SPECTRAL PROPERTIES OF POLYHARMONIC OPERATORS WITH LIMIT-PERIODIC POTENTIAL IN DIMENSION TWO.

YULIA KARPESHINA[1] AND YOUNG-RAN LEE[1]

In memory of our colleague and friend
Robert M. Kauffman.

ABSTRACT. We consider a polyharmonic operator $H = (-\Delta)^l + V(x)$ in dimension two with $l \geq 6$ and a limit-periodic potential $V(x)$. We prove that the spectrum of H contains a semiaxis and there is a family of generalized eigenfunctions at every point of this semiaxis with the following properties. First, the eigenfunctions are close to plane waves $e^{i\langle \vec{k}, \vec{x}\rangle}$ at the high energy region. Second, the isoenergetic curves in the space of momenta $\vec{k}$ corresponding to these eigenfunctions have a form of slightly distorted circles with holes (Cantor type structure).

1. INTRODUCTION

We study an operator

$$H = (-\Delta)^l + V(x) \tag{1.1}$$

in two dimensions, where $l \geq 6$, $V(x)$ is a limit-periodic potential:

$$V(x) = \sum_{r=1}^{\infty} V_r(x), \tag{1.2}$$

$\{V_r\}_{r=1}^{\infty}$ being a family of periodic potentials with doubling periods and decreasing L_∞-norms; namely, V_r has orthogonal periods $2^{r-1}\vec{b_1}$, $2^{r-1}\vec{b_2}$ and $\|V_r\|_\infty < \hat{C}exp(-2^{\eta r})$ for some $\eta > 2 + 64/(2l-11)$. Without loss of generality we assume that $\hat{C} = 1$ and $\int_{Q_r} V_r(x)dx = 0$, Q_r being the elementary cell of periods corresponding to $V_r(x)$.

The one-dimensional analog of (1.1), (1.2) with $l = 1$ is already thoroughly investigated. It is proven in [1]–[7] that the spectrum of the operator $H_1 u = -u'' + Vu$ is a Cantor type set. It has a positive Lebesgue measure [1, 6]. The spectrum is absolutely continuous [1, 2], [5]–[9]. Generalized eigenfunctions can be represented in the form of $e^{ikx}u(x)$, $u(x)$ being limit-periodic [5, 6, 7]. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11]–[14]. Properties of eigenfunctions of discrete multidimensional limit-periodic Schrödinger operators are studied in [15]. As to the continuum multidimensional case, it is proven [14], that the integrated density of states for (1.1) is the limit of densities of states for periodic operators. Here we study properties of the spectrum and eigenfunctions of (1.1), (1.2) in the high energy region. We prove the following results for the case $d = 2$, $l \geq 6$.

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(1) The spectrum of the operator (1.1), (1.2) contains a semiaxis. A proof of the analogous result by different means is to appear in the forthcoming paper [16]. In [16], more general case, $8l > d+3$, $d \neq 1(\mathrm{mod}4)$, is considered, however, under additional restriction on the potential: the lattices of periods of all periodic potentials V_r need to contain a nonzero vector $\vec{\gamma}$ in common, i.e., $V(x)$ is $\vec{\gamma}$-periodic.

(2) There are generalized eigenfunctions $\Psi_\infty(\vec{k},\vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k},\vec{x})$ of the equation $H\Psi_\infty = \lambda_\infty\Psi_\infty$ which can be described by the formula:

$$\Psi_\infty(\vec{k},\vec{x}) = e^{i\langle\vec{k},\vec{x}\rangle}\left(1+u_\infty(\vec{k},\vec{x})\right), \tag{1.3}$$

$$\|u_\infty\|_{L_\infty(\mathbb{R}^2)} \underset{|\vec{k}|\to\infty}{=} O\left(|\vec{k}|^{-\gamma_1}\right), \quad \gamma_1 > 0, \tag{1.4}$$

where $u_\infty(\vec{k},\vec{x})$ is a limit-periodic function:

$$u_\infty(\vec{k},\vec{x}) = \sum_{r=1}^{\infty} u_r(\vec{k},\vec{x}), \tag{1.5}$$

$u_r(\vec{k},\vec{x})$ being periodic with periods $2^{r-1}\vec{b_1},\ 2^{r-1}\vec{b_2}$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k},\vec{x})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_\infty(\vec{k}) \underset{|\vec{k}|\to\infty}{=} |\vec{k}|^{2l} + O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2 > 0. \tag{1.6}$$

The "non-resonant" set $\mathcal{G}_\infty$ of vectors $\vec{k}$, for which (1.3) – (1.6) hold, is an extensive Cantor type set: $\mathcal{G}_\infty = \cap_{n=1}^{\infty}\mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathbb{R}^2$. Each $\mathcal{G}_n$ has a finite number of holes in each bounded region. More and more holes appears when n increases, however holes added at each step are of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

$$|\mathcal{G}_\infty \cap \mathbf{B_R}| \underset{R\to\infty}{=} |\mathbf{B_R}|\big(1+O(R^{-\gamma_3})\big), \quad \gamma_3 > 0, \tag{1.7}$$

where $\mathbf{B_R}$ is the disk of radius R centered at the origin, $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k}\in\mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda\right\},$$

is proven to be a slightly distorted circle with infinite number of holes. It can be described by the formula:

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k} : \vec{k} = \varkappa_\infty(\lambda,\vec{\nu})\vec{\nu},\ \vec{\nu}\in\mathcal{B}_\infty(\lambda)\right\}, \tag{1.8}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (1.3). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda\to\infty$:

$$L\big(\mathcal{B}_\infty(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right), \tag{1.9}$$

here and below $L(\cdot)$ is a length of a curve. The value $\varkappa_\infty(\lambda,\vec\nu)$ in (1.8) is the "radius" of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec\nu$. The function $\varkappa_\infty(\lambda,\vec\nu)-\lambda^{1/2l}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$. It is proven that the deviation is small:

$$\varkappa_\infty(\lambda,\vec\nu) \underset{\lambda\to\infty}{=} \lambda^{1/2l}+O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4>0. \tag{1.10}$$

We will prove absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec k,\vec x)$ in a forthcoming paper.

To prove the results listed above we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method. This paper is inspired by [17, 18, 19], where the method is used for periodic problems. In [17] the KAM method is applied to classical Hamiltonian systems. In [18, 19] the technique developed in [17] is applied for semiclassical approximation for multidimensional periodic Schrödinger operators at high energies.

We consider a sequence of operators

$$H_0=(-\Delta)^l, \qquad H^{(n)}=H_0+\sum_{r=1}^{M_n}V_r, \quad n\geq 1, \quad M_n\to\infty \text{ as } n\to\infty.$$

Obviously, $\|H-H^{(n)}\|\to 0$ as $n\to\infty$ and $H^{(n)}=H^{(n-1)}+W_n$ where $W_n=\sum_{r=M_{n-1}+1}^{M_n}V_r$. We treat each operator $H^{(n)}$, $n\geq 1$, as a perturbation of the previous operator $H^{(n-1)}$. Every operator $H^{(n)}$ is periodic, however the periods go to infinity as $n\to\infty$. We will show that there is a λ_*, $\lambda_*=\lambda_*(V)$, such that the semiaxis $[\lambda_*,\infty)$ is contained in the spectra of **all** operators $H^{(n)}$. For every operator $H^{(n)}$ there is a set of eigenfunctions (corresponding to the semiaxis) close to plane waves: for any $\vec k$ in an extensive subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec k,\vec x)$ of the differential equation $H^{(n)}\Psi_n=\lambda^{(n)}\Psi_n$, which can be represented by the formula:

$$\Psi_n(\vec k,\vec x)=e^{i\langle\vec k,\vec x\rangle}\left(1+\tilde u_n(\vec k,\vec x)\right), \quad \|\tilde u_n\|_{L_\infty(\mathbb{R}^2)} \underset{|\vec k|\to\infty}{=} O(|\vec k|^{-\gamma_1}), \quad \gamma_1>0, \tag{1.11}$$

where $\tilde u_n(\vec k,\vec x)$ has periods $2^{M_n-1}b_1, 2^{M_n-1}b_2$.[1] The corresponding eigenvalue $\lambda^{(n)}(\vec k)$ is close to $|\vec k|^{2l}$:

$$\lambda^{(n)}(\vec k) \underset{|\vec k|\to\infty}{=} |\vec k|^{2l}+O\left(|\vec k|^{-\gamma_2}\right), \quad \gamma_2>0.$$

The non-resonant set $\mathcal{G}_n$ is proven to be extensive in $\mathbb{R}^2$:

$$|\mathcal{G}_n\cap\mathbf{B_R}| \underset{R\to\infty}{=} |\mathbf{B_R}|\big(1+O(R^{-\gamma_3})\big). \tag{1.12}$$

The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for non-resonant eigenvalue $\lambda^{(n)}(\vec k)$:

$$\mathcal{D}_n(\lambda)=\left\{\vec k\in\mathcal{G}_n:\lambda^{(n)}(\vec k)=\lambda\right\}.$$

This set is proven to be a slightly distorted circle with a finite number of holes (see Fig. 1, 2), the set $\mathcal{D}_1(\lambda)$ being strictly inside the circle of the radius $\lambda^{1/2l}$ for sufficiently large λ. The set $\mathcal{D}_n(\lambda)$ can be described by the formula:

$$\mathcal{D}_n(\lambda)=\left\{\vec k:\vec k=\varkappa_n(\lambda,\vec\nu)\vec\nu,\ \vec\nu\in\mathcal{B}_n(\lambda)\right\}, \tag{1.13}$$

[1]Obviously, $\tilde u_n(\vec k,\vec x)$ is simply related to functions $u_r(\vec k,\vec x)$ used in (1.5): $\tilde u_n(\vec k,\vec x)=\sum_{r=M_{n-1}+1}^{M_n}u_r(\vec k,\vec x)$.

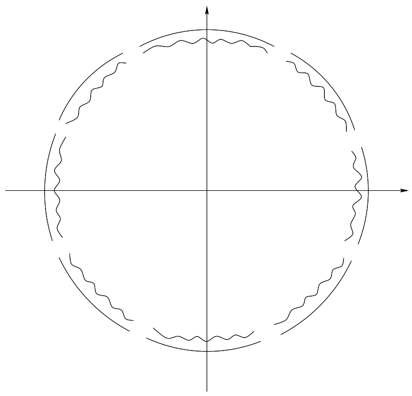

FIGURE 1. Distorted circle with holes, $\mathcal{D}_1(\lambda)$

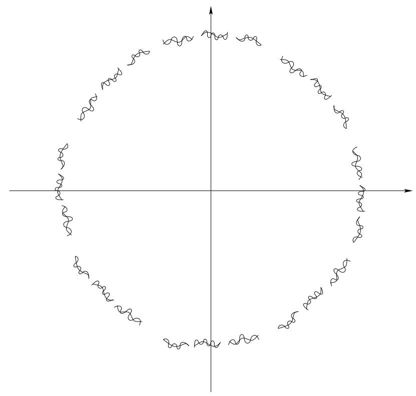

FIGURE 2. Distorted circle with holes, $\mathcal{D}_2(\lambda)$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (1.11). It has an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\big(\mathcal{B}_n(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right). \tag{1.14}$$

The set $\mathcal{B}_n(\lambda)$ has only a finite number of holes, however their number is growing with n. More and more holes of a smaller and smaller size are added at each step. The value $\varkappa_n(\lambda,\vec{\nu}) - \lambda^{1/2l}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small:

$$\varkappa_n(\lambda,\vec{\nu}) = \lambda^{1/2l} + O\left(\lambda^{-\gamma_4}\right), \qquad \frac{\partial \varkappa_n(\lambda,\vec{\nu})}{\partial\varphi} = O\left(\lambda^{-\gamma_5}\right) \quad \gamma_4,\gamma_5 > 0, \tag{1.15}$$

φ being an angle variable, $\vec{\nu} = (\cos\varphi, \sin\varphi)$.

On each step more and more points are excluded from the non-resonant sets $\mathcal{G}_n$, thus $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is defined as the limit set: $\mathcal{G}_\infty = \cap_{n=1}^{\infty}\mathcal{G}_n$. It has an infinite number of holes, but nevertheless satisfies the relation (1.7). For every $\vec{k} \in \mathcal{G}_\infty$ and every n, there is a generalized eigenfunction of $H^{(n)}$ of the type (1.11). It is proven that the sequence of $\Psi_n(\vec{k},\vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ when $\vec{k} \in \mathcal{G}_\infty$. The function $\Psi_\infty(\vec{k},\vec{x}) = \lim_{n\to\infty}\Psi_n(\vec{k},\vec{x})$ is a generalized eigenfunction of H. It can be written in the form (1.3) – (1.5). Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda^{(n)}(\vec{k})$ as $n \to \infty$.

It is shown that $\{\mathcal{B}_n(\lambda)\}_{n=1}^{\infty}$ is a decreasing sequence of sets, on each step more and more directions being excluded. We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$:

$$\mathcal{B}_\infty(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{B}_n(\lambda). \tag{1.16}$$

This set has a Cantor type structure on the unit circle. It is proven that $\mathcal{B}_\infty(\lambda)$ has an asymptotically full measure on the unit circle (see (1.9)). We prove that the sequence $\varkappa_n(\lambda,\vec{\nu})$, $n = 1, 2, ...,$, describing the isoenergetic curves $\mathcal{D}_n(\lambda)$, quickly converges as $n \to \infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense (1.8), where $\varkappa_\infty(\lambda,\vec{\nu}) = \lim_{n\to\infty}\varkappa_n(\lambda,\vec{\nu})$ for every

$\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. It is shown that the derivatives of the functions $\varkappa_n(\lambda, \vec{\nu})$ (with respect to the angle variable on the unit circle) have a limit as $n \to \infty$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial \varkappa_\infty(\lambda,\vec{\nu})}{\partial \varphi}$. It follows from (1.15) that

$$\frac{\partial \varkappa_\infty(\lambda, \vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right). \tag{1.17}$$

Thus, the limit curve $\mathcal{D}_\infty(\lambda)$ has a tangent vector in spite of its Cantor type structure, the tangent vector being the limit of corresponding tangent vectors for $\mathcal{D}_n(\lambda)$ as $n \to \infty$. The curve $\mathcal{D}_\infty(\lambda)$ looks as a slightly distorted circle with infinite number of holes.

The main technical difficulty overcome is construction of non-resonance sets $\mathcal{B}_n(\lambda)$ for every fixed sufficiently large λ, $\lambda > \lambda_0(V)$, where $\lambda_0(V)$ is the same for all n. The set $\mathcal{B}_n(\lambda)$ is obtained by deleting a "resonant" part from $\mathcal{B}_{n-1}(\lambda)$. Definition of $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$ includes Bloch eigenvalues of $H^{(n-1)}$. To describe $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$ one has to use not only non-resonant eigenvalues of the type (1.6), but also resonant eigenvalues, for which no suitable formulae are known. Absence of formulae causes difficulties in estimating the size of $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$. To deal with this problem we start with introducing an angle variable $\varphi \in [0, 2\pi)$, $\vec{\nu} = (\cos\varphi, \sin\varphi) \in S_1$ and consider sets $\mathcal{B}_n(\lambda)$ in terms of this variable. Next, we show that the resonant set $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$ can be described as the set of zeros of determinants of the type $Det\big(I + S_n(\varphi)\big)$, $S_n(\varphi)$ being a trace type operator,

$$I + S_n(\varphi) = \Big(H^{(n-1)}\big(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\big) - \lambda - \epsilon\Big)\Big(H_0\big(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\big) + \lambda\Big)^{-1},$$

where $\vec{\varkappa}_{n-1}(\varphi)$ is a vector-function describing $\mathcal{D}_{n-1}(\lambda)$: $\vec{\varkappa}_{n-1}(\varphi) = \varkappa_{n-1}(\lambda, \vec{\nu})\vec{\nu}$. To obtain $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$ we take all values of ϵ in a small interval and vectors $\vec{b}$ in a finite set, $\vec{b} \neq 0$. Further, we extend our considerations to a complex neighborhood $\mathit{\Phi}_0$ of $[0, 2\pi)$. We show that the determinants are analytic functions of φ in $\mathit{\Phi}_0$, and, by this, reduce the problem of estimating the size of the resonance set to a problem in complex analysis. We use theorems for analytic functions to count zeros of the determinants and to investigate how far zeros move when ε changes. It enables us to estimate the size of the zero set of the determinants, and, hence, the size of the non-resonance set $\mathit{\Phi}_n \subset \mathit{\Phi}_0$, which is defined as a non-zero set for the determinants. Proving that the non-resonance set $\mathit{\Phi}_n$ is sufficiently large, we obtain estimates (1.12) for $\mathcal{G}_n$ and (1.14) for $\mathcal{B}_n$, the set $\mathcal{B}_n$ being the real part of $\mathit{\Phi}_n$. To obtain $\mathit{\Phi}_n$ we delete from $\mathit{\Phi}_0$ more and more holes of smaller and smaller radii at each step. Thus, the non-resonance set $\mathit{\Phi}_n \subset \mathit{\Phi}_0$ has a structure of Swiss Cheese (Fig. 7, 8, pages 26, 45). Deleting resonance set from $\mathit{\Phi}_0$ at each step of the recurrent procedure we call a "Swiss Cheese Method". The essential difference of our method from those applied in similar situations before (see e.g. [17]–[20]) is that we construct a non-resonance set not only in the whole space of a parameter ($\vec{k} \in \mathbb{R}^2$ here), but also on isoenergetic curves $\mathcal{D}_n(\lambda)$ in the space of the parameter, when λ is sufficiently large. Estimates for the size of non-resonance sets on a curve require more subtle technical considerations than those sufficient for description of a non-resonant set in the whole space of the parameter.

The restriction $l \geq 6$ is technical, it is needed only for the first two steps of the recurrent procedure. The requirement for super exponential decay of $\|V_r\|$ as $r \to \infty$ is more essential, since it is needed to ensure convergence of the recurrent procedure. It is not essential that the potential V_r has doubling periods, the periods of the type $q^{r-1}\vec{b}_1$, $q^{r-1}\vec{b}_2, q \in \mathbb{N}$, can be treated in the same way. First consideration of the problem was given in [23].

The plan of the paper is the following. Sections 2 – 6 describe steps of the recurrent procedure. Discussion of convergence of the procedure and proofs of the results 1 – 3, listed above, are in Section 7.

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2. The First Approximation

2.1. **Operator $H^{(1)}$.** We introduce the first operator $H^{(1)}$, which corresponds to a partial sum in the series (1.2):

$$H^{(1)} = (-\Delta)^l + W_1, \qquad W_1 = \sum_{r=1}^{M_1} V_r, \tag{2.1}$$

where M_1 is chosen in such a way that $2^{M_1} \approx k^{s_1}$ [2] for a $k > 1$, $s_1 = (2l - 11)/32$. Obviously, the periods of W_1 are $(a_1, 0) = 2^{M_1-1}(b_1, 0)$ and $(0, a_2) = 2^{M_1-1}(0, b_2)$, and $a_1 \approx k^{s_1} b_1/2$, $a_2 \approx k^{s_1} b_2/2$. Note that $\|W_1\|_\infty \leq \sum_{r=1}^{M_1} \|V_r\|_\infty = O(1)$ as $k \to \infty$.

It is well-known (see e.g. [21]) that spectral analysis of a periodic operator $H^{(1)}$ can be reduced to analysis of a family of operators $H^{(1)}(t)$, $t \in K_1$, where K_1 is the elementary cell of the dual lattice, $K_1 = [0, 2\pi a_1^{-1}) \times [0, 2\pi a_2^{-1})$. The vector t is called *quasimomentum*. An operator $H^{(1)}(t)$, $t \in K_1$, acts in $L_2(Q_1)$, Q_1 being the elementary cell of the periods of the potential, $Q_1 = [0, a_1] \times [0, a_2]$. The operator $H^{(1)}(t)$ is described by formula (2.1) and the quasiperiodic conditions for a function itself and its derivatives:

$$u(a_1, x_2) = \exp(it_1 a_1) u(0, x_2), \quad u(x_1, a_2) = \exp(it_2 a_2) u(x_1, 0), \tag{2.2}$$

$$u_{x_1}^{(j)}(a_1, x_2) = \exp(it_1 a_1) u_{x_1}^{(j)}(0, x_2), \quad u_{x_2}^{(j)}(x_1, a_2) = \exp(it_2 a_2) u_{x_2}^{(j)}(x_1, 0),$$

$0 < j < 2l$. Each operator $H^{(1)}(t)$, $t \in K_1$, has a discrete bounded below spectrum $\Lambda^{(1)}(t)$:

$$\Lambda^{(1)}(t) = \cup_{n=1}^{\infty} \lambda_n^{(1)}(t), \ \lambda_n^{(1)}(t) \to_{n \to \infty} \infty.$$

The spectrum $\Lambda^{(1)}$ of the operator $H^{(1)}$ is the union of the spectra of the operators $H^{(1)}(t)$: $\Lambda^{(1)} = \cup_{t \in K_1} \Lambda(t) = \cup_{n \in N, t \in K_1} \lambda_n^{(1)}(t)$. The functions $\lambda_n^{(1)}(t)$ are continuous, so $\Lambda^{(1)}$ has a band structure:

$$\Lambda^{(1)} = \cup_{n=1}^{\infty} [q_n^{(1)}, Q_n^{(1)}], \ q_n^{(1)} = \min_{t \in K_1} \lambda_n^{(1)}(t), \ Q_n^{(1)} = \max_{t \in K_1} \lambda_n^{(1)}(t). \tag{2.3}$$

Eigenfunctions of $H^{(1)}(t)$ and $H^{(1)}$ are simply related. Extending all the eigenfunctions of the operators $H^{(1)}(t)$ quasiperiodically (see (2.2)) to $\mathbb{R}^2$, we obtain a complete system of generalized eigenfunctions of $H^{(1)}$.

Let $H_0^{(1)}$ be the operator (1.1) corresponding to $V = 0$. We consider that it has periods a_1, a_2 and that operators $H_0^{(1)}(t)$, $t \in K_1$, are defined in $L_2(Q_1)$. Eigenfunctions of an operator $H_0^{(1)}(t)$, $t \in K_1$, are plane waves satisfying (2.2). They are naturally indexed by points of $\mathbb{Z}^2$: $\Psi_j^0(t, x) = |Q_1|^{-1/2} \exp i\langle \vec{p}_j(t), x\rangle$, $j \in \mathbb{Z}^2$, the eigenvalue corresponding to $\Psi_j^0(t, x)$ being equal to

[2] We write $a(k) \approx b(k)$ when the inequalities $\frac{1}{2} b(k) \leq a(k) \leq 2b(k)$ hold.

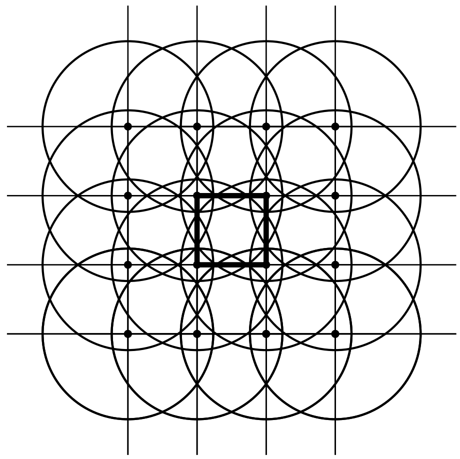

FIGURE 3. The isoenergetic surface $S_0(\lambda)$ of the free operator $H_0^{(1)}$

$p_j^{2l}(t)$, here and below $\vec{p}_j(t) = 2\pi j/a + t$, $2\pi j/a = (2\pi j_1/a_1, 2\pi j_2/a_2)$, $j \in \mathbb{Z}^2$, $|Q_1| = a_1 a_2$ and $p_j^{2l}(t) = |\vec{p}_j(t)|^{2l}$.

Next, we introduce an isoenergetic surface[3] $S_0(\lambda)$ of the free operator $H_0^{(1)}$. A point t belongs to $S_0(\lambda)$ if and only if $H_0^{(1)}(t)$ has an eigenvalue equal to λ, i.e., there exists a $j \in \mathbb{Z}^2$ such that $p_j^{2l}(t) = \lambda$. This surface can be obtained as follows: the circle of radius $k = \lambda^{\frac{1}{2l}}$ centered at the origin is divided into pieces by the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$, and then all pieces are translated in a parallel manner into the cell K_1 of the dual lattice. We also can get $S_0(\lambda)$ by drawing sufficiently many circles of radii k centered at the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$ and by looking at the figure in the cell K_1. As a result of any of these two procedures we obtain a circle of radius k "packed into the bag K_1" as it is shown in the Fig. 3. Note that each piece of $S_0(\lambda)$ can be described by an equation $p_j^{2l}(t) = \lambda$ for a fixed j. If $t \in S_0(\lambda)$, then j can be uniquely defined from the last equation, unless t is not the point of a self-intersection of the isoenergetic surface. A point t is a self-intersection of $S_0(\lambda)$ if and only if

$$p_q^{2l}(t) = p_j^{2l}(t) = k^{2l} \tag{2.4}$$

for at least on pair of indices $q,\ j,\ q \neq j$.

Note that any vector $\vec{\varkappa}$ in $\mathbb{R}^2$ can be uniquely represented in the form $\vec{\varkappa} = \vec{p}_j(t)$, where $j \in \mathbb{Z}^2$ and $t \in K_1$. Let $\mathcal{K}_1$ be the parallel shift into K_1: $\mathcal{K}_1 : \mathbb{R}^2 \to K_1$, $\mathcal{K}_1\vec{p}_j(t) = t$. Suppose $\Omega \subset \mathbb{R}^2$. In order to obtain $\mathcal{K}_1\Omega$ it is necessary to partition Ω by the lattice with nodes at the points $\vec{p}_q(0)$, $q \in \mathbb{Z}^2$ and to shift all parts in a parallel manner into a single cell. It is obvious that $|\mathcal{K}_1\Omega| \leq |\Omega|$ for any Ω. If Ω is a smooth curve, then

$$L(\mathcal{K}_1\Omega) \leq L(\Omega). \tag{2.5}$$

[3]"surface" is a traditional term. In our case, it is a curve.

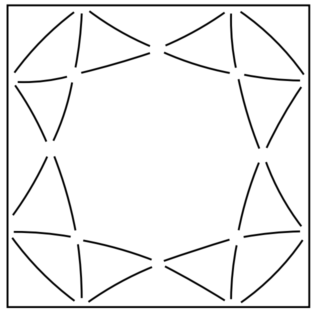

FIGURE 4. The first non-resonance set $\chi_1(\lambda,\delta)$

For any pair of sets Ω_1 and Ω_2, $\mathcal{K}_1\big(\Omega_1\cup\Omega_2\big)=\mathcal{K}_1\Omega_1\cup\mathcal{K}_1\Omega_2$. Obviously, $\mathcal{K}_1S_k=S_0(\lambda)$ and $L\big(S_0(\lambda)\big)=L(S_k)=2\pi k$, $k=\lambda^{\frac{1}{2l}}$, S_k being the circle of radius k centered at the origin.

The operator $H^{(1)}$ has the following matrix representation in the basis of plane waves: $H^{(1)}(t)_{mq}=p_m^{2l}(t)\delta_{mq}+w_{m-q}$, [4] here and below w_{m-q} are Fourier coefficients of W_1, the coefficient w_0 being equal to zero.

2.2. **Perturbation Formulae.** In this section we consider operator $H^{(1)}(t)$ as a perturbation of the free operator $H_0^{(1)}(t)$. We show that for every sufficiently large λ there is a "non-resonant" subset $\chi_1(\lambda)$ of $S_0(\lambda)$ such that perturbation series for an eigenvalue and a spectral projection of $H^{(1)}(t)$ converge when $t\in\chi_1(\lambda)$. The set $\chi_1(\lambda)$ is obtained by deleting small neighborhoods of selfintersections of $S_0(\lambda)$, see Fig. 4. The selfintersections are described by (2.4) and correspond to degenerated eigenvalues of $H_0^{(1)}(t)$. The size of the neighborhood is $k^{-1-4s_1-\delta}$, $k=\lambda^{\frac{1}{2l}}$, δ being a small positive number. The set $\chi_1(\lambda)$ is sufficiently large: its relative measure with respect to $S_0(\lambda)$ tends to 1 as $\lambda\to\infty$. The precise formulation of these results is given in the next Geometric Lemma. The lemma is proven by elementary geometric considerations in [22].

Lemma 2.1 (Geometric Lemma). *For an arbitrarily small positive δ, $2\delta<2l-2-4s_1$, and sufficiently large λ, $\lambda>\lambda_0(\delta)$, there exists a nonresonance set $\chi_1(\lambda,\delta)\subset S_0(\lambda)$ such that:*

(1) *For any point $t\in\chi_1$ the following conditions hold:*
   (a) *There exists a unique $j\in\mathbb{Z}^2$ such that $p_j(t)=k$, $k=\lambda^{\frac{1}{2l}}$.*
   (b) *The inequality holds:*
$$\min_{i\neq j}|p_j^2(t)-p_i^2(t)|>2k^{-4s_1-\delta}. \tag{2.6}$$
(2) *For any t in the $(2k^{-1-4s_1-2\delta})$-neighborhood of the nonresonance set in $\mathbb{C}^2$, there exists a unique $j\in\mathbb{Z}^2$ such that*
$$|p_j^2(t)-k^2|<5k^{-4s_1-2\delta} \tag{2.7}$$
and the inequality (2.6) holds.
(3) *The nonresonance set χ_1 has an asymptotically full measure on $S_0(\lambda)$ in the following sense:*
$$\frac{L\left(S_0(\lambda)\setminus\chi_1(\lambda,\delta)\right)}{L\left(S_0(\lambda)\right)}=_{\lambda\to\infty}O(k^{-\delta/2}). \tag{2.8}$$

[4] δ_{mq} is the Kronecker symbol.

Corollary 2.2. *If t belongs to the $(2k^{-1-4s_1-2\delta})$-neighborhood of the nonresonance set $\chi_1(\lambda,\delta)$ in $\mathbb{C}^2$, then, for any z lying on the circle*

$$C_1 = \{z : |z - k^{2l}| = k^{2l-2-4s_1-\delta}\} \tag{2.9}$$

and any i in $\mathbb{Z}^2$, the following inequality holds:

$$2|p_i^{2l}(t) - z| > k^{2l-2-4s_1-\delta}. \tag{2.10}$$

Corollary 2.2. Let $t \in \chi_1(\lambda,\delta)$. Taking into account the relation $p_j^{2l}(t) = k^{2l}$ and the definition of C_1, we see that

$$|p_j^{2l}(t) - z| = k^{2l-2-4s_1-\delta}. \tag{2.11}$$

Thus, the estimate (2.10) is valid for $i = j$. From (2.6), it follows that $\min_{i\neq j} |p_j^{2l}(t) - p_i^{2l}(t)| > 2k^{2l-2-4s_1-\delta}$. Using the last inequality and (2.11), we obtain inequality (2.10) for $i \neq j$. It is easy to see that all the estimates are stable under a perturbation of t of order $2k^{-1-4s_1-2\delta}$. Therefore, the estimate (2.10) can be extended to the $(2k^{-1-4s_1-2\delta})$-neighborhood of χ_1. ∎

Let $E_j(t)$ be the spectral projection of the free operator, corresponding to the eigenvalue $p_j^{2l}(t)$: $(E_j)_{rm} = \delta_{jr}\delta_{jm}$. In the $(2k^{-1-4s_1-2\delta})$-neighborhood of $\chi_1(\lambda,\delta)$, we define functions $g_r^{(1)}(k,t)$ and operator-valued functions $G_r^{(1)}(k,t)$, $r = 1, 2, \cdots$ as follows:

$$g_r^{(1)}(k,t) = \frac{(-1)^r}{2\pi i r} \mathrm{Tr} \oint_{C_1} ((H_0^{(1)}(t) - z)^{-1} W_1)^r dz, \tag{2.12}$$

$$G_r^{(1)}(k,t) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_1} ((H_0^{(1)}(t) - z)^{-1} W_1)^r (H_0^{(1)}(t) - z)^{-1} dz. \tag{2.13}$$

To find $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$ it is necessary to compute the residues of a rational function of a simple structure, whose numerator does not depend on z, while the denominator is a product of factors of the type $(p_i^{2l}(t) - z)$. For all t in the nonresonance set within C_1 the integrand has a single pole at the point $z = k^{2l} = p_j^{2l}(t)$. By computing the residue at this point, we obtain explicit expressions for $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$. For example, $g_1^{(1)}(k,t) = 0$,

$$\begin{aligned} g_2^{(1)}(k,t) &= \sum_{q\in\mathbb{Z}^2, q\neq 0} |w_q|^2 (p_j^{2l}(t) - p_{j+q}^{2l}(t))^{-1} \\ &= \sum_{q\in\mathbb{Z}^2, q\neq 0} \frac{|w_q|^2 (2p_j^{2l}(t) - p_{j+q}^{2l}(t) - p_{j-q}^{2l}(t))}{2(p_j^{2l}(t) - p_{j+q}^{2l}(t))(p_j^{2l}(t) - p_{j-q}^{2l}(t))}, \end{aligned} \tag{2.14}$$

$$G_1^{(1)}(k,t)_{rm} = \frac{w_{j-m}}{p_j^{2l}(t) - p_m^{2l}(t)} \delta_{rj} + \frac{w_{r-j}}{p_j^{2l}(t) - p_r^{2l}(t)} \delta_{mj}, \quad G_1^{(1)}(k,t)_{jj} = 0. \tag{2.15}$$

It is not difficult to show that $g_2^{(1)}(k,t) > 0$ for sufficiently large λ. For technical reasons it is convenient to introduce parameter α in front of the potential W_1. Namely, $H_\alpha^{(1)} = (-\Delta)^l + \alpha W_1$, $0 \leq \alpha \leq 1$, $H_1^{(1)} \equiv H^{(1)}$.

Theorem 2.3. *Suppose t belongs to the $(2k^{-1-4s_1-2\delta})$-neighborhood in K_1 of the nonresonance set $\chi_1(\lambda,\delta)$, $0 < 2\delta < 2l - 2 - 4s_1$. Then for sufficiently large λ, $\lambda > \lambda_0(\|V\|, b_1, b_2, \delta)$, for all*

α, $-1 \le \alpha \le 1$, there exists a single eigenvalue of the operator $H_\alpha^{(1)}(t)$ in the interval $\varepsilon_1(k,\delta) := (k^{2l} - k^{2l-2-4s_1-\delta}, k^{2l} + k^{2l-2-4s_1-\delta})$. It is given by the series:

$$\lambda_j^{(1)}(\alpha,t) = p_j^{2l}(t) + \sum_{r=2}^{\infty} \alpha^r g_r^{(1)}(k,t), \tag{2.16}$$

converging absolutely in the disk $|\alpha| \le 1$, where the index j is determined according to articles 1(a) and 2 of Geometric Lemma. The spectral projection, corresponding to $\lambda_j^{(1)}(\alpha,t)$, is given by the series:

$$E_j^{(1)}(\alpha,t) = E_j + \sum_{r=1}^{\infty} \alpha^r G_r^{(1)}(k,t), \tag{2.17}$$

which converges in the class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha| \le 1$.

For coefficients $g_r^{(1)}(k,t)$, $G_r^{(1)}(k,t)$ the following estimates hold:

$$|g_r^{(1)}(k,t)| < k^{2l-2-4s_1-\gamma_0 r-\delta}, \quad \|G_r^{(1)}(k,t)\|_1 < k^{-\gamma_0 r}, \tag{2.18}$$

where $\gamma_0 = 2l - 2 - 4s_1 - 2\delta$.

Corollary 2.4. *For the perturbed eigenvalue and its spectral projection the following estimates are valid:*

$$|\lambda_j^{(1)}(\alpha,t) - p_j^{2l}(t)| \le 2\alpha^2 k^{2l-2-4s_1-2\gamma_0-\delta}, \tag{2.19}$$

$$\|E_j^{(1)}(\alpha,t) - E_j\|_1 \le 2|\alpha| k^{-\gamma_0}. \tag{2.20}$$

Remark 2.5. The theorem states that $\lambda_j^{(1)}(\alpha,t)$ is a single eigenvalue in the interval $\varepsilon_1(k,\delta)$. This means that $|\lambda_j^{(1)}(\alpha,t) - k^{2l}| < k^{2l-2-4s_1-\delta}$. Formula (2.19) provides a stronger estimate on the location of $\lambda_j^{(1)}(\alpha,t)$, the right-hand side of (2.19) being smaller than the size of ε_1.

Proof. The proof of the theorem is based on expanding the resolvent $(H_\alpha^{(1)}(t) - z)^{-1}$ in a perturbation series for z, belonging to the contour C_1 about the unperturbed eigenvalue $p_j^{2l}(t)$. Then integrating the resolvent yields the formulae for the perturbed eigenvalue and its spectral projection.

It is obvious that

$$(H_\alpha^{(1)}(t) - z)^{-1} = (H_0^{(1)}(t) - z)^{-1}(I - \alpha A_1)^{-1}, \quad A_1 := -W_1(H_0^{(1)}(t) - z)^{-1}. \tag{2.21}$$

Suppose $z \in C_1$. Using (2.10), we obtain:

$$\|(H_0^{(1)}(t) - z)^{-1}\| \le 2k^{-2l+2+4s_1+\delta}, \quad \|A_1\| \le 2\|W_1\| k^{-\gamma_0-\delta}. \tag{2.22}$$

Thus, $\|A_1\| \ll 1$ for sufficiently large k. Expanding $(I - \alpha A_1)^{-1}$ in a series in powers of αA_1 we obtain:

$$(H_\alpha^{(1)}(t) - z)^{-1} - (H_0^{(1)}(t) - z)^{-1} = \sum_{r=1}^{\infty} \alpha^r (H_0^{(1)}(t) - z)^{-1} A_1^r. \tag{2.23}$$

Since $2l - 2 - 2s_1 > 0$ it is not hard to show that $(H_0^{(1)}(t) - z)^{-1} \in \mathbf{S_1}$. Taking into account estimates (2.22), we see that the series (2.23) converges in the class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha| \le 1$. Substituting the series into the formula for a spectral projection

$E_j^{(1)}(\alpha,t) = -\frac{1}{2\pi i}\oint_{C_1}(H_\alpha^{(1)}(t)-z)^{-1}dz$ and integrating termwise, we arrive at (2.17). Formula (2.16) is obtained in the analogous way. Estimates (2.18) follow from (2.22), for details see [22]. ∎

Next, we show that the series (2.16), (2.17) can be extended as holomorphic functions of t in a complex neighborhood of χ_1; they can be differentiated with respect to t any number of times with retaining their asymptotic character.

Let us introduce the notations:

$$T(m) := \frac{\partial^{|m|}}{\partial t_1^{m_1}\partial t_2^{m_2}}, \quad m=(m_1,m_2), \quad |m| := m_1+m_2, \quad m! := m_1!m_2!, \ T(0)f := f.$$

Lemma 2.6. *The coefficients $g_r^{(1)}(k,t)$, and $G_r^{(1)}(k,t)$ can be continued as holomorphic functions of two variables from the real $(2k^{-1-4s_1-2\delta})$-neighborhood of the nonresonance set χ_1 to its complex $(2k^{-1-4s_1-2\delta})$-neighborhood and the following estimates hold in the complex neighborhood:*

$$|T(m)g_r^{(1)}(k,t)| < m!k^{2l-2-4s_1-\delta-\gamma_0 r+|m|(1+4s_1+2\delta)}, \tag{2.24}$$

$$\|T(m)G_r^{(1)}(k,t)\|_1 < m!k^{-\gamma_0 r+|m|(1+4s_1+2\delta)}. \tag{2.25}$$

Proof. Since (2.10) is valid in the complex $(2k^{-1-4s_1-2\delta})$-neighborhood of the nonresonance set, it is not hard to see that the coefficients $g_r^{(1)}(k,t)$, and $G_r^{(1)}(k,t)$ can be continued from the real $(2k^{-1-4s_1-2\delta})$-neighborhood of t to the complex $(2k^{-1-4s_1-2\delta})$-neighborhood as holomorphic functions of two variables, and inequalities (2.18) are hereby preserved. Estimating by means of the Cauchy integral formula, the value of the derivative with respect to t in terms of the value of the function itself on the boundary of the $(2k^{-1-4s_1-2\delta})$-neighborhood of t (formulas (2.18)), we obtain (2.24) and (2.25). ∎

From this lemma the following theorem easily follows.

Theorem 2.7. *The series (2.16), (2.17) can be continued as holomorphic functions of two variables from the real $(2k^{-1-4s_1-2\delta})$-neighborhood of the nonresonance set χ_1 to its complex $(2k^{-1-4s_1-2\delta})$-neighborhood and the following estimates hold in the complex neighborhood:*

$$|T(m)(\lambda_j^{(1)}(\alpha,t)-p_j^{2l}(t))| < 2m!\alpha^2 k^{2l-2-4s_1-\delta-2\gamma_0+|m|(1+4s_1+2\delta)}, \tag{2.26}$$

$$\|T(m)(E_j^{(1)}(\alpha,t)-E_j)\|_1 < 2m!\alpha k^{-\gamma_0+|m|(1+4s_1+2\delta)}. \tag{2.27}$$

Corollary 2.8.

$$\nabla\lambda_j^{(1)}(\alpha,t) = 2lk^{2l-2}\vec{k} + o\left(k^{2l-1}\right), \qquad \vec{k} = \vec{p}_j(t), \tag{2.28}$$

$$T(m)\lambda_j^{(1)}(\alpha,t) < 4l^2k^{2l-2}, \quad \textit{if} \quad |m| = 2. \tag{2.29}$$

The next lemma will be used in the second step of approximation where the operator $H^{(1)}(t)$ will play a role of the initial (unperturbed) operator.

