Applications of M. G. Krein's Theory of Regular Symmetric Operators to Sampling Theory *†‡

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Abstract

The classical Kramer sampling theorem establishes general conditions that allow the reconstruction of functions by orthogonal sampling formulas. One major task in sampling theory is to find concrete, non trivial realizations of this theorem. In this paper we provide a new approach to this subject on the basis of the M. G. Krein's theory of simple regular symmetric operators, with deficiency indices $(1, 1)$, for obtaining Kramer-type sampling formulas. We show that these formulas have the form of Lagrange interpolation series. Concerning the case of entire operators, we also characterize the space of functions reconstructible by our sampling formulas.

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1 Introduction

It has been argued recently that sampling theory might be the bridge that would allow to reconcile the continuous nature of physical fields with the need of discretization of space-time, as required by a yet-to-be-formulated theory of quantum gravity [12, 13, 14]. This idea, which is partially developed in [12] for the one-dimensional case, introduces the use of simple regular symmetric operators with deficiency indices $(1,1)$ to obtain orthogonal sampling formulas. It is remarkable that the class of operators under consideration in [12] had already been studied in detail by M. G. Krein [17, 18, 19] more than 60 years ago.

The conjunction of the ideas of [12] and Krein's theory of symmetric operators with equal deficiency indices suggests the means of treating sampling theory in a new and straightforward way. By introducing this new approach, we put in a mathematically rigorous framework the ideas in [12] and propose a general method for obtaining analytical Kramer-type sampling formulas associated with the self-adjoint extensions of a simple regular symmetric operator with deficiency indices $(1,1)$.

A seminal result in sampling theory is the Whittaker-Shannon-Kotel'nikov sampling theorem [15, 21, 23]. This theorem states that functions that belong to a Paley-Wiener space may be uniquely reconstructed from their values at certain discrete sets of points. A general approach to Whittaker-Shannon-Kotel'nikov type formulas was developed by Kramer [16] based on the following result. Given a finite interval $I=[a,b]$, let $K(y,x)\in L^2(I,dy)$ for all $x\in\mathbb{R}$. Assume that there exists a sequence $\{x_n\}_{n\in\mathbb{Z}}$ for which $\{K(y,x_n)\}_{n\in\mathbb{Z}}$ is an orthogonal basis of $L^2(I,dy)$. Let f be any function of the form

$$f(x):=\langle K(\cdot,x),g(\cdot)\rangle_{L^2(I)}$$

for some $g\in L^2(I,dy)$, where the brackets denotes the inner product in $L^2(I,dy)$. Then f can be reconstructed from its samples $\{f(x_n)\}_{n\in\mathbb{Z}}$ by the formula

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{\langle K(\cdot,x),K(\cdot,x_n)\rangle_{L^2(I)}}{\|K(\cdot,x_n)\|^2_{L^2(I)}}f(x_n),\qquad x\in\mathbb{R}.$$

The search for concrete realizations of the Kramer sampling theorem, which in some cases has proven to be a difficult task, has motivated a large amount of literature. See, for instance, [3, 8, 9, 10, 24, 25] and references therein. On the basis of the approach proposed in the present work, we obtain Kramer-type analytic sampling formulas for functions belonging to linear sets of analytic functions determined by a given regular simple symmetric operator. A central idea in Krein's theory is the fact that any simple symmetric operator with deficiency indices $(1,1)$, acting in a certain Hilbert space $\mathcal{H}$, defines a bijective mapping from $\mathcal{H}$ onto a space of scalar functions of one complex variable having certain analytic properties. In this space of functions one can introduce an inner product. In the particular case when the starting point is an entire operator (see Definition 2), the space of functions turns out to be a Hilbert space of entire functions known as a de Branges space [4].

Summing up, in this paper we present an original technique in sampling theory based on Krein's theory of regular symmetric operators. Concretely, we prove an analytic sampling formula valid for functions belonging to linear sets of analytic functions associated to symmetric operators of the kind already mentioned. We then focus our attention to entire operators to provide a characterization of the corresponding spaces of entire functions. As a byproduct, we also provide rigorous proofs to some of the results formally obtained in [12]. In a sense this work may be considered as introductory of the theoretical framework. Further rigorous results related to [12], as well as other applications to sampling theory, will be discussed in subsequent papers.

In relation with the articles on sampling theory mentioned above, our results are close to those of [8]. There are, of course, several differences. First, the sampling theory developed in [8] is based upon the spectral properties of a single self-adjoint operator (since [8] considers only symmetric operators having zero is their resolvent sets); as already mentioned, our sampling theory is grounded on the family of self-adjoint extensions of a simple regular symmetric operator with deficiency indices $(1, 1)$. Second, we mainly use techniques of operator theory, whereas [8] relies mostly on complex variable methods complemented with some elementary results of functional analysis. Finally, the results of [8] do not fit easily in the framework proposed in [12] for which our results indeed give some rigorous justification.

This paper is organized as follows. In Section 2, we introduce some concepts of the theory of regular symmetric operators. In Section 3, we state and prove a sampling formula. In Section 4, we provide a characterization of the Hilbert spaces associated with the class of operators under consideration, paying particular attention to the case of entire operators. Finally, in Section 5, we discuss some examples.

