Theory of Point Processes and Some Basic Notions in Energy Level Statistics *

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Abstract

Although it is not explicitly stated in physics literature, energy level statistics is based upon the hypothesis that the spectrum of a quantum Hamiltonian, after suitably normalized (unfolded), and observed in thermodynamic or semiclassical limit, looks like a typical realization of a stationary point process. In this survey article, we present a purely phenomenological view of energy level statistics based on the theory of point processes.

Contents

*Dedicated to Professor S.A. Molchanov on the occasion of his 65th birthday

1 Introduction

Energy level statistics is a way to observe the discrete spectra of quantum Hamiltonians, in which one is interested in statistical fluctuation of energy levels (eigenvalues) looked at as a whole, rather than the location of individual levels nor the asymptotic or "average" distribution of levels. It originates in the study by nuclear physicists in early 60's ([32]), in which the excitation spectra of heavy nuclei were modelled by the spectra of random matrices. This led to an extensive study by Dyson, Mehta et al of random matrices. Their theory was also applied to fluctuation of spectra of small metallic particles ([9]). Later, Berry and Tabor ([3]) proposed to apply energy level statistics to semi-classical Hamiltonians in order to distinguish between regular and irregular spectra, a classification of discrete spectra of quantum Hamiltonians made by Percival ([31]) according to the property of the underlying classical Hamiltonian dynamics describing the system being integrable or non-integrable. They argued that the spectrum of a quantum Hamiltonian which are obtained through quantization of a classical completely integrable Hamiltonian looks, after suitably normalized, like a typical realization of the Poisson point process. (This assertion is now called the "Berry-Tabor conjecture"[16].) They also conjectured that if a classical Hamiltonian dynamical system has chaotic behavior, then the spectrum of its quantization has fluctuation property similar to random matrices. This conjecture was later confirmed numerically ([5]), but it is yet to be clarified mathematically. Fluctuation of spectra was investigated also for one-dimensional random Schrödinger operators ([27], [28], [33], [34]), and for multi-dimensional Anderson tight binding models ([22]). In connection to Anderson transition, it is believed that the spectrum of random Schrödinger operator, considered on a large finite domain, looks like Poisson point process in the energy region where Anderson localization takes place, while in the energy region where delocalization of wave functions is expected, it looks like the spectrum of a random matrix.

In order to give a meaning to the vague notion of "fluctuation of levels", suppose we are given a (random) Hamiltonian H with discrete spectrum

$$\cdots < E_n < E_{n+1} < \cdots < E_{n+m} < \cdots \ .$$

Among various statistics of the spectrum, one is often interested in spacings between consecutive levels, $S_i = E_{i+1} - E_i$, and wishes to know their statistical characteristics. (see e.g. §1.4 of Mehta [19].) But the following questions immediately arise:

1. The "density" of levels is in general different in different energy regions. For example, if $\{E_i\}$ are eigenvalues of the Dirichlet Laplacian $-\Delta_D$ on a bounded domain $\Omega \subset \mathbf{R}^d$, the cumulative distribution function
$$N(E) := \sum_i 1_{\{E_i \le E\}}$$
has the asymptotic behavior $N(E) \sim \bar{N}(E) := CE^{d/2}$ $(E \to \infty)$ with a positive constant C. Hence for $d > 2$, levels are distributed more densely for higher energy region, while for $d = 1$, they are distributed more sparsely. Thus in general, S_i's cannot have the same statistical nature.

2. When one is observing the spectrum in the high energy region, the exact numbering of levels is practically meaningless.

3. Even if the density of levels is uniform over the entire energy axis, it is not obvious if S_i's have the same probability distribution so that one can discuss the "typical" level spacing.

In physics literature, the first obstacle is overcome by "unfolding" of the spectrum. In the case of Dirichlet Laplacian for example, this amounts to considering the sequence $\{\lambda_i\}_i$, where $\lambda_i := \bar{N}(E_i)$, instead of $\{E_i\}$. By doing so, we obtain a sequence which has asymptotically uniform distribution. In fact, as $L \to \infty$,

$$\sum_i 1_{\{\lambda_i \le L\}} = \sum_i 1_{\{\bar{N}(E_i) \le L\}} = N(\bar{N}^{-1}(L)) \sim \bar{N}(\bar{N}^{-1}(L)) = L \ .$$

In §5 of this article, we shall discuss the notion of unfolding in a slightly generalized setting.

In order that level statistics makes sense, the unfolded levels $\{\lambda_i\}_i$ are expected to be distributed homogeneously over $\mathbf{R}^1$. Namely, the statistical nature of levels should not depend on the choice of the origin: for any choice of a real number λ, the sequence

$$\cdots, \lambda_n - \lambda,\ \lambda_{n+1} - \lambda,\ \ldots, \lambda_{n+m} - \lambda, \ldots$$

as a whole should look like

$$\cdots,\ \lambda_n,\ \lambda_{n+1},\ \ldots,\ \lambda_{n+m}, \cdots .$$

Thus we should look at the levels $\{\lambda_i\}$ not one by one, but as a (random) configuration on $\mathbf{R}^1$. In particular, the exact numbering of levels is not essential, and we are thus led to formulate the basic assumption of energy level statistics, which is not clearly stated in physics literature:

> After suitably unfolded, the discrete spectrum of a quantum Hamiltonian looks like a typical realization of a stationary point process on $\mathbf{R}^1$.

The purpose of the present article is to present a phenomenological theory of basic notions in statistics of energy spectrum, viewed as a realization of a fictitious stationary point process on $\mathbf{R}$. Much of materials are taken from author's notes ([24], [25], [26]) published in preliminary forms, with additional new comments and improvements.

The outline of the article is as follows: In §2, we summarize briefly some basic facts about stationary point processes on the real line, especially the Palm-Khinchin formula. Relations among various statistics such as $E(k, L)$, $F(k, L)$ or $P(S)$ (see e.g. [2], [19]) are derived as direct corollaries of Palm-Khinchin formula. In §3, we assume that our point process is ergodic. Then as a result of ergodic theorem, various statistics taken along the energy axis coincide with their counterparts taken on the ensemble (namely the probability space at the basis). It will be stressed that even if the unfolded spectrum $\{\lambda_i\}$ is realized as a sample of ergodic point process, the probability distribution of spacings $s_i := \lambda_{i+1} - \lambda_i$ can differ from i to i, and that *the* spacing should be observed not under the probability measure $\mathbf{P}$ given at the beginning, but under the Palm distribution $\hat{\mathbf{P}}$ which is naturally associated to $\mathbf{P}$ when $\{\lambda_i\}$ has unit mean density. In the next section §4, we consider a deterministic family of sequences of real numbers, which we consider samples of a fictitious stationary point process. This time, there is no ensemble average, but one can consider various statistics taken along $\mathbf{R}$ and deduce relations among them. In particular, a deterministic analogue of Palm-Khinchin formula is obtained. The level statistics discussed in this section is what is usually done in quantum chaology. In §5, we shall discuss two notions of unfolding. While the first one is rather well defined and applied in physics literature (see e.g. [13]), the second idea, which is essentially due to Berry and Tabor ([3]), is not commonly used. Unfortunately, the present author is unable to clarify the general relationship of these two notions. For some examples, these two notions of unfolding give the same results, but there are also some cases where they lead to different mathematical problems. In the final section §6, we give sufficient conditions for repulsion and clustering of levels in a wide sense.

2 Stationary point processes on the real line; statistics over the ensemble

2.1 Generalities and Palm-Khinchin formula

Let $M_p(\mathbf{R})$ be the totality of integer valued Radon measures on $\mathbf{R}$, and let $\mathcal{M}_p(\mathbf{R})$ be the σ-field on $M_p(\mathbf{R})$generated by mappings

$$M_p(\mathbf{R}) \ni N \mapsto N(I) \in \mathbf{Z}_+$$

for all bounded intervals $I \subset \mathbf{R}$. $\mathcal{M}_p(\mathbf{R})$ is equal to the Borel σ-field generated by the topology of vague convergence of Radon measures in $M_p(\mathbf{R})$ (see [7], [30]). This topology is known to be metrizable.

Definition 1 *A random variable N_ω defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in $(M_p(\mathbf{R}), \mathcal{M}_p(\mathbf{R}))$ is called a point process on $\mathbf{R}$. It is said to be* stationary*, if its probability law induced on the space $(M_p(\mathbf{R}), \mathcal{M}_p(\mathbf{R}))$ is invariant under the translation $N(\cdot) \mapsto N(\cdot + x)$ for all $x \in \mathbf{R}$.*

To each sample N_ω of a point process, we can associate sequences of random variables $\{x_n(\omega)\}_{n=-\infty}^{\infty}$ and $\{m_n(\omega)\}_{n=-\infty}^{\infty}$ so that

$$N_\omega(dx) = \sum_{n\in\mathbf{Z}} m_n(\omega)\delta_{x_n(\omega)}(dx) \ ,$$

$\delta_a(dx)$ being the unit mass concentrated at a. Here the random variables $m_n(\omega) \geq 1$ are integer valued, and $x_n(\omega)$'s satisfy

$$\cdots < x_{-1}(\omega) < x_0(\omega) \leq 0 < x_1(\omega) < x_2(\omega) < \cdots \ . \tag{1}$$

$m_n(\omega)$ is called the multiplicity of the point $x_n(\omega)$. The point process N_ω is said to be *simple* if $m_n(\omega) = 1$ almost surely for all $n \in \mathbf{Z}$. In the following, we shall assume that $m := \mathbf{E}[N(0,1]] < \infty$, and call it the mean density of the point process N_ω. (For notational simplicity, we have written $N(0,1]$ instead of $N((0,1])$ or $N_\omega((0,1])$.)

Now let us summarize some of the fundamental properties of stationary point processes which are derived directly from the definition. For detailed proofs, the reader is refered to the book by Daley and Vere-Jones [7].

Proposition 1 *For any $t \in \mathbf{R}$ and any $x \geq 0$, one has $\mathbf{E}[N(t, t+x]] = mx$.*

Indeed, by the stationarity we have $\mathbf{E}[N(t,t+x]] = \mathbf{E}[N(0,x]]$, which we shall denote by $M(x)$. Then for any $x, y \geq 0$,

$$M(x+y) = \mathbf{E}[N(0,x+y]] = \mathbf{E}[N(0,x]] + \mathbf{E}[N(x,x+y]] = M(x) + M(y) \ .$$

Under the assumption $m = M(1) < \infty$, $M(x)$ is right continuous in x, hence we must have $M(x) = mx$ by a well known argument.

Proposition 2 *The limit*

$$\lambda := \lim_{h\searrow 0} \frac{1}{h}\mathbf{P}(N(0,h] > 0)$$

exists and satisfies $\lambda \leq m$. λ is called the intensity *of the stationary point process N.*

For the proofs of limit theorems of this kind, the following lemma is repeatedly used:

Lemma 1 *For any function $g(x)$ defined on $[0,\infty)$ which satisfies the conditions*

1. $\lim_{x\searrow 0} g(x) = g(0) = 0$;
2. (sub-additivity) $g(x+y) \leq g(x) + g(y)$ $(x, y \geq 0)$,

the limit

$$\alpha := \lim_{x\searrow 0} \frac{1}{x} g(x) \leq \infty$$

exists and satisfies

$$\alpha = \sup_{x>0} \frac{1}{x} g(x) \ .$$

For example, Proposition 2 is easily proved if you consider the function

$$\phi(x) := \mathbf{P}(N(0,x] > 0) \ ,$$

which obviously satisfies the conditions of the lemma and $\phi(x) \leq mx$. Hence the desired limit λ exists and we have $\lambda \leq m$.

Proposition 3 *If the stationary point process N_ω is simple, then $\lambda = m$.*

From a given stationary point process

$$N_\omega(dx) = \sum_{n\in\mathbf{Z}} m_n(\omega)\delta_{x_n(\omega)}(dx) \ ,$$

we obtain a new point process

$$N^*_\omega(dx) := \sum_{n\in\mathbf{Z}} \delta_{x_n(\omega)}(dx) \ ,$$

which is stationary and simple. Obviously N and N^* have the same intensity λ, and by Proposition 3, λ is equal to the mean density of N^*.

Proposition 4 *Under the assumption $m < \infty$, a stationary point process N is simple if and only if it satisfies the condition*

$$\mathbf{P}(N(0,h] \geq 2) = o(h) \quad (h \searrow 0) \ .$$

As an important example of a point process, let us introduce the stationary Poisson point process:

Definition 2 *A point process N is called the stationary Poisson point process with parameter $c > 0$, if it satisfies the following two conditions:*

1. *For any finite family $A_1, \ldots, A_n$ of disjoint Borel subsets of $\mathbf{R}$, $N(A_1), \ldots, N(A_n)$ are independent random variables;*
2. *If A is a bounded Borel subset of $\mathbf{R}$ with Lebesgue measure $|A|$, then $N(A)$ obeys the Poisson distribution with mean $c|A|$, namely*

$$\mathbf{P}(N(A) = k) = e^{-c|A|}\frac{(c|A|)^k}{k!} \ , \quad k = 0, 1, 2, \ldots \ .$$

It is easy to see that a stationary Poisson point process N is "stationary" in the sense of Definition 1. Moreover since

$$\mathbf{P}(N(0,h] \geq 2) = 1 - e^{-ch} - ch \cdot e^{-ch} = \mathcal{O}(h^2)$$

as $h \searrow 0$, we see that N is simple by Proposition 4, which has been not quite obvious from Definition 2. Of course, we have $\lambda = m = c$.

Proposition 5 *For $k = 1, 2, \ldots$, the limits*

$$\pi_k := \lim_{h\searrow 0} \mathbf{P}(N(0,h] = k| \ N(0,h] > 0)$$

exist, and satisfy $\sum_{k\geq 1} \pi_k = 1$.

Intuitively, π_k is the conditional probability that a point has multiplicity k, given that the origin is occupied by that point. When the stationary point process N is simple, Proposition 2 and 4 show that $\pi_k = 0$ for $k \geq 2$.

Proposition 6 *For each $x > 0$ and $k = 0, 1, 2, \ldots$, the limit*

$$Q_k(x) := \lim_{h\searrow 0} \mathbf{P}(N(0,x] \leq k| \ N(-h,0] > 0)$$

exists, and is right-continuous and non-decreasing as a function of x.