Lemma 2.9. *For any z on the circle C_1 given by (2.9) and t in the complex $(2k^{-1-4s_1-2\delta})$-neighborhood of χ_1,*

$$\|(H^{(1)}(t)-z)^{-1}\| \le 4k^{-2l+2+4s_1+\delta}. \tag{2.30}$$

Proof. Considering the following Hilbert relation

$$(H^{(1)}(t)-z)^{-1}=(H_0^{(1)}(t)-z)^{-1}+(H_0^{(1)}(t)-z)^{-1}(-W_1)(H^{(1)}(t)-z)^{-1}$$

and (2.10), i.e., $\|(H_0^{(1)}(t)-z)^{-1}\|\leq 2k^{-2l+2+4s_1+\delta}$, we obtain

$$\|(H^{(1)}(t)-z)^{-1}\|\leq\frac{\|(H_0^{(1)}(t)-z)^{-1}\|}{1-\|(H_0^{(1)}(t)-z)^{-1}W_1\|}\leq 2\|(H_0^{(1)}(t)-z)^{-1}\|\leq 4k^{-2l+2+4s_1+\delta}.$$

∎

2.3. **Nonresonant part of the Isoenergetic Set of $H^{(1)}$.** Let $S_1(\lambda)$ [5] be the isoenergetic surface of the operator $H_\alpha^{(1)}$

$$S_1(\lambda)=\{t\in K_1:\exists n\in\mathbb{N}\text{ s.t. }\lambda_n^{(1)}(\alpha,t)=\lambda\},\tag{2.31}$$

where $\{\lambda_n^{(1)}(\alpha,t)\}_{n=1}^{\infty}$ is the complete set of eigenvalues of $H_\alpha^{(1)}(t)$. Next, we construct a "nonresonance" subset $\chi_1^*(\lambda)$ of $S_1(\lambda)$, which corresponds to nonresonance eigenvalues $\lambda_j^{(1)}(\alpha,t)$ given by the perturbation series.

Let us note that for every t belonging to the non-resonant set $\chi_1(\lambda,\delta)$ described by Lemma 2.1, there is a single $j\in\mathbb{Z}^2$ such that $p_j(t)=k$, $k=\lambda^{\frac{1}{2l}}$. This means that the function $t\to\vec{p}_j(t)$ maps $\chi_1(\lambda,\delta)$ into the circle S_k. The image of $\chi_1(\lambda,\delta)$ in S_k we denote by $\mathcal{D}_0(\lambda)_{nonres}$. Obviously,

$$\chi_1(\lambda,\delta)=\mathcal{K}_1\mathcal{D}_0(\lambda)_{nonres},\tag{2.32}$$

where $\mathcal{K}_1$ establishes one-to-one relation between two sets. Let $\mathcal{B}_1(\lambda)$ is a set of unit vectors corresponding to $\mathcal{D}_0(\lambda)_{nonres}$:

$$\mathcal{B}_1(\lambda)=\{\vec{\nu}\in S_1:k\vec{\nu}\in\mathcal{D}_0(\lambda)_{nonres}\}.$$

It is easy to see that $\mathcal{B}_1(\lambda)$ is a unit circle with holes, centered at the origin. We denote by $\Theta_1(\lambda)$ the set of angles φ in polar coordinates, corresponding to $\mathcal{B}_1(\lambda)$:

$$\Theta_1(\lambda)=\{\varphi\in[0,2\pi):\ (\cos\varphi,\sin\varphi)\in\mathcal{B}_1(\lambda)\}.\tag{2.33}$$

Let $\vec{\varkappa}\in\mathcal{D}_0(k)_{nonres}$. Then there is $j\in\mathbb{Z}^2$, $t\in\chi_1(\lambda,\delta)$ such that $\vec{\varkappa}=\vec{p}_j(t)$.[6] Obviously, $t=\mathcal{K}_1\vec{\varkappa}$ and, by (2.32), $t\in\chi_1(\lambda,\delta)$. According to Theorem 2.3, for sufficiently large k, there exists an eigenvalue of the operator $H_\alpha^{(1)}(t)$, $t=\mathcal{K}_1\vec{\varkappa}$, $0\leq\alpha\leq 1$, given by (2.16). It is convenient here to denote $\lambda_j^{(1)}(\alpha,t)$ by $\lambda^{(1)}(\alpha,\vec{\varkappa})$, we can do this since there is one-to-one correspondence between $\vec{\varkappa}$ and the pair (t,j). We rewrite (2.16) in the form:

$$\lambda^{(1)}(\alpha,\vec{\varkappa})=\varkappa^{2l}+f_1(\alpha,\vec{\varkappa}),\quad \varkappa=|\vec{\varkappa}|,\quad f_1(\alpha,\vec{\varkappa})=\sum_{r=2}^{\infty}\alpha^r g_r^{(1)}(\vec{\varkappa}),\tag{2.34}$$

$g_r^{(1)}(\vec{\varkappa})$ being defined by (2.12) with j and t such that $\vec{p}_j(t)=\vec{\varkappa}$, and $f_1(\alpha,\vec{\varkappa})$ satisfying the estimates:

$$|f_1(\alpha,\vec{\varkappa})|\leq 2\alpha^2k^{2l-2-4s_1-2\gamma_0-\delta},\tag{2.35}$$

$$|T(m)f_1(\alpha,\vec{\varkappa})|\leq 2\alpha^2k^{2l-2-4s_1-2\gamma_0-\delta+|m|(1+4s_1+2\delta)}.\tag{2.36}$$

[5] $S_1(\lambda)$ definitely depends on αW_1, however we omit this to keep the notation simple.

[6] Usually the vector $\vec{p}_j(t)$ is denoted by $\vec{k}$, the corresponding plane wave being $e^{\langle\vec{k},x\rangle}$. We use less common notation $\vec{\varkappa}$, since we have already other k's in the text.

By Theorem 2.3, the formulae (2.34) – (2.35) hold in the $(2k^{-1-4s_1-2\delta})$-neighborhood of $\mathcal{D}_0(\lambda)_{nonres}$, i.e., they hold for any $\varkappa\vec{\nu}$ such that $\vec{\nu}\in\mathcal{B}_1(\lambda)$, $|\varkappa-k|<k^{-1-4s_1-2\delta}$. We define $\mathcal{D}_1(\lambda)$ as the level set of the function $\lambda^{(1)}(\alpha,\vec{\varkappa})$ in this neighborhood:

$$\mathcal{D}_1(\lambda):=\{\vec{\varkappa}=\varkappa\vec{\nu}:\vec{\nu}\in\mathcal{B}_1(\lambda),\ |\varkappa-k|<k^{-1-4s_1-2\delta},\ \lambda^{(1)}(\alpha,\vec{\varkappa})=\lambda\}. \tag{2.37}$$

We prove in Lemma 2.11 that $\mathcal{D}_1(\lambda)$ is a distorted circle with holes, which is close to and inside of the circle of radius k, Fig. 1. First, we prove that the equation $\lambda^{(1)}(\alpha,\vec{\varkappa})=\lambda$ is solvable with respect to $\varkappa=|\vec{\varkappa}|$ for any $\vec{\nu}=\frac{\vec{\varkappa}}{\varkappa}\in\mathcal{B}_1(\lambda)$.

Lemma 2.10. *For every $\vec{\nu}\in\mathcal{B}_1(\lambda)$ and every α, $0\le\alpha\le1$ and sufficiently large λ, there is a unique $\varkappa_1=\varkappa_1(\lambda,\vec{\nu})$ in the interval $I_1:=[k-k^{-1-4s_1-2\delta},k+k^{-1-4s_1-2\delta}]$, $k^{2l}=\lambda$, such that*

$$\lambda^{(1)}(\alpha,\varkappa_1\vec{\nu})=\lambda. \tag{2.38}$$

Furthermore, $|\varkappa_1-k|\le ck^{-1-4s_1-2\gamma_0-\delta}$.

Proof. 1) Let us show that there exists such an $\varkappa_1$. Given $\vec{\nu}\in\mathcal{B}_1(\lambda)$, by Theorem 2.3, there exists an eigenvalue $\lambda^{(1)}(\alpha,\varkappa\vec{\nu})$ given by (2.34) for all $\varkappa$ in the interval I_1 with a sufficiently large k. Obviously, $\lambda^{(1)}(0,k\vec{\nu})=p_j^{2l}(t_0)=k^{2l}=\lambda$, $\lambda^{(1)}(0,\varkappa\vec{\nu})=p_j^{2l}(t)=\varkappa^{2l}$, $\vec{p}_j(t_0)=k\vec{\nu}$, $\vec{p}_j(t)=\varkappa\vec{\nu}$, where j, t, and t_0 are uniquely determined by $\varkappa\vec{\nu}$ and $k\vec{\nu}$. Let $\mathcal{L}^{(1)}(\alpha,\vec{\nu}):=\{\lambda=\lambda^{(1)}(\alpha,\varkappa\vec{\nu}):\varkappa\in I_1\}$. Using the definition of I_1 and considering that $\lambda^{(1)}(0,\varkappa\vec{\nu})=\varkappa^{2l}$, we easily obtain $\mathcal{L}^{(1)}(0,\vec{\nu})\supset[k^{2l}-\delta_1,k^{2l}+\delta_1]$, $\delta_1=c_1k^{2l-2-4s_1-2\delta}$, $0<c_1\ne c_1(k)$. Since $\lambda^{(1)}(\alpha,\varkappa\vec{\nu})$ is continuous in $\varkappa$ and (2.35) is valid, we have $\mathcal{L}^{(1)}(\alpha,\vec{\nu})\supset[k^{2l}-\delta_1+\delta_2,k^{2l}+\delta_1-\delta_2]$, $\delta_2=c_2k^{2l-2-4s_1-2\gamma_0-\delta}$. Since $2\gamma_0-\delta>0$, we have $\mathcal{L}^{(1)}(\alpha,\vec{\nu})\supset[k^{2l}-\delta_1/2,k^{2l}+\delta_1/2]$. Thus for every $\vec{\nu}\in\mathcal{B}_1(\lambda)$, there exists a $\varkappa_1$ such that $\lambda^{(1)}(\alpha,\varkappa_1\vec{\nu})=k^{2l}$, $\varkappa_1\in I_1$.

2) Now we show that there is only one $\varkappa_1$ satisfying (2.38) in the interval I_1. Differentiating (2.34) with respect to $\varkappa$ and using (2.36), we get:

$$\frac{\partial\lambda^{(1)}(\alpha,\varkappa\vec{\nu})}{\partial\varkappa}=2l\varkappa^{2l-1}+O(k^{2l-1-2\gamma_0+\delta}). \tag{2.39}$$

Therefore,

$$\frac{\partial\lambda^{(1)}(\alpha,\varkappa\vec{\nu})}{\partial\varkappa}\ge 2lk^{2l-1}\big(1+o(1)\big), \tag{2.40}$$

and this implies that $\lambda^{(1)}(\alpha,\varkappa\vec{\nu})$ is monotone with respect to $\varkappa$ in I_1. Thus, there is a single $\varkappa_1\in I_1$ satisfying (2.38).

3) Now let us estimate $|\varkappa_1-k|$. By (2.34) and (2.35) we have

$$\lambda^{(1)}(\alpha,k\vec{\nu})=k^{2l}+f_1(\alpha,k\vec{\nu})=k^{2l}+O(k^{2l-2-4s_1-2\gamma_0-\delta}). \tag{2.41}$$

Using (2.41), (2.38), (2.40), and the Mean Value Theorem, we obtain

$$ck^{2l-2-4s_1-2\gamma_0-\delta}\ge|\lambda^{(1)}(\alpha,\varkappa_1\vec{\nu})-\lambda^{(1)}(\alpha,k\vec{\nu})|\ge$$

$$\min_{\varkappa\in I_1}\left|\frac{\partial\lambda^{(1)}(\alpha,\varkappa\vec{\nu})}{\partial\varkappa}\right||\varkappa_1-k|\ge ck^{2l-1}|\varkappa_1-k|,$$

and hence we arrive at $|\varkappa_1-k|\le ck^{-1-4s_1-2\gamma_0-\delta}$. ∎

Here and below, we use the notations $\varkappa_1(\varphi):=\varkappa_1(\lambda,\vec{\nu})$, $\vec{\nu}=(\cos\varphi,\sin\varphi)$, $\vec{\varkappa}_1(\varphi)=\varkappa_1(\varphi)\vec{\nu}$, $h_1(\varphi)=\varkappa_1(\varphi)-k$.

Lemma 2.11. (1) *The set $\mathcal{D}_1(\lambda)$ is a distorted circle with holes which is strictly inside the circle of the radius k: it can be described by the formula:*

$$\mathcal{D}_1(\lambda) = \{\vec{\varkappa} \in \mathbb{R}^2 : \vec{\varkappa} = \varkappa_1(\varphi)\vec{\nu}, \quad \vec{\nu} \in \mathcal{B}_1(\lambda)\}, \tag{2.42}$$

where $\varkappa_1(\varphi) = k + h_1(\varphi)$ and $h_1(\varphi)$ obeys the inequalities:

$$|h_1| < ck^{-1-4s_1-2\gamma_0-\delta}, \qquad \left|\frac{\partial h_1}{\partial \varphi}\right| < ck^{-2\gamma_0+1}, \tag{2.43}$$

$h_1(\varphi) < 0$ when $W_1 \neq 0$.

(2) *The total length of $\mathcal{B}_1(\lambda)$ satisfies the estimate:*

$$L\left(\mathcal{B}_1\right) = 2\pi(1 + O(k^{-\delta/2})). \tag{2.44}$$

(3) *Function $h_1(\varphi)$ can be extended as a holomorphic function of φ to the complex $(2k^{-2-4s_1-2\delta})$-neighborhood of each connected component of $\Theta_1(\lambda)$ and estimates (2.43) hold.*

(4) *The curve $\mathcal{D}_1(\lambda)$ has a length which is asymptotically close to that of the whole circle in the following sense:*

$$L\big(\mathcal{D}_1(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi k\big(1 + O(k^{-\delta/2})\big), \ \lambda = k^{2l}. \tag{2.45}$$

Proof. (1) Let $\vec{\varkappa} \in \mathcal{D}_1(\lambda)$, then $\vec{\varkappa} = \varkappa_1\vec{\nu}$. Using Lemma 2.10, we get $|h_1(\varphi)| = |\varkappa_1 - k| \leq ck^{-1-4s_1-2\gamma_0-\delta}$, and $\dfrac{\partial h_1}{\partial \varphi} = \dfrac{\partial \varkappa_1}{\partial \varphi}$. From (2.34) and the definition of $\mathcal{D}_1(\lambda)$, we have

$$\lambda = \varkappa_1^{2l} + f_1(\varkappa_1 \cos\varphi, \varkappa_1 \sin\varphi), \tag{2.46}$$

and, therefore, by implicit differentiation,

$$\left|\frac{\partial h_1}{\partial \varphi}\right| = \left|\frac{\partial \varkappa_1}{\partial \varphi}\right| = \left|\frac{\frac{\partial f_1}{\partial \varphi}}{2l\varkappa_1^{2l-1} + \langle \nabla f, \vec{\nu}\rangle}\right|. \tag{2.47}$$

Using (2.36), we easily obtain $|\nabla f_1| \leq ck^{2l-2\gamma_0-1}$, and, hence, (2.43) for the derivative. It is easy to show that $g_2^{(1)}(k,t)$ given by (2.14) is positive when $W_1(x)$ is non-zero, and, moreover, $f_1(\alpha, \vec{\varkappa}) > 0$. It easily follows that $h_1 < 0$.

(2) By definition, $\mathcal{B}_1(\lambda)$ is the set of directions corresponding to $\mathcal{D}_0(\lambda)_{nonres}$, the latter set being a subset of the sphere of radius k. Formula (2.32) establishes one-to-one correspondence between $\chi_1(\lambda, \delta)$ and $\mathcal{D}_0(\lambda)_{nonres}$, their length being equal. Considering (2.8), we obtain $L\left(\mathcal{D}_0(\lambda)_{nonres}\right) = 2\pi k\big(1 + O(k^{-\delta/2})\big)$. Hence, (2.44) holds.

(3) This part of the lemma easily follows from the Implicit Function Theorem applied to (2.46), the above estimate for $|\nabla f_1|$ and the fact that $f_1(\alpha, \vec{\varkappa})$ is a holomorphic function of two variables (Theorem 2.7).

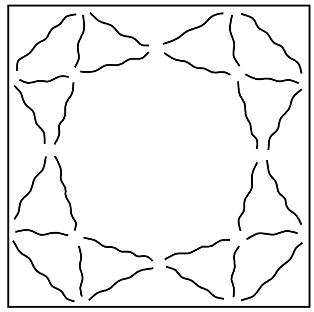

FIGURE 5. The set $\chi_1^*(\lambda,\delta)$

(4) Considering that $L(\mathcal{D}_1(\lambda)) = \int_{\Theta_1(\lambda)} \sqrt{\varkappa_1^2(\varphi)+\varkappa_1'(\varphi)^2}\,d\varphi$ and taking into account (2.44) and (2.47), we obtain

$$\begin{aligned} L(\mathcal{D}_0(\lambda)) - L(\mathcal{D}_1(\lambda)) &= \int_{[0,2\pi]\setminus\Theta_1(\lambda)} k\,d\varphi + \int_{\Theta_1}\Big(k - \sqrt{\varkappa_1^2(\varphi)+\varkappa_1'(\varphi)^2}\Big)\,d\varphi \\ &\le \int_{[0,2\pi]\setminus\Theta_1(\lambda)} k\,d\varphi + \int_{\Theta_1(\lambda)}\big(h_1(\varphi)+h_1'(\varphi)\big) \\ &< ck\cdot k^{-\delta/2} + O(k^{-1-4s_1-2\gamma_0-\delta}) + O(k^{-2\gamma_0+1}) = k\cdot O(k^{-\delta/2}), \end{aligned}$$

here and below, $\mathcal{D}_0(\lambda)$, corresponding to the free operator, is the perfect circle of radius $\lambda^{1/2l}$. Hence, (2.45) is valid.

∎

Next, we define a nonresonance subset $\chi_1^*(\lambda)$ of isoenergetic set $S_1(\lambda)$ as the parallel shift of $\mathcal{D}_1(\lambda)$ into K_1 (Fig. 5):

$$\chi_1^*(\lambda) := \mathcal{K}_1\mathcal{D}_1(\lambda). \tag{2.48}$$

Lemma 2.12. *The set $\chi_1^*(\lambda)$ belongs to the $(ck^{-1-4s_1-2\gamma_0-\delta})$-neighborhood of $\chi_1(\lambda)$ in K_1. If $t\in\chi_1^*(\lambda)$, then the operator $H_\alpha^{(1)}(t)$ has a simple eigenvalue $\lambda_n^{(1)}(\alpha,t)$, $n\in\mathbb{N}$, equal to λ, no other eigenvalues being in the interval $\varepsilon_1(k,\delta)$, $\varepsilon_1(k,\delta) := (k^{2l}-k^{2l-2-4s_1-\delta}, k^{2l}+k^{2l-2-4s_1-\delta})$. This eigenvalue is given by the perturbation series (2.16), j being uniquely defined by t from the relation $p_j^{2l}(t)\in\varepsilon_1(k,\delta)$.*

Proof. By Lemma 2.11, $\mathcal{D}_1(\lambda)$ is in the $\big(ck^{-1-4s_1-2\gamma_0-\delta}\big)$-neighborhood of $\mathcal{D}_0(\lambda)$. Considering that $\chi_1(\lambda) = \mathcal{K}_1\mathcal{D}_0(\lambda)$ and $\chi_1^*(\lambda) = \mathcal{K}_1\mathcal{D}_1(\lambda)$, we immediately obtain that $\chi_1^*(\lambda)$ is in the $(ck^{-1-4s_1-2\gamma_0-\delta})$-neighborhood of $\chi_1(\lambda)$. The size of this neighborhood is less than $k^{-1-4s_1-2\delta}$, hence Theorem 2.3 holds in it, i.e. for any $t\in\chi_1^*(\lambda)$ there is a single eigenvalue of $H_\alpha^{(1)}(t)$ in the interval $\varepsilon_1(k,\delta)$. Since $\chi_1^*(\lambda)\subset S_1(\lambda)$, this eigenvalue is equal to λ. By the theorem, the eigenvalue is given by the series (2.16), j being uniquely defined by t from the relation $p_j^{2l}(t)\in\varepsilon_1(k,\delta)$. ∎

Lemma 2.13. *Formula (2.48) establishes one-to-one correspondence between $\chi_1^*(\lambda)$ and $\mathcal{D}_1(\lambda)$.*

Remark 2.14. From geometric point of view this means that $\chi_1^*(\lambda)$ does not have self-intersections.

Proof. Suppose there is a pair $\vec{\varkappa}_{1,1}, \vec{\varkappa}_{1,2} \in \mathcal{D}_1(\lambda)$, such that $\mathcal{K}_1\vec{\varkappa}_{1,1} = \mathcal{K}_1\vec{\varkappa}_{1,2} = t$, $t \in \chi_1^*(\lambda)$. By the definition (2.37) of $\mathcal{D}_1(\lambda)$, we have $\lambda^{(1)}(\alpha, \vec{\varkappa}_{1,1}) = \lambda^{(1)}(\alpha, \vec{\varkappa}_{1,2}) = \lambda$, i.e., the eigenvalue λ of $H_\alpha^{(1)}(t)$ is not simple. This contradicts to the previous lemma. ∎

3. The Second Approximation

3.1. **The Operator $H_\alpha^{(2)}$.** Choosing $s_2 = 2s_1$, we define the second operator $H_\alpha^{(2)}$ by the formula:

$$H_\alpha^{(2)} = H^{(1)} + \alpha W_2, \quad (0 \le \alpha \le 1), \qquad W_2 = \sum_{r=M_1+1}^{M_2} V_r, \tag{3.1}$$

where $H^{(1)}$ is defined by (2.1), M_2 is chosen in such a way that $2^{M_2} \approx k^{s_2}$. Obviously, the periods of W_2 are $2^{M_2-1}(b_1, 0)$ and $2^{M_2-1}(0, b_2)$. We will write them in the form: $N_1(a_1, 0)$ and $N_1(0, a_2)$, where a_1, a_2 are the periods of W_1 and $N_1 = 2^{M_2-M_1}$, $\frac{1}{4}k^{s_2-s_1} < N_1 < 4k^{s_2-s_1}$. Note that

$$\|W_2\|_\infty \le \sum_{r=M_1+1}^{M_2} \|V_r\|_\infty \le \sum_{r=M_1+1}^{M_2} \exp(-2^{\eta r}) < \exp(-k^{\eta s_1}). \tag{3.2}$$

3.1.1. *Multiple Periods of $W_1(x)$.* The operator $H^{(1)} = H_0 + W_1(x)$ has the periods a_1, a_2. The corresponding family of operators, $\{H^{(1)}(t)\}_{t\in K_1}$, acts in $L_2(Q_1)$, where $Q_1 = [0, a_1] \times [0, a_2]$ and $K_1 = [0, 2\pi/a_1) \times [0, 2\pi/a_2)$. Eigenvalues of $H^{(1)}(t)$ are denoted by $\lambda_n^{(1)}(t)$, $n \in \mathbb{N}$, and its spectrum by $\Lambda^{(1)}(t)$. Now let us consider the same $W_1(x)$ as a periodic function with the periods N_1a_1, N_1a_2. Obviously, the definition of the operator $H^{(1)}$ does not depend on the way how we define the periods of W_1. However, the family of operators $\{H^{(1)}(t)\}_{t\in K_1}$ does change, when we replace the periods a_1, a_2 by N_1a_1, N_1a_2. The family of operators $\{H^{(1)}(t)\}_{t\in K_1}$ has to be replaced by a family of operators $\{\tilde{H}^{(1)}(\tau)\}_{\tau\in K_2}$ acting in $L_2(Q_2)$, where $Q_2 = [0, N_1a_1] \times [0, N_1a_2]$ and $K_2 = [0, 2\pi/N_1a_1) \times [0, 2\pi/N_1a_2)$. We denote eigenvalues of $\tilde{H}^{(1)}(\tau)$ by $\tilde{\lambda}_n^{(1)}(\tau)$, $n \in \mathbb{N}$ and its spectrum by $\tilde{\Lambda}^{(1)}(\tau)$. The next lemma establishes a connection between spectra of operators $H^{(1)}(t)$ and $\tilde{H}^{(1)}(\tau)$. It easily follows from Bloch theory (see e.g. [21]).

Lemma 3.1. *For any $\tau \in K_2$,*

$$\tilde{\Lambda}^{(1)}(\tau) = \bigcup_{p\in P} \Lambda^{(1)}(t_p), \tag{3.3}$$

where

$$P = \{p = (p_1, p_2) \in \mathbb{Z}^2 : 0 \le p_1 \le N_1 - 1,\ 0 \le p_2 \le N_1 - 1\} \tag{3.4}$$

and $t_p = (t_{p,1}, t_{p,2}) = (\tau_1 + 2\pi p_1/N_1a_1, \tau_2 + 2\pi p_2/N_1a_2) \in K_1$, see Fig. 6.

We defined isoenergetic set $S_1(\lambda) \subset K_1$ of $H^{(1)}$ by formula (2.31). Obviously, this definition is directly associated with the family of operators $H^{(1)}(t)$ and, therefore, with periods a_1, a_2, which we assigned to $W_1(x)$. Now, assuming that the periods are equal to N_1a_1, N_1a_2, we give an analogous definition of the isoenergetic set $\tilde{S}_1(\lambda)$ in K_2: $\tilde{S}_1(\lambda) := \{\tau \in K_2 : \exists n \in \mathbb{N} : \ \tilde{\lambda}_n^{(1)}(\tau) = \lambda\}$. By Lemma 3.1, $\tilde{S}_1(\lambda)$ can be expressed as follows:

$$\tilde{S}_1(\lambda) = \Big\{\tau \in K_2 : \exists n \in \mathbb{N},\ p \in P :\ \lambda_n^{(1)}\big(\tau + 2\pi p/N_1a\big) = \lambda\Big\},$$

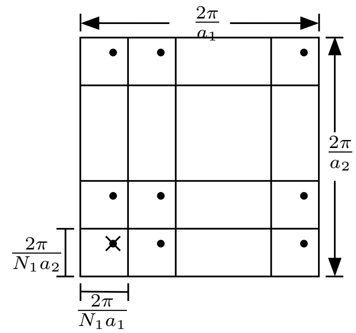

FIGURE 6. Relation between τ ($\times$) and t_p ($\cdot$)

$2\pi p/N_1 a = \left(\frac{2\pi p_1}{N_1 a_1}, \frac{2\pi p_2}{N_1 a_2}\right)$. The relation between $S_1(\lambda)$ and $\tilde{S}_1(\lambda)$ can be easily understood from the geometric point of view as $\tilde{S}_1(\lambda) = \mathcal{K}_2 S_1(\lambda)$, where $\mathcal{K}_2$ is the parallel shift into K_2, i.e., $\mathcal{K}_2 : \mathbb{R}^2 \to K_2$, $\mathcal{K}_2(\tau + 2\pi m/N_1 a) = \tau$, $m \in \mathbb{Z}^2$, $\tau \in K_2$. Thus, $\tilde{S}_1(\lambda)$ is obtained from $S_1(\lambda)$ by cutting $S_1(\lambda)$ into pieces of the size K_2 and shifting them together in K_2.

Definition 3.2. We say that τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $n, \hat{n} \in N$, $n \neq \hat{n}$ such that $\tilde{\lambda}_n^{(1)}(\tau) = \tilde{\lambda}_{\hat{n}}^{(1)}(\tau) = \lambda$.

Remark 3.3. By Lemma 3.1, τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $p, \hat{p} \in P$ and a pair $n, \hat{n} \in N$ such that $|p-\hat{p}|+|n-\hat{n}| \neq 0$ and $\lambda_n^{(1)}(\tau+2\pi p/N_1 a) = \lambda_{\hat{n}}^{(1)}(\tau+2\pi\hat{p}/N_1 a) = \lambda$.

Now let us recall that the isoenergetic set $S_1(\lambda)$ consists of two parts: $S_1(\lambda) = \chi_1^*(\lambda) \cup \big(S_1(\lambda) \setminus \chi_1^*(\lambda)\big)$, where $\chi_1^*(\lambda)$ is the first non-resonance set given by (2.48). Obviously $\mathcal{K}_2\chi_1^*(\lambda) \subset \mathcal{K}_2 S_1(\lambda) = \tilde{S}_1(\lambda)$ and can be described by the formula: $\mathcal{K}_2\chi_1^*(\lambda) = \{\tau \in K_2 : \ \exists p \in P : \tau + 2\pi p/N_1 a \in \chi_1^*(\lambda)\}$. Let us consider only those self-intersections of $\tilde{S}_1$ which belong to $\mathcal{K}_2\chi_1^*(\lambda)$, i.e., we consider the points of intersection of $\mathcal{K}_2\chi_1^*(\lambda)$ both with itself and with $\tilde{S}_1(\lambda) \setminus \mathcal{K}_2\chi_1^*(\lambda)$.

Lemma 3.4. *A self-intersection τ of $\tilde{S}_1(\lambda)$ belongs to $\mathcal{K}_2\chi_1^*(\lambda)$ if and only if there are a pair $p, \hat{p} \in P$, $p \neq \hat{p}$ and a pair $n, \hat{n} \in N$ such that $\tau + 2\pi p/N_1 a \in \chi_1^*(\lambda)$ and $\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) = \lambda_{\hat{n}}^{(1)}(\tau+2\pi\hat{p}/N_1 a) = \lambda$, the eigenvalue $\lambda_n^{(1)}(\tau+2\pi p/N_1 a)$ being given by the series (2.16) with $t = \tau+2\pi p/N_1 a$ and j uniquely defined by t from the relation $p_j^{2l}(t) \in \varepsilon_1$, $\varepsilon_1 = (k^{2l}-k^{2l-2-4s_1-\delta}, k^{2l}+k^{2l-2-4s_1-\delta})$.*

Proof. Suppose τ is a point of self-intersection of $\tilde{S}_1(\lambda)$ belonging to $\mathcal{K}_2\chi_1^*(\lambda)$. Since, $\tau \in \mathcal{K}_2\chi_1^*(\lambda)$, there is a $p \in P$ such that $\tau + 2\pi p/N_1 a \in \chi_1^*(\lambda)$. By Lemma 2.12, there is an eigenvalue $\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) = \lambda$, which is given by the series (2.16) with $t = \tau + 2\pi p/N_1 a$ and j uniquely defined by t from the relation $p_j^{2l}(t) \in \varepsilon_1$. By the same lemma $\lambda_n^{(1)}(\tau + 2\pi p/N_1 a)$ is a unique eigenvalue of $H^{(1)}(\tau + 2\pi p/N_1 a)$ in ε_1. This means

$$\left|\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi p/N_1 a)\right| \geq k^{2l-2-4s_1-\delta}, \quad \text{when } \ \hat{n} \neq n. \tag{3.5}$$

Since τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, $\lambda_n^{(1)}(\tau+2\pi p/N_1a)=\lambda_{\hat{n}}^{(1)}(\tau+2\pi\hat{p}/N_1a)=\lambda$ for some $\hat{n},\hat{p}$: $|p-\hat{p}|+|n-\hat{n}|\neq 0$. Considering the above inequality, we obtain that the equality holds for some $\hat{p}\neq p$. The converse part of the lemma is trivial. ∎

To obtain a new non-resonance set $\chi_2(\lambda)$ we remove from $\mathcal{K}_2\chi_1^*(\lambda)$, shortly speaking, a neighborhood of its intersections with the whole $\tilde{S}_1(\lambda)$. More precisely, we remove from $\mathcal{K}_2\chi_1^*(\lambda)$ the following set:

$$\Omega_1(\lambda)=\{\tau\in\mathcal{K}_2\chi_1^*(\lambda):\exists n,\hat{n}\in\mathbb{N},\ p,\hat{p}\in P,\ p\neq\hat{p}:\ \lambda_n^{(1)}(\tau+2\pi p/N_1a)=\lambda,$$
$$\tau+2\pi p/N_1a\in\chi_1^*(\lambda),\ |\lambda_n^{(1)}(\tau+2\pi p/N_1a)-\lambda_{\hat{n}}^{(1)}(\tau+2\pi\hat{p}/N_1a)|\leq\epsilon_1\},\quad (3.6)$$

where $\epsilon_1=e^{-\frac{1}{4}k^{\eta s_1}}$. We define $\chi_2(\lambda)$ by the formula:

$$\chi_2(\lambda)=\mathcal{K}_2\big(\chi_1^*(\lambda)\big)\setminus\Omega_1(\lambda).\quad (3.7)$$

3.2. **Perturbation Formulae.** Before proving the main result, we formulate the Geometric Lemma:

Lemma 3.5 (Geometric Lemma). *For an arbitrarily small positive δ, $7\delta<2l-11-16s_1$ and sufficiently large λ, $\lambda>\lambda_0(V,\delta)$, there exists a non-resonance set $\chi_2(\lambda,\delta)\subset\mathcal{K}_2\chi_1^*$ such that:*

(1) *For any $\tau\in\chi_2$, the following conditions hold:*
  (a) *There exists a unique $p\in P$ such that $\tau+2\pi p/N_1a\in\chi_1^*$.* [7]
  (b) *The following relation holds:*
$$\lambda_j^{(1)}(\tau+2\pi p/N_1a)=k^{2l},$$
  where $\lambda_j^{(1)}(\tau+2\pi p/N_1a)$ is given by the perturbation series (2.16) with $\alpha=1$, j being uniquely defined by $t=\tau+2\pi p/N_1a$ as it is described in Part 2 of Geometric Lemma for the previous step.
  (c) *The eigenvalue $\lambda_j^{(1)}(\tau+2\pi p/N_1a)$ is a simple eigenvalue of $\tilde{H}^{(1)}(\tau)$ and its distance from all other eigenvalues $\lambda_{\hat{n}}^{(1)}(\tau+2\pi\hat{p}/N_1a)$, $\hat{n}\in\mathbb{N}$ of $\tilde{H}_1(\tau)$ is greater than $\epsilon_1=e^{-\frac{1}{4}k^{\eta s_1}}$:*
$$|\lambda_j^{(1)}(\tau+2\pi p/N_1a)-\lambda_{\hat{n}}^{(1)}(\tau+2\pi\hat{p}/N_1a)|>\epsilon_1.\quad (3.8)$$

(2) *For any τ in the $(\epsilon_1k^{-2l+1-\delta})$-neighborhood in $\mathbb{C}^2$ of χ_2, there exists a unique $p\in P$ such that $\tau+2\pi p/N_1a$ is in the $(\epsilon_1k^{-2l+1-\delta})$-neighborhood in $\mathbb{C}^2$ of χ_1^* and*
$$|\lambda_j^{(1)}(\tau+2\pi p/N_1a)-k^{2l}|<\epsilon_1k^{-\delta},\quad (3.9)$$
j being uniquely defined by $\tau+2\pi p/N_1a$ as it is described in Part 2 of Geometric Lemma for the previous step.

(3) *The second non-resonance set χ_2 has an asymptotically full measure in χ_1^* in the following sense:*
$$\frac{L(\mathcal{K}_2\chi_1^*\setminus\chi_2))}{L(\chi_1^*)}<k^{-2-2s_1}.\quad (3.10)$$

The proof of the lemma will be presented in section 3.3.

[7] From geometric point of view this means that $\chi_2(\lambda)$ does not have self-intersections.

Corollary 3.6. *If τ belongs to the complex $(\epsilon_1 k^{-2l+1-\delta})-$neighborhood of the second non-resonance set $\chi_2(\lambda,\delta)$, then for any z lying on the circle $C_2=\{z:|z-k^{2l}|=\epsilon_1/2\}$, the following inequalities hold:*

$$\|(\tilde{H}^{(1)}(\tau)-z)^{-1}\| \leq \frac{4}{\epsilon_1}, \tag{3.11}$$

$$\|(\tilde{H}^{(1)}(\tau)-z)^{-1}\|_1 \leq \frac{c_0 k^{2+2s_2}}{\epsilon_1}, \quad c_0=32b_1b_2. \tag{3.12}$$

Corollary is proven in Appendix A.1.

Remark 3.7. Note that every point $2\pi m/N_1a$ ($m\in\mathbb{Z}^2$) of a dual lattice corresponding to the larger periods N_1a_1, N_1a_2 can be uniquely represented in the form $2\pi m/N_1a=2\pi j/a+2\pi p/N_1a$, where $m=N_1j+p$ and $2\pi j/a$ is a point of a dual lattice for periods a_1, a_2, while $p\in P$ is responsible for refining the lattice.

Let us consider a normalized eigenfunction $\psi_n(t,x)$ of $H^{(1)}(t)$ in $L_2(Q_1)$. We extend it quasiperiodically to Q_2, renormalize in $L_2(Q_2)$ and denote the new function by $\tilde{\psi}_n(\tau,x)$, $\tau=\mathcal{K}_2t$. The Fourier representations of $\psi_n(t,x)$ in $L_2(Q_1)$ and $\tilde{\psi}_n(\tau,x)$ in $L_2(Q_2)$ are simply related. If we denote Fourier coefficients of $\psi_n(t,x)$ with respect to the basis of exponential functions $|Q_1|^{-1/2}e^{i\langle t+2\pi j/a,x\rangle}$, $j\in\mathbb{Z}^2$, in $L_2(Q_1)$ by C_{nj}, then, the Fourier coefficients $\tilde{C}_{nm}$ of $\tilde{\psi}_n(\tau,x)$ with respect to the basis of exponential functions $|Q_2|^{-1/2}e^{i\langle\tau+2\pi m/N_1a,x\rangle}$, $m\in\mathbb{Z}^2$, in $L_2(Q_2)$ are given by the formula:

$$\tilde{C}_{nm}=\begin{cases} C_{nj}, & \text{if } m=jN_1+p;\\ 0, & \text{otherwise,}\end{cases}$$

p being defined from the relation $t=\tau+2\pi p/N_1a,\ p\in P$. Hence, matrices of the projections on $\psi_n(t,x)$ and $\tilde{\psi}_n(\tau,x)$ with respect to the above bases are simply related:

$$(\tilde{E}_n)_{j\hat{j}}=\begin{cases} (E_n)_{m\hat{m}}, & \text{if } m=jN_1+p,\ \hat{m}=\hat{j}N_1+p;\\ 0, & \text{otherwise,}\end{cases}$$

$\tilde{E}_n$ and E_n being projections in $L_2(Q_2)$ and $L_2(Q_1)$, respectively.

Let us denote by $\tilde{E}_j^{(1)}\big(\tau+2\pi p/N_1a\big)$ the spectral projection $E_j^{(1)}(\alpha,t)$ (see (2.17)) with $\alpha=1$ and $t=\tau+2\pi p/N_1a$, "extended" from $L_2(Q_1)$ to $L_2(Q_2)$.

By analogy with (2.12), (2.13), we define functions $g_r^{(2)}(k,\tau)$ and operator-valued functions $G_r^{(2)}(k,\tau)$, $r=1,2,\cdots$, as follows:

$$g_r^{(2)}(k,\tau)=\frac{(-1)^r}{2\pi ir}\mathrm{Tr}\oint_{C_2}\Big(\big(\tilde{H}^{(1)}(\tau)-z\big)^{-1}W_2\Big)^r dz, \tag{3.13}$$

$$G_r^{(2)}(k,\tau)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_2}\Big(\big(\tilde{H}^{(1)}(\tau)-z\big)^{-1}W_2\Big)^r\big(\tilde{H}^{(1)}(\tau)-z\big)^{-1}dz. \tag{3.14}$$

We consider the operators $H_\alpha^{(2)}=H^{(1)}+\alpha W_2$ and the family $H_\alpha^{(2)}(\tau)$, $\tau\in K_2$, acting in $L_2(Q_2)$.

Theorem 3.8. *Suppose τ belongs to the $(\epsilon_1k^{-2l+1-\delta})$-neighborhood in K_2 of the second non-resonance set $\chi_2(\lambda,\delta)$, $0<7\delta<2l-11-16s_1$, $\epsilon_1=e^{-\frac{1}{4}k^{\eta s_1}}$. Then, for sufficiently large λ,*

$\lambda > \lambda_0(V,\delta)$ and for all α, $0 \le \alpha \le 1$, there exists a unique eigenvalue of the operator $H_\alpha^{(2)}(\tau)$ in the interval $\varepsilon_2(k) := (k^{2l} - \epsilon_1/2, k^{2l} + \epsilon_1/2)$. It is given by the series:

$$\lambda_{\tilde{j}}^{(2)}(\alpha,\tau) = \lambda_j^{(1)}\big(\tau + 2\pi p/N_1 a\big) + \sum_{r=1}^{\infty} \alpha^r g_r^{(2)}(k,\tau), \qquad \tilde{j} = j + p/N_1, \tag{3.15}$$

converging absolutely in the disk $|\alpha| \le 1$, where $p \in P$ and $j \in \mathbb{Z}^2$ are described as in Geometric Lemma 3.5. The spectral projection corresponding to $\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)$ is given by the series:

$$E_{\tilde{j}}^{(2)}(\alpha,\tau) = \tilde{E}_j^{(1)}\big(\tau + 2\pi p/N_1 a\big) + \sum_{r=1}^{\infty} \alpha^r G_r^{(2)}(k,\tau), \tag{3.16}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha| \le 1$.

