}^d [D(P_{\mathrm{f}\mu}) \cap D(A_{\hat{\varphi}}(0)_\mu)].$$

Since $\tilde{Q}_P$ is a densely defined nonnegative quadratic form, there exists a positive self-adjoint operator $K_\mathrm{F}(P)$ such that $\tilde{Q}_P(\Psi, \Phi) = (K_\mathrm{F}(P)^{1/2}\Psi, K_\mathrm{F}(P)^{1/2}\Phi)$.

Theorem 2.3 *(1) It follows that*

$$\int_{\mathbb{R}^d}^{\oplus} K_\mathrm{F}(P) dP \cong K. \tag{2.4}$$

(2) Assume that $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$. Then $K(P)$ is essentially self-adjoint and

$$\int_{\mathbb{R}^d}^{\oplus} \overline{K(P)} dP \cong K. \tag{2.5}$$

Theorem 2.3 (2) is proved by using a functional integral representation in Section 3.3.2. We here prove Theorem 2.2 and Theorem 2.3 (1).

The fiber decomposition of H will be achieved through the unitary operator $U : L^2(\mathbb{R}^d_x) \otimes \mathcal{F}_\mathrm{b} \to L^2(\mathbb{R}^d_\xi) \otimes \mathcal{F}_\mathrm{b}$ given by

$$U := (\mathbb{F} \otimes 1) \int_{\mathbb{R}^d}^{\oplus} \exp\left(ix \cdot P_\mathrm{f}\right) dx, \tag{2.6}$$

where $\mathbb{F} : L^2(\mathbb{R}^d_x) \to L^2(\mathbb{R}^d_\xi)$ denotes the Fourier transformation on $L^2(\mathbb{R}^d)$. Actually for $\Psi \in L^2(\mathbb{R}^d_x) \otimes \mathcal{F}_\mathrm{b}$,

$$(U\Psi)(\xi) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{ix\cdot P_\mathrm{f}} \Psi(x) dx,$$

where $\int \cdots dx$ denotes the $\mathcal{F}_\mathrm{b}$-valued integral in the strong topology. Let Q_μ denote the multiplication operator in $L^2(\mathbb{R}^d)$ defined by $(Q_\mu f)(\xi) := \xi_\mu f(\xi)$, $\mu = 1, ..., d$. Set

$$\mathcal{F}_{\mathrm{b}\infty} := \text{ L.H. } \{a^*(f_1) \cdots a^*(f_n)\Omega | f_j \in C_0^\infty(\mathbb{R}^d), j = 1, ..., n, n = 1, 2, ..\} \cup \{\mathbb{C}\Omega\}$$

and $\mathcal{D} := C_0^\infty(\mathbb{R}^d) \hat{\otimes} \mathcal{F}_{\mathrm{b}\infty}$, where L.H.$\{\cdots\}$ denotes the linear hull of $\{\cdots\}$ and $\hat{\otimes}$ is the algebraic tensor product, i.e., the set of vectors of the form $\sum_{j=1}^{\text{finite}} \alpha_j f_j \otimes \phi_j$, $\alpha_j \in \mathbb{C}$, $f_j \in C_0^\infty(\mathbb{R}^d)$ and $\phi_j \in \mathcal{F}_{\mathrm{b}\infty}$. We define L by

$$L := \overline{\frac{1}{2}(Q \otimes 1 - 1 \otimes P_\mathrm{f} - e1 \otimes A_{\hat{\varphi}}(0))^2 + 1 \otimes H_\mathrm{f} \lceil_{\mathcal{D}}}.$$

Lemma 2.4 *(1) L is self-adjoint on $D((\overline{Q \otimes 1 - 1 \otimes P_\mathrm{f}})^2) \cap D(1 \otimes H_\mathrm{f}))$,*
(2) $UHU^{-1} = L$ on $D((\overline{Q \otimes 1 - 1 \otimes P_\mathrm{f}})^2) \cap D(1 \otimes H_\mathrm{f}))$.

Proof: Not that $e^{ix\cdot P_\mathrm{f}} a^*(f) e^{-ix\cdot P_\mathrm{f}} = a^*(e^{ik\cdot x} f)$ and $e^{ix\cdot P_\mathrm{f}} a(f) e^{-ix\cdot P_\mathrm{f}} = a(e^{-ik\cdot x} f)$. Hence we have $UH\Phi = LU\Phi$ for $\Phi \in \mathcal{D}$. Since $\mathcal{D}$ is a core of H and L is closed, we obtain that U maps $D(H)$ onto $D(L)$ with $UHU^{-1} = L$. Then L is self-adjoint on $UD(H)$. Since $UD(-\Delta \otimes 1) = D((\overline{Q \otimes 1 - 1 \otimes P_\mathrm{f}})^2)$ and $UD(1 \otimes H_\mathrm{f}) = D(1 \otimes H_\mathrm{f})$, we have $UD(H) = D((\overline{Q \otimes 1 - 1 \otimes P_\mathrm{f}})^2) \cap D(1 \otimes H_\mathrm{f})$. Thus (1) and (2) follow. **qed**

Proof of Theorem 2.2 and Theorem 2.3 (1)
Proof: The quadratic form $Q_P(\Psi, \Phi)$ is given by

$$Q_P(\Psi, \Phi) := \frac{1}{2} \sum_{\mu=1}^d ((P - P_\mathrm{f} - eA_{\hat{\varphi}}(0))_\mu \Psi, (P - P_\mathrm{f} - eA_{\hat{\varphi}}(0))_\mu \Phi) + (H_\mathrm{f}^{1/2}\Psi, H_\mathrm{f}^{1/2}\Phi),$$

$$D(Q_P) := \cap_{\mu=1}^d [D(P_{\mathrm{f}\mu}) \cap D(A_{\hat{\varphi}}(0)_\mu)] \cap D(H_\mathrm{f}^{1/2}).$$

Since Q_P is a densely defined nonnegative quadratic form, there exists a positive self-adjoint operator $H_\mathrm{F}(P)$ such that $Q_P(\Psi, \Phi) = (H_\mathrm{F}(P)^{1/2}\Psi, H_\mathrm{F}(P)^{1/2}\Phi)$. Define the self-adjoint operator $\tilde{H}$ acting in $\mathcal{H}$ by $\tilde{H} := \int_{\mathbb{R}^d}^{\oplus} H_{\mathrm{F}}(P)dP$. For $\phi \in \mathcal{D}$, $U\phi(P) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot P}e^{ix\cdot P_{\mathrm{f}}}\phi(x)dx \in D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$. Then for $\psi,\phi \in \mathcal{D}$, we have

$$\begin{aligned}(U\psi, LU\phi)_{\mathcal{H}} &= \int_{\mathbb{R}^d} dP(U\psi(P), [\frac{1}{2}(P-P_{\mathrm{f}}-eA_{\hat{\varphi}}(0))^2 + H_{\mathrm{f}}]U\phi(P))_{\mathcal{F}_{\mathrm{b}}} \\ &= \int_{\mathbb{R}^d} dP(U\psi(P), H_{\mathrm{F}}(P)U\phi(P))_{\mathcal{F}_{\mathrm{b}}} = (U\psi, \tilde{H}U\phi)_{\mathcal{H}}.\end{aligned}$$

Hence $U^{-1}LU = U^{-1}\tilde{H}U$ on $\mathcal{D}$ and then $H = U^{-1}\tilde{H}U$ on $\mathcal{D}$ by Lemma 2.4. Since $\mathcal{D}$ is a core of H and $\tilde{H}$ is self-adjoint, we can see that U maps $D(H)$ onto $D(\tilde{H})$ with $UHU^{-1} = \tilde{H}$. Then $\int_{\mathbb{R}^d}^{\oplus} H_{\mathrm{F}}(P)dP \cong H$ is obtained. The proof of Theorem 2.3 (1) is similar. The proof of self-adjointness of $H_{\mathrm{F}}(P)$ below is due to [39]. In [30, 31] it is proved that there exists a constant C such that

$$\|H_0F\|_{\mathcal{H}} \le C\|(H+1)F\|_{\mathcal{H}}, \quad F \in D(H). \tag{2.7}$$

We define $H_0(P)$ by $H_0(P) := \frac{1}{2}(P-P_{\mathrm{f}})^2 + H_{\mathrm{f}}$ and $D(H_0(P)) = D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$. Note that $H_0(P)$ is self-adjoint and $\int_{\mathbb{R}^d}^{\oplus} H_0(P)dP = H_0$. Then we have for $F \in \mathcal{H}$ such that $(UF)(P) = f(P)\Phi$ with $f \in C_0^\infty(\mathbb{R}^d)$ and $\Phi \in \mathcal{F}_{\mathrm{b}\infty}$ by (2.7),

$$\int_{\mathbb{R}^d} |f(P)|^2\|H_0(P)\Phi\|^2_{\mathcal{F}_{\mathrm{b}}}dP \le C^2\int_{\mathbb{R}^d} f(P)^2\|(H_{\mathrm{F}}(P)+1)\Phi\|^2_{\mathcal{F}_{\mathrm{b}}}dP.$$

Here since $f \in C_0^\infty(\mathbb{R}^d)$ is arbitrary, we see that

$$\|H_0(P)\Phi\|_{\mathcal{F}_{\mathrm{b}}} \le C\|(H_{\mathrm{F}}(P)+1)\Phi\|_{\mathcal{F}_{\mathrm{b}}} \tag{2.8}$$

for almost everywhere $P \in \mathbb{R}^d$. Since the both-hand sides of (2.8) are continuous in P, (2.8) holds for all $P \in \mathbb{R}^d$. (2.8) implies that $H_0(P)(H_{\mathrm{F}}(P)+1)^{-1}$ is bounded and then $H_0(P)e^{-tH_{\mathrm{F}}(P)}$ is bounded, which implies that $e^{-tH_{\mathrm{F}}(P)}$ leaves $D(H_0(P)) = D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$ invariant. Then $H_{\mathrm{F}}(P)$ is essentially self-adjoint on $D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$ by [46, Theorem X.49]. Moreover (2.8) yields that $H_{\mathrm{F}}(P)$ is closed on $D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$. Hence $H_{\mathrm{F}}(P)$ is self-adjoint on $D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$. By the fundamental inequality derived by (2.2):

$$\|H_{\mathrm{F}}(P)\Phi\|_{\mathcal{F}_{\mathrm{b}}} \le C_1\|H_0(P)\Phi\|_{\mathcal{F}_{\mathrm{b}}} + C_2\|\Phi\|_{\mathcal{F}_{\mathrm{b}}}, \quad \Phi \in D(H_0),$$

we can see that $H_{\mathrm{F}}(P)$ is essentially self-adjoint on any core of $H_0(P)$, where C_1 and C_2 are some constants. Since $H(P) = H_{\mathrm{F}}(P)$ on $D(H_{\mathrm{f}})\cap D(P_{\mathrm{f}}{}^2)$, Theorem 2.2 follows. Then the proof is complete. **qed**

Corollary 2.5 *Let $\Lambda > 0$ and $\hat{\varphi}_\Lambda(k) := \begin{cases} 1/\sqrt{(2\pi)^3}, & |k| < \Lambda, \\ 0, & |k| \ge \Lambda. \end{cases}$ Then $H(P)$ with $\hat{\varphi}$ replaced by $\hat{\varphi}_\Lambda$ is self-adjoint for arbitrary $P \in \mathbb{R}^d$, $e \in \mathbb{R}$ and $\Lambda > 0$.*

Proof: Since $\hat{\varphi}$ satisfies (A), the corollary follows. **qed**

2.3 Rotation invariance, helicity and degeneracy of ground states

In this subsection we discuss the rotation invariance of H and $H(P)$. For simplicity we set $d = 3$, and add a spin to H and $H(P)$. Namely let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be 2×2 Pauli matrices such that $\sigma_\mu\sigma_\nu + \sigma_\nu\sigma_\mu = 2\delta_{\mu\nu}$. We define two operators H_σ and $H_\sigma(P)$ acting on $\mathbb{C}^2 \otimes \mathcal{H}$ and $\mathbb{C}^2 \otimes \mathcal{F}_{\mathrm{b}}$, respectively, by

$$H_\sigma := 1 \otimes H + \sum_{\mu=1}^{3} \sigma_\mu \otimes H_{\mathrm{S}\mu},$$
$$H_\sigma(P) := 1 \otimes H(P) + \sum_{\mu=1}^{3} \sigma_\mu \otimes H_{\mathrm{S}}(0)_\mu,$$

where $H_{\mathrm{S}\mu} := -\dfrac{e}{2}\displaystyle\int_{\mathbb{R}^3}^{\oplus} B_\mu(x)dx$ with $B(x) := \mathrm{rot}_x A_{\hat{\varphi}}(x)$, and $H_{\mathrm{S}\mu}(0) := -\dfrac{e}{2}B_\mu(0)$. In the similar way to the proof of Theorem 2.2 it can be shown that H_σ and $H_\sigma(P)$ are self-adjoints operator on $\mathbb{C}^2 \otimes D(H)$ and $\mathbb{C}^2 \otimes [D({P_{\mathrm{f}}}^2) \cap D(H_{\mathrm{f}})]$, respectively. Let $R \in SO(3)$ and $\hat{k} = k/|k|$. The relationship between two orthogonal bases $e(Rk, 1), e(Rk, 2), \hat{Rk}$ and $Re(k, 1), Re(k, 2), R\hat{k}$ in $\mathbb{R}^3$ at k is as follows:

$$\begin{pmatrix} e(Rk,1) \\ e(Rk,2) \\ \hat{Rk} \end{pmatrix} = \begin{pmatrix} \cos\theta 1_3 & -\sin\theta 1_3 & 0 \\ \sin\theta 1_3 & \cos\theta 1_3 & 0 \\ 0 & 0 & 1_3 \end{pmatrix} \begin{pmatrix} Re(k,1) \\ Re(k,2) \\ R\hat{k} \end{pmatrix}, \tag{2.9}$$

where 1_3 denotes the 3×3 identity matrix and

$$\theta := \theta(R, k) := \arccos(Re(k, 1) \cdot e(Rk, 1)). \tag{2.10}$$

Let $R = R(\phi, n) \in SO(3)$ be the rotation around $n \in S^2 := \{k \in \mathbb{R}^3 | |k| = 1\}$ with angle $\phi \in \mathbb{R}$ and $\det R = 1$. Let $\ell_k := k \times (-i\nabla_k) = (\ell_{k1}, \ell_{k2}, \ell_{k3})$ be the triplet of angular momentum operators in $L^2(\mathbb{R}^3_k)$. Then

$$e^{i\theta(R,k)X} e^{i\phi n\cdot\ell_k} \begin{pmatrix} e(k,1) \\ e(k,2) \end{pmatrix} = \begin{pmatrix} Re(k,1) \\ Re(k,2) \end{pmatrix}, \tag{2.11}$$

where

$$X = -i\begin{pmatrix} 0 & -1_3 \\ 1_3 & 0 \end{pmatrix} : \begin{array}{ccc} \mathbb{R}^3 \oplus \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \oplus \mathbb{R}^3 \\ x \oplus y & \longmapsto & -i(-y \oplus x). \end{array}$$

In general, the polarization vectors of photons with momentum k is arbitrary given, but form a right-handed system at k. To discuss a rotation symmetry of H_σ and $H_\sigma(P)$, we introduce *coherent* polarization vectors to some direction. Assumption (P) is as follows.