Now we can state the celebrated Palm-Khinchin formula:

Theorem 1 (the Palm-Khinchin formula) *If N is a simple stationary point process (so that $\lambda = m$) such that*

$$\mathbf{P}(N(-\infty, 0] = N(0, \infty) = \infty) = 1 \ , \tag{2}$$

then one has

$$\mathbf{P}(N(0, x] \leq k) = \lambda \int_x^\infty q_k(u) du \ , \ \ k = 0, 1, 2, \ldots, \tag{3}$$

where we have set

$$q_k(x) := Q_k(x) - Q_{k-1}(x) = \lim_{h \searrow 0} \mathbf{P}(N(0, x] = k| \ N(-h, 0] > 0) \ ,$$

with the convention $Q_{-1}(x) \equiv 0$. Moreover for $k \geq 1$,

$$R_k(x) := 1 - \sum_{j=0}^{k-1} q_j(x) = 1 - Q_{k-1}(x) \tag{4}$$

is a probability distribution function on $(0, \infty)$ with mean

$$\int_0^\infty x d_x R_k(x) = \frac{k}{\lambda} \ . \tag{5}$$

Note that if we let $x = 0$ in (3), we get

$$\lambda \int_0^\infty q_k(u) du = 1 \ , \tag{6}$$

and hence

$$\mathbf{P}(N(0, x] > k) = \lambda \int_0^x q_k(u) du \ . \tag{7}$$

We omit the proof of this theorem as well as of Propositions 3 – 6, leaving them to [7]. We only note that the fact $R_k(\infty) = 1$ is a consequence of the condition (2). In fact, we can write intuitively

$$R_k(x) = 1 - \mathbf{P}(N(0, x] \leq k - 1| \ N(\{0\}) > 0) = \mathbf{P}(N(0, x] \geq k| \ N(\{0\}) > 0) \ .$$

If we let $x \to \infty$, we should have

$$R_k(\infty) = \mathbf{P}(N(0, \infty) \geq k| \ N(\{0\}) > 0) \ .$$

According to (2), this probability should be equal to 1 for all k.

2.2 Factorial moment measures of a point process

Given a (not necessarily stationary) point process $N_\omega(dx)$ on $\mathbf{R}$, we define its k-th factorial $N_\omega^{[k]}(dx_1, \ldots, dx_k)$ by

$$N_\omega^{[k]}(B) := (N_\omega \times \cdots \times N_\omega)(B \cap \mathbf{R}_k^*) \ , \tag{8}$$

where B runs through $\mathcal{B}(\mathbf{R}^k)$, the totality of k- dimensional Borel sets, and we have set

$$\mathbf{R}_k^* := \{(x_1, \ldots, x_k) \in \mathbf{R}^k | \ x_i \neq x_j \ (i \neq j)\} \ .$$

When the point process N_ω is simple and is written in the form

$$N_\omega(dx) = \sum_{n \in \mathbf{Z}} \delta_{x_n(\omega)}(dx) \ , \tag{9}$$

with $\{x_n(\omega)\}$ satisfying (1), then we have, for any intervals $I_1, \dots, I_k$,

$$\begin{aligned} & N_\omega^{[k]}(I_1 \times \cdots \times I_k) \\ & = \sharp\{(x_{i_1}(\omega), \dots, x_{i_k}(\omega)) |\ x_{i_j}(\omega) \text{ are distinct and } x_{i_j}(\omega) \in I_j\ (j = 1, \dots, k)\} \ . \end{aligned} \tag{10}$$

Also we define the k-th factorial moment measure $m^{[k]}(dx_1, \dots, dx_k)$ as the expectation of $N^{[k]}$, namely

$$m^{[k]}(B) := \mathbf{E}[N^{[k]}(B)] \ , \quad B \in \mathcal{B}(\mathbf{R}^k) \ . \tag{11}$$

Suppose $B \in \mathcal{B}(\mathbf{R}^k)$ is of the form

$$B = B_1^{k_1} \times B_2^{k_2} \times \cdots \times B_\ell^{k_\ell} \ , \tag{12}$$

where $B_1, \dots, B_\ell \in \mathcal{B}(\mathbf{R})$ are disjoint and $k_1 + \cdots + k_\ell = k$. Then it is easy to see that

$$N_\omega^{[k]}(B) = N_\omega(B_1)^{[k_1]} N_\omega(B_2)^{[k_2]} \cdots N_\omega(B_\ell)^{[k_\ell]} \ , \tag{13}$$

where for integers $n \geq 0$ and $j \geq 1$, we let

$$n^{[j]} = \begin{cases} n(n-1)\cdots(n-j+1) & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \ . \end{cases}$$

As an important example, consider the case where N_ω is the Poisson point process with parameter c. When $B \in \mathcal{B}(\mathbf{R}^k)$ is of the form (12), then by (13) and by the definition of the Poisson point process, one has

$$\mathbf{E}[N^{[k]}(B_1^{k_1} \times \cdots \times B_\ell^{k_\ell})] = \mathbf{E}[N(B_1)^{[k_1]}] \cdots \mathbf{E}[N(B_1)^{[k_\ell]}] = \prod_{j=1}^{\ell} (c|B_j|)^{k_j} \ . \tag{14}$$

This can be rephrased as follows: when the one-dimensional Borel sets $B_1, B_2, \dots, B_k$ are such that B_i and B_j $(i \neq j)$ are either disjoint or identical, then

$$\mathbf{E}[N^{[k]}(B_1 \times \cdots \times B_k)] = c^k \prod_{j=1}^{k} |B_j| \ . \tag{15}$$

This can be used to prove the following

Proposition 7 *If N is the stationary Poisson point process with parameter c, then for any sequence $A_1, A_2, \dots, A_k \in \mathcal{B}(\mathbf{R})$ of k Borel sets, one has*

$$m^{[k]}(A_1 \times \cdots \times A_k) = \mathbf{E}[N^{[k]}(A_1 \times \cdots \times A_k)] = c^k \prod_{j=1}^{k} |A_j| \ .$$

Proof. Let $\{B_1, \dots, B_m\}$ be the disjoint partition of the set $\cup_{i=1}^n A_i$ generated by $\{A_1, \dots, A_n\}$, so that we can write

$$A_i = \bigcup_{p \in J_i} B_p$$

with a $J_i \subset \{1, 2, \dots, m\}$. Then we obtain

$$\begin{aligned} \mathbf{E}[N^{[k]}(A_1 \times \cdots \times A_k)] &= \mathbf{E}\left[N^{[k]} \left(\bigcup_{(p_1, \dots, p_k) \in J_1 \times \cdots \times J_k} (B_{p_1} \times \cdots \times B_{p_k}) \right) \right] \\ &= \sum_{(p_1, \dots, p_k) \in J_1 \times \cdots \times J_k} \mathbf{E}[N^{[k]}(B_{p_1} \times \cdots \times B_{p_k})] \\ &= \sum_{(p_1, \dots, p_k) \in J_1 \times \cdots \times J_k} c^k |B_{p_1}| \cdot |B_{p_2}| \cdots |B_{p_k}| = c^k \prod_{j=1}^{k} |A_j| \end{aligned}$$

as desired.

2.3 Applications to level statistics

As we already noted, the unfolded spectrum of a quantum Hamiltonian is often implicitly assumed to be a realization $\{x_i\}$ of some stationary point process. In the context of quantum chaos, this identification is often something which might be called an intentional confusion. For example in [2], the authors define $E(k,S)$, where $k = 0,1,2,\dots$ and $S > 0$, to be "the probability to find k (unfolded) levels in a randomly chosen interval of length L". They even write it as $E(k,S,\hat{x})$ in order to indicate that this interval is of the form $(x, x+S]$ with x randomly chosen from $(\hat{x}, c\hat{x}]$ $(c > 1)$ according to the uniform distribution. This is actually a right way to consider the statistics of levels, because the quantal spectrum under consideration does not have the true stochasticity. (We shall return to the discussion of this situation in §4.) But immediately, they omit $\hat{x}$ "to keep the notation simple", and continue their discussion as if the spectrum is a truely random sequence of levels. In this subsection, we shall assume *explicitly* that the unfolded quantal spectrum under consideration is actually represented by a simple stationary point process N_ω defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Thus we consider $E(k,S)$ to be the probability to find k levels in a *fixed* interval $(x, x+S]$ of length S. By the stationarity, this probability does not depend on the choice of x, so we define

$$E(k,S) := \mathbf{P}(N(0,S] = k) \; . \tag{16}$$

Following [2] (see also [12]), we also consider $F(k,S)$, "the probability to count k levels in an interval of length S which starts at an arbitrary quantal level x_i", and $P(k,S)$, "the probability to obtain exactly k levels in an interval of length S beginning at an arbitrary quantal level x_i and ending at the quantal level x_{i+k+1}". $F(k,S)$ should be interpreted as the conditional probability to count k levels in the interval $(0,S]$, given that the origin 0 is occupied by a level. Since the event $\{N(\{0\}) = 1\}$ has probability zero, we are led to define this conditional probability as

$$F(k,S) := \lim_{h \searrow 0} \mathbf{P}(N(0,S] = k |\; N(-h,0] > 0) \; , \tag{17}$$

or $F(k,S) := q_k(S)$ in the notation of Theorem 1. To give a meaning to $P(k,S)$, we interpret $P(k,S)dS$ as the conditional probability, given that the origin is occupied by a level, say x_i, to find x_{i+k+1} between S and $S + dS$. This leads us to consider

$$\begin{aligned} \mathbf{P}(k,S)dS \;\; &= \;\; \lim_{h \searrow 0} \mathbf{P}(N(0,S] \le k |\; N(-h,0] > 0) \\ &\quad - \lim_{h \searrow 0} \mathbf{P}(N(0,S+dS] \le k |\; N(-h,0] > 0) \; . \end{aligned}$$

In view of this, we define

$$P(k,S)dS := -dQ_k(S) \; . \tag{18}$$

When $k = 0$, we shall write $E(S)$, $F(S)$, and $P(S)$ respectively for $E(0,S)$, $F(0,S)$, and $P(0,S)$. $E(S)$ is often called the "gap probability". $P(S)$, when it exists, is the probability density of the spacing to the level which lies next to the one occupying the origin, because $F(S)$ is the probability that this spacing is greater than S.

We now apply the Palm-Khinchin formula, to obtain

$$\begin{aligned} E(k,S) &= \mathbf{P}(N(0,S] \le k) - \mathbf{P}(N(0,S] \le k-1) \\ &= \lambda \int_S^\infty \{q_k(x) - q_{k-1}(x)\} dx \\ &= \lambda \int_S^\infty \{F(k,x) - F(k-1,x)\} dx \; , \end{aligned}$$

for $k \ge 1$, and

$$E(S) = E(0,S) = \lambda \int_S^\infty F(0,x) dx$$

for $k = 0$, where λ is the intensity of the point process N. Summing over k, we get

$$\lambda \int_S^\infty F(k,x)dx = \sum_{j=0}^{k} E(j,S) \ ,$$

or

$$F(k,S) = -\frac{1}{\lambda}\frac{d^+}{dS}\sum_{j=0}^{k} E(j,S) \ , \tag{19}$$

which is formula (13) of [2]. Recall that $F(k,S)$ is right-continuous in S, so that differentiation from the left is possible. If both sides of (19) are again differentiable in S, we can write

$$\begin{aligned} P(k,S) &= -\frac{d}{dS}\sum_{i=0}^{k} F(i,S) = \frac{1}{\lambda}\frac{d^2}{dS^2}\sum_{i=0}^{k}\sum_{j=0}^{i} E(j,S) \\ &= \frac{1}{\lambda}\frac{d^2}{dS^2}\sum_{i=0}^{k}(k-i+1)E(i,S) \ . \end{aligned} \tag{20}$$

In particular, letting $k = 0$,

$$P(S) = \frac{1}{\lambda}\frac{d^2}{dS^2}E(S) \ . \tag{21}$$

The following relations are also valid:

$$\sum_{k=0}^{\infty} E(k,S) = 1 \ ; \tag{22}$$

$$\sum_{k=0}^{\infty} F(k,S) = 1 \ ; \tag{23}$$

$$\int_0^\infty P(k,S)dS = 1 \ ; \tag{24}$$

$$\int_0^\infty F(k,S)dS = \frac{1}{\lambda} \ ; \tag{25}$$

$$\int_0^\infty SP(k,S)dS = \frac{k+1}{\lambda} \ . \tag{26}$$

(22) and (24) are obvious from the definitions of $E(k,S)$ and $P(k,S)dS$. (25) and (26) follow from (6) and (5) respectively. (23) can be verified in the following way. Since $F(k,S) = Q_k(S) - Q_{k-1}(S)$ and $Q_{-1}(S) = 0$, we have

$$\sum_{k=0}^{\infty} F(k,S) = \lim_{k\to\infty} Q_k(S) = \sup_{k\geq 0} Q_k(S) \ .$$

On the other hand, by the definition of $Q_k(S)$ (Propsition 6) and $\mathbf{P}(N(-h,0] > 0) \sim \lambda h$, we can write

$$Q_k(S) = \lim_{h\searrow 0}\frac{1}{\lambda h}\mathbf{P}(N(0,S] \leq k,\ N(-h,0] > 0) \ .$$

Now it is not difficult to show that $\mathbf{P}(N(0,S] \leq k,\ N(-h,0] > 0)$, as a function of $h > 0$, satisfies the condition of Lemma 1. (The proof of Propsition 6 depends on this fact.) Hence

$$Q_k(S) = \sup_{h>0}\frac{1}{\lambda h}\mathbf{P}(N(0,S] \leq k,\ N(-h,0] > 0) \ .$$

Interchanging $\sup_k$ and $\sup_{h>0}$, we see

$$\begin{aligned}\sum_{k=0}^{\infty} F(k,S) &= \sup_{h>0}\sup_{k} \frac{1}{\lambda h}\mathbf{P}(N(0,S] \le k,\ N(-h,0] > 0) \\ &= \sup_{h>0} \frac{1}{\lambda h}\mathbf{P}(N(-h,0] > 0) = 1 \ ,\end{aligned}$$

by Proposition 2.

Let us now look for a representation of the *number variance* $\Sigma^2(S)$ in terms of $F(k,S)$ or $P(k,S)$. $\Sigma^2(S)$ is defined as the variance of the random variable $N(0,S]$, i.e.

$$\Sigma^2(S) := \mathbf{E}\left[N(0,S]^2\right] - (\lambda S)^2 \ . \tag{27}$$

We can compute, using Theorem 1 and $\lambda S = \mathbf{E}[N(0,S]]$,

$$\begin{aligned}\mathbf{E}\left[N(0,S]^2\right] - \lambda S &= \sum_{k=2}^{\infty} k(k-1)\mathbf{P}(N(0,S] = k) \\ &= \sum_{k=2}^{\infty} k(k-1)\left\{\mathbf{P}(N(0,S] > k-1) - \mathbf{P}(N(0,S] > k)\right\} \\ &= 2\sum_{k=1}^{\infty} k\mathbf{P}(N(0,S] > k) \\ &= 2\lambda\int_0^S \sum_{k=1}^{\infty} kq_k(u)du \ .\end{aligned}$$

Thus we obtain

$$\Sigma^2(S) = \lambda S - (\lambda S)^2 + 2\lambda\int_0^S \left(\sum_{k=1}^{\infty} kF(k,u)\right) du \ ,$$

or

$$\Sigma^2(S) = \lambda S + 2\lambda\int_0^S \left(\sum_{k=1}^{\infty} kF(k,u) - \lambda u\right) du \ .$$

Formally, this can be rewritten as

$$\Sigma^2(S) = \lambda S + 2\lambda\int_0^S (S-u)\left(\sum_{k=0}^{\infty} P(k,u) - \lambda\right) du \ . \tag{28}$$

Here

$$\rho_2(u) := \sum_{k=0}^{\infty} P(k,u) \tag{29}$$

and

$$Y_2(S) := \rho_2(u) - \lambda \ , \tag{30}$$

which are in general singular, are often called *two level correlation* and *two level cluster function* respectively.