The following estimates hold for coefficients $g_r^{(2)}(k,\tau)$, $G_r^{(2)}(k,\tau)$, $r \ge 1$:

$$\big|g_r^{(2)}(k,\tau)\big| < \frac{3\epsilon_1}{2}(4\epsilon_1^3)^r, \qquad \big\|G_r^{(2)}(k,\tau)\big\|_1 < 6r(4\epsilon_1^3)^r. \tag{3.17}$$

Corollary 3.9. *The following estimates hold for the perturbed eigenvalue and its spectral projection:*

$$\Big|\lambda_{\tilde{j}}^{(2)}(\alpha,\tau) - \lambda_j^{(1)}\big(\tau + 2\pi p/N_1 a\big)\Big| \le 12\alpha\epsilon_1^4, \tag{3.18}$$

$$\Big\|E_{\tilde{j}}^{(2)}(\alpha,\tau) - \tilde{E}_j^{(1)}\big(\tau + 2\pi p/N_1 a\big)\Big\|_1 \le 48\alpha\epsilon_1^3. \tag{3.19}$$

Remark 3.10. The theorem states that $\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)$ is a single eigenvalue in the interval $\varepsilon_2(k,\delta)$. This means that $\big|\lambda_{\tilde{j}}^{(2)}(\alpha,\tau) - k^{2l}\big| < \epsilon_1/2$. Formula (3.18) provides a stronger estimate on the location of $\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)$.

Proof. The proof of the theorem is analogous to that of Theorem 2.3 and it is based on expanding the resolvent $(H_\alpha^{(2)}(\tau) - z)^{-1}$ in a perturbation series for $z \in C_2$. Integrating the resolvent yields the formulae for an eigenvalue of $H_\alpha^{(2)}$ and its spectral projection. In fact, it is obvious that

$$(H_\alpha^{(2)}(\tau) - z)^{-1} = (\tilde{H}^{(1)}(\tau) - z)^{-1}(I - \alpha A_2)^{-1}, \quad A_2 := -W_2(\tilde{H}^{(1)}(\tau) - z)^{-1}. \tag{3.20}$$

Suppose $z \in C_2$. Using Corollary 3.6 and the estimate $\|W_2\| < e^{-k^{\eta s_1}} = \epsilon_1^4$, we obtain:

$$\|(\tilde{H}^{(1)}(\tau) - z)^{-1}\| \le \frac{4}{\epsilon_1}, \quad \|A_2\| \le \frac{4\|W_2\|}{\epsilon_1} < 4\epsilon_1^3 < 1. \tag{3.21}$$

The last inequality makes it possible to expand $(I - \alpha A_2)^{-1}$ in the series in powers of αA_2. Integrating the series for the resolvent, we obtain formulae (3.15), (3.16). Estimates (3.17) follow from the estimates (3.21). ∎

Next, we show that the series (3.15), (3.16) can be extended as holomorphic functions of τ in a complex neighborhood of χ_2; they can be differentiated any number of times with respect to τ and retain their asymptotic character.

Lemma 3.11. *The following estimates hold for the coefficients $g_r^{(2)}(k,\tau)$ and $G_r^{(2)}(k,\tau)$ in the complex $(\frac{1}{2}\epsilon_1 k^{-2l+1-\delta})$-neighborhood of the non-resonance set χ_2:*

$$|T(m)g_r^{(2)}(k,\tau)| < m!\cdot 3\cdot 2^{2r-1+|m|}\epsilon_1^{3r+1-|m|}k^{|m|(2l-1+\delta)}, \tag{3.22}$$

$$\|T(m)G_r^{(2)}(k,\tau)\|_1 < m!\cdot 3r\cdot 2^{2r+1+|m|}\epsilon_1^{3r-|m|}k^{|m|(2l-1+\delta)}. \tag{3.23}$$

Proof. Since (3.11) is valid in the complex $(\epsilon_1 k^{-2l+1-\delta})$- neighborhood of the second non-resonance set, it is not hard to see that the coefficients $g_r^{(2)}(k,\tau)$ and $G_r^{(2)}(k,\tau)$ can be continued from the real $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood of τ to the complex $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood as holomorphic functions of two variables and inequalities (3.17) are hereby preserved. Estimating, by means of the Cauchy integral formula, the value of the derivative with respect to τ in terms of the value of the function itself on the boundary of the $(\frac{1}{2}\epsilon_1 k^{-2l+1-\delta})$-neighborhood of τ (formulas (3.17)), we obtain (3.22) and (3.23). ∎

From this lemma the following theorem easily follows.

Theorem 3.12. *Under the conditions of Theorem 3.8 the series (3.15), (3.16) can be continued as holomorphic functions of two variables from the real $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood of the non-resonance set χ_2 to its complex $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood and the following estimates hold in the complex neighborhood:*

$$\left|T(m)\left(\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)-\lambda_j^{(1)}(\tau+2\pi p/N_1 a)\right)\right| < \alpha C_m \epsilon_1^{4-|m|}k^{|m|(2l-1+\delta)}, \tag{3.24}$$

$$\left\|T(m)\left(E_{\tilde{j}}^{(2)}(\alpha,\tau)-\tilde{E}_j^{(1)}(\tau+2\pi p/N_1 a)\right)\right\|_1 < \alpha C_m \epsilon_1^{3-|m|}k^{|m|(2l-1+\delta)}, \tag{3.25}$$

here and below $C_m = 48m!2^{|m|}$.

Corollary 3.13.

$$\nabla\lambda_j^{(2)}(\alpha,\tau) = 2lk^{2l-2}\vec{k}+o(k^{2l-1}), \quad \vec{k}=\vec{p}_j(\tau+2\pi p/N_1 a). \tag{3.26}$$

$$T(m)\lambda_j^{(2)}(\alpha,\tau) = O\big(k^{2l-2}\big) \quad \text{if} \quad |m|=2. \tag{3.27}$$

The next lemma will be used in the third step of approximation where the operator $H^{(2)}(\tau) = H_1^{(2)}(\tau)$ will play a role of the initial (unperturbed) operator.

Lemma 3.14. *For any z on the circle C_2 and τ in the complex $(\epsilon_1 k^{-2l+1-\delta})-$ neighborhood of χ_2,*

$$\|(H^{(2)}(\tau)-z)^{-1}\| \le \frac{8}{\epsilon_1}. \tag{3.28}$$

Proof. Considering the Hilbert relation

$$(H^{(2)}(\tau)-z)^{-1} = (\tilde{H}^{(1)}(\tau)-z)^{-1} + (\tilde{H}^{(1)}(\tau)-z)^{-1}(-W_2)(H^{(2)}(\tau)-z)^{-1},$$

and the estimate (3.11), together with the estimate $\|W_2\| < \epsilon_1^4$, we obtain:

$$\|(H^{(2)}(\tau)-z)^{-1}\| \le \frac{\|(\tilde{H}^{(1)}(\tau)-z)^{-1}\|}{1-\|(\tilde{H}^{(1)}(\tau)-z)^{-1}W_2\|} \le 2\|(\tilde{H}^{(1)}(\tau)-z)^{-1}\| \le \frac{8}{\epsilon_1}. \tag{3.29}$$

∎

3.3. **Proof of the Geometric Lemma.**

3.3.1. *Proof of Statement 1.* At the end of Section 3.1.1 (page 18) we defined the second non-resonance set, $\chi_2(\lambda) \subset K_2$, by the formula $\chi_2(\lambda) = \mathcal{K}_2(\chi_1^*) \setminus \Omega_1$, the set Ω_1 being given by (3.6). Suppose $\tau \in \chi_2(\lambda)$. Since $\chi_2(\lambda) \subset \mathcal{K}_2(\chi_1^*)$, there is a $p \in P$ such that $\tau + 2\pi p/N_1a \in \chi_1^*$. By Lemma 2.12, there is an eigenvalue $\lambda_n^{(1)}(\tau + 2\pi p/N_1a) = \lambda$, which is given by perturbation series (2.16). By the same lemma $\lambda_n^{(1)}(\tau+2\pi p/N_1a)$ is a unique eigenvalue of $H^{(1)}(\tau + 2\pi p/N_1a)$ in $\varepsilon_1 = (k^{2l} - k^{2l-2-4s_1-\delta}, k^{2l} + k^{2l-2-4s_1-\delta})$. This means

$$\left|\lambda_n^{(1)}(\tau + 2\pi p/N_1a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi p/N_1a)\right| \geq k^{2l-2-4s_1-\delta}, \quad \text{when} \quad \hat{n} \neq n. \tag{3.30}$$

Using the definition of Ω_1, we obtain:

$$|\lambda_n^{(1)}(\tau + 2\pi p/N_1a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1a)| > \epsilon_1 \quad \text{when} \quad p \neq \hat{p}.$$

Combining the last two inequalities, we obtain Statement 1(c). It remains to prove that p is defined uniquely by the relation $\tau + 2\pi p/N_1a \in \chi_1^*$. Suppose it is not so, and there is $\hat{p} \neq p$: $\tau + 2\pi\hat{p}/N_1a \in \chi_1^*$. Applying Lemma 2.12 to both points in χ_1^*, we obtain $\lambda_n^{(1)}(\tau + 2\pi p/N_1a) = \lambda_{\hat{n}}^{(1)}(\tau+2\pi\hat{p}/N_1a)$, the eigenvalues being given by perturbation series. The last equality contradicts to (3.30).

3.3.2. *Proof of the Statement 2.* If τ is in the complex $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood of χ_2, then there exists a τ_0 in χ_2 such that $|\tau - \tau_0| < \epsilon_1 k^{-2l+1-\delta}$ and τ_0 satisfies Statement 1 of the lemma. Obviously $\tau + 2\pi p/N_1a$ is in the $(k^{-1-4s_1-2\delta})$-neighborhood of χ_1, where Theorem 2.3 holds. By Corollary 3.13,

$$\left|\lambda_j^{(1)}(\tau + 2\pi p/N_1a) - \lambda_j^{(1)}(\tau_0 + 2\pi p/N_1a)\right| \leq \sup_{\tilde{\tau}} \left|\nabla_\tau \lambda_j^{(1)}(\tilde{\tau} + 2\pi p/N_1a)\right| |\tau - \tau_0| = O(\epsilon_1 k^{-\delta}).$$

Considering that $\lambda_j^{(1)}(\tau_0 + 2\pi p/N_1a) = k^{2l}$, we get (3.9).

3.3.3. *Determinants. Intersections and Quasi-intersections. Description of the set Ω_1 in terms of determinants.* We considered self-intersections of $\tilde{S}_1(\lambda)$ belonging to $\mathcal{K}_2\chi_1^*$ (see Definition 3.2 and Lemma 3.4, page 17). Here we describe self-intersections as zeros of determinants of operators of the type $I + A$, $A \in \mathbf{S_1}$ (see e.g. [21]). In fact, let us represent the operator $(H^{(1)}(t) - \lambda)(H_0(t) + \lambda)^{-1}$ in the form $I + A_1$:

$$(H^{(1)}(t) - \lambda)(H_0(t) + \lambda)^{-1} = I + A_1(t), \quad A_1(t) = (W_1 - 2\lambda)(H_0(t) + \lambda)^{-1}. \tag{3.31}$$

Obviously, $A_1(t) \in \mathbf{S_1}$.

Remark 3.15. From properties of determinants and the definition of $S_1(\lambda)$ it easily follows that the isoenergetic set $S_1(\lambda)$ is the zero set of $\det\Big(I + A_1(t)\Big)$ in K_1.

Remark 3.16. Matrices $H_0(t)$, $H^{(1)}(t)$ and $A_1(t)$ can be extended as holomorphic functions of two variables to a vicinity of $\mathbb{R}^2$. If $\vec{y} = \vec{p}_j(0) + t$, then matrices $H_0(\vec{y})$, $H^{(1)}(\vec{y})$ and $A_1(\vec{y})$ differ from $H_0(t)$, $H^{(1)}(t)$ and $A_1(t)$ only by a shift of indices: $H_0(\vec{y})_{mm} = H_0(t)_{m+j,m+j}$, etc. Hence, $\|(H^{(1)}(\vec{y}) - z)^{-1}\| = \|(H^{(1)}(t) - z)^{-1}\|$ and $\det\big(I + A_1(\vec{y})\big) = \det\big(I + A_1(t)\big)$.

Next, we recall that the set $\mathcal{D}_1(\lambda)$ can be described in terms of vectors $\vec{\varkappa}(\varphi) = \varkappa(\lambda, \vec{\nu})\vec{\nu}$, $\vec{\nu} = (\cos\varphi, \sin\varphi)$, $\varphi \in \Theta_1$, see (2.33), (2.42). By definition, $\chi_1^*(\lambda) = \mathcal{K}_1\mathcal{D}_1(\lambda)$. By Lemma 2.12, page 15, $\chi_1^*(\lambda)$ does not have self-intersections, i.e., for every $t \in \chi_1^*(\lambda)$, there is a single $\vec{\varkappa}(\varphi) \in \mathcal{D}_1(\lambda)$, such that $t = \mathcal{K}_1\vec{\varkappa}(\varphi)$. Next, if $\tau \in \mathcal{K}_2\chi_1^*(\lambda)$, then there is $p \in P$ such that $\tau + 2\pi p/N_1 a \in \chi_1^*(\lambda)$. Note that p is not uniquely defined by τ, since $\mathcal{K}_2\chi_1^*(\lambda)$ may have self-intersections. Hence, every $\tau \in \mathcal{K}_2\chi_1^*(\lambda)$ can be represented as $\tau = \mathcal{K}_2\vec{\varkappa}(\varphi)$, here $\vec{\varkappa}(\varphi)$ is not necessary uniquely defined. The next lemma describes self-intersection of $\tilde{S}_1$ belonging to $\mathcal{K}_2\chi_1^*(\lambda)$ as zeros of a group of determinants.

Lemma 3.17. *If τ is a point of self-intersection of $\tilde{S}_1(\lambda)$ belonging to $\mathcal{K}_2\chi_1^*(\lambda)$, then τ can be represented in the form $\tau = \mathcal{K}_2\vec{\varkappa}(\varphi)$, where $\varphi \in \Theta_1$ and satisfies the equation*

$$\det\Big(I + A_1\big(\vec{y}(\varphi)\big)\Big) = 0, \quad \vec{y}(\varphi) = \vec{\varkappa}(\varphi) + \vec{b}, \quad \vec{b} = 2\pi p/N_1 a, \tag{3.32}$$

for some $p \in P \setminus \{0\}$. Conversely, if (3.32) is satisfied for some $p \in P \setminus \{0\}$, then $\tau = \mathcal{K}_2\vec{\varkappa}(\varphi)$ is a point of self-intersection.

Proof. Suppose, τ is a point of self-intersection of $\tilde{S}_1(\lambda)$ belonging to $\mathcal{K}_2\chi_1^*(\lambda)$. By Lemma 3.4, page 17, there is $\tilde{p} \in P$ (here we use $\tilde{p}$ instead of p in Lemma 3.4), such that $\tau + 2\pi\tilde{p}/N_1 a \in \chi_1^*$, and $\hat{p} \in P$, $\hat{p} \neq \tilde{p}$, such that $\tau + 2\pi\hat{p}/N_1 a \in S_1(\lambda)$. This means that

$$\det\Big(I + A_1\big(\tau + 2\pi\tilde{p}/N_1 a\big)\Big) = 0 \quad \text{and} \quad \det\Big(I + A_1\big(\tau + 2\pi\hat{p}/N_1 a\big)\Big) = 0. \tag{3.33}$$

By Lemma 2.13, page 15, there is a single $\vec{\varkappa}(\varphi) \in \mathcal{D}_1(\lambda)$: $\mathcal{K}_1\vec{\varkappa} = \tau + 2\pi\tilde{p}/N_1 a$ (this means $\mathcal{K}_2\vec{\varkappa} = \tau$). According to Remark 3.16 and formulae (3.33), $\det\Big(I + A_1(\vec{\varkappa})\Big) = 0$, $\det\Big(I + A_1(\vec{\varkappa} + \vec{b})\Big) = 0$, where $\vec{b} = \big[2\pi(\hat{p} - \tilde{p})/N_1 a\big]_{\text{mod } K_1}$. Thus, we have obtained (3.32).

Suppose now that (3.32) holds for some $p \in P$. Let us show that $\tau = \mathcal{K}_2\vec{\varkappa}(\varphi)$ is a point of self-intersection. Let $\tilde{p} \in P$: $\tau + 2\pi\tilde{p}/N_1 a = \mathcal{K}_1\vec{\varkappa}$; such $\tilde{p}$, obviously, exists, since $\mathcal{K}_1\vec{\varkappa} \in K_1$, $\tau \in K_2$. Since $\vec{\varkappa}(\varphi) \in \mathcal{D}_1(\lambda)$, we have $\tau + 2\pi\tilde{p}/N_1 a \in \chi_1^* \subset S_1(\lambda)$. Therefore, there is $n \in \mathbb{N}$:

$$\lambda_n^{(1)}(\tau + 2\pi\tilde{p}/N_1 a) = \lambda. \tag{3.34}$$

Next, it follows from (3.32) and Remarks 3.15, 3.16 that $\mathcal{K}_1\vec{y}(\varphi) \in S_1(\lambda)$. Hence, there is $\hat{\tau} \in K_2$, $\hat{p} \in P$ such that $\hat{\tau} + 2\pi\hat{p}/N_1 a = \mathcal{K}_1\vec{y}(\varphi) \in S_1(\lambda)$. This means that there is an $\hat{n} \in \mathbb{N}$ such that

$$\lambda_{\hat{n}}^{(1)}(\hat{\tau} + 2\pi\hat{p}/N_1 a) = \lambda. \tag{3.35}$$

Note that $\tau - \hat{\tau} = \mathcal{K}_2\vec{y}(\varphi) - \mathcal{K}_2\vec{\varkappa}(\varphi) = \mathcal{K}_2\vec{b} = 0$, i.e., $\tau = \hat{\tau}$. Considering this equality together with (3.34), (3.35), we see that τ is a point of self-intersection, provided that $p \neq \hat{p}$. It remains to show that $p \neq \hat{p}$. In fact, $\tau + 2\pi p/N_1 a = \mathcal{K}_1\vec{\varkappa}$, $\tau + 2\pi\hat{p}/N_1 a = \mathcal{K}_1\vec{y}$, hence $2\pi(\hat{p} - p)/N_1 a = \mathcal{K}_1\big(\vec{y} - \vec{\varkappa}\big) = \mathcal{K}_1\vec{b} = \vec{b} \neq 0$, the last equality holding, because $\vec{b} \in K_1$. By Lemma 3.4, τ is a point of self-intersection of $\tilde{S}_1$ belonging to $\mathcal{K}_2\chi_1^*(\lambda)$. ∎

Definition 3.18. Let Φ_1 be the complex $\big(k^{-2-4s_1-2\delta}\big)$-neighborhood of Θ_1.

By Lemma 2.11, $\vec{\varkappa}(\varphi)$ is an analytic function in Φ_1, and, hence, $\det\Big(I + A\big(\vec{y}(\varphi)\big)\Big)$, $\vec{y} = \vec{\varkappa}(\varphi) + \vec{b}$, $\vec{b} \in K_1$, is analytic too.

Definition 3.19. We say that $\varphi \in \Phi_1$ is a quasi-intersection of $\mathcal{K}_2\chi_1^*$ with $\tilde{S}_1(\lambda)$ if the relation (3.32) holds for some $p \in P \setminus \{0\}$.

Thus, real intersections correspond to real zeros of the determinant, while quasi-intersections may have a small imaginary part (quasi-intersections include intersections).

Next we describe the non-resonance set in terms of determinants.

Lemma 3.20. *If $\tau \in \Omega_1$, then $\tau = \mathcal{K}_2\vec{\varkappa}(\varphi)$ where $\varphi \in \Theta_1$ and satisfies the equation*

$$\det\left(\frac{H^{(1)}\big(\vec{y}(\varphi)\big) - k^{2l} - \epsilon}{H_0\big(\vec{y}(\varphi)\big) + k^{2l}}\right) = 0, \quad \vec{y}(\varphi) = \vec{\varkappa}(\varphi) + \vec{b}, \quad \vec{b} = 2\pi p/N_1 a, \tag{3.36}$$

for some $p \in P \setminus \{0\}$ and $|\epsilon| < \epsilon_1$. Conversely, if (3.36) is satisfied for some $p \in P \setminus \{0\}$, and $|\epsilon| < \epsilon_1$, then $\tau = \mathcal{K}_2\vec{\varkappa}(\varphi)$ belongs to Ω_1.

Proof. The proof is analogous to the proof of the previous lemma after we notice that the set of points τ satisfying the inequality $\left|\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1 a)\right| < \epsilon_1$ can be described as the set of points satisfying $\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1 a) = \epsilon$ for some $|\epsilon| < \epsilon_1$. ∎

Let us discuss separately the case $W_1 = 0$. If φ is a point of intersection, then (3.32) means that

$$|\vec{\varkappa}(\varphi) + 2\pi j/a + 2\pi p/N_1 a|^2 = k^2 \tag{3.37}$$

for some $j \in \mathbb{Z}^2$ and $p \in P$, and vise versa. Thus, (3.32) means that $\vec{\varkappa}(\varphi)$ is a point of intersection of the circle of the radius k centered at the origin ($|\vec{\varkappa}(\varphi)|^2 = k^2$ when $W_1 = 0$) and the circle of the same radius centered at a point $2\pi j/a + 2\pi p/N_1 a$ of a dual lattice corresponding to the periods $N_1 a_1$, $N_1 a_2$. If there is a quasi-intersection, then (3.37) holds for some $\varphi \in \Phi_1$. We will refer to such φ as a point of quasi-intersection of two circles. Geometrically, such terminology is justified, since validity of (3.37) for some φ with a small imaginary part means, that two circles come close together, even if they do not intersect.

We denote by ω_1 the set of $\varphi \in \Theta_1$ corresponding to Ω_1, i.e., $\omega_1 = \{\varphi \in \Theta_1 : \mathcal{K}_2\vec{\varkappa}_1(\varphi) \in \Omega_1\} \subset [0, 2\pi)$.

3.3.4. *Complex resonant set.* Further we consider a complex resonance set $\omega_1^*(\lambda)$, which is the set of zeros of the determinants (3.36) in Φ_1 ($p \in P \setminus \{0\}$, $|\epsilon| < \epsilon_1$). By Lemma 3.20, $\omega_1 = \omega_1^* \cap \Theta_1$. We prefer to consider quasi-intersections, instead of intersections, and the complex resonance set, instead of just the real one, for the following reason: the determinants (3.32) and (3.36), involved in the definitions of quasi-intersections and the complex resonance set ω_1^* are holomorphic functions of φ in Φ_1. We can apply complex analysis theorem to these determinants. Rouché's theorem is particularly important here, since it states stability of zeros of a holomorphic function with respect to small perturbations of the function. We take the determinant (3.32) as a holomorphic function, its zeros being quasi-intersections: the initial determinant corresponds to the case $W_1 = 0$, the perturbation is obtained by "switching on" a potential W_1. Since there is no analog of Rouché's theorem (e.g. see [24]) for real functions on the real axis, introducing the region Φ_1 and analytic extension of the determinants into this region is in the core of our considerations. We will use also a well-known inequality for the determinants (e.g. see [21]):

$$\Big|\det(I + A) - \det(I + B)\Big| \le \|A - B\|_1 \exp\Big(\|A\|_1 + \|B\|_1 + 1\Big), \ A, B \in \mathbf{S}_1. \tag{3.38}$$

Note that $\omega_1^*=\cup_{p\in P\setminus\{0\}}\omega_{1,p}^*$, where $\omega_{1,p}^*$ corresponds to a fixed p in (3.36); and similarly, $\omega_1=\cup_{p\in P\setminus\{0\}}\omega_{1,p}$. We fix $p\in P$ and study each $\omega_{1,p}^*$ separately. We start with the case $W_1=0$. The corresponding determinant (3.32) is

$$\det\Big(I+A_0\big(\vec{y}_0(\varphi)\big)\Big),\ I+A_0\big(\vec{y}_0(\varphi)\big)=\Big(H_0\big(\vec{y}_0(\varphi)\big)-\lambda\Big)\Big(H_0\big(\vec{y}_0(\varphi)\big)+\lambda\Big)^{-1}, \tag{3.39}$$

$\vec{y}_0(\varphi)=k(\cos\varphi,\sin\varphi)+\vec{b}$. This determinant can be investigated by elementary means. We easily check, that the number of zeros of this determinant in Φ_1 does not exceed $c_0k^{2+2s_1}$ (Proposition 3.21, Corollary 3.22). The resolvent $\big(H_0\big(\vec{y}_0(\varphi)\big)-k^{2l}\big)^{-1}$ has poles at zeros of the determinant. The resolvent norm at $\varphi\in\Phi_1$ can be estimated by the distance, which φ has to the nearest zero of the determinant (Lemma 3.23). Next, we introduce a set $\mathcal{O}(\vec{b})$, which is the union of all disks of the radius $r=k^{-4-6s_1-3\delta}$ surrounding zeros of the determinant (3.39) in Φ_1. Obviously, any $\varphi\in\Phi_1\setminus\mathcal{O}(\vec{b})$ is separated from zeros of the determinant (3.39) by the distance no less than r. This estimate on the distance yields an estimate for the norm of the resolvent $\Big(H_0\big(\vec{y}_0(\varphi)\big)-k^{2l}\Big)^{-1}$, when $\varphi\in\Phi_1\setminus\mathcal{O}(\vec{b})$ (Remark 3.28). Further, we introduce a potential W_1. Our goal in this section is to prove that the number of zeros of each determinant (3.36) is preserved in each connected component $\Gamma(\vec{b})$ of $\mathcal{O}(\vec{b})$, when we switch from the case $W_1=0, A_1=A_0$ to the case of nonzero W_1 and from $\epsilon=0$ to $|\epsilon|<\epsilon_1$. We also show that estimates for the resolvent are stable under such change when $\varphi\in\Phi_1\setminus\mathcal{O}(\vec{b})$. We "switch on" potential W_1 in two steps. First, we replace $\vec{y}_0(\varphi)$ by $\vec{y}(\varphi)$, i.e., we consider $\det\Big(I+A_0\big(\vec{y}(\varphi)\big)\Big)$ and $\Big(H_0\big(\vec{y}(\varphi)\big)-k^{2l}\Big)^{-1}$ in Φ_1. We take into account that $\vec{y}(\varphi)-\vec{y}_0(\varphi)$ is small and holomorphic in Φ_1 (Lemma 2.11), use (3.38) on the boundary of Γ and apply Rouché's theorem. This leads us to the conclusion, that the number of zeros of the determinant in $\Gamma(\vec{b})$ is preserved, when we replace $\vec{y}_0(\varphi)$ by $\vec{y}(\varphi)$ (Lemma 3.29). Applying Hilbert relation for resolvents, we show that estimates for the resolvent in $\Phi_1\setminus\mathcal{O}(\vec{b})$ are also stable under such change (Lemma 3.29, Corollary 3.30). In the second step we replace $H_0\big(\vec{y}(\varphi)\big)$ by $H^{(1)}\big(\vec{y}(\varphi)\big)$ and prove similar results (Lemmas 3.31, 3.34, Corollaries 3.32, 3.36). Further we consider the determinant (3.36). Obviously it is equal to (3.32) when $\epsilon=0$. Since, ϵ is very small, its properties are similar to those of (3.32). In particular, it has the same number of zeros (for a fixed ϵ) as (3.32) at each connected component $\Gamma(\vec{b})$ of $\mathcal{O}(\vec{b})$ (Lemma 3.38). From this lemma we see that $\omega_{1,p}^*\subset\mathcal{O}(\vec{b})$, $\vec{b}=2\pi p/N_1a$. Taking the real parts of the sets, we conclude: $\omega_{1,p}\subset\mathcal{O}(\vec{b})\cap\Theta_1$. We show that $\mathcal{O}(\vec{b})$ is formed by no more than $c_0k^{2+2s_1}$ disks of the radius $r=k^{-4-6s_1-3\delta}$. From this we easily obtain, that the total length of $\omega_{1,p}$ does not exceed $k^{-2-4s_1-3\delta}$. Since the set P contains no more than $4k^{2s_2-2s_1}$ elements, $s_2=2s_1$, the total length of Ω_1 does not exceed $k^{-2-2s_1-3\delta}$. This estimate is proven in Section 3.3.5.

Let us introduce new notations:

$$\mathcal{O}_*=\cup_{p\in P\setminus\{0\}}\mathcal{O}\big(2\pi p/N_1a\big),\qquad \Phi_2=\Phi_1\setminus\mathcal{O}_*, \tag{3.40}$$

Φ_1 being given by Definitions 3.18. Note that the set $\mathcal{O}_*$ is formed by small disks centered at points $\varphi_{m,p}^{\pm}\in\Phi_1$, which are quasi-intersections of the circle of the radius k around the origin with the circles of the same radius k around points $\vec{p}_m(0)+2\pi p/N_1a$, $p\neq 0$, that is around points of the dual lattice corresponding to the periods N_1a_1, N_1a_2. We will show (Corollary 3.39) that $\omega_1^*\subset\mathcal{O}_*$ and $\omega_1\subset\mathcal{O}_*\cap\Theta_1$. From now on we call $\mathcal{O}_*$ the first complex resonance set in Φ_1 and

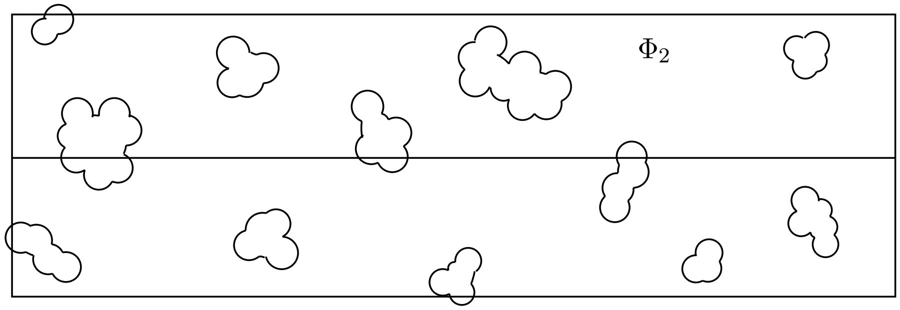

FIGURE 7. The set Φ_2.

$\mathcal{O}_* \cap \Theta_1$ the first resonance set in Θ_1. Obviously, to obtain Φ_2 we produce round holes in each connected component of Φ_1. The set Φ_2 has a structure of Swiss cheese (Fig. 7); we will add more holes of a smaller size at each step of approximation.

In the definition of the non-resonance set we take $\vec{b} = 2\pi p/N_1 a$, $p \neq 0$. Let us consider now arbitrary $\vec{b} \in K_1$ and investigate $\mathcal{O}(\vec{b})$.[8] We denote by b_0 be the distance from $\vec{b} \in K_1$ to the nearest vertex of K_1 :

$$b_0 = \min_{m=(0,0),(0,1),(1,0),(1,1)} \left|\vec{b} - 2\pi m/a\right| \tag{3.41}$$

Further we assume $b_0 \neq 0$, i.e., $\vec{b}$ is not a vertex of K_1. We denote by Φ_0 a narrow strip in the complex plane around the interval $[0, 2\pi)$:

$$\Phi_0 = \left\{\varphi : \Re\varphi \in [0, 2\pi), |\Im\varphi| < k^{-2-4s_1}\right\}.$$

Obviously, $\Phi_1 \subset \Phi_0$. Let $|\vec{y}_0(\varphi) - \vec{x}_0|_*^2$ be an analytic function of φ defined by the formula: $|\vec{y}_0(\varphi) - \vec{x}_0|_*^2 := (k\cos\varphi + b_1 - x_{01})^2 + (k\sin\varphi + b_2 - x_{02})^2$, $\vec{x}_0$ being a vector in $\mathbb{R}^2$. Note that $|\vec{y}_0(\varphi) - \vec{b}|_*^2 = k^2\cos^2\varphi + k^2\sin^2\varphi = k^2$. Let $|\vec{y}_0(\varphi) - \vec{x}_0|_*^{2l} := \left(|\vec{y}_0(\varphi) - \vec{x}_0|_*^2\right)^l$. Obviously, $\det\Big(I + A_0\big(\vec{y}_0(\varphi)\big)\Big) = 0$ if and only if $|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} = k^{2l}$ for some $m \in \mathbb{Z}^2$.

Proposition 3.21. *Suppose $\vec{b}$ is not a vertex of K_1. If $p_m(0) < 4k$, then the number of solutions of the equation*

$$|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} = k^{2l} \tag{3.42}$$

in Φ_0 does not exceed two. The solutions of (3.42) are, in fact, the solutions $\varphi_m^{\pm}$ of the equation

$$|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^2 = k^2. \tag{3.43}$$

If the distance from $\varphi \in \Phi_0$ to the nearest solution $\varphi_m^{\pm}$ of (3.43) is greater than ϵ_0, $\epsilon_0 > 0$, then

$$\left||\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - k^{2l}\right| > c_l b_0 \epsilon_0^2 k^{2l-1}, \tag{3.44}$$

c_l being a positive number depending only on l. If $p_m(0) \geq 4k$, then the equation (3.42) does not have solutions in Φ_0. Moreover,

$$\left||\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - k^{2l}\right| > \frac{1}{2^l} p_m^{2l}(0). \tag{3.45}$$

[8] We need arbitrary $\vec{b}$ for next steps of approximation.

The proof is elementary and based on the fact that (3.43) is a quadratic equation with respect to $e^{i\varphi}$.

Corollary 3.22. *For each $\vec{b}$, the number of $\varphi \in \Phi_0$ satisfying (3.42) for at least one $m \in \mathbb{Z}^2$ does not exceed $c_0 k^{2+2s_1}$, $c_0 = 32b_1b_2$.*

Corollary 3.22. The equation (3.42) can have solutions in Φ_0 only if $p_m(0) < 4k$. The number of such m's does not exceed $16b_1b_2k^{2+2s_1}$. For a fixed m there exist no more than 2 solutions, $\varphi_m^{\pm}$. This gives the statement of the corollary. ∎

According to the definition at the end of Section 3.3.3, points $\varphi_m^{\pm}$ are points of quasi-intersections of the circle of the radius k centered at the origin and the circle of the same radius centered at $-\vec{p}_m(0) - \vec{b}$.

Lemma 3.23. *If $\varphi \in \Phi_0$ satisfies the inequality* $\min_{m \in \mathbb{Z}^2} |\varphi - \varphi_m^{\pm}| \geq \epsilon_0$ *for some* $0 < \epsilon_0 < 1$, *then the following estimates hold:*

$$\left\| \Big(H_0\big(\vec{y}_0(\varphi)\big) - k^{2l} \Big)^{-1} \right\| \leq c_l k^{-2l+1} b_0^{-1} \epsilon_0^{-2}, \tag{3.46}$$

$$\left\| \Big(H_0\big(\vec{y}_0(\varphi)\big) - k^{2l} \Big)^{-1} \right\|_1 \leq c_l k^{-2l+3+2s_1} b_0^{-1} \epsilon_0^{-2}. \tag{3.47}$$

Proof. Estimate (3.46) immediately follows from (3.44) and (3.45). Considering that $H_0\big(\vec{y}_0(\varphi)\big)$ is a diagonal operator and summarizing (3.44), (3.45) over m, we arrive at (3.47). ∎

Now let us construct contours in $\Phi_1 \subset \Phi_0$ whose distance from all the solutions of the equations $|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} = k^{2l}$, $m \in \mathbb{Z}^2$, is at least $k^{-4-6s_1-3\delta}$. We recall that there are no more than two solutions $\varphi_m^{\pm}$ in Φ_0 if $p_m(0) < 4k$, and there are no solutions if $p_m(0) \geq 4k$ (Proposition 3.21).

Definition 3.24. For each φ_m^{+} (or φ_m^{-}) in Φ_0 we make an open disk of the radius $k^{-4-6s_1-3\delta}$ centered at φ_m^{+} (or φ_m^{-}) and denote it by $\mathcal{O}_m^{+}$ (or $\mathcal{O}_m^{-}$). Let $\mathcal{O}'$ the union of all such disks: $\mathcal{O}' = \cup_{m,\pm}\mathcal{O}_m^{\pm}$. We denote by $\mathcal{O}$ the part of $\mathcal{O}'$ related to Φ_1. Namely, $\mathcal{O}$ is the union of connected components of $\mathcal{O}'$ not being completely outside Φ_1. To stress that $\mathcal{O}$ depends on $\vec{b}$, we often will write $\mathcal{O}(\vec{b})$.

Remark 3.25. Obviously, $\mathcal{O}(\vec{b})$ consists of no more than $c_0k^{2+2s_1}$ disks (see Corollary 3.22). Thus set $\mathcal{O}_*$ given by (3.40) contains no more than $4c_0k^{2+2s_2}$ disks, since P contains less than $4k^{s_2-s_1}$ elements.

Definition 3.26. Let $\Gamma(\vec{b})$ be a connected component of $\mathcal{O}(\vec{b})$ and $\gamma = \partial\Gamma$.

Remark 3.27. The set $\mathcal{O}(\vec{b})$ consists of several components $\Gamma(\vec{b})$. Since the number of points $\varphi_m^{\pm}$ in Φ_0 does not exceed $c_0k^{2+2s_1}$, the size of any Γ is less than $c_0k^{2+2s_1} \cdot 2k^{-4-6s_1-3\delta} = 2c_0k^{-2-4s_1-\mathbf{3}\delta}$. Clearly it is much smaller than the size of Φ_1 (see Definition 3.18). Some $\Gamma(\vec{b})$ are strictly inside Φ_1, while others can have their parts outside. However, any $\Gamma(\vec{b})$ is in the $(2c_0k^{-2-4s_1-\mathbf{3}\delta})-$ neighborhood of Φ_1. This neighborhood has properties completely analogous to those of Φ_1, since $ck^{-2-4s_1-\mathbf{3}\delta} << k^{-2-4s_1-\mathbf{2}\delta}$, the latter value being the size of Φ_1. Hence, even if a component $\Gamma(\vec{b})$ is not strictly inside Φ_1 it has the same properties as those inside Φ_1. Further we consider $\Gamma(\vec{b}) \subset \Phi_1$, $\mathcal{O}(\vec{b}) \subset \Phi_1$.

Remark 3.28. For any $\varphi \in \Phi_1 \setminus \mathcal{O}(\vec{b})$ and any $\varphi_m^{\pm}$ the estimate $|\varphi - \varphi_m^{\pm}| \geq k^{-4-6s_1-3\delta}$ holds. Therefore, the estimates (3.44), (3.46) and (3.47) hold with $\epsilon_0 = k^{-4-6s_1-3\delta}$ for such φ.

Obviously, zeros of $\det\Big(I + A_0\big(\vec{y}(\varphi)\big)\Big)$ are described by the equations $|\vec{\varkappa}(\varphi) + \vec{b} + \vec{p}_m(0)|_*^{2l} = k^{2l}$, $m \in \mathbb{Z}^2$, $\varphi \in \Phi_1$, which can be rewritten as $|\vec{y}_0(\varphi) + h_1(\varphi)\vec{\nu} + \vec{p}_m(0)|_*^{2l} = k^{2l}$, the functions $\vec{\varkappa}(\varphi) \equiv \varkappa(\lambda, \vec{\nu})$ and $h_1(\varphi) \equiv h_1(\lambda, \vec{\nu})$ being defined in Lemma 2.11. It is easy to see that the vectors $\vec{\varkappa}(\varphi)$ satisfying this equation with a real φ are just intersections of $\mathcal{D}_1(\lambda)$ with the circle of radius k centered at $-\vec{b} - \vec{p}_m(0)$. Since $\vec{y}(\varphi)$ differ from $\vec{y}_0(\varphi)$ by a small value $h_1(\varphi)$, the estimates (3.44), (3.45) and (3.46) (3.47) with $\epsilon_0 = k^{-4-6s_1-3\delta}$ are preserved for $\varphi \in \Phi_1 \setminus \mathcal{O}$, when we replace $\vec{y}_0(\varphi)$ by $\vec{y}(\varphi)$. We prove this result in the following lemma.

Lemma 3.29. *If $b_0 > k^{7+8s_1-2\gamma_0+6\delta}$, $\gamma_0 = 2l - 2 - 4s_1 - 2\delta$, then:*

1. *The number of zeros of $\det\Big(I + A_0\big(\vec{y}(\varphi)\big)\Big)$ in any $\Gamma(\vec{b})$ is the same as that of $\det\Big(I + A_0\big(\vec{y}_0(\varphi)\big)\Big)$.*
2. *The following estimates hold for $\varphi \in \Phi_1 \setminus \mathcal{O}(\vec{b})$:*

$$\left\| \Big(H_0\big(\vec{y}(\varphi)\big) - k^{2l}\Big)^{-1} \right\| < c b_0^{-1} k^{-2l+9+12s_1+6\delta}, \tag{3.48}$$

$$\left\| \Big(H_0\big(\vec{y}(\varphi)\big) - k^{2l}\Big)^{-1} \right\|_1 < c b_0^{-1} k^{-2l+11+14s_1+6\delta}. \tag{3.49}$$

Corollary 3.30. *There are no zeros of $\det\Big(I + A_0\big(\vec{y}(\varphi)\big)\Big)$ in $\Phi_1 \setminus \mathcal{O}(\vec{b})$.*

Lemma 3.29. Note that $\det\Big(I + A_0\big(\vec{y}_0(\varphi)\big)\Big) = 0$ if and only if $|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} = k^{2l}$ for at least one $m \in \mathbb{Z}^2$. The analogous statement holds for $\det\Big(I + A_0\big(\vec{y}(\varphi)\big)\Big)$. It is easy to show (Appendix A.2) that

$$\Big| |\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - |\vec{y}(\varphi) + \vec{p}_m(0)|_*^{2l} \Big| < \frac{1}{2} \Big| |\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - k^{2l} \Big| \tag{3.50}$$

for any φ in $\Phi_1 \setminus \mathcal{O}(\vec{b})$ and $m \in \mathbb{Z}^2$. By Rouché's theorem [24], the number of zeros of $|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - k^{2l}$ in every $\Gamma(\vec{b})$ is the same as that of $|\vec{y}(\varphi) + \vec{p}_m(0)|_*^{2l} - k^{2l}$. Therefore, the first statement of the lemma is proved.