(P) *There exists* $(n, w) \in S^2 \times \mathbb{Z}$ *such that polarization vectors* $e(\cdot, 1)$ *and* $e(\cdot, 2)$ *satisfy for* $R = R(n, \phi) \in SO(3)$ *and for* $\hat{k} \neq n$,

$$\begin{pmatrix} e(Rk,1) \\ e(Rk,2) \end{pmatrix} = \begin{pmatrix} \cos(\phi w)1_3 & -\sin(\phi w)1_3 \\ \sin(\phi w)1_3 & \cos(\phi w)1_3 \end{pmatrix} \begin{pmatrix} Re(k,1) \\ Re(k,2) \end{pmatrix}, \quad \phi \in \mathbb{R}. \tag{2.12}$$

Assume (P). Then

$$e^{i\phi(wX + n\cdot \ell_k)} \begin{pmatrix} e(k,1) \\ e(k,2) \end{pmatrix} = \begin{pmatrix} Re(k,1) \\ Re(k,2) \end{pmatrix}. \tag{2.13}$$

We show some examples for polarization vectors satisfying (P).

Example 2.6 *Let* $n_z = (0,0,1)$. *Given polarization vectors* $e(\hat{k}_0, 1)$ *and* $e(\hat{k}_0, 2)$ *for* $\hat{k}_0 \in \mathbb{S} = \{(\sqrt{1-z^2}, 0, z) \in S^2 | -1 \leq z \leq 1\}$. *For* $\hat{k} = (\hat{k}_1, \hat{k}_2, \hat{k}_3)$, *there exists* $0 \leq \phi < 2\pi$ *such that* $R(n_z, \phi)\hat{k}_0 = \hat{k}$, *where* $k_0 = (\sqrt{1 - \hat{k}_3^2}, 0, \hat{k}_3) \in \mathbb{S}$. *Define*

$$\begin{pmatrix} e(k,1) \\ e(k,2) \end{pmatrix} := \begin{pmatrix} \cos\phi 1_3 & -\sin\phi 1_3 \\ \sin\phi 1_3 & \cos\phi 1_3 \end{pmatrix} \begin{pmatrix} R(n_z,\phi)e(\hat{k}_0,1) \\ R(n_z,\phi)e(\hat{k}_0,2) \end{pmatrix}. \tag{2.14}$$

It is checked that $e(k,j)$ *satisfies (2.12) with* $(n_z, 1) \in S^2 \times \mathbb{Z}$.

Example 2.7 *Let* $n \in S^3$ *and* $e(k,1) := \hat{k} \cdot n / \sin\theta$ *and* $e(k,2) := (k/|k|) \times e(k,2)$, *where* $\theta = \arccos(\hat{k} \cdot n)$. *Then* $e(k,j)$ *satisfies (2.12) with* $(n, 0) \in S^2 \times \mathbb{Z}$.

Assume (P) with some $(n, w) \in S^2 \times \mathbb{Z}$. We define $S_{\rm f} := d\Gamma(w\mathbb{X})$ and $L_{\rm f} := d\Gamma(\ell_k)$. Here

$$\mathbb{X} := -i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \begin{array}{ccc} L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) & \longrightarrow & L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \\ f \oplus g & \longmapsto & -i(-g \oplus f). \end{array} \tag{2.15}$$

$S_{\rm f}$ is called the helicity of the field[1] and $L_{\rm f}$ the angular momentum of the field. Define $J_{\rm f}$ and $J_{\rm p}$ by

$$J_{\rm f} := n \cdot L_{\rm f} + S_{\rm f}, \quad J_{\rm p} := n \cdot \ell_x + \frac{1}{2} n \cdot \sigma$$

and set

$$J_{\rm total} := J_{\rm p} \otimes 1 + 1 \otimes J_{\rm f}$$

which acts in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}_{\rm b}$.

Lemma 2.8 *Assume (P) and that* $\hat{\varphi}(R(n,\phi)k) = \hat{\varphi}(k)$ *for* $\phi \in \mathbb{R}$. *Then* H_σ *is rotation invariant around* n, *i.e.,*

$$e^{i\phi J_{\rm total}} H_\sigma e^{-i\phi J_{\rm total}} = H_\sigma, \quad \phi \in \mathbb{R}. \tag{2.16}$$

[1] It is written informally as $S_{\rm f} = i \int w(a^*(k,2)a(k,1) - a^*(k,1)a(k,2))dk$

Proof: By $e^{i\phi J} = e^{i\phi S_{\rm f}} e^{i\phi n\cdot L_{\rm f}}$ and (2.13), we see that

$$
\begin{aligned}
& e^{i\phi J_{\rm f}} H_{\rm f} e^{-i\phi J_{\rm f}} = H_{\rm f}, && (2.17)\\
& e^{i\phi J_{\rm f}} P_{{\rm f}\mu} e^{-i\phi J_{\rm f}} = (R(n,\phi) P_{\rm f})_\mu, && (2.18)\\
& e^{i\phi J_{\rm f}} A_{\hat{\varphi}_\mu}(x) e^{-i\phi J_{\rm f}} = (R(n,\phi) A_{\hat{\varphi}}(R(n,\phi)^{-1}x))_\mu, && (2.19)\\
& e^{i\phi J_{\rm f}} B_\mu(x) e^{-i\phi J_{\rm f}} = (R(n,\phi) B(R(n,\phi)^{-1}x))_\mu. && (2.20)
\end{aligned}
$$

Since

$$
\begin{aligned}
& e^{i\phi n\cdot \ell_x} x_\mu e^{-i\phi n\cdot\ell_x} = (R(n,\phi)x)_\mu,\\
& e^{i\phi n\cdot \ell_x} (-i\nabla_x)_\mu e^{-i\phi n\cdot\ell_x} = (R(n,\phi)(-i\nabla_x))_\mu,\\
& e^{i\phi n\cdot (1/2)\sigma} \sigma_\mu e^{-i\phi n\cdot(1/2)\sigma} = (R(n,\phi)\sigma)_\mu,
\end{aligned}
$$

we can see (2.16) by (2.17)-(2.20). **qed**

Note that $\sigma(n\cdot(\ell_x + (1/2)\sigma)) = \mathbb{Z}_{1/2}$, $\sigma(n\cdot L_{\rm f}) = \mathbb{Z}$ and $\sigma(S_{\rm f}) = \mathbb{Z}$, since

$$
\sigma(d\Gamma(h)) = \overline{\{0\} \bigcup \cup_{n=1}^\infty \{\lambda_1 + \cdots + \lambda_n | \lambda_i \in \sigma(h), j = 1,..,n\}}. \tag{2.21}
$$

Then $\sigma(J_{\rm total}) = \mathbb{Z}_{1/2}$ and we have the theorem below.

Theorem 2.9 *We assume the same assumptions as in Lemma 2.8. Then*

$$
\mathbb{C}^2 \otimes \mathcal{H} = \bigoplus_{z\in\mathbb{Z}_{1/2}} \mathcal{H}(z), \qquad H_\sigma = \bigoplus_{z\in\mathbb{Z}_{1/2}} H_\sigma(z).
$$

Here $\mathcal{H}(z)$ is the subspace of $\mathbb{C}^2\otimes\mathcal{H}$ spanned by eigenvectors of $J_{\rm total}$ with eigenvalue $z \in \mathbb{Z}_{1/2}$ and $H_\sigma(z) = H_\sigma\lceil_{\mathcal{H}(z)}$.

Proof: This follows from Lemma 2.8 and the fact that $\sigma(J_{\rm total}) = \mathbb{Z}_{1/2}$. **qed**

Next we study $H_\sigma(P)$.

Lemma 2.10 *Let $\hat{\varphi}(k)$ be rotation invariant. Then for arbitrary polarization vectors, $H_\sigma(P)$ is unitarily equivalent to $H_\sigma(R^{-1}P)$ for arbitrary $R \in SO(3)$.*

Proof: It is enough to show the lemma for an arbitrary $R = R(m,\phi)$, $m \in S^2$ and $\phi \in \mathbb{R}$. For arbitrary polarization vectors $e(\cdot,1)$ and $e(\cdot,2)$, we define $h_{\rm f} = d\Gamma(\theta(R,\cdot)\mathbb{X})$, where $\theta(R,k) = \arccos(e(Rk,1), Re(k,1))$ is given in (2.10) and $\mathbb{X}$ in (2.15). Thus we can see that

$$
\begin{aligned}
& e^{ih_{\rm f}} e^{i\phi m\cdot L_{\rm f}} H_{\rm f} e^{-i\phi m\cdot L_{\rm f}} e^{-ih_{\rm f}} = H_{\rm f},\\
& e^{ih_{\rm f}} e^{i\phi m\cdot L_{\rm f}} P_{{\rm f}\mu} e^{-i\phi m\cdot L_{\rm f}} e^{-ih_{\rm f}} = (R(m,\phi)P_{\rm f})_\mu,\\
& e^{ih_{\rm f}} e^{i\phi m\cdot L_{\rm f}} A_{\hat{\varphi}_\mu}(0) e^{-i\phi m\cdot L_{\rm f}} e^{-ih_{\rm f}} = (R(m,\phi)A_{\hat{\varphi}}(0))_\mu,\\
& e^{ih_{\rm f}} e^{i\phi m\cdot L_{\rm f}} B_\mu(0) e^{-i\phi m\cdot L_{\rm f}} e^{-ih_{\rm f}} = (R(m,\phi)B(0))_\mu.
\end{aligned}
$$

From these identities it follows that

$$e^{ih_{\rm f}}e^{i\phi m\cdot((1/2)\sigma\otimes 1+1\otimes L_{\rm f})}H_\sigma(P)e^{-i\phi m\cdot((1/2)\sigma)\otimes 1+1\otimes L_{\rm f})}e^{-ih_{\rm f}}=H_\sigma(R(m,\phi)^{-1}P). \tag{2.22}$$

Thus the lemma follows. **qed**

Let $E_\sigma(P,e^2):=\inf\sigma(H_\sigma(P))$. An immediate consequence of Lemma 2.10 is as follows.

Corollary 2.11 *Let $\hat\varphi$ be rotation invariant. Then $E_\sigma(RP,e^2)=E_\sigma(P,e^2)$ for arbitrary $R\in SO(3)$.*

Theorem 2.12 *Assume (P) and that $\hat\varphi(k)=\hat\varphi(R(n,\phi)k)$ for $\phi\in\mathbb{R}$. Then $H_\sigma(P)$ is unitarily equivalent to $H_\sigma(|P|n)$ and, $\mathbb{C}^2\otimes\mathcal{F}_{\rm b}$ and $H_\sigma(|P|n)$ are decomposed as*

$$\mathbb{C}^2\otimes\mathcal{F}_{\rm b}=\bigoplus_{z\in\mathbb{Z}_{1/2}}\mathcal{F}_{\rm b}(z),\qquad H_\sigma(P)\cong H_\sigma(|P|n)=\bigoplus_{z\in\mathbb{Z}_{1/2}}H_\sigma(P,z). \tag{2.23}$$

Here $\mathcal{F}_{\rm b}(z)$ is the subspace spanned by eigenvectors of $n\cdot((1/2)\sigma\otimes 1+1\otimes J_{\rm f})$ with eigenvalue $z\in\mathbb{Z}_{1/2}$ and $H_\sigma(P,z)=H_\sigma(P)\lceil_{\mathcal{F}_{\rm b}(z)}$.

Proof: The fact $H_\sigma(P)\cong H_\sigma(|P|n)$ follows from Lemma 2.10. Since

$$e^{i\phi n\cdot((1/2)\sigma\otimes 1+1\otimes J_{\rm f})}H_\sigma(|P|n)e^{-i\phi n\cdot((1/2)\sigma\otimes 1+1\otimes J_{\rm f})}=H_\sigma(|P|n),\quad \phi\in\mathbb{R}, \tag{2.24}$$

follows from (2.22), (2.23) is obtained. **qed**

Let $\hat\varphi$ be rotation invariant and $\tilde H_\sigma(P)$ be the Hamiltonian with polarization vectors given in Example 2.7 with $n=n_z$, i.e.,

$$e(k,1)=(-k_2,k_1,0)/\sqrt{k_1^2+k_2^2},\quad e(k,2)=(k/|k|)\times e(k,1).$$

Then Sasaki [48] proved that $H_\sigma(P)$ with arbitrary polarization vectors is unitarily equivalent to $\tilde H_\sigma(P)$. In particular $H_\sigma(P)\cong\tilde H_\sigma(P)=\bigoplus_{z\in\mathbb{Z}_{1/2}}\tilde H_\sigma(P,z)$. Moreover $\tilde H_\sigma(P,z)\cong\tilde H_\sigma(P,-z)$ for $z\in\mathbb{Z}_{1/2}$. Let M denote the multiplicity of ground state of $H_\sigma(P)$. In [35], the lower bound, $M\ge 2$, is proven for a sufficiently small coupling constant. Sasaki [48] gives an immediate consequence of $\tilde H_\sigma(P,z)\cong\tilde H_\sigma(P,-z)$.

Corollary 2.13 *Let $\hat\varphi$ be rotation invariant. Then M is an even number. In particular $M\ge 2$, whenever ground states exist.*

Let us remove a spin from $H_\sigma(P)$. Assume (P) and that $\hat\varphi(R(n,\phi)k)=\hat\varphi(k)$ for $\phi\in\mathbb{R}$. Then, as is seen in (2.24), as well as $H_\sigma(|P|n)$, $H(|P|n)$ is also rotation invariant around n, i.e.,

$$e^{i\phi n\cdot J_{\rm f}}H(|P|n)e^{-i\phi n\cdot J_{\rm f}}=H(|P|n),\quad \phi\in\mathbb{R}.$$

We have

$$\mathcal{F}_{\rm b} = \bigoplus_{z\in\mathbb{Z}} {\mathcal{F}_{\rm b}}^0(z), \quad H(P) \cong H(|P|n) = \bigoplus_{z\in\mathbb{Z}} H(P,z) \tag{2.25}$$

where ${\mathcal{F}_{\rm b}}^0(z)$ denotes the subspace spanned by eigenvectors associated with eigenvalue $z \in \mathbb{Z}$ of $n \cdot J_{\rm f}$. It is shown that the ground state of $H(P)$ is unique for an arbitrary $e \in \mathbb{R}$ in the case of $P = 0$, and for a sufficiently small $|e|$ in the case of $P \neq 0$. See Section 3 and [35].