Example 1 *Let N be the stationary Poisson point process with parameter $c > 0$. Then by the independence (see Definition 2),*

$$F(k,S) = q_k(S) = \mathbf{P}(N(0,S] = k) = E(k,S) = e^{-cS}\frac{(cS)^k}{k!} \ ; \tag{31}$$

$$P(k,S) = -\frac{d}{dS}\sum_{i=0}^{k} F(i,S) = ce^{-cS}\frac{(cS)^k}{k!} \ , \tag{32}$$

and hence

$$\rho_2(u) = c \ ; \quad Y_2(u) = 0 \ ; \quad \Sigma_2(S) = cS \ . \tag{33}$$

Example 2 *For each ω in the probability space $(\Omega, \mathcal{B}[0,1), \mathbf{P})$, where $\Omega = [0,1)$, $\mathcal{B}[0,1)$ is the Borel σ-field on $[0,1)$, and $\mathbf{P}$ is the Lebesgue measure, define*

$$N_\omega(dx) := \sum_{n=-\infty}^{\infty} \delta_{(n+\omega)/\lambda}(dx) \ , \tag{34}$$

where $\lambda > 0$. The resulting point process is obviously simple and stationary. It is just the random translation of the lattice $\lambda^{-1}\mathbf{Z}$.

For any bounded measurable function $f(x)$ with compact support, one has

$$\mathbf{E}\left[\int_{-\infty}^{\infty} f(x) N_\omega(dx)\right] = \int_0^1 \sum_{n=-\infty}^{\infty} f(\lambda^{-1}(n+\omega))d\omega = \lambda \int_{-\infty}^{\infty} f(x)dx \ .$$

In particular letting $f(x) = 1_{(0,1]}(x)$,

$$\mathbf{E}[N(0,1]] = \lambda \ .$$

Now fix $k \geq 0$ and $x > 0$. Then for sufficiently small $h > 0$, $N_\omega(-h, 0] > 0$ implies $N_\omega(0, x] \leq k$ when $x < (k+1)/\lambda$, and $N_\omega(0, x] > k$ when $x \geq (k+1)/\lambda$. On the other hand, $\mathbf{P}(N(-h, 0] > 0) = \lambda h$ for small $h > 0$. Hence

$$Q_k(x) = \lim_{h \searrow 0} \frac{1}{\lambda h} \mathbf{P}(N(0,x] \leq k, N(-h,0] > 0) = 1_{[0,(k+1)/\lambda)}(x) \ . \tag{35}$$

Thus we get

$$F(k,S) = q_k(S) = 1_{[k/\lambda,(k+1)/\lambda)}(S) \ ; \quad P(k,S) = \delta\left(S - \frac{k+1}{\lambda}\right) \ . \tag{36}$$

By direct computation, one can verify

$$\Sigma^2(S) = \{\lambda S\}(1 - \{\lambda S\}) \ , \tag{37}$$

where $\{\lambda S\}$ is the fractional part of λS.

2.4 Superposition of independent point processes

In several situations, we can consider the spectrum of a quantum system as a superposition of spectra of independent or very weakly interacting subsystems (see e.g. [4], [13], [14], and also [27], [22]). In our language, this amounts to consider a point process of the form

$$N(dx) = N_1(dx) + N_2(dx) + \cdots + N_p(dx) \ , \tag{38}$$

where $N_1(dx)$, $N_2(dx)$, $\ldots, N_p(dx)$ are stationary point processes which are statistically independent from each other. Let m_i, m, λ_i, λ be respectively the mean density and the intensity N_i and N respectively. Also let $\Sigma_i^2(S)$, $\Sigma^2(S)$, $Y_2^{(i)}(u)$, $Y_2(u)$ be the corresponding number variances and two level cluster functions, and let $E_i(S)$, $E(S)$, $F_i(S)$, $F(S)$ be defined similarly.

From (38), we obviously have

$$m = m_1 + m_2 + \cdots + m_p \tag{39}$$

and

$$\Sigma^2(S) = \Sigma_i^2(S) + \Sigma_2^2(S) + \cdots + \Sigma_p^2(S) \tag{40}$$

from the independence. For small $h > 0$, we have from (38), the independence, and Propsition 2,

$$\begin{aligned}\mathbf{P}(N(0,h] > 0) &= 1 - \mathbf{P}(N(0,h] = 0) = 1 - \prod_{i=1}^{p} \mathbf{P}(N_i(0,h] = 0) \\ &= 1 - \prod_{i=1}^{p}(1 - \lambda_i h + o(h)) = \left(\sum_{i=1}^{p} \lambda_i\right) h + o(h) \ ,\end{aligned}$$

whence

$$\lambda = \sum_{i=1}^{p} \lambda_i \ . \tag{41}$$

Now suppose $N_1, \ \ldots, N_p$ are all simple, so that $\mathbf{P}(N_i(0,h] \geq 2) = o(h)$ as $h \searrow 0$ for $i = 1, \ldots, p$. Since

$$\{N(0,h] \geq 2\} \subset \bigcup_{1 \leq i \leq p} \{N_i(0,h] \geq 2\} \cup \bigcup_{1 \leq i < j \leq p} \{N_i(0,h] \geq 1, \ N_j(0,h] \geq 1\} \ ,$$

we see that

$$\begin{aligned}\mathbf{P}(N(0,h] \geq 2) &\leq \sum_{i=1}^{p} \mathbf{P}(N_i(0,h] \geq 2) + \sum_{i<j} \mathbf{P}(N_i(0,h] \geq 1)\mathbf{P}(N_j(0,h] \geq 1) \\ &= o(h) + \mathcal{O}(h^2) = o(h) \ ,\end{aligned}$$

so that by Proposition 4, the superposition N is also simple. In this case, if we compare (40) with (28) and (29), we obtain (as in [14])

$$\lambda Y_2(S) = \sum_{i=1}^{p} \lambda_i Y_2^{(i)}(u) \ . \tag{42}$$

If we apply (19) to

$$E(S) = \mathbf{P}(N(0,S] = 0) = \prod_{i=1}^{p} \mathbf{P}(N_i(0,S] = 0) = \prod_{i=1}^{p} E_i(S) \ , \tag{43}$$

we get

$$F(S) = \sum_{i=1}^{p} \frac{\lambda_i}{\lambda} F_i(S) \prod_{j \neq i} E_i(S) \ . \tag{44}$$

(Compare [4], [13].)

Remark 1 *It is natural to consider the case where the system is decomposed into infinitely many non-interacting small subsystems, so that the spectrum of the system is a superposition of a large number of subspectra. To model this situation, suppose for each $n = 1, 2, \ldots$, we are given mutually independent point processes $N_1^{(n)}(dx), \ldots, N_{p_n}^{(n)}(dx)$ such that $p_n \to \infty$ and that*

$$\lim_{n \to \infty} \max_{1 \leq i \leq p_n} \mathbf{P}(N_i^{(n)}(I) \geq 1) = 0 \tag{45}$$

holds for each bounded interval I. Such a triangular array of point processes is called infinitesimal. *The class of all point processes which are obtained as a weak limit of the superposition*

$$N^{(n)}(dx) = N_1^{(n)}(dx) + \cdots + N_{p_n}^{(n)}(dx) \tag{46}$$

from an infinitesimal triangular array coincides with the class of all infinitely divisible point processes (see [17]), among which the Poisson point process is only a special example. That is, independent superposition of a large number of sparse point processes does not necessarily imply the limiting Poissonian nature, as is *often naively supposed. In physics literature, the possibility of obtaining non-Poissonian limits was pointed out in a recent work of Makino and Tasaki ([13]).*

It is known ([17]) that when each constituent $N_j^{(n)}(dx)$ is a simple stationary point process, the superposition (46) converges weakly to a Poisson point process as $n \to \infty$, if and only if one has

$$\lim_{n\to\infty} \sum_{j=1}^{p_n} \lambda_j^{(n)} \{1 - Q_0^{(n,j)}(u)\} = 0 \tag{47}$$

for any $u > 0$, where $\lambda_j^{(n)}$ is the intensity of $N_j^{(n)}(dx)$ and

$$Q_0^{(n,j)}(u) := \lim_{h \searrow 0} \mathbf{P}(N_j^{(n)}(0,u] = 0 |\ N_j^{(n)}(-h, 0] > 0) \ .$$

The condition (47) is exactly the same as "Case 1"which is formulated in [13].

3 Ergodic point processes; statistics along the sequence

3.1 Ergodic point processes on R

In this section, we strengthen the assumption on the point process N_ω, and suppose that our probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is equipped with a measure preserving *ergodic* flow $\{\theta_t;\ t \in \mathbf{R}\}$ such that

$$N_{\theta_t \omega}(A) = N_\omega(t + A) \tag{48}$$

holds for all $t \in \mathbf{R}$ and any Borel set $A \in \mathcal{B}(\mathbf{R})$. Here $t + A = \{t + x;\ x \in A\}$ is the translation of the set A. A point process satisfying the condition (48) is called a $\{\theta_t\}$- ergodic point process on the probability space $(\Omega, \mathcal{F}, \mathbf{P}; \{\theta_t\})$. As before, we shall assume $m := \mathbf{E}[N(0, 1]] < \infty$.

Example 3 *Suppose N is a stationary Poisson point process with parameter $c > 0$, and let P_N be its law induced on the measurable space $(M_p(\mathbf{R}), \mathcal{M}_p(\mathbf{R}))$. Let us then forget the original probability space on which $N = N_\omega$ was given, and take $(M_p(\mathbf{R}), \mathcal{M}_p(\mathbf{R}), P_N)$ as our new probability space, the mapping $M_p(\mathbf{R}) \ni \omega \mapsto N_\omega$ being the identity, and the tranformation $\theta_t \omega$ being the spatial translation of $\omega \in M_p(\mathbf{R})$. Then the flow $\{\theta_t\}$ is ergodic, and the new point process satisfies (48) and the conditions of Definition 1 with the same parameter $c > 0$. Thus a stationary Poisson point process can always be supposed to be ergodic.*

Example 4 *Define the transformations $\{\theta_t;\ t \in \mathbf{R}\}$ on the unit circle $\Omega = [0, 1)$ by $\theta_t \omega = \{t + \omega\}$. It is well known that $\{\theta_t;\ t \in \mathbf{R}\}$ is an ergodic flow on the probability space $(\Omega, \mathcal{B}[0, 1), \mathbf{P})$ of Example 2, and it is easy to see that the point process defined by (34) is a $\{\theta_t\}$-ergodic point process.*

Let N_ω be an ergodic point process, which we consider as representing an unfolded quantal spectrum. The individual ergodic theorem then tells us that various statistics taken along the spectral sequence has non-random limits with probability one. In the following proposition, we formulate some of such limiting relations, for later comparison with level statistics in §4:

Proposition 8 *(i) With probability one, we have*

$$\lim_{L\to\infty} \frac{1}{L} N_\omega(0, L] = m \ . \tag{49}$$

(ii) For any $n \geq 1$, any bounded intervals $I_1, \ldots, I_n$, and any integers $k_1, \ldots, k_n \geq 0$, define

$$F_\omega := \{t \geq 0;\ N_\omega(t + I_j) = k_j \ ,\ j = 1, \ldots, n\} \ .$$

Then we have with probability one

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L 1_{F_\omega}(t) dt = \mathbf{P}(N(I_j) = k_j,\ j = 1, \ldots, n) \ . \tag{50}$$

In particular, for any $k = 0, 1, \ldots$ and $c > 0$, the limit

$$\pi_k(c) := \lim_{L\to\infty} \frac{1}{L} \int_0^L 1_{\{N_\omega(t,t+c]=k\}} dt \tag{51}$$

exists almost surely, and is equal to $\mathbf{P}(N(0,c] = k)$.
(ii) For any $k \geq 1$ and any bounded intervals $I_1, \ldots, I_n$, we have

$$\begin{aligned} & \lim_{L\to\infty} \frac{1}{L} \int_0^L N_\omega^{[k]} \left((t + I_1) \times \cdots \times (t + I_k)\right) dt \\ = & \lim_{L\to\infty} \frac{1}{L} \int_0^L N_{\theta_t\omega}^{[k]}(I_1 \times \cdots \times I_k) dt = m^{[k]}(I_1 \times \cdots \times I_k) \ . \end{aligned} \tag{52}$$

In particular, for any $k = 0, 1, \ldots$ and $c > 0$, the limit

$$\mu_k(c) := \lim_{L\to\infty} \frac{1}{L} \int_0^L \left(N_\omega(t, t + c]\right)^{[k]} dt \tag{53}$$

exists almost surely, and is equal to $\mathbf{E}\left[(N_\omega(0,c])^{[k]}\right]$, the k-th factorial moment of the random variable $N_\omega(0,c]$.

3.2 The Palm measure

To each ergodic point process N_ω defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P}; \{\theta_t\})$, there corresponds a measure $\hat{\mathbf{P}}$ on $(\Omega, \mathcal{F})$ called the Palm measure of the point process N_ω. The Palm measure clarifies the basic structure of a given ergodic point process. Although $\hat{\mathbf{P}}$ and $\mathbf{P}$ are mutually singular, it will turn out that the level spacing should be observed under $\hat{\mathbf{P}}$, but not under $\mathbf{P}$.

The basic materials of this section are taken from [30], to which the reader is referred for detailed proofs.