To prove the second statement, we use the Hilbert relation:

$$\begin{aligned}\Big(H_0\big(\vec{y}(\varphi)\big) - k^{2l}\Big)^{-1} &= \Big(H_0\big(\vec{y}_0(\varphi)\big) - k^{2l}\Big)^{-1} \\ &+ \Big(H_0\big(\vec{y}(\varphi)\big) - k^{2l}\Big)^{-1} \Big(H_0\big(\vec{y}_0(\varphi)\big) - H_0\big(\vec{y}(\varphi)\big)\Big) \Big(H_0\big(\vec{y}_0(\varphi)\big) - k^{2l}\Big)^{-1}.\end{aligned}$$

By Lemma 3.23 with $\epsilon_0 = k^{-4-6s_1-3\delta}$,

$$\left\| \Big(H_0\big(\vec{y}_0(\varphi)\big) - k^{2l}\Big)^{-1} \right\| < c b_0^{-1} k^{-2l+9+12s_1+6\delta},$$

$$\left\| \Big(H_0\big(\vec{y}_0(\varphi)\big) - k^{2l}\Big)^{-1} \right\|_1 < c b_0^{-1} k^{-2l+11+14s_1+6\delta}.$$

We easily obtain from (3.50) that

$$\left\|\Big(H_0\big(\vec{y}_0(\varphi)\big)-H_0\big(\vec{y}(\varphi)\big)\Big)\Big(H_0\big(\vec{y}_0(\varphi)\big)-k^{2l}\Big)^{-1}\right\|<1/2. \tag{3.51}$$

Now, using a standard argument as in Lemma 2.9, we arrive at (3.48) and (3.49). ∎

Lemma 3.31. *If $b_0>k^{-2l+9+12s_1+7\delta}$, then the following estimates hold for any $\varphi\in\Phi_1\setminus\mathcal{O}(\vec{b})$:*

$$\left\|(H^{(1)}(\vec{y}(\varphi))-k^{2l})^{-1}\right\|<cb_0^{-1}k^{-2l+9+12s_1+6\delta}, \tag{3.52}$$

$$\left\|(H^{(1)}(\vec{y}(\varphi))-k^{2l})^{-1}\right\|_1<cb_0^{-1}k^{-2l+11+14s_1+6\delta}. \tag{3.53}$$

Corollary 3.32. *The determinant of $I+A_1\big(\vec{y}(\varphi)\big)$ has no zeros in $\Phi_1\setminus\mathcal{O}(\vec{b})$.*

Proof. We use Hilbert relation $\Big(H^{(1)}\big(\vec{y}(\varphi)\big)-k^{2l}\Big)^{-1}=$

$$\Big(H_0\big(\vec{y}(\varphi)\big)-k^{2l}\Big)^{-1}+\Big(H^{(1)}\big(\vec{y}(\varphi)\big)-k^{2l}\Big)^{-1}(-W_1)\Big(H_0\big(\vec{y}(\varphi)\big)-k^{2l}\Big)^{-1}.$$

Since $b_0>k^{-2l+9+12s_1+7\delta}>k^{7+8s_1-2\gamma_0+6\delta}$ $(\gamma_0=2l-2-4s_1-2\delta)$, the previous lemma can be applied. It yields:

$$\left\|\Big(H_0\big(\vec{y}(\varphi)\big)-k^{2l}\Big)^{-1}\right\|=O\left(b_0^{-1}k^{-2l+9+12s_1+6\delta}\right)=o(1).$$

Using a standard argument as in Lemma 2.9, we arrive at (3.52) and (3.53). ∎

Remark 3.33. Obviously (3.52) and (3.53) hold for any $H_\alpha^{(1)}=H_0+\alpha W_1$, $0\le\alpha\le1$ uniformly in α.

Further, we will use the notation $\frac{T_1}{T_2}=T_1T_2^{-1}$ for a pair of operators T_1, T_2.

Lemma 3.34. *If $b_0>k^{-2l+9+12s_1+7\delta}$, then the number of zeros of $\det\Big(I+A_1\big(\vec{y}(\varphi)\big)\Big)$ in any $\Gamma(\vec{b})$ is the same as that of $\det\Big(I+A_0\big(\vec{y}(\varphi)\big)\Big)$.*

Proof. Let us consider $\det\left(\dfrac{H_0\big(\vec{y}(\varphi)\big)+\alpha_1W_1-k^{2l}}{H_0\big(\vec{y}(\varphi)\big)+\alpha_2W_1-k^{2l}}\right)$, where $0\le\alpha_1\le\alpha_2\le1$. Obviously,

$$\det\left(\frac{H_0\big(\vec{y}(\varphi)\big)+\alpha_1W_1-k^{2l}}{H_0\big(\vec{y}(\varphi)\big)+\alpha_2W_1-k^{2l}}\right)=\det\left(I+\frac{(\alpha_1-\alpha_2)W_1}{H_0\big(\vec{y}(\varphi)\big)+\alpha_2W_1-k^{2l}}\right).$$

Using (3.53), which is uniform with respect to α, we obtain, by Rouché's theorem, that this determinant has the same number of poles and zeros if $|\alpha_2-\alpha_1|$ is sufficiently small: $|\alpha_2-\alpha_1|<b_0k^{2l-11-14s_1-7\delta}$. We make a finite number of steps between $\alpha=0$ and $\alpha=1$. Indeed, $\det\left(\dfrac{H_0\big(\vec{y}(\varphi)\big)+\alpha_1W_1-k^{2l}}{H_0\big(\vec{y}(\varphi)\big)+\alpha_2W_1-k^{2l}}\right)$ has the same number of zeros and poles and so does $\det\left(\dfrac{H_0\big(\vec{y}(\varphi)\big)+\alpha_2W_1-k^{2l}}{H_0\big(\vec{y}(\varphi)\big)+\alpha_3W_1-k^{2l}}\right)$. Both determinants can be represented as fractions of holomorphic functions: $\dfrac{f_1(\varphi)}{f_2(\varphi)}$ and $\dfrac{g_1(\varphi)}{g_2(\varphi)}$, respectively. The number of zeros being the same for the

pairs f_1, f_2 and g_1, g_2. Note that $\det\left(\frac{H_0\big(\vec{y}(\varphi)\big)+\alpha_1 W_1-k^{2l}}{H_0\big(\vec{y}(\varphi)\big)+\alpha_3 W_1-k^{2l}}\right)=\frac{f_1g_1}{f_2g_2}$. Obviously, it also has the same number of zeros and poles. Hence, $\det\left(\frac{H_0\big(\vec{y}(\varphi)\big)+W_1-k^{2l}}{H_0\big(\vec{y}(\varphi)\big)-k^{2l}}\right)$ has the same number of zeros and poles inside $\Gamma(\vec{b})$. Considering that the last determinant is the quotient of $\det\Big(I+A_1\big(\vec{y}(\varphi)\big)\Big)$ and $\det\Big(I+A_0\big(\vec{y}(\varphi)\big)\Big)$ and these two determinants are holomorphic functions, we obtain, that they have the same number of zeros inside $\Gamma(\vec{b})$. ∎

Lemma 3.35. (1) *If $b_0>k^{-2l+9+12s_1+7\delta}$, then the number of zeros of $\det\Big(I+A_1\big(\vec{y}(\varphi)\big)\Big)$ in any $\Gamma(\vec{b})$ is the same as that of $\det\Big(I+A_0\big(\vec{y}_0(\varphi)\big)\Big)$.*

(2) *The distance from a zero of $\det\Big(I+A_1\big(\vec{y}(\varphi)\big)\Big)$ to the nearest zero of $\det\Big(I+A_0\big(\vec{y}_0(\varphi)\big)\Big)$ does not exceed $r/2$.*

Corollary 3.36. *The total number of zeros of $\det\Big(I+A_1\big(\vec{y}(\varphi)\big)\Big)$ in $\mathcal{O}(\vec{b})$ does not exceed $c_0k^{2+2s_1}$.*

The Corollary immediately follows from the first statement of the lemma, the definition of $\mathcal{O}(\vec{b})$ as a collection of disks surrounding zeros of $\det\Big(I+A_0\big(\vec{y}_0(\varphi)\big)\Big)$ and Remark 3.25, page 27.

Corollary 3.37. *The number of self-intersections of $\mathcal{K}_2\chi_1^*(\lambda)$ does not exceed $c_0k^{2+2s_1}$ and $L\left(\mathcal{K}_2\chi_1^*(\lambda)\right)=L\left(\chi_1^*(\lambda)\right)$.*

The latter equality holds, since $\mathcal{K}_2\chi_1^*(\lambda)$ has only a finite number of self-intersections.

Lemma 3.35. The first statement immediately follows Lemmas 3.29 (Part 1) and 3.34. Next, note that Lemmas 3.29 – 3.34 hold not only for $\mathcal{O}(\vec{b})$, but also for a set $\tilde{\mathcal{O}}(\vec{b})$, constructed from disks of twice smaller radius $r/2$ with the same centers, since all estimates in the lemmas preserved under such change up to some insignificant constants. This proves that all zeros of $\det\Big(I+A_1\big(\vec{y}(\varphi)\big)\Big)$ in $\mathcal{O}(\vec{b})$ are, in fact, in the smaller set $\tilde{\mathcal{O}}(\vec{b})$, i.e., the second statement of the lemma holds. ∎

Let us recall that $\omega_{1,p}^*$ is the set of zeros in $\mathit{\Phi}_1$ of the determinant (3.36) when $|\epsilon|<\epsilon_1$, here $p\in P\setminus\{0\}$ and fixed. The set $\omega_{1,p}$ is the real part of $\omega_{1,p}^*$: $\omega_{1,p}=\omega_{1,p}^*\cap\Theta_1$.

Lemma 3.38. *If $b_0>k^{-2l+9+12s_1+7\delta}$, $|\epsilon|<b_0k^{2l-11-14s_1-7\delta}$, then*

(1) *The determinant of $\frac{H^{(1)}(\vec{y}(\varphi))-k^{2l}-\epsilon}{H_0(\vec{y}(\varphi))+k^{2l}}$ has no zeros in $\mathit{\Phi}_1\setminus\mathcal{O}(\vec{b})$.*

(2) *The number of zeros of $\det\left(\frac{H^{(1)}(\vec{y}(\varphi))-k^{2l}-\epsilon}{H_0(\vec{y}(\varphi))+k^{2l}}\right)$ in $\Gamma(\vec{b})$ is the same as that of $\det\left(\frac{H^{(1)}(\vec{y}(\varphi))-k^{2l}}{H_0(\vec{y}(\varphi))+k^{2l}}\right)$.*

Corollary 3.39. *If $\vec{b}=2\pi p/N_1a$, $\omega_{1,p}^*\subset\mathcal{O}(\vec{b})$, $\quad\omega_{1,p}\subset\mathcal{O}(\vec{b})\cap\Theta_1$.*

Proof. To prove Lemmas 3.31 and 3.34, we used the estimate $\|W_1\| = O(1)$. Obviously, $\|W_1-\epsilon\| = O(1)$. Hence, the analogous lemmas hold when we replace W_1 by $W_1-\epsilon$, i.e., when we consider $H^{(1)}(\vec{y}(\varphi)) - k^{2l} - \epsilon$ instead of just $H^{(1)}(\vec{y}(\varphi)) - k^{2l}$. Modified Lemmas 3.31 and 3.34 immediately yield the statement of the present lemma. ∎

3.3.5. *Proof of statement 3 of Geometric Lemma.* By Corollary 3.39, $\omega_{1,p} \subset \mathcal{O}(\vec{b}) \cap \Theta_1$, $\vec{b} = 2\pi p/N_1 a$. It is easy to see that $b_0 \geq c_b k^{-s_2}$, $c_b = \frac{1}{4}\min\{b_1, b_2\}$. Such $\vec{b}$ satisfies conditions of Lemmas 3.29 – 3.38 for any $p \in P \setminus \{0\}$. Taking into account that that $\mathcal{O}(\vec{b})$ is formed by no more than $c_0 k^{2+2s_1}$ disks of the radius $r = k^{-4-6s_1-3\delta}$, we obtain that the total length of $\omega_{1,p}$ does not exceed $2c_0 k^{-2-4s_1-3\delta}$. Since the set P contains no more than $4k^{2s_2-2s_1}$ elements and $s_2 = 2s_1$, the total length of ω_1 does not exceed $8c_0 k^{-2-2s_1-3\delta}$. Therefore, the length of Ω_1 does not exceed $20\pi c_0 k^{-1-2s_1-3\delta}$. Considering that $\chi_1^*(\lambda)$ has a length $2\pi k\big(1+o(1)\big)$, we obtain Statement 3.

3.4. **Non-resonant part of the isoenergetic set of $H_\alpha^{(2)}$.** Let $S_2(\lambda)$ be an isoenergetic set of the operator $H_\alpha^{(2)}$: $S_2(\lambda) = \{\tau \in K_2 : \exists n \in \mathbb{N} :\ \lambda_n^{(2)}(\alpha,\tau) = \lambda\}$, here $\{\lambda_n^{(2)}(\alpha,\tau)\}_{n=1}^{\infty}$ is the spectrum of $H_\alpha^{(2)}(\tau)$. Now we construct a non-resonance subset $\chi_2^*(\lambda)$ of $S_2(\lambda)$. It corresponds to non-resonance eigenvalues $\lambda_{\tilde{j}}^{(2)}(\tau)$ given by the perturbation series (3.15). We start with a definition of $\mathcal{D}_1(\lambda)_{nonres}$. Recall that $\chi_2 \subset \mathcal{K}_2\chi_1^*(\lambda)$ (see the Geometric Lemma) and $\chi_1^*(\lambda) = \mathcal{K}_1\mathcal{D}_1(\lambda)$, see (2.48). Hence, $\chi_2 \subset \mathcal{K}_2\mathcal{D}_1(\lambda)$. Let $\mathcal{D}_1(\lambda)_{nonres}$ be the preimage of χ_2 in $\mathcal{D}_1(\lambda)$:

$$\mathcal{D}_1(\lambda)_{nonres} = \{\vec{\varkappa} \in \mathcal{D}_1(\lambda) : \mathcal{K}_2\vec{\varkappa} \in \chi_2\}. \tag{3.54}$$

Lemma 3.40. *The formula $\mathcal{K}_2\mathcal{D}_1(\lambda)_{nonres} = \chi_2$ establishes one-to-one correspondence between $\mathcal{D}_1(\lambda)_{nonres}$ and χ_2.*

Proof. Suppose there is a pair $\vec{\varkappa}_1, \vec{\varkappa}_2 \in \mathcal{D}_1(\lambda)_{nonres}$ such that $\mathcal{K}_2\vec{\varkappa}_1 = \mathcal{K}_2\vec{\varkappa}_2 = \tau$, $\tau \in \chi_2$. We introduce also $t_1 = \mathcal{K}_1\vec{\varkappa}_1$ and $t_2 = \mathcal{K}_1\vec{\varkappa}_2$. The definition (2.48) of $\chi_1^*(\lambda)$ implies that $t_1, t_2 \in \chi_1^*(\lambda)$, since $\vec{\varkappa}_1, \vec{\varkappa}_2 \in \mathcal{D}_1(\lambda)_{nonres} \subset \mathcal{D}_1(\lambda)$. Clearly, $\mathcal{K}_2 t_1 = \mathcal{K}_2 t_2 = \tau$ and, hence, $t_1 = \tau + 2\pi p_1/N_1 a$, $t_2 = \tau + 2\pi p_2/N_1 a$ for some $p_1, p_2 \in P$. Now, by Part 1a of Geometric Lemma 3.5, $p_1 = p_2$, and, therefore, $t_1 = t_2$. Next, by Lemma 2.13, $\vec{\varkappa}_1 = \vec{\varkappa}_2$. ∎

We define $\mathcal{B}_2(\lambda)$ as the set of directions corresponding to $\mathcal{D}_1(\lambda)_{nonres}$:

$$\mathcal{B}_2(\lambda) = \{\vec{\nu} \in \mathcal{B}_1(\lambda) : \varkappa_1(\lambda,\vec{\nu})\vec{\nu} \in \mathcal{D}_1(\lambda)_{nonres}\},$$

where $\varkappa_1(\lambda,\vec{\nu})$ is defined by Lemma 2.11, $\varkappa_1(\lambda,\vec{\nu}) \equiv \varkappa_1(\varphi)$. Note that $\mathcal{B}_2(\lambda)$ is a unit circle with holes, centered at the origin, and $\mathcal{B}_2(\lambda) \subset \mathcal{B}_1(\lambda)$. We denote by $\Theta_2(\lambda)$ the set of angles φ, corresponding to $\mathcal{B}_2(\lambda)$:

$$\Theta_2(\lambda) = \{\varphi \in [0,2\pi) :\ (\cos\varphi, \sin\varphi) \in \mathcal{B}_2(\lambda)\}, \qquad \Theta_2 \subset \Theta_1.$$

Let $\vec{\varkappa} \in \mathcal{D}_1(\lambda)_{nonres}$. By (3.54), $\tau \equiv \mathcal{K}_2\vec{\varkappa} \in \chi_2(\lambda)$. According to Theorem 3.8, for sufficiently large λ, there exists an eigenvalue of the operator $H_\alpha^{(2)}(\tau)$, given by (3.15). It is convenient here to denote $\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)$ by $\lambda^{(2)}(\alpha,\vec{\varkappa})$. We can do this, since, by Lemma 3.40, there is one-to-one correspondence between $\vec{\varkappa} \in \mathcal{D}_1(\lambda)_{nonres}$ and the pair $(\tau,\tilde{j})$, $\vec{\varkappa} = 2\pi\tilde{j}/a + \tau$. We rewrite (3.15) in the form:

$$\lambda^{(2)}(\alpha,\vec{\varkappa}) = \lambda^{(1)}(\vec{\varkappa}) + f_2(\alpha,\vec{\varkappa}), \quad f_2(\alpha,\vec{\varkappa}) = \sum_{r=1}^{\infty} \alpha^r g_r^{(2)}(\vec{\varkappa}), \tag{3.55}$$

here $g_r^{(2)}(\vec{\varkappa})$ is given by (3.13). The function $f_2(\alpha,\vec{\varkappa})$ satisfies the estimates:

$$|f_2(\alpha,\vec{\varkappa})| \le 12\alpha\epsilon_1^4, \tag{3.56}$$

$$|\nabla f_2(\alpha,\vec{\varkappa})| \le 24\alpha\epsilon_1^3 k^{2l-1+\delta}. \tag{3.57}$$

By Theorems 3.8 and 3.12, the formulas (3.55) – (3.57) hold even in $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood of $\mathcal{D}_1(\lambda)_{nonres}$, i.e., for any $\vec{\varkappa} = \varkappa\vec{\nu}$ such that $\vec{\nu} \in \mathcal{B}_2(\lambda)$ and $|\varkappa - \varkappa_1(\lambda,\vec{\nu})| < \epsilon_1 k^{-2l+1-\delta}$. We define $\mathcal{D}_2(\lambda)$ as a level set for $\lambda^{(2)}(\alpha,\vec{\varkappa})$ in this neighborhood:

$$\mathcal{D}_2(\lambda) := \left\{\vec{\varkappa} = \varkappa\vec{\nu} : \vec{\nu} \in \mathcal{B}_2,\ \ \left|\varkappa - \varkappa_1(\lambda,\vec{\nu})\right| < \epsilon_1 k^{-2l+1-\delta},\ \lambda^{(2)}(\alpha,\vec{\varkappa}) = \lambda\right\}. \tag{3.58}$$

Next two lemmas are to prove that $\mathcal{D}_2(\lambda)$ is a distorted circle with holes.

Lemma 3.41. *For every $\vec{\nu} \in \mathcal{B}_2$ and every α, $0 \le \alpha \le 1$, there is a unique $\varkappa = \varkappa_2(\lambda,\vec{\nu})$ in the interval $I_2 := \left[\varkappa_1(\lambda,\vec{\nu}) - \epsilon_1 k^{-2l+1-\delta}, \varkappa_1(\lambda,\vec{\nu}) + \epsilon_1 k^{-2l+1-\delta}\right]$ such that*

$$\lambda^{(2)}(\alpha,\varkappa_2\vec{\nu}) = \lambda. \tag{3.59}$$

Furthermore,

$$|\varkappa_2(\lambda,\vec{\nu}) - \varkappa_1(\lambda,\vec{\nu})| \le 2\alpha\epsilon_1^4 k^{-2l+1}. \tag{3.60}$$

Proof. Let us show that there exists a $\varkappa_2(\lambda,\vec{\nu})$ satisfying (3.59). Since $\varkappa_1\vec{\nu} \in \mathcal{D}_1(\lambda)$ when $\vec{\nu} \in \mathcal{B}_2(\lambda)$, for all $\varkappa$ in the interval I_2, there is an eigenvalue $\lambda^{(2)}(\alpha,\varkappa\vec{\nu})$ given by (3.55). In particular, $\lambda^{(2)}\big(0,\varkappa_1\vec{\nu}\big) = \lambda^{(1)}\big(\varkappa_1\vec{\nu}\big) = \lambda$. Let $\mathcal{L}^{(2)}(\alpha,\vec{\nu}) := \{\lambda = \lambda^{(2)}(\alpha,\varkappa\vec{\nu}) : \varkappa \in I_2\}$. It is an interval, since $\lambda^{(2)}(\alpha,\varkappa\vec{\nu})$ is a continuous function of $\varkappa$. Considering that $\lambda^{(1)}\big(\varkappa\vec{\nu}\big)$ is a continuous function and $\nabla\lambda^{(1)}\big(\varkappa\vec{\nu}\big) = 2l\varkappa^{2l-1}\vec{\nu}\big(1+o(1)\big)$, we readily obtain: $\mathcal{L}^{(2)}(0,\vec{\nu}) \supset [k^{2l} - c_1\epsilon_1 k^{-\delta}, k^{2l} + c_1\epsilon_1 k^{-\delta}]$, $0 < c_1 \ne c_1(k)$. Since $\lambda^{(2)}(\alpha,\varkappa\vec{\nu})$ is continuous in $\varkappa$ and (3.56) holds, we have $\mathcal{L}^{(2)}(\alpha,\vec{\nu}) \supset [k^{2l} - c_1\epsilon_1 k^{-\delta} + c_2\epsilon_1^4, k^{2l} + c_1\epsilon_1 k^{-\delta} - c_2\epsilon_1^4]$, $0 < c_2 \ne c_2(k)$. The inequality $\epsilon_1^4 \ll \epsilon_1 k^{-\delta}$ yields: $\Lambda^{(2)}(\alpha,\vec{\nu}) \supset [k^{2l} - c_1\epsilon_1 k^{-\delta}/2, k^{2l} + c_1\epsilon_1 k^{-\delta}/2]$. Thus for every $\vec{\nu} \in \mathcal{B}_2(\lambda)$, there exists a $\varkappa_2(\lambda,\vec{\nu}) \in I_2$ such that $\lambda^{(2)}(\alpha,\varkappa_2\vec{\nu}) = \lambda$.

Using (3.26), we obtain

$$\frac{\partial\lambda^{(2)}(\alpha,\varkappa\vec{\nu})}{\partial\varkappa} = 2lk^{2l-1}\big(1+o(1)\big), \tag{3.61}$$

and this implies that $\lambda^{(2)}(\alpha,\varkappa\vec{\nu})$ is monotone with respect to $\varkappa$ in I_2. Hence, there is only one $\varkappa_2(\lambda,\vec{\nu})$ satisfying (3.59) in I_2.

Now let us estimate $|\varkappa_2(\lambda,\vec{\nu}) - \varkappa_1(\lambda,\vec{\nu})|$. By (3.55), we have $\lambda^{(2)}(\alpha,\varkappa_1\vec{\nu}) = \lambda^{(1)}\big(\varkappa_1\vec{\nu}\big) + f_2\big(\alpha,\varkappa_1\vec{\nu}\big) = \lambda + f_2\big(\alpha,\varkappa_1\vec{\nu}\big)$. Using (3.56) and (3.59), we easily obtain: $\big|\lambda^{(2)}\big(\alpha,\varkappa_2\vec{\nu}\big) - \lambda^{(2)}\big(\alpha,\varkappa_1\vec{\nu}\big)\big| < 12\alpha\epsilon_1^4$. Applying the Mean Value Theorem at the left-hand side and using (3.61), we arrive at the inequality: $lk^{2l-1}|\varkappa_2 - \varkappa_1| < 12\alpha\epsilon_1^4$. The estimate (3.60) immediately follows. ∎

Let us recall that $\mathit{\Phi}_2$ is the second complex non-resonant set given by (3.40), see Fig. 7. Further, we use the notations: $\varkappa_2(\varphi) \equiv \varkappa_2(\lambda,\vec{\nu})$, $h_2(\varphi) = \varkappa_2(\varphi) - \varkappa_1(\varphi)$, $\vec{\varkappa}_2(\varphi) = \varkappa_2(\varphi)\vec{\nu}$.

Lemma 3.42. (1) *The set $\mathcal{D}_2(\lambda)$ is a distorted circle with holes: it can be described by the formula:*

$$\mathcal{D}_2(\lambda) = \left\{\vec{\varkappa} \in \mathbb{R}^2 : \vec{\varkappa} = \vec{\varkappa}_2(\varphi),\ \ \varphi \in \Theta_2(\lambda)\right\}, \tag{3.62}$$

where $\varkappa_2(\varphi) = \varkappa_1(\varphi) + h_2(\varphi)$, $\varkappa_1(\varphi)$ *is the "radius" of* $\mathcal{D}_1(\lambda)$ *and* $h_2(\varphi)$ *satisfies the estimates*

$$|h_2| \le 2\alpha\epsilon_1^4 k^{-2l+1}, \quad \left|\frac{\partial h_2}{\partial \varphi}\right| \le 4\alpha\epsilon_1^3 k^{1+\delta}. \tag{3.63}$$

(2) *The total length of* $\mathcal{B}_2(\lambda)$ *satisfies the estimate:*

$$L\left(\mathcal{B}_1 \setminus \mathcal{B}_2\right) < 4\pi k^{-2-2s_1}. \tag{3.64}$$

(3) *The function* $\varkappa_2(\varphi)$ *can be extended as a holomorphic function of* φ *into the complex non-resonance set* Φ_2*, estimates (3.63) being preserved.*

(4) *The curve* $\mathcal{D}_2(\lambda)$ *has a length which is asymptotically close to that of* $\mathcal{D}_1(\lambda)$ *in the following sense:*

$$L\Big(\mathcal{D}_2(\lambda)\Big) \underset{\lambda\to\infty}{=} L\Big(\mathcal{D}_1(\lambda)\Big)\Big(1 + O\big(k^{-2-2s_1}\big)\Big). \tag{3.65}$$

Proof. (1) By Lemma 3.41, $|h_2| = |\varkappa_2 - \varkappa_1| \le 2\alpha\epsilon_1^4 k^{-2l+1}$. Let us prove the second estimate in (3.63). Considering that $\lambda = \lambda^{(2)}\left(\alpha, \vec{\varkappa}_2(\varphi)\right)$, $\lambda = \lambda^{(1)}\left(\vec{\varkappa}_1(\varphi)\right)$, differentiating both equations with respect to φ and subtracting the results, we obtain:

$$\frac{\partial\lambda^{(2)}}{\partial\varkappa}\left(\alpha, \vec{\varkappa}_2\right)\frac{\partial h_2}{\partial\varphi} = F(\lambda,\varphi), \tag{3.66}$$

$$F(\lambda,\varphi) = \tag{3.67}$$

$$\Big(\nabla\lambda^{(1)}\left(\vec{\varkappa}_1\right) - \nabla\lambda^{(2)}\left(\alpha, \vec{\varkappa}_2\right)\Big)\frac{\partial\varkappa_1}{\partial\varphi} + \Big\langle \varkappa_1\nabla\lambda^{(1)}\left(\vec{\varkappa}_1\right) - \varkappa_2\nabla\lambda^{(2)}\left(\alpha, \vec{\varkappa}_2\right), \vec{\mu}\Big\rangle,$$

where $\vec{\mu} = (-\sin\varphi, \cos\varphi)$. It is easy to check (Appendix A.3) that

$$\big|F(\lambda,\varphi)\big| < 27\alpha k^{2l+\delta}\epsilon_1^3. \tag{3.68}$$

Then, the second estimate in (3.63) easily follows from formulae (3.66) and estimates (3.61) and (3.68).

(2) Considering that Θ_1 is the set of angles corresponding to $\mathcal{D}_1(\lambda)$ (Lemma 2.11) and Θ_2 is the set of angles corresponding to $\mathcal{D}_1(\lambda)_{nonres}$, we obtain

$$L\Big(\mathcal{D}_1(\lambda)\setminus\mathcal{D}_1(\lambda)_{nonres}\Big) = \int_{\Theta_1\setminus\Theta_2}\sqrt{\varkappa_1^2(\varphi) + \varkappa_1'(\varphi)^{\,2}}\,d\varphi. \tag{3.69}$$

Using (2.43), we easily show that the integrand is close to k. Hence,

$$L\Big(\mathcal{D}_1(\lambda)\setminus\mathcal{D}_1(\lambda)_{nonres}\Big) = L\left(\Theta_1\setminus\Theta_2\right)k\big(1+o(1)\big). \tag{3.70}$$

Let us show that

$$L\Big(\mathcal{D}_1(\lambda)\setminus\mathcal{D}_1(\lambda)_{nonres}\Big) < 2\pi k^{-1-2s_1}\big(1+o(1)\big). \tag{3.71}$$

In fact, by Lemma 3.40, $L\big(\mathcal{D}_1(\lambda)_{nonres}\big) = L(\chi_2)$. By Lemma 2.13, $L\big(\mathcal{D}_1(\lambda)\big) = L(\chi_1^*)$. By Corollary 3.37, $L\big(\chi_1^*\big) = L\big(\mathcal{K}_2\chi_1^*\big)$, and, therefore, $L\big(\mathcal{D}_1(\lambda)\big) = L(\mathcal{K}_2\chi_1^*)$. It follows: $L\big(\mathcal{D}_1(\lambda)\setminus\mathcal{D}_1(\lambda)_{nonres}\big) = L\big(\mathcal{D}_1(\lambda)\big) - L\big(\mathcal{D}_1(\lambda)_{nonres}\big) = L(\mathcal{K}_2\chi_1^*) - L(\chi_2) = L\big(\mathcal{K}_2\chi_1^*\setminus\chi_2\big)$, the last equality being valid since χ_2 is a subset of $\mathcal{K}_2\chi_1^*$. Using the estimate (3.10) in the Geometric Lemma, we obtain

$$L\Big(\mathcal{D}_1(\lambda)\setminus\mathcal{D}_1(\lambda)_{nonres}\Big) < L\Big(\mathcal{D}_1(\lambda)\Big)k^{-2-2s_1}. \tag{3.72}$$

Now considering that $L\big(\mathcal{D}_1(\lambda)\big) = 2\pi k\big(1+o(1)\big)$ (Lemma 2.11), we arrive at the estimate (3.71). Combining it with (3.70), we get (3.64).

(3) By the third statement of Lemma 2.11, the function $\varkappa_1(\varphi)$ is holomorphic in Φ_1 and $\lambda^{(1)}\big(\varkappa_1(\varphi)\big) = \lambda$. If $\varphi \in \Theta_2$, then $\lambda^{(2)}\big(\varkappa_1(\varphi)\big)$ is defined by the series (3.55), since $\varkappa_1(\varphi) \in \mathcal{D}_1(\lambda)_{nonres}$ for any $\varphi \in \Theta_2$. Now we show that $\lambda^{(2)}\big(\varkappa_1(\varphi)\big)$ can be analytically extended from Θ_2 to $\Phi_1 \setminus \mathcal{O}$. It is enough to check that the series (3.55) converges in $\Phi_1 \setminus \mathcal{O}$. We need an estimate for $g_r^{(2)}\big(\varkappa_1(\varphi)\big)$, which is similar to (3.17). Indeed, considering formula (3.13) for $g_r^{(2)}$ and taking into account that $\|W_2\| < \epsilon_1^4$, we easily conclude that it suffices to prove the estimate:

$$\sup_{z\in C_2}\left\|\left(\tilde{H}^{(1)}\big(\vec{\varkappa}_1(\varphi)\big) - z\right)^{-1}\right\| < \frac{1}{\epsilon_1}. \tag{3.73}$$

From the definition of $\tilde{H}^{(1)}$, we see:

$$\left\|\left(\tilde{H}^{(1)}\big(\vec{\varkappa}_1(\varphi)\big) - z\right)^{-1}\right\| = \max_{p\in P}\left\|\left(H^{(1)}\big(\vec{\varkappa}_1(\varphi) + 2\pi p/N_1 a\big) - z\right)^{-1}\right\|.$$

To estimate the right-hand side, we start with the case $p = 0$. We obtain an upper bound on $\|\big(H^{(1)}\big(\vec{\varkappa}_1(\varphi)\big) - z\big)^{-1}\|$ using results of Section 2. Indeed, $\mathcal{K}_1\vec{\varkappa}_1(\varphi)$ belongs to $\chi_1^*(\lambda)$, which is in the complex $(k^{-1-4s_1-2\delta})$-neighborhood of the first non-resonance set $\chi_1(\lambda)$. Hence, we can apply Lemma A.1 from Appendix A.4 with $t = \mathcal{K}_1\vec{\varkappa}_1(\varphi)$. By this lemma, $\big(H^{(1)}\big(\vec{\varkappa}_1(\varphi)\big) - z\big)^{-1}$ has a single pole z_0 inside the contour C_1. It is easy to see that $z_0 = \lambda$, since $\lambda^{(1)}\big(\vec{\varkappa}_1(\varphi)\big) = \lambda$. Again, by Lemma A.1,

$$\left\|(z - k^{2l})\left(H^{(1)}\big(\vec{\varkappa}_1(\varphi)\big) - z\right)^{-1}\right\| < 16 \tag{3.74}$$

for all z inside C_1. Considering $z \in C_2$, that is, $|z - k^{2l}| = \epsilon_1/2$, we easily obtain from (3.74) that

$$\left\|\left(H^{(1)}\big(\vec{\varkappa}_1(\varphi)\big) - z\right)^{-1}\right\| < \frac{32}{\epsilon_1}, \quad z \in C_2. \tag{3.75}$$

Next, we consider the case $p \in P \setminus \{0\}$. We use Lemma 3.31 with $\vec{b} = 2\pi p/N_1 a$, $p \neq 0$. It is possible to replace k^{2l} in (3.52) by any $z \in C_2$, since the difference between z and k^{2l} is small: $|z - k^{2l}| = \epsilon_1/2$, $\epsilon_1 = o\left(b_0 k^{2l-9-12s_1-6\delta}\right)$. Thus,

$$\left\|\left(H^{(1)}\big(\vec{\varkappa}_1(\varphi) + 2\pi p/N_1 a\big) - z\right)^{-1}\right\| < \frac{2}{\epsilon_1} \tag{3.76}$$

for any $p \in P \setminus \{0\}$, $z \in C_2$. Combining (3.75) with the last estimate, we arrive at (3.73), which means that the series for $\lambda^{(2)}\big(\alpha, \vec{\varkappa}_1(\varphi)\big)$ converges. It is easy to see that estimates (3.75), (3.76) are stable in a vicinity of $\vec{\varkappa}_1(\varphi)$, namely, they hold at all points $\varkappa\vec{\nu}$: $|\varkappa - \varkappa_1(\varphi)| < \epsilon_1 k^{-2l+1-\delta}$. Hence, the perturbation series for the eigenvalue converges for such $\varkappa$ too, the following estimates being valid:

$$\left|\lambda^{(2)}(\alpha, \varkappa\vec{\nu}) - \lambda^{(1)}(\varkappa\vec{\nu})\right| < 12\alpha\epsilon_1^4, \tag{3.77}$$

$$\nabla\lambda^{(2)}(\alpha, \varkappa\vec{\nu}) = 2lk^{2l-2}\vec{k} + o(k^{2l-1}). \tag{3.78}$$

Next, we want to show that the equation $\lambda^{(2)}(\alpha, \varkappa\vec{\nu}) = \lambda$ has a solution $\varkappa_2(\varphi)$[9] for every $\varphi \in \Phi_1 \setminus \mathcal{O}$. It easily follows from (2.34) – (2.36) that

$$\frac{\partial \lambda^{(1)}(\varkappa\vec{\nu})}{\partial \varkappa} = 2lk^{2l-1}(1+o(1)), \quad \frac{\partial^2 \lambda^{(1)}(\varkappa\vec{\nu})}{\partial \varkappa^2} = O(k^{2l-2}) \tag{3.79}$$

for any $\varphi \in \Phi_1$ and $\varkappa : |\varkappa - k| < k^{-1-4s_1-2\delta}$. Using these estimates we readily obtain that $|\lambda^{(1)}(\varkappa\vec{\nu}) - \lambda| > k^{2l-1}\epsilon_1^4$ for all $\varkappa$ on the circle $|\varkappa - \varkappa_1(\varphi)| = \epsilon_1^4$. Considering (3.77) and applying Rouché's theorem, we obtain that there is a single solution $\varkappa_2(\varphi)$ of the equation $\lambda^{(2)}(\alpha, \varkappa\vec{\nu}) = \lambda$ such that $|\varkappa_2(\varphi) - \varkappa_1(\varphi)| \leq \epsilon_1^4$. Applying (3.78) and Implicit function theorem, we obtain that $\varkappa_2(\varphi)$ is a holomorphic function of φ in $\Phi_1 \setminus \mathcal{O}$ and estimates (3.63) hold.

Let us make here a remark for the future. Convergence of the series for the resolvent $\left(H^{(2)}\left(\vec{\varkappa}_1(\varphi)\right) - z\right)^{-1}$, $z \in C_2$, following from (3.73), means that the resolvent has a single pole $z = \lambda^{(2)}(\vec{\varkappa}_1(\varphi)\vec{\nu})$ inside C_2. Similar result holds when we replace $\vec{\varkappa}_1(\varphi)$ by $\vec{\varkappa}_2(\varphi)$, since $\vec{\varkappa}_1(\varphi)$ and $\vec{\varkappa}_2(\varphi)$ are close: $\left|\vec{\varkappa}_2(\varphi) - \vec{\varkappa}_1(\varphi)\right| = o(\epsilon_1)$. Considering that $\lambda^{(2)}(\vec{\varkappa}_2(\varphi)) = \lambda$, we obtain that $(z-\lambda)\left(H^{(2)}\left(\vec{\varkappa}_2(\varphi)\right) - z\right)^{-1}$ is holomorphic inside C_2 and the estimate similar to (3.74) holds:

$$\left\|(z-\lambda)\left(H^{(2)}\left(\vec{\varkappa}_2(\varphi)\right) - z\right)^{-1}\right\| < 32. \tag{3.80}$$

(4) First, we show that

$$L\big(\mathcal{D}_2(\lambda)\big) \underset{\lambda\to\infty}{=} L\big(\mathcal{D}_1(\lambda)_{nonres}\big)\Big(1 + O(k^{-2-2s_1})\Big). \tag{3.81}$$

Second, we check that

$$L\big(\mathcal{D}_1(\lambda)_{nonres}\big) \underset{\lambda\to\infty}{=} L\big(\mathcal{D}_1(\lambda)\big)\Big(1 + O(k^{-2-2s_1})\Big). \tag{3.82}$$

From these two estimates (3.65) follows. In fact, considering that

$$L\big(\mathcal{D}_2(\lambda)\big) = \int_{\Theta_2} \sqrt{\varkappa_2^2 + \varkappa_2'^{\,2}}\, d\varphi, \quad L\big(\mathcal{D}_1(\lambda)_{nonres}\big) = \int_{\Theta_2} \sqrt{\varkappa_1^2 + \varkappa_1'^{\,2}}\, d\varphi,$$

and taking into account that $h_2(\varphi) = \varkappa_2(\varphi) - \varkappa_1(\varphi)$ satisfies (3.63), we easily get (3.81). Formula (3.82) easily follows from (3.72).

∎

Now define the non-resonance set $\chi_2^*(\lambda)$ in $S_2(\lambda)$ by

$$\chi_2^*(\lambda) := \mathcal{K}_2\mathcal{D}_2(\lambda). \tag{3.83}$$

Lemma 3.43. *The set $\chi_2^*(\lambda)$ belongs to the $\left(2\alpha\epsilon_1^4 k^{-2l+1}\right)$-neighborhood of $\chi_2(\lambda)$ in K_2. If $\tau \in \chi_2^*(\lambda)$, then the operator $H_\alpha^{(2)}(\tau)$ has a simple eigenvalue equal to λ. This eigenvalue is given by the perturbation series (3.15), where $p \in P, j \in \mathbb{Z}^2$ are described as in Geometric Lemma 3.5, part 2.*

[9] In fact, it should be $\varkappa_2(\alpha, \varphi)$; we omit α for shortness.

Proof. By Lemma 3.42, $\mathcal{D}_2(\lambda)$ is in the $\left(2\alpha\epsilon_1^4 k^{-2l+1}\right)$ neighborhood of $\mathcal{D}_1(\lambda)_{nonres}$. Considering that $\chi_2^*(\lambda)=\mathcal{K}_2\mathcal{D}_2(\lambda)$ and $\chi_2(\lambda)=\mathcal{K}_2\mathcal{D}_1(\lambda)_{nonres}$ (see (3.54)), we immediately obtain that $\chi_2^*(\lambda)$ is in the $(2\alpha\epsilon_1^4 k^{-2l+1})$-neighborhood of $\chi_2(\lambda)$. The size of this neighborhood is less than $\epsilon_1 k^{-2l+1-\delta}$, hence Theorem 3.1 holds in it, i.e., for any $\tau\in\chi_2^*(\lambda)$ there is a single eigenvalue of $H_\alpha^{(2)}(\tau)$ in the interval $\varepsilon_2(k,\delta)$. Since $\chi_2^*(\lambda)\subset S_2(\lambda)$, this eigenvalue is equal to λ. By the theorem, the eigenvalue is given by the series (3.15), where $p\in P, j\in\mathbb{Z}^2$ are described as in Geometric Lemma 3.5, part 2. ∎

Lemma 3.44. *Formula (3.83) establishes one-to-one correspondence between $\chi_2^*(\lambda)$ and $\mathcal{D}_2(\lambda)$.*

Remark 3.45. From geometric point of view this means that $\chi_2^*(\lambda)$ does not have self-intersections.