2 Preliminaries

Most of the mathematical background required in this work is based on Krein's theory of entire operators as accounted in the expository book [11]. In this section we review some definitions and results from the theory of symmetric operators and introduce the notation.

Let $\mathcal{H}$ denote a separable Hilbert space whose inner product $\langle \cdot, \cdot \rangle$ will be assumed anti-linear in its first argument.

Definition 1. A closed symmetric operator A with domain and range in $\mathcal{H}$ is called:

(a) *simple* if it does not have non-trivial invariant subspaces on which A is self-adjoint,

(b) *regular* if every point in $\mathbb{C}$ is a point of regular type for A, that is, if the operator $(A - zI)$ has bounded inverse for every $z \in \mathbb{C}$.

By [11, Theorem 1.2.1], an operator A is simple if and only if

$$\bigcap_{z:\operatorname{Im} z\neq 0} \operatorname{Ran}(A - zI) = \{0\}.$$

Moreover, as shown in [11, Propositions 1.3.3, 1.3.5 and 1.3.6], a closed symmetric operator with finite deficiency indices (m, m) is regular if and only if some (hence every) self-adjoint extension of A within $\mathcal{H}$ has only discrete spectrum of multiplicity at most m. Also,

$$\dim [\operatorname{Ran}(A - zI)]^{\perp} = \dim [\operatorname{Ker}(A^* - \overline{z}I)] = m \tag{2.1}$$

for all $z \in \mathbb{C}$. For this kind of operators it is also known that, for given any $x \in \mathbb{R}$, there is exactly one self-adjoint extension within $\mathcal{H}$ such that x is in its spectrum.

Remark 1. In what follows, whenever we consider self-adjoint extensions of a symmetric operator A, we will always mean the self-adjoint restrictions of A^*, in other words, the self-adjoint extensions of A within $\mathcal{H}$ (cf. Naimark's theory on generalized self-adjoint extensions [2, Appendix 1]).

We will denote by $\operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ the class of regular, simple, closed symmetric operators, defined on $\mathcal{H}$, with deficiency indices $(1, 1)$.

Let $A_\sharp$ be some self-adjoint extension of $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$. The generalized Cayley transform is defined as

$$(A_\sharp - wI)(A_\sharp - zI)^{-1}.$$

for every $w \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \operatorname{Sp}(A_\sharp)$. This operator has several properties [11, pag. 9]. We only mention the following:

$$(A_\sharp - wI)(A_\sharp - zI)^{-1} : \operatorname{Ker}(A^* - wI) \to \operatorname{Ker}(A^* - zI) \tag{2.2}$$

one-to-one and onto.

A complex conjugation on $\mathcal{H}$ is a bijective anti-linear operator $C : \mathcal{H} \to \mathcal{H}$ such that $C^2 = I$ and $\langle C\eta, C\varphi\rangle = \langle \varphi, \eta\rangle$ for all $\eta, \varphi \in \mathcal{H}$. A symmetric operator A is said to be real with respect to a complex conjugation C if $C \operatorname{Dom}(A) \subseteq \operatorname{Dom}(A)$ and $CA\varphi = AC\varphi$ for every $\varphi \in \operatorname{Dom}(A)$. Clearly, the condition $C \operatorname{Dom}(A) \subseteq \operatorname{Dom}(A)$, along with $C^2 = 1$, implies that $C \operatorname{Dom}(A) = \operatorname{Dom}(A)$. Consequently, A^* is also real with respect to C.

3 Sampling formulae

Let us consider an operator $A \in \operatorname{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$. Let $A_\sharp$ be some self-adjoint extension of A. Given $z_0 \in \mathbb{C} \setminus \operatorname{Sp}(A_\sharp)$ and $\psi_0 \in \operatorname{Ker}(A^* - z_0 I)$, define

$$\psi(z) := (A_\sharp - z_0 I)(A_\sharp - zI)^{-1}\psi_0 = \psi_0 + (z - z_0)(A_\sharp - zI)^{-1}\psi_0 \tag{3.1}$$

for every $z \in \mathbb{C} \setminus \mathrm{Sp}(A_\sharp)$. This vector-valued function is analytic in the resolvent set of $A_\sharp$ which, by (2.2), takes values in $\mathrm{Ker}\,(A^* - zI)$ when evaluated at z. Moreover, $\psi(z)$ has simple poles at the points of $\mathrm{Sp}(A_\sharp) = \mathrm{Sp}_{\mathrm{disc}}(A_\sharp)$. By looking at the first equality in (3.1), it is clear that the dependency of $\psi(z)$ on z_0 and ψ_0 is rather unessential. Indeed, for any other $z_0' \in \mathbb{C} \setminus \mathrm{Sp}(A_\sharp)$, we can take $\psi_0' = (A_\sharp - z_0 I)\,(A_\sharp - z_0' I)^{-1}\,\psi_0$ in $\mathrm{Ker}\,(A^* - z_0' I)$ and thus $\psi(z) = \psi_0' + (z - z_0')\,(A_\sharp - zI)^{-1}\,\psi_0'$ by the first resolvent identity.