Theorem 2 (Existence and definition of the Palm measure) *If N_ω is a $\{\theta_t\}$-ergodic point process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P}; \{\theta_t\})$, then there is a unique measure $\hat{\mathbf{P}}$ on $(\Omega, \mathcal{F})$ with the following property: for any jointly measurable function $f(\omega, s) \geq 0$ on $\Omega \times \mathbf{R}$,*

$$\int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(ds) f(\theta_s\omega, s) = \int_\Omega \hat{\mathbf{P}}(d\omega) \int_\mathbf{R} ds f(\omega, s) \ . \tag{54}$$

If we let $f(\omega, s) = g(\theta_{-s}\omega, s)$ with $g(\omega, s) \geq 0$ jointly measurable, the defining condition (54) of the Palm measure is equivalent to

$$\int_\Omega \mathbf{P}(d\omega) \int_\mathbf{R} N_\omega(ds) g(\omega, s) = \int_\Omega \hat{\mathbf{P}}(d\omega) \int_\mathbf{R} ds g(\theta_{-s}\omega, s) \ . \tag{55}$$

In particular, taking $g(\omega, s) = 1_A(s)$, with $A \in \mathcal{B}(\mathbf{R})$, we get

$$\mathbf{E}[N(A)] = |A| \hat{\mathbf{P}}(\Omega) \ .$$

Letting $A = (0, 1]$, we obtain

Proposition 9

$$\hat{\mathbf{P}}(\Omega) = \mathbf{E}[N(0, 1]] = m \ .$$

Hence $\hat{\mathbf{P}}$ is a finite measure if $m < \infty$, but is a probability measure only if $m = 1$.

Proposition 10 *The Palm measure $\hat{\mathbf{P}}$ is concentrated on the set*

$$\hat{\Omega} := \{\omega \in \Omega |\ N_\omega(\{0\}) > 0\} \ .$$

To see this, take a strictly positive function u with $\int_{\mathbf{R}} u(s)ds = 1$. Then for $k = 0, 1, \ldots$,

$$\begin{aligned}
\hat{\mathbf{P}}(N(\{0\}) = k) &= \int_\Omega \hat{\mathbf{P}}(d\omega) \int_{\mathbf{R}} ds 1_{\{N_\omega(\{0\})=k\}} u(s) \\
&= \int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(ds) 1_{\{N_{\theta_s\omega}(\{0\})=k\}} u(s) \\
&= \int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(ds) 1_{\{N_\omega(\{s\})=k\}} u(s) \ .
\end{aligned}$$

This shows $\hat{\mathbf{P}}(N(\{0\}) = 0) = 0$, and when N is simple, also $\hat{\mathbf{P}}(N(\{0\}) \geq 2) = 0$.

For non-negative continuous functions $\varphi(x)$ on $\mathbf{R}$, we set

$$\mathcal{L}_N(\varphi) := \mathbf{E}[\exp(-N(\varphi))] \ , \tag{56}$$

where

$$N_\omega(\varphi) := \int_{\mathbf{R}} \varphi(x) N_\omega(dx) \ ,$$

and call it the Laplace functional of the (not necessarily stationary) point process N. The probability law of a point process is uniquely determined by its Laplace functional.

By definition, a sequence $\{N^{(n)}\}$ of point processes converges weakly to a point process N if their law induced on the metric space $M_p(\mathbf{R})$ converge weakly to that of N. It is well known that this is equivalent to the convergence of $\mathcal{L}_{N^{(n)}}(\varphi)$ to $\mathcal{L}_N(\varphi)$ for all φ in the class $C_0^+(\mathbf{R})$, the totality of non-negative continuous functions with a compact support. The following proposition then tells us that the conditional law $\mathbf{P}(N \in \bullet |\ N(-h, 0] > 0)$ of a simple ergodic point process N converges weakly to the probability law $\hat{\mathbf{P}}(N \in \bullet)/\hat{\mathbf{P}}(\Omega)$ as $h \searrow 0$.

Proposition 11 *Let a point process N_ω be $\{\theta_t\}$-ergodic and simple, so that $\hat{\mathbf{P}}(\Omega) = m = \lambda < \infty$. Then for any $\varphi \in C_0^+(\mathbf{R})$, one has*

$$\frac{1}{\hat{P}(\Omega)} \int_\Omega \exp(-N_\omega(\varphi)) \hat{\mathbf{P}}(d\omega) = \lim_{h \searrow 0} \int_\Omega \exp(-N_\omega(\varphi)) \mathbf{P}(d\omega |\ N(-h, 0] > 0) \ . \tag{57}$$

Proof. If we set $H(\omega) := \exp[-N_\omega(\varphi)]$, then $H(\theta_t\omega)$ is continuous in t. Hence letting $g(\omega, s) = H(\omega)1_{(-h,0]}(s)$ in (55), we get

$$\frac{1}{\lambda h} \int_\Omega \mathbf{P}(d\omega) H(\omega) N_\omega(-h, 0] = \frac{1}{\lambda h} \int_\Omega \hat{\mathbf{P}}(d\omega) \int_{-h}^0 ds H(\theta_{-s}\omega) \ .$$

Since N_ω is simple, we have $m = \lambda$, $\mathbf{P}(N(-h, 0] > 0) \sim \lambda h = \mathbf{E}[N(-h, 0]]$ as $h \searrow 0$. Hence $N_\omega(-h, 0]$ in the above equality can be replaced by $1_{\{N_\omega(-h,0]>0\}}$ in the limit $h \searrow 0$, and the conclusion follows.

Remark 2 *If we let $f(\omega, s) = 1_{(-\infty,0]}(s) 1_{\{N_\omega(0,x-s]=k\}}$ in (54), we obtain the equality*

$$\mathbf{P}(N(0, x] \leq k) = \int_x^\infty \hat{\mathbf{P}}(N(0, s] = k) ds \ . \tag{58}$$

Since the integrand on the right hand side of (58) is easily seen to be right continuous, we can conclude from Palm-Khinchin formula (Theorem 1) that

$$\hat{\mathbf{P}}(N(0,x]=k)=\lambda q_k(x):=\lambda \lim_{h\searrow 0}\mathbf{P}(N(0,x]=k|\ N(-h,0]>0) \tag{59}$$

holds for $k=0,1,2,\ldots$ and $x\geq 0$. (Note that $q_k(x)$ is also right continuous by Proposition 6.) This is not a direct consequence of the weak convergence of $\mathbf{P}(N\in\bullet|\ N(-h,0]>0)$ to $\hat{\mathbf{P}}(N\in\bullet)/\lambda$. In fact, weak convergence of a sequence of point process is in general equivalent to the convergence

$$\mathbf{P}(N^{(n)}(I_j)=k_j\ ,\ j=1,2,\ldots,p)\longrightarrow \mathbf{P}(N(I_j)=k_j\ ,\ j=1,2,\ldots,p)$$

for all $p\geq 1$, all bounded intervals $I_1,\ldots,I_p$ such that $\mathbf{P}(N(\partial I_j)>0)=0$, and for all integers $k_1,\ldots,k_p$. But $\hat{\mathbf{P}}$ always has a mass at the end point 0 of $(0,x]$.

Example 5 (Palm measure of the stationary Poisson point process) *Let N_ω be a stationary Poisson point process with parameter $c>0$ which is $\{\theta_t\}$-ergodic. (See Example 3.) Then Proposition 11 can be applied to obtain its Palm measure in the following way. If we define for $h>0$,*

$$N_\omega^{1,h}(\cdot):=N_\omega(\cdot\cap(-h,0])\ ;\quad N_\omega^{2,h}(\cdot):=N_\omega(\cdot\setminus(-h,0])\ ,$$

then $N_\omega^{1,h}$ and $N_\omega^{2,h}$ are independent. Hence for each $\varphi\in C_0^+(\mathbf{R})$,

$$\begin{aligned}
&\int_\Omega \exp[-N_\omega(\varphi)]\mathbf{P}(d\omega|\ N(-h,0]>0)\\
=\ &\int_\Omega \exp[-N_\omega^{1,h}(\varphi)]\exp[-N_\omega^{2,h}(\varphi)]\mathbf{P}(d\omega|\ N(-h,0]>0)\\
=\ &\mathbf{E}[\exp[-N_\omega^{2,h}(\varphi)]\mathbf{E}[1_{\{N(-h,0]>0\}}\exp[-N^{1,h}(\varphi)]]/\mathbf{P}(N(-h,0]>0)\ .
\end{aligned}$$

It is easy to see that this converges to

$$\frac{1}{c}e^{-\varphi(0)}\hat{\mathbf{E}}[\exp[-N(\varphi)]]\ .$$

as $h\searrow 0$. Hence, under the probability measure $\frac{1}{c}\hat{\mathbf{P}}$, N has the distribution of the stationary Poisson point process with an extra non-random point placed at the origin.

Example 6 *For the point process treated in Example 2 and 4, it is easy to see that $\hat{\mathbf{P}}(d\omega)=\lambda\delta_0(d\omega)$. Under $\frac{1}{\lambda}\hat{\mathbf{P}}$, N is the non-random lattice $\lambda\mathbf{Z}$.*

In the remaining part of this section, we assume that our ergodic point process N_ω is simple and has the representation (9), with $x_n(\omega)$'s satisfying (1). Its Palm measure is concentrated on the set

$$\hat{\Omega}:=\{\omega\in\Omega|\ x_0(\omega)=0\}\ .$$

In terms of $x_0(\omega)$ and $x_1(\omega)$, the defining condition (54) can be rephrased in the following way:

Proposition 12 *For any jointly measurable function $g(\omega,s)\geq 0$ on $\Omega\times[0,\infty)$,*

$$\int_\Omega \mathbf{P}(d\omega)g(\theta_{x_0(\omega)}\omega,-x_0(\omega))=\int_{\hat{\Omega}}\hat{\mathbf{P}}(d\omega)\int_0^{x_1(\omega)}dsg(\omega,s)\ . \tag{60}$$

In particular, if we let $g(\omega,s)=f(\omega)$, we get

$$\int_\Omega \mathbf{P}(d\omega)f(\theta_{x_0(\omega)}\omega)=\int_{\hat{\Omega}}\hat{\mathbf{P}}(d\omega)x_1(\omega)f(\omega)\ , \tag{61}$$

and letting $g(\omega,s)=f(\theta_s\omega)$ and noting $g(\theta_{x_0(\omega)}\omega,-x_0(\omega))=f(\omega)$, we obtain

$$\int_\Omega \mathbf{P}(d\omega)f(\omega)=\int_{\hat{\Omega}}\hat{\mathbf{P}}(d\omega)\int_0^{x_1(\omega)}dsf(\theta_s\omega)\ . \tag{62}$$

Proof. The function $a_0(\omega, s) = 1_{\{x_0(\omega) \le s < x_1(\omega)\}}$ satisfies the relation

$$a_0(\theta_s\omega, -s) = a_0(\omega, s) \ .$$

If we take

$$f(\omega, s) = a_0(\omega, -s)g(\omega, -s)$$

in (54), we get

$$\int_\Omega \mathbf{P}(d\omega) \int_{\mathbf{R}} N_\omega(ds) a_0(\theta_s\omega, -s) g(\theta_s\omega, -s) = \int_{\hat\Omega} \hat{\mathbf{P}}(d\omega) \int_{\mathbf{R}} ds a_0(\omega, -s) g(\omega, -s) \ .$$

But

$$\begin{aligned} \int_{\mathbf{R}} N_\omega(ds) a_0(\theta_s\omega, -s) g(\theta_s\omega, -s) &= \int_{\mathbf{R}} N_\omega(ds) a_0(\omega, s) g(\theta_s\omega, -s) \\ &= g(\theta_{x_0(\omega)}\omega, -x_0(\omega)) \ , \end{aligned}$$

and since $x_0(\omega) = 0$ for $\omega \in \hat\Omega$ on which $\hat{\mathbf{P}}$ is concentrated, we have

$$\int_{\mathbf{R}} ds a_0(\omega, -s) g(\omega, -s) = \int_0^{x_0(\omega)} ds g(\omega, s) \ .$$

Hence we obtain (60).

As corollaries, we obtain the following formulae for the probability distributions of x_0 and x_1 ([30]):

Proposition 13 *If we define the measure $\hat F(dv)$ on $(0, \infty)$ by $\hat F(dv) = \hat{\mathbf{P}}(x_1 \in dv)$, then*

(0)

$$\int_0^\infty v \hat F(dv) = 1 \ ; \tag{63}$$

(i)

$$\mathbf{P}(-x_0 \in ds,\ x_1 - x_0 \in dv) = 1_{[0,v)}(ds) \hat F(dv) \ ; \tag{64}$$

(ii)

$$\mathbf{P}(x_1 - x_0 \in dv) = v \hat F(dv) \ ; \tag{65}$$

(iii)

$$\mathbf{P}(-x_0 \in ds) = \mathbf{P}(x_1 \in ds) = p(s) ds \ , \ \textit{where } p(s) = \hat F((s, \infty)) \ . \tag{66}$$

In particular, the distributions of x_0 and x_1 are absolutely continuous.

Proof. Applying (60) to the function of the form $g(\omega, s) = h(x_1(\omega) - x_0(\omega), s)$, with $h(v, s)$ an arbitrary non-negative Borel function on $\mathbf{R}_+^2$, we obtain, after noting $g(\theta_{x_0(\omega)}\omega, s) = g(\omega, s)$ and that $x_0(\omega) = 0$ for $\omega \in \hat\Omega$,

$$\mathbf{E}[h(x_1(\omega) - x_0(\omega), -x_0(\omega))] = \int_0^\infty ds \int_{\hat\Omega} \hat{\mathbf{P}}(d\omega) 1_{[0, x_1(\omega))}(s) h(x_1(\omega), s) \ , \tag{67}$$

which proves the assertion (i). When $h \equiv 1$, (67) reduces to the assertion (0), and when $h(v, s) = \psi(v)$, with $\psi \ge 0$ Borel, then we have from (67)

$$\mathbf{E}[\psi(x_1(\omega) - x_0(\omega))] = \int_{\hat\Omega} \hat{\mathbf{P}}(d\omega)\ x_1(\omega) \psi(x_1(\omega)) \ ,$$

proving (ii). On the other hand if h is of the form $h(v,s) = \varphi(s)$, we get

$$\mathbf{E}[\varphi(-x_0(\omega))] = \int_0^\infty ds\varphi(s)\hat{\mathbf{P}}(x_1 > s) \ ,$$

or

$$\mathbf{P}(-x_0 \in ds) = \hat{F}((s,\infty))ds \ .$$

Finally, let $h(v,s) = 1_{(u,\infty)}(v-s)$, where $u > 0$ is arbitrarily fixed. Then we have

$$g(\theta_{x_0(\omega)}\omega, -x_0(\omega)) = 1_{(u,\infty)}(x_1(\omega))$$

for $\omega \in \Omega$, and

$$g(\omega, s) = 1_{(u+s,\infty)}(x_1(\omega))$$

for $\omega \in \hat{\Omega}$ and $s \geq 0$. Thus (67) reduces to

$$\mathbf{P}(x_1 > u) = \int_u^\infty \hat{\mathbf{P}}(x_1 > s)ds \ ,$$

which completes the proof of (iii).