Proof. Suppose there is a pair $\vec{\varkappa},\vec{\varkappa}^*\in\mathcal{D}_2(\lambda)$ such that $\mathcal{K}_2\vec{\varkappa}=\mathcal{K}_2\vec{\varkappa}^*=\tau$, $\tau\in\chi_2^*(\lambda)$. By the definition (3.58) of $\mathcal{D}_2(\lambda)$, we have $\lambda^{(2)}(\alpha,\vec{\varkappa})=\lambda^{(2)}(\alpha,\vec{\varkappa}^*)=\lambda$, i.e., the eigenvalue λ of $H_\alpha^{(2)}(\tau)$ is not simple. This contradicts to the previous lemma. ∎

3.5. **Preparation for the Next Approximation.**

3.5.1. *Contracted set $\mathcal{O}_s(\vec{b})$.* Let us recall that $\mathcal{O}(\vec{b})$ is the union of disks of the radius $r=k^{-4-6s_1-3\delta}$ centered at the zeros of the unperturbed determinant $\det\bigl(I+A_0(\vec{y}_0(\varphi))\bigr)$, see Definition 3.24. We proved that the "perturbed" determinant $\det\bigl(I+A_1(\vec{y}(\varphi)\bigr)$ has the same number of zeros inside each $\Gamma(\vec{b})$ as the unperturbed one, when b_0 is big enough. To prepare the next approximation, we contract the set $\Gamma(\vec{b})$ around zeros of the perturbed determinant. First, we consider the disks with the radius $r'=rk^{-2-4s_2-\delta}$ centered at each zero $\varphi_{j,0}$ of the perturbed determinant $\det\bigl(I+A_1(\vec{y}(\varphi)\bigr)$ in the set $\mathcal{O}(\vec{b})$. Obviously each of these disks is in $\mathcal{O}(\vec{b})$, since $r'<<r$ and the distance between zeros of the perturbed and unperturbed determinants is smaller than $r/2$ (Lemma 3.35, Part 2). Next, we take the union of these smaller disks and denote it by $\mathcal{O}_s(\vec{b})$, here the index s stands for "small". Clearly, $\mathcal{O}_s(\vec{b})\subset\mathcal{O}(\vec{b})$. We denote by $\Gamma_s(\vec{b})$ a connected component of these new disks which intersects with Φ_1 and by $\gamma_s(\vec{b})$ its boundary, $\gamma_s=\partial\Gamma_s$. Let us remind that in all our previous considerations $\vec{b}$ satisfies the conditons $b_0>k^{-2l+9+12s_1+7\delta}$.[10]

Lemma 3.46. (1) *The set $\mathcal{O}_s(\vec{b})$ contains the same number of disks as $\mathcal{O}(\vec{b})$. This number does not exceed $J=c_0k^{2+2s_1}$.*
(2) *For any $\varphi\in\Phi_1\setminus\mathcal{O}_s(\vec{b})$,*

$$\left\|\Bigl(H^{(1)}(\vec{y}(\varphi))-k^{2l}\Bigr)^{-1}\right\|<cb_0^{-1}k^{-2l+9+12s_1+6\delta}(2rJ/r')^J,\tag{3.84}$$

$$\left\|\Bigl(H^{(1)}(\vec{y}(\varphi))-k^{2l}\Bigr)^{-1}\right\|_1<cb_0^{-1}k^{-2l+11+14s_1+6\delta}(2rJ/r')^J.\tag{3.85}$$

Corollary 3.47. *If $b_0>k^{-2l+9+12s_1+7\delta}$,*

$$\left\|\Bigl(H^{(1)}(\vec{y}(\varphi))-k^{2l}\Bigr)^{-1}\right\|<\frac{1}{\epsilon_1},\tag{3.86}$$

[10]Indeed, we had two restrictions on b_0: $b_0>k^{7+8s_1-2\gamma_0+6\delta}$ (Lemma 3.29) and $b_0>k^{-2l+9+12s_1+7\delta}$ (Lemmas 3.31–3.38). Recalling that $\gamma_0=2l-2-4s_1-2\delta$, we get $7+8s_1-2\gamma_0+6\delta<-2l+9+12s_1+7\delta$, we can combine two restrictions into $b_0>k^{-2l+9+12s_1+7\delta}$.

$$\left\|\left(H^{(1)}(\vec{y}(\varphi))-k^{2l}\right)^{-1}\right\|_1<\frac{c_0k^{2+2s_1}}{\epsilon_1}. \tag{3.87}$$

Corollary 3.47. We use the condition on η: $\eta>2+64/(2l-11)=2+2/s_1$ and the formulae for $\epsilon_1=e^{-\frac{1}{4}k^{\eta s_1}}$, $r/r'=k^{2+4s_2+\delta}$. Simple computation shows that the right-hand side in (3.84) does not exceed $1/\epsilon_1$. Hence, the corollary holds. ∎

Lemma 3.46. By construction, the set $\mathcal{O}_s(\vec{b})$ contains the same number of disks as $\mathcal{O}(\vec{b})$. Since the number of disks in $\mathcal{O}(\vec{b})$ does not exceed $J=c_0k^{2+2s_1}$ (see the remark after Definition 3.24), the same is true for $\mathcal{O}_s(\vec{b})$.

If $\varphi\in\Phi_1\setminus\mathcal{O}(\vec{b})$, then the estimates just follow from Lemma 3.31. Thus, we have to consider $\mathcal{O}\setminus\mathcal{O}_s$. Note that $(H^{(1)}(\vec{y}(\varphi))-k^{2l})^{-1}\det\left(\frac{H^{(1)}(\vec{y}(\varphi))-k^{2l}}{H_0(\vec{y}(\varphi))+k^{2l}}\right)$ is holomorphic in every connected component $\Gamma(\vec{b})$ of $\mathcal{O}(\vec{b})$. We denote zeros of $\det\left(\frac{H^{(1)}(\vec{y}(\varphi))-k^{2l}}{H_0(\vec{y}(\varphi))+k^{2l}}\right)$ in Γ by $\varphi_{j,0},\ j=1,\cdots,J^*$, $J^*\leq J$. Obviously, $(H^{(1)}(\vec{y}(\varphi))-k^{2l})^{-1}\prod_{j=1}^{J^*}(\varphi-\varphi_{j,0})$ is holomorphic in Γ. For any φ on γ,

$$\left\|(H^{(1)}(\vec{y}(\varphi))-k^{2l})^{-1}\prod_{j=1}^{J^*}(\varphi-\varphi_{j,0})\right\|\leq\left\|(H^{(1)}(\vec{y}(\varphi))-k^{2l})^{-1}\right\|\prod_{j=1}^{J^*}|\varphi-\varphi_{j,0}|.$$

Considering that $|\varphi-\varphi_{j,0}|\leq 2rJ^*$ for any $\varphi\in\gamma$ and the estimate (3.52), we get

$$\|(H^{(1)}(\vec{y}(\varphi))-k^{2l})^{-1}\prod_{j=1}^{J^*}(\varphi-\varphi_{j,0})\|<cb_0^{-1}k^{-2l+9+12s_1+6\delta}(2rJ^*)^{J^*}. \tag{3.88}$$

By the maximum principle the inequality (3.88) holds for any $\varphi\in\Gamma$. Note that any $\varphi\in\Gamma\setminus\Gamma_s$ satisfies the inequality $|\varphi-\varphi_{j,0}|\geq r'$ and, hence,

$$\left\|\left(H^{(1)}(\vec{y}(\varphi))-k^{2l}\right)^{-1}\right\|<cb_0^{-1}k^{-2l+9+12s_1+6\delta}(2rJ^*/r')^{J^*}$$

holds for $\varphi\in\Gamma\setminus\Gamma_s$. Considering that $r/r'>1$ and $J^*\leq J$, we obtain (3.84). To prove (3.85), we use the following Hilbert relation:

$$\left(H^{(1)}(\vec{y}(\varphi))-k^{2l}\right)^{-1}=\\ \left(H_0(\vec{y}(\varphi))+k^{2l}\right)^{-1}+\left(H_0(\vec{y}(\varphi))+k^{2l}\right)^{-1}(-W_1+2k^{2l})\left(H^{(1)}(\vec{y}(\varphi))-k^{2l}\right)^{-1}. \tag{3.89}$$

Summarizing the terms of a diagonal operator, we easily get

$$\left\|\left(H_0(\vec{y}(\varphi))+k^{2l}\right)^{-1}\right\|_1<ck^{-2l+2+2s_1}. \tag{3.90}$$

Substituting (3.84) and the last estimate into (3.89), we obtain (3.85). ∎

3.5.2. *The set $\mathcal{O}_s(\vec{b})$ for small b_0.* Everything we considered so far is valid for $\vec{b}$ obeying the inequality $b_0 > k^{-2l+9+12s_1+7\delta}$, here b_0 is the distance from $\vec{b}$ to the nearest vertex of K_1. We showed in Section 3.3.5 that $\vec{b} = 2\pi p/N_1 a$, $p \in P \setminus \{0\}$, satisfies the requirement. However, in the next section and later, b_0 will be taken smaller, since the reciprocal lattice is getting finer with each step. To prepare for this, let us consider $\vec{b}$ being close to a vertex of K_1:

$$0 < b_0 \leq k^{-2l+9+12s_1+7\delta}. \tag{3.91}$$

We show that for such $\vec{b}$ the resolvent $\big(H^{(1)}(\vec{y}(\varphi)) - k^{2l}\big)^{-1}$ has no more than two poles $\varphi^{\pm}$ in Φ_1 and even its small neighborhood. We surround these poles by two contours $\gamma^{\pm}$ and obtain estimates for $\big(H^{(1)}(\vec{y}(\varphi)) - k^{2l}\big)^{-1}$ when φ is outside $\gamma^{\pm}$.

In fact, it easily follows from (3.91) that: $b_0 = o(k^{-1-4s_1-2\delta})$ as $k \to \infty$. Let $\tilde{\Phi}_1$ be the $(\frac{1}{2}k^{-2-4s_1-2\delta})$ neighborhood of Φ_1. Suppose $|\vec{b}| = b_0$, i.e., the closest vertex of K_1 for $\vec{b}$ is $(0,0)$. The perturbation series (2.34) converge for both $\lambda^{(1)}(\vec{\varkappa}_1(\varphi))$ and $\lambda^{(1)}(\vec{y}(\varphi))$ when $\varphi \in \tilde{\Phi}_1$, and both functions are holomorphic in $\tilde{\Phi}_1$, because $\vec{\varkappa}_1(\varphi)$ and $\vec{y}(\varphi)$ are in the complex $(2k^{-1-4s_1-2\delta})$-neighborhood of $\mathcal{D}_0(\lambda)_{nonres}$ for such φ (Lemma 2.11). Note that $\lambda^{(1)}(\vec{\varkappa}_1(\varphi)) = k^{2l}$ for all $\varphi \in \tilde{\Phi}_1$. We base our further considerations on these perturbation series expansions. For $\vec{b}$ being close to a vertex $\vec{e}$ other than $(0,0)$, we take $\vec{y}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b} - \vec{e}$.

We define $\varphi_b \in [0, 2\pi)$ by the formula $\vec{b} = b_0(\cos\varphi_b, \sin\varphi_b)$ when $|\vec{b}| = b_0$, and by the analogous formula $\vec{b} - \vec{e} = b_0(\cos\varphi_b, \sin\varphi_b)$ when $\vec{b}$ is close to a vertex $\vec{e}$ other than $(0,0)$.

Lemma 3.48. *If $\vec{b}$ satisfies (3.91) and $|\epsilon_0| < b_0 k^{2l-3-4s_1-3\delta}$, then the equation*

$$\lambda^{(1)}(\vec{y}(\varphi)) = k^{2l} + \epsilon_0 \tag{3.92}$$

has no more than two solutions, $\varphi^{\pm}_{\epsilon_0}$, in $\tilde{\Phi}_1$. They satisfy the inequality

$$\big|\varphi^{\pm}_{\epsilon_0} - (\varphi_b \pm \pi/2)\big| < \frac{1}{8}k^{-2-4s_1-2\delta}. \tag{3.93}$$

Proof. Suppose $W_1 = 0$ and $|\vec{b}| = b_0$, i.e., the closest vertex of K_1 for $\vec{b}$ is $(0,0)$. Then the equation (3.92) has the form $|k\vec{\nu} + \vec{b}|^{2l} = k^{2l} + \epsilon_0$. It is easy to show that it has two solutions $\varphi^{\pm}_{\epsilon_0}$ satisfying (3.93). Applying perturbative arguments and Rouché's theorem, we prove the lemma for nonzero W_1. A detailed proof is in Appendix A.5. In the case when $\vec{b}$ is close to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. ∎

Lemma 3.49. *Suppose $\vec{b}$ satisfies (3.91) and $\varphi \in \tilde{\Phi}_1$ obeys the inequality analogous to (3.93): $|\varphi - (\varphi_b \pm \pi/2)| < k^{-2-4s_1-2\delta}$. Then,*

$$\frac{\partial}{\partial\varphi}\lambda^{(1)}(\vec{y}(\varphi)) =_{k\to\infty} \pm 2lb_0k^{2l-1}\big(1 + o(1)\big). \tag{3.94}$$

Proof. Let $W_1 = 0$ and $|\vec{b}| = b_0$. Then $\lambda^{(1)}(\vec{y}(\varphi)) = |k\vec{\nu} + \vec{b}|^{2l}$ and

$$\frac{\partial}{\partial\varphi}|k\vec{\nu} + \vec{b}|^{2l} = 2l|k\vec{\nu} + \vec{b}|^{2l-2}\langle k\vec{\nu} + \vec{b}, k\vec{\mu}\rangle, \quad \vec{\mu} = (-\sin\varphi, \cos\varphi).$$

For φ close to $\varphi_b \pm \pi/2$, we have $\langle \vec{b}, \vec{\mu}\rangle = \pm b_0(1 + o(1))$. Considering also that $\langle \vec{\mu}, \vec{\nu}\rangle = 0$, we obtain $\frac{\partial}{\partial\varphi}|k\vec{\nu} + b|^{2l} = \pm 2lb_0k^{2l-1}(1 + o(1))$. Applying perturbative arguments, we get a similar formula for nonzero W_1. For a detailed proof see Appendix A.6. In the case when $\vec{b}$ is close to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. ∎

Definition 3.50. Let $\Gamma_s^{\pm}(\vec{b})$ be open disks centered at $\varphi_0^{\pm} \in \tilde{\Phi}_1$ [11] with the radius r' introduced in the previous subsection, $r' = rk^{-2-4s_2-\delta}$; $\gamma_s^{\pm}(\vec{b})$ be their boundary circles and $\mathcal{O}_s(\vec{b}) = \Gamma_s^+ \cup \Gamma_s^-$.

Lemma 3.51. *For any φ in $\Phi_1 \setminus \mathcal{O}_s(\vec{b})$,*

$$|\lambda^{(1)}(\vec{y}(\varphi)) - k^{2l}| \geq b_0 k^{2l-1-\delta} r'. \tag{3.95}$$

Proof. Suppose (3.95) does not hold for some φ in $\Phi_1 \setminus \mathcal{O}_s(\vec{b})$. This means that φ satisfies equation (3.92) with some ε_0: $|\varepsilon_0| < b_0 k^{2l-1} r' < b_0 k^{2l-3-4s_1-3\delta}$. By Lemma 3.48, φ obeys (3.93). Thus φ could be either $\varphi_{\epsilon_0}^+$ or $\varphi_{\epsilon_0}^-$. Without loss of generocity, assume $\varphi = \varphi_{\epsilon_0}^+$. Applying (3.94) in the segment between φ and φ_0^+, we obtain: $\varepsilon_0 = \lambda^{(1)}(\vec{y}(\varphi)) - k^{2l} = \pm 2 l b_0 k^{2l-1}\big(1+o(1)\big)\left(\varphi - \varphi_0^+\right)$. It immediately follows that $\left|\varphi - \varphi_0^+\right| < r'$, i.e., $\varphi \in \Gamma_s^+ \subset O_s(\vec{b})$, which contradicts the assumption $\varphi \in \Phi_1 \setminus \mathcal{O}_s(\vec{b})$. ∎

Lemma 3.52. *For any $\varphi \in \Phi_1 \setminus \mathcal{O}_s(\vec{b})$,*

$$\left\| \left(H^{(1)}(\vec{y}(\varphi)) - k^{2l}\right)^{-1} \right\| < \frac{16}{b_0 r' k^{2l-1-\delta}}, \tag{3.96}$$

$$\left\| \left(H^{(1)}(\vec{y}(\varphi)) - k^{2l}\right)^{-1} \right\|_1 < \frac{16 c_0 k^{2+2s_1}}{b_0 r' k^{2l-1-\delta}}. \tag{3.97}$$

Corollary 3.53. *If $\epsilon_1 k^{-2l+1-2\delta} < b_0 < k^{-2l+9+12s_1+7\delta}$ and $\varphi \in \Phi_1 \setminus \mathcal{O}_s(\vec{b})$, then*

$$\left\| \left(H^{(1)}(\vec{y}(\varphi)) - k^{2l}\right)^{-1} \right\| < \frac{1}{\epsilon_1^2}, \tag{3.98}$$

$$\left\| \left(H^{(1)}(\vec{y}(\varphi)) - k^{2l}\right)^{-1} \right\|_1 < \frac{c_0 k^{2+2s_1}}{\epsilon_1^2}. \tag{3.99}$$

Corollary follows from the condition on b_0 and the estimate $r' k^{-3\delta} > 16\epsilon_1$, which is obviously valid for sufficiently large k.

Proof. Since $\vec{y}(\varphi)$ is in the $(2k^{-1-4s_1-2\delta})$−neighborhood of $\mathcal{D}_0(\lambda)_{nonres}$, we can apply Lemma A.1 from Appendix A.4. By this lemma,

$$\left\| \left(\lambda^{(1)}(\vec{y}(\varphi)) - k^{2l}\right)\left(H^{(1)}(\vec{y}(\varphi)) - k^{2l}\right)^{-1} \right\| \leq 16.$$

Using (3.95), we easily get (3.96). Using (3.96) and considering (3.89), (3.90), we obtain (3.97). ∎

[11] $\varphi_0^{\pm} = \varphi_{\epsilon_0}^{\pm}$, when $\epsilon_0 = 0$.

4. The Third Approximation

4.1. The Operator $H_\alpha^{(3)}$.

Choosing $s_3 = 2s_2$, we define the third operator $H_\alpha^{(3)}$ by the formula:

$$H_\alpha^{(3)} = H^{(2)} + \alpha W_3, \quad (0 \le \alpha \le 1), \qquad W_3 = \sum_{r=M_2+1}^{M_3} V_r,$$

where M_3 is chosen in such a way that $2^{M_3} \approx k^{s_3}$. Obviously, the periods of W_3 are $2^{M_3-1}(b_1, 0)$ and $2^{M_3-1}(0, b_2)$. We write them in the form: $N_2N_1(a_1, 0)$ and $N_2N_1(0, a_2)$, here $N_2 = 2^{M_3-M_2}$, $\frac{1}{4}k^{s_3-s_2} < N_2 < 4k^{s_3-s_2}$. Note that

$$\|W_3\|_\infty \le \sum_{r=M_2+1}^{M_3} \|V_r\|_\infty \le \sum_{r=M_2+1}^{M_3} \exp(-2^{\eta r}) < \exp(-k^{\eta s_2}).$$

4.2. Multiple Periods of $W_2(x)$.

The operator, $H^{(2)} = H_1 + W_2(x)$, has the periods N_1a_1, N_1a_2. The corresponding family of operators, $\{H^{(2)}(\tau)\}_{\tau \in K_2}$, acts in $L_2(Q_2)$, where $Q_2 = [0, N_1a_1] \times [0, N_1a_2]$ and $K_2 = [0, 2\pi/N_1a_1) \times [0, 2\pi/N_1a_2)$. Since now on we denote quasimomentum t from the first step by $t^{(1)}$, quasimomentum τ from the second step by $t^{(2)}$. Correspondingly, quasimomentum for $H_\alpha^{(3)}$ we denote by $t^{(3)}$. Eigenvalues of $H^{(2)}\left(t^{(2)}\right)$ are denoted by $\lambda_n^{(2)}\left(t^{(2)}\right)$, $n \in \mathbb{N}$ and its spectrum by $\Lambda^{(2)}\left(t^{(2)}\right)$.

Next, let us consider $W_2(x)$ as a periodic function with the periods $N_2N_1a_1$, $N_2N_1a_2$. When changing the periods, the family of operators $\left\{H^{(2)}\left(t^{(2)}\right)\right\}_{t^{(2)} \in K_2}$ is replaced by the family of operators $\left\{\tilde{H}^{(2)}\left(t^{(3)}\right)\right\}_{t^{(3)} \in K_3}$, acting in $L_2(Q_3)$, where $Q_3 = [0, N_2N_1a_1] \times [0, N_2N_1a_2]$ and $K_3 = [0, 2\pi/N_2N_1a_1) \times [0, 2\pi/N_2N_1a_2)$. We denote eigenvalues of $\tilde{H}^{(2)}\left(t^{(3)}\right)$ by $\tilde{\lambda}_n^{(2)}\left(t^{(3)}\right)$, $n \in \mathbb{N}$, and its spectrum by $\tilde{\Lambda}^{(2)}\left(t^{(3)}\right)$. We denote now by $P^{(1)}$ the set P, introduced by (3.4), its elements being $p^{(1)}$. By Bloch theory (see e.g.[21]), for any $t^{(3)} \in K_3$,

$$\tilde{\Lambda}^{(2)}\left(t^{(3)}\right) = \bigcup_{p^{(2)} \in P^{(2)}} \Lambda^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right), \tag{4.1}$$

where $P^{(2)} = \left\{p^{(2)} = \left(p_1^{(2)}, p_2^{(2)}\right) \in \mathbb{Z}^2 : 0 \le p_1^{(2)} \le N_2 - 1,\ 0 \le p_2^{(2)} \le N_2 - 1\right\}$, $2\pi p^{(2)}/N_2N_1a = \left(2\pi p_1^{(2)}/a_1, 2\pi p_2^{(2)}/a_2\right)/N_2N_1$. An isoenergetic set $\tilde{S}_2(\lambda_0) \subset K_3$ of the operator $\tilde{H}^{(2)}$ is defined by the formula:

$$\begin{aligned}\tilde{S}_2(\lambda) &= \left\{t^{(3)} \in K_3 : \exists n \in \mathbb{N} :\ \tilde{\lambda}_n^{(2)}\left(t^{(3)}\right) = \lambda\right\} \\ &= \left\{t^{(3)} \in K_3 : \exists n \in \mathbb{N},\ p^{(2)} \in P^{(2)} : \lambda_n^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right) = \lambda\right\}.\end{aligned}$$

Obviously, $\tilde{S}_2 = \mathcal{K}_3 S_2$, where $\mathcal{K}_3$ is the parallel shift into K_3, that is, $\mathcal{K}_3 : \mathbb{R}^2 \to K_3$, $\mathcal{K}_3\left(t^{(3)} + 2\pi m/N_2N_1a\right) = t^{(3)}$, $m \in \mathbb{Z}^2$, $t^{(3)} \in K_3$. We denote index j, introduced in Part 1 of Geometric Lemma 2.1, by $j^{(1)}$ and $\tilde{j}$, introduced in (3.15), by $j^{(2)}$, $j^{(2)} = j^{(1)} + p^{(1)}/N_1$.

4.3. **Perturbation Formulae.** The results of this section are analogous to those in the previous step, the index 2 just being replaced by 3. Let us start with establishing a lower bound for k. Since $\eta s_1 > 2 + 2s_1$, there is a number $k_* > e$ such that

$$\hat{C}(1+s_1)k^{2+2s_1}\ln k < k^{\eta s_1}, \hat{C} = 400l(c_0+1)^2, \tag{4.2}$$

for any $k > k_*$. Assume also, that k_* is sufficiently large to ensure validity of all estimates in the first two steps for any $k > k_*$.

Lemma 4.1 (Geometric Lemma)**.** *For a sufficiently large λ, $\lambda > k_*^{2l}$, there exists a non-resonance set $\chi_3(\lambda,\delta) \subset \mathcal{K}_3\chi_2^*$ such that:*

(1) *For any point $t^{(3)} \in \chi_3$, the following conditions hold:*
   (a) *There exists a unique $p^{(2)} \in P^{(2)}$ such that $t^{(3)} + 2\pi p^{(2)}/N_2N_1a \in \chi_2^*$.*
   (b) *The following relation holds:*

$$\lambda^{(2)}_{j^{(2)}}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right) = k^{2l},$$

   where $\lambda^{(2)}_{j^{(2)}}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right)$ is given by the perturbation series (3.15) with $\alpha = 1$ and $j^{(2)} = j + p/N_1$, here j and p are defined by the point $\tau = t^{(3)} + 2\pi p^{(2)}/N_2N_1a$ as it is described in Part 2 of the Geometric Lemma for the previous step.
   (c) *The eigenvalue $\lambda^{(2)}_{j^{(2)}}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right)$ is a simple eigenvalue of $\tilde{H}^{(2)}\left(t^{(3)}\right)$ and its distance to all other eigenvalues $\lambda^{(2)}_n\left(t^{(3)} + 2\pi \hat{p}^{(2)}/N_2N_1a\right)$ of $\tilde{H}^{(2)}\left(t^{(3)}\right)$ is greater than $\epsilon_2 = e^{-\frac{1}{4}k^{\eta s_2}}$:*

$$\left|\lambda^{(2)}_{j^{(2)}}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right) - \lambda^{(2)}_n\left(t^{(3)} + 2\pi \hat{p}^{(2)}/N_2N_1a\right)\right| > \epsilon_2. \tag{4.3}$$

(2) *For any $t^{(3)}$ in the $(\epsilon_2 k^{-2l+1-\delta})$-neighborhood in $\mathbb{C}^2$ of χ_3, there exists a unique $p^{(2)} \in P^{(2)}$ such that $t^{(3)} + 2\pi p^{(2)}/N_2N_1a$ is in the $(\epsilon_2 k^{-2l+1-\delta})$-neighborhood in $\mathbb{C}^2$ of χ_2^* and*

$$\left|\lambda^{(2)}_{j^{(2)}}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right) - k^{2l}\right| < \epsilon_2 k^{-\delta}, \tag{4.4}$$

$j^{(2)} = j + p/N_1$, here j and p are defined by the point $\tau = t^{(3)} + 2\pi p^{(2)}/N_2N_1a$ as it is described in Part 2 of the Geometric Lemma for the previous step.

(3) *The third nonresonance set χ_3 has an asymptotically full measure on χ_2^* in the following sense:*

$$\frac{L\left(\mathcal{K}_3\chi_2^* \setminus \chi_3\right)}{L\left(\chi_2^*\right)} < k^{-4-2s_1-2s_2}. \tag{4.5}$$

The proof of the lemma is analogous to that for Geometric Lemma in the second step. It will be presented in Section 4.4.

Corollary 4.2. *If $t^{(3)}$ belongs to the complex $(\epsilon_2 k^{-2l+1-\delta})-$neighborhood of $\chi_3(\lambda,\delta)$, then for any z lying on the circle $C_3 = \{z : |z - k^{2l}| = \epsilon_2/2\}$, the following inequalities hold:*

$$\left\|\left(\tilde{H}^{(2)}(t^{(3)}) - z\right)^{-1}\right\| < \frac{4}{\epsilon_2}, \tag{4.6}$$

$$\|(\tilde{H}^{(2)}(t^{(3)}) - z)^{-1}\|_1 < \frac{4c_0k^{2+2s_3}}{\epsilon_2}. \tag{4.7}$$

Remark 4.3. Every point $2\pi q/N_2N_1a$ ($q\in\mathbb{Z}^2$) of the dual lattice for periods $N_2N_1a_1$, $N_2N_1a_2$ can be uniquely represented in the form: $2\pi q/N_2N_1a=2\pi m/N_1a+2\pi p^{(2)}/N_2N_1a$, where $m\in\mathbb{Z}^2$, $p^{(2)}\in P^{(2)}$. Note that $2\pi m/N_1a$ is a point of a dual lattice for periods N_1a_1, N_1a_2 and $p^{(2)}\in P^{(2)}$ is responsible for refining the lattice. By Remark 3.7, $2\pi q/N_2N_1a$ also can be uniquely represented as $2\pi q/N_2N_1a=2\pi j/a+2\pi p^{(1)}/N_1a+2\pi p^{(2)}/N_2N_1a$, here $j\in\mathbb{Z}^2$, $p^{(1)}\in P^{(1)}$, $p^{(2)}\in P^{(2)}$.

Let us consider a normalized eigenfunction $\psi_n\left(t^{(2)},x\right)$ of $H^{(2)}\left(t^{(2)}\right)$ in $L_2(Q_2)$. We extend it quasiperiodically to $L_2(Q_3)$, renormalize and denote the new function by $\tilde\psi_n\left(t^{(3)},x\right)$, $t^{(3)}=\mathcal{K}_3t^{(2)}$. The Fourier representations of $\psi_n\left(t^{(2)},x\right)$ in $L_2(Q_2)$ and $\tilde\psi_n(t^{(3)},x)$ in $L_2(Q_3)$ are simply related. If we denote Fourier coefficients of $\psi_n\left(t^{(2)},x\right)$ with respect to the basis $|Q_2|^{-1/2}e^{i\left(2\pi m/N_1a+t^{(2)},x\right)}$, $m\in\mathbb{Z}^2$, in $L_2(Q_2)$ by C_{nm}, then, obviously, the Fourier coefficients $\tilde C_{nq}$ of $\tilde\psi_n\left(t^{(3)},x\right)$ with respect to the basis $|Q_3|^{-1/2}e^{i\left(2\pi q/N_2N_1a+t^{(3)},x\right)}$, $q\in\mathbb{Z}^2$, in $L_2(Q_3)$ are given by the formula

$$\tilde C_{nq}=\begin{cases}C_{nm}, & \text{if } q=mN_2+p^{(2)};\\ 0, & \text{otherwise,}\end{cases}$$

$p^{(2)}$ being defined from the relation $t^{(2)}=t^{(3)}+2\pi p^{(2)}/N_1N_2a$. Correspondingly, matrices of the projections on $\psi_n(\tau,x)$ and $\tilde\psi_n(t^{(3)},x)$ with respect to the above bases are simply related:

$$(\tilde E_n)_{q\hat q}=\begin{cases}(E_n)_{m\hat m}, & \text{if } q=mN_2+p^{(2)},\ \hat q=\hat mN_2+p^{(2)};\\ 0, & \text{otherwise,}\end{cases}$$

$\tilde E_n$ and E_n being projections in $L_2(Q_3)$ and $L_2(Q_2)$, respectively.

We define functions $g_r^{(3)}(k,t^{(3)})$ and operator-valued functions $G_r^{(3)}(k,t^{(3)})$, $r=1,2,\cdots$, as follows:

$$g_r^{(3)}\big(k,t^{(3)}\big)=\frac{(-1)^r}{2\pi ir}\mathrm{Tr}\oint_{C_3}\left(\left(\tilde H^{(2)}\big(t^{(3)}\big)-z\right)^{-1}W_3\right)^r dz,$$

$$G_r^{(3)}\big(k,t^{(3)}\big)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_3}\left(\left(\tilde H^{(2)}\big(t^{(3)}\big)-z\right)^{-1}W_3\right)^r\left(\tilde H^{(2)}\big(t^{(3)}\big)-z\right)^{-1}dz.$$

Theorem 4.4. *Suppose $t^{(3)}$ belongs to the $(\epsilon_2k^{-2l+1-\delta})$-neighborhood in K_3 of the third non-resonance set $\chi_3(\lambda,\delta)$, $0<7\delta<2l-11-16s_1$. Then, for sufficiently large λ, $\lambda>k_*^{2l}$ and for all α, $0\le\alpha\le1$, there exists a unique eigenvalue of the operator $H_\alpha^{(3)}\left(t^{(3)}\right)$ in the interval $\varepsilon_3(k):=(k^{2l}-\epsilon_2/2,k^{2l}+\epsilon_2/2)$, $\epsilon_2=e^{-\frac14k^{\eta s_2}}$. It is given by the series:*

$$\lambda_{j^{(3)}}^{(3)}\big(\alpha,t^{(3)}\big)=\lambda_{j^{(2)}}^{(2)}\left(t^{(3)}+2\pi p^{(2)}/N_2N_1a\right)+\sum_{r=1}^{\infty}\alpha^rg_r^{(3)}\big(k,t^{(3)}\big),\tag{4.8}$$

converging absolutely in the disk $|\alpha|\le1$, where $j^{(3)}:=j^{(2)}+p^{(2)}/N_2N_1$, $p^{(2)}$, $j^{(2)}$ being described in Geometric Lemma 4.1. The spectral projection, corresponding to $\lambda_{j^{(3)}}^{(3)}(\alpha,t^{(3)})$, is given by the series:

$$E_{j^{(3)}}^{(3)}\big(\alpha,t^{(3)}\big)=\tilde E_{j^{(2)}}^{(2)}\left(t^{(3)}+2\pi p^{(2)}/N_2N_1a\right)+\sum_{r=1}^{\infty}\alpha^rG_r^{(3)}\big(k,t^{(3)}\big),\tag{4.9}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha|\le1$.

The following estimates hold for coefficients $g_r^{(3)}(k,t^{(3)})$, $G_r^{(3)}(k,t^{(3)})$*:*

$$\left\|g_r^{(3)}\left(k,t^{(3)}\right)\right| < \frac{3\epsilon_2}{2}\left(4\epsilon_2^3\right)^r, \qquad \left\|G_r^{(3)}\left(k,t^{(3)}\right)\right\|_1 < 6r\left(4\epsilon_2^3\right)^r. \tag{4.10}$$

Corollary 4.5. *The following estimates hold for the perturbed eigenvalue and its spectral projection:*

$$\left|\lambda_{j^{(3)}}^{(3)}\left(\alpha,t^{(3)}\right)-\lambda_{j^{(2)}}^{(2)}\left(t^{(3)}+2\pi p^{(2)}/N_2N_1a\right)\right| \le 12\alpha\epsilon_2^4, \tag{4.11}$$

$$\left\|E_{j^{(3)}}^{(3)}\left(\alpha,t^{(3)}\right)-\tilde{E}_{j^{(2)}}^{(2)}\left(t^{(3)}+2\pi p^{(2)}/N_2N_1a\right)\right\|_1 \le 48\alpha\epsilon_2^3. \tag{4.12}$$

The series (4.8), (4.9) can be extended as holomorphic functions of $t^{(3)}$ in the complex $\left(\frac{1}{2}\epsilon_2 k^{-2l+1-\delta}\right)$-neighborhood of χ_3; they can be differentiated any number of times with respect to $t^{(3)}$ and retain their asymptotic character. The results analogous to Lemma 3.11 – Lemma 3.14 hold.

4.4. **Proof of the Geometric Lemma.** We define the third nonresonance set, $\chi_3(\lambda)\subset\mathcal{K}_3\chi_2^*(\lambda)$ as follows: $\chi_3(\lambda)=\mathcal{K}_3\chi_2^*(\lambda)\setminus\Omega_2(\lambda)$,

$$\begin{aligned}\Omega_2(\lambda)=\Big\{t^{(3)}\in\mathcal{K}_3\chi_2^*:\exists n,\hat n\in\mathbb{N},\ p^{(2)},\hat p^{(2)}\in P^{(2)},\quad p^{(2)}\neq\hat p^{(2)}:\\ \lambda_n^{(2)}\left(t^{(3)}+2\pi p^{(2)}/N_2N_1a\right)=\lambda,\quad t^{(3)}+2\pi p^{(2)}/N_2N_1a\in\chi_2^*(\lambda),\\ \left|\lambda_n^{(2)}\left(t^{(3)}+2\pi p^{(2)}/N_2N_1a\right)-\lambda_{\hat n}^{(2)}\left(t^{(3)}+2\pi\hat p^{(2)}/N_2N_1a\right)\right|\le\epsilon_2\Big\}.\end{aligned} \tag{4.13}$$

Proofs of Statements 1 and 2 of Geometric Lemma are identical to those for the second step up to the shift of indices by 1.

From now on we denote a vector $\vec b\in K_1$, introduced in the second step, by $\vec b^{(1)}$. Naturally, we denote the set $\mathcal{O}(\vec b)$ constructed in the previous step (Section 3.3.4) by $\mathcal{O}^{(1)}(\vec b^{(1)})$. Correspondingly, $\Gamma(\vec b)$ is $\Gamma^{(1)}(\vec b^{(1)})$, $\mathcal{O}_*=\mathcal{O}_*^{(1)}$ (see (3.40)), and $\mathcal{O}_s(\vec b)$ is $\mathcal{O}_s^{(1)}(\vec b^{(1)})$. Note that $\mathcal{O}_s^{(1)}(\vec b^{(1)})$ is defined for an arbitrarily small $\vec b^{(1)}$, see Sections 3.5.1 and 3.5.2. Let us recall that the complex non-resonant set Φ_2 is given by (3.40).

Definition 4.6. By analogy with the definition of $\omega_1^*(\lambda)$ (Section 3.3.4), we consider a complex resonance set $\omega_2^*(\lambda)$, which is the set of $\varphi\in\Phi_2$ satisfying

$$\det\left(\frac{H^{(2)}\big(\vec y^{(2)}(\varphi)\big)-k^{2l}-\epsilon}{H_0\big(\vec y^{(2)}(\varphi)\big)+k^{2l}}\right)=0,\quad \vec y^{(2)}(\varphi)=\vec\varkappa_2(\varphi)+\vec b^{(2)},\quad \vec b^{(2)}=\frac{2\pi p^{(2)}}{N_2N_1a}, \tag{4.14}$$

for some $p^{(2)}\in P^{(2)}\setminus\{0\}$ and $|\epsilon|<\epsilon_2$. Let $\omega_2(\lambda)$ be the set of $\varphi\in\Theta_2$ corresponding to $\Omega_2(\lambda)$. Considering as in the proof of Lemma 3.20, it is easy to show that $\omega_2=\omega_2^*\cap\Theta_2$.

By analogy with (3.41), let us consider an arbitrary $\vec b^{(2)}\in K_2$ and its distance $b_0^{(2)}$ to the closest vertex of K_2:

$$b_0^{(2)}=\min_{m=(0,0),(0,1),(1,0),(1,1)}\left|\vec b^{(2)}-2\pi m/N_1a\right|.$$

Now we construct $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)\subset \mathit{\Phi}_2$, assuming $b_0^{(2)}>0$. Indeed, for each $\vec{b}^{(2)}+2\pi p^{(1)}/N_1a$, $p^{(1)}\in P^{(1)}$, we take the set $\mathcal{O}_s^{(1)}\big(\vec{b}^{(2)}+2\pi p^{(1)}/N_1a\big)$ described in Section 3.5 and consider a union:

$$\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)=\cup_{p^{(1)}\in P^{(1)}}\mathcal{O}_s^{(1)}\left(\vec{b}^{(2)}+2\pi p^{(1)}/N_1a\right). \tag{4.15}$$

A connected component of $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ we denote by $\Gamma^{(2)}\left(\vec{b}^{(2)}\right)$ and its boundary by $\gamma^{(2)}\left(\vec{b}^{(2)}\right)$.

Remark 4.7. The set $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ consists of disks of the radius r' around all zeros of the determinant $\det\Big(I+\tilde{A}_1\big(\vec{y}^{(1)}(\varphi)\big)\Big)$, here and below:

$$\vec{y}^{(1)}(\varphi)=\vec{\varkappa}_1(\varphi)+\vec{b}^{(2)}, \tag{4.16}$$
$$I+\tilde{A}_1\big(\vec{y}^{(1)}\big)=\Big(\tilde{H}^{(1)}\big(\vec{y}^{(1)}\big)-\lambda\Big)\Big(\tilde{H}_0\big(\vec{y}^{(1)}\big)+\lambda\Big)^{-1}.$$

This remark easily follows from the fact that each $\mathcal{O}_s^{(1)}\big(\vec{b}^{(2)}+2\pi p^{(1)}/N_1a\big)$ is built around zeros of $\det\Big(I+A_1\left(\vec{\varkappa}_1(\varphi)+\vec{b}^{(2)}+2\pi p^{(1)}/N_1a\right)\Big)$, and $\tilde{A}_1\big(\vec{y}^{(1)}(\varphi)\big)$ is a direct sum of the operators $A_1\left(\vec{\varkappa}^{(1)}+\vec{b}^{(2)}+2\pi p^{(1)}/N_1a\right)$, $p^{(1)}\in P^{(1)}$.

