Perturbation theory of $\mathcal{PT}$ symmetric Hamiltonians

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Abstract

In the framework of perturbation theory the reality of the perturbed eigenvalues of a class of $\mathcal{PT}$symmetric Hamiltonians is proved using stability techniques. We apply this method to $\mathcal{PT}$symmetric unperturbed Hamiltonians perturbed by $\mathcal{PT}$symmetric additional interactions.

1 Introduction

Perturbation theory has played a very important role in the past ([1]) in the study of non Hermitian Hamiltonians with $\mathcal{PT}$ symmetry ([2]). In [3, 4, 5] it was applied starting from a selfadjoint Hamiltonian in order to investigate the perturbation of its spectrum by a $\mathcal{PT}$ symmetric interaction. In the present paper we try to present more general results concerning in general the perturbation of a non selfadjoint $\mathcal{PT}$ symmetric Hamiltonian:

$$H \neq H^* , \quad H\mathcal{PT} = \mathcal{PT}H \, .$$

This extension of perturbation theory will be shown to have non trivial aspects even when one restricts oneself to unperturbed discrete simple levels with real energies. Our aim is to provide a consistent and selfcontained framework where well established results and new developments can be discussed coherently. The main original aspects that we present are:

1. extension of perturbation theory for $\mathcal{PT}$ symmetric Hamiltonians in order to prove the reality of the spectrum of a class of $\mathcal{PT}$ symmetric operators perturbed by a $\mathcal{PT}$ symmetric interaction;

2. the interaction does not need to be bounded relative to the unperturbed Hamiltonian;

3. the results concerning the reality of the eigenvalues are achieved by using the stability theory developed by Hunziker and Vock (HV-stability theory) in [6].

The paper is organized as follows. In Section 2 we give a general presentation of perturbation theory for non selfadjoint $\mathcal{PT}$ symmetric operators and a review of the stability theory for eigenvalues. In Section 3 we give the technical results leaving the proof to Section 4. Some open problems and further perspectives are outlined in the Conclusions (Section 5).

2 General formalism in perturbation theory and review of results

An operator is called $\mathcal{PT}$ symmetric if it is invariant under the combined action of a reflection operator $\mathcal{P}$ and the antilinear complex conjugation operator $\mathcal{T}$. One basic issue is to prove the reality of the spectrum. Results in this direction have been obtained in [7, 8] by means of ODE techniques, and they have been recently extended in [3, 4, 5] (see also [9] for a brief review) in the framework of perturbation theory by perturbing selfadjoint Hamiltonians. The main goal of this paper is to extend part of these results to the case of $\mathcal{PT}$ symmetric Hamiltonians obtained by perturbing a $\mathcal{PT}$ symmetric operator, not necessarily selfadjoint. Thus, for the convenience of the reader, we first recall the results of [3, 4, 5], concerning $\mathcal{PT}$ symmetric operators in a Hilbert space $(\mathcal{H}, < u, v >)$ of the form

$$H_g = H_0 + igW\,, \quad g \in \mathbf{R}\,,$$

where H_0 is selfadjoint, with domain $D \subset \mathcal{H}$, and W is a symmetric operator, relatively bounded with respect to H_0. Moreover there exist a unitary involution $\mathcal{P} : \mathcal{H} \to \mathcal{H}$ and an antilinear involution $\mathcal{T} : \mathcal{H} \to \mathcal{H}$, both mapping D to D, such that

$$H_0\mathcal{P} = \mathcal{P}H_0,\ \mathcal{P}W = -W\mathcal{P},\ \mathcal{T}H_0 = H_0\mathcal{T},\ \mathcal{T}W = W\mathcal{T}\,. \tag{2.1}$$