Now we state a theorem which describes the fundamental structure of ergodic simple point processes. For its proof, we refer the reader to [30] or [11].

Theorem 3 *Let $N_\omega = \sum_{n\in\mathbf{Z}} \delta_{x_n(\omega)}$ be a simple, $\{\theta_t\}$-ergodic point process. Then for each $n \in \mathbf{Z}$, the mapping θ_{x_n}, defined by $\theta_{x_n(\omega)}\omega$, maps Ω onto $\hat{\Omega}$. If we denote by $\hat{\theta}_n$ the restriction of θ_{x_n} to $\hat{\Omega}$, then $\{\hat{\theta}_n |\ n \in \mathbf{Z}\}$ is an ergodic group of measure preserving transformation on $(\hat{\Omega}, \frac{1}{\lambda}\hat{\mathbf{P}})$.*

Let $\tau_n(\omega) := x_n(\omega) - x_{n-1}(\omega)$ ($n \in \mathbf{Z}$). Then obviously $\tau_n = \tau \circ \hat{\theta}_1^{n-1}$ holds on $\hat{\Omega}$, for all $n \in \mathbf{Z}$. By the theorem above, $\{\tau_n\}$, which is the sequence of spacings between points $\{x_n(\omega)\}$, is stationary and ergodic under the probability measure $\frac{1}{\lambda}\hat{\mathbf{P}}$.

Example 7 *When the random variables $\{\tau_n\}$ are independent, the point process N_ω is called a* renewal process. *The stationary Poisson point process (Example 3) and the randomly shifted periodic lattice (Examples 2 and 4) are both renewal processes. For the case of the Poisson point process, the distribution of τ_n under $\frac{1}{\lambda}\hat{\mathbf{P}}$ is the exponential distribution $\lambda e^{-\lambda t}dt$.*

Remark 3 *If N_ω is a renewal process, the distributions of τ_n for $n \neq 1$ under $\mathbf{P}$ and $\frac{1}{\lambda}\hat{\mathbf{P}}$ coincide. But according to (ii) of Proposition 13, $\tau_1 = x_1 - x_0$ and τ_n ($n \neq 1$) have different distribution in general. When $\{x_n\}$ is Poissonian, this phenomena is called the "waiting time paradox"([7]). In fact, the only renewal process such that τ_1 and τ_n ($n \neq 1$) are identically distributed under $\mathbf{P}$ is the randomly shifted lattice $\lambda^{-1}\mathbf{Z}$ discussed in Example 2, 4 and 6. This is seen by solving the equation*

$$\frac{1}{\lambda}\hat{F}(dv) = v\hat{F}(dv) \ .$$

3.3 The level spacing distribution

Let $N_\omega = \sum_{n\in\mathbf{Z}} \delta_{x_n(\omega)}$ be a simple, ergodic point process. In the previous section, we saw that the spacings $\tau_n(\omega) := x_n(\omega) - x_{n-1}(\omega)$ of points are identically distributed under the probability measure $\frac{1}{\lambda}\hat{\mathbf{P}}$. But under the original probability measure $\mathbf{P}$, the distribution of τ_n can be different for different n. To see this, first note that $\tau_n(\omega) = \tau_n(\theta_{x_0}\omega)$. By (61) of Proposition 12, for any $t \geq 0$,

$$P(\tau_n(\omega) > t) = P(\tau_n \circ \theta_{x_0} > t) = \int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega)x_1(\omega)1_{\{\tau_n(\omega)>t\}} \ . \tag{68}$$

If N is not a renewal process, x_1 and τ_n $(n \neq 1)$ are correlated under $\hat{\mathbf{P}}$. Hence the right hand side of (68) can be different for different n.

This poses a question: Suppose we are given a sample

$$\cdots < x_{-1}(\omega) < x_0(\omega) \leq 0 < x_1(\omega) < x_2(\omega) < \cdots$$

of an ergodic point process, which we regard as an unfolded quantal spectrum. If you wish to determine the level spacing distribution experimentally, then what you naturally do would be to collect a large set of data $\tau_k(\omega) = x_k(\omega) - x_{k-1}(\omega)$, $1 \leq k \leq n$, of level spacings, and then draw a histogram of their frequency, expecting that it will approximately represent some smooth curve when the data size n is large enough and the histogram is fine enough. Mathematically speaking, this amounts to considering the empirical distribution

$$F_\omega^n(dt) := \frac{1}{n}\sum_{k=1}^{n} \delta_{\tau_k(\omega)}(dt) \ , \tag{69}$$

and expecting this to converge, as $n \to \infty$, to some fixed distribution $F(dt)$ almost surely. But if the random variables τ_n are observed under the probability measure $\mathbf{P}$, they are not identically distributed. Hence mixing these to form empirical distributions seems meaningless. On the other hand, if the data $\{\tau_n\}$ are observed under $\frac{1}{\lambda}\hat{\mathbf{P}}$, they form a stationary ergodic sequence. In this case, the ergodic theorem works well, and $F_\omega^n(dt)$ converges to $\frac{1}{\lambda}\hat{\mathbf{P}}(\tau_1 \in dt)$ as $n \to \infty$, $\hat{\mathbf{P}}$-almost surely. But $\hat{\mathbf{P}}$ and $\mathbf{P}$ are mutually singular. Which of them should be considered as the real law governing $\{\tau_n\}$?

Fortunately, this puzzle is easily solved:

Proposition 14 *The empirical distribution $F_\omega^n(dt)$ defined by (69) converges weakly, as $n \to \infty$, to $\frac{1}{\lambda}\hat{\mathbf{P}}(\tau_1 \in dt)$ for $\mathbf{P}$-almost all $\omega \in \Omega$.*

Proof. For each $t \geq 0$, define

$$A_t = \left\{ \omega \in \Omega \;\middle|\; \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} 1_{\{\tau_k(\omega)>t\}} = \frac{1}{\lambda}\hat{\mathbf{P}}(\tau_1 > t) \right\} \ .$$

As was already pointed out, $\tau_k = \tau_1 \circ \hat{\theta}_1^{k-1}$ holds on $(\hat{\Omega}, \frac{1}{\lambda}\hat{\mathbf{P}})$, $\hat{\theta}_1$ being an ergodic automorphism. Hence by the individual ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} 1_{\{\tau_k(\omega)>t\}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} 1_{\{\tau_1>t\}} \circ \hat{\theta}_1^{k-1}(\omega) = \frac{1}{\lambda}\hat{\mathbf{P}}(\tau_1 > t)$$

holds for $\frac{1}{\lambda}\hat{\mathbf{P}}$-almost all $\omega \in \hat{\Omega}$, namely

$$\frac{1}{\lambda}\hat{\mathbf{P}}(\hat{\Omega} \cap A_t) = 1 \ .$$

It remains to show $\mathbf{P}(A_t) = 1$. But since $\omega \in A$ if and only if $\theta_{x_0}\omega \in A$, we have from Proposition 12,

$$\begin{aligned} \mathbf{P}(A) &= \int_\Omega \mathbf{P}(d\omega)1_A(\omega) = \int_\Omega \mathbf{P}(d\omega)1_A(\theta_{x_0}\omega) \\ &= \int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega)x_1(\omega)1_A(\omega) = \int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega)x_1(\omega) = 1 \ , \end{aligned}$$

the last equality following from (0) of Proposition 13.

Remark 4 *By the Palm-Khinchin formula, we have*

$$\mathbf{P}(N(0,x] = 0) = \int_x^\infty \hat{\mathbf{P}}(N(0,s] = 0)ds = \int_x^\infty \hat{\mathbf{P}}(x_1 > t)dt \ .$$

Thus if we put $\pi_0(x) := \mathbf{P}(N(0,x]=0)$, and $\rho(t) := \hat{\mathbf{P}}(x_1 > t)$, we get

$$\pi_0(x) = \int_x^\infty \rho(t)dt = 1 - \int_0^x \rho(t)dt \ , \tag{70}$$

or

$$\rho(x) = -\frac{d^+}{dx}\pi_0(x) \ . \tag{71}$$

The consideration in the proof of Proposition 14 can be easily generalized to yield the following assertion:

Proposition 15 *For any non-negative measurable function f on $(\Omega, \mathcal{F}, \mathbf{P})$, one has*

$$\lim_{L\to\infty} \frac{1}{L}\int_{(0,L]} N_\omega(dt) f(\theta_t\omega) = \int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega) f(\omega) \tag{72}$$

for $\mathbf{P}$-almost all $\omega \in \Omega$.

Proof. If we let

$$A = \left\{ \omega \in \Omega \,\middle|\, \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n f(\theta_{x_k}\omega) = \frac{1}{\lambda}\int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega) f(\omega) \right\} \ ,$$

then by the ergodic theorem applied to $(\hat{\Omega}, \hat{\mathbf{P}}, \hat{\theta}_1)$, we have $\hat{\mathbf{P}}(\hat{\Omega}\cap A) = \lambda$. But since $\omega \in A$ is equivalent to $\theta_{x_0}\omega \in A$, the argument in Proposition 14 gives $\mathbf{P}(A) = 1$. On the other hand, we have

$$\lim_{L\to\infty} \frac{1}{L} N_\omega(0,L] = \lambda$$

$\mathbf{P}$-almost surely. Hence for $\mathbf{P}$-almost all $\omega \in \Omega$, we have $N_\omega(0,L] \to \infty$ and

$$\begin{aligned}
\lim_{L\to\infty} \frac{1}{L}\int_{(0,L]} N_\omega(dx) f(\theta_t\omega) &= \lim_{L\to\infty} \frac{1}{L}\sum_j 1_{(0,L]}(x_j(\omega)) f(\theta_{x_j}\omega) \\
&= \lim_{L\to\infty} \frac{N_\omega(0,L]}{L} \cdot \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n f(\theta_{x_j}\omega) \\
&= \lambda \cdot \frac{1}{\lambda}\int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega) f(\omega) = \int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega) f(\omega) \ .
\end{aligned}$$

Let us apply this proposition to the limiting k-level correlation, for later comparison with level statistics in §4 and 6.

Proposition 16 *For realizations $\{x_n(\omega)\}$ of a simple ergodic point process, define the k-level correlation*

$$\begin{aligned}
R_L^\omega(I_2,\ldots,I_k) := \frac{1}{L}\sharp\{(x_{i_1}(\omega),\ldots,x_{i_k}(\omega)) \quad | \quad & x_{i_1}(\omega) \le L,\ i_1,\ldots,i_k \text{ are distinct and} \\
& x_{i_j}(\omega) - x_{i_1}(\omega) \in I_j \ , j = 2,\ldots,k\} \ ,
\end{aligned} \tag{73}$$

where $L > 0$ and $I_2,\ldots,I_k$ are any bounded intervals. Then

$$\lim_{L\to\infty} R_L^\omega(I_2,\ldots,I_k) = \int_{\hat{\Omega}} \hat{\mathbf{P}}(d\omega) N_\omega^{[k-1]}(I_2^*\times\cdots\times I_k^*) =: R_k(I_2,\ldots,I_k) \tag{74}$$

holds for $\mathbf{P}$-almost all $\omega \in \Omega$, where we have set $I_j^* := I_j \setminus \{0\}$.

Proof. Noting (10), we can write

$$\begin{aligned} LR_L^\omega(I_2,\ldots,I_k) &= \sum_i 1_{(0,L]}(x_i(\omega))\sharp\{(x_{i_2}(\omega),\ldots,x_{i_k}(\omega))|\ i_2,\ldots,i_k \text{ are distinct,} \\ &\qquad x_{i_j}(\omega)\neq x_{i_1}(\omega),\ x_{i_j}(\omega)\in I_j+x_{i_j}(\omega),\ j=2,\ldots,k\} \\ &= \int N_\omega(ds)1_{(0,L]}(s)N_\omega^{[k-1]}\left((I_2+s)\setminus\{s\}\right)\times\cdots\times\left((I_k+s)\setminus\{s\}\right)) \\ &= \int N_\omega(ds)1_{(0,L]}(s)N_{\theta_s\omega}^{[k-1]}(I_2^*\times\cdots\times I_k^*)\ , \end{aligned}$$

from which the conclusion follows by Proposition 15.

Remark 5 *In case $I_2=\cdots=I_k=(0,c]$, let us write $R_k(I_2,\ldots,I_k)=R_k(c)$. By the Palm-Khinchin formula, we can compute*

$$\begin{aligned} \frac{1}{k!}\mathbf{E}\left[(N(0,L])^{[k]}\right] &= \sum_{n\geq k}\binom{n-1}{k-1}\mathbf{P}(N(0,c]\geq n) \\ &= \sum_{n\geq k}\binom{n-1}{k-1}\int_0^c\hat{\mathbf{P}}(N(0,s]=n-1)ds \\ &= \frac{1}{(k-1)!}\int_0^c\int_{\hat\Omega}\hat{\mathbf{P}}(d\omega)(N_\omega(0,s])^{[k-1]}\ , \end{aligned}$$

namely

$$\mu_k(c)=k\int_0^c R_k(s)ds\ . \tag{75}$$

See [6] for related formulas.

Example 8 *If N_ω is the stationary Poisson point process with parameter $\lambda>0$, then by Example 5 and Proposition 7,*

$$R_k(I_2,\ldots,I_k)=\lambda\int_\Omega\mathbf{P}(d\omega)N_\omega^{[k-1]}(I_2\times\cdots\times I_k)=\lambda^k\prod_{j=2}^k|I_j|\ . \tag{76}$$

In particular, $R_k(c)=\lambda^kc^{k-1}$.

Remark 6 *If $\{x_j(\omega)\}$ is a renewal process, then $R_2(c)=\lambda^2c$ implies that N is Poissonian. Indeed, if we set*

$$F(c)=\frac{1}{\lambda}\hat{P}(N(0,c]>0)=\frac{1}{\lambda}\hat{P}(x_{n+1}-x_n\leq c)\ ,$$

which does not depend on n , then

$$\begin{aligned} \frac{1}{\lambda}R_2(c) &= \frac{1}{\lambda}\int_{\hat\Omega}\hat{\mathbf{P}}(d\omega)N_\omega(0,c]=\frac{1}{\lambda}\sum_{n=1}^\infty\hat{\mathbf{P}}(N(0,c]\geq n) \\ &= \frac{1}{\lambda}\sum_{n=1}^\infty\hat{\mathbf{P}}\left(\sum_{j=0}^{n-1}(x_{j+1}-x_j)\leq c\right)=\sum_{n=1}^\infty(F^{n*})(c)\ , \end{aligned}$$

where F^{n*} is the n-fold convolution of F. Hence if $\mathcal{L}(\xi)=\int_0^\infty e^{-\xi c}dF(c)$ is the Laplace transform of $dF(c)$, then from $R_2(c)=\lambda^2c$, we get

$$\sum_{n=1}^\infty(\mathcal{L}(\xi))^n=\frac{\mathcal{L}(\xi)}{1-\mathcal{L}(\xi)}=\frac{1}{\lambda}\int_0^\infty e^{-\xi c}dR_2(c)=\frac{\lambda}{\xi}\ ,$$

namely $\mathcal{L}(\xi)=\lambda/(\lambda+\xi)$. Hence $dF(c)=\lambda e^{-\lambda c}dc$. But the Poisson point process is the only renewal process for which $x_{n+1}-x_n$ obeys the exponential distribution.