Remark 4.8. Let us estimate the number of disks in $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ and the size of a connected component $\Gamma^{(2)}\left(\vec{b}^{(2)}\right)$. Indeed, the set $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ contains no more than $4c_0k^{2+2s_2}$ disks, since each $\mathcal{O}_s^{(1)}$ consists of no more than $c_0k^{2+2s_1}$ disks (Lemma 3.46, Definition 3.50), and $P^{(1)}$ contains no more than $4k^{2(s_2-s_1)}$ elements. The set $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ is formed by disks of the radius r', here $r'=rk^{-2-4s_2-\delta}$, r being the radius of disks constituting $\mathcal{O}_*^{(1)}$, $r=k^{-4-6s_1-3\delta}$. Hence, the size of each connected component of $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ does not exceed $8c_0r'k^{2+2s_2}=8c_0rk^{-\delta-2s_2}$. Thus, the size of $\Gamma^{(2)}\left(\vec{b}^{(2)}\right)$ is much smaller than the radius of disks constituting $\mathcal{O}_*^{(1)}$. Here, even if a component $\Gamma^{(2)}\left(\vec{b}^{(2)}\right)$ is not strictly inside $\mathit{\Phi}_2$, it has the same properties as a component inside $\mathit{\Phi}_2$. Further, we consider $\Gamma^{(2)}\left(\vec{b}^{(2)}\right)\subset\mathit{\Phi}_2$, $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)\subset\mathit{\Phi}_2$.

By analogy with (3.40), we introduce new notations:

$$\mathcal{O}_*^{(2)}=\cup_{p^{(2)}\in P^{(2)}\setminus\{0\}}\mathcal{O}^{(2)}\big(2\pi p^{(2)}/N_2N_1a\big),\qquad \mathit{\Phi}_3=\mathit{\Phi}_2\setminus\mathcal{O}_*^{(2)}. \tag{4.17}$$

We will show (Corollary 4.20) that $\omega_2^*\subset\mathcal{O}_*^{(2)}$ and $\omega_2\subset\mathcal{O}_*^{(2)}\cap\Theta_2$. From now on we call $\mathcal{O}_*^{(2)}$ the second (complex) resonance set in $\mathit{\Phi}_2$ and $\mathcal{O}_*^{(2)}\cap\Theta_2$ the second resonance set in Θ_2. From (4.17) and (4.15) we see:

$$\mathcal{O}_*^{(2)}=\cup_{p^{(1)}\in P^{(1)},p^{(2)}\in P^{(2)}\setminus\{0\}}\mathcal{O}_s^{(1)}\Big(2\pi p^{(1)}/N_1a+2\pi p^{(2)}/N_2N_1a\Big), \tag{4.18}$$

each $\mathcal{O}_s^{(1)}\Big(2\pi p^{(1)}/N_1a+2\pi p^{(2)}/N_2N_1a\Big)$ being a union of disks with centers at zeros of the determinants $\det\Big(I+A_1\Big(\vec{\varkappa}^{(1)}+2\pi p^{(1)}/N_1a+2\pi p^{(2)}/N_2N_1a\Big)\Big)$, the size of disks being r'.

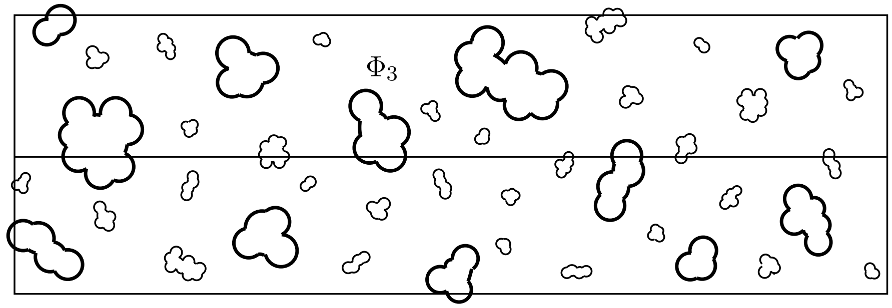

FIGURE 8. The set Φ_3.

Remark 4.9. The set $\mathcal{O}_*^{(2)}$ consists of no more than $16c_0k^{2+2s_3}$ disks of the radius r', since each $\mathcal{O}_s^{(1)}$ is formed by no more than $c_0k^{2+2s_1}$ disks and $P^{(1)}$, $P^{(2)}$ contains no more than $4k^{2(s_2-s_1)}$ and $4k^{2(s_3-s_2)}$ elements, respectively.

Obviously, $O_*^{(2)}$ contains more disks than $\mathcal{O}_*^{(1)}$, however, disks in $O_*^{(2)}$ are of a smaller size than ones in $\mathcal{O}_*^{(1)}$. Disks in $\mathcal{O}_*^{(1)}$ are centered at zeros of unperturbed determinants, while smaller disks in $O_*^{(2)}$ are centered about zeros of the perturbed determinant (see Remark 4.7). The shift of centers is of principle importance: we can reduce the size of disks, since they are more precisely targeted. If $W_1 = 0$, then $O_*^{(2)}$ is just a union of disks centered at quasi-intersections of the circle of radius k, centered at the origin, with circles of the same radius k, centered at points $2\pi j/a + 2\pi p^{(1)}/N_1a + 2\pi p^{(2)}/N_2N_1a$, $p^{(2)} \neq 0$, of the dual lattice for periods $N_2N_1a_1, N_2N_1a_2$.

To obtain Φ_3 we delete $\mathcal{O}_*^{(2)}$ from Φ_2. Thus, the set Φ_3 has a structure of Swiss cheese (Fig. 8); we will add more holes of a smaller size at each step of approximation.

By analogy with (3.31), we define the operator $A_2\left(\vec{y}^{(2)}(\varphi)\right)$ by the formula:

$$I + A_2\big(\vec{y}^{(2)}(\varphi)\big) = \Big(H^{(2)}\big(\vec{y}^{(2)}(\varphi)\big) - \lambda\Big)\Big(\tilde{H}_0\big(\vec{y}^{(2)}(\varphi)\big) + \lambda\Big)^{-1},$$

$\vec{y}^{(2)}(\varphi)$ being given in Definition 4.6, $\tilde{H}_0\left(\vec{y}^{(2)}(\varphi)\right)$ acting in $L_2(Q_2)$. The goal is to show that $\det\Big(I + \tilde{A}_1\big(\vec{y}^{(1)}(\varphi)\big)\Big)$ and $\det\Big(I + A_2\big(\vec{y}^{(2)}(\varphi)\big)\Big)$ have the same number of zeros inside each $\Gamma^{(2)}\big(\vec{b}^{(2)}\big)$ and that the resolvents for $\tilde{H}^{(1)}\big(\vec{y}^{(1)}(\varphi)\big)$ and $H^{(2)}\big(\vec{y}^{(2)}(\varphi)\big)$ satisfy similar estimates when $\varphi \in \Phi_1 \setminus \mathcal{O}^{(2)}\big(\vec{b}^{(2)}\big)$.

Lemma 4.10. *If $\vec{b}^{(2)} \in K_2$ and $b_0^{(2)} > \epsilon_1 k^{-2l+1-2\delta}$, then*

(1) *For any $\varphi \in \Phi_1 \setminus \mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$:*

$$\left\|\Big(\tilde{H}^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) - k^{2l}\Big)^{-1}\right\| < \frac{1}{\epsilon_1^2} \tag{4.19}$$

$$\left\|\Big(\tilde{H}^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) - k^{2l}\Big)^{-1}\right\|_1 < \frac{4c_0k^{2+2s_2}}{\epsilon_1^2} \tag{4.20}$$

(2) *The number of zeros of $\det\Big(I + \tilde{A}_1\left(\vec{y}^{(1)}(\varphi)\right)\Big)$ in Φ_1 does not exceed $4c_0k^{2+2s_2}$.*

Corollary 4.11. *There are no zeros of* $\det\Big(I+\tilde{A}_1\left(\vec{y}^{(1)}(\varphi)\right)\Big)$ *in* $\Phi_1\setminus\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$.

Proof. (1) Considering the relation between $H^{(1)}$ and $\tilde{H}^{(1)}$, we easily see

$$\left\|\left(\tilde{H}^{(1)}(\vec{y}^{(1)}(\varphi))-k^{2l}\right)^{-1}\right\|=\max_{p^{(1)}\in P^{(1)}}\left\|\left(H^{(1)}\left(\vec{y}^{(1)}(\varphi)+2\pi p^{(1)}/N_1a\right)-k^{2l}\right)^{-1}\right\|.$$

The condition $\varphi\in\Phi_1\setminus\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ yields $\varphi\in\Phi_1\setminus\mathcal{O}^{(1)}_s\left(\vec{b}^{(2)}+2\pi p^{(1)}/N_1a\right)$ for every $p^{(1)}\in P^{(1)}$. If $\vec{b}^{(2)}+2\pi p^{(1)}/N_1a$ satisfies $b_0\geq k^{-2l+9+12s_1+7\delta}$, where b_0 is the distance from $\vec{b}^{(2)}+2\pi p^{(1)}/N_1a$ to the nearest vertex of K_1, then we use Lemma 3.46 and Corollary 3.47:

$$\left\|\left(H^{(1)}\left(\vec{y}^{(1)}(\varphi)+2\pi p^{(1)}/N_1a\right)-k^{2l}\right)^{-1}\right\|<\frac{1}{\epsilon_1}<\frac{1}{\epsilon_1^2}. \tag{4.21}$$

Suppose $\vec{b}^{(2)}+2\pi p^{(1)}/N_1a$ satisfies $b_0<k^{-2l+9+12s_1+7\delta}$. It is easy to see that $b_0>\epsilon_1k^{-2l+1-2\delta}$, since $b_0^{(2)}>\epsilon_1k^{-2l+1-2\delta}$ by the hypothesis of the lemma. Using Corollary 3.53 to obtain an estimate similar to (4.21). Thus, (4.19) is proved. The estimate (4.20) can be checked in the analogous way from (3.87) and (3.99), when we take into account that $P^{(1)}$ contains less than $4k^{2s_2-2s_1}$ elements.

(2) The determinant of $I+\tilde{A}_1\left(\vec{y}^{(1)}(\varphi)\right)$ is equal to zero if and only if at least one of the determinants $\det\Big(I+A_1\left(\vec{y}^{(1)}(\varphi)+2\pi p^{(1)}/N_1a\right)\Big)$, $p^{(1)}\in P^{(1)}$, is zero. Since each of them has no more than $c_0k^{2+2s_1}$ zeros (Corollary 3.36, Definition 3.50) and $P^{(1)}$ contains no more than $4k^{2(s_2-s_1)}$ elements, the total number of zeros of $\det\Big(I+\tilde{A}_1\left(\vec{y}^{(1)}(\varphi)\right)\Big)$ does not exceed $4c_0k^{2+2s_2}$.

∎

The following five lemmas are the analogs of Lemmas 3.29 – 3.38.

Lemma 4.12. *If* $\vec{b}^{(2)}\in K_2$ *and* $b_0^{(2)}>\epsilon_1k^{-2l+1-2\delta}$, *then:*

(1) *The number of zeros of* $\det\Big(I+\tilde{A}_1(\vec{y}^{(2)}(\varphi))\Big)$ *inside* $\Gamma^{(2)}\left(\vec{b}^{(2)}\right)$ *is the same as that of* $\det\Big(I+\tilde{A}_1(\vec{y}^{(1)}(\varphi))\Big)$.

(2) *For any* $\varphi\in\Phi_2\setminus\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$, *the following inequalities hold:*

$$\left\|\left(\tilde{H}^{(1)}\left(\vec{y}^{(2)}(\varphi)\right)-k^{2l}\right)^{-1}\right\|<\frac{2}{\epsilon_1^2}, \tag{4.22}$$

$$\left\|\left(\tilde{H}^{(1)}(\vec{y}^{(2)}(\varphi))-k^{2l}\right)^{-1}\right\|_1<\frac{8c_0k^{2+2s_2}}{\epsilon_1^2}. \tag{4.23}$$

Corollary 4.13. *There are no zeros of* $\det\Big(I+\tilde{A}_1\left(\vec{y}^{(2)}(\varphi)\right)\Big)$ *in* $\Phi_2\setminus\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$.

Proof. (1) Direct calculation gives:

$$\det\Big(I+\tilde{A}_1(\vec{y}^{(2)})\Big)\Big/\det\Big(I+\tilde{A}_1(\vec{y}^{(1)})\Big)=\det(I+T).$$

$$T=\Big(W_1-2\lambda\Big)\Big(\tilde H_0(\vec y^{(2)})+\lambda\Big)^{-1}\Big(\tilde H_0(\vec y^{(1)})-\tilde H_0(\vec y^{(2)})\Big)\Big(\tilde H^{(1)}(\vec y^{(1)})-\lambda\Big)^{-1}.$$

The estimate (3.63) yields: $\left|\vec y^{(2)}(\varphi)-\vec y^{(1)}(\varphi)\right|<2\alpha\epsilon_1^4k^{-2l+1}$. Using the last estimate we readily show that

$$\left\|\Big(\tilde H_0(\vec y^{(2)})+\lambda\Big)^{-1}\Big(\tilde H_0(\vec y^{(1)})-\tilde H_0(\vec y^{(2)})\Big)\right\|<2l\alpha\epsilon_1^4k^{-2l}. \tag{4.24}$$

Considering (4.20), we obtain

$$\|T\|_1\le 4k^{2l}\left(4l\epsilon_1^4k^{-2l}\right)\left(4c_0k^{2+2s_2}/\epsilon_1^2\right)\le 64lc_0\epsilon_1^2k^{2+2s_2}<1. \tag{4.25}$$

Hence, $|\det(I+T)-1|<1$ for any $\varphi\in\gamma^{(2)}\left(\vec b^{(2)}\right)$. Applying Rouché's Theorem, we complete the proof.

(2) Let us consider the Hilbert relation:

$$\Big(\tilde H^{(1)}(\vec y^{(2)}(\varphi))-\lambda\Big)^{-1}=\Big(\tilde H^{(1)}(\vec y^{(1)}(\varphi))-\lambda\Big)^{-1}+T_*\Big(\tilde H^{(1)}(\vec y^{(2)}(\varphi))-\lambda\Big)^{-1},$$

$$T_*=\Big(\tilde H^{(1)}(\vec y^{(1)}(\varphi))-\lambda\Big)^{-1}\Big(\tilde H_0(\vec y^{(1)}(\varphi))-\tilde H_0(\vec y^{(2)}(\varphi))\Big).$$

To estimate $\|T_*\|$, we represent T_* in the form $T_*=(I+T_1)T_2$, here

$$T_1=\Big(\tilde H^{(1)}(\vec y^{(1)}(\varphi))-\lambda\Big)^{-1}\Big(-W_1+2\lambda\Big),$$

$$T_2=\Big(\tilde H_0(\vec y^{(1)}(\varphi))+\lambda\Big)^{-1}\Big(\tilde H_0(\vec y^{(1)}(\varphi))-\tilde H_0(\vec y^{(2)}(\varphi))\Big).$$

Using (4.19), we get $\|T_1\|\le 4\lambda\epsilon_1^{-2}$. The norm of T_2 can be bounded as (4.24). Multiplying the estimates for $\|T_1\|$, $\|T_2\|$, we obtain: $\|T_*\|<8\alpha l\varepsilon_1^2<1/2$. Substituting the last estimate into the Hilbert relation and considering (4.19), we arrive at (4.22). Using (4.20), we prove (4.23) in the analogous way.

∎

Lemma 4.14. *If $\vec b^{(2)}\in K_2$ and $b_0^{(2)}>\epsilon_1k^{-2l+1-2\delta}$, then the following estimates hold for any $\varphi\in\Phi_2\setminus\mathcal O^{(2)}\left(\vec b^{(2)}\right)$:*

$$\left\|\Big(H^{(2)}(\vec y^{(2)}(\varphi))-\lambda\Big)^{-1}\right\|<\frac{4}{\epsilon_1^2}, \tag{4.26}$$

$$\left\|\Big(H^{(2)}(\vec y^{(2)}(\varphi))-\lambda\Big)^{-1}\right\|_1<\frac{16c_0k^{2+2s_2}}{\epsilon_1^2}. \tag{4.27}$$

Corollary 4.15. *There are no zeros of* $\det\Big(I+A_2\left(\vec y^{(2)}(\varphi)\right)\Big)$ *in $\Phi_2\setminus\mathcal O^{(2)}\left(\vec b^{(2)}\right)$.*

Proof. The proof is analogous to that of Lemma 3.31; we use Hilbert relation, the estimates (4.22), (4.23) and the inequality $\|W_2\|<\epsilon_1^4$, which immediately follows from (3.2) and the definition of ϵ_1. ∎

Lemma 4.16. *If $\vec b^{(2)}\in K_2$ and $b_0^{(2)}>\epsilon_1k^{-2l+1-2\delta}$, then the number of zeros of* $\det\left(I+A_2(\vec y^{(2)}(\varphi))\right)$ *inside $\Gamma^{(2)}(\vec b^{(2)})$ is the same as that of* $\det\Big(I+\tilde A_1(\vec y^{(2)}(\varphi))\Big)$.

Proof. Using (4.23), we obtain that for any $\varphi \in \gamma^{(2)}$:

$$\left|\det\left(\frac{H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right)-k^{2l}}{\tilde{H}^{(1)}\left(\vec{y}^{(2)}(\varphi)\right)-k^{2l}}\right)-\det I\right| = \left|\det\left(I+\frac{W_2}{\tilde{H}^{(1)}\left(\vec{y}^{(2)}(\varphi)\right)-k^{2l}}\right)-\det I\right|$$
$$\leq 2\|W_2\|\left\|\left(\tilde{H}^{(1)}\left(\vec{y}^{(2)}(\varphi)\right)-k^{2l}\right)^{-1}\right\|_1 \leq 2\epsilon_1^4\cdot\frac{4c_0k^{2+2s_2}}{\epsilon_1^2} = 8c_0\epsilon_1^2k^{2+2s_2} \ll 1.$$

By Rouché's Theorem, we complete the proof. ∎

Lemma 4.17. (1) *If $\vec{b}^{(2)} \in K_2$ and $b_0^{(2)} > \epsilon_1 k^{-2l+1-2\delta}$, then the number of zeros of* $\det\left(I + A_2(\vec{y}^{(2)}(\varphi))\right)$ *in any $\Gamma^{(2)}(\vec{b}^{(2)})$ is the same as that of* $\det\left(I+\tilde{A}_1(\vec{y}^{(1)}(\varphi))\right)$.

(2) *The distance from a zero of* $\det\left(I + A_2(\vec{y}^{(2)}(\varphi))\right)$ *to the nearest zero of* $\det\left(I+\tilde{A}_1(\vec{y}^{(1)}(\varphi))\right)$ *does not exceed $r^{(2)}/2$.*

Corollary 4.18. *The number of zeros of* $\det\left(I + A_2(\vec{y}^{(2)}(\varphi))\right)$ *does not exceed $4c_0k^{2+2s_2}$.*

The corollary is the combination of the first part of the lemma with the second part of Lemma 4.10.

Lemma 4.16. The first statement immediately follows from Lemmas 4.12 (Part 1) and 4.16. Next, note that Lemmas 4.10 – 4.16 hold not only for $\mathcal{O}^{(2)}(\vec{b}^{(2)})$, but also for a set $\tilde{\mathcal{O}}^{(2)}(\vec{b}^{(2)})$, constructed from disks of twice smaller radius $r^{(2)}/2$ with the same centers, since all estimates in the lemmas preserved under such change up to some insignificant constants. This proves that all zeros of $\det\left(I + A_2(\vec{y}^{(2)}(\varphi))\right)$ in $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ are, in fact, in the smaller set $\tilde{\mathcal{O}}^{(2)}(\vec{b}^{(2)})$, i.e., the second statement of the lemma holds. ∎

Lemma 4.19. *If $\vec{b}^{(2)} \in K_2$ and $b_0^{(2)} \geq \epsilon_1 k^{-2l+1-2\delta}$, $|\epsilon| < \epsilon_1^4$, then*

(1) *The determinant of* $\dfrac{H^{(2)}(\vec{y}^{(2)}(\varphi))-k^{2l}-\epsilon}{H_0(\vec{y}^{(2)}(\varphi))+k^{2l}}$ *has no zeros in $\Phi_2 \setminus \mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$.*

(2) *The number of zeros of* $\det\left(\dfrac{H^{(2)}(\vec{y}^{(2)}(\varphi))-k^{2l}-\epsilon}{H_0(\vec{y}^{(2)}(\varphi))+k^{2l}}\right)$ *in $\Gamma^{(2)}\left(\vec{b}^{(2)}\right)$ is the same as that of* $\det\left(\dfrac{H^{(2)}(\vec{y}^{(2)}(\varphi))-k^{2l}}{H_0(\vec{y}^{(2)}(\varphi))+k^{2l}}\right)$.

Corollary 4.20. *The following relation hold: $\omega_2^* \subset \mathcal{O}_*^{(2)}$.*

Lemma 4.19 and Corollary 4.20. Proof of the lemma is completely analogous to that of the previous two lemmas up to replacement of the estimate $\|W_2\| < \epsilon_1^4$ by $\|W_2+\epsilon\| < 2\epsilon_1^4$. Corollary follows from the lemma and Definition 4.6. In fact, $\omega_2^*(\lambda) = \cup_{p^{(2)}\in P^{(2)}\setminus\{0\}}\omega_{2,p^{(2)}}$, where $\omega_{2,p^{(2)}}$ is the set of zeros of the determinant (4.14) for a fixed $p^{(2)}$. By Lemma 4.19 (Part 1), $\omega_{2,p^{(2)}} \subset \mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$, $\vec{b}^{(2)} = \frac{2\pi p^{(2)}}{N_2N_1a}$, since $\epsilon_2 < \epsilon_1^4$. Note that such $\vec{b}^{(2)}$ satisfies the condition of the lemma, since $N_2N_1a_{1,2} \leq 2k^{s_3}$. Using the definition (4.17) of $\mathcal{O}_*^{(2)}$ and the formula $\omega_2^*(\lambda) = \cup_{p^{(2)}\in P^{(2)}\setminus\{0\}}\omega_{2,p^{(2)}}$ yields the corollary. ∎

From now on we denote the radius r of the disks in $\mathcal{O}_*^{(1)}$ by $r^{(1)}$ and the radius r' of the disks in $\mathcal{O}^{(2)}$ by $r^{(2)}$.

Lemma 4.21. *(proof of Statement 3) Under conditions of Geometric lemma 4.1 , estimate (4.5) holds.*

Proof. By Corollary 4.20, $\omega_2^* \subset \mathcal{O}_*^{(2)}$. Hence, $\omega_2 \subset \mathcal{O}_*^{(2)} \cap \Theta_2$, the set ω_2 being given in Definition 4.6. Considering that that $\mathcal{O}_*^{(2)}$ is formed by no more than $16c_0k^{2+2s_3}$ disks of the radius $r^{(2)}$ (see Remark 4.9), we obtain that the total length of ω_2 does not exceed $32c_0k^{2+2s_3}r^{(2)}$. Taking into account that $L(\Omega_2) \approx kL(\omega_2)$ and $r^{(2)} = r^{(1)}k^{-2-4s_2-\delta} = k^{-6-6s_1-4s_2-4\delta}$, $s_3 = 2s_2 = 4s_1$, we obtain that the the length of Ω_2 does not exceed $64c_0k^{-3-2s_1-2s_2-4\delta}$. Considering that $\chi_2^*(\lambda)$ has a length $2\pi k(1 + o(1))$ we arrive at Statement 3. ∎

4.5. **Nonresonant part of the isoenergetic set of $H_\alpha^{(3)}$.** This section is analogous to Section 3.4 for the second step. Indeed, let $S_3(\lambda)$ be an isoenergetic set of the operator $H_\alpha^{(3)}$: $S_3(\lambda) = \{t^{(3)} \in K_3 : \exists n \in \mathbb{N} : \lambda_n^{(3)}(\alpha, t^{(3)}) = \lambda\}$, here $\{\lambda_n^{(3)}(\alpha, t^{(3)})\}_{n=1}^{\infty}$ is the spectrum of $H_\alpha^{(3)}(t^{(3)})$. Now we construct a non-resonance subset $\chi_3^*(\lambda)$ of $S_3(\lambda)$. It corresponds to non-resonance eigenvalues $\lambda_{j^{(3)}}^{(3)}(t^{(3)})$ given by the perturbation series (4.8). We start with a definition of $\mathcal{D}_2(\lambda)_{nonres}$. Recall that $\chi_3 \subset \mathcal{K}_3\chi_2^*(\lambda)$ (see Geometric Lemma 4.1) and $\chi_2^*(\lambda) = \mathcal{K}_2\mathcal{D}_2(\lambda)$, see (3.83). Hence, $\chi_3 \subset \mathcal{K}_3\mathcal{D}_2(\lambda)$. Let $\mathcal{D}_2(\lambda)_{nonres}$ be the preimage of χ_3 in $\mathcal{D}_2(\lambda)$:

$$\mathcal{D}_2(\lambda)_{nonres} = \{\vec{\varkappa} \in \mathcal{D}_2(\lambda) : \mathcal{K}_3\vec{\varkappa} \in \chi_3\}. \tag{4.28}$$

The following lemma is analogous to Lemma 3.40 in the second step.

Lemma 4.22. *The formula $\mathcal{K}_3\mathcal{D}_2(\lambda)_{nonres} = \chi_3$ establishes one-to-one correspondence between $\mathcal{D}_2(\lambda)_{nonres}$ and χ_3.*

Proof. The proof is analogous to that of Lemma 3.40 up to the shift of indices by 1, i.e., $\chi_2 \to \chi_3$, $\chi_1^*(\lambda) \to \chi_2^*(\lambda)$, $\tau = t^{(2)} \to t^{(3)}$; we use formula (3.83) instead of (2.48), Part 1a of the Geometric Lemma for the third step instead of Part 1a of the Geometric Lemma for the second step, and Lemma 3.44 instead of 2.13. ∎

We define $\mathcal{B}_3(\lambda)$ as the set of directions corresponding to $\mathcal{D}_2(\lambda)_{nonres}$:

$$\mathcal{B}_3(\lambda) = \{\vec{\nu} \in \mathcal{B}_2(\lambda) : \varkappa_2(\lambda, \vec{\nu})\vec{\nu} \in \mathcal{D}_2(\lambda)_{nonres}\}$$

where $\varkappa_2(\lambda, \vec{\nu})$ is defined by Lemma 3.42, $\varkappa_2(\lambda, \vec{\nu}) \equiv \varkappa_2(\varphi)$. Note that $\mathcal{B}_3(\lambda)$ is a unit circle with holes centered at the origin and $\mathcal{B}_3(\lambda) \subset \mathcal{B}_2(\lambda)$. We denote by $\Theta_3(\lambda)$ the set of angles φ corresponding to $\mathcal{B}_3(\lambda)$:

$$\Theta_3(\lambda) = \{\varphi \in [0, 2\pi) : \ (\cos\varphi, \sin\varphi) \in \mathcal{B}_3(\lambda)\}, \qquad \Theta_3 \subset \Theta_2. \tag{4.29}$$

We define $\mathcal{D}_3(\lambda)$ as a level set for $\lambda^{(3)}(\alpha, \vec{\varkappa})$ in the $\left(\epsilon_2 k^{-2l+1-\delta}\right)$-neighborhood of $\mathcal{D}_2(\lambda)_{nonres}$:

$$\mathcal{D}_3(\lambda) := \left\{\vec{\varkappa} = \varkappa\vec{\nu} : \vec{\nu} \in \mathcal{B}_3(\lambda), \ \ \left|\varkappa - \varkappa_2(\lambda, \vec{\nu})\right| < \epsilon_2 k^{-2l+1-\delta}, \ \lambda^{(3)}(\alpha, \vec{\varkappa}) = \lambda\right\}.$$

Next two lemmas are to prove that $\mathcal{D}_3(\lambda)$ is a distorted circle with holes. Their formulations and proofs are analogous to those of Lemmas 3.41 and 3.42.

Lemma 4.23. *For every $\vec\nu \in \mathcal{B}_3(\lambda)$ and every α, $0 \le \alpha \le 1$, there is a unique $\varkappa = \varkappa_3(\lambda,\vec\nu)$ in the interval $I_3 := \big[\varkappa_2(\lambda,\vec\nu) - \epsilon_2 k^{-2l+1-\delta}, \varkappa_2(\lambda,\vec\nu) + \epsilon_2 k^{-2l+1-\delta}\big]$ such that*

$$\lambda^{(3)}(\alpha, \varkappa_3\vec\nu) = \lambda. \tag{4.30}$$

Furthermore,

$$|\varkappa_3(\lambda,\vec\nu) - \varkappa_2(\lambda,\vec\nu)| \le 2\alpha\epsilon_2^4 k^{-2l+1}. \tag{4.31}$$

Let us recall that $\mathit{\Phi}_3$ is the third complex non-resonant set given by (4.17), see Fig. 8. Further we use the notations $\varkappa_3(\varphi) \equiv \varkappa_3(\lambda,\vec\nu)$, $h_3(\varphi) \equiv \varkappa_3(\varphi) - \varkappa_2(\varphi)$, $\vec\varkappa_3(\varphi) = \varkappa_3(\varphi)\vec\nu$.

Lemma 4.24. (1) *The set $\mathcal{D}_3(\lambda)$ is a distorted circle with holes: it can be described by the formula:*

$$\mathcal{D}_3(\lambda) = \big\{\vec\varkappa \in \mathbb{R}^2 : \vec\varkappa = \vec\varkappa_3(\varphi), \quad \varphi \in \Theta_3(\lambda)\big\}, \tag{4.32}$$

where $\varkappa_3(\varphi) = \varkappa_2(\varphi) + h_3(\varphi)$, $\varkappa_2(\varphi)$ is the "radius" of $\mathcal{D}_2(\lambda)$ and $h_3(\varphi)$ satisfies the estimates

$$|h_3| < 2\alpha\epsilon_2^4 k^{-2l+1}, \quad \left|\frac{\partial h_3}{\partial\varphi}\right| \le 4\alpha\epsilon_2^3 k^{1+\delta}. \tag{4.33}$$

(2) *The total length of $\mathcal{B}_3(\lambda)$ satisfies the estimate:*

$$L\left(\mathcal{B}_2 \setminus \mathcal{B}_3\right) < 4\pi k^{-4-2s_1-2s_2}. \tag{4.34}$$

(3) *The function $\varkappa_3(\varphi)$ can be extended as a holomorphic function of φ to the complex non-resonce set $\mathit{\Phi}_3$, estimates (4.33) being preserved.*

(4) *The curve $\mathcal{D}_3(\lambda)$ has a length which is asymptotically close to that of $\mathcal{D}_2(\lambda)$ in the following sense:*

$$L\Big(\mathcal{D}_3(\lambda)\Big) \underset{\lambda\to\infty}{=} L\Big(\mathcal{D}_2(\lambda)\Big)\Big(1 + O(k^{-4-2s_1-2s_2})\Big). \tag{4.35}$$

Proof. The proof is analogous to that of Lemma 3.42. Note only that, in Part 2, when proving convergence of the series for the resolvent $\big(H^{(3)}(\vec\varkappa_2(\varphi)) - z\big)^{-1}$, we use the estimate

$$\sup_{z\in C_3}\left\|\Big(\tilde H^{(2)}(\vec\varkappa_2(\varphi)) - z\Big)^{-1}\right\| < \frac{64}{\epsilon_2}, \tag{4.36}$$

analogous to (3.73), the operator $\tilde H^{(2)}$ acting in $L_2(Q_3)$. The estimate (4.36) follows from (4.26) for $\vec b = 2\pi p^{(2)}/N_2N_1 a$, $p^{(2)} \ne 0$, and estimate (3.80), which takes care about the case $p^{(2)} = 0$. As a side result of these considerations, we obtain an estimate analogous to (3.80) for the new resolvent $\big(H^{(3)}(\vec\varkappa_3(\varphi)) - z\big)^{-1}$. ∎

We define the non-resonance set, $\chi_3^*(\lambda)$ in $S_3(\lambda)$ by the formula analogous to (3.83):

$$\chi_3^*(\lambda) := \mathcal{K}_3\mathcal{D}_3(\lambda). \tag{4.37}$$

The following lemmas are analogous to Lemmas 3.43 and 3.44.

Lemma 4.25. *The set $\chi_3^*(\lambda)$ belongs to the $\big(2\alpha\epsilon_2^4 k^{-2l+1}\big)$-neighborhood of $\chi_3(\lambda)$ in K_3. If $t^{(3)} \in \chi_3^*(\lambda)$, then the operator $H_\alpha^{(3)}(t^{(3)})$ has a simple eigenvalue equal to λ. This eigenvalue is given by the perturbation series (4.8).*

Lemma 4.26. *Formula (4.37) establishes one-to-one correspondence between $\chi_3^*(\lambda)$ and $\mathcal{D}_3(\lambda)$.*

Remark 4.27. From geometric point of view this means that $\chi_3^*(\lambda)$ does not have self-intersections.

4.6. **Preparation for the Next Approximation.**

4.6.1. *Contracted set* $\mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right)$. For any $\vec{b}^{(2)}$ such that $b_0^{(2)} > \epsilon_1 k^{-2l+1-2\delta}$, we have constructed the set $\mathcal{O}^{(2)}(\vec{b}^{(2)}) \subset \mathit{\Phi}_2$, which surrounds zeros of the determinant $\det\Big(I+\tilde{A}_1\left(\vec{y}^{(1)}(\varphi)\right)\Big)$, $\vec{y}^{(1)}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b}^{(2)}$, see Remark 4.7. The set consists of disks of the radius $r^{(2)} = r^{(1)}k^{-2-4s_2-\delta}$, $r^{(1)} = k^{-4-6s_1-3\delta}$ and contains no more than $4c_0k^{2+2s_2}$ disks, see Remark 4.8. We proved that the new determinant $\det\Big(I + A_2\big(\vec{y}^{(2)}(\varphi)\big)\Big)$, $\vec{y}^{(2)}(\varphi) = \vec{\varkappa}_2(\varphi) + \vec{b}^{(2)}$, has the same number of zeros inside each $\Gamma^{(2)}(\vec{b}^{(2)})$ as $\det\Big(I + \tilde{A}_1\big(\vec{y}^{(1)}(\varphi)\big)\Big)$ (Lemma 5.11, Part 1) and both have no zeros in $\in \mathit{\Phi}_2 \setminus \mathcal{O}^{(2)}(\vec{b}^{(2)})$ (Corollaries 4.11, 4.15). To prepare the next approximation, we contract the set $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$ around the zeros of the new determinant $\det\Big(I + A_2\big(\vec{y}^{(2)}(\varphi)\big)\Big)$. We consider the disks with the radius $r^{(3)} = r^{(2)}k^{-2-4s_3-\delta}$ centered at each zero $\varphi_{j,0}^{(2)}$ of the determinant $\det\Big(I + A_2\big(\vec{y}^{(2)}(\varphi)\big)\Big)$. Obviously each of these disks is in $\mathcal{O}^{(2)}\big(\vec{b}^{(2)}\big)$, since $r^{(3)} << r^{(2)}$ and the distance between zeros of the old and new determinants is smaller than $r^{(2)}/2$ (Lemma 5.11, Part 2). Next, we take the union of these smaller disks and denote it by $\mathcal{O}_s^{(2)}\big(\vec{b}^{(2)}\big)$. We denote by $\Gamma_s^{(2)}\big(\vec{b}\big)$ a connected component of these new disks, which intersects with $\mathit{\Phi}_2$, and by $\gamma_s^{(2)}\big(\vec{b}^{(2)}\big)$ its boundary, $\gamma_s^{(2)} = \partial\Gamma_s^{(2)}$.

Lemma 4.28. (1) *The set* $\mathcal{O}_s^{(2)}\big(\vec{b}^{(2)}\big)$ *contains the same number of disks as* $\mathcal{O}^{(2)}\big(\vec{b}^{(2)}\big)$. *This number does not exceed* $4c_0k^{2+2s_2}$.

(2) *For any* $\varphi \in \mathit{\Phi}_2 \setminus \mathcal{O}_s^{(2)}\big(\vec{b}^{(2)}\big)$,

$$\left\|\Big(H^{(2)}\big(\vec{y}^{(2)}(\varphi)\big) - k^{2l}\Big)^{-1}\right\| \leq \frac{4}{\epsilon_1^2}\left(\frac{2r^{(2)}J^{(2)}}{r^{(3)}}\right)^{J^{(2)}}, \quad J^{(2)} = 4c_0k^{2+2s_2}, \tag{4.38}$$

$$\left\|\Big(H^{(2)}\big(\vec{y}^{(2)}(\varphi)\big) - k^{2l}\Big)^{-1}\right\|_1 \leq \frac{4J^{(2)}}{\epsilon_1^2}\left(\frac{2r^{(2)}J^{(2)}}{r^{(3)}}\right)^{J^{(2)}}. \tag{4.39}$$

The proof is analogous to that of Lemma 3.46, Lemma 4.14 being used instead of Lemma 3.31. Considering the expressions for $r^{(2)}, r^{(3)}, J^{(2)}, \epsilon_2$, we easily obtain that the right-hand part of (4.38) does not exceed $1/\epsilon_2$. Hence, the following corollary holds:

Corollary 4.29. *If* $b_0^{(2)} > \epsilon_1 k^{-2l+1-2\delta}$, *then*

$$\left\|\Big(H^{(2)}\big(\vec{y}^{(2)}(\varphi)\big) - k^{2l}\Big)^{-1}\right\| < \frac{1}{\epsilon_2}, \tag{4.40}$$

$$\left\|\Big(H^{(2)}\big(\vec{y}^{(2)}(\varphi)\big) - k^{2l}\Big)^{-1}\right\|_1 < \frac{4c_0k^{2+2s_2}}{\epsilon_2}. \tag{4.41}$$

4.6.2. *The set* $\mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right)$ *for small* $\vec{b}^{(2)}$. Everything we considered so far is valid if $b_0^{(2)} > \epsilon_1 k^{-2l+1-2\delta}$. However, in the next chapter and later, $b_0^{(2)}$ is taken smaller, since the reciprocal lattice is getting finer with each step. To prepare for this, let us consider $\vec{b}^{(2)}$ being close to a vertex of K_2: $0 < b_0^{(2)} \leq \epsilon_1 k^{-2l+1-2\delta}$. We show that for such $\vec{b}^{(2)}$, the resolvent $\left(H^{(2)}\big(\vec{y}^{(2)}(\varphi)\big) - k^{2l}\right)^{-1}$ has no more than two poles in Φ_2 and even its small neighborhood. We surround these poles by two contours $\gamma_s^{\pm\,(2)}$ and prove an estimate for the norm of $\left(H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) - k^{2l}\right)^{-1}$ when φ is outside these contours.

Let $\tilde{\Phi}_2$ be the $\frac{r^{(1)}}{2}$-neighborhood of Φ_2. It is easy to see that $\tilde{\Phi}_2 \subset \tilde{\Phi}_1$. Suppose $|\vec{b}^{(2)}| = b_0^{(2)}$, i.e., the closest vertex of K_2 for $\vec{b}^{(2)}$ is $(0,0)$. We consider the functions $\lambda^{(1)}(\vec{y}^{(1)}(\varphi))$ and $\lambda^{(2)}(\vec{y}^{(2)}(\varphi))$ defined by perturbation series (2.34) and (3.55) for $\varphi \in \tilde{\Phi}_2$. The convergence of these series can be easily justified. In fact, by Lemmas 2.11 and 3.42, $\vec{\varkappa}_1(\varphi)$ and $\vec{\varkappa}_2(\varphi)$ are holomorphic functions of φ in Φ_2. It is easy to show that the lemmas hold even in $\tilde{\Phi}_2$, since all estimates involved are preserved in $\tilde{\Phi}_2$. The perturbation series (2.34) and (3.55) converge for $\lambda^{(1)}(\vec{\varkappa}_1(\varphi))$ and $\lambda^{(2)}(\vec{\varkappa}_2(\varphi))$, respectively, because $\vec{\varkappa}_1(\varphi)$ and $\vec{\varkappa}_2(\varphi)$ are in small neighborhoods of $\mathcal{D}_0(\lambda)_{nonres}$ and $\mathcal{D}_1(\lambda)_{nonres}$ (see (2.75), (3.75)). Since the estimates involved are stable with respect to a change of $\vec{\varkappa}_{1,2}$ not exceeding $\epsilon_1 k^{-2l+1-\delta}$, the perturbation series for $\lambda^{(1)}\big(\vec{y}^{(1)}(\varphi)\big)$ and $\lambda^{(2)}\big(\vec{y}^{(2)}(\varphi)\big)$ also converge, both functions being holomorphic in $\tilde{\Phi}_2$. We base our further considerations on these perturbation series expansions. For $\vec{b}^{(2)}$ being close to a vertex $\vec{e}$ other than $(0,0)$, we take $\vec{y}^{(2)}(\varphi) = \vec{\varkappa}_2(\varphi) + \vec{b}^{(2)} - \vec{e}$.

From now on, we denote the solutions $\varphi^{\pm}_{\epsilon_0}$ of the equation $\lambda^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) = k^{2l} + \varepsilon_0$, introduced in Lemma 3.48, by $\varphi^{\pm(1)}_{\epsilon_0}$. As in the previous subsection, we choose $r^{(3)} = r^{(2)} k^{-2-4s_3-\delta}$ here.

Lemma 4.30. *If* $0 < b_0^{(2)} \leq \epsilon_1 k^{-2l+1-2\delta}$ *and* $|\epsilon_0| < b_0^{(2)} k^{2l-1-\delta} r^{(3)}$, *then the equation*

$$\lambda^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) = k^{2l} + \epsilon_0 \tag{4.42}$$

has no more than two solutions $\varphi^{\pm\,(2)}_{\epsilon_0}$ *in* $\tilde{\Phi}_2$. *For any* $\varphi^{\pm\,(2)}_{\epsilon_0}$ *there is* $\varphi_0^{\pm(1)} \in \tilde{\Phi}_1$ *such that*

$$\left|\varphi^{\pm(2)}_{\epsilon_0} - \varphi_0^{\pm(1)}\right| < r^{(3)}/4, \tag{4.43}$$

here and below $\varphi_0^{\pm(1)}$ *is* $\varphi^{\pm(1)}_{\epsilon_0}$ *for* $\epsilon_0 = 0$.