Corollary 2.14 *Let $\hat{\varphi}(R(n,\phi)k) = \hat{\varphi}(k)$ and polarization vectors be in Example 2.7. Assume that $H(|P|n)$ has a unique ground state $\varphi_{\rm g}(|P|n)$. Then $(n\cdot J_{\rm f})\varphi_{\rm g}(|P|n) = 0$, i.e., $\varphi_{\rm g}(|P|n) \in {\mathcal{F}_{\rm b}}^0(0)$ in the decomposition (2.25).*

Proof: Since the ground state $\varphi_{\rm g}(|P|n)$ is unique, $\varphi_{\rm g}(|P|n)$ has to belong to some ${\mathcal{F}_{\rm b}}^0(z)$. Then it has to be $z = 0$ since $H(|P|n, z) \cong H(|P|n, -z)$, $z \in \mathbb{Z}$, by [48]. **qed**

3 Functional integral representations

In the quantum mechanics the functional integral representation of the semigroup $e^{-th(a)}$ with

$$h(a) := \frac{1}{2}(-i\nabla - a)^2 + V$$

on $L^2(\mathbb{R}^d)$ for real multiplication operators $a = (a_1, \cdots, a_d)$ and V is given through a stochastic integral. Let $(b(t))_{t\geq 0} = (b_1(t), \cdots, b_d(t))_{t\geq 0}$ be the d-dimensional Brownian motion starting at 0 on a probability space $(W, \mathcal{B}, db)$. Set $X_s := x + b(s)$, $x \in \mathbb{R}^d$, and $dX := dx \otimes db$. Then it is known that

$$(f, e^{-th(a)}g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d\times W} \overline{f(X_0)} g(X_t) e^{-i\int_0^t a(X_s)\circ db_s} dX, \tag{3.1}$$

where $\int_0^t a(X_s)\circ db(s) := \sum_{\mu=1}^d \int_0^t a_\mu(X_s) db_\mu(s) + \frac{1}{2}\int_0^t (\nabla\cdot a)(X_s) ds$. For the functional integral representation of the semigroup generated by the Pauli-Fierz Hamiltonian H we also need a stochastic integral but a Hilbert space-valued one. We quickly review a functional integral representation of e^{-tH} in the next subsection.

3.1 Functional integral representations for e^{-tH}

3.1.1 Gaussian random variables $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$

Let $\mathcal{A}_0(f)$ be a Gaussian random process on a probability space (Q_0, Σ_0, μ_0) indexed by real $f = (f_1, ..., f_d) \in \overset{d}{\oplus} L^2(\mathbb{R}^d)$ with mean zero:

$$\int_{Q_0} \mathcal{A}_0(f) d\mu_0 = 0, \tag{3.2}$$

and the covariance:

$$\int_{Q_0} \mathcal{A}_0(f)\mathcal{A}_0(g)d\mu_0 = q_0(f,g), \tag{3.3}$$

where

$$q_0(f,g) := \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\int_{\mathbb{R}^d} \delta^{\perp}_{\alpha\beta}(k)\overline{\hat{f}_\alpha(k)}\hat{g}_\beta(k)dk.$$

The existence of probability space (Q_0, Σ_0, μ_0) and Gaussian random variable $\mathcal{A}_0(f)$ satisfying (3.2) and (3.3) are governed by the Minlos theorem [49, Theorem I.10]. In a similar way, thanks to the Minlos theorem, we can construct two other Gaussian random variables. Let $\mathcal{A}_1(f)$ indexed by real $f \in \overset{d}{\oplus} L^2(\mathbb{R}^{d+1})$ and $\mathcal{A}_2(f)$ by real $f \in \overset{d}{\oplus} L^2(\mathbb{R}^{d+2})$ be Gaussian random processes on probability spaces (Q_1, Σ_1, μ_1) and (Q_2, Σ_2, μ_2), respectively, with mean zero and covariances given by

$$\int_{Q_1} \mathcal{A}_1(f)\mathcal{A}_1(g)d\mu_1 = q_1(f,g), \tag{3.4}$$

$$\int_{Q_2} \mathcal{A}_2(f)\mathcal{A}_2(g)d\mu_2 = q_2(f,g), \tag{3.5}$$

where

$$q_1(f,g) := \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\int_{\mathbb{R}^{d+1}} \delta^{\perp}_{\alpha\beta}(k)\overline{\hat{f}_\alpha(k,k_0)}\hat{g}_\beta(k,k_0)dkdk_0,$$

$$q_2(f,g) := \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\int_{\mathbb{R}^{d+1+1}} \delta^{\perp}_{\alpha\beta}(k)\overline{\hat{f}_\alpha(k,k_0,k_1)}\hat{g}_\beta(k,k_0,k_1)dkdk_0dk_1.$$

Note that $\mathcal{A}_\#(f)$, $\# = 0,1,2$, is real linear in f. We extend it for $f = f_{\mathrm{R}} + if_{\mathrm{I}}$ with $f_{\mathrm{R}} = (f+\bar{f})/2$ and $f_{\mathrm{I}} = (f-\bar{f})/(2i)$ as $\mathcal{A}_\#(f) = \mathcal{A}_\#(f_{\mathrm{R}}) + i\mathcal{A}_\#(f_{\mathrm{I}})$. The n-particle subspace $L^2_n(Q_\#)$ of $L^2(Q_\#)$ is defined by

$$L^2_n(Q_\#) = \overline{\text{L.H.}\{: \mathcal{A}_\#(f_1)\cdots\mathcal{A}_\#(f_n) : | f_j \in L^2(\mathbb{R}^{d+\#}), j = 1,...,n\}}.$$

Here $: X :$ denotes the Wick product of X [46] defined recursively as

$$\begin{aligned}
&: \mathcal{A}_\#(f) := \mathcal{A}_\#(f),\\
&: \mathcal{A}_\#(f)\mathcal{A}_\#(f_1)\cdots\mathcal{A}_\#(f_n) :=: \mathcal{A}_\#(f_1)\cdots\mathcal{A}_\#(f_n) :\\
&\qquad -\sum_{j=1}^{n} q_\#(f,f_j) : \mathcal{A}_\#(f_1)\cdots\widehat{\mathcal{A}_\#(f_j)}\cdots\mathcal{A}_\#(f_n) :,
\end{aligned}$$

where $\hat{Y}$ denotes neglecting Y. The identity $L^2(Q_\#) = \oplus_{n=0}^{\infty} L^2_n(Q_\#)$ is known as the Wiener-Ito decomposition.

3.1.2 Factorization of semigroups

Let $T : L^2(\mathbb{R}^{d+\#}) \to L^2(\mathbb{R}^{d+\#})$ be a linear contraction operator. Then the linear operator $\Gamma_\#(T) : L^2(Q_\#) \to L^2(Q_\#)$ is defined by

$$\Gamma_\#(T)1 = 1, \quad \Gamma_\#(T) : \mathcal{A}_\#(f_1)\cdots\mathcal{A}_\#(f_n) :=: \mathcal{A}_\#([T]_d f_1)\cdots\mathcal{A}_\#([T]_d f_n) : .$$

Since the linear hull of vectors of the form $: \mathcal{A}_\#(f_1)\cdots\mathcal{A}_\#(f_n) :$ is dense in $L^2(Q_\#)$, we can extend $\Gamma_\#(T)$ to the contraction operator on $L^2(Q_\#)$. We denote its extension by the same symbol. In particular since $\{\Gamma_\#(e^{ith})\}_{t\in\mathbb{R}}$ with a self-adjoint operator h on $L^2(\mathbb{R}^d)$ is a strongly continuous one-parameter unitary group, by the Stone theorem, there exists a self-adjoint operator $d\Gamma_\#(h)$ on $L^2(Q_\#)$ such that $\Gamma_\#(e^{ith}) = e^{itd\Gamma_\#(h)}$, $t \in \mathbb{R}$. We set $N_\# := d\Gamma_\#(1)$. Let h be a multiplication operator in $L^2(\mathbb{R}^d)$. We define the families of isometries,

$$j_s, \xi_t = \xi_t(h) : L^2(\mathbb{R}^d) \xrightarrow{j_s} L^2(\mathbb{R}^{d+1}) \xrightarrow{\xi_t} L^2(\mathbb{R}^{d+2}), \quad s, t \in \mathbb{R}, \tag{3.6}$$

by

$$\hat{j_s f}(k, k_0) := \frac{e^{-isk_0}}{\sqrt{\pi}} \left(\frac{\omega(k)}{\omega(k)^2 + |k_0|^2}\right)^{1/2} \hat{f}(k), \quad (k, k_0) \in \mathbb{R}^d \times \mathbb{R}, \tag{3.7}$$

$$\hat{\xi_t f}(k, k_0, k_1) := \frac{e^{-itk_1}}{\sqrt{\pi}} \left(\frac{h(k)}{h(k)^2 + |k_1|^2}\right)^{1/2} \hat{f}(k, k_0), \quad (k, k_0, k_1) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}.$$

By a direct computation we can see that

$$j_s^* j_t = e^{-|t-s|\omega(-i\nabla)} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad s, t \in \mathbb{R}, \tag{3.8}$$

$$\xi_s^* \xi_t = e^{-|t-s|(h(-i\nabla)\otimes 1)} : L^2(\mathbb{R}^{d+1}) \to L^2(\mathbb{R}^{d+1}), \quad s, t \in \mathbb{R}. \tag{3.9}$$

Here $\omega(-i\nabla)$ is defined by $\omega(-i\nabla)f = (\omega\hat{f})^\vee$ and $h(-i\nabla) \otimes 1$ is an operator defined on $L^2(\mathbb{R}^{d+1})$ under the identification $L^2(\mathbb{R}^{d+1}) \cong L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R})$. Let us define the families of operators J_s and $\Xi_t = \Xi_t(h)$, $s, t \in \mathbb{R}$;

$$L^2(Q_0) \xrightarrow{J_s} L^2(Q_1) \xrightarrow{\Xi_t} L^2(Q_2)$$

by

$$J_s 1 = 1, \quad J_s : \mathcal{A}_0(f_1)\cdots\mathcal{A}_0(f_n) :=: \mathcal{A}_1([j_s]_d f_1)\cdots\mathcal{A}_1([j_s]_d f_n) :, \quad s \in \mathbb{R},$$

$$\Xi_t 1 = 1, \quad \Xi_t : \mathcal{A}_1(f_1)\cdots\mathcal{A}_1(f_n) :=: \mathcal{A}_2([\xi_t]_d f_1)\cdots\mathcal{A}_2([\xi_t]_d f_n) :, \quad t \in \mathbb{R}.$$

Both J_s and Ξ_t can be extended to contraction operators in the similar manner as $\Gamma_\#(T)$. Those extensions are denoted by the same symbols. We have by (3.8) and (3.9)

$$J_s^* J_t = e^{-|t-s|d\Gamma_0(\omega(-i\nabla))} : L^2(Q_0) \to L^2(Q_0), \quad s, t \in \mathbb{R}, \tag{3.10}$$

$$\Xi_s^* \Xi_t = e^{-|t-s|d\Gamma_1(h(-i\nabla)\otimes 1)} : L^2(Q_1) \to L^2(Q_1), \quad s, t \in \mathbb{R}. \tag{3.11}$$

Figure 1: $S = e^{-|t-s|d\Gamma_1(h(-i\nabla)\otimes 1))}$, $T = e^{-|t-s|d\Gamma_0(\omega(-i\nabla))}$, $U = e^{-|t-s|d\Gamma_0(h(-i\nabla))}$

3.1.3 Functional integrals

Define

$$\mathcal{A}_{\#,\mu}(f) = \mathcal{A}_{\#}(\oplus_{\ell=1}^d \delta_{\ell\mu} f), \quad f \in L^2(\mathbb{R}^{d+\#}), \quad \mu = 1, ..., d.$$

We set

$$A_\mu(\hat{f}) := \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int e_\mu(k,j)(a^*(k,j)\hat{f}(k) + a(k,j)\hat{f}(-k))dk$$

for $f \in L^2(\mathbb{R}^d)$, $\mu = 1, ..., d$. It is well known that $L^2(Q_0)$ is unitarily equivalent to $\mathcal{F}_\mathrm{b}$ with $1 \cong \Omega$, $\mathcal{A}_{0,\mu}(f) \cong A_\mu(\hat{f})$ and $d\Gamma_0(h(-i\nabla)) \cong d\Gamma(h)$. In particular

$$d\Gamma_0(-i\nabla) \cong P_\mathrm{f}, \quad d\Gamma_0(\omega(-i\nabla)) \cong H_\mathrm{f} \tag{3.12}$$

hold. Since $\mathcal{H} \cong \int_{\mathbb{R}^d}^\oplus \mathcal{F}_\mathrm{b} dx$, we can see that

$$\mathcal{H} \cong \int_{\mathbb{R}^d}^\oplus L^2(Q_0)dx, \tag{3.13}$$

i.e., $F \in \mathcal{H}$ can be regarded as an $L^2(Q_0)$-valued L^2-function on $\mathbb{R}^d$. In what follows we use identification (3.12) and (3.13) without notices. Note that in the Fock representation the test function $\hat{f}$ of $A_\mu(\hat{f})$ is taken in the momentum representation, but in the Schrödinger representation, f of $\mathcal{A}_{0,\mu}(f)$ in the position representation. We can see that

$$H \cong \frac{1}{2}(-i\nabla \otimes 1 - e\mathcal{A}_0^{\tilde{\varphi}})^2 + V \otimes 1 + 1 \otimes H_\mathrm{f}.$$

Here $\mathcal{A}_0^{\tilde{\varphi}} := (\mathcal{A}_{0,1}^{\tilde{\varphi}}, ..., \mathcal{A}_{0,d}^{\tilde{\varphi}})$ with

$$\mathcal{A}_{0,\mu}^{\tilde{\varphi}} := \int_{\mathbb{R}^d}^\oplus \mathcal{A}_{0,\mu}(\tilde{\varphi}(\cdot - x))dx, \quad \mu = 1, ..., d,$$

and $\tilde{\varphi} := (\hat{\varphi}/\sqrt{\omega})^\vee$. By the Feynman-Kac formula (3.1) with $a = (0, \cdots, 0)$ and the fact $J_0^* J_t = e^{-tH_\mathrm{f}}$ we can see that

$$(F, e^{-tH_0}G)_\mathcal{H} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s)ds}(J_0 F(X_0), J_t G(X_t))_{L^2(Q_1)} dX.$$

Adding the minimal perturbation: $-i\nabla_\mu \otimes 1 \to -i\nabla_\mu \otimes 1 - e\mathcal{A}_0^{\hat{\varphi}}$, we can see in [28] the functional integral representation below.