Given $z_1 \in \mathbb{C} \setminus \mathrm{Sp}(A_\sharp)$, let us choose some $\mu \in \mathcal{H}$ such that $\langle \psi(\overline{z_1}), \mu \rangle \neq 0$. The inner product $\langle \psi(\overline{z}), \mu \rangle$ then defines an analytic function in $\mathbb{C} \setminus \mathrm{Sp}(A_\sharp)$, having zeroes at a countable set S_μ devoid of accumulation points in $\mathbb{C}$. The set S_μ is defined as the subset of $\mathbb{C}$ for which $\mathcal{H}$ can not be written as the direct sum of $\mathrm{Ran}(A - zI)$ and $\mathrm{Span}\{\mu\}$. Following Krein, we call the element μ a *gauge* [19].

Let us define

$$\xi(z) := \frac{\psi(\overline{z})}{\langle \mu, \psi(\overline{z}) \rangle} \tag{3.2}$$

for all $z \in \mathbb{C} \setminus S_\mu$.

Lemma 1. *The vector-valued function $\xi(z)$ is analytic in $\mathbb{C} \setminus S_\mu$ and has simple poles at points of S_μ. Moreover, it does not depend on the self-adjoint extension of A used to define $\psi(z)$.*

Proof. The first statement holds by rather obvious reasons, notice only that the poles of $\langle \psi(\overline{z}), \mu \rangle$ coincide with those of $\psi(\overline{z})$. Let us pay attention to the second statement. Consider two self-adjoint extensions $A_\sharp$ and $A_\sharp'$. We have

$$\psi(z) = (A_\sharp - z_0 I)\,(A_\sharp - zI)^{-1}\,\psi_0 \quad \text{and} \quad \psi'(z) = \left(A_\sharp' - z_0 I\right)\left(A_\sharp' - zI\right)^{-1}\psi_0.$$

Since both $\psi(z)$ and $\psi'(z)$ belong to $\mathrm{Ker}\,(A^* - zI)$ and the dimension of this subspace is always equal to one, it follows that $\psi'(z) = g(z)\psi(z)$ for every $z \notin \mathrm{Sp}(A_\sharp) \cup \mathrm{Sp}(A_\sharp')$, where $g(z)$ is a scalar function. Inserting this identity into (3.2) yields $\xi(z) = \xi'(z)$. $\square$

For every $z \in \mathbb{C} \setminus S_\mu$ we have the decomposition $\mathcal{H} = \mathrm{Ran}\,(A - zI) \dotplus \mathrm{Span}\,\{\mu\}$, in which case every element $\varphi \in \mathcal{H}$ can be written as

$$\varphi = [\varphi - \widehat{\varphi}(z)\mu] + \widehat{\varphi}(z)\mu,$$

where $\varphi - \widehat{\varphi}(z) \in \mathrm{Ran}\,(A - zI)$. A simple computation shows that the non-orthogonal projection $\widehat{\varphi}(z)$ is given by

$$\widehat{\varphi}(z) := \frac{\langle \psi(\overline{z}), \varphi \rangle}{\langle \psi(\overline{z}), \mu \rangle} = \langle \xi(z), \varphi \rangle \tag{3.3}$$

whenever $z \in \mathbb{C} \setminus S_\mu$; it is otherwise not defined. Indeed, the function $\widehat{\varphi}(z)$ is analytic in $\mathbb{C} \setminus S_\mu$ and meromorphic in $\mathbb{C}$ for every $\varphi \in \mathcal{H}$. We note that $\widehat{\mu}(z) \equiv 1$.

Let us denote the linear map $\varphi \mapsto \widehat{\varphi}(z)$ as Φ_μ and the linear space of functions given by (3.3) as $\Phi_\mu\mathcal{H}$. Since the operator A is simple, it follows that Φ_μ is injective and, therefore, is an isomorphism from $\mathcal{H}$ onto $\Phi_\mu\mathcal{H}$ [11, Theorem 1.2.2]. Moreover, Φ_μ transforms A into the multiplication operator on $\Phi_\mu\mathcal{H}$, that is,

$$\widehat{(A\varphi)}(z) = \left(\Phi_\mu A \Phi_\mu^{-1}\widehat{\varphi}\right)(z) = z\widehat{\varphi}(z), \quad \varphi \in \mathrm{Dom}(A).$$

Proposition 1. *Assume $S_\mu \cap \mathbb{R} = \emptyset$. Let $\{x_n\}$ be the spectrum of any self-adjoint extension $A_\sharp$ of A. Then, for any analytic function $f(z)$ that belongs to $\Phi_\mu\mathcal{H}$, we have*

$$f(z) = \sum_{x_n \in \mathrm{Sp}(A_\sharp)} \frac{\langle \xi(z), \xi(x_n)\rangle}{\left\|\xi(x_n)\right\|^2} f(x_n), \tag{3.4}$$

The convergence in (3.4) is uniform over compact subsets of $\mathbb{C} \setminus S_\mu$.