Proof. Let $|\vec{b}^{(2)}| = b_0^{(2)}$. First, we expand $\lambda^{(2)}(\vec{y}^{(2)}(\varphi))$ and $\lambda^{(1)}(\vec{y}^{(1)}(\varphi))$ near the point $\vec{b}^{(2)} = 0$ and consider that $\lambda^{(2)}(\vec{\varkappa}_2(\varphi)) = \lambda^{(1)}(\vec{\varkappa}_1(\varphi)) = \lambda$. Then, using (3.57), the estimates of the type (2.29), (3.27) and (3.60), we check that

$$\left|\lambda^{(2)}(\vec{y}^{(2)}(\varphi)) - \lambda^{(1)}(\vec{y}^{(1)}(\varphi))\right| < b_0^{(2)}\epsilon_1 \tag{4.44}$$

in $\tilde{\Phi}_2$ and even in its $r^{(3)}$-neighborhood, the neighborhood being a subset of $\tilde{\Phi}_1$. Suppose (4.42) holds for some $\varphi \in \tilde{\Phi}_2$. By (4.44), $\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) = k^{2l} + \varepsilon_0'$, $\varepsilon_0' = O\left(b_0^{(2)} k^{2l-1-\delta} r^{(3)}\right)$. Hence, φ satisfies conditions of Lemmas 3.48 and 3.49. Surrounding φ by a circle C of the radius $r^{(3)}/4$, we see that $|\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) - k^{2l} - \varepsilon_0'| = \frac{1}{2} l k^{2l-1} b_0^{(2)} r^{(3)}\,(1 + o(1)) > |\varepsilon_0'|$ on this circle. Applying Rouché's theorem, we obtain that there is a solution of $\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) = k^{2l}$ inside this circle. Thus, any solution of (4.42) is in the circle of the radius $r^{(3)}/4$ surrounding $\varphi_0^{\pm(1)}$, the point $\varphi_0^{\pm(1)}$ being in the $(r^{(3)}/4)$-neighborhood of $\tilde{\Phi}_2$. It remains to check that (4.42) has no more than two solutions and no more than one in a vicinity of each $\varphi_0^{\pm(1)}$. We construct the disk of the radius $r^{(3)}/4$ centered at $\varphi_0^{\pm(1)}$, described above, and note that $|\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) - k^{2l}| = \frac{1}{2}lk^{2l-1}b_0^{(2)}r^{(3)}(1+o(1))$ on the circle. Using (4.44) and Rouché's Theorem, we obtain that there is only one solution of (4.42) in the disk. If $\vec{b}$ is close to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. ∎

Lemma 4.31. *Suppose* $0 < b_0^{(2)} \leq \epsilon_1 k^{-2l+1-2\delta}$ *and* $\varphi \in \tilde{\Phi}_2$ *obeys the inequality analogous to (4.43):* $\left|\varphi - \varphi_0^{\pm(1)}\right| < r^{(3)}$. *Then,*

$$\frac{\partial}{\partial\varphi}\lambda^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) =_{k\to\infty} \pm 2lb_0^{(2)}k^{2l-1}\big(1+o(1)\big). \tag{4.45}$$

The proof is completely analogous to that of Lemma 3.49.

Definition 4.32. Let $\Gamma_s^{\pm(2)}\left(\vec{b}^{(2)}\right)$ be the open disks centered at $\varphi_0^{\pm(2)}$ with radius $r^{(3)}$; $\gamma_s^{\pm(2)}\left(\vec{b}^{(2)}\right)$ be their boundary circles and $\mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right) = \Gamma_s^{+(2)} \cup \Gamma_s^{-(2)}$.

Lemma 4.33. *For any* φ *in* $\Phi_2 \setminus \mathcal{O}_s^{(2)}(\vec{b}^{(2)})$,

$$|\lambda^{(2)}(\vec{y}^{(2)}(\varphi)) - k^{2l}| \geq b_0^{(2)}k^{2l-1-\delta}r^{(3)}. \tag{4.46}$$

The proof is analogous to that of Lemma 3.51.

Lemma 4.34. *For any* $\varphi \in \Phi_2 \setminus \mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right)$,

$$\left\|\left(H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) - k^{2l}\right)^{-1}\right\| < \frac{16}{b_0^{(2)}r^{(3)}k^{2l-1-\delta}}, \tag{4.47}$$

$$\left\|\left(H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) - k^{2l}\right)^{-1}\right\|_1 < \frac{64c_0k^{2+2s_2}}{b_0^{(2)}r^{(3)}k^{2l-1-\delta}}. \tag{4.48}$$

Corollary 4.35. *If* $\epsilon_2 k^{-2l+1-2\delta} < b_0^{(2)} \leq \epsilon_1 k^{-2l+1-2\delta}$, *and* $\varphi \in \Phi_2 \setminus \mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right)$, *then*

$$\left\|\left(H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) - k^{2l}\right)^{-1}\right\| < \frac{1}{\epsilon_2^2}, \tag{4.49}$$

$$\left\|\left(H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) - k^{2l}\right)^{-1}\right\|_1 < \frac{4c_0k^{2+2s_2}}{\epsilon_2^2}. \tag{4.50}$$

The corollary follows from the condition on $b_0^{(2)}$, for the formula $r^{(3)} = r^{(2)}k^{-2-4s_3-\delta} = k^{-8-6s_1-4s_2-4s_3-5\delta}$ and estimate (4.2). The proof of the lemma is analogous to that of Lemma 3.52.

5. The n-th Step of Approximation. Swiss Cheese Method.

5.1. **Perturbation Formulae.** On the n-th step, $n \geq 4$, we choose $s_n = 2s_{n-1}$ define the operator $H_\alpha^{(n)}$ by the formula:

$$H_\alpha^{(n)} = H^{(n-1)} + \alpha W_n, \quad (0 \leq \alpha \leq 1), \qquad W_n = \sum_{r=M_{n-1}+1}^{M_n} V_r,$$

where M_n is choosen in such a way that $2^{M_n} \approx k^{s_n}$. Obviously, the periods of W_n are $2^{M_n-1}(b_1,0)$ and $2^{M_n-1}(0,b_2)$. We write them in the form: $N_{n-1} \cdot ... \cdot N_1(a_1,0)$ and $N_{n-1} \cdot ... \cdot N_1(0,a_2)$, here $N_{n-1} = 2^{M_n - M_{n-1}}$, $\frac{1}{4}k^{s_n - s_{n-1}} < N_{n-1} < 4k^{s_n-s_{n-1}}$. Note that $\|W_n\|_\infty \leq \sum_{r=M_{n-1}+1}^{M_n} \|V_r\|_\infty < \exp(-k^{\eta s_{n-1}})$.

The n-th step is analogous to the third step up to replacement the indices 3 by n, 2 by $n-1$, the product $N_2 N_1$ by $N_{n-1} \cdot ... \cdot N_1$, etc.

We note that k, satisfying (4.2), obeys the analogous condition for the n-th step:

$$\hat{C}(1+s_{n-2})k^{2+2s_{n-2}} \ln k < k^{\eta s_{n-2}} \tag{5.1}$$

with the same constant $\hat{C}$. The inequality (5.1) can be obtained from (4.2) by induction. This is an important fact: it means that the lower bound for k does not grow with n, i.e., all steps hold uniformly in k for $k > k_*$, k_* being introduced by (4.2). Further we assume $k > k_*$.

The formulation of the geometric lemma for n-th step is the same as that for Step 3 up to a shift of indices, we skip it here for shortness. Note only that in the lemma we use the set $\chi_{n-1}^*(\lambda)$ to define $\chi_n(\lambda)$. In fact, we started with the definition of $\chi_1(\lambda)$ and then use it to define $\chi_1^*(\lambda)$ (Step 1). Considering $\chi_1^*(\lambda)$, we constructed $\chi_2(\lambda)$ (Lemma 3.5) and later used it to define $\chi_2^*(\lambda)$ (Section 3.4). Using $\chi_2^*(\lambda)$, we introduced $\chi_3(\lambda)$ (Lemma 4.1) and, then $\chi_3^*(\lambda)$ (Section 4.5). Thus, the process goes like $\chi_1 \to \chi_1^* \to \chi_2 \to \chi_2^* \to \chi_3 \to \chi_3^*$. Here we consider the set $\chi_{n-1}^*(\lambda)$ defined by (4.37) for $n=4$ and by (5.18) for $n>4$. [12] The estimate (4.5) for n-th step takes the form:

$$\frac{L\left(\mathcal{K}_n \chi_{n-1}^* \setminus \chi_n\right)}{L\left(\chi_{n-1}^*\right)} < k^{-S_n}, \qquad S_n = 2\sum_{i=1}^{n-1}(1+s_i). \tag{5.2}$$

It is easy to see that $S_n = 2(n-1) + (2^n-2)\,s_1$ and $S_n \approx 2^n s_1 \approx s_n$ for large n. The formulation of the main results (perturbation formulae) for n-th step is the same as for the third step up to the shift of indices. The formula for the resonance set Ω_{n-1} and non-resonance set χ_n are analogous to those for Ω_2, χ_2 (see (4.13)).

5.2. **Proof of Geometric Lemma.** Proofs of the first and second statements of Geometric Lemma are simple and completely analogous to those in the second step and third step. We skip them here for shortness. Proof of the third statement is completely analogous to that for the third step. For the purpose of rigor we present a short version of it.

[12] Strictly speaking we assume that there is a subset $\chi_{n-1}^*(\lambda)$ of the isoenergetic surface $S_{n-1}(\lambda)$ of $H^{(n-1)}$ such that perturbation series of the type (4.8), (4.9) converges for $t^{(n-1)} \in \chi_{n-1}^*(\lambda)$ and $\chi_{n-1}^*(\lambda)$ has properties described in Section 4.5 up to replacement of 3 by $n-1$. In particular, we assume that $\chi_{n-1}^*(\lambda) = \mathcal{K}_{n-1}\mathcal{D}_{n-1}(\lambda)$, where $\mathcal{D}_{n-1}(\lambda)$ satisfies the analog of Lemma 4.24 and that the analogs of Lemmas 4.25 and 4.26 hold too. Here, $\mathcal{K}_{n-1}$ is the parallel shift into K_{n-1}. Further in this section we describe the next set $\chi_n^*(\lambda)$ which has analogous properties.

In the second and third steps we defined the sets $\mathcal{O}^{(1)}\left(\vec{b}^{(1)}\right)$ and $\mathcal{O}^{(2)}\left(\vec{b}^{(2)}\right)$ (see Definition 3.24 and formula (4.15)). Considering (4.15), we define $\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$, $\vec{b}^{(n-1)}\in K_{n-1}$, $n\geq 4$, by the analogous formula

$$\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)=\cup_{p^{(n-2)}\in P^{(n-2)}}\mathcal{O}_s^{(n-2)}\left(\vec{b}^{(n-1)}+2\pi p^{(n-2)}/\hat{N}_{n-2}a\right), \tag{5.3}$$

here and below $\hat{N}_{n-2}\equiv N_{n-2}\cdot \dots \cdot N_1$,

$$P^{(m)}=\{p^{(m)}=\left(p_1^{(m)},p_2^{(m)}\right),\ 0\leq p_1^{(m)}\leq N_m-1, 0\leq p_2^{(m)}\leq N_m-1\};$$

the set $\mathcal{O}_s^{(m)}(\vec{b}^{(m)})$, $m\geq 1$, is a collection of disks of the radius $r^{(m+1)}=r^{(m)}k^{-2-4s_{m+1}-\delta}$ around zeros of the determinant $\det\Big(I+A_m\left(\vec{y}^{(m)}(\varphi)\right)\Big)$ in Φ_m, here $\vec{y}^{(m)}(\varphi)=\vec{\varkappa}_m(\varphi)+\vec{b}^{(m)}$, $\vec{\varkappa}_m(\varphi)$ being defined by Lemmas 2.11 ($m=1$), 3.42 ($m=2$), 4.24 ($m=3$) and 5.17 ($m\geq 4$). Let

$$\mathcal{O}_*^{(n-1)}=\cup_{p^{(n-1)}\in P^{(n-1)}\setminus\{0\}}\mathcal{O}^{(n-1)}\Big(2\pi p^{(n-1)}/\hat{N}_{n-1}a\Big),\quad \Phi_n=\Phi_{n-1}\setminus\mathcal{O}_*^{(n-1)}. \tag{5.4}$$

We denote by $\Gamma^{(n-1)}\left(\vec{b}^{(n-1)}\right)$ a connected component of $\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$. Note that the complex non-resonance set Φ_m is defined by the recurrent formula analogous to (4.17).

Lemma 5.1. *The set $\mathcal{O}_s^{(m)}\left(\vec{b}^{(m)}\right)$, $\vec{b}^{(m)}\in K_m$ contains no more than $4^{m-1}c_ok^{2+2s_m}$ disks.*

Corollary 5.2. *The set $\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$ contains no more than $4^{n-2}c_ok^{2+2s_{n-1}}$ disks.*

Corollary 5.3. *The set $\mathcal{O}_*^{(n-1)}$ contains no more than $4^{n-1}c_ok^{2+2s_n}$ disks.*

The lemma is proved by induction procedure. For $m=2$ it holds by Part 1 of Lemma 4.28 and by Definition 4.32. For $m>2$ it follows from Corollary 5.21 and Definition 5.26. Rigorously speaking, we assume for now that the lemma holds for $m\leq n-2$. At the end of this section, we obtain the analog of the lemma for the next step ($n\to n+1$), this will complete the induction procedure. Corollaries 5.2 and 5.3 are based on the fact that $P^{(n-1)}$ contains no more than $4k^{2(s_n-s_{n-1})}$ elements and a similar estimate holds for $P^{(n-2)}$.

Obviously, Φ_n has a structure of Swiss cheese, more and more holes of smaller and smaller radii appear at each step of approximation (Fig. 8). Note that the disks are more and more precisely "targeted" at each step of approximation. At the n-th step the disks of $\mathcal{O}_*^{(n-1)}$ are centered around the zeros of the determinants

$$\det\Big(I+A_{n-2}\left(\vec{\varkappa}_{n-2}(\varphi)+2\pi p^{(n-2)}/\hat{N}_{n-2}a+2\pi p^{(n-1)}/\hat{N}_{n-1}a\right)\Big),$$

where $p^{(n-2)}\in P^{(n-2)}$, $p^{(n-1)}\in P^{(n-1)}$, the corresponding operator $H^{(n-2)}$ being closer and closer to the operator H at each step of approximation. If $W_1=W_2=\dots=W_{n-2}=0$, then $\mathcal{O}_*^{(n-1)}$ is just a union of circles centered at quasi-intersections of the circle of radius k, centered at the origin, with circles of the same radius centered at points $2\pi j/a+2\pi p^{(1)}/N_1a+\dots+2\pi p^{(n-1)}/\hat{N}_{n-1}a$, which are points of the dual lattice corresponding to the periods $\hat{N}_{n-1}a_1,\hat{N}_{n-1}a_2$.

The following is a proof of third statement of Geometric Lemma for n-th step, namely, estimate (5.2). It is completely analogous to that in the third step, we only have to provide that condition (5.1) is sufficient for all necessary estimates. We start with an analog of Lemma 4.10: we shift indices $1 \to n-2, 2 \to n-1$ and replace the coefficient $4c_0$ in the estimate (4.20) by $4^{n-2}c_0$. The operator $\tilde{H}^{(n-2)}$ is $H^{(n-2)}$ extended from Q_{n-2} to Q_{n-1}. By analogy with (4.16), $\vec{y}^{(n-2)}(\varphi) = \vec{\varkappa}_{n-2}(\varphi) + \vec{b}^{(n-1)}$, the function being defined in Φ_{n-2}. Thus, we have the following lemma and a corollary:

Lemma 5.4. *If $\vec{b}^{(n-1)} \in K_{n-1}$ and $b_0^{(n-1)} > \epsilon_{n-2} k^{-2l+1-2\delta}$, then*

(1) *For any $\varphi \in \Phi_{n-2} \setminus \mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$:*

$$\left\| \left(\tilde{H}^{(n-2)}\left(\vec{y}^{(n-2)}(\varphi)\right) - k^{2l}\right)^{-1} \right\| < \frac{1}{\epsilon_{n-2}^2} \tag{5.5}$$

$$\left\| \left(\tilde{H}^{(n-2)}\left(\vec{y}^{(n-2)}(\varphi)\right) - k^{2l}\right)^{-1} \right\|_1 < \frac{4^{n-2}c_0 k^{2+2s_{n-1}}}{\epsilon_{n-2}^2} \tag{5.6}$$

(2) *The number of zeros of* $\det\Big(I+\tilde{A}_{n-2}\big(\vec{y}^{(n-2)}(\varphi)\big)\Big)$ *in Φ_{n-2} does not exceed $4^{n-2}c_0k^{2+2s_{n-1}}$.*

Corollary 5.5. *There are no zeros of* $\det\Big(I+\tilde{A}_{n-2}\big(\vec{y}^{(n-2)}(\varphi)\big)\Big)$ *in* $\Phi_{n-2} \setminus \mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$.

Proof. Proof is analogous to that of Lemma 4.10. We use Corollary 4.29 for $n=4$ and Corollary 5.23 for $n>4$, as well as Corollary 4.35 ($n=4$) and Corollary 5.29 ($n>4$).[13] Part 2 of the lemma follows from the definition of $\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$ and Corollary 5.2. ∎

Next, we repeat Lemmas 4.12 – 4.21 up to replacement of indices. Proofs are completely analogous to those for Step 3. In Lemma 4.21 we show that the total length of Ω_{n-1} does not exceed k^{-S_n}, here Ω_{n-1} is the analog of Ω_1, Ω_2, see (3.6), (4.13). Let $\vec{y}^{(n-1)}(\varphi) = \vec{\varkappa}_{n-1}(\varphi)+\vec{b}^{(n-1)}$, the function being defined in Φ_{n-1}.

Lemma 5.6. *If $\vec{b}^{(n-1)} \in K_{n-1}$ and $b_0^{(n-1)} > \epsilon_{n-2} k^{-2l+1-2\delta}$, then:*

(1) *The number of zeros of* $\det\Big(I+\tilde{A}_{n-2}\big(\vec{y}^{(n-1)}(\varphi)\big)\Big)$ *in any* $\Gamma^{(n-1)}\left(\vec{b}^{(n-1)}\right)$ *is the same as that of* $\det\Big(I+\tilde{A}_{n-2}\big(\vec{y}^{(n-2)}(\varphi)\big)\Big)$.

(2) *For any $\varphi \in \Phi_{n-1} \setminus \mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$, the following inequalities hold:*

$$\left\| \left(\tilde{H}^{(n-2)}\big(\vec{y}^{(n-1)}(\varphi)\big) - k^{2l}\right)^{-1} \right\| < \frac{2}{\epsilon_{n-2}^2}, \tag{5.7}$$

$$\left\| \left(\tilde{H}^{(n-2)}\big(\vec{y}^{(n-1)}(\varphi)\big) - k^{2l}\right)^{-1} \right\|_1 < \frac{2\cdot 4^{n-2}c_0 k^{2+2s_{n-1}}}{\epsilon_{n-2}^2}. \tag{5.8}$$

Corollary 5.7. *There are no zeros of* $\det\Big(I+\tilde{A}_{n-2}\big(\vec{y}^{(n-1)}(\varphi)\big)\Big)$ *in* $\Phi_{n-1} \setminus \mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$.

[13] Rigorously speaking, we have to assume that the lemma holds for n-th step. For $(n+1)$-th step it follows from Corollaries 5.23 and 5.29.

Proof. The proof is analogous to that of Lemma 4.12. In Part 1 we use the estimate $\left|\vec{y}^{(n-1)}(\varphi) - \vec{y}^{(n-2)}(\varphi)\right| < 2\alpha\epsilon_{n-2}^4 k^{-2l+1}$, which follows from assumed properties of χ_{n-1}^*, see (4.33) with $3 \to n-1$. We apply (5.6) instead of (4.20) and produce an analog of (4.25), the estimate $16 \cdot 4^{n-2} c_0 \epsilon_{n-2}^2 k^{2+2s_{n-1}} < 1$ following from (5.1). In Part 2 we use the obvious estimate $\epsilon_{n-2}^2 < 1/4$. ∎

Lemma 5.8. *If $\vec{b}^{(n-1)} \in K_{n-1}$ and $b_0^{(n-1)} > \epsilon_{n-2} k^{-2l+1-2\delta}$, then the following estimates hold for any $\varphi \in \Phi_{n-1} \setminus \mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$:*

$$\left\| \left(H^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) - k^{2l} \right)^{-1} \right\| < \frac{4}{\epsilon_{n-2}^2}, \tag{5.9}$$

$$\left\| \left(H^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) - k^{2l} \right)^{-1} \right\|_1 < \frac{4^{n-1} c_0 k^{2+2s_{n-1}}}{\epsilon_{n-2}^2}. \tag{5.10}$$

Corollary 5.9. *There are no zeros of* $\det\left(I + A_{n-1}\left(\vec{y}^{(n-1)}(\varphi)\right)\right)$ *in* $\Phi_{n-1} \setminus \mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$.

Lemma 5.10. *If $\vec{b}^{(n-1)} \in K_{n-1}$ and $b_0^{(n-1)} > \epsilon_{n-2} k^{-2l+1-2\delta}$, then the number of zeros of* $\det\Big(I + A_{n-1}\big(\vec{y}^{(n-1)}(\varphi)\big)\Big)$ *in any* $\Gamma^{(n-1)}\left(\vec{b}^{(n-1)}\right)$ *is the same as that of* $\det\Big(I + \tilde{A}_{n-2}\big(\vec{y}^{(n-1)}(\varphi)\big)\Big)$.

Lemma 5.11.
(1) *If $\vec{b}^{(n-1)} \in K_{n-1}$ and $b_0^{(n-1)} > \epsilon_{n-2} k^{-2l+1-2\delta}$, then the number of zeros of* $\det\Big(I + A_{n-1}\big(\vec{y}^{(n-1)}(\varphi)\big)\Big)$ *in any* $\Gamma^{(n-1)}(\vec{b}^{(n-1)})$ *is the same as that of* $\det\Big(I + \tilde{A}_{n-2}\big(\vec{y}^{(n-2)}(\varphi)\big)\Big)$.
(2) *The distance from a zero of* $\det\Big(I + A_{n-1}\big(\vec{y}^{(n-1)}(\varphi)\big)\Big)$ *to the nearest zero of* $\det\Big(I + \tilde{A}_{n-2}\big(\vec{y}^{(n-2)}(\varphi)\big)\Big)$ *does not exceed $r^{(n-1)}/2$.*

Corollary 5.12. *The number of zeros of* $\det\left(I + A_{n-1}\left(\vec{y}^{(n-1)}(\varphi)\right)\right)$ *does not exceed* $4^{n-2} c_0 k^{2+2s_{n-1}}$.

The corollary is the combination of the first part of the lemma with the second part of Lemma 5.4. The following is an analog of Definition 4.6.

Definition 5.13. A complex resonance set $\omega_{n-1}^*(\lambda)$ is the set of $\varphi \in \Phi_{n-1}$ satisfying

$$\det\left(\frac{H^{(n-1)}\big(\vec{y}^{(n-1)}(\varphi)\big) - k^{2l} - \epsilon}{H_0\big(\vec{y}^{(n-1)}(\varphi)\big) + k^{2l}} \right) = 0, \quad \vec{y}^{(n-1)}(\varphi) = \vec{\varkappa}_{n-1}(\varphi) + \vec{b}^{(n-1)}, \tag{5.11}$$

for some $\vec{b}^{(n-1)} = 2\pi p^{(n-1)}/\hat{N}_{n-1}a$, $p^{(n-1)} \in P^{(n-1)} \setminus \{0\}$, and $|\epsilon| < \epsilon_{n-1}$. We denote by $\omega_{n-1}(\lambda)$ the set of $\varphi \in \Theta_{n-1}$ corresponding to $\Omega_{n-1}(\lambda)$. Considering as in the proof of Lemma 3.20, it is easy to show that $\omega_{n-1} = \omega_{n-1}^* \cap \Theta_{n-1}$.

Lemma 5.14. *If $\vec{b}^{(n-1)} \in K_{n-1}$ and $b_0^{(n-1)} \geq \epsilon_{n-2} k^{-2l+1-2\delta}$, $|\epsilon| < \epsilon_{n-2}^4$, then*

(1) *The determinant of* $\dfrac{H^{(n-1)}\big(\vec{y}^{(n-1)}(\varphi)\big)-k^{2l}-\epsilon}{H_0\big(\vec{y}^{(n-1)}(\varphi)\big)+k^{2l}}$ *has no zeros in* $\Phi_{n-1}\setminus \mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$.

(2) *The number of zeros of* $\det\left(\dfrac{H^{(n-1)}\big(\vec{y}^{(n-1)}(\varphi)\big)-k^{2l}-\epsilon}{H_0\big(\vec{y}^{(n-1)}(\varphi)\big)+k^{2l}}\right)$ *in any* $\Gamma^{(n-1)}\left(\vec{b}^{(n-1)}\right)$ *is the same as that of* $\det\left(\dfrac{H^{(n-1)}\big(\vec{y}^{(n-1)}(\varphi)\big)-k^{2l}}{H_0\big(\vec{y}^{(n-1)}(\varphi)\big)+k^{2l}}\right)$.

Corollary 5.15. *The following relation holds:* $\omega^*_{n-1}\subset \mathcal{O}^{(n-1)}_*$.

Lemma 5.16. *The estimate (5.2) holds.*

Proof. By Corollary 5.15, $\omega^*_{n-1}\subset \mathcal{O}^{(n-1)}_*$. Hence, $\omega_{n-1}\subset \mathcal{O}^{(n-1)}_*\cap\Theta_{n-1}$. By Corollary 5.3, $\mathcal{O}^{(n-1)}_*$ is formed by no more than $4^{n-1}c_0k^{2+2s_n}$ disks of the radius $r^{(n-1)}=r^{(n-2)}k^{-2-4s_{n-1}-\delta}$. Hence, the total length of ω_{n-1} does not exceed $2\cdot 4^{n-1}c_0r^{(n-2)}k^{-\delta}$. Considering the recurrent formula for $r^{(m+1)}$, $r^{(m+1)}=r^{(m)}k^{-2-4s_{m+1}-\delta}$, we easily get that $4^nc_0r^{(n-2)}<k^{-S_n}$, $n>1$. Therefore, the length of Ω_{n-1} does not exceed $\frac{k}{2}k^{-S_n}$. Considering that $\chi^*_{n-1}(\lambda)$ has a length $2\pi k\big(1+o(1)\big)$, we obtain (5.2). ∎

5.3. **Nonresonant part of the isoenergetic set of** $H^{(n)}_\alpha$**.** Now we construct a nonresonance subset $\chi^*_n(\lambda)$ of the isoenergetic surface $S_n(\lambda)$ of $H^{(n)}_\alpha$ in K_n, $S_n(\lambda)\subset K_n$. It corresponds to nonresonance eigenvalues given by perturbation series. The sets $\chi^*_1(\lambda)$, $\chi^*_2(\lambda)$, $\chi^*_3(\lambda)$ are defined in the previous steps as well as the non-resonance sets $\chi_1(\lambda)$, $\chi_2(\lambda)$, $\chi_3(\lambda)$. Let us recall that we started with the definition of $\chi_1(\lambda)$ and then use it to define $\chi^*_1(\lambda)$. Considering $\chi^*_1(\lambda)$, we constructed $\chi_2(\lambda)$ (Step 2). Next, we defined $\chi^*_2(\lambda)$, using $\chi_2(\lambda)$. Using $\chi^*_2(\lambda)$, we introduced $\chi_3(\lambda)$ and, then $\chi^*_3(\lambda)$. Thus, the process looks like $\chi_1\to\chi^*_1\to\chi_2\to\chi^*_2\to\chi_3\to\chi^*_3$. The geometric lemma in this section gives us χ_4 and every next χ_n if χ^*_{n-1} is defined. To ensure the reccurent procedure we show now how to define $\chi^*_n(\lambda)$ using $\chi_n(\lambda)$.

As in previous sections, we start with the definition of $\mathcal{D}_{n-1}(\lambda)_{nonres}$. First note that χ^*_{n-1} is defined by the formula $\chi^*_{n-1}=\mathcal{K}_{n-1}\mathcal{D}_{n-1}$, where $\mathcal{D}_{n-1}$ is a distorted circle with holes, and the shift gives one-to-one correspondence between χ^*_{n-1} and $\mathcal{D}_{n-1}$ (it is proven for $n=2,3,4$ and we we assume this here for $n>4$ to provide an induction procedure). By analogy with previous steps, we define $\mathcal{D}_{n-1}(\lambda)_{nonres}$ as the preimage of χ_n in $\mathcal{D}_{n-1}(\lambda)$:

$$\mathcal{D}_{n-1}(\lambda)_{nonres}=\{\vec{\varkappa}\in\mathcal{D}_{n-1}(\lambda):\mathcal{K}_n\vec{\varkappa}\in\chi_n\},\tag{5.12}$$

$\mathcal{D}_{n-1}(\lambda)_{nonres}$ being defined since $\chi_n\subset\chi^*_{n-1}$. We define $\mathcal{B}_n$ as the set of directions corresponding to $\mathcal{D}_{n-1}(\lambda)_{nonres}$:

$$\mathcal{B}_n(\lambda)=\{\vec{\nu}\in S_1:\varkappa_{n-1}(\lambda,\vec{\nu})\vec{\nu}\in\mathcal{D}_{n-1}(\lambda)_{nonres}\},$$

where $\varkappa_{n-1}(\lambda,\vec{\nu})$ is the "radius" of the distorted circle $\mathcal{D}_{n-1}(\lambda)$ defined by the analog of Lemma 4.24 for step $n-1$. Note that $\mathcal{B}_n$ is a unit circle with holes centered at the origin and $\mathcal{B}_n(\lambda)\subset\mathcal{B}_{n-1}(\lambda)$. We denote by Θ_n the set of angles φ in polar coordinates, corresponding to $\mathcal{B}_n$:

$$\Theta_n=\{\varphi\in[0,2\pi):\ (\cos\varphi,\sin\varphi)\in\mathcal{B}_n\},\qquad \Theta_n\subset\Theta_{n-1}.\tag{5.13}$$

We define $\mathcal{D}_n(\lambda)$ as a level set for $\lambda^{(n)}(\alpha,\vec{\varkappa})$ in this neighborhood:

$$\mathcal{D}_n(\lambda)=\left\{\vec{\varkappa}=\varkappa\vec{\nu}:\vec{\nu}\in\mathcal{B}_n,\left|\varkappa-\varkappa_{n-1}(\lambda,\vec{\nu})\right|<\epsilon_{n-1}k^{-2l+1-\delta},\lambda^{(n)}(\alpha,\vec{\varkappa})=\lambda\right\}.$$

Considering as in the previous step, we prove the analogs of Lemmas 4.23 and 4.24. For shortness, we provide here only the second lemma. By analogy with previous sections, we shorten notations here: $\varkappa_{n-1}(\varphi)\equiv\varkappa_{n-1}(\lambda,\vec{\nu})$, $\vec{\varkappa}_{n-1}(\varphi)\equiv\varkappa_{n-1}(\lambda,\vec{\nu})\vec{\nu}$.

Lemma 5.17. (1) *The set $\mathcal{D}_n(\lambda)$ is a distorted circle with holes: it can be described by the formula:*

$$\mathcal{D}_n(\lambda)=\left\{\vec{\varkappa}\in\mathbb{R}^2:\vec{\varkappa}=\vec{\varkappa}_n(\varphi),\quad\varphi\in\Theta_n(\lambda)\right\},\tag{5.14}$$

where $\varkappa_n(\varphi)=\varkappa_{n-1}(\varphi)+h_n(\varphi)$, and $h_n(\varphi)$ satisfies the estimates

$$|h_n|<2\alpha\epsilon_{n-1}^4k^{-2l+1},\qquad\left|\frac{\partial h_n}{\partial\varphi}\right|\le4\alpha\epsilon_{n-1}^3k^{1+\delta}.\tag{5.15}$$

(2) *The total length of $\mathcal{B}_n(\lambda)$ satisfies the estimate:*

$$L\left(\mathcal{B}_{n-1}\setminus\mathcal{B}_n\right)<4\pi k^{-S_n}.\tag{5.16}$$

(3) *Function $\varkappa_n(\varphi)$ can be extended as a holomorphic function of φ to the complex non-resonance set Φ_n, estimates (5.15) being preserved.*

(4) *The curve $\mathcal{D}_n(\lambda)$ has a length which is asymptotically close to that of $\mathcal{D}_{n-1}(\lambda)$ in the following sense:*

$$L\Big(\mathcal{D}_n(\lambda)\Big)\underset{\lambda\to\infty}{=}L\Big(\mathcal{D}_{n-1}(\lambda)\Big)\Big(1+O(k^{-S_n})\Big).\tag{5.17}$$

Now define the nonresonance set, $\chi_n^*(\lambda)$ in $S_n(\lambda)$ by the formula analogous to (4.37). Indeed,

$$\chi_n^*(\lambda):=\mathcal{K}_n\mathcal{D}_n(\lambda).\tag{5.18}$$

The following lemmas are analogous to Lemmas 4.25 and 4.26.

Lemma 5.18. *The set $\chi_n^*(\lambda)$ belongs to the $\left(2\alpha\epsilon_{n-1}^4k^{-2l+1}\right)$-neighborhood of $\chi_n(\lambda)$ in K_n. If $t^{(n)}\in\chi_n^*(\lambda)$, then the operator $H_\alpha^{(n)}(t^{(n)})$ has a simple eigenvalue equal to λ. This eigenvalue is given by the perturbation series analogous to (4.8).*

Lemma 5.19. *Formula (5.18) establishes one-to-one correspondence between $\chi_n^*(\lambda)$ and $\mathcal{D}_n(\lambda)$.*

5.4. **Preparation for the Next Approximation.**

5.4.1. *Contracted set $\mathcal{O}_s^{(n-1)}\big(\vec{b}^{(n-1)}\big)$.* We constructed the set $\mathcal{O}^{(n-1)}\big(\vec{b}^{(n-1)}\big)$, which surrounds zeros of the determinants

$$\det\Big(I+A_{n-2}\left(\vec{\varkappa}_{n-2}(\varphi)+\vec{b}^{(n-1)}+2\pi p^{(n-2)}/\hat{N}_{n-2}a\right)\Big),\quad p^{(n-2)}\in P^{(n-2)},$$

or, shorter, zeros of the determinant $\det\Big(I+\tilde{A}_{n-2}\left(\vec{y}^{(n-2)}(\varphi)\right)\Big)$, here $\vec{y}^{(n-2)}(\varphi)=\vec{\varkappa}_{n-2}(\varphi)+\vec{b}^{(n-1)}$, $\tilde{A}_{n-2}$ corresponds to $\tilde{H}^{(n-2)}$. The set consists of disks of the radius $r^{(n-1)}$ and contains no more than $4^{n-2}c_0k^{2+2s_{n-1}}$ disks. We proved that the new determinant $\det\Big(I+A_{n-1}\big(\vec{y}^{(n-1)}(\varphi)\big)\Big)$, $\vec{y}^{(n-1)}(\varphi)=\vec{\varkappa}_{n-1}(\varphi)+\vec{b}^{(n-1)}$, has the same number of zeros as the old one inside each $\Gamma^{(n-1)}\big(\vec{b}^{(n-1)}\big)$

(Lemma 5.11); and both have no zeros in $\in \mathit{\Phi}_{n-1} \setminus \mathcal{O}^{(n-1)}\big(\vec b^{(n-1)}\big)$ (Corollaries 5.5, 5.9). To prepare the next approximation, we contract the set $\mathcal{O}_s^{(n-1)}\big(\vec b^{(n-1)}\big)$ around the zeros of the new determinant. We consider the disks with the radius $r^{(n)} = r^{(n-1)}k^{-2-4s_n-\delta}$ centered at zeros of the new determinant. Obviously each of these disks is in $\mathcal{O}^{(n-1)}\big(\vec b^{(n-1)}\big)$, since $r^{(n)} << r^{(n-1)}$ and the distance between zeros of the old and new determinants is smaller than $r^{(n-1)}/2$ (Lemma 5.11). Next, we take the union of these smaller disks and denote it by $\mathcal{O}_s^{(n-1)}\big(\vec b^{(n-1)}\big)$. Obviously, the following lemma holds.

Lemma 5.20. *The set $\mathcal{O}_s^{(n-1)}\big(\vec b^{(n-1)}\big)$ belongs to $\mathcal{O}^{(n-1)}\big(\vec b^{(n-1)}\big)$ and consists of the same number of disks as $\mathcal{O}^{(n-1)}\big(\vec b^{(n-1)}\big)$.*

Corollary 5.21. *The set $\mathcal{O}_s^{(n-1)}\big(\vec b^{(n-1)}\big)$ contains no more than $4^{n-2}c_0k^{2+2s_{n-1}}$ disks.*

The corollary follows from the lemma and Corollary 5.2.

Lemma 5.22. *For any $\varphi \in \mathit{\Phi}_{n-1} \setminus \mathcal{O}_s^{(n-1)}\big(\vec b^{(n-1)}\big)$,*

$$\left\|\left(H^{(n-1)}\big(\vec y^{(n-1)}(\varphi)\big) - k^{2l}\right)^{-1}\right\| \leq \frac{4}{\epsilon_{n-2}^2}\left(\frac{2r^{(n-1)}J^{(n-1)}}{r^{(n)}}\right)^{J^{(n-1)}}, \tag{5.19}$$

$$\left\|\left(H^{(n-1)}\big(\vec y^{(n-1)}(\varphi)\big) - k^{2l}\right)^{-1}\right\|_1 \leq \frac{4J^{(n-1)}}{\epsilon_{n-2}^2}\left(\frac{2r^{(n-1)}J^{(n-1)}}{r^{(n)}}\right)^{J^{(n-1)}}, \tag{5.20}$$

$$J^{(n-1)} = 4^{n-2}c_0k^{2+2s_{n-1}}, \quad \epsilon_{n-2} = e^{-\frac{1}{4}k^{\eta s_{n-2}}}.$$

The proof is analogous to that of Lemma 3.46. Using the recurrent formula for $r^{(n)}$, condition (5.1) and a simple induction procedure, we easily obtain that the right-hand part of (5.19) does not exceed $1/\epsilon_{n-1}$. Hence, the following is true:

Corollary 5.23.

$$\left\|\left(H^{(n-1)}\big(\vec y^{(n-1)}(\varphi)\big) - k^{2l}\right)^{-1}\right\| < \frac{1}{\epsilon_{n-1}} \tag{5.21}$$

$$\left\|\left(H^{(n-1)}\big(\vec y^{(n-1)}(\varphi)\big) - k^{2l}\right)^{-1}\right\|_1 \leq \frac{J^{(n-1)}}{\epsilon_{n-1}}. \tag{5.22}$$

5.4.2. *The set $\mathcal{O}_s^{(n-1)}\big(\vec b^{(n-1)}\big)$ for small $\vec b^{(n-1)}$.* Everything we considered so far is valid if $b_0^{(n-1)} > \epsilon_{n-2}k^{-2l+1-2\delta}$. However, at each step $b_0^{(n-1)}$ is taken smaller, since the reciprocal lattice is getting finer. For this reason we now consider $\vec b^{(n-1)}$ being close to a vertex of K_{n-1}: $0 < b_0^{(n-1)} \leq \epsilon_{n-2}k^{-2l+1-2\delta}$. As in previous steps, we show that for such $\vec b^{(n-1)}$ the resolvent $\left(H^{(n-1)}\left(\vec y^{(n-1)}(\varphi)\right) - k^{2l}\right)^{-1}$ has no more than two poles in $\mathit{\Phi}_{n-1}$ and even its small neighborhood. We surround these poles by two contours $\gamma_s^{\pm\ (n-1)}$ and obtain an estimate for the norm of $\left(H^{(n-1)}\left(\vec y^{(n-1)}(\varphi)\right) - k^{2l}\right)^{-1}$ when φ is outside these contours.[14]

[14]If $|\vec b^{(n-1)}| = b_0^{(n-1)}$, then $\vec y^{(n-1)}(\varphi) = \vec\varkappa_{n-1}(\varphi) + \vec b^{(n-1)}$. For $\vec b^{(n-1)}$ being close to a vertex $\vec e$ of K_{n-1} other than $(0,0)$, we take $\vec y^{(n-1)}(\varphi) = \vec\varkappa_{n-1}(\varphi) + \vec b^{(n-1)} - \vec e$.

By analogy with Step 3, $\tilde{\Phi}_{n-1}$ is the $\frac{r^{(n-2)}}{2}$-neighborhood of Φ_{n-1}. Obviously, $\tilde{\Phi}_{n-1} \subset \tilde{\Phi}_{n-2}$. We assume that the analog of Lemma 4.30 holds when we replace the index (2) by $(n-2)$. Namely, we assume that for any $|\epsilon_0| < b_0^{(n-2)} k^{2l-1-\delta} r^{(n-1)}$ the equation $\lambda^{(n-2)}(\vec{y}^{(n-2)}(\varphi)) = k^{2l} + \epsilon_0$ has no more than two solutions $\varphi_{\epsilon_0}^{\pm(n-2)}$ in $\tilde{\Phi}_{n-2}$ and the inequality $|\varphi_{\epsilon_0}^{\pm(n-2)} - \varphi_0^{\pm(n-3)}| < r^{(n-1)}/4$ holds, here $\varphi_0^{\pm(n-3)}$ is $\varphi_{\epsilon_0}^{\pm(n-3)}$ for $\epsilon_0 = 0$. We also assume that the estimate analogous to (4.45) holds when we replace (2) by $(n-2)$. Let us formulate the corresponding results for the next step of approximation $(n-2 \to n-1)$. The formulations and proofs of the following four lemmas and are analogous to those for Lemmas 4.30 – 4.34. Definition 5.26 is analogous to Definition 4.32, and Corollary 5.29 is an analog of Corollary 4.35.