Proposition 3.1 *Let $F, G \in \mathcal{H}$. Then*

$$(F, e^{-tH}G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s)ds}(J_0F(X_0), e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x))}J_tG(X_t))_{L^2(Q_1)}dX, \quad (3.14)$$

where

$$\mathcal{K}_1^{[0,t]}(x) := \oplus_{\mu=1}^d \int_0^t j_s\tilde{\varphi}(\cdot - X_s)db_\mu(s) \in \oplus^d L^2(\mathbb{R}^{d+1}),$$

and

$$(F, e^{-tK}G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s)ds}(F(X_0), e^{-ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(x))}G(X_t))_{L^2(Q_0)}dX, \quad (3.15)$$

where

$$\mathcal{K}_0^{[0,t]}(x) = \oplus_{\mu=1}^d \int_0^t \tilde{\varphi}(\cdot - X_s)db_\mu(s) \in \oplus^d L^2(\mathbb{R}^d).$$

Define two Gaussian random processes

$$\{\mathcal{A}_{\mu,s}(f)\}_{s\in\mathbb{R}, f\in L^2(\mathbb{R}^d)} \text{ on } (Q_1, \Sigma_1, \mu_1) \text{ by } \mathcal{A}_{\mu,s}(f) := \mathcal{A}_{1,\mu}(j_sf),$$
$$\{\mathcal{A}_{\mu,s,t}(f)\}_{(s,t)\in\mathbb{R}^2, f\in L^2(\mathbb{R}^d)} \text{ on } (Q_2, \Sigma_2, \mu_2) \text{ by } \mathcal{A}_{\mu,s,t}(f) := \mathcal{A}_{2,\mu}(\xi_sj_tf).$$

By (3.4), (3.5), (3.8) and (3.9), it is directly seen that

$$\int_{Q_1} \mathcal{A}_{\alpha,s}(f)\mathcal{A}_{\beta,t}(g)d\mu_1 = \frac{1}{2}\int_{\mathbb{R}^d} e^{-|s-t|\omega(k)}\delta_{\alpha\beta}^\perp(k)\overline{\hat{f}(k)}\hat{g}(k)dk, \quad (3.16)$$

$$\int_{Q_2} \mathcal{A}_{\alpha,s,t}(f)\mathcal{A}_{\beta,s',t'}(g)d\mu_2 = \frac{1}{2}\int_{\mathbb{R}^d} e^{-|s-s'|h(k)}e^{-|t-t'|\omega(k)}\delta_{\alpha\beta}^\perp(k)\overline{\hat{f}(k)}\hat{g}(k)dk. \quad (3.17)$$

Then the identity $\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x)) = \displaystyle\sum_{\mu=1}^d \int_0^t \mathcal{A}_{\mu,s}(\tilde{\varphi}(\cdot - X_s))db_\mu(s)$ follows.

3.2 Functional integral representations for $e^{-tH(P)}$

We shall construct the functional integral representation of $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}}$.

Lemma 3.2 *Let $\Psi, \Phi \in \mathcal{F}_\mathrm{b}$. Then $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}}$ and $(\Psi, e^{-tK_\mathrm{F}(P)}\Phi)_{\mathcal{F}_\mathrm{b}}$ are continuous in $P \in \mathbb{R}^d$.*

Proof: We prove the lemma for $f(P) := (\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}}$. That of $(\Psi, e^{-tK(P)}\Phi)_{\mathcal{F}_\mathrm{b}}$ is similar. We have

$$f(P) - f(P') = \int_0^t (e^{-(t-s)H(P)}\Psi, (H(P) - H(P'))e^{-sH(P')}\Phi)_{\mathcal{F}_\mathrm{b}}ds$$
$$= \frac{1}{2}\sum_{\mu=1}^d (P - P')_\mu \int_0^t (e^{-(t-s)H(P)}\Psi, (P + P' - 2P_\mathrm{f} - 2eA_{\hat{\varphi}}(0))_\mu e^{-sH(P')}\Phi)_{\mathcal{F}_\mathrm{b}}ds.$$

The integral on the right-hand side above is locally finite for P and P'. Then it follows that $\lim_{P'\to P} f(P') = f(P)$ follows. **qed**

For $\Psi \in L^2(Q_0)$, we set $\Psi_t := J_t e^{-iP_{\rm f}\cdot b(t)}\Psi$, $t \geq 0$.

Theorem 3.3 *Let $\Psi, \Phi \in \mathcal{F}_{\rm b}$. Then*

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}} = \int_W (\Psi_0, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} e^{iP\cdot b(t)} db, \tag{3.18}$$

where $\mathcal{K}_1^{[0,t]}(0) := \oplus_{\mu=1}^d \int_0^t j_s \tilde{\varphi}(\cdot - b(s)) db_\mu(s)$, *and*

$$(\Psi, e^{-tK_{\rm F}(P)}\Phi)_{\mathcal{F}_{\rm b}} = \int_W (\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_0^{[0,t]}(0))} e^{-iP_{\rm f}\cdot b(t)}\Phi)_{L^2(Q_0)} e^{iP\cdot b(t)} db, \tag{3.19}$$

where $\mathcal{K}_0^{[0,t]}(0) := \oplus_{\mu=1}^d \int_0^t \tilde{\varphi}(\cdot - b(s)) db_\mu(s)$.

Proof: Set $F_s = \rho_s \otimes \Psi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{{\rm b}\infty}$ and $G_{s'} = \rho_{s'} \otimes \Phi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{{\rm b}\infty}$, where ρ_s is the heat kernel:

$$\rho_s(x) = (2\pi s)^{-d/2} e^{-|x|^2/(2s)}, \quad s > 0. \tag{3.20}$$

By the fact that $H = U^{-1}\left(\int_{\mathbb{R}^d}^\oplus H(P)dP\right)U$ and $Ue^{-i\xi\cdot P_{\rm T}}U^{-1} = \int_{\mathbb{R}^d}^\oplus e^{-i\xi\cdot P}dP$, we have

$$(F_s, e^{-tH}e^{-i\xi\cdot P_{\rm T}}G_{s'})_{\mathcal{H}} = \int_{\mathbb{R}^d} dP((UF_s)(P), e^{-tH(P)}e^{-i\xi\cdot P}(UG_{s'})(P))_{\mathcal{F}_{\rm b}}, \quad \xi \in \mathbb{R}^d.$$

Here $(UF_s)(P) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot P} e^{ix\cdot P_{\rm f}}\rho_s(x)\Psi dx$. Note that

$$\lim_{s\to 0}(UF_s)(P) = \frac{1}{\sqrt{(2\pi)^d}}\Psi \tag{3.21}$$

strongly in $\mathcal{F}_{\rm b}$ for each $P \in \mathbb{R}^d$. Hence we have by the Lebesgue dominated convergence theorem,

$$\lim_{s\to 0}(F_s, e^{-tH}e^{-i\xi\cdot P_{\rm T}}G_{s'})_{\mathcal{F}_{\rm b}} = \frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d} dP(\Psi, e^{-tH(P)}e^{-i\xi\cdot P}(UG_{s'})(P))_{\mathcal{F}_{\rm b}}. \tag{3.22}$$

On the other hand we see that by (3.14)

$$\begin{aligned}
&\lim_{s\to 0}(F_s, e^{-tH}e^{-i\xi\cdot P_{\rm T}}G_{s'})_{\mathcal{H}} \\
&= \lim_{s\to 0}\int_{W\times\mathbb{R}^d}(J_0F_s(X_0), e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x))}J_t e^{-i\xi\cdot P_{\rm T}}G_{s'}(X_t))_{L^2(Q_1)}dX \\
&= \lim_{s\to 0}\int_{W\times\mathbb{R}^d}\rho_s(X_0)\rho_{s'}(X_t-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x))}J_t e^{-i\xi\cdot P_{\rm f}}\Phi)_{L^2(Q_1)}dX \\
&= \int_W \rho_{s'}(b(t)-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_{\rm f}}\Phi)_{L^2(Q_1)}db.
\end{aligned} \tag{3.23}$$

Here we used that $e^{-i\xi\cdot\tilde{P}_T}(\rho(X_t)\otimes\Phi) = \rho(X_t-\xi)\otimes e^{-i\xi\cdot P_{\rm f}}\Phi$. The third equality of (3.23) is due to the Lebesgue dominated convergence theorem. Then we obtained that from (3.22) and (3.23)

$$\begin{aligned}&\frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d} e^{-i\xi\cdot P}(\Psi, e^{-tH(P)}(UG_{s'})(P))_{\mathcal{F}_{\rm b}}dP\\&=\int_W \rho_{s'}(b(t)-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_{\rm f}}\Phi)_{L^2(Q_1)}db.\end{aligned}\tag{3.24}$$

Since

$$\int_{\mathbb{R}^d}\|e^{-tH(P)}UG_{s'}(P)\|^2_{\mathcal{F}_{\rm b}}dP \le \int_{\mathbb{R}^d}\|UG_{s'}(P)\|^2_{\mathcal{F}_{\rm b}}dP = \|G_{s'}\|^2_{\mathcal{H}}<\infty,$$

we have $(\Psi, e^{-tH(\cdot)}(UG_{s'})(\cdot))_{\mathcal{F}_{\rm b}}\in L^2(\mathbb{R}^d)$ for $s'\neq 0$. Then taking the inverse Fourier transform of the both-hand sides of (3.24) with respect to P, we have

$$\begin{aligned}&(\Psi, e^{-tH(P)}(UG_{s'})(P))_{\mathcal{F}_{\rm b}}\\&=\frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d}d\xi e^{iP\cdot\xi}\int_W db\rho_{s'}(b(t)-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_{\rm f}}\Phi)_{L^2(Q_1)}\\&=\frac{1}{\sqrt{(2\pi)^d}}\int_W db\int_{\mathbb{R}^d}d\xi e^{iP\cdot\xi}\rho_{s'}(b(t)-\xi)(J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_{\rm f}}\Phi)_{L^2(Q_1)}\end{aligned}\tag{3.25}$$

for almost every $P\in\mathbb{R}^d$. The second equality of (3.25) is due to Fubini's lemma. The right-hand side of (3.25) is continuous in P, and the left-hand side is also continuous by Lemma 3.2. Then (3.25) is true for all $P\in\mathbb{R}^d$. Taking $s'\to 0$ on the both-hand sides of (3.25), we have by the Lebesgue dominated convergence theorem and (3.21),

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}} = \int_W (J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-iP_{\rm f}\cdot b(t)}\Phi)_{L^2(Q_1)}e^{iP\cdot b(t)}db = (3.18).$$

Thus the theorem follows for $\Psi,\Phi\in\mathcal{F}_{{\rm b}\infty}$. Let $\Psi,\Phi\in\mathcal{F}_{\rm b}$, and $\Psi_n,\Phi_n\in\mathcal{F}_{{\rm b}\infty}$ such that $\Psi_n\to\Psi$ and $\Phi_n\to\Phi$ strongly as $n\to\infty$. Since

$$|(J_0\Psi_n, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-iP_{\rm f}\cdot b(t)}\Phi_n)_{L^2(Q_1)}|\le\|\Psi_n\|_{\mathcal{F}_{\rm b}}\|\Phi_n\|_{\mathcal{F}_{\rm b}}\le c$$

with some constant c independent of n, we have by the Lebesgue dominated convergence theorem

$$\begin{aligned}&\lim_{n\to\infty}\int_W (J_0\Psi_n, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-iP_{\rm f}\cdot b(t)}\Phi_n)_{L^2(Q_1)}e^{iP\cdot b(t)}db\\&=\int_W (J_0\Psi, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-iP_{\rm f}\cdot b(t)}\Phi)_{L^2(Q_1)}e^{iP\cdot b(t)}db,\end{aligned}$$

and it is immediate that $\lim_{n\to\infty}(\Psi_n, e^{-tH(P)}\Phi_n)_{\mathcal{F}_{\rm b}} = (\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_{\rm b}}$. Hence (3.18) is proved. (3.19) is similarly proven through (3.15) and the fact $\int_{\mathbb{R}^d}^{\oplus}K_{\rm F}(P)dp\cong K$. **qed**

3.3 Applications

Let $L^2_{\rm fin}(Q_\#) := \bigcup_{N=0}^{\infty} [\oplus_{n=0}^N L^2_n(Q_\#)]$ and T a self-adjoint operator on $L^2(\mathbb{R}^{d+\#})$. Let us define operator $\Pi_{\#,\mu}(Tf)$ on $L^2_{\rm fin}(Q_\#)$ by

$$\Pi_{\#,\mu}(Tf) := i[d\Gamma_\#(T), \mathcal{A}_{\#,\mu}(f)], \quad f \in D(T). \tag{3.26}$$

In the case where f is real-valued, $\Pi_{\#,\mu}(Tf)$ is a symmetric operator and $L^2_{\rm fin}(Q_\#)$ is the set of analytic vectors of $\Pi_{\#,\mu}(f)$. Then $L^2_{\rm fin}(Q_\#)$ is a core of $\Pi_{\#,\mu}(f)$. The self-adjoint extension of $\Pi_{\#,\mu}(f)$ with real f is denoted by the same symbol.

3.3.1 Ergodic properties

Let $\mathcal{K}_+ := \{\Psi \in L^2(Q_0) | \Psi \geq 0\}$ be the positive cone and set $\mathcal{K}_+^0 := \{\Psi \in \mathcal{K}_+ | \Psi > 0\}$. It is well known [49, Theorem I.12] that $e^{iP_{\rm f}\cdot v}\mathcal{K}_+ \subset \mathcal{K}_+$ for $v \in \mathbb{R}^d$.