Proof. Fix some arbitrary self-adjoint extension $A_\sharp$ of A. Take another self-adjoint extension $A'_\sharp \neq A_\sharp$ to define $\psi(z)$, that is, $\psi(z) = \left(A'_\sharp - z_0 I\right)\left(A'_\sharp - zI\right)^{-1}\psi_0$. Arrange the elements of $\mathrm{Sp}\,(A_\sharp)$ in a sequence $\{x_n\}_{n\in M}$, where M is a countable indexing set, and let η_n be an eigenstate of $A_\sharp$ corresponding to x_n, i. e., $A_\sharp\eta_n = x_n\eta_n$. Since $A^* \supset A_\sharp$, it follows that $\eta_m \in \mathrm{Ker}\,(A^* - x_m I)$, where furthermore $\dim\left[\mathrm{Ker}\,(A^* - x_m I)\right] = 1$. On the other hand, since x_m is not a pole of $\psi(z)$, $\psi(x_m)$ is well defined and belongs also to $\mathrm{Ker}\,(A^* - x_m I)$. Therefore, up to a factor, $\eta_m = \xi(x_m)$.

Pick a sequence $\{M_k\}_{k\in\mathbb{N}}$ of subsets of M such that $M_k \subset M_{k+1}$ and $\cup_k M_k = M$. Consider any analytic function $f(z) \in \Phi_\mu\mathcal{H}$. Since Φ_μ is injective, there exists a unique $\varphi \in \mathcal{H}$ such that $\widehat{\varphi}(z) = f(z)$. Clearly,

$$\left|\widehat{\varphi}(z) - \sum_{n\in M_k} \frac{\langle \xi(z), \xi(x_n)\rangle}{\left\|\xi(x_n)\right\|^2}\langle \xi(x_n), \varphi\rangle\right| \leq \left\|\xi(z)\right\| \left\|\varphi - \sum_{n\in M_k} \frac{1}{\left\|\xi(x_n)\right\|^2}\langle \xi(x_n), \varphi\rangle\, \xi(x_n)\right\|.$$

The second factor in the r. h. s. of this inequality does not depend on z and obviously tends to zero as $k \to \infty$. Since $\xi(z)$ is analytic on $\mathbb{C} \setminus S_\mu$, the uniform convergence of (3.4) has been proven. □

Remark 2. Krein asserts that, for any operator in $\mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$, one can always choose μ so that $S_\mu \cap \mathbb{R} = \emptyset$ [18, Theorem 8].

We notice that the sequence $\{x_n\}$ could be replaced by any sequence $\{z_n\}$ of complex numbers for which $\{\xi(z_n)\}$ is a sequence of orthogonal elements that span $\mathcal{H}$ (for a related discussion, see [9]). In our case, the question of whether such a sequence exists or not is answered to the affirmative by invoking the self-adjoint extensions of the operator A.

Below we show that the interpolation formula (3.4) is indeed a Lagrange interpolation series.

Proposition 2. *Under the hypotheses of Proposition 1, there exists a complex function $G(z)$, analytic in $\mathbb{C} \setminus S_\mu$ (hence $\mathbb{R}$) and having simple zeroes at $\{x_n\}_{n\in I} = \operatorname{Sp}(A_\sharp)$, such that*

$$f(z) = \sum_{n\in M} \frac{G(z)}{(z-x_n)G'(x_n)} f(x_n),$$

for every $f(z) \in \Phi_\mu \mathcal{H}$.

Proof. Given $\{x_n\}_{n\in M} = \operatorname{Sp}(A_\sharp)$, set $\psi(z) = (A_\sharp - z_0 I)\,(A_\sharp - zI)^{-1}\,\psi_0$, for some z_0 : $\operatorname{Im} z_0 \neq 0$ and $\psi_0 \in \operatorname{Ker}(A^* - z_0 I)$. Define

$$G(z) := \frac{1}{\langle \psi(\overline{z}), \mu \rangle}.$$

This function has simples zeroes at the poles of $\psi(\overline{z})$, that is, at points of the set $\{x_n\}_{n\in M}$. Also, $\xi(z) = \overline{G(z)}\psi(\overline{z})$. Moreover, we can write

$$\psi(\overline{z}) = \frac{\eta(\overline{z})}{\overline{z} - x_n} \quad \text{and} \quad G(z) = (\overline{z} - x_n)F(\overline{z}),$$

where $\eta(z) := (\overline{z} - x_n)\,(A_\sharp - z_0 I)\,(A_\sharp - zI)^{-1}\,\psi_0$ is analytic at $z = x_n$, $\eta(x_n) \neq 0$, and $F(x_n) = G'(x_n)$. Thus, a straightforward computation shows that

$$\frac{\langle \xi(z), \xi(x_n)\rangle}{\|\xi(x_n)\|^2} = \frac{G(z)}{(z-x_n)G'(x_n)} \frac{\langle \eta(\overline{z}), \eta(x_n)\rangle}{\|\eta(x_n)\|^2}$$

so it remains to verify that the last factor above equals one. By the Cauchy integral formula, we have

$$\begin{aligned}
\eta(x_n) &= \frac{1}{2\pi i} \oint_{|w-x_n|=\epsilon} \frac{\eta(w)}{w - x_n}\, dw \\
&= \frac{1}{2\pi i} \oint_{|w-x_n|=\epsilon} (A_\sharp - z_0 I)\,(A_\sharp - wI)^{-1}\,\psi_0\, dw \\
&= -\,(A_\sharp - z_0 I) \left(\frac{1}{2\pi i} \oint_{|w-x_n|=\epsilon} (wI - A_\sharp)^{-1}\, dw \right) \psi_0 \\
&= -\,(x_n - z_0)\, P_n \psi_0,
\end{aligned}$$