4 Statistics for asymptotically uniformly distributed sequences

Suppose we are given a finite or infinite sequence

$$0 \le x_1(L) < x_2(L) < \cdots < x_n(L) < \cdots$$

depending on a parameter L, which we shall let tend to infinity. We think of this sequence $\{x_j(L)\}_j$ as the unfolded spectrum of a quantum Hamiltonian depending on a parameter, or more directly a realization of some fictitious simple ergodic point process with mean density (or intensity) λ, if it is asymptotically uniformly distributed in the following sense:

Condition 1 *There is a positive constant $A \le \infty$ and $\lambda < \infty$ such that for any $0 \le \alpha < A$,*

$$\sharp\{j \ge 1 |\ x_j(L) \le \alpha L\} \sim \lambda \alpha L \quad (L \to \infty)\ . \tag{77}$$

In particular, when the sequence $\{x_j(L)\}_j$ does not depend on L and satisfies

$$\sharp\{j \ge 1 |\ x_j \le L\} \sim \lambda L \quad (L \to \infty)\ , \tag{78}$$

then it satisfies Condition 1.

For each $c > 0$ and for each interval $I = (a, b]$, we consider the following quantity (statistics):

$$N_L(I) := \sharp\{j \ge 1 |\ x_j(L) \in I\}\ ; \tag{79}$$

$$\pi_k(L; I) := \frac{1}{L}\int_0^L 1_{\{N_L(t+I)=k\}}dt \quad (k \ge 0)\ ; \tag{80}$$

$$\mu_k(L; I) := \frac{1}{L}\int_0^L N_L(t+I)^{[k]}dt = \sum_{j\ge k} j^{[k]}\pi_j(L; I) \quad (k \ge 1)\ ; \tag{81}$$

$$\rho(L; c) := \frac{1}{L}\sharp\{j \ge 1 |\ x_j(L) \le L,\ x_{j+1}(L) - x_j(L) > c\} \quad (c > 0)\ ; \tag{82}$$

$$\begin{aligned}&R_k(L; c)\\ &:= \frac{1}{L}\sharp\{(x_{i_1}(L), \ldots, x_{i_k}(L))|\ i_1, \ldots, i_k \text{ are distinct},\ x_{i_1}(L) \le L,\ x_{i_j}(L) - x_{i_1}(L) \in (0, c]\}\\ &\quad (c > 0,\ k \ge 2)\ .\end{aligned} \tag{83}$$

As an immediate consequence of Condition 1, we obtain the following

Proposition 17 *For any $c > 0$,*

$$\mu_1(c) := \lim_{L\to\infty} \frac{1}{L}\int_0^L N_L(t, t+c]\ dt = \lim_{L\to\infty} \mu_1(L; (0, c]) = \lambda c\ . \tag{84}$$

Proof. By the definition,

$$\mu_1(L; (0, c]) = \frac{1}{L}\sum_j \int_0^L 1_{(t,t+c]}(x_j(L))dt = \frac{1}{L}\sum_j \int_0^L 1_{[x_j(L)-c, x_j(L))}(t)dt\ . \tag{85}$$

But when $L > c > 0$, one has

$$\int_0^L 1_{[x_j(L)-c,x_j(L))}(t)dt = \begin{cases} c, & \text{if } c \le x_j(L) \le L \\ 0, & \text{if } x_j(L) > L + c\ , \end{cases}$$

while
$$\int_0^L 1_{[x_j(L)-c,x_j(L))}(t)dt \le c\ , \quad \text{if } x_j(L) < c \quad \text{or} \quad L < x_j(L) \le L + c\ .$$
Hence for each fixed $c > 0$, it is obvious from Condition 1 that
$$\mu_1(L;(0,c]) \sim c\frac{1}{L}\sharp\{j \ge 1|\ x_j(L) \le L\} \quad \text{as } L \to \infty\ ,$$
so that
$$\mu_1(c) = \lim_{L\to\infty} \mu_1(L;(0,c]) = \lambda c\ ,$$
which was to be proved.

Except for this trivial case, Condition 1 does not imply that any of the statistics (80)–(83) converges to some limit as $L \to \infty$. But when some of these convergences occur, we can prove equalities between them, which may be called deterministic analogues of the Palm-Khinchin formula, as the following two propositions show. (Compare Remarks 4 and 5.)

Proposition 18 *Let $k \ge 2$ and assume that*
$$\sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)<c} (c - x_{j_k}(L)) = o(L) \quad (L \to \infty) \tag{86}$$
holds for any $c > 0$.
(i) If the limit
$$\mu_k(c) := \lim_{L\to\infty} \mu_k(L;(0,c]) \tag{87}$$
exists for each $c > 0$ and is differentiable with respect to c, then
$$R_k(c) := \lim_{L\to\infty} R_k(L;c) \tag{88}$$
also exists and one has
$$R_k(c) = \frac{1}{k}\frac{d}{dc}\mu_k(c)\ . \tag{89}$$
(ii) If on the other hand, $R_k(c)$ exists for each $c > 0$, then $\mu_k(c)$ also exists and
$$\mu_k(c) = k\int_0^L R_k(c')dc'\ . \tag{90}$$

Remark 7 *The condition (86) is obviously satisfied when the sequence $\{x_j(L)\}_j$ does not depend on L.*

Proof of Propsition 18. Since
$$\begin{aligned}\frac{1}{k!}(N_L(t,t+c])^{[k]} &= \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)} 1_{(t,t+c]}(x_{j_1}(L))\cdots 1_{(t,t+c]}(x_{j_k}(L)) \\ &= \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)} 1_{[x_{j_1}(L)-c,x_{j_1}(L))}(t)\cdots 1_{[x_{j_k}(L)-c,x_{j_k}(L))}(t)\ ,\end{aligned}$$

we can compute

$$
\begin{aligned}
\frac{L}{k!}\mu_k(L;(0,c]) &= \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)} \left|[0,L]\cap\bigcap_{p=1}^{k}[x_{j_p}(L)-c,x_{j_p}(L))\right| \\
&= \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)} \{(L\wedge x_{j_1}(L))-(x_{j_k}(L)-c)_+\}_+ \\
&= \sum_{\substack{x_{j_1}(L)<\cdots<x_{j_k}(L);\\ x_{j_1}(L)\le L,x_{j_k}(L)\ge c}} \{c-x_{j_k}(L)+x_{j_1}(L)\}_+ \\
&\quad + \sum_{L<x_{j_1}(L)<\cdots<x_{j_k}(L)} \{c-x_{j_k}(L)+L\}_+ + \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)<c} x_{j_1}(L) \\
&= \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} \{c-x_{j_k}(L)+x_{j_1}(L)\}_+ \\
&\quad + \left[\sum_{\substack{x_{j_1}(L)<\cdots<x_{j_k}(L);\\ x_{j_1}(L)\le L<x_{j_k}(L)}} \{c-x_{j_k}(L)+x_{j_1}(L)\}_+ + \sum_{L<x_{j_1}(L)<\cdots<x_{j_k}(L)} \{c-x_{j_k}(L)+L\}_+\right] \\
&\quad + \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)<c} [x_{j_1}(L)-\{c-x_{j_k}(L)+x_{j_1}(L)\}_+]\ .
\end{aligned}
$$

The third term equals

$$
-\sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)<c} (c-x_{j_k}(L))\ ,
$$

and is $o(L)$ as $L\to\infty$ by the assumption. On the other hand, the second term, which is enclosed in the big bracket and which we denote by $M(L)$, can be estimated in two ways. First of all, we have

$$
\begin{aligned}
M(L) &\le \sum_{\substack{x_{j_1}(L)<\cdots<x_{j_k}(L);\\ x_{j_1}(L)\le L<x_{j_k}(L)\ \ or\ \ L<x_{j_1}(L)<\cdots<x_{j_k}(L)\le L+c}} (c-x_{j_k}(L)+x_{j_1}(L))_+ \\
&= \sum_{\substack{x_{j_1}(L)<\cdots<x_{j_k}(L);\\ x_{j_1}(L)\le L<x_{j_k}(L)\ \ or\ \ L<x_{j_1}(L)<\cdots<x_{j_k}(L)\le L+c}} \int_{-\infty}^{\infty} 1_{(t,t+c]}(x_{j_1}(L))\cdots 1_{(t,t+c]}(x_{j_k}(L))dt\ .
\end{aligned}
$$

Suppose $x_{j_1}(L)\le L<x_{j_k}(L)$. Then we have $1_{(t,t+c]}(x_{j_1}(L))=0$ for $t>L+c$ and $1_{(t,t+c]}(x_{j_k}(L))=0$ for $t<L-c$. If, on the other hand, $L<x_{j_1}(L)<\cdots<x_{j_k}(L)\le L+c$, then we have $1_{(t,t+c]}(x_{j_k}(L))=0$ for $t>L+c$ and $1_{(t,t+c]}(x_{j_1}(L))=0$ for $t<L-c$. Hence

$$
\begin{aligned}
M(L) &\le \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)} \int_{L-c}^{L+c} 1_{(t,t+c]}(x_{j_1}(L))\cdots 1_{(t,t+c]}(x_{j_k}(L))dt \\
&= \frac{1}{k!}\int_{L-c}^{L+c} (N_L(t,t+c])^{[k]}dt\ ,
\end{aligned}
$$

which is of $o(L)$ $(L\to\infty)$ when $\mu_k(c)$ exists.

Secondly we also have the inequality

$$
\begin{aligned}
M(L) &\le c \sum_{\substack{x_{j_1}(L)<\cdots<x_{j_k}(L);\\ x_{j_1}(L)\le L<x_{j_k}(L)\ \ or\ \ L<x_{j_1}(L)<\cdots<x_{j_k}(L)\le L+c}} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}} \\
&\le c \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L+c} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}} - c \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}}\ .
\end{aligned}
$$

Obviously, the right hand side is of $o(L)$ as $L \to \infty$ when $R_k(c)$ exists. Therefore, under either of the conditions of (i) and (ii), one has

$$\frac{1}{k!} L\mu_k(L;(0,c]) = \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} \{c - x_{j_k}(L) + x_{j_1}(L)\}_+ + o(L) \ . \tag{91}$$

Let us now prove the assertion (i). From the above remark and

$$\{c+\delta - x_{j_k}(L) + x_{j_1}(L)\}_+ - \{c - x_{j_k}(L) + x_{j_1}(L)\}_+ \ge \delta 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}} \ ,$$

we see

$$\frac{1}{k!}(\mu_k(c+\delta) - \mu_k(c)) \ge \delta \limsup_{L\to\infty} \frac{1}{L} \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}} \ .$$

Similarly, from the inequality

$$\{c - x_{j_k}(L) + x_{j_1}(L)\}_+ - \{c-\delta - x_{j_k}(L) + x_{j_1}(L)\}_+ \le \delta 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}} \ ,$$

it follows that

$$\frac{1}{k!}(\mu_k(c) - \mu_k(c-\delta)) \le \delta \liminf_{L\to\infty} \frac{1}{L} \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}} \ .$$

Under the assumption of the differentiability of $\mu_k(c)$, we can let $\delta \searrow 0$, to conclude

$$\lim_{L\to\infty} \frac{1}{L} \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c\}} = \frac{1}{k!}\frac{d}{dc}\mu_k(c) \ .$$

But the limit on the left hand side is easily seen to coincide with

$$\frac{1}{(k-1)!} \lim_{L\to\infty} R_k(L;c)$$

and hence

$$R_k(c) = \frac{1}{k}\frac{d}{dc}\mu_k(c) \ ,$$

as was to be proved.

Let us turn to the proof of (ii). Let $\ell \in \mathbf{N}$, and write $\delta = c/\ell$ and $\Delta = x_{j_k}(L) - x_{j_1}(L)$. Then we have

$$\delta \sum_{m=1}^{\ell} 1_{\{c\ge \Delta + m\delta\}} \le (c-\Delta)_+ \le \delta \sum_{m=0}^{\ell} 1_{\{c\ge \Delta + m\delta\}} \ ,$$

and hence

$$\begin{aligned} & \frac{c}{\ell}\sum_{m=1}^{\ell} \frac{1}{L} \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c - mc/\ell\}} \\ \le \ & \frac{1}{L} \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} \{c - x_{j_k}(L) + x_{j_1}(L)\}_+ \\ \le \ & \frac{c}{\ell}\sum_{m=0}^{\ell} \frac{1}{L} \sum_{x_{j_1}(L)<\cdots<x_{j_k}(L)\le L} 1_{\{x_{j_k}(L)-x_{j_1}(L)\le c - mc/\ell\}}. \end{aligned}$$

Letting $L \to \infty$ and noting (91), we obtain

$$\begin{aligned}
\frac{1}{(k-1)!}\frac{c}{\ell}\sum_{m=1}^{\ell} R_k(c(1-\frac{m}{\ell})) &\leq \frac{1}{k!}\liminf_{L\to\infty} \mu_k(L;(0,c]) \\
&\leq \frac{1}{k!}\limsup_{L\to\infty} \mu_k(L;(0,c]) \\
&\leq \frac{1}{(k-1)!}\frac{c}{\ell}\sum_{m=0}^{\ell} R_k(c(1-\frac{m}{\ell}))\ , \qquad (92)
\end{aligned}$$

for all $\ell \in \mathbf{N}$. Finally letting $\ell \to \infty$, we arrive at

$$\frac{1}{k!}\mu_k(c) = \frac{1}{k!}\lim_{L\to\infty}\mu_k(L;(0,c]) = c\int_0^1 R_k(c(1-x))dx = \int_0^c R_k(c')dc'\ ,$$

or

$$\mu_k(c) = k\int_0^c R_k(c')dc'\ ,$$

completing the proof.