Proposition 3.4 *For real $f \in L^2(\mathbb{R}^{d+1})$, it follows that $J_0^* e^{i\Pi_{1,\mu}(f)} J_t[\mathcal{K}_+ \setminus \{0\}] \subset \mathcal{K}_+^0$, i.e., $J_0^* e^{i\Pi_{1,\mu}(f)} J_t$ is positivity improving.*

Proof: See [15, 16] for $f = 0$ and [29] for $f \neq 0$. **qed**

We define

$$\vartheta := \exp\left(i\frac{\pi}{2}N\right).$$

Theorem 3.5 *In the Schrödinger representation, $\vartheta e^{-tH(0)}\vartheta^{-1}$ is positivity improving.*

Proof: Let $\Psi, \Phi \in \mathcal{K}_+ \setminus \{0\}$. It is seen by the functional integral representation in Theorem 3.3 that

$$\begin{aligned}(\Psi, \vartheta e^{-tH(0)}\vartheta^{-1}\Phi)_{\mathcal{F}_{\rm b}} &= \int_W (\Psi_0, e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} db \\ &= \int_W (\Psi, J_0^* e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_{\rm f}\cdot b(t)}\Phi)_{L^2(Q_0)} db. \end{aligned} \tag{3.27}$$

Here we used the facts that $J_t e^{-iP_{\rm f}\cdot b(t)} e^{-i(\pi/2)N} = e^{-i(\pi/2)\tilde{N}} J_t e^{-iP_{\rm f}\cdot b(t)}$ and

$$e^{i(\pi/2)\tilde{N}} e^{-ie\mathcal{A}_1(f)} e^{-i(\pi/2)\tilde{N}} = e^{-ie\Pi_1(f)},$$

where $\tilde{N} = d\Gamma_1(1)$. By Proposition 3.4, $J_0^* e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_{\rm f}\cdot b(t)}$ is positivity improving for each $b \in W$. Namely the integrand in (3.27) is strictly positive for each $b \in W$. Hence the right-hand side of (3.27) is strictly positive, which implies that $\vartheta e^{-tH(0)}\vartheta^{-1}\mathcal{K}_+ \setminus \{0\} \subset \mathcal{K}_+^0$. Thus the theorem follows. **qed**

Corollary 3.6 *The ground state $\varphi_{\rm g}(0)$ of $H(0)$ is unique up to multiple constants, if it exists, and it can be taken as $\vartheta\varphi_{\rm g}(0) > 0$ in the Schrödinger representation.*

Proof: Theorem 3.5 implies that the ground state of $\vartheta H(0)\vartheta^{-1}$ is unique and that we can take a strictly positive ground state, by an infinite dimensional version of the Perron-Frobenius theorem for a positivity improving operator. See [23]. Since ϑ is unitary, the corollary follows. **qed**

Corollary 3.7 [Two diamagnetic inequalities] *It follows that*

$$|(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}}| \leq (|\Psi|, e^{-t(\frac{1}{2}P_\mathrm{f}{}^2+H_\mathrm{f})}|\Phi|)_{L^2(Q_0)}, \tag{3.28}$$
$$|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_\mathrm{b}}| \leq (|\Psi|, \vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}. \tag{3.29}$$

Proof: When L is positivity preserving, it holds that $|L\Psi| \leq L|\Psi|$. We have

$$|(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}}| \leq \int_W (J_0|\Psi|, J_t e^{-iP_\mathrm{f}\cdot b(t)}|\Phi|)_{L^2(Q_1)}db = (|\Psi|, e^{-t(\frac{1}{2}P_\mathrm{f}{}^2+H_\mathrm{f})}|\Phi|)_{L^2(Q_0)}$$

where we used that $b(t)$ is Gaussian with $\int |b_\mu(t)|^2 db = 1/2$. Thus (3.28) follows. We have

$$(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_\mathrm{b}} = \int_W (\Psi_0, e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} e^{iP\cdot b(t)}db. \tag{3.30}$$

Then it follows that

$$\begin{aligned}
|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_\mathrm{b}}| &\leq \int_W (|\Psi|_0, e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}|\Phi|_t)_{L^2(Q_1)}db \\
&= (|\Psi|, \vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}.
\end{aligned}$$

Hence (3.29) follows. **qed**

Let $E(P, e^2) = \inf \sigma(H(P))$.

Corollary 3.8 *(1)* $0 = E(0,0) \leq E(0,e^2) \leq E(P,e^2)$*, (2) Assume that the ground state* $\varphi_\mathrm{g}(0)$ *of* $H(0)$ *exists for* $e \in [0, e_0)$ *with some* $e_0 > 0$*. Then* $E(0,e^2)$ *is concave, continuous and monotonously increasing function on* e^2*, (3)* $E(0,e^2) \leq \inf\sigma(H)$*.*

Proof: (3.29) implies $|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Psi)_{\mathcal{F}_\mathrm{b}}| \leq e^{-tE(0,e^2)}\|\Psi\|^2_{\mathcal{F}_\mathrm{b}}$. Since ϑ is unitary, (1) follows. Let $\varphi_\mathrm{g}(0)$ be the ground state of $H(0)$. Thus by Corollary 3.6, $\vartheta\varphi_\mathrm{g}(0) > 0$, and hence $(1, \varphi_\mathrm{g}(0))_{L^2(Q_0)} \neq 0$ by $\vartheta^{-1}1 = 1$. Thus

$$\begin{aligned}
E(0,e^2) &= \lim_{t\to\infty} -\frac{1}{t}\log(\Omega, e^{-tH(0)}\Omega)_{\mathcal{F}_\mathrm{b}} = \lim_{t\to\infty} -\frac{1}{t}\log\int_W (1, e^{-ie\mathcal{A}(\mathcal{K}_1^{[0,t]}(0))}1)_{L^2(Q_1)}db \\
&= \lim_{t\to\infty} -\frac{1}{t}\log\int_W e^{-\frac{e^2}{2}q_0(\mathcal{K}_1^{[0,t]}(0),\mathcal{K}_1^{[0,t]}(0))}db.
\end{aligned}$$

Since $e^{-\frac{e^2}{2}q_0(\mathcal{K}_1^{[0,t]}(0),\mathcal{K}_1^{[0,t]}(0))}$ is log convex on e^2, $E(0,e^2)$ is concave. Then $E(0,e^2)$ is continuous on $(0, e_0)$. Since $E(0,e^2)$ is also continuous at $e^2 = 0$ by the fact that $H(0)$ converges as $e^2 \to 0$ in the uniform resolvent sense, $E(0,e^2)$ is continuous on $[0, e_0)$.

Then $E(0,e^2)$ can be expressed as $E(0,e^2)=\int_0^{e^2}\phi(t)dt$ with some positive function ϕ. Thus $E(0,e^2)$ is monotonously increasing on e^2. Then (2) is obtained. We have

$$(F,(1\otimes\vartheta)e^{-tH}(1\otimes\vartheta^{-1})G)_{\mathcal{H}}=\int_{\mathbb{R}^d}dP(F(P),\vartheta e^{-tH(P)}\vartheta^{-1}G(P)))_{\mathcal{F}_{\rm b}}.$$

Then by (3.29) it is seen that

$$|(F,(1\otimes\vartheta)e^{-tH}(1\otimes\vartheta^{-1})F)_{\mathcal{H}}|\le e^{-tE(0,e^2)}\int_{\mathbb{R}^d}dP\|F(P)\|^2_{\mathcal{F}_{\rm b}}=e^{-tE(0,e^2)}\|F\|^2_{\mathcal{H}}.$$

Thus (3) follows. **qed**

Remark 3.9 *(1) The uniqueness of the ground state of $H(P)$ is shown in [35] for a sufficiently small $|e|$. The result in Corollary 3.6 is valid for arbitrary values of coupling constants but $P=0$. (2) In [39], a weaker statement $\vartheta e^{-tH(0)}\vartheta^{-1}\mathcal{K}_+\subset\mathcal{K}_+$ is shown, and Corollary 3.8 (1) is also obtained.*

3.3.2 Invariant domains and essential self-adjointness of $K(P)$

Lemma 3.10 *Assume that $\omega^{3/2}\hat\varphi\in L^2(\mathbb{R}^d)$. Then*

$$e^{-tK_{\rm F}(P)}[D({P_{\rm f}}^2)\cap D(H_{\rm f})]\subset D({P_{\rm f}}^2)\cap D(H_{\rm f}). \tag{3.31}$$

Proof: We have for $f\in\overset{d}{\oplus}L^2(\mathbb{R}^{d+1})$,

$$\begin{aligned}e^{ie\mathcal{A}_0(f)}H_{\rm f}e^{-ie\mathcal{A}_0(f)} &= H_{\rm f}-ie[H_{\rm f},\mathcal{A}_0(f)]+\frac{1}{2}(-ie)^2[[H_{\rm f},\mathcal{A}_0(f)],\mathcal{A}_0(f)]\\ &= H_{\rm f}-e\Pi_0([\omega]_d f)-e^2q_0([\omega]_d f,f).\end{aligned}$$

From the Burkholder type inequality [30]: for $\mu=1,...,d$,

$$\int_W db\left\|\omega^{n/2}\int_0^t\tilde\varphi(\cdot-X_s)db_\mu(s)\right\|^2_{L^2(\mathbb{R}^d)}\le\frac{(2m)!}{2^m}t^m\|\omega^{(n-1)/2}\hat\varphi\|^2_{L^2(\mathbb{R}^d)}, \tag{3.32}$$

and $\|\Pi_0(f)\Phi\|_{L^2(Q)}\le c\sum_{\alpha=1}^d(\|f_\alpha/\sqrt{\omega}\|_{L^2(\mathbb{R}^d)}+\|f_\alpha\|_{L^2(\mathbb{R}^d)})\|(H_{\rm f}+1)^{1/2}\Phi\|_{L^2(Q)}$ by (2.2) with some constant c, it follows that for $\Psi\in D(H_{\rm f})$,

$$\int_W\|e^{ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))}H_{\rm f}e^{-ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))}J_t\Psi\|^2_{L^2(Q_0)}db\le C\|(H_{\rm f}+1)\Psi\|^2_{\mathcal{F}_{\rm b}} \tag{3.33}$$

with some constant C. Thus we see that by means of functional integral representations (3.19),

$$\begin{aligned}&|(H_{\rm f}\Psi,e^{-tK_{\rm F}(P)}\Phi)_{\mathcal{F}_{\rm b}}|\\ &\le\int_W\|e^{ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))}\Psi\|_{L^2(Q_0)}\|e^{ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))}H_{\rm f}e^{-ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))}e^{-iP_{\rm f}\cdot b(t)}\Phi\|_{L^2(Q_0)}db\\ &\le C'\|\Psi\|_{\mathcal{F}_{\rm b}}\|(H_{\rm f}+1)\Phi\|_{\mathcal{F}_{\rm b}}\end{aligned} \tag{3.34}$$

with some constant C'. We can also see that

$$
\begin{aligned}
& e^{ie\mathcal{A}_0(f)} {P_{\mathrm{f}}}^2 e^{-ie\mathcal{A}_0(f)} \\
&= \sum_{\mu=1}^d \left(P_{\mathrm{f}\mu} - ie[P_{\mathrm{f}\mu}, \mathcal{A}_0(f)] + \frac{1}{2}(-ie)^2 [[P_{\mathrm{f}\mu}, \mathcal{A}_0(f)], \mathcal{A}_0(f)] \right)^2 \\
&= \sum_{\mu=1}^d \Big\{ P_{\mathrm{f}\mu}^2 - e P_{\mathrm{f}\mu} \Pi_0([-i\nabla_\mu]_d f) - e \Pi_0([-i\nabla_\mu]_d f) P_{\mathrm{f}\mu} + e^2 \Pi_0([-i\nabla_\mu]_d f)^2 \\
& -e^2 q_0([-i\nabla_\mu]_d f, f) P_{\mathrm{f}\mu} + e^3 \Pi_0([-i\nabla_\mu]_d f) q_0([-i\nabla_\mu]_d f, f) + e^4 q_0([-i\nabla_\mu]_d f, f)^2 \Big\}.
\end{aligned}
$$

We have for $\Psi \in D({P_{\mathrm{f}}}^2) \cap D(H_{\mathrm{f}})$

$$
\begin{aligned}
& \|P_{\mathrm{f}\mu} \Pi_0([-i\nabla_\mu]_d f)\Psi\| \leq c_1 \|({P_{\mathrm{f}}}^2 + H_{\mathrm{f}} + 1)\Psi\|, && (3.35) \\
& \|\Pi_0([-i\nabla_\mu]_d f) P_{\mathrm{f}\mu}\Psi\| \leq c_2 \|({P_{\mathrm{f}}}^2 + H_{\mathrm{f}} + 1)\Psi\|, && (3.36) \\
& \|\Pi([-i\nabla_\mu]_d f)^2 \Psi\| \leq c_3 \|(H_{\mathrm{f}} + 1)\Psi\|, && (3.37) \\
& \|q_0([-i\nabla_\mu]_d f, f) P_{\mathrm{f}}\Psi\| \leq c_4 \|P_{\mathrm{f}}\Psi\|, && (3.38) \\
& \|\Pi_0([-i\nabla_\mu]_d f) q_0([-i\nabla_\mu]_d f, f)\Psi\| \leq c_5 \|H_{\mathrm{f}}^{1/2}\Psi\|, && (3.39) \\
& \|q_0([-i\nabla_\mu]_d f, f)^2 \Psi\| \leq c_6 \|\Psi\|, && (3.40)
\end{aligned}
$$

where we used $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$ in (3.35). Thus, together with Burkholder type inequality (3.32), the integration of $c_1, ..., c_6$ in (3.35) - (3.40) with f replaced by $\mathcal{K}_0^{[0,t]}(0)$ over db is suppressed from upper and

$$
\begin{aligned}
& |({P_{\mathrm{f}}}^2\Psi, e^{-tK_{\mathrm{F}}(P)}\Phi)_{\mathcal{F}_{\mathrm{b}}}| \\
& \leq \int_W \|e^{ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))}\Psi\|_{L^2(Q_0)} \|e^{ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))} {P_{\mathrm{f}}}^2 e^{-ie\mathcal{A}_0(\mathcal{K}_0^{[0,t]}(0))} e^{-iP_{\mathrm{f}}\cdot b(t)}\Phi\|_{L^2(Q_0)} db \\
& \leq C' \|\Psi\|_{\mathcal{F}_{\mathrm{b}}} \|(H_{\mathrm{f}} + {P_{\mathrm{f}}}^2 + 1)\Phi\|_{\mathcal{F}_{\mathrm{b}}} \qquad (3.41)
\end{aligned}
$$

with some constant C'. Here we used that $\|H_{\mathrm{f}} e^{-iP_{\mathrm{f}}\cdot b(t)}\Psi\| = \|H_{\mathrm{f}}\Psi\|$. By (3.34) and (3.41) the lemma is obtained. **qed**