where P_n denotes the orthoprojector onto the eigenspace associated to x_n. Therefore,

$$\begin{aligned}
\langle \eta(\overline{z}), \eta(x_n)\rangle &= -\,(x_n - z_0)\,\langle \eta(\overline{z}), P_n\psi_0\rangle \\
&= -\,(x_n - z_0)\,(z - x_n)\,\left\langle P_n\,(A_\sharp - z_0 I)\,(A_\sharp - zI)^{-1}\,\psi_0, \psi_0 \right\rangle \\
&= |x_n - z_0|^2\,\langle P_n\psi_0, \psi_0\rangle\,.
\end{aligned}$$

Finally, it is clear that $\langle \eta(x_n), \eta(x_n)\rangle = |x_n - z_0|^2\,\langle P_n\psi_0, \psi_0\rangle$. $\square$

Remark 3. Notice that the function $G(z)$ is defined up to a factor. In particular, one may adjust it so that $G'(x_k) = 1$, where x_k is a fixed eigenvalue of $A_\sharp$. Thus, a computation like the one in the proof above shows that

$$G(z) = (z - x_k)\frac{\langle \xi(z), \xi(x_k)\rangle}{\|\xi(x_k)\|^2}. \tag{3.5}$$

This identity may be useful in some applications; see Example 2 below.

4 Spaces of analytic functions

In this section we characterize the set of functions given by the mapping Φ_μ

Let $\mathcal{R} \subset \mathcal{H}$ be the linear space of elements φ for which $\widehat{\varphi}(z)$ is analytic on $\mathbb{R}$. As a consequence of [11, Corollary 1.2.1], it follows that

$$\langle \varphi, \eta\rangle = \int_{-\infty}^{\infty} \overline{\widehat{\varphi}(x)}\,\widehat{\eta}(x)\,dm(x) \tag{4.1}$$

for any $\varphi, \eta \in \mathcal{R}$ and $m(x) = \langle E_x\mu, \mu\rangle$, where E_x is any spectral function of the operator A. That is, $\Phi_\mu\mathcal{R}$ is a linear space of analytic functions in $\mathbb{C} \setminus S_\mu$ such that their restriction to $\mathbb{R}$ belong to $L^2(\mathbb{R}, dm)$; in short,

$$\Phi_\mu\mathcal{R}|_{\mathbb{R}} \subset L^2(\mathbb{R}, dm).$$

Moreover, in this restricted sense Φ_μ is an isometry from $\mathcal{R}$ into $L^2(\mathbb{R}, dm)$.

The following theorem is due to Krein [18, Theorem 3]

Theorem 1 (Krein). *For $A \in Sym_R^{(1,1)}(\mathcal{H})$, assume that $\overline{\mathcal{R}} = \mathcal{H}$. Consider a distribution function $m(x) = \langle E_x\mu, \mu\rangle$, where E_x is a spectral function of A. Then the map Φ generates a bijective isometry from $\mathcal{H}$ onto $L^2(\mathbb{R}, dm)$ if and only if E_x is orthogonal.*

This theorem deserves some comments. For entire operators, $\Phi_\mu\mathcal{H}$ is a linear subset of entire functions. When E_x is orthogonal it occurs that

$$\Phi_\mu\mathcal{H}|_{\mathbb{R}} = L^2(\mathbb{R}, dm)$$

in the usual sense of equivalence classes. Thus, every function in $L^2(\mathbb{R}, dm)$ is, up to a set of measure zero with respect to $m(x)$, the restriction to $\mathbb{R}$ of a unique entire function that is the image under Φ_μ of one and only element belonging to $\mathcal{H}$. In passing, we notice that every orthogonal spectral function of the operator A is the restriction to $\mathrm{Dom}(A)$ of the spectral function of some of its self-adjoint extensions within $\mathcal{H}$. Since these self-adjoint extensions have only discrete spectrum, the inner product in $L^2(\mathbb{R}, dm)$, with $m(x) = \langle E_x\mu, \mu\rangle$, reduces to an expression like

$$\int_{-\infty}^{\infty} \overline{f(x)}g(x)\,dm(x) = \sum_k c_k\overline{f(x_k)}g(x_k)$$

whenever E_x is orthogonal. That is, the equivalence classes in these spaces are quite broad.

The following corollary is partly a straightforward consequence of Theorem 1. Notice that $S_\mu \cap \mathbb{R} = \emptyset$ implies $\mathcal{R} = \mathcal{H}$.

Corollary 1. *Let $A \in Sym_R^{(1,1)}(\mathcal{H})$ and choose a gauge μ for this operator. Assume that $S_\mu \cap \mathbb{R} = \emptyset$. Let E_x be one of its orthogonal spectral functions. Then the linear space of functions $\widehat{\mathcal{H}}_\mu := \Phi_\mu \mathcal{H}$, equipped with the inner product*

$$\langle f(\cdot), g(\cdot)\rangle := \int_{-\infty}^{\infty} \overline{f(x)} g(x)\, dm(x) \qquad \text{where } m(x) = \langle E_x \mu, \mu\rangle\,, \tag{4.2}$$

is a reproducing kernel Hilbert space, with reproducing kernel $k(z,w) := \langle \xi(z), \xi(w)\rangle$.