Proposition 19 *Suppose* $\pi_0(c) := \lim_{L\to\infty}\pi_0(L;(0,c])$ *exists for each* $c > 0$ *and is differentiable with respect to* c*. Then* $\rho(c) = \lim_{L\to\infty}\rho(L;c)$ *also exists, and is given by*

$$\rho(c) = -\frac{d}{dc}\pi_0(c)\ . \qquad (93)$$

Conversely if $\rho(c)$ *exists for each* $c > 0$*, then* $\pi_0(c)$ *also exists, and is given by*

$$\pi_0(c) = 1 - \int_0^c \rho(c')dc'\ . \qquad (94)$$

Proof. If $x_j(L) \leq L$ and $x_{j+1}(L) - x_j(L) > c$, then for any $\delta \in (0,c)$, we have the implication

$$t \in [x_j(L)-\delta, x_j(L)) \implies N_L(t;\delta) > 0\ ,\ N_L(t+\delta; c-\delta) = 0\ ,$$

the intervals $[x_j(L)-\delta, x_j(L))$ corresponding to these j's being disjoint. Hence if we set

$$n(c;L) = \sharp\{j \geq 1 \mid x_j(L) \leq L\ ,\ x_{j+1}(L) - x_j(L) > c\}\ ,$$

then it holds that

$$\begin{aligned}
\delta n(L;c) &= \sum_{x_j(L)\leq L\ ;\ x_{j+1}(L)-x_j(L)>c} \int_{-\delta}^{L} 1_{[x_j(L)-\delta, x_j(L))}(t)dt \\
&\leq \int_{-\delta}^{L} 1_{\{N_L(t,t+\delta]>0,\ N_L(t+\delta,t+c]=0\}}dt \\
&= \int_{-\delta}^{L}\left[1_{\{N_L(t+\delta,t+c]=0\}} - 1_{\{N_L(t,t+c]=0\}}\right]dt \\
&= \int_0^{L+\delta} 1_{\{N_L(t,t+c-\delta]=0\}}dt - \int_{-\delta}^{L} 1_{\{N_L(t,t+c]=0\}}dt \\
&\leq \delta + \int_0^{L} 1_{\{N_L(t,t+c-\delta]=0\}}dt - \int_0^{L} 1_{\{N_L(t,t+c]=0\}}dt\ ,
\end{aligned}$$

namely

$$\delta\rho(L;c) \le \frac{\delta}{L} + \pi_0(L;(0,c-\delta]) - \pi_0(L;(0,c]) \ .$$

Letting $L \to \infty$, we get

$$\delta \limsup_{L\to\infty} \rho(L;c) =\le \pi_0(c-\delta) - \pi_0(c) \ .$$

Since $\delta > 0$ is arbitrary, this shows

$$\limsup_{L\to\infty} \rho(L;c) \le -\frac{d}{dc}\pi_0(c) \ .$$

On the other hand, if $N_L(t,t+\delta] > 0$ and $N_L(t+\delta, t+c+\delta) = 0$, then there is an $x_j(L) \in (t,t+\delta]$ such that $x_{j+1}(L) - x_j(L) > c$. Hence the set

$$\{t \in [0, L-\delta] \mid N_L(t,t+\delta] > 0 \ , \ N_L(t+\delta, t+c+\delta] = 0\}$$

is a union of finitely many disjoint intervals I_ℓ each of length no greater than δ , and for each I_ℓ , there corresponds an $x_j(L) \le L$ satisfying $x_{j+1}(L) - x_j(L) > c$. Hence the number of these intervals does not exceed $n(c;L)$, and so

$$\begin{aligned}
\delta n(c;L) &\ge \sum_\ell |I_\ell| \\
&= \int_0^{L-\delta} 1_{\{N_L(t,t+\delta]>0, \ N_L(t+\delta,t+c+\delta]=0\}} dt \\
&= \int_0^{L-\delta} \left[1_{\{N_L(t+\delta,t+c+\delta]=0\}} - 1_{\{N_L(t,t+c+\delta]=0\}}\right] dt \\
&= \int_\delta^{L} 1_{\{N_L(t,t+c]=0\}} dt - \int_0^{L-\delta} 1_{\{N_L(t,t+c+\delta]=0\}} dt \\
&\ge -\delta + L\pi_0(L;(0,c]) - L\pi_0(L;(0,c+\delta]) \ .
\end{aligned}$$

Dividing both sides by L and letting $L \to \infty$, we get

$$\delta \liminf_{L\to\infty} \rho(L;c) \ge \pi_0(c) - \pi_0(c+\delta) \ ,$$

and since $\delta > 0$ is arbitrary, we obtain

$$\liminf_{L\to\infty} \rho(L;c) \ge -\frac{d}{dc}\pi_0(c) \ ,$$

which completes the proof of the first assertion.

Conversely suppose that the limit $\rho(c)$ exists for each $c > 0$. As we have already seen, the following two inequalities are valid for any $L > 0$ and $\delta \in (0,c)$:

$$\delta\rho(L;c) \le \frac{\delta}{L} + \pi_0(L;(0,c-\delta]) - \pi_0(L;(0,c]) \ ; \tag{95}$$

$$\delta\rho(L;c) \ge -\frac{\delta}{L} + \pi_0(L;(0,c]) - \pi_0(L;(0,c+\delta]) \ . \tag{96}$$

Now let $0 < c_1 < c_2$ be given and let $\ell \in \mathbf{N}$ be large enough so that $\delta := (c_2 - c_1)/\ell < c_1$. If in (95) we take $c = c_1 + k\delta$, $k = 1, 2, \ldots, \ell$, then

$$\delta\rho(L;c_1+k\delta) \le \frac{\delta}{L} + \pi_0(L;(0,c_1+(k-1)\delta]) - \pi_0(L;(0,c_1+k\delta]) \ .$$

Summing over k, one obtains

$$\delta \sum_{k=1}^{\ell} \rho(L; c_1 + k\delta) \leq \frac{c_2 - c_1}{L} + \pi_0(L; (0, c_1]) - \pi_0(L; (0, c_2]) \ .$$

Letting first $L \to \infty$ and then $\ell \to \infty$, we arrive at

$$\limsup_{L\to\infty} \pi_0(L; (0, c_2]) + \int_{c_1}^{c_2} \rho(c')dc' \leq 1 \ . \tag{97}$$

In the same way, taking $c = c_1 + k\delta$, $k = 0, 1, \ldots, \ell - 1$, in (96), and summing over k, we obtain

$$\liminf_{L\to\infty} \pi_0(L; (0, c_2]) + \int_{c_1}^{c_2} \rho(c')dc' \geq \liminf_{L\to\infty} \pi_0(L; (0, c_1]) \ . \tag{98}$$

Denote $\underline{\pi}_0(c) := \liminf_{L\to\infty} \pi_0(L; (0, c])$. We shall now verify that $\underline{\pi}_0(c) \nearrow 1$ as $c \searrow 0$. Suppose, on the contrary, that

$$\alpha := \lim_{c\searrow 0} \underline{\pi}_0(c) < 1 \ .$$

Then for any $\beta \in (\alpha, 1)$ and $c > 0$, we must have

$$\limsup_{L\to\infty} \frac{1}{L} \int_0^L 1_{\{N_L(t,t+c]>0\}} dt > 1 - \beta \ .$$

Now the set

$$\{t \in (0, L]| \ N_L(t, t + c] > 0\}$$

is a disjoint union of intervals $I_1, \ldots, I_m$, and by the definition of this set, we see

$$\sharp\{j \geq 1| \ x_j(L) \leq L\} \geq \frac{1}{c} \sum_{j=1}^{m} |I_j| \ ,$$

and hence

$$\lambda = \lim_{L\to\infty} \frac{1}{L} \sharp\{j \geq 1| \ x_j(L) \leq L\} \geq \frac{1}{c} \limsup_{L\to\infty} \frac{1}{L} \sum_{j=1}^{m} |I_j| \geq \frac{1 - \beta}{c} \ .$$

Letting $c \searrow 0$, we arrive at a contradiction.

Having shown $\lim_{c\searrow 0} \underline{\pi}_0(c) = 1$, we can let $c_1 \searrow 0$ in (97) and (98), to obtain

$$\limsup_{L\to\infty} \pi_0(L; (0, c]) + \int_0^c \rho(c')dc' \leq 1 \leq \liminf_{L\to\infty} \pi_0(L; (0, c]) + \int_0^c \rho(c')dc' \ , \tag{99}$$

which show the existence of $\pi_0(c)$ and the relation

$$\pi_0(c) = 1 - \int_0^c \rho(c')dc' \ .$$

Remark 8 *Proposition 19 above shows the close relationship between the "two possible ways of describing the randomness of the spectrum", proposed by Sinai in the Introduction of [38].*

5 Two notions of "unfolding"

5.1 "Unfoldable" family of sequences

In general, energy levels of a quantum Hamiltonian have a different local density in different parts of the energy axis. Hence, as we noted in the introduction, in order to discuss the fluctuating properties of energy levels, it is necessary to "unfold" the quantal spectrum under consideration, so that the transformed sequence has uniform asymptotic distribution. To formulate this notion, let us first define a class of spectra which can be unfolded.

Suppose we are given a finite or infinite increasing sequence $\{E_j(h); j \geq 1\}$ of real numbers depending on a parameter $h > 0$. We shall say that this family is *unfoldable* in the limit $h \searrow 0$ if the following condition is satisfied:

Condition 2 *There is an $\alpha > 0$ and a non-decreasing function $\nu(E) \geq 0$ on $\mathbf{R}$ with $\lim_{E\to-\infty} \nu(E) = 0$ such that for any $E \in \mathbf{R}$, one has*

$$N_h(E) := \sharp\{j \geq 1;\ E_j(h) \leq E\} \sim \nu(E) h^{-\alpha} \tag{100}$$

as $h \searrow 0$.

Below we give some examples of quantum Hamiltonians whose spectrum is unfoldable in the sense defined above.

Example 9 *Let $H(\hbar) := -\hbar^2\Delta_D$, where Δ_D is the Laplacian on a connected, bounded domain $D \subset \mathbf{R}^d$ with smooth boundary, considered under the Dirichlet boundary condition. If $0 < e_1 < e_2 < \cdots$ are the eigenvalues of $H(1) = -\Delta_D$, then the spectrum $\{E_j(\hbar)\}_j$ of $H(\hbar)$ is given by $E_j(\hbar) = \hbar^2 e_j$. As is well known (Weyl's asymptotic), there is a universal constant $c_d > 0$, depending only on the dimension of the space, such that*

$$N_1(E) := \sharp\{j \geq 1;\ e_j \leq E\} \sim c_d|D|E^{d/2}\ , \quad E \to \infty\ . \tag{101}$$

Hence for any fixed $E > 0$, we have

$$N_\hbar(E) := \sharp\{j \geq 1;\ E_j(\hbar) \leq E\} = N_1(E/\hbar^2) \sim (c_d|D|E^{d/2})\hbar^{-d} \tag{102}$$

as $E \to \infty$. Thus $\{E_j(\hbar)\}_j$ is unfoldable in the (semiclassical) limit $\hbar \searrow 0$ with $\alpha = d$ and $\nu(E) = c_d|D|(E \vee 0)^{d/2}$.

Example 10 (Regular spectra ([3], [31])) *Suppose that the classical Hamiltonian system associated to $H(p,q)$ is completely integrable. Then one can transform the momentum-coordinate variable $(p,q) \in \mathbf{R}^d \times \mathbf{R}^d$ into the action–angle variable $(I,\varphi) = (I_1,\ldots,I_d;\varphi_1,\ldots,\varphi_d)$, and $H(p,q) = H(I)$ depends only on the action variable (see [1]). Following Percival ([31]), Berry and Tabor ([3]), we shall say that $H(\hbar)$, the quantization of $H(p,q)$, has regular spectrum if its energy levels are (approximately) given by quantizing the action variables $I_1,\ldots,I_d$ appearing in $H(I)$. Namely, we suppose that the energy levels of $H(\hbar)$ are*

$$E_{\mathbf{n}}(\hbar) = H\left(\hbar(\mathbf{n} + \frac{1}{4}\alpha)\right) \quad ; \quad \mathbf{n} = (n_1,\ldots,n_d) \in \mathbf{Z}_+^d\ , \tag{103}$$

where $\alpha \in \mathbf{Z}_+^d$ is called the Maslov index. (In the following discussion, we put $\alpha = \mathbf{0}$ as in [3].) This procedure is called the EBK quantization after the names of Einstein, Brillouin and Keller, and it gives the exact spectrum in such concrete examples as the harmonic oscillators or free motion on flat tori ([36], [39]).

Suppose that $H(I)$, as a function on $\mathbf{R}_+^d$ satisfies the following conditions:

1. *$H(r\mathbf{x})$ is increasing in $r \geq 0$ for any $\mathbf{x} \in \mathbf{R}_+^d \setminus \{\mathbf{0}\}$;*
2. *$H(\mathbf{0}) = 0$;*

3. $\{\mathbf{x};\ H(\mathbf{x}) \le E\}$ *is compact for any* $E > 0$.

Then $\{E_{\mathbf{n}}(\hbar);\ \mathbf{n} \in \mathbf{Z}_+^d\}$ *is unfoldable in the limit* $\hbar \searrow 0$ *with* $\alpha = d$ *and*

$$\nu(E) = \int \cdots \int_{\mathbf{R}_+^d} 1_{\{H(I) \le E\}} dI_1 \cdots dI_d \ . \tag{104}$$

Example 11 (Ergodic Schrödinger operators) *Let* $\{V_\omega(x); x \in \mathbf{R}^d \text{ or } \mathbf{Z}^d\}$ *be an ergodic random field defined on some probability space* $(\Omega, \mathcal{F}, \mathbf{P})$. *Consider the random Schrödinger operator* $H_\omega := -\Delta + V_\omega(x)$ *and its restriction* H_ω^L *to the cube* $[-L/2, L/2]^d$ *under Dirichlet boundary condition. Here* Δ *denotes the continuum or discrete Laplacian. Let* $\{E_j^\omega(L)\}_j$ *be the spectrum of* H_ω^L. *Then under mild conditions on the random potential* $\{V_\omega(x)\}_x$, *there exists a non-decreasing function* $\mathcal{N}(E) \ge 0$ *(the integrated density of states) such that* $\mathcal{N}(-\infty) = 0$ *and that for* $\mathbf{P}$*-almost all* $\omega \in \Omega$, *one has*

$$\lim_{L \to \infty} \frac{1}{|\Lambda|} \sharp\{j \ge 1;\ E_j^\omega(L) \le E\} = \mathcal{N}(E) \tag{105}$$

for all $E \in \mathbf{R}$. *(See [40] for a review of recent results.) This shows that for* $\mathbf{P}$*-almost all* $\omega \in \Omega$, *the spectrum* $\{E_j^\omega(L)\}_j$ *is unfoldable in the (thermodynamic) limit* $h := 1/L \searrow 0$, *with* $\alpha = d$ *and* $\nu(E) = \mathcal{N}(E)$.

5.2 The unfolding of the first kind

In general, the purpose of unfolding is to transform a given family of unfoldable sequences of levels into another family of sequences $\{x_j(L)\}_j$ which is asymptotically uniformly distributed so as to satisfy Condition 1. In this subsection, we assume that the function $\nu(E)$ appearing in Condition 2 is continuous in E.