Then H_g is $\mathcal{PT}$ symmetric, i.e. $\mathcal{PT}H_g = H_g\mathcal{PT}$, for $g \in \mathbf{R}$. An example is provided by $\mathcal{PT}$ symmetric Schrödinger operators in $L^2(\mathbf{R}^d)$, $d \geq 1$, where H_0 is the selfadjoint realization of $-\Delta + V$, $\mathcal{T}$ is complex conjugation, i.e. $(\mathcal{T}\psi)(x) = \overline{\psi(x)}$, and $\mathcal{P}$ is the parity operation defined by

$$(\mathcal{P}\psi)(x) = \psi((-1)^{j_1}x_1, ..., (-1)^{j_d}x_d)\,, \quad \psi \in L^2(\mathbf{R}^d) \tag{2.2}$$

where $j_k = 0, 1$, and $j_k = 1$ for at least one $1 \leq k \leq d$. Here $-\Delta$ denotes the d-dimensional Laplace operator; V and W are real valued functions, $\mathcal{P}$-even and $\mathcal{P}$-odd respectively: $\mathcal{P}V = V, \mathcal{P}W = -W$. The following theorem, proved in [3, 4], provides a result in the case of bounded perturbation.

Theorem 2.1 *Let H_0 be a selfadjoint operator in $\mathcal{H}$ and W a symmetric operator in $\mathcal{H}$, satisfying the above assumption (2.1). Assume further that H_0 is bounded below, W is bounded and that the spectrum of H_0 is discrete. Let $\sigma(H_0) = \{E_j : j = 0, 1, \ldots\}$ denote the incresing sequence of distinct eigenvalues of H_0. Finally, let $\delta := \inf_{j\geq 0} [E_{j+1} - E_j]/2$ and assume that $\delta > 0$. Then the following results hold:*

(i) if for each degenerate eigenvalue of H_0 the corresponding eigenvectors have the same $\mathcal{P}$-parity, i.e. they are either all $\mathcal{P}$-even or all $\mathcal{P}$-odd, then $\sigma(H_g) \subset \mathbf{R}$, if $g \in \mathbf{R}$, $|g| < \delta/\|W\|$;

(ii) if H_0 has an eigenvalue E with multiplicity 2 whose corresponding eigenvectors have opposite $\mathcal{P}$-parity, i.e. one is $\mathcal{P}$-even and the other one is $\mathcal{P}$-odd, then H_g has a pair of non real complex conjugate eigenvalues near E for $|g|$ small, $g \in \mathbf{R}$.

The proof of the reality of the spectrum under suitable conditions has been extended to the case of relatively bounded perturbation in [5] and is recalled in the following

Theorem 2.2 *Let*

$$H_g = -\frac{d^2}{dx^2} + V(x) + igW(x) \quad \text{in } L^2(\mathbf{R}),$$

where $V(x)$ is a real-valued even polynomial of degree $2l$, with $\lim_{|x|\to\infty} V(x) = +\infty$ and $W(x)$ is a real-valued odd polynomial of degree $2r-1$, with $l > 2r$. Then there exists $g_0 > 0$ such that $\sigma(H_g) \subset \mathbf{R}$ for $|g| < g_0$.

Remark 2.3 The above theorems enlarge the class of $\mathcal{PT}$symmetric Hamiltonians with real spectrum provided in [7, 8].

The question that we want to address in this paper is the folllowing. Let now H_0 be $\mathcal{PT}$ symmetric (not necessarily selfadjoint) with real spectrum and let W be $\mathcal{PT}$ symmetric as well. We will provide conditions on H_0 and W in order to guarantee that the perturbed eigenvalues of $H(\epsilon) := H_0 + \epsilon W$, ϵ real, are real for $|\epsilon|$ small. Therefore we shall not prove that the whole spectrum of $H(\epsilon)$ is real, but that at least the eigenvalues generated by the unperturbed real ones stay real. First of all notice that if H_0 has degenerate eigenvalues there may be problems even in the selfadjoint case, as stated in Theorem 2.1(ii). In general we have

$$m_g(\lambda) \leq m_a(\lambda) \tag{2.3}$$

where

$$m_g(\lambda) = \dim\{u : (H_0 - \lambda)u = 0\} \tag{2.4}$$

is the geometric multiplicity of $\lambda \in \mathbf{C}$ and

$$m_a(\lambda) = \dim\{u : (H_0 - \lambda)^n u = 0\,, \text{for some } n \in \mathbf{N}\} \tag{2.5}$$