Proof of Theorem 2.3 (2)

Proof: By Lemma 3.10 we see that $K_{\mathrm{F}}(P)$ is essentially self-adjoint on $D({P_{\mathrm{f}}}^2) \cap D(H_{\mathrm{f}})$ by [46, Theorem X.49]. Since $K_{\mathrm{F}}(P) = K(P)$ on $D(H_{\mathrm{f}}) \cap D({P_{\mathrm{f}}}^2)$, the desired result is obtained. **qed**

Lemma 3.11 *Let $\Phi \in D((N+1)^\alpha)$ with $\alpha \in \mathbb{N}$. Then $e^{-tH(P)}\Phi \in D(N^\alpha)$ with the inequality $\|N^\alpha e^{-tH(P)}\Phi\|_{\mathcal{F}_{\mathrm{b}}} \leq C\|(N+1)^\alpha\Phi\|_{\mathcal{F}_{\mathrm{b}}}$, where C is a constant independent of P.*

Proof: Let $\Psi, \Phi \in D(N^\alpha)$. Set $\mathcal{A} = \mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))$ and $\tilde{N} = d\Gamma_1(1)$ for simplicity. Then

$$
(N^\alpha\Psi, e^{-tH(P)}\Phi) = \int_W (\tilde{N}^\alpha\Psi_0, e^{-ie\mathcal{A}}\Phi_t)_{L^2(Q_1)} e^{iP\cdot b(t)} db. \qquad (3.42)
$$

Since

$$\begin{aligned} e^{ie\mathcal{A}}\tilde{N}^\alpha e^{-ie\mathcal{A}}\Phi &= \left(\tilde{N} - ie[\tilde{N},\mathcal{A}] + \frac{1}{2!}(-ie)^2[[\tilde{N},\mathcal{A}],\mathcal{A}]\right)^\alpha \Phi \\ &= \left(\tilde{N} - e\Pi_1(\mathcal{K}_1^{[0,t]}(0)) - e^2 q_1(\mathcal{K}_1^{[0,t]}(0),\mathcal{K}_1^{[0,t]}(0))\right)^\alpha \Phi. \quad (3.43) \end{aligned}$$

The right-hand side of (3.43) is suppressed as $\|\text{r.h.s.(3.43)}\|_{L^2(Q_1)} \le c\|(N+1)^\alpha\Phi\|_{\mathcal{F}_\mathrm{b}}$ by the Burkholder type inequality

$$\int_W db \left\| \int_0^t j_s\tilde{\varphi}(\cdot - X_s)db_\mu(s)\right\|^2_{L^2(\mathbb{R}^{d+1})} \le \frac{(2m)!}{2^m} t^m \|\hat{\varphi}/\sqrt{\omega}\|^2_{L^2(\mathbb{R}^d)}. \quad (3.44)$$

Then

$$\begin{aligned} &|(N^\alpha\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_\mathrm{b}}| \le \int_W |(\Psi_0, \tilde{N}^\alpha e^{-ie\mathcal{A}}\Phi_t)_{L^2(Q_1)}|db \\ &\le \int_W \|e^{ie\mathcal{A}}\Psi_0\|_{L^2(Q_1)}\|e^{ie\mathcal{A}}\tilde{N}^\alpha e^{-ie\mathcal{A}}\Phi_t\|_{L^2(Q_1)}db \le c\|\Psi\|_{\mathcal{F}_\mathrm{b}}\|(N+1)^\alpha\Phi\|_{\mathcal{F}_\mathrm{b}}. \end{aligned}$$

Hence the lemma follows. **qed**

4 The n point Euclidean Green functions

In this section we extend functional integral representations derived in the previous section to the n point Euclidean Green functions. We fix $0 = s_0 \le s_1 \le \cdots \le s_{m-1} \le s_m = s$ and $0 = t_0 \le t_1 \le \cdots \le t_{m-1} \le t_m = t$. For notational simplicity we define for objects (operators or vectors) T_j, $j = 1, ..., n$,

$$\prod_{j=1}^n T_j := T_1 T_2 \cdots T_n.$$

We introduce the set $\mathcal{F}_\mathrm{b}^\infty$ of bounded operators on $\mathcal{F}_\mathrm{b}$ by

$$\mathcal{F}_\mathrm{b}^\infty := \{\Phi(A(f_1),\cdots,A(f_n)) | \Phi \in L^\infty(\mathbb{R}^n), f_j \in \oplus^d L^2(\mathbb{R}^d), j = 1,...,n, n \ge 0\}.$$

We identify bounded multiplication operator $\Phi(A(f_1),\cdots,A(f_n))$ on $\mathcal{F}_\mathrm{b}$ and bounded multiplication operator $\Phi(\mathcal{A}_0(f_1),\cdots,\mathcal{A}_0(f_n))$ on $L^2(Q_0)$.

4.1 In the case of H

Theorem 4.1 *Let $K = 1 \otimes d\Gamma(h)$ with a multiplication operator h in $L^2(\mathbb{R}^d)$. Let $F_j = f_j \otimes \Phi_j \in L^\infty(\mathbb{R}^d) \otimes \mathcal{F}_\mathrm{b}^\infty$, $j = 1, ..., m-1$, with $\Phi_j = \Phi_j(A(f_1^j),\cdots,A(f_{n_j}^j))$, and $F_0, F_m \in \mathcal{H}$. Then*

$$\begin{aligned} &(F_0, \prod_{j=1}^m e^{-(s_j - s_{j-1})K} e^{-(t_j - t_{j-1})H} F_j)_\mathcal{H} \\ &= \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s)ds}(\hat{F}_0(X_0), e^{-ie\mathcal{A}_2(\mathcal{K}_2(x))}\prod_{j=1}^m \hat{F}_j(X_{t_j}))_{L^2(Q_2)} dX, \quad (4.1) \end{aligned}$$

where $\hat{F}_j(x) := \Xi_{s_j} J_{t_j} F_j(x) = f_j(x)\Phi_j(\mathcal{A}_2(\xi_{s_j} j_{t_j} f_1^j), \cdots, \mathcal{A}_2(\xi_{s_j} j_{t_j} f_{n_j}^j))$ *and*

$$\mathcal{K}_2(x) := \oplus_{\mu=1}^d \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \xi_{s_j} j_s \tilde{\varphi}(\cdot - X_s) db_\mu(s).$$

Proof: Set $K_0 = \frac{1}{2}(-i\nabla\otimes 1 - e\mathcal{A}_0^{\tilde{\hat{\varphi}}})^2$ and assume that $V \in C_0^\infty(\mathbb{R}^d)$ in a moment. By the Trotter-Kato product formula [37], we have $e^{-tH} = s - \lim_{n\to\infty} \left(e^{-\frac{t}{n}V} e^{-\frac{t}{n}K_0} e^{-\frac{t}{n}H_{\rm f}}\right)^n$. Set $a_n = t_n - t_{n-1}$ and $b_n = s_n - s_{n-1}$, $n = 1, ..., m$, for notational convenience. Thus

$$\begin{aligned} \text{l.h.s. } (4.1) = \lim_{n\to\infty} (F, e^{-b_1 K} \left(e^{-\frac{a_1}{n}V} e^{-\frac{a_1}{n}K_0} e^{-\frac{a_1}{n}H_{\rm f}}\right)^n F_1 e^{-b_2 K} \left(e^{-\frac{a_2}{n}V} e^{-\frac{a_2}{n}K_0} e^{-\frac{a_2}{n}H_{\rm f}}\right)^n \cdots \\ \cdots F_{m-1} e^{-b_m K} \left(e^{-\frac{a_m}{n}V} e^{-\frac{a_m}{n}K_0} e^{-\frac{a_m}{n}H_{\rm f}}\right)^n G)_{L^2(Q_0)}. \end{aligned} \tag{4.2}$$

Define $Q_s : \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned} &(Q_0 F)(x) := F(x), \\ &(Q_s F)(x) := \int_{\mathbb{R}^d} p_s(|x-y|) e^{-i(e/2)\sum_{\mu=1}^d \mathcal{A}_{0,\mu}(\tilde{\varphi}(\cdot - x) + \tilde{\varphi}(\cdot - y))\cdot(x_\mu - y_\mu)} F(y) dy, \quad s \neq 0. \end{aligned}$$

Here $p_s(x)$ is the heat kernel given in (3.20). Then it is established in [28] that

$$s - \lim_{n\to\infty} (Q_{t/2^n})^{2^n} = e^{-tK_0}. \tag{4.3}$$

Let $E_s := J_s J_s^*$ and define $Q_{[a,b]} := \text{L.H.}\{F \in L^2(Q_1) | F \in \text{Ran} E_s, s \in [a,b]\}$. We denote the smallest σ field generated by $Q_{[a,b]}$ by $\Sigma_{[a,b]}$. Let $a \le b \le c \le d$ and assume that Ψ is measurable with respect to $\Sigma_{[a,b]}$ and Φ with respect to $\Sigma_{[c,d]}$. Then it is known as Markov property [49] of E_s on $L^2(Q_1)$ that

$$(\Psi, E_s \Phi)_{L^2(Q_1)} = (\Psi, \Phi)_{L^2(Q_1)} \tag{4.4}$$

for $b \le s \le c$. We note that for $F = f \otimes \Phi(\mathcal{A}_0(f_1), \cdots, \mathcal{A}_0(f_n)) \in L^\infty(\mathbb{R}^d) \otimes \mathcal{F}_{\rm b}^\infty$, the identity

$$J_s F J_s^* = (J_s F) E_s = E_s (J_s F) E_s \tag{4.5}$$

holds as an operator, where $J_s F$ on the right-hand side of (4.5) is

$$J_s F = f \otimes \Phi(\mathcal{A}_1(j_s f_1), \cdots, \mathcal{A}_1(j_s f_n)).$$

In particular it follows that

$$J_s e^{-i\mathcal{A}_{0,\mu}(f)} J_s^* = E_s e^{-i\mathcal{A}_{1,\mu}(j_s f)} E_s \tag{4.6}$$

as an operator. Substituting (4.3), $e^{-|s-t|H_{\rm f}} = J_s^* J_t$ and $J_s e^{-td\Gamma_1(h)} = e^{-t\tilde{K}} J_s$, where $\tilde{K} := d\Gamma_1(h\otimes 1)$, into (4.2), we can obtain that

$$\begin{aligned} \text{l.h.s } (4.1) = \int_{\mathbb{R}^d \times W} dX e^{-\int_0^t V(X_s) ds} (J_0 F_0(X_0), e^{-b_1 \tilde{K}} e^{-ie\mathcal{A}_1(K_{t_0}^{t_1})} (J_{t_1} F_1(X_{t_1})) e^{-b_2 \tilde{K}} \cdots \\ \cdots (J_{t_{m-1}} F_{m-1}(X_{t_m})) e^{-b_m \tilde{K}} e^{-ie\mathcal{A}_1(K_{t_m}^{t_{m-1}})} J_{t_m} F_m(X_{t_m}))_{L^2(Q_1)}, \end{aligned}$$

where we used (4.5), (4.6) and the Markov property (4.4) of E_s, and set simply $K_u^v = \mathcal{K}_1^{[u,v]}(x) = \oplus_{\mu=1}^d \int_u^v j_s\tilde{\varphi}(\cdot - X_s)db_\mu(s)$. Factorizing $e^{-b_j\tilde{K}}$ as $\Xi_{s_j}^*\Xi_{s_{j-1}}$ and using the Markov property of $\Xi_s\Xi_s^*$ on $L^2(Q_2)$ again, we have

$$\text{l.h.s. } (4.1) = \int_{\mathbb{R}^d\times W} dX e^{-\int_0^t V(X_s)ds}(\Xi_0 J_0 F_0(X_0), e^{-ie\mathcal{A}_2(\xi_{s_1}K_{t_0}^{t_1})}e^{-ie\mathcal{A}_2(\xi_{s_2}K_{t_1}^{t_2})}\cdots$$
$$\cdots e^{-ie\mathcal{A}_2(\xi_{s_m}K_{t_{m-1}}^{t_m})}\prod_{j=1}^m(\Xi_{s_j}J_{t_j}F_j(X_{t_j})))_{L^2(Q_2)} = \text{r.h.s. } (4.1).$$

Hence the theorem follows for $V \in C_0^\infty(\mathbb{R}^d)$. By a simple limiting argument on V, we can get the theorem. **qed**

Remark 4.2 *By the proof of Theorem 3.5, we can see that $F_1, ..., F_{m-1}$ in Theorem 3.5 can be extended for more general bounded multiplication operators such as the form $F(x) = e^{-ix\cdot P_{\rm f}}f(x)\otimes\Psi(A(f_1),\cdots,A(f_n))e^{ix\cdot P_{\rm f}} = f(x)\otimes\Psi(A(e^{-ikx}f_1),\cdots,A(e^{-ikx}f_n))$. This facts will be used in the next subsection.*

4.2 In the case of $H(P)$

Theorem 4.3 *Let $K = d\Gamma(h)$ with a multiplication operator h in $L^2(\mathbb{R}^d)$. We assume that $\Phi_0, \Phi_m \in \mathcal{F}_{\rm b}$ and $\Phi_j \in \mathcal{F}_{\rm b}^\infty$ for $j = 1, ..., m-1$ with $\Phi_j = \Phi_j(A(f_1^j),\cdots,A(f_{n_j}^j))$. Then for $P_0,\cdots,P_{m-1} \in \mathbb{R}^d$,*

$$(\Phi_0, \prod_{j=1}^m e^{-(s_j-s_{j-1})K}e^{-(t_j-t_{j-1})H(P_{j-1})}\Phi_j)_{\mathcal{F}_{\rm b}}$$
$$= \int_W (\hat{\Phi}_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat{\Phi}_j)_{L^2(Q_2)}e^{+i\sum_{j=1}^m(b(t_j)-b(t_{j-1}))P_{j-1}}db, \tag{4.7}$$

where

$$\hat{\Phi}_j := \Xi_{s_j}J_{t_j}e^{-iP_{\rm f}\cdot b(t_j)}\Phi_j = \Phi_j(\mathcal{A}_2(\xi_{s_j}j_{t_j}f_1^j(\cdot - b(t_j))),\cdots,\mathcal{A}_2(\xi_{s_j}j_{t_j}f_{n_j}^j(\cdot - b(t_j))))$$

and

$$\mathcal{K}_2(0) := \oplus_{\mu=1}^d\sum_{j=1}^m\int_{t_{j-1}}^{t_j}\xi_{s_j}j_s\tilde{\hat{\varphi}}(\cdot - b(s))db_\mu(s).$$