Proof. We only verify the last statement. Given $f(z) = \langle \xi(z), \varphi\rangle \in \widehat{\mathcal{H}}_\mu$, we have

$$\langle k(\cdot, w), f(\cdot)\rangle = \int_{-\infty}^{\infty} \overline{k(x,w)} f(x)\, dm(x) = \langle \xi(w), \varphi\rangle = f(w),$$

where the second equality follows from (4.1). ☐

Notice that the linear space $\widehat{\mathcal{H}}_\mu$ depends only on the choice of gauge μ. By Corollary 1, for those gauges that obeys $S_\mu \cap \mathbb{R} = \emptyset$, $\widehat{\mathcal{H}}_\mu$ may be endowed with different Hilbert space structures, one for each orthogonal spectral function of the operator A. By (4.1), all these Hilbert spaces are however isometrically equivalent.

Irrespective of any Hilbert space structure, $\widehat{\mathcal{H}}_\mu$ possesses the following properties.

Proposition 3. *Let w be a non real zero of $f(z) \in \widehat{\mathcal{H}}_\mu$. Then the function $g(z) := f(z)(z - \overline{w})(z - w)^{-1}$ is also in $\widehat{\mathcal{H}}_\mu$ and $\|g(\cdot)\| = \|f(\cdot)\|$.*

Proof. Suppose that w is a non real zero of $f(z)$. Since $f(z) = \langle \xi(z), \varphi\rangle$ for some $\varphi \in \mathcal{H}$, it follows that φ is orthogonal to $\psi(\overline{w})$ and therefore $\varphi \in \operatorname{Ran}(A - wI)$. Note that

$$\begin{aligned} f(z) &= \left\langle \xi(z), (A - wI)(A - wI)^{-1} \varphi\right\rangle \\ &= \left\langle [A^* - \overline{z} + (\overline{z} - \overline{w}) I]\, \xi(z), (A - wI)^{-1} \varphi\right\rangle \\ &= (z - w) \left\langle \xi(z), (A - wI)^{-1} \varphi\right\rangle . \end{aligned}$$

In these computations we have used that $\xi(z) \in \operatorname{Ker}(A^* - \overline{z})$. Moreover,

$$\begin{aligned} \left\langle \xi(z), (A - \overline{w}I)(A - wI)^{-1} \varphi\right\rangle &= \left\langle [A^* - \overline{z}I + (\overline{z} - w) I]\, \xi(z), (A - wI)^{-1} \varphi\right\rangle \\ &= (z - \overline{w}) \left\langle \xi(z), (A - wI)^{-1} \varphi\right\rangle \\ &= \frac{z - \overline{w}}{z - w} f(z). \end{aligned}$$

Then $f(z)(z - \overline{w})(z - w)^{-1} \in \widehat{\mathcal{H}}_\mu$. The equality of norms follows from the fact that the Cayley transform $(A - \overline{w}I)(A - wI)^{-1}$ is an isometry. ☐

Proposition 4. *The evaluation functional, defined by $f(\cdot) \mapsto f(z)$, is continuous.*

Proof. Let $f(z), g(z) \in \widehat{\mathcal{H}}_\mu$. Then $f(z) = \langle \xi(z), \varphi \rangle$ and $g(z) = \langle \xi(z), \eta \rangle$ for some $\varphi, \eta \in \mathcal{H}$, and furthermore

$$|f(w) - g(w)| = |\langle \xi(w), \varphi - \eta \rangle| \leq \|\xi(w)\| \, \|\varphi - \eta\| = \|\xi(w)\| \, \|f(\cdot) - g(\cdot)\| \, .$$

In other words, this result follows from the fact that $\widehat{\mathcal{H}}_\mu$ is a reproducing kernel Hilbert space. □

Definition 2. An operator $A \in \mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ is called *entire* if there exists a so-called *entire gauge* $\mu \in \mathcal{H}$ such that $\widehat{\varphi}(z)$ is an entire function for every $\varphi \in \mathcal{H}$. Equivalently, A is entire if $\mathcal{H} = \operatorname{Ran}(A - zI) \dotplus \operatorname{Span}\{\mu\}$ for all $z \in \mathbb{C}$.

Notice that if A is entire, $\xi(z)$ is a vector-valued entire function and $\widehat{\mathcal{H}}_\mu$ is a Hilbert space of entire functions.

Definition 3. A Hilbert space of entire functions is called a *de Branges space* if, for every $f(z)$ in that space, the following conditions holds:

(i) for every non real zero w of $f(z)$, the function $f(z)(z - \overline{w})(z - w)^{-1}$ belongs to the Hilbert space and has the same norm as $f(z)$,

(ii) the function $f^*(z) := \overline{f(\overline{z})}$ belongs to the Hilbert space and also has the same norm as $f(z)$;

and furthermore,

(iii) for every $w : \operatorname{Im} w \neq 0$, the linear functional $f(\cdot) \mapsto f(w)$ is continuous.