is the algebraic multiplicity of λ. Notice that in the finite dimensional case (i.e. H_0 and W are matrices) $m_a(\lambda)$ is the multiplicity of λ as a root of the characteristic polynomial of H_0. If H_0 is selfadjoint then $m_g(\lambda) = m_a(\lambda)$, for all $\lambda \in \mathbf{C}$. However in the general (non selfadjoint) $\mathcal{PT}$symmetric case it is not enough to assume that $m_g(\lambda) = 1$ in order to guarantee that the eigenvalues are simple, i.e. that there is no degeneracy, i.e. that $m_a(\lambda) = 1$. In fact there may be "exceptional points", caused by the presence of Jordan blocks.

Example 2.4 Set

$$H_0 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6}$$

and let $\mathcal{T}$ be complex conjugation. Then H_0 is $\mathcal{PT}$ symmetric. Its characteristic polynomial $\det(H_0 - \lambda I) = \lambda^2$ has just one root $\lambda_0 = 0$ with $m_g(\lambda_0) = 1$ and $m_a(\lambda_0) = 2$. If we take

$$W = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

as a perturbation, then $H(\epsilon) = H_0 + \epsilon W$ has a pair of non-real complex conjugate eigenvalues $\lambda(\epsilon) = \pm i\sqrt{\epsilon(\epsilon + 2)}$ for $\epsilon > 0$.

To avoid such difficulties we analize here only the non-degenerate case, i.e. we assume that

$$m_a(\lambda) = 1 \tag{2.7}$$

for all eigenvalues λ of H_0. The Hamiltonians H_0 that we perturb are one-dimensional $\mathcal{PT}$symmetric Schrödinger operators with real simple spectrum, for example the Hamiltonians H_g provided by the above Theorems 2.1 and 2.2 for $|g|$ small.

Remark 2.5 Perturbation theory for matrices in the non-degenerate case provides a straightforward result because:

(1) the stability of the unperturbed eigenvalues (see Definition 2.6 below) is guaranteed by the boundedness of the perturbation W (any matrix W is a bounded operator). This implies that near any unperturbed eigenvalue λ of H_0 there is one and only one eigenvalue $\lambda(\epsilon)$ of $H(\epsilon)$ for $|\epsilon|$ small and $\lim_{\epsilon\to 0} \lambda(\epsilon) = \lambda$. Thus $m_a(\lambda(\epsilon)) = 1$;

(2) the eigenvalues of $\mathcal{PT}$symmetric operators come in pairs of complex conjugates, i.e. if $\lambda(\epsilon)$ is an eigenvalue of $H(\epsilon)$, then $\overline{\lambda(\epsilon)}$ is an eigenvalue of $H(\epsilon)$ too.

Therefore, combining (1) and (2), $\lambda(\epsilon) \in \mathbf{R}$.

We will provide new results on the reality of the perturbed eigenvalues ($\lambda(\epsilon)$) in the case of unbounded (not necessarily relatively bounded) perturbation. As observed in Remark 2.5, a crucial issue is the stability of the unperturbed eigenvalues, i.e. the problem is reduced to a stability result for the eigenvalues of H_0. Such result is immediate if the perturbation W is bounded relative to H_0 (see e.g. [10, 11]). Indeed, in this case the Rayleigh-Schrödinger perturbation expansion (RSPE) is convergent, for $|\epsilon|$ small, to the (real) perturbed eigenvalues. However there can be stability even if the RSPE does not exist. We provide here a brief reminder on stability (see [10] and [6]).