In particular, in the case of $P_0 = \cdots = P_{m-1} = P$, it follows that

$$(\Phi_0, \prod_{j=1}^m e^{-(s_j-s_{j-1})K}e^{-(t_j-t_{j-1})H(P)}\Phi_j)_{\mathcal{F}_{\rm b}} = \int_W(\hat{\Phi}_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat{\Phi}_j)_{L^2(Q_2)}e^{iP\cdot b(t)}db.$$

Proof: Let $\xi_0, \xi_1, \cdots, \xi_{m-1} \in \mathbb{R}^d$ and $l_0, l_1, \cdots, l_{m-1} > 0$. Set $F_j(x) = \rho_{l_j}(x)\Phi_j(x)$, where $\Phi_j(x) = e^{-ix\cdot P_\mathrm{f}}\Phi_j e^{ix\cdot P_\mathrm{f}} = \Phi_j(\mathcal{A}_0(f_1^j(\cdot - x)), \cdots, \mathcal{A}_0(f_{n_j}^j(\cdots - x)))$, and ρ_s is the heat kernel given in (3.20). Then

$$UF_j\Psi = (\hat{\rho}_{l_j}\Phi_j) * (U\Psi) = \left[\int_{\mathbb{R}^d} \hat{\rho}_{l_j}(\cdot - y)(U\Psi)(y)dy\right]\Phi_j \tag{4.8}$$

follows, where U is given in (2.6) and $\hat{\rho}$ the Fourier transform of ρ. For notational convenience we set, for $j = 1, ..., m$,

$$O_j(P_{j-1}) := e^{-(s_j - s_{j-1})K}e^{-(t_j - t_{j-1})H(P_{j-1})}, \quad O_j := e^{-(s_j - s_{j-1})1\otimes K}e^{-(t_j - t_{j-1})H}.$$

Then the left-hand side of (4.7) can be presented as $(\Phi_0, \prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_\mathrm{b}}$ We note that $[O_j(P), e^{-i\eta\cdot P_\mathrm{T}}] = 0$, $P, \eta \in \mathbb{R}^d$. Set $\tilde{F}_j(P) = \hat{\rho}_{l_j}(P)\Phi_j$. We see that by (4.8)

$$\begin{aligned}
&(F_0, e^{-i\xi_0\cdot P_\mathrm{T}}O_1F_1e^{-i\xi_1\cdot P_\mathrm{T}}O_2F_2\cdots e^{-i\xi_{m-1}\cdot P_\mathrm{T}}O_mF_m)_{\mathcal{F}_\mathrm{b}}\\
&= (UF_0, Ue^{-i\xi_0\cdot P_\mathrm{T}}O_1F_1e^{-i\xi_1\cdot P_\mathrm{T}}O_2F_2\cdots e^{-i\xi_{m-1}\cdot P_\mathrm{T}}O_mF_m)_{\mathcal{F}_\mathrm{b}}\\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} dP_0(\tilde{F}_0(P_0), O_1(P_0)[\tilde{F}_1 * (UO_2e^{-i\xi_1\cdot P_\mathrm{T}}\cdots F_m)](P_0))_{\mathcal{F}_\mathrm{b}}e^{-i\xi_0\cdot P_0}\\
&= [(2\pi)^{-d/2}]^2\int_{\mathbb{R}^d} dP_0\int_{\mathbb{R}^d} dP_1(\tilde{F}_0(P_0), O_1(P_0)\tilde{F}_1(P_0 - P_1)O_2(P_1)\\
&\qquad\qquad \times[\tilde{F}_2 * (UO_3e^{-i\xi_2\cdot P_\mathrm{T}}\cdots F_m)](P_1))_{\mathcal{F}_\mathrm{b}}e^{-i\xi_0 P_0}e^{-i\xi_1 P_1}\\
&\qquad\qquad\vdots\\
&\qquad\qquad\vdots\\
&= [(2\pi)^{-d/2}]^m\int_{(\mathbb{R}^d)^m} dP\bar{\hat{\rho}}_{l_0}(P_0)\hat{\rho}_{l_m}(P_{m-1})\prod_{j=1}^{m-1}\hat{\rho}_{l_j}(P_{j-1} - P_j)\\
&\qquad\qquad\qquad\qquad \times(\Phi_0, \prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_\mathrm{b}}e^{-i\xi\cdot P},
\end{aligned}\tag{4.9}$$

where $P = (P_0, \cdots, P_{m-1}) \in (\mathbb{R}^d)^m$ and $\xi = (\xi_0, \cdots, \xi_{m-1}) \in (\mathbb{R}^d)^m$. On the other hand, we can see that

$$\begin{aligned}
&(F_0, e^{-i\xi_0\cdot P_\mathrm{T}}O_1F_1e^{-i\xi_1\cdot P_\mathrm{T}}O_2F_2\cdots e^{-i\xi_{m-1}\cdot P_\mathrm{T}}O_mF_m)_{\mathcal{F}_\mathrm{b}}\\
&= (F_0, e^{-i\xi_0\cdot P_\mathrm{T}}O_1F_1e^{+i\xi_0 P_\mathrm{T}}e^{-i(\xi_0+\xi_1)\cdot P_\mathrm{T}}O_2F_2e^{+i(\xi_0+\xi_1)\cdot P_\mathrm{T}}\cdots e^{-i(\xi_0+\cdots+\xi_{m-1})\cdot P_\mathrm{T}}O_mF_m)_{\mathcal{F}_\mathrm{b}}\\
&= (\dot{F}_0(0), O_1\dot{F}_1(\xi_0)O_2\dot{F}_2(\xi_0+\xi_1)\cdots O_m\dot{F}_m(\xi_0+\cdots+\xi_{m-1}))_{\mathcal{F}_\mathrm{b}}\\
&= \int_{\mathbb{R}^d\times W}\rho_{l_0}(X_{t_0})\rho_{l_1}(X_{t_1} - \xi_0)\cdots\rho_{l_m}(X_{t_m} - \xi_0 - \cdots - \xi_{m-1})\\
&\qquad\qquad \times(\hat{\Phi}_0(X_0), e^{-ie\mathcal{A}_2(\mathcal{K}_2(x))}\prod_{j=1}^m\hat{\Phi}_j(X_{t_j}))_{L^2(Q_2)}dX,
\end{aligned}\tag{4.10}$$

where $\dot{F}_j(\xi) = \rho_{l_j}(x - \xi)\Phi_j(x)$, $\hat{\Phi}_j(x) = \Phi_j(\mathcal{A}_2(\xi_{s_j}j_{t_j}f_1^j(\cdot - x)), \cdots, \mathcal{A}_2(\xi_{s_j}j_{t_j}f_{n_j}^j(\cdot - x)))$, and the third identity above is due to Theorem 4.1 and Remark 4.2. Hence by (4.9) and (4.10) we obtain that

$$[(2\pi)^{-d/2}]^m \int_{(\mathbb{R}^d)^m} e^{-i\xi\cdot P}\bar{\hat{\rho}}_{l_0}(P_0)\hat{\rho}_{l_m}(P_{m-1})\prod_{j=1}^{m-1}\hat{\rho}_{l_j}(P_{j-1}-P_j)(\Phi_0,\prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_{\rm b}}$$
$$=\int_{\mathbb{R}^d\times W}\rho_{l_0}(X_{t_0})\left(\prod_{j=1}^{m-1}\rho_{l_j}(X_{t_j}-\sum_{i=0}^{j-1}\xi_i)\right)(\hat{\Phi}_0(X_0),e^{-ie\mathcal{A}_2(\mathcal{K}_2(x))}\prod_{j=1}^m\hat{\Phi}_j(X_{t_j}))_{L^2(Q_2)}dX. \tag{4.11}$$

Since $(\Phi_0,\prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_{\rm b}}$ is bounded with respect to $P\in(\mathbb{R}^d)^m$, the integrand of the left-hand side of (4.11) satisfies that

$$\bar{\hat{\rho}}_{l_0}(P_0)\hat{\rho}_{l_m}(P_{m-1})\prod_{j=1}^{m-1}\hat{\rho}_{l_j}(P_{j-1}-P_j)(\Phi_0,\prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_{\rm b}}\in L^2((\mathbb{R}^d)^m,dP).$$

By this, we can take the inverse Fourier transform of the both-hand sides of (4.11) in the L^2-sense. Then we obtain that

$$\bar{\hat{\rho}}_{l_0}(P_0)\hat{\rho}_{l_m}(P_{m-1})\prod_{j=1}^{m-1}\hat{\rho}_{l_j}(P_{j-1}-P_j)(\Phi_0,\prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_{\rm b}}$$
$$=[(2\pi)^{-d/2}]^m\int_{(\mathbb{R}^d)^m}d\xi e^{+i\xi\cdot P}\int_{\mathbb{R}^d\times W}\rho_{l_0}(X_{t_0})\left(\prod_{j=1}^{m-1}\rho_{l_j}(X_{t_j}-\sum_{i=0}^{j-1}\xi_i)\right)$$
$$\times(\hat{\Phi}_0(X_0),e^{-ie\mathcal{A}_2(\mathcal{K}_2(x))}\prod_{j=1}^m\hat{\Phi}_j(X_{t_j}))_{L^2(Q_2)}dX \tag{4.12}$$

for almost every $P\in(\mathbb{R}^d)^m$. Since the both-hand sides of (4.12) are continuous in $P\in(\mathbb{R}^d)^m$ by Lemma 3.2. Thus (4.12) is true for all $P\in(\mathbb{R}^d)^m$. Take the limit, $l_m\to0\cdots l_0\to0$, on the both-hand sides of (4.12). On the left-hand side of (4.12) we have

$$\lim_{l_m\to0}\cdots\lim_{l_0\to0}\bar{\hat{\rho}}_{l_0}(P_0)\hat{\rho}_{l_m}(P_{m-1})\prod_{j=1}^{m-1}\hat{\rho}_{l_j}(P_{j-1}-P_j)(\Phi_0,\prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_{\rm b}}$$
$$=[(2\pi)^{-d/2}]^m(\Phi_0,\prod_{j=1}^m O_j(P_{j-1})\Phi_j)_{\mathcal{F}_{\rm b}}. \tag{4.13}$$

We note that $\hat{\Phi}_j(X_{t_j})\lceil_{x=0}=\hat{\Phi}_j$ by the definition of $\hat{\Phi}_j$. Then we see that, on the right-hand side of (4.12) without coefficient $[(2\pi)^{-d/2}]^m$,

$$\lim_{l_m\to0}\cdots\lim_{l_0\to0}\int_{(\mathbb{R}^d)^m}d\xi e^{+i\xi\cdot P}\int_{\mathbb{R}^d\times W}dX\rho_{l_0}(X_{t_0})\left(\prod_{j=1}^{m}\rho_{l_j}(X_{t_j}-\sum_{i=0}^{j-1}\xi_i)\right)$$
$$\times(\hat{\Phi}_0(X_0),e^{-ie\mathcal{A}_2(\mathcal{K}_2(x))}\prod_{j=1}^m\hat{\Phi}_j(X_{t_j}))$$

$$
\begin{aligned}
&= \lim_{l_m\to 0}\cdots\lim_{l_1\to 0}\int_{(\mathbb{R}^d)^m} d\xi e^{+i\xi\cdot P}\int_W db\left(\prod_{j=1}^m \rho_{l_j}(b(t_j)-\sum_{i=0}^{j-1}\xi_i)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times(\hat\Phi_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat\Phi_j))\\
&= \lim_{l_m\to 0}\cdots\lim_{l_2\to 0}\int_{(\mathbb{R}^d)^{m-1}} d\xi_1\cdots d\xi_{m-1}e^{+i(b(t_1)-b(t_0))P_0+i\sum_{j=1}^{m-1}\xi_j\cdot P_j}\\
&\qquad\times\int_W db\left(\prod_{j=2}^m \rho_{l_j}(b(t_j)-b(t_1)-\sum_{i=1}^{j-1}\xi_i)\right)(\hat\Phi_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat\Phi_j)\\
&= \lim_{l_m\to 0}\cdots\lim_{l_3\to 0}\int_{(\mathbb{R}^d)^{m-2}} d\xi_2\cdots d\xi_{m-1}e^{+i(b(t_1)-b(t_0))P_0+i(b(t_2)-b(t_1))P_1+i\sum_{j=2}^{m-1}\xi_j\cdot P_j}\\
&\qquad\times\int_W db\left(\prod_{j=3}^m \rho_{l_j}(b(t_j)-b(t_2)-\sum_{i=2}^{j-1}\xi_i)\right)(\hat\Phi_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat\Phi_j)\\
&\qquad\qquad\qquad\qquad\vdots\\
&\qquad\qquad\qquad\qquad\vdots\\
&= \int_W dbe^{+i\sum_{j=1}^m(b(t_j)-b(t_{j-1}))P_{j-1}}(\hat\Phi_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat\Phi_j). \qquad (4.14)
\end{aligned}
$$

From (4.13) and (4.14) the theorem follows. **qed**

Corollary 4.4 *Let $\Psi,\Phi\in\mathcal{F}_{\mathrm{b,fin}}$. Then*

$$
\begin{aligned}
&(\Psi, e^{-t_1H(P)}A(f_1)e^{-(t_2-t_1)H(P)}A(f_2)\cdots A(f_{n-1})e^{-(t_n-t_{n-1})H(P)}\Phi)_{\mathcal{F}_{\mathrm{b}}}\\
&= \int_W(\Psi_0, e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(0))}\left(\prod_{j=1}^{n-1}\mathcal{A}_1(j_{t_j}f(\cdot-b(t_j)))\right)\Phi_t)_{L^2(Q_1)}e^{iP\cdot b(t)}db. \qquad (4.15)
\end{aligned}
$$

Proof: We show an outline of a proof. We note that the left-hand side of (4.15) is well defined by Lemma 3.11. First we can see by Theorem 4.3 that

$$
\begin{aligned}
&(\Psi, e^{-t_1H(P)}e^{is_1A(f_1)}e^{-(t_2-t_1)H(P)}e^{is_2A(f_2)}\cdots e^{is_{n-1}A(f_{n-1})}e^{-(t_n-t_{n-1})H(P)}\Phi)_{\mathcal{F}_{\mathrm{b}}}\\
&= \int_W(\Psi_0, e^{-ie\mathcal{A}_1(\mathcal{K}_1(0)^{[0,t]})}\left(\prod_{j=1}^{n-1}e^{is_j\mathcal{A}_1(j_{t_j}f(\cdot-b(t_j)))}\right)\Phi_t)_{L^2(Q_1)}e^{iP\cdot b(t)}db. \qquad (4.16)
\end{aligned}
$$

By Lemma 3.11, $e^{-t_1H(P)}\Psi\in C^\infty(N)$. Then $e^{is_1A(f_1)}e^{-t_1H(P)}\Psi$ is strongly differentiable at $s_1=0$ with

$$
\frac{d}{ds_1}e^{is_1A(f_1)}e^{-t_1H(P)}\Psi\lceil_{s_1=0}= iA(f_1)e^{-t_1H(P)}\Psi,
$$

and $e^{is_2A(f_1)}e^{-(t_2-t_1)H(P)}A(f_1)e^{-t_1H(P)}\Psi$ is also differentiable at $s_2=0$ with

$$
\frac{d}{ds_2}e^{is_2A(f_2)}e^{-(t_2-t_1)H(P)}A(f_1)e^{-t_1H(P)}\Psi = A(f_2)e^{-(t_2-t_1)H(P)}A(f_1)e^{-t_1H(P)}\Psi.
$$

Repeating this procedure on the left-hand side of (4.16), we can get the left-hand side of (4.15). It is also seen that the right-hand side of (4.16) is also differentiable at $(s_1, ..., s_{n-1}) = (0, ..., 0)$ with the right-hand side of (4.15) as a result. Thus the corollary follows. **qed**

4.3 Applications

We shall show some application of Theorem 4.3, by which we can construct a sequence of measures on W converging to $(\varphi_{\rm g}(P), T\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}}$ for some bounded operator T. In particular $T = e^{-\beta N}$ and $T = e^{-iA(f)}$ are taken as examples. In [35] it is proved that $H(P)$ has a unique ground state $\varphi_{\rm g}(P)$ and $(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}} \neq 0$ for a sufficiently small e.