There is an extensive literature concerning the properties of de Branges spaces. We refer to [4] for more details.

It can be shown that $\widehat{\mathcal{H}}_\mu$ is a de Branges space for certain choices of the entire gauge μ. There are some evidence indicating that Krein noticed this fact [11, pag. 209]. Also, some hints supporting this assertion has been given by de Branges himself [5]. We however could not find any formal proof of this statement. Thus, for the sake of completeness and for the lack of a proper reference, we provide a proof below. In any case, we do not claim any originality regarding this matter.

Remark 4. As a consequence of [11, Lemma 2.7.1], given any self-adjoint extension $A_\sharp$ of an operator $A \in \mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$, one can always find a complex conjugation C for which $A_\sharp$ is real. If follows from the proof of the cited lemma that $C\psi(z) = \psi(\overline{z})$ when $\psi(z)$ is written in terms of the real self-adjoint extension. Moreover, by [11, Theorem 2.7.1], the operator A is also real with respect to C. Since by [22, Corollary 2.5] all the self-adjoint extensions of A are real, it follows that $C\psi(z) = \psi(\overline{z})$ for every realization of $\psi(z)$.

If furthermore A is entire, then an entire gauge μ may be chosen such that $C\mu = \mu$ (see [17, Theorem 1] and also [11, Section 2.7.7]).

Proposition 5. *Assume that an entire operator A is real with respect to some complex conjugation C and let μ be a real entire gauge. Then the associated Hilbert space $\widehat{\mathcal{H}}_\mu$ is a de Branges space.*

Proof. In view of Propositions 3 and 4 we only have to verify (ii). By Remark 4 we know that $C\psi(\overline{z}) = \psi(z)$ thence $C\xi(z) = \xi(\overline{z})$. Now consider any $f(z) = \langle \xi(z), \varphi \rangle$. Clearly $f^*(z) := \langle \xi(z), C\varphi \rangle$ also belongs to $\widehat{\mathcal{H}}_\mu$. Furthermore,

$$f^*(z) = \langle \xi(z), C\varphi \rangle = \overline{\langle C\xi(z), \varphi \rangle} = \overline{\langle \xi(\overline{z}), \varphi \rangle} = \overline{f(\overline{z})}.$$

Since C is an isometry, the equality of norms follows. $\square$

Notice that we only have used the simultaneous reality of the entire operator and its entire gauge μ in showing that $\widehat{\mathcal{H}}_\mu$ obeys (ii) of Definition 3. Indeed, this condition is also necessary.

Proposition 6. *If $\widehat{\mathcal{H}}_\mu$ is a de Branges space there is a complex conjugation C with respect to which both A and μ are real.*

Proof. Let $\widehat{C} : \widehat{\mathcal{H}}_\mu \to \widehat{\mathcal{H}}_\mu$ be defined by $(\widehat{C}f)(z) = \overline{f(\overline{z})}$, for every $f(z) \in \widehat{\mathcal{H}}$. Clearly, $\widehat{C}$ is a complex conjugation. Moreover, the self-adjoint multiplication operator $\widehat{A}$, defined on the maximal domain in $\widehat{\mathcal{H}}$ by $(\widehat{A}f)(z) = zf(z)$, is real with respect to $\widehat{C}$. Now define $C := \Phi_\mu^{-1}\widehat{C}\Phi_\mu$. By construction, C is a complex conjugation in $\mathcal{H}$. Since $(\widehat{C}\widehat{\mu})(z) \equiv 1 \equiv \widehat{\mu}(z)$, it follows that μ is real with respect to C. Finally, it not difficult to see that $\Phi_\mu^{-1}\widehat{A}\Phi_\mu$ is a self-adjoint extension of A real with respect to C. The considerations of Remark 4 complete the proof. $\square$

5 Examples

Example 1. Consider the semi-infinite Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \tag{5.1}$$

where $b_k > 0$ and $q_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Fix an orthonormal basis $\{\delta_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$. Let J be the operator in $\mathcal{H}$ whose matrix representation with respect to $\{\delta_k\}_{k\in\mathbb{N}}$ is (5.1). Thus, J is the minimal closed operator satisfying

$$\langle \delta_n, J\delta_n \rangle = q_n\,, \quad \langle \delta_{n+1}, J\delta_n \rangle = \langle \delta_n, J\delta_{n+1} \rangle = b_n\,, \quad \forall n \in \mathbb{N}\,.$$

(Consult [2] for a discussion on matrix representation of unbounded symmetric operators.) It is well known that J may have only deficiency indices $(1,1)$ or $(0,0)$ [1]. A classical result is that if J has deficiency indices $(1,1)$, then the orthogonal polynomials of the first kind $P_k(z)$ associated with (5.1) are such that

$$\sum_{k=0}^{\infty} |P_k(z)|^2 < \infty$$

uniformly in any compact domain of the complex plane [1] . Therefore, for any $z \in \mathbb{C}$, $\pi(z) = \sum_{k=1}^{\infty} P_{k-1}(z)\delta_k$ is in $\mathcal{H}$. By construction, $\pi(z)$ is in the one-dimensional space $\operatorname{Ker}(J^* - zI)$. It is also known that, when the deficiency indices are (1,1), J is an entire operator and δ_1 is an entire gauge for J [11, Section 3.1.1 and Theorem 3.1.2].