Definition 2.6 *A discrete eigenvalue λ of H_0 is stable with respect to (w.r.t.) the family $H(\epsilon) = H_0 + \epsilon W$ if*

(i) for any small $r > 0$,

$$\Gamma_r = \{z : \mid z - \lambda \mid = r\} \subset \rho(H(\epsilon))\,, \quad \text{as } \epsilon \to 0$$

where $\rho(H(\epsilon)) := \mathbf{C} - \sigma(H(\epsilon))$ is the resolvent set of $H(\epsilon)$;

(ii) $\lim_{\epsilon \to 0} \parallel P(\epsilon) - P(0) \parallel = 0$ *where*

$$P(\epsilon) = (2\pi i)^{-1} \oint_{\Gamma_r} (z - H(\epsilon))^{-1} dz$$

is the spectral projection of $H(\epsilon)$ corresponding to the part of the spectrum enclosed in Γ_r, and $H(0) := H_0$.

Since (ii) implies that

$$\dim P(\epsilon) = \dim P(0) \ (= m_a(\lambda)) \tag{2.8}$$

for $|\epsilon|$ small, λ is the limit of a group of perturbed eigenvalues with the same total algebraic multiplicity (see [10]). If W is bounded relative to H_0 there exist $a, b > 0$ such that

$$\parallel Wu \parallel \leq b \parallel H_0 u \parallel + a \parallel u \parallel\,, \quad \forall u \in D(H_0). \tag{2.9}$$

Then

$$\parallel (z_0 - H(\epsilon))^{-1} - (z_0 - H_0)^{-1} \parallel \to 0\,, \quad \text{as } \epsilon \to 0 \tag{2.10}$$

for some $z_0 \notin \sigma(H_0)$ and this implies (ii).

Remark 2.7 Eigenvalues may be stable even if (2.10) fails. As an example let $H(\epsilon) = p^2+x^2+\epsilon x^4$, $\epsilon > 0$, in $\mathcal{H} = L^2(\mathbf{R})$ be the Hamiltonian of the even anharmonic oscillator. Here $p^2 = -d^2/dx^2$. Then the eigenvalues of the harmonic oscillator $H(0)$ are stable w.r.t. $H(\epsilon)$, $\epsilon \geq 0$, in spite of the fact that (2.10) fails and the RSPE is divergent. In other words, the continuity of the eigenvalues at $\epsilon = 0$ holds although analitycity fails. In this case we only have strong resolvent convergence, i.e.:

$$\lim_{\epsilon \to 0} (z_0 - H(\epsilon))^{-1} u = (z_0 - H(0))^{-1} u\,, \quad \forall u \in \mathcal{H} \tag{2.11}$$

for some $z_0 \notin \sigma(H(0))$. Then (2.11) yields the strong convergence of the projections $P(\epsilon)$, i.e.

$$\lim_{\epsilon \to 0} P(\epsilon) u = P(0) u\,, \quad \forall u \in \mathcal{H} \tag{2.12}$$

which implies

$$\mathrm{dim} P(\epsilon) \geq \mathrm{dim} P(0)\,, \quad \text{as } \epsilon \to 0\,. \tag{2.13}$$

So $H(\epsilon)$ may have more eigenvalues than $H(0)$ in the circle Γ_r. This happens for instance for the double-well operator

$$H(\epsilon) = p^2 + x^2(1-\epsilon x)^2\,, \quad \text{in } L^2(\mathbf{R}). \tag{2.14}$$

In fact near any eigenvalue of the harmonic oscillator $H(0)$ there are two eigenvalues of $H(\epsilon)$ for $|\epsilon|$ small.

3 Statement of the results

Let $H(\epsilon) = p^2 + V + \epsilon W, \epsilon \geq 0$, denote the closed operator in $L^2(\mathbf{R})$ with $C_0^\infty(\mathbf{R})$ as a core, defined by

$$H(\epsilon) u = -u'' + Vu + \epsilon W u\,, \quad \forall u \in D(H(\epsilon))\,, \tag{3.1}$$

where $V = V_+ + iV_-$, $W = W_+ + iW_-$ and V_+, V_-, W_+, W_- are real valued functions in $L^\infty_{loc}(\mathbf{R})$. Moreover V_+, W_+ are $\mathcal{P}$-even and bounded below, V_-, W_- are $\mathcal{P}$-odd and

$$\lim_{\substack{|x| \to \infty \\ \epsilon \to 0}} |V(x) + \epsilon W(x)| = +\infty. \tag{3.2}$$