Corollary 4.5 *We suppose that $H(P)$ has the unique ground state $\varphi_{\rm g}(P)$ and it satifies $(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}} \neq 0$. Then for $\beta > 0$,*

$$(\varphi_{\rm g}(P), e^{-\beta N}\varphi_{\rm g}(P)) = \lim_{t\to\infty} \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP\cdot b(2t)} d\mu_{2t},$$

where $D(t) := q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0))$ and μ_{2t} is a measure on W given by

$$d\mu_{2t} := \frac{1}{Z} e^{-(e^2/2)q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0))} db$$

with the normalizing constant Z such that $\int_W e^{iP\cdot b(2t)} d\mu_{2t} = 1$.

Proof: We define the family of isometries $\xi_s = \xi_s(1)$, $s \in \mathbb{R}$, by (3.6). By Theorem 4.3 we have

$$\begin{aligned} (e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}} &= \int_W db e^{iP\cdot b(2t)} (1, e^{-ie\mathcal{A}_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))} 1)_{L^2(Q_2)} \\ &= \int_W db e^{iP\cdot b(2t)} e^{-(e^2/2)q_2(\xi_0\mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))}. \end{aligned}$$

Noticing that $q_2(\xi_s f, \xi_t g) = e^{-|s-t|} q_1(f, g)$, we have

$$q_2(\xi_0\mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0)) = q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0)) - (1 - e^{-\beta}) q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0)).$$

Then

$$\frac{(e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)}{(e^{-tH(P)}\Omega, e^{-tH(P)}\Omega)} = \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP\cdot b(2t)} d\mu_{2t}. \tag{4.17}$$

The corollary follows from the fact

$$s - \lim_{t\to\infty} \frac{e^{-tH(P)}\Omega}{\|e^{-tH(P)}\Omega\|_{\mathcal{F}_{\rm b}}} = \frac{(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}}}{|(\varphi_{\rm g}(P), \Omega)_{\mathcal{F}_{\rm b}}|} \varphi_{\rm g}(P)$$

and (4.17). **qed**

Remark 4.6 *It is informally written as*

$$\begin{aligned} & q_1(\mathcal{K}_1^{[S,T]}(0), \mathcal{K}_1^{[S',T']}(0)) \\ & = \frac{1}{2}\sum_{\alpha,\beta=1}^d \int_S^T db_\alpha(s) \int_{S'}^{T'} db_\beta(s') \int_{\mathbb{R}^d} \delta^\perp_{\alpha\beta}(k) \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-|s-s'|\omega(k)} e^{-ik(b(s)-b(s'))} dk. \end{aligned}$$

There are some discussions on double stochastic integrals mentioned above in [52].

Corollary 4.7 *Assume the same assumptions as in Corollary 4.5. Then*

$$(\varphi_{\rm g}(P), e^{-iA(f)}\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}} = \lim_{t\to\infty}\int_W e^{-eq_1(\mathcal{K}_1^{[0,2t]}(0), f^t) - \frac{1}{2}q_0(f,f)} e^{iP\cdot b(2t)} d\mu_{2t}, \tag{4.18}$$

where $f^t := \oplus_{\mu=1}^d j_t f_\alpha(\cdot - b(t))$.

Proof: We have by Theorem 4.3

$$\begin{aligned} (\varphi_{\rm g}(P), e^{-iA(f)}\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}} &= \lim_{t\to\infty} \frac{(e^{-tH(P)}\Omega, e^{-iA(f)}e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}}}{(e^{-tH(P)}\Omega, e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}}} \\ &= \lim_{t\to\infty} \frac{1}{Z}\int_W db e^{iP\cdot b(2t)} (1, e^{-i(e\mathcal{A}_1(\mathcal{K}_1^{[0,2t]}(0)) + \mathcal{A}_1(j_t f))} 1)_{L^2(Q_1)} \\ &= \lim_{t\to\infty} \frac{1}{Z}\int_W db e^{iP\cdot b(2t)} e^{-\frac{1}{2}q_1(e\mathcal{K}_1^{[0,2t]}(0) + f^t)}. \end{aligned}$$

Note that $q_1(f^t, f^t) = q_0(f,f)$. Then the corollary follows. **qed**

Remark 4.8 $q_1(\mathcal{K}_1^{[0,2t]}(0), f^t)$ *is informally given by*

$$q_1(\mathcal{K}_1^{[0,2t]}(0), f^t) = \frac{1}{2}\sum_{\alpha,\beta=1}^d \int_0^{2t} db_\alpha(s) \int_{\mathbb{R}^d} \delta^\perp_{\alpha\beta}(k) \frac{\overline{\hat{\varphi}(k)}}{\sqrt{\omega(k)}} \hat{f}_\beta(k) e^{ik\cdot(b(s)-b(t))} e^{-|s-t|\omega(k)} dk.$$

Acknowledgment I thank Sasaki for a useful comment on a rotation invariance. This work is partially done in "Open Quantum System" held in Ervin Schrödinger Institute (ESI) on March of 2005, and "Feynman-Kac Formulas and their Applications" in the Wolfgang Pauli Institute (WPI) of Universität Vienna on March of 2006. The author thanks these kind hospitalities. This work is financially supported by Grant-in-Aid for Science Research (C) 17540181 from JSPS.

References

[1] A. Arai, A particle-field Hamiltonian in relativistic quantum electrodynamics, *J. Math. Phys.* **41** (2000), 4271–4283.

[2] L. Amour, B. Grébert and J. C. Guillot, L'électron habillé non relativiste dans un champ magn'etique, *C. R. Acad. Math. Sci. Paris, Ser.* I **340** (2005), 421–426.

[3] L. Amour, B. Grébert and J. C. Guillot, The dressed nonrelativistic electron in a magnetic field, arXiv:math-ph/0412069, 2004, preprint.

[4] L. Amour, B. Grébert and J. C. Guillot, The dressed mobile atoms and ions, arXiv:math-ph/0507052, 2005, preprint.

[5] V. Bach, T. Chen, F. Fröhlich and I. M. Sigal, The renormalized electron mass in non-relativistic quantum electrodynamics, arXiv.math-ph/0507043, 2005, preprint.

[6] J. M. Barbaroux, H. Linde and S. Vugalter, Quantitative estimates on the enhanced binding for the Pauli-Fierz operator, mp-arc 05-283, 2005, preprint.

[7] V. Betz, Existence of Gibbs measures relative to Brownian motion, *Markov Processes and Related Fields*, **9** (2003), 85-102.

[8] V. Betz and J. Lőrinczi, Uniqueness of Gibbs measures relative to Brownian motion, *Annal, Inst. Henri Poincaré* **5** (2003), 877-889.

[9] V. Betz and H. Spohn, A central limit theorem for Gibbs measures relative to Brownian motion, *Prob. Theor. Related Topics* **131** (2005), 459 - 478.

[10] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos and H. Spohn, Gibbs measure associated with particle-field system, *Rev. Math. Phys.*, **14** (2002), 173–198.

[11] C. Borell, Positivity improving operators and hypercontractivity, *Math. Z.* **180** (1982), 225–234.

[12] J. Cannon, Quantum field theoretic properties of a model of Nelson:domain and eigenvector stability for perturbed linear operators, *J. Funct. Anal.* **8** (1971), 101-152.

[13] T. Chen, Operator-theoretic infrared renormalization and construction of dressed 1-particle states in non-relativistic QED, mp-arc 01-301, preprint, 2001.

[14] T. Chen, Infrared renormalization in non-relativistic QED for the endpoint case, arXiv:math-ph/0601010, 2006, preprint.

[15] W. Faris, Invariant cones and uniqueness of the ground state for Fermion systems, *J. Math. Phys.* **13** (1972), 1285–1290.

[16] W. Faris and B. Simon, Degenerate and nondegenerate ground states for Schrödinger operators, *Duke Math. J.* **42** (1975), 559–567.

[17] C. Fefferman, J. Fröhlich, G. M. Graf, Stability of ultraviolet-cutoff quantum electrodynamics with non-relativistic matter, *Commun. Math. Phys.* **190** (1997), 309–330.

[18] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Inst. Henri Poincaré* **19** (1973), 1–103.

[19] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschritte der Physik* **22** (1974), 159–198.

[20] J. Fröhlich and Y. M. Park, Correlation inequalities and thermodynamic limit for classical and quantum continuous systems II. Bose-Einstein and Fermi-Dirac statistics, *J. Stat. Phys.* **23** (1980), 701–753.

[21] J. Fröhlich, M. Griesemer and B. Schlein, Asymptotic completeness for Compton scattering. *Commun. Math. Phys.* **252** (2004), 415–476.

[22] B. Gerlach and H. Löwen, Analytical properties of polaron system or: Do polaronic phase transitions exist or not?, *Rev. Modern Phys.* **63** (1991), 63–89.

[23] J. Glimm and A. Jaffe, The $\lambda(\phi^4)_2$ quantum field theory without cutoffs. I. *Phys. Rev.* **176** (1968), 1945–1951.

[24] L. Gross, A noncommutative extension of the Perron-Frobenius theorem, *Bull. Amer. Math. Soc.* **77** (1971), 343–347.

[25] L. Gross, Existence and uniqueness of physical ground states, *J. Funct. Anal.* **10** (1972), 52–109.

[26] C. Hainzl and R. Seiringer, Mass renormalization and energy level shift in non-relativistic QED, *Adv. Theor. Math. Phys.* **6** (2002), 847-871.

[27] M. Hirokawa, An expression of the ground state energy of the Spin-Boson model, *J. Funct. Anal.* **162** (1999), 178–218.

[28] F. Hiroshima, Functional integral representations of quantum electrodynamics, *Rev. Math. Phys.* **9** (1997), 489–530.

[29] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics II, *J. Math. Phys.* **41** (2000), 661-674.

[30] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585–613.

[31] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. Henri Poincaré*, **3** (2002), 171–201.

[32] F. Hiroshima, Multiplicity of ground states in quantum field models:applications of asymptotic fields, *J. Funct. Anal.* **224** (2005) 431–470

[33] F. Hiroshima, Tightness of measures with double stochastic integrals on a path space, in preparation.

[34] F. Hiroshima and K. R. Ito, Mass Renormalization in Non-relativistic Quantum Electrodynamics with spin 1/2 mp-arc 04-406, preprint 2004, Effective mass of nonrelativistic quantum electrodynamics, mp-arc 05-439, preprint 2005.

[35] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz model with spin, *Adv. Theor. Math. Phys.* **5** (2001), 1091–1104.

[36] F. Hiroshima and H. Spohn, Mass renormalization in nonrelativistic QED, *J. Math. Phys.* **46** 042302 (2005).

[37] T. Kato and K. Masuda, Trotter's product formula for nonlinear semigroups generated by the subdifferentiables of convex functionals, *J. Math. Soc. Japan* **30** (1978), 169-178.

[38] E. Lieb and M. Loss, A bound on binding energies and a mass renormalization in models of quantum electrodynamics, *J. Stat. Phys.* **108** (2002),

[39] M. Loss, T. Miyao and H. Spohn, Lowest energy states in nonrelativistic QED: atoms and ions in motion, mp-arc 06-136, preprint, 2006.

[40] J. Lőrinczi and R. A. Minlos, Gibbs measures for Brownian paths under the effect of an external and a small pair potential, *J. Stat. Phys.* **105** (2001), 605–647.

[41] H. Löwen, Spectral properties of an optical polaron in a magnetic field, *J. Math. Phys.* **29** (1988), 1498–1504.

[42] H. Löwen, Analytic behavior of the ground-state energy and pinning transitions for a bound polaron, *J. Math. Phys.* **29** (1988), 1505–1513.

[43] J. S. Mϕller, The translation invariant massive Nelson model: I. The bottom of the spectrum, *Ann. Henri Poincaré* **6** (2005), 1091–1135.

[44] J. S. Mϕller, On the essential spectrum of the translation invariant Nelson model, mp-arc 05-195 2005, preprint.

[45] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[46] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II*, Academic Press, 1975.

[47] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1978.

[48] I. Sasaki, Ground state of a model in relativistic quantum electrodynamics with a fixed total momentum, math-ph/0606029, preprint 2006.

[49] B. Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*, Princeton University Press 1974.

[50] A. D. Sloan, A nonperturbative approach to nondegeneracy of ground states in quantum field theory, polaron models, *J. Funct. Anal.* **16** (1974), 161–191.

[51] H. Spohn, The polaron at large total momentum, *J. Phys. A: Math. Gen.* **21** (1988), 1199–1211.

[52] H. Spohn, Effective mass of the polaron: A functional integral approach, *Ann. Phys.* **175** (1987), 278–318.

[53] H. Spohn, Dynamics of charged particles and their radiation field, Cambridge University Press, 2004.

[54] H. Spohn and S. Teufel, Semiclassical motion of dressed electrons, mp-arc 00-396, 2000, preprint.