Let us find $\xi(z)$ for the operator J. Taking into account (3.2), $\langle \delta_1, \xi(z) \rangle = 1$ and $\langle \delta_1, \pi(\overline{z}) \rangle = 1$ for all $z \in \mathbb{C}$. Then, since both $\pi(\overline{z})$ and $\xi(z)$ are in $\operatorname{Ker}(J^* - \overline{z}I)$ and δ_1 is entire, $\pi(\overline{z}) = \xi(z)$ for all $z \in \mathbb{C}$. Thus, for any φ in $\mathcal{H}$ we have $\varphi = \sum_{k=1}^{\infty} \varphi_k \delta_k$ and $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}_{\delta_1}$ is then given by

$$\widehat{\varphi}(z) := \langle \pi(\overline{z}), \varphi \rangle = \sum_{k=1}^{\infty} P_k(z)\varphi_k \,, \quad z \in \mathbb{C} \,.$$

Clearly, if $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}_{\delta_1}$, $\overline{\widehat{\varphi}(\overline{z})} \in \widehat{\mathcal{H}}_{\delta_1}$. Whence, in virtue of Propositions 3 and 4, our space $\widehat{\mathcal{H}}_{\delta_1}$ is a de Branges space and then, by Proposition 6, δ_1 is real with respect to $C = \Phi_{\delta_1}^{-1} \widehat{C} \Phi_{\delta_1}$ ($\widehat{C}$ is the conjugation in $\widehat{\mathcal{H}}_{\delta_1}$ given in the proof of Proposition 6).

Taking into account that $\|\pi(\overline{z})\| \equiv 1$, formula (3.4) is written in this case as

$$\begin{aligned} f(z) &= \sum_{x_n \in \operatorname{Sp}(J_\sharp)} \langle \pi(\overline{z}) \pi(x_n) \rangle f(x_n) \\ &= \sum_{x_n \in \operatorname{Sp}(J_\sharp)} f(x_n) \sum_{k=1}^{\infty} P_k(z) P_k(x_n) \,, \quad z \in \mathbb{C} \,, \end{aligned}$$

where $J_\sharp$ is certain self-adjoint extension of J.

In a different setting, sampling formulas obtained on the basis of Jacobi operators have been studied before [6, 7].

Example 2. The entire operator used here has been taken from [11] and is a particular case of an example given by Krein in [20].

Consider a non-decreasing bounded function $s(t)$ such that

$$s(-\infty) = 0 \quad \text{and} \quad s(t-0) = s(t) \,.$$

Fix a function defined for any x in the real interval $(-a, a)$ by

$$F(x) := \int_{-\infty}^{\infty} e^{ixt} ds(t) \,.$$

In the linear space $\widetilde{\mathcal{L}}$ of continuous functions in $[0, a)$ vanishing in some left neighborhood of a, we define a sesquilinear form as follows

$$(g, f) := \int_0^a \int_0^a F(x-t) f(x) \overline{g(t)} dx dt \,. \tag{5.2}$$

This form is a quasi-scalar product, i. e., the existence of elements f in $\widetilde{\mathcal{L}}$ such that $f \neq 0$ and nevertheless $(f, f) = 0$ is not excluded.

Denote by $\mathcal{D}$ the set of continuously differentiable functions $f \in \widetilde{\mathcal{L}}$ such that $f(0) = 0$ and define in $\mathcal{D}$ the differential operator $\widetilde{A}$ by the rule $\widetilde{A}f := i\partial_x f$. It is not difficult to show that $(g, \widetilde{A}f) = (\widetilde{A}g, f)$ and $\mathcal{D}$ is quasi-dense in $\widetilde{\mathcal{L}}$. Now, proceeding as in [11, Section 2.8.2], one defines the space $\mathcal{L}$ as follows

$$\mathcal{L} = \widetilde{\mathcal{L}} \setminus \widetilde{0} \,, \qquad \widetilde{0} = \{f \in \widetilde{\mathcal{L}} : (f, f) = 0\} \,.$$

In $\mathcal{L}$ we define an inner product by

$$\langle \eta, \varphi \rangle := (g, f) \,, \tag{5.3}$$

where φ and η are equivalence classes containing f and g, respectively. Let $\mathcal{H}$ be the completion of $\mathcal{L}$ and consider in it the operator A such that, for the equivalence class φ containing $f \in \mathcal{D}$, $A\varphi$ is the equivalence class containing $\widetilde{A}f$. It can be shown that A is an entire operator and that

$$\widehat{\varphi}(z) = \langle \xi(z), \varphi \rangle = \int_0^a e^{izt} f(t) dt \,,$$

where $f \in \varphi$. This identity, together with (5.2) and (5.3), determines $\xi(z)$ completely [11, Section 3.2.2].

Notice that, in this example, the entire gauge associated to $\xi(z)$ remains unknown. This is not however an issue since the sampling kernel can be computed anyway by resorting to expression (3.5).

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