Here $(\mathcal{P}u)(x) = u(-x), \forall u \in L^2(\mathbf{R})$. Let us further assume that the spectrum of $H(\epsilon)$, denoted $\sigma(H(\epsilon))$, is discrete for $\epsilon \in [0, \epsilon_0]$, i.e. it consists of a sequence of isolated eigenvalues with finite algebraic multiplicity:

$$\sigma(H(\epsilon)) = \{E_j(\epsilon) : j = 0, 1, \ldots\}. \tag{3.3}$$

For the unperturbed eigenvalues $(E_j(0))$ we will adopt the simplified notation $E_j := E_j(0), j = 0, 1, \ldots$

Theorem 3.1 *Under the above assumptions the following statements hold:*

(1) each eigenvalue E_j of $H(0)$ is stable w.r.t. the family $H(\epsilon), \epsilon > 0$. In particular, if E_j is simple, i.e. $m_a(E_j) = 1$, and real there exists $\epsilon_j > 0$ such that for $0 < \epsilon < \epsilon_j$, $H(\epsilon)$ has exactly one eigenvalue $E_j(\epsilon)$ close to E_j:

$$\lim_{\epsilon \to 0} E_j(\epsilon) = E_j$$

and $E_j(\epsilon)$ is real;

(2) there are no "dying eigenvalues", i.e. if $E(\epsilon) \in \sigma(H(\epsilon))$ and $\lim_{\epsilon \to 0} E(\epsilon) = E$, then E is an eigenvalue of $H(0)$.

In the following examples all the above conditions are satisfied. Moreover, the eigenvalues of $H(0)$ are real and simple.

Example 3.2 The unperturbed Hamiltonian $H(0)$ can be any of the operators H_g satisfying the above Theorems 2.1 and 2.2, for instance $H(0) = p^2 + x^{2n} + ig \sin x$, $H(0) = p^2 + x^{2n} + igx/(x^2 + 1)$, $H(0) = p^2 + x^{2l} + igx^{2q-1}$, with $|g|$ small, $n, l, q \in \mathbf{N}$ and $l > 2q$. Another example is provided by $H(0) = p^2 + ix^{2k+1}$, $k \in \mathbf{N}$. In all these cases the perturbation W can be taken of the form $W = W_+ + iW_-$ where $W_+ = \exp(x^2)$, or W_+ is an even polynomial function diverging positively at infinity, and W_- is an odd polynomial function bounded relative to W_+. If $H(0) = p^2 + ix^{2k+1}$, $k \in \mathbf{N}$, the polynomial W_- must diverge positively at $+\infty$ in order to guarantee that condition (3.2) is satisfied.

Corollary 3.3 *Assume that the eigenvalues $E_j, j \in \mathbf{N}$, of $H(0)$ are all simple and real. Then all the perturbed eigenvalues $E_j(\epsilon), 0 < \epsilon < \epsilon_j$, of $H(\epsilon)$ are real and simple. Moreover (non-real) complex eigenvalues of $H(\epsilon)$ cannot accumulate at finite points but only at infinity.*

Remark 3.4 *1.* The question of the reality of the whole spectrum of $H(\epsilon)$ is still open, because there may be (possibly complex) eigenvalues of $H(\epsilon)$ diverging to infinity as $\epsilon \to 0$.

2. Hamiltonians satisfying Corollary 3.3 are provided in the above Example 3.2.

If the perturbation W is bounded relative to $H(0)$ then the RSPE near any E_j converges. We have therefore the following

Corollary 3.5 *If the unperturbed eigenvalues E_j are simple and real and W is bounded relative to $H(0)$, then the RSPE near E_j is real for all $j = 0, 1, \ldots$. More precisely, for $0 < \epsilon < \epsilon_j$*

$$E_j(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \ , \quad a_0 = E_j \tag{3.4}$$

and $a_n \in \mathbf{R}, \forall n \in \mathbf{N}$.