Lemma 5.24. *If* $0 < b_0^{(n-1)} \leq \epsilon_{n-2} k^{-2l+1-2\delta}$ *and* $|\epsilon_0| < b_0^{(n-1)} k^{2l-1-\delta} r^{(n)}$, *then the equation*

$$\lambda^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) = k^{2l} + \epsilon_0 \tag{5.23}$$

has no more than two solutions, $\varphi_{\epsilon_0}^{\pm\ (n-1)}$, *in* $\tilde{\Phi}_{n-1}$. *For any* $\varphi_{\epsilon_0}^{\pm\ (n-1)}$ *there is* $\varphi_0^{\pm\ (n-2)} \in \tilde{\Phi}_{n-2}$ *such that*

$$|\varphi_{\epsilon_0}^{\pm(n-1)} - \varphi_0^{\pm(n-2)}| < r^{(n)}/4. \tag{5.24}$$

Lemma 5.25. *Suppose* $0 < b_0^{(n-1)} \leq \epsilon_{n-2} k^{-2l+1-2\delta}$ *and* $\varphi \in \tilde{\Phi}_{n-1}$ *obeys the inequality analogous to (5.24):* $|\varphi - \varphi_0^{\pm(n-2)}| < r^{(n)}$. *Then,*

$$\frac{\partial}{\partial\varphi}\lambda^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) =_{k\to\infty} \pm 2l b_0^{(n-1)} k^{2l-1}\big(1 + o(1)\big). \tag{5.25}$$

Definition 5.26. Let $\Gamma_s^{\pm(n-1)}\big(\vec{b}^{(n-1)}\big)$ be the open disks centered at $\varphi_0^{\pm(n-2)}$ with radius $r^{(n)}$; $\gamma_s^{\pm(n-1)}\big(\vec{b}^{(n-1)}\big)$ be their boundary circles and $\mathcal{O}_s^{(n-1)}\big(\vec{b}^{(n-1)}\big) = \Gamma_s^{+(n-1)} \cup \Gamma_s^{-(n-1)}$.

Lemma 5.27. *For any* φ *in* $\Phi_{n-1} \setminus \mathcal{O}_s^{(n-1)}(\vec{b}^{(n-1)})$,

$$\left|\lambda^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) - k^{2l}\right| \geq b_0^{(n-1)} k^{2l-1-\delta} r^{(n)}. \tag{5.26}$$

Lemma 5.28. *For any* $\varphi \in \Phi_{n-1} \setminus \mathcal{O}_s^{(n-1)}\big(\vec{b}^{(n-1)}\big)$,

$$\left\|\left(H^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) - k^{2l}\right)^{-1}\right\| < \frac{16}{b_0^{(n-1)} r^{(n)} k^{2l-1-\delta}}, \tag{5.27}$$

$$\left\|\left(H^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) - k^{2l}\right)^{-1}\right\|_1 < \frac{16 \cdot 4^{n-2} c_0 k^{2+2s_{n-1}}}{b_0^{(n-1)} r^{(n)} k^{2l-1-\delta}}. \tag{5.28}$$

Corollary 5.29. *If* $\epsilon_{n-1} k^{-2l+1-2\delta} < b_0^{(n-1)} \leq \epsilon_{n-2} k^{-2l+1-2\delta}$, *and* φ *belongs to* $\Phi_{n-1} \setminus \mathcal{O}_s^{(n-1)}\left(\vec{b}^{(n-1)}\right)$, *then*

$$\left\|\left(H^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) - k^{2l}\right)^{-1}\right\| < \frac{1}{\epsilon_{n-1}^2}, \tag{5.29}$$

$$\left\|\left(H^{(n-1)}\left(\vec{y}^{(n-1)}(\varphi)\right) - k^{2l}\right)^{-1}\right\|_1 < \frac{4^{n-2} c_0 k^{2+2s_{n-1}}}{\epsilon_{n-1}^2}. \tag{5.30}$$

The corollary follows from the condition on $b_0^{(n-1)}$, estimate (5.1) and the formula $r^{(n)} = r^{(n-1)} k^{-2-4s_n-\delta}$.

6. Limit-Isoenergetic Set and Eigenfunctions

6.1. **Limit-Isoenergetic Set and Proof of the Bethe-Sommerfeld Conjecture.** At every step n we constructed a set $\mathcal{B}_n(\lambda)$, $\mathcal{B}_n(\lambda) \subset \mathcal{B}_{n-1}(\lambda) \subset S_1(\lambda)$, and a function $\varkappa_n(\lambda, \vec{\nu})$, $\vec{\nu} \in \mathcal{B}_n(\lambda)$, with the following properties. The set $\mathcal{D}_n(\lambda)$ of vectors $\vec{\varkappa} = \varkappa_n(\lambda, \vec{\nu})\vec{\nu}$, $\vec{\nu} \in \mathcal{B}_n(\lambda)$, is a slightly distorted circle with holes, see Fig.1, Fig.2, formula (1.13) and Lemmas 2.11, 3.42, 4.24, 5.17. For any $\vec{\varkappa}_n(\lambda, \vec{\nu}) \in \mathcal{D}_n(\lambda)$ there is a single eigenvalue of $H^{(n)}(\vec{\varkappa}_n)$ equal to λ and given by a perturbation series. [15] Let $\mathcal{B}_\infty(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{B}_n(\lambda)$. Since $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ for every n, $\mathcal{B}_\infty(\lambda)$ is a unit circle with infinite number of holes, more and more holes of smaller and smaller size appearing at each step.

Lemma 6.1. *The length of $\mathcal{B}_\infty(\lambda)$ satisfies estimate (1.9) with $\gamma_3 = \delta/2$.*

Proof. Using (2.44), (3.64), (4.34) and (5.16) and consideirng that $S_n \approx 2^n s_1$, we easily conclude that $L(\mathcal{B}_n) = \left(1 + O(k^{-\delta/2})\right)$, $k = \lambda^{1/2l}$ uniformly in n. Since $\mathcal{B}_n$ is a decreasing sequence of sets, (1.9) holds. ∎

Let us consider $\varkappa_\infty(\lambda, \vec{\nu}) = \lim_{n \to \infty} \varkappa_n(\lambda, \vec{\nu})$, $\quad \vec{\nu} \in \mathcal{B}_\infty(\lambda)$.

Lemma 6.2. *The limit $\varkappa_\infty(\lambda, \vec{\nu})$ exists for any $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$ and the following estimates hold when $n \geq 1$:*

$$|\varkappa_\infty(\lambda, \vec{\nu}) - \varkappa_n(\lambda, \vec{\nu})| < 4\epsilon_n^4 k^{-2l+1}, \quad \epsilon_n = \exp(-\frac{1}{4}k^{\eta s_n}), \quad s_n = 2^{n-1}s_1. \tag{6.1}$$

Corollary 6.3. *For every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$ estimate (1.10) holds, where $\gamma_4 = (4l - 3 - 4s_1 - 3\delta)/2l > 0$.*

The lemma easily follows from the estimates (3.63), (4.33) and (5.15). To obtain corollary we use (2.43) and take into account that $\gamma_0 = 2l - 2 - 4s_1 - 2\delta$.

Estimates (3.63), (4.33) and (5.15) justify convergence of the series $\sum_{m=1}^{\infty} \frac{\partial h_n}{\partial \varphi}$, and hence, of the sequence $\frac{\partial \varkappa_n}{\partial \varphi}$. We denote the limit of this sequence by $\frac{\partial \varkappa_\infty}{\partial \varphi}$.

Lemma 6.4. *The estimate (1.17) with $\gamma_5 = (4l - 5 - 8s_1 - 4\delta)/2l > 0$, holds for any $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$.*

We define $\mathcal{D}_\infty(\lambda)$ by (1.8). Clearly, $\mathcal{D}_\infty(\lambda)$ is a slightly distorted circle of radius k with infinite number of holes. We can assign a tangent vector $\frac{\partial \varkappa}{\partial \varphi}\vec{\nu} + \varkappa\vec{\mu}$, $\vec{\mu} = (-\sin\varphi, \cos\varphi)$ to the curve $\mathcal{D}_\infty(\lambda)$, this tangent vector being the limit of corresponding tangent vectors for curves $\mathcal{D}_n(\lambda)$ at points $\vec{\varkappa}_n(\lambda, \vec{\nu})$ as $n \to \infty$.

Remark 6.5. We easily see from (6.1), that any $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$ belongs to the $\left(4\epsilon_n^4 k^{-2l+1}\right)$-neighborhood of $\mathcal{D}_n(\lambda)$. Applying perturbation formulae for n-th step, we easily obtain that there is an eigenvalue $\lambda^{(n)}(\vec{\varkappa})$ of $H^{(n)}(\vec{\varkappa})$ satisfying the estimate $\lambda^{(n)}(\vec{\varkappa}) = \lambda + \delta_n$, $\delta_n = O\left(\epsilon_n^4\right)$, the eigenvalue $\lambda^{(n)}(\vec{\varkappa})$ being given by a perturbation series of the type (3.55). Hence, for every $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$ there is a limit:

$$\lim_{n \to \infty} \lambda^{(n)}(\vec{\varkappa}) = \lambda. \tag{6.2}$$

Theorem 6.6 (Bethe-Sommerfeld Conjecture)**.** *The spectrum of operator H contains a semi-axis.*

[15] The operator $H^{(n)}(\vec{\varkappa})$ is defined for every $\vec{\varkappa} \in \mathbb{R}^2$ as explained in Remark 3.16, page 22. The perturbation series is given by a formula analogous to (3.55), which coincides with (3.15) up to a shift of indices corresponding to the parallel shift of $\vec{\varkappa}$ into K_n.

Proof. By Remark 6.5, there is a point of the spectrum of H_n in the δ_n-neighborhood of λ for every $\lambda > k_*^{2l}$, k_* being introduced by (4.2). Since $\|H_n - H\| < \epsilon_n^4$, there is a point of the spectrum of H in the δ_n^*-neighborhood of λ, $\delta_n^* = \delta_n + \epsilon_n^4$. Since it is true for every n and the spectrum of H is closed, λ is in the spectrum of H. ∎

6.2. **Generalized Eigenfunctions of H.** A plane wave is usually denoted by $e^{i\langle \vec{k}, x\rangle}$, $\vec{k} \in \mathbb{R}^2$. Here we use $\vec{\varkappa}$ instead of $\vec{k}$ to comply with our previous notations. We show that for every $\vec{\varkappa}$ in a set

$$\mathcal{G}_\infty = \cup_{\lambda > \lambda_*} \mathcal{D}_\infty(\lambda), \quad \lambda_* = k_*^{2l},$$

there is a solution $\Psi_\infty(\vec{\varkappa}, x)$ of the equation for eigenfunctions:

$$(-\Delta)^{2l}\Psi_\infty(\vec{\varkappa}, x) + V(x)\Psi_\infty(\vec{\varkappa}, x) = \lambda_\infty(\vec{\varkappa})\Psi_\infty(\vec{\varkappa}, x), \tag{6.3}$$

which can be represented in the form

$$\Psi_\infty(\vec{\varkappa}, x) = e^{i\langle \vec{\varkappa}, x\rangle}\Big(1 + u_\infty(\vec{\varkappa}, x)\Big), \qquad \big\|u_\infty(\vec{\varkappa}, x))\big\|_{L_\infty(\mathbb{R}^2)} < c|\vec{\varkappa}|^{-\gamma_1}, \tag{6.4}$$

where $u_\infty(\vec{\varkappa}, x)$ is a limit-periodic function, $\gamma_1 = 2l - 4 - 7s_1 - 2\delta > 0$; the eigenvalue $\lambda_\infty(\vec{\varkappa})$ satisfies the asymptotic formula:

$$\lambda_\infty(\vec{\varkappa}) = |\vec{\varkappa}|^{2l} + O(|\vec{\varkappa}|^{-\gamma_2}), \quad \gamma_2 = 2l - 2 - 4s_1 - 3\delta > 0. \tag{6.5}$$

We also show that the set $\mathcal{G}_\infty$ satisfies (1.7).

In fact, by (6.1), any $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$ belongs to the $(\epsilon_n k^{-2l+1-\delta})$-neighborhood of $\mathcal{D}_n(\lambda)$. Applying the perturbation formulae proved in the previous sections, we obtain the following inequalities:

$$\big\|E^{(1)}(\vec{\varkappa}) - E^{(0)}(\vec{\varkappa})\big\|_1 < 2k^{-\gamma_0}, \qquad \big\|E^{(n+1)}(\vec{\varkappa}) - \tilde{E}^{(n)}(\vec{\varkappa})\big\|_1 < 48\epsilon_n^3, \quad n \ge 1, \tag{6.6}$$

$$\big|\lambda^{(1)}(\vec{\varkappa}) - |\vec{\varkappa}|^{2l}\big| < 2k^{-\gamma_2}, \qquad \big|\lambda^{(n+1)}(\vec{\varkappa}) - \lambda^{(n)}(\vec{\varkappa})\big| < 12\epsilon_n^4, \quad n \ge 1, \tag{6.7}$$

where $E^{(n+1)}$, $\tilde{E}^{(n)}$ are one-dimensional spectral projectors in $L_2(Q_{n+1})$ corresponding to potentials W_{n+1} and W_n, respectively; $\lambda^{(n+1)}(\vec{\varkappa})$ is the eigenvalue corresponding to $E^{(n+1)}(\vec{\varkappa})$, $E^{(0)}(\vec{\varkappa})$ corresponds to $V = 0$ and the periods a_1, a_2. This means that for properly chosen eigenfunctions $\Psi_{n+1}(\vec{\varkappa}, x)$:

$$\|\Psi_1 - \Psi_0\|_{L_2(Q_1)} < 4k^{-\gamma_0}|Q_1|^{1/2}, \quad \Psi_0(x) = e^{i\langle \vec{\varkappa}, x\rangle}, \tag{6.8}$$

$$\|\Psi_{n+1} - \tilde{\Psi}_n\|_{L_2(Q_{n+1})} < 100\epsilon_n^3|Q_{n+1}|^{1/2}, \tag{6.9}$$

where $\tilde{\Psi}_n$ is Ψ_n extended quasi-periodically from Q_n to Q_{n+1}. Eigenfunctions Ψ_n, $n \ge 1$, are chosen to obey two conditions. First, $\|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2}$; [16] second $\mathrm{Im}(\Psi_n, \tilde{\Psi}_{n-1}) = 0$, here $(\cdot, \cdot)$ is an inner product in $L_2(Q_n)$. These two conditions, obviously, provide a unique choice of each Ψ_n. Considering that Ψ_{n+1} and $\tilde{\Psi}_n$ satisfy equations for eigenfunctions and taking into account (6.7), (6.9) we obtain: $\|\Psi_{n+1} - \tilde{\Psi}_n\|_{W_2^{2l}(Q_{n+1})} < ck^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}$, $n \ge 1$ and, hence, $\|\Psi_{n+1} - \tilde{\Psi}_n\|_{L_\infty(Q_{n+1})} < ck^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}$. Since Ψ_{n+1} and $\tilde{\Psi}_n$ obey the same quasiperiodic conditions, the same inequality holds in the whole space $\mathbb{R}^2$:

$$\|\Psi_{n+1} - \Psi_n\|_{L_\infty(\mathbb{R}^2)} < ck^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n \ge 1, \tag{6.10}$$

[16] The condition $\|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2}$ implies $\|\tilde{\Psi}_n\|_{L_2(Q_{n+1})} = |Q_{n+1}|^{1/2}$.

where Ψ_{n+1}, Ψ_n are quasiperiodically extended to $\mathbb{R}^2$. Obviously, we have a Cauchy sequence in $L_\infty(\mathbb{R}^2)$. Let $\Psi_\infty(\vec{\varkappa}, x) = \lim_{n\to\infty} \Psi_n(\vec{\varkappa}, x)$. This limit is defined pointwise uniformly in x and in $W^{2l}_{2,loc}(\mathbb{R}^2)$.

Theorem 6.7. *For every sufficiently large λ and $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$ the sequence of functions $\Psi_n(\vec{\varkappa}, x)$ converges in $L_\infty(\mathbb{R}^2)$ and $W^{2l}_{2,loc}(\mathbb{R}^2)$. The limit function $\Psi_\infty(\vec{\varkappa}, x)$, $\Psi_\infty(\vec{\varkappa}, x) = \lim_{n\to\infty} \Psi_n(\vec{\varkappa}, x)$, satisfies the equation*

$$(-\Delta)^{2l}\Psi_\infty(\vec{\varkappa}, x) + V(x)\Psi_\infty(\vec{\varkappa}, x) = \lambda\Psi_\infty(\vec{\varkappa}, x). \tag{6.11}$$

It can be represented in the form

$$\Psi_\infty(\vec{\varkappa}, x) = e^{i\langle \vec{\varkappa}, x\rangle}\big(1 + u_\infty(\vec{\varkappa}, x)\big), \tag{6.12}$$

where $u_\infty(\vec{\varkappa}, x)$ is a limit-periodic function:

$$u_\infty(\vec{\varkappa}, x) = \sum_{n=1}^{\infty} \tilde{u}_n(\vec{\varkappa}, x), \tag{6.13}$$

$\tilde{u}_n(\vec{\varkappa}, x)$ being a periodic function with the periods $2^{M_n-1}b_1, 2^{M_n-1}b_2$, $2^{M_n} \approx k^{2^{n-1}s_1}$,

$$\|\tilde{u}_1\|_{L_\infty(\mathbb{R}^2)} < ck^{-\gamma_1}, \quad \gamma_1 = 2l - 4 - 7s_1 - 2\delta > 0, \tag{6.14}$$

$$\|\tilde{u}_n\|_{L_\infty(\mathbb{R}^2)} < ck^{2l}\epsilon^3_{n-1}|Q_n|^{1/2}, \quad n \geq 2. \tag{6.15}$$

Corollary 6.8. *Function $u_\infty(\vec{\varkappa}, x)$ obeys the estimate (6.4).*

Remark 6.9. If V is sufficiently smooth, say $V \in C^1(R)$, then estimate (6.14) and, hence, (6.4) can be improved by replacing γ_1 by γ_0.

Proof. Let us show that Ψ_∞ is a limit-periodic function. Obviously, $\Psi_\infty = \Psi_0 + \sum_{n=0}^\infty(\Psi_{n+1} - \Psi_n)$, the series converging in $L_\infty(\mathbb{R}^2)$ by (6.10). Introducing the notation $u_{n+1} = e^{-i\langle\vec{\varkappa}, x\rangle}(\Psi_{n+1} - \Psi_n)$, we arrive at (6.12), (6.13). Note that $\tilde{u}_n$ is periodic with the periods $2^{M_n-1}b_1, 2^{M_n-1}b_2$. Estimate (6.15) follows from (6.10). We check (6.14). Indeed, by (6.8), Fourier coefficients $(u_1)_j$, $j \in \mathbb{Z}^2$, satisfy the estimate $|(\tilde{u}_1)_j| < ck^{-\gamma_0}|Q_1|^{1/2} < ck^{-\gamma_0+s_1}$. This estimate can be easily improved for $j : p_j(0) > 2k$: $|(\tilde{u}_1)_j| < c|j|^{-2l}$. Summarizing these inequalities and taking into account that the number of $j : p_j(0) \leq 2k$ does not exceed $c_0k^{2+2s_1}$, we arrive at (6.14). It remains to prove (6.11). Indeed, $\Psi_n(\vec{\varkappa}, x)$, $n \geq 1$, satisfy equations for eigenfunctions: $H^{n)}\Psi_n = \lambda^{(n)}(\vec{\varkappa})\Psi_n$. Considering that $\Psi_n(\vec{\varkappa}, x)$ converges to $\Psi(\vec{\varkappa}, x)$ in $W^2_{2l,loc}$ and relation (6.2), we arrive at (6.11). ∎

Theorem 6.10. *Formulae (6.3), (6.4) and (6.5) hold for every $\vec{\varkappa} \in \mathcal{G}_\infty$. The set $\mathcal{G}_\infty$ is Lebesgue measurable and satisfies (1.7) with $\gamma_3 = \delta/2$.*

Proof. By Theorem 6.7, (6.3), (6.4) hold, where $\lambda_\infty(\vec{\varkappa}) = \lambda$ for $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$. Using (1.10), which is proven in Corollary 6.3, with $\varkappa_\infty = |\vec{\varkappa}|$, we easily obtain (6.5). It remains to prove (1.7). Let us consider a small region $U_n(\lambda_0)$ around an isoenergetic surface $\mathcal{D}_n(\lambda_0)$, $\lambda_0 > k_*^{2l}$. Namely, $U_n(\lambda_0) = \cup_{|\lambda-\lambda_0|<r_n}\mathcal{D}_n(\lambda)$, $r_n = \epsilon_{n-1}k^{-2\delta}$, $k = \lambda_0^{1/2l}$. Considering an estimate of the type (3.61) for $\lambda^{(n)}(\vec{\varkappa})$, which holds in the $\big(\epsilon_{n-1}k^{-2l+1-2\delta}\big)$-neighborhood of $\mathcal{D}_n(\lambda_0)$, we see that $U_n(\lambda_0)$ is an open set (a distorted ring with holes) and the width of the ring is of order $\epsilon_{n-1}k^{-2l+1-2\delta}$. Hence, $|U_n(\lambda_0)| = 2\pi k r_n\big(1 + o(k^{-\delta/2})\big)$. It easily follows from Lemma 5.17 that $U_{n+1} \subset U_n$. Definition of $\mathcal{D}_\infty(\lambda_0)$ yield: $\mathcal{D}_\infty(\lambda_0) = \cap_{n=1}^\infty U_n(\lambda_0)$. Hence, $\mathcal{G}_\infty = \cap_{n=1}^\infty \mathcal{G}_n$, where

$\mathcal{G}_n = \cup_{\lambda>\lambda_*}\mathcal{D}_n(\lambda)$. Considering that $U_{n+1} \subset U_n$ for every $\lambda_0 > \lambda_*$, we obtain $\mathcal{G}_{n+1} \subset \mathcal{G}_n$. Hence, $|\mathcal{G}_\infty \cap \mathbf{B_R}| = \lim_{n\to\infty} |\mathcal{G}_n \cap \mathbf{B_R}|$. Summarizing volumes of the regions U_n, we easily conclude $|\mathcal{G}_n \cap \mathbf{B_R}| = |\mathbf{B_R}|\left(1 + O(R^{-\delta/2})\right)$ uniformly in n. Thus, we have obtained (1.7) with $\gamma_3 = \delta/2$. ∎

Appendix A. Appendices.

A.1. **Proof of Corollary 3.6.** Let $\tau_0 \in \chi_2$. Taking into account the relation $\lambda_j^{(1)}(\tau_0+2\pi p/N_1a) = k^{2l}$ and the definition of C_2, we see that $|\lambda_j^{(1)}(\tau_0 + 2\pi p/N_1a) - z| = \epsilon_1/2$. Using (3.8) and the last equality, we easily obtain: $|\lambda_n^{(1)}(\tau_0 + 2\pi \hat{p}/N_1a) - z| \geq \epsilon_1/2$ for $\lambda_n^{(1)}(\tau_0 + 2\pi\hat{p}/N_1a) \neq \lambda_j^{(1)}(\tau_0 + 2\pi p/N_1a)$. Therefore, for any $z \in C_2$,

$$\|(\tilde{H}^{(1)}(\tau_0) - z)^{-1}\| \leq 2/\epsilon_1, \tag{A.1}$$

i.e., (3.11) is proved for $\tau_0 \in \chi_2$. Now we consider τ in the complex $(\epsilon_1 k^{-2l+1-\delta})$−neighborhood of χ_2. By Hilbert relation,

$$(\tilde{H}^{(1)}(\tau) - z)^{-1} = (\tilde{H}^{(1)}(\tau_0) - z)^{-1} + T_1T_2(\tilde{H}^{(1)}(\tau) - z)^{-1},$$
$$T_1 = (\tilde{H}_0(\tau_0) - z)^{-1}(\tilde{H}_0(\tau_0) + k^{2l}), \quad T_2 = (\tilde{H}_0(\tau_0) + k^{2l})^{-1}(\tilde{H}_0(\tau_0) - \tilde{H}_0(\tau)).$$

Suppose we have checked that $\|T_1T_2\| < k^{-\delta}$. Then, using (A.1) and a standard argument as in Lemma 2.9, we easily arrive at (3.11). The estimate $\|T_1\| < 4k^{2l}/\epsilon_1$, easily follows from (A.1). The estimate $\|T_2\| < 2l\epsilon_1 k^{-2l-\delta}$ easily follows from $|\tau - \tau_0| < \epsilon_1 k^{-2l+1-\delta}$. Thus, $\|T_1T_2\| < 8lk^{-\delta}$ and, hence, (3.11) is proved. Estimate (3.12) follows from the Hilbert relation

$$(\tilde{H}^{(1)}(\tau) - z)^{-1} = (\tilde{H}_0(\tau) + k^{2l})^{-1} + (\tilde{H}_0(\tau) + k^{2l})^{-1}(-W_1 + k^{2l} + z)(\tilde{H}^{(1)}(\tau) - z)^{-1},$$

an elementary estimate $\|(\tilde{H}_0(\tau) + k^{2l})^{-1}\|_1 < b_1b_2k^{-2l+2+2s_2}$ and (3.11).

A.2. **Proof of estimate (3.50).** Note that

$$\Big||\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^2 - |\vec{y}(\varphi) + \vec{p}_m(0)|_*^2\Big| =$$
$$\Big||\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^2 - |\vec{y}_0(\varphi) + \vec{h}_1(\varphi) + \vec{p}_m(0)|_*^2\Big| = O\Big(|h_1| \cdot |\vec{y}_0(\varphi) + \vec{p}_m(0)| + |h_1|^2\Big). \tag{A.2}$$

If $p_m(0) < 4k$, then, obviously, $|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^2 = O(k^2)$. Considering (2.43), we arrive at

$$\Big||\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - |\vec{y}(\varphi) + \vec{p}_m(0)|_*^{2l}\Big| = O(|h_1| \cdot k^{2l-1}) = O(k^{2l-2-4s_1-2\gamma_0-\delta}).$$

Combining this relation with (3.44) where $\epsilon_0 = k^{-4-6s_1-3\delta}$, we obtain

$$\left|\frac{|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - |\vec{y}(\varphi) + \vec{p}_m(0)|_*^{2l}}{|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - k^{2l}}\right|$$
$$\leq \frac{c|h_1|k^{2l-1}}{b_0k^{2l-9-12s_1-6\delta}} = O(b_0^{-1}|h_1|k^{8+12s_1+6\delta}) = O(b_0^{-1}k^{7+8s_1-2\gamma_0+5\delta}) = o(1),$$

the last estimate following from the restriction on b_0 in the statement of the lemma. Thus, (3.50) holds for $p_m(0) < 4k$. Suppose $p_m(0) \geq 4k$. Using (3.45) and (A.2), we easily obtain:

$$\left|\frac{|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - |\vec{y}(\varphi) + \vec{p}_m(0)|_*^{2l}}{|\vec{y}_0(\varphi) + \vec{p}_m(0)|_*^{2l} - k^{2l}}\right| \leq \frac{|h_1|}{p_m(0)} = O(k^{-2-4s_1-2\gamma_0-\delta}).$$

This means (3.50) holds in this case too.

A.3. **Proof of estimate (3.68).** We use (3.67). First, we represent $\nabla\lambda^{(1)}(\vec{\varkappa}_1) - \nabla\lambda^{(2)}(\alpha, \vec{\varkappa}_2)$ as the sum of A_1 and A_2:

$$A_1 = \nabla\lambda^{(1)}(\varkappa_1\vec{\nu}) - \nabla\lambda^{(2)}(\alpha, \varkappa_1\vec{\nu}), \quad A_2 = \nabla\lambda^{(2)}(\alpha, \varkappa_1\vec{\nu}) - \nabla\lambda^{(2)}(\alpha, \varkappa_2\vec{\nu}).$$

Considering (3.57), we obtain $|A_1| < 24\alpha\epsilon_1^3 k^{2l-1+\delta}$. Using the Mean Value Theorem and (3.27), we get: $|A_2| < ck^{2l-2}|\varkappa_2 - \varkappa_1|$. Applying (3.60) yields: $|A_2| < c\alpha\epsilon_1^4 k^{-1}$. Adding the estimates for A_1 and A_2 we arrive at:

$$\left|\nabla\lambda^{(1)}(\varkappa_1\vec{\nu}) - \nabla\lambda^{(2)}(\alpha, \varkappa_2\vec{\nu})\right| < 25\alpha\epsilon_1^3 k^{2l-1+\delta}. \tag{A.3}$$

Similarly,

$$\left|\varkappa_1\nabla\lambda^{(1)}(\varkappa_1\vec{\nu}) - \varkappa_2\nabla\lambda^{(2)}(\alpha, \varkappa_2\vec{\nu})\right| < 26\alpha\epsilon_1^3 k^{2l+\delta}. \tag{A.4}$$

By Lemma 2.11, $\frac{\partial\varkappa_1}{\partial\varphi} = \frac{\partial h_1}{\partial\varphi} = O(k^{-2\gamma_0+1})$. Substituting (A.3), (A.4) and the last estimate into the formula for $F(\lambda, \vec{\nu})$, we obtain (3.68).

A.4. **Lemma A.1.**

Lemma A.1. *If t is in the complex $(2k^{-1-4s_1-2\delta})$-neighborhood of the nonresonance set $\chi_1(\lambda, \delta)$, then the operator-function $\left(H^{(1)}(t) - z\right)^{-1}$ has a single pole z_0 inside the contour C_1 and*

$$\left\|(z - z_0)\left(H^{(1)}(t) - z\right)^{-1}\right\| < 16, \tag{A.5}$$

for every z inside C_1.

Proof. By (3.38),

$$\left|\det\frac{H^{(1)}(t) - z}{H_0(t) - z} - 1\right| < 2\|W_1\|\left\|(H_0(t) - z)^{-1}\right\|_1 \tag{A.6}$$

for every z on the contour C_1. Using the estimate (2.10) and considering that $H_0(t)$ is a diagonal operator, we easily obtain $\left\|(H_0(t) - z)^{-1}\right\|_1 < k^{-2l+4+6s_1+\delta}$. It immediately follows that the right-hand part of (A.6) is less then 1. Applying Rouché's theorem, we conclude that the determinant has the same number of zeros and poles inside C_1. Considering that $(H_0(t) - z)^{-1}$ has a single pole, we obtain that $\left(H^{(1)}(t) - z\right)^{-1}$ has a single pole too. We denote it by z_0. Obviously, $(z - z_0)\left(H^{(1)}(t) - z\right)^{-1}$ is holomorphic inside C_1. The definition of C_1, given in Corollary 2.2, yields: $|z - z_0| < 2k^{2l-2-4s_1-\delta}$. Using the last inequality and estimate (2.30), we obtain (A.5) for any $z \in C_1$. By the Maximum principle it can be extended inside C_1. ∎

A.5. **Proof of Lemma 3.48.** By Lemma 2.11, Part 2, the function $\vec{\varkappa}_1(\varphi)$ is holomorphic in the $(k^{-2-4s_1-2\delta})-$neighborhood of Φ_1 and $\lambda^{(1)}(\vec{\varkappa}_1(\varphi)) = k^{2l}$. Hence, the equation (3.92) is equivalent to $\lambda^{(1)}(\vec{y}(\varphi)) = \lambda^{(1)}(\vec{y}(\varphi) - \vec{b}) + \epsilon_0$. We use perturbation formula (2.34): $|\vec{y}(\varphi)|_*^{2l} + f_1(\vec{y}(\varphi)) = |\vec{y}(\varphi) - \vec{b}|_*^{2l} + f_1(\vec{y}(\varphi) - \vec{b}) + \epsilon_0$. This equation can be rewritten as

$$\begin{aligned}\left(2\langle\vec{y}(\varphi), \vec{b}\rangle_* - |\vec{b}|^2\right)\left(|\vec{y}(\varphi)|_*^{2l-2} + \cdots + |\vec{y}(\varphi) - \vec{b}|_*^{2l-2}\right)& \\ + f_1(\vec{y}(\varphi)) - f_1(\vec{y}(\varphi) - \vec{b}) - \epsilon_0 = 0,&\end{aligned} \tag{A.7}$$

where $\langle \vec{y}, \vec{b}\rangle_* = y_1 b_1 + y_2 b_2$. Using the notation $\vec{b} = b_0(\cos\varphi_b, \sin\varphi_b)$, dividing both sides of the equation (A.7) by $2b_0 k\Big(|\vec{y}(\varphi)|_*^{2l-2} + \dots + |\vec{y}(\varphi) - \vec{b}|_*^{2l-2}\Big)$, and considering that $\vec{y}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b} = (k+h_1)\vec{\nu} + \vec{b}$, we obtain:

$$\cos(\varphi - \varphi_b) - \epsilon_0 g_1(\varphi) + g_2(\varphi) = 0, \tag{A.8}$$

where $g_1(\varphi) = \Big[2b_0 k\Big(|\vec{y}(\varphi)|_*^{2l-2} + \dots + |\vec{y}(\varphi) - \vec{b}|_*^{2l-2}\Big)\Big]^{-1}$ and

$$g_2(\varphi) = \frac{\langle \vec{h}_1(\varphi), \vec{b}\rangle_*}{b_0 k} + \frac{b_0}{2k} + \Big(f_1(\vec{y}(\varphi)) - f_1(\vec{y}(\varphi) - \vec{b})\Big) g_1(\varphi), \quad \vec{h}_1(\varphi) = h_1(\varphi)\vec{\nu}.$$

Obviously $g_1 = O\left(b_0^{-1} k^{-2l+1}\right)$. Let us estimate $g_2(\varphi)$. Using the inequality (2.43) for h_1, and considering that $b_0 = o(k^{-1-4s_1-2\delta})$, we easily obtain:

$$\left|\frac{\langle \vec{h}_1(\varphi), \vec{b}\rangle_*}{b_0 k}\right| \le \frac{2h_1}{k} = O(k^{-2-4s_1-2\gamma_0-\delta}), \quad \frac{b_0}{2k} = o(k^{-2-4s_1-2\delta}).$$

By (2.26),

$$\Big|f_1(\vec{y}(\varphi)) - f_1(\vec{y}(\varphi) - \vec{b})\Big| \le \sup|\nabla f_1| b_0 = O(b_0 k^{2l-1-2\gamma_0+\delta}) = O(b_0 k^{-2l+3+8s_1+5\delta}),$$

and therefore, $\Big|\Big(f_1(\vec{y}(\varphi)) - f_1(\vec{y}(\varphi) - \vec{b})\Big) g_1(\varphi)\Big| = O(k^{-4l+4+8s_1+5\delta})$. Since $s_1 = \dfrac{2l-11}{32}$, we have $g_2(\varphi) = o(k^{-2-4s_1-2\delta})$. Using $g_1(\varphi) = O\left(b_0^{-1} k^{-2l+1}\right)$ and $\epsilon_0 < b_0 k^{2l-3-4s_1-3\delta}$, we obtain $\epsilon_0 g_1(\varphi) = o(k^{-2-4s_1-2\delta})$. Thus,

$$g_2(\varphi) - \epsilon_0 g_1(\varphi) = o(k^{-2-4s_1-2\delta}). \tag{A.9}$$

Suppose that $\varphi_b \pm \dfrac{\pi}{2}$ is in the $\left(\frac{1}{8}k^{-2-4s_1-2\delta}\right)$-neighborhood of $\tilde{\mathit{\Phi}}_1$. We draw two circles $C_\pm$ centered at $\varphi_b \pm \dfrac{\pi}{2}$ with the radius $\frac{1}{8}k^{-2-4s_1-2\delta}$. They are both inside the complex $\frac{3}{4}k^{-2-4s_1-2\delta}$-neighborhood of $\mathit{\Phi}_1$, the perturbation series converging and the estimate (A.9) holds. For any φ on $C_\pm$, $|\varphi - (\varphi_b \pm \pi/2)| = \frac{1}{8}k^{-2-4s_1-2\delta}$ and, therefore, $|\cos(\varphi - \varphi_b)| > \frac{1}{16}k^{-2-4s_1-2\delta} > |g_2(\varphi) - \epsilon_0 g_1(\varphi)|$ for any $\varphi \in C_\pm$. By Rouché's Theorem, there is only one solution of the equation (A.8) inside each $C_\pm$. If $\varphi_b + \pi/2$ is not in the $(\frac{1}{8}k^{-2-4s_1-2\delta})-$neighborhood of $\tilde{\mathit{\Phi}}_1$, then $|\cos(\varphi - \varphi_b)| > \frac{1}{16}k^{-2-4s_1-2\delta}$ in $\mathit{\Phi}_1$ and, hence, equation (A.8) has no solution. Thus, there are at most two solutions in $\tilde{\mathit{\Phi}}_1$ and $|\varphi_{\epsilon_0}^\pm - (\varphi_b \pm \pi/2)| < \frac{1}{8}k^{-2-4s_1-2\delta}$.

A.6. **Proof of Lemma 3.49.** Using the perturbation formula (2.34), we obtain:

$$\begin{aligned}\frac{\partial}{\partial\varphi}\lambda^{(1)}\big(\vec{y}(\varphi)\big) = \frac{\partial}{\partial\varphi}\Big[\lambda^{(1)}\big(\vec{y}(\varphi)\big) - k^{2l}\Big] = \frac{\partial}{\partial\varphi}\Big[\lambda^{(1)}\big(\vec{y}(\varphi)\big) - \lambda^{(1)}\big(\vec{y}(\varphi) - \vec{b}\big)\Big] = \\ \Big\langle \nabla_{\vec{y}}\lambda^{(1)}\big(\vec{y}(\varphi)\big) - \nabla_{\vec{y}}\lambda^{(1)}\big(\vec{y}(\varphi) - \vec{b}\big), \frac{\partial}{\partial\varphi}\vec{y}(\varphi)\Big\rangle_* = \\ \Big\langle \nabla\big|\vec{y}(\varphi)\big|_*^{2l} - \nabla\big|\vec{y}(\varphi) - \vec{b}\big|_*^{2l}, (k+h_1)\vec{\mu} + h_1'\vec{\nu}\Big\rangle_* + \\ \Big\langle \nabla f_1\big(\vec{y}(\varphi)\big) - \nabla f_1\big(\vec{y}(\varphi) - \vec{b}\big), (k+h_1)\vec{\mu} + h_1'\vec{\nu}\Big\rangle_*,\end{aligned} \tag{A.10}$$

where $\vec{\nu}=(\cos\varphi,\sin\varphi)$ and $\vec{\mu}=\vec{\nu}'=(-\sin\varphi,\cos\varphi)$. Note that

$$\begin{aligned}\nabla\big|\vec{y}(\varphi)\big|_*^{2l}-\nabla\big|\vec{y}(\varphi)-\vec{b}\big|_*^{2l}&=2l\big|\vec{y}(\varphi)\big|_*^{2l-2}\vec{y}(\varphi)-2l\big|\vec{y}(\varphi)-\vec{b}\big|_*^{2l-2}\big(\vec{y}(\varphi)-\vec{b}\big)\\&=2l\big|\vec{y}(\varphi)\big|_*^{2l-2}\vec{b}+2l\Big[\big|\vec{y}(\varphi)\big|_*^{2l-2}-\big|\vec{y}(\varphi)-\vec{b}\big|_*^{2l-2}\Big]\big(\vec{y}(\varphi)-\vec{b}\big).\end{aligned}\quad\text{(A.11)}$$

Substituting (A.11) into (A.10), we get $\frac{\partial}{\partial\varphi}\lambda_j^{(1)}\big(\vec{y}(\varphi)\big)=T_1+T_2+T_3$,

$$\begin{aligned}T_1&=2l\big|\vec{y}(\varphi)\big|_*^{2l-2}\left\langle\vec{b},(k+h_1)\vec{\mu}+h_1'\vec{\nu}\right\rangle_*,\\T_2&=2l\left[\big|\vec{y}(\varphi)\big|_*^{2l-2}-\big|\vec{y}(\varphi)-\vec{b}\big|_*^{2l-2}\right]\left\langle\vec{y}(\varphi)-\vec{b},(k+h_1)\vec{\mu}+h_1'\vec{\nu}\right\rangle_*,\\T_3&=\left\langle\nabla f_1\big(\vec{y}(\varphi)\big)-\nabla f_1\big(\vec{y}(\varphi)-\vec{b}\big),(k+h_1)\vec{\mu}+h_1'\vec{\nu}\right\rangle_*.\end{aligned}$$

Considering that φ is close to $\varphi_b\pm\pi/2$, we readily obtain: $\langle\vec{b},\vec{\nu}\rangle_*=o(b_0)$, $\langle\vec{b},\vec{\mu}\rangle_*=\pm b_0(1+o(1))$. Using also estimates (2.43) for h_1, we get $T_1=\pm 2lb_0k^{2l-1}(1+o(1))$. Substituting $\vec{y}(\varphi)-\vec{b}=(k+h_1(\varphi))\vec{\nu}$ into the formula for T_2 and taking into account that $\langle\vec{\mu},\vec{\nu}\rangle_*=0$, we arrive at the estimate $T_2=o\left(b_0k^{2l-1}\right)$. By (2.36) with $|m|=2$, $\left|\nabla f_1\big(\vec{y}(\varphi)\big)-\nabla f_1\big(\vec{y}(\varphi)-\vec{b}\big)\right|=O(b_0k^{-2l+4+12s_1+7\delta})$. Hence, $T_3=o\left(b_0k^{2l-1}\right)$. Adding the estimates for T_1,T_2,T_3, we get (3.94).

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Department of Mathematics, University of Alabama at Birmingham, 1300 University Boulevard, Birmingham, AL 35294.

E-mail address: `karpeshi@math.uab.edu`

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801.

E-mail address: `yrlee4@math.uiuc.edu`