Remark 3.6 If W is not relatively bounded w.r.t. $H(0)$, the RSPE although divergent can be Borel summable to $E_j(\epsilon)$. In this case the coefficients of the RSPE are real. This happens for instance for the Hamiltonians $H(0)$ and the perturbations W of polynomial type described in Example 3.2.

Remark 3.7 The stability theory developed in [6] allows one to prove a result similar to that stated in Theorem 3.1 also in presence of essential spectrum provided that

$$\mathrm{dist}(E_j, \sigma_{ess}(H(\epsilon))) \geq c > 0 \ , \quad \text{as } \epsilon \to 0. \tag{3.5}$$

where $\sigma_{ess}(H(\epsilon))$ is the complement in $\sigma(H(\epsilon))$ of the discrete spectrum of $H(\epsilon)$.

4 Proof of Theorem 3.1

Although it is a straightforward application of the HV-stability theory developed in [6], for a pedagogical purpose and for the convenience of the reader we describe here the main steps. The only "continuity condition" required by the HV-theory and guaranteed by the assumptions of the theorem is:

$$\lim_{\epsilon \to 0} H(\epsilon) u = H(0) u \, , \quad \forall u \in C_0^{\infty}(\mathbf{R}). \tag{4.1}$$

Condition (4.1) implies the strong convergence of the resolvents (2.11) which yields (2.13) and is not enough, as already remarked, to ensure stability. One key ingredient is the numerical range of $H(\epsilon)$:

$$N(\epsilon) := \{ < u, H(\epsilon) u >: u \in D(H(\epsilon)), \|u\| = 1 \} \tag{4.2}$$

which contains the eigenvalues of $H(\epsilon)$ and in the present case is contained in the right half-plane:

$$\sigma(H(\epsilon)) \subset N(\epsilon) \subset \mathcal{R}_+ := \{ z : \mathrm{Re}\, z \geq 0 \}. \tag{4.3}$$

Next we need to introduce the set of uniform boundedness of the resolvents:

$$\mathcal{D} := \{z : (z - H(\epsilon))^{-1} \,\text{exists and is uniformly bounded as}\, \epsilon \to 0\}.$$

The sets $\mathcal{D}$ and $N(\epsilon)$ are closely related to each other (see e.g. [10]), since $\mathbf{C} - N(\epsilon) \subset \mathcal{D}$. In particular for $z \in \mathcal{R}_- := \{z : \mathrm{Re}\, z < 0\}$, we have

$$\|(z - H(\epsilon))^{-1}\| \leq \mathrm{dist}(z, N(\epsilon))^{-1} \leq |Rez|^{-1}, \quad \forall \epsilon > 0. \tag{4.4}$$

Thus $z \in \mathcal{D}$. The HV-theory allows us to prove that $\mathcal{D}$ is much wider than $\mathcal{R}_-$. Indeed we can prove that the following alternative holds:

(1') if $E \in \sigma(H(0))$ then E is stable w.r.t. $H(\epsilon)$, $\epsilon > 0$;

(2') if $E \notin \sigma(H(0))$ then $E \in \mathcal{D}$.

Thus, $\mathcal{D}$ coincides with the complement of the spectrum of $H(0)$. Notice that $(2')$ implies statement (2) of the theorem. Moreover a first step in the proof of $(1')$ consists in showing that the circle Γ_r is contained in $\mathcal{D}$, and this guarantees the possibility to construct the projection $P(\epsilon)$ for all ϵ sufficiently small (see the above Definition 2.6). Finally, a third key ingredient, on which the proofs of both $(1')$ and $(2')$ are based, is represented by the so called "characteristic sequences" of $(z - H(\epsilon))$ or "Weyl-type sequences", i.e. sequences (ϵ_n, u_n) such that:

$$\epsilon_n \to 0,\ u_n \in D(H(\epsilon_n)),\ \|u_n\| = 1,\ u_n \xrightarrow{w} 0,\ \|(z - H(\epsilon_n))u_n\| \to 0\,. \tag{4.5}$$

Now it is easy to check that for a suitable constant $a > 0$ we have

$$< u, p^2 u > \leq a(\mathrm{Re}\ < u, H(\epsilon)u > + < u, u >)\,, \tag{4.6}$$

$\forall u \in D(H(\epsilon))$, $\forall \epsilon \in [0, \epsilon_0]$. By making use of (4.6) one can prove that a sequence of type (4.5) generates another characteristic sequence (ϵ_n, v_n) which is "supported at infinity", i.e. such that

$$v_n(x) = 0\,, \quad \text{for } |x| \leq n\,,\ \forall n \in \mathbf{N}. \tag{4.7}$$

Since $\lim_{|x|\to\infty} |V(x)| = +\infty$ by the above assumption (3.2), we first assume that $\lim_{|x|\to\infty} V_+(x) = +\infty$. Now the idea is to prove $(2')$, i.e. that $\sigma(H(0)) \cup \mathcal{D} = \mathbf{C}$, by contradiction. More precisely, one can show that if $z \notin \sigma(H(0)) \cup \mathcal{D}$, then a characteristic sequence (ϵ_n, v_n) of $(z - H(\epsilon))$ exists and it satisfies both (4.5) and (4.7). But this contradicts the fact that for all $z \in \mathbf{C}$

$$\lim_{\substack{n\to\infty \\ \epsilon\to 0}} d_n(z, \epsilon) = +\infty \tag{4.8}$$

where

$$d_n(z,\epsilon) = \inf\{\|(z - H(\epsilon)) : u \in D(H(\epsilon))\,, \|u\| = 1\,, u(x) = 0\,\text{for}\,|x| \leq n\}\,. \tag{4.9}$$

Indeed we have

$$d_n(z,\epsilon) \geq \mathrm{dist}(z, N_n(\epsilon)) \tag{4.10}$$

where

$$N_n(\epsilon) = \{< u, H(\epsilon)u >: u \in D(H(\epsilon)), \|u\| = 1\,, u(x) = 0\,\text{for}\,|x| \leq n\} \tag{4.11}$$

is the so called "numerical range at infinity". Now, for all $u \in D(H(\epsilon))$ such that $\|u\| = 1$ and $u(x) = 0$ for $|x| \leq n$, we have

$$\begin{aligned} |z- < u, H(\epsilon)u >| \ &\geq \ |< u, H(\epsilon)u >| - |z| \\ \geq |\mathrm{Re} < u, H(\epsilon)u >| - |z| \ &\geq < u, V_+u > -|z| - c \end{aligned} \tag{4.12}$$

as $\epsilon \to 0$, for some constant $c > 0$. Thus, (4.8) follows from (4.10) and the fact that $\lim_{|x|\to\infty} V_+(x) = +\infty$ by assumption. Hence $(2')$ is proved. Now let $E \in \sigma(H(0))$. In order to prove that E is stable w.r.t. $H(\epsilon)$, $\epsilon > 0$, we recall that it is enough to prove (2.8), i.e. in view of (2.13), that

$$\mathrm{dim}P(\epsilon) \leq \mathrm{dim}P(0)\,, \quad \text{as}\,\epsilon \to 0\,. \tag{4.13}$$

Again the proof is by contradiction. In fact if we assume that $\mathrm{dim}P(\epsilon_n) > \mathrm{dim}P(0)$, for $\epsilon_n \to 0$, a characteristic sequence of $(E - H(\epsilon))$ which satisfies (4.7) can be found, and exactly as before this contradicts (4.8). Thus the theorem is proved under the assumption that $\lim_{|x|\to\infty} V_+(x) = +\infty$. If this assumption is not satisfied then it follows from (3.2) that $\lim_{\substack{|x|\to\infty \\ \epsilon\to 0}} |V_-(x) + \epsilon W_-(x)| = +\infty$. Without loss of generality, since V_- and W_- are $\mathcal{P}$-odd, we may assume that $\lim_{\substack{x\to+\infty \\ \epsilon\to 0}}(V_-(x)+\epsilon W_-(x)) = +\infty$ and $\lim_{\substack{x\to-\infty \\ \epsilon\to 0}}(V_-(x) + \epsilon W_-(x)) = -\infty$. This immediately implies that there exists $c \geq 0$ such that

$$\lim_{\substack{n\to\infty \\ \epsilon\to 0}} d_n(z,\epsilon) = 0\,, \quad \forall z \in \mathcal{R}_+\,, \,\mathrm{Re}\, z \geq c$$

and therefore (4.8) cannot be used to generate a contradiction with (4.5) and (4.7). Nevertheless the problem is overcome as follows (see also [12] where an analogous problem was handled in a similar fashion). First of all one can prove that any characteristic sequence (ϵ_n, u_n) generates another characteristic sequence (ϵ_n, v_n) which is supported either at $+\infty$ or at $-\infty$, i.e. such that either

$$v_n(x) = 0\,, \quad \text{for}\; x \leq n\,,\; \forall n \in \mathbf{N} \tag{4.14}$$

or

$$v_n(x) = 0\,, \quad \text{for } x \geq -n\,,\ \forall n \in \mathbf{N}\,. \tag{4.15}$$

Next we introduce the numerical range at $+\infty$, $N_n^+(\epsilon)$, and the numerical range at $-\infty$, $N_n^-(\epsilon)$, defined by (4.11) where the condition $u(x) = 0$ for $|x| \leq n$ is replaced by $u(x) = 0$ for $x \leq n$, and $u(x) = 0$ for $x \geq -n$ respectively. More precisely:

$$N_n^+(\epsilon) = \{< u, H(\epsilon)u >: u \in D(H(\epsilon)), \|u\| = 1\,, u(x) = 0\, \text{for } x \leq n\}$$

and

$$N_n^-(\epsilon) = \{< u, H(\epsilon)u >: u \in D(H(\epsilon)), \|u\| = 1\,, u(x) = 0\, \text{for } x \geq -n\}\,.$$

Then for all $z \in \mathbf{C}$ we have

$$\lim_{\substack{n\to\infty \\ \epsilon\to 0}} d_n^{\pm}(z,\epsilon) = +\infty \tag{4.16}$$

where

$$d_n^+(z,\epsilon) = \inf\{\|(z - H(\epsilon)) : u \in D(H(\epsilon))\,, \|u\| = 1\,, u(x) = 0\, \text{for } x \leq n\}\,. \tag{4.17}$$

and similarly for $d_n^-(z,\epsilon)$. Then the existence of a characteristic sequence satisfying either (4.14) or (4.15) contradicts (4.16) and this concludes the proof of the theorem.

5 Conclusions

The reality of the spectrum of a $\mathcal{PT}$symmetric Hamiltonian has been proved for a class of polynomial Hamiltonians and for solvable classes of potentials generated by solvable selfadjoint problems by a complex coordinate shift. By a suitable use of the perturbation theory we have summarized in this paper that it is conceivable that the class of $\mathcal{PT}$symmetric Hamiltonians with real spectrum can be considerably enlarged in so far as one starts from a $\mathcal{PT}$symmetric Hamiltonian with real spectrum and adds a $\mathcal{PT}$ symmetric perturbation. This analysis does not pretend to be exhaustive in the sense that the perturbed eigenvalues of $H(\epsilon)$ are real but there is no guarantee that the whole spectrum is real. Therefore the connection with pseudo-Hermiticity deserves further investigation. Moreover the pseudo-Hermiticity condition $H = \eta H^* \eta^{-1}$ (see e.g. [13, 14]) of the unperturbed Hamiltonian could generate a perturbative approach to pseudo-Hermiticity for larger classes of Hamiltonians and of the operator η. Another open question is a detailed analysis of the degenerate case, i.e. the case when the unperturbed eigenvalues are diabolic points ([15]) or exceptional points ([16, 17]); the latter are typical of non diagonalizable Hamiltonians.

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