For each $t \in (0, \nu(+\infty))$, define

$$\nu^{-1}(t) := \inf\{E;\ \nu(E) > t\} \ . \tag{106}$$

Then we have $\nu(\nu^{-1}(t)) = t$ and the equivalence

$$\nu(E) \le t \ \Leftrightarrow \ E \le \nu^{-1}(t) \ .$$

If we let $e_j(h) := \nu(E_j(h))$, then for each $t > 0$,

$$\begin{aligned}\sharp\{j \ge 1;\ e_j(h) \le t\} &= \sharp\{j \ge 1;\ E_j(h) \le \nu^{-1}(t)\} \\ &= N_h(\nu^{-1}(t)) \\ &\sim \nu(\nu^{-1}(t)) h^{-\alpha} = t h^{-\alpha}\end{aligned}$$

as $h \searrow 0$ by Condition 2. Hence if we let $L = h^{-\alpha}$, and $x_j(L) = L e_j(L^{-1/\alpha})$, then we have for any $\alpha > 0$,

$$\begin{aligned}\sharp\{j \ge 1;\ x_j(L) \le \alpha L\} &= \sharp\{j \ge 1;\ e_j(L^{-1/\alpha}) \le \alpha\} \\ &\sim \alpha (L^{-1/\alpha})^{-\alpha} = \alpha L\end{aligned}$$

as $L \to \infty$, so that the parametrized sequence $\{x_j(L)\}_j$ satisfies Condition 1 with $\lambda = 1$.

5.3 The unfolding of the second kind

The unfolding formulated above is what is usually practiced in quantum chaos (see e.g. [13]), and looks like a quite natural idea in that it leads one to directly observe energy levels, although under a non-linear transformation $e_j = \nu(E_j)$. But with regard to this, we notice a seemingly strange remark made by Berry and Tabor in their celebrated paper [3]. They argue that the energy E itself

> is an unsuitable parameter to describe the levels, for two reasons. The first is that the *mean level density* $\bar{n}(E)$ depends on energy.$\cdots$ The second reason is that the statistics of the levels might themselves depend on the region of energy being studied.

As a solution to this problem, Berry and Tabor propose to observe the quantum system at a fixed energy value E, and to "quantize the Planck's constant $\hbar$". They apply this procedure only to regular spectra discussed in Example 10, but in a general setting, it would be formulated in the following way: In addition to Condition 2, let us assume that $E_j(h)$ is a non-decreasing function of $h > 0$ for each j. Let us fix E such that $E_0 < E < E_1$, where

$$E_0 := \inf\{E;\ \nu(E) > 0\} \geq -\infty \tag{107}$$

and

$$E_1 := \sup\{E;\ \nu(E) < \nu(\infty)\} \leq \infty\ . \tag{108}$$

Now define

$$h_j(E) := \inf\{h > 0;\ E_j(h) > E\} \tag{109}$$

for each $j \geq 1$, where we let $h_j(E) = 0$ if $E_j(h) > E$ for all $h > 0$, and $h_j(E) = +\infty$ if $E_j(h) \leq E$ for all $h > 0$. Actually, the last two possibilities are ruled out by the following argument: Since $\nu(E) > 0$, Condition 2 implies in particular that

$$\lim_{h \searrow 0} \sharp\{j \geq 1;\ E_j(h) \leq E\} = \infty\ . \tag{110}$$

If $h_j(E) = 0$ for some i, then $E_j(h) > E$ for all $j \geq i$ and $h > 0$. Hence

$$\sharp\{j \geq 1;\ E_j(h) \leq E\} \leq i - 1\ ,$$

contradicting (110), and so $h_j(E) > 0$ must hold for all $j \geq 1$. On the other hand, since $E < E_1$, there is an $E' > E$ such that $\nu(E) < \nu(E') < \nu(\infty)$. Again by Condition 2, we have

$$\lim_{h \searrow 0} \sharp\{j \geq 1;\ E < E_j(h) \leq E'\} = \infty\ . \tag{111}$$

This shows that for only finitely many j's, one can have

$$\sup_{h>0} E_j(h) \leq E\ .$$

Thus there is a $j_0 \geq 1$ such that $h_j(E) < \infty$ for all $j \geq j_0$. We now define

$$x_j = x_j(E) := \nu(E) h_j(E)^{-\alpha} \tag{112}$$

for $j \geq j_0$. In the sequel we assume $j_0 = 1$. By the definition, we have the following equivalence and implication for each $L > 0$:

$$\begin{aligned} x_j < L \quad &\Leftrightarrow \quad h_j(E) > (\nu(E)/L)^{1/\alpha} \\ &\Rightarrow \quad E_j\left((\nu(E)/L)^{1/\alpha}\right) \leq E \\ &\Rightarrow \quad h_j(E) \geq (\nu(E)/L)^{1/\alpha} \Leftrightarrow x_j \leq L\ . \end{aligned}$$

This shows

$$\sharp\{j \geq 1;\ x_j < L\} \leq \sharp\left\{j \geq 1;\ E_j\left((\nu(E)/L)^{1/\alpha}\right) \leq E\right\} \leq \sharp\{j \geq 1;\ x_j \leq L\}\ .$$

But by Condition 2, we have

$$\lim_{L\to\infty} \frac{1}{L} \sharp\left\{j \geq 1;\ E_j\left((\nu(E)/L)^{1/\alpha}\right) \leq E\right\} = \lim_{L\to\infty} \frac{1}{L} \nu(E) \left\{\left(\frac{\nu(E)}{L}\right)^{1/\alpha}\right\}^{-\alpha} = 1\ ,$$

so that

$$\limsup_{L\to\infty} \frac{1}{L} \sharp\{j \geq 1;\ x_j < L\} \leq 1 \leq \liminf_{L\to\infty} \sharp\{j \geq 1;\ x_j \leq L\}\ ,$$

namely

$$\sharp\{j \geq 1;\ x_j \leq L\} \sim L\ , \quad L \to \infty\ .$$

Since $\{x_j\}$ itself does not depend on L, we obtain a sequence satisfying Condition 1 with $\lambda = 1$.

5.4 Examples of unfolding

Unfortunately, the present author is unable to clarify the general relationship between unfoldings of first and second kinds. For the time being, let us content ourselves by discussing several examples of quantal spectra to which both kinds of unfoldings can be applied.

5.4.1 The quantum billiard

For the Laplacian examined in Example 9, we have $E_j(\hbar) = \hbar^2 e_j$, $\alpha = d$, and $\nu(E) = c_d|D|E^{d/2}$. Since $\nu(E)$ is continuous and $E_j(\hbar)$ is increasing in $\hbar$, both kinds of unfoldings can be applied. To perform the unfolding of the first kind, we put as in 5.2,

$$e_j(\hbar) := \nu(E_j(\hbar)) = c_d|D|\hbar^d e_j^{d/2} \ ; \tag{113}$$

and

$$x_j(L) := Le_j(L^{-1/d}) = c_d|D|e_j^{d/2} \ . \tag{114}$$

The resulting sequence $\{x_j(L)\}$ does not depend on the parameter L.

To perform the unfolding of the second kind, let us fix an $E > 0$. Then we put

$$\hbar_j(E) := \inf\{\hbar;\ \hbar^2 e_j = E\} = (E/e_j)^{1/2} \tag{115}$$

and

$$x_j(E) := \nu(E)\hbar_j(E)^{-d} = c_d|D|E^{d/2}(E/e_j)^{-d/2} = c_d|D|e_j^{d/2} \ . \tag{116}$$

Thus both kinds of unfoldings yield the same sequence.

5.4.2 A one-dimensional Schrödinger operator with δ-potentials in the semi-classical limit

Consider the one-dimensional Schrödinger operator

$$H_v(\hbar) = -\hbar^2 \frac{d^2}{dx^2} \ + \ v \sum_{s=1}^{n} \delta(x - x_s) \ , \quad 0 \leq x \leq 1 \tag{117}$$

under Dirichlet boundary conditions at $x = 0, 1$. Here $v > 0$, and

$$0 \equiv x_0 < x_1 < \cdots < x_n < x_{n+1} \equiv 1 \ .$$

Let $\{E_j(\hbar)\}_j$ be the spectrum of $H_v(\hbar)$. Then we have for each $E > 0$

$$N_\hbar(E) := \sharp\{j \geq 1;\ E_j(\hbar) \leq E\} \sim \frac{1}{\pi}\sqrt{E}\hbar^{-1} \ , \quad \hbar \searrow 0 \ . \tag{118}$$

Thus Condition 2 is satisfied with $\alpha = 1$ and $\nu(E) = \frac{1}{\pi}\sqrt{E}$, which is continuous. Moreover, we can verify that each energy level $E_j(\hbar)$ is non-decreasing in $\hbar > 0$ by the min-max principle. Hence both kinds of unfoldings are applicable to our spectrum $\{E_j(\hbar)\}$: For each $L > 0$, the unfolded levels of the first kind are given by

$$x_j(L) := L\nu(E_j(L^{-1})) = \frac{L}{\pi}\sqrt{E_j(L^{-1})} \ , \tag{119}$$

while for each fixed $E > 0$, the unfolded levels of the second kind are given by

$$x_j(E) := \nu(E)\hbar_j(E)^{-1} = \frac{\sqrt{E}}{\pi}\hbar_j(E)^{-1} \ , \tag{120}$$

where $\hbar_j(E)$ is defined by the equation $E_j(\hbar) = E$.

It was shown in [21] that E is an energy level of $H_v(\hbar)$, if and only if it is a zero of the function

$$\Gamma_\hbar(E) := \sum_{p=0}^{n} \left(\frac{\hbar\sqrt{E}}{v}\right)^{n-p} \sum_{1\le j_1<\cdots<j_p\le n} \prod_{s=1}^{p+1} \sin\left\{(x_{j_s} - x_{j_{s-1}})\frac{\sqrt{E}}{\hbar}\right\} . \tag{121}$$

This shows on one hand that for a fixed $L > 0$, $\xi > 0$ is an unfolded level of the first kind if and only if it is a zero of the function $G_L(\xi) := \Gamma_{L^{-1}}\left((\pi\xi/L)^2\right)$, while on the other hand $\zeta > 0$ is an unfolded level of the second kind if and only if it is a zero of the function $\gamma_E(\zeta) := \Gamma_{\sqrt{E}/\pi\zeta}(E)$. If we define

$$f(z) := \prod_{s=1}^{n+1} \sin\left\{\pi(x_s - x_{s-1})z\right\} , \tag{122}$$

then we can write

$$G_L(\xi) = f(\xi) + g_L(\xi) \text{ and } \quad \gamma_E(\zeta) = f(\zeta) + \delta_E(\zeta) , \tag{123}$$

where $g_L(\xi)$ is small for large L and for $\xi = \mathcal{O}(L)$, while $\delta_E(\zeta)$ is small for large values of ζ. Hence by the argument in [21] using Rouché's theorem, the statistics for the zeros of the functions $G_L(\xi)$ and $\gamma_E(\zeta)$ are both reduced to the statistics for the zeros of the function $f(z)$, which are

$$\left\{\frac{m}{x_s - x_{s-1}} \ ; \ m \ge 1 \ , \ s = 0, 1, \ldots, n\right\} . \tag{124}$$

Thus two kinds of unfoldings applied to the spectrum $\{E_j(\hbar)\}$ lead to the same level statistics.

To proceed, consider as in §4 the counting function

$$N(t, t+c] := \sum_{s=0}^{n} \sum_{m=1}^{\infty} 1_{(t,t+c]}\left(\frac{m}{x_s - x_{s-1}}\right) , \tag{125}$$

and the probability

$$\pi_k(L; (0, c]) := \frac{1}{L}\int_0^L 1_{\{N(t,t+c]=k\}} dt \tag{126}$$

for $k = 0, 1, \ldots$, $c > 0$, and $L > 0$. Then we have

$$M_c \le N(t, t+c] \le M_c + n + 1 , \tag{127}$$

where

$$M_c := \sum_{s=1}^{n+1} [c(x_s - x_{s-1})] .$$

Moreover if $x_s - x_{s-1}$, $s = 1, 2, \ldots, n+1$, are rationally independent, one can show ([21]) that

$$\begin{aligned} \pi_k(c) &:= \lim_{L\to\infty} \pi_k(L; (0, c]) \\ &= \sum_{A\in\mathcal{F}(k-M_c)} \left(\prod_{s\in A}\{c(x_s - x_{s-1})\}\right)\left(\prod_{s\in A^c}(1 - \{c(x_s - x_{s-1})\})\right) \end{aligned} \tag{128}$$

for each $k = M_c, M_c + 1, \ldots, M_c + n + 1$. Here $\{a\}$ is the fractional part of a, and $\mathcal{F}(j)$ is the totality of subsets of $\{0, 1, \ldots, n\}$ having cardinality j. In particular, if $c > 0$ is small, then $M_c = 0$ and

$$\pi_0(c) = \prod_{s=1}^{n+1} (1 - c(x_s - x_{s-1})) . \tag{129}$$

We can now apply Proposition 19, to obtain

$$\rho(c) = -\frac{d}{dc}\pi_0(c) = \left(\sum_{s=1}^{n+1} \frac{x_s - x_{s-1}}{1 - c(x_s - x_{s-1})}\right) \prod_{s=1}^{n} (1 - c(x_s - x_{s-1})) \ . \tag{130}$$

For large n and for typical configuration $\{x_s\}$ of the δ-potentials, $\pi_k(c)$, $k = 0, 1, \ldots$, and $\rho(c)$ are different from, but very close to being Poissonian. In fact, if $X_1, X_2, \ldots$ are independent random variables which are uniformly distributed on $(0, 1)$, and if $0 < x_1^{(n)} < \cdots < x_n^{(n)} < 1$ is a rearrangement of $X_1, \ldots, X_n$ according to their magnitude, then one can show ([21]) that with probability one,

$$\lim_{n\to\infty} \pi_k(c; x_1^{(n)}, \ldots, x_n^{(n)}) = e^{-c}\frac{c^k}{k!} \ , \quad k = 0, 1, \ldots \ , \tag{131}$$

and

$$\lim_{n\to\infty} \rho(c; x_1^{(n)}, \ldots, x_n^{(n)}) = e^{-c} \ . \tag{132}$$

5.4.3 Regular spectra

Under the assumptions on the classical Hamiltonian of Example 10, $\nu(E)$ is a continuous function of E, and $E_{\mathbf{n}}(\hbar) = H(\hbar\mathbf{n})$ is increasing in $\hbar$ for each $\mathbf{n}$. Hence both kinds of unfolding are applicable to our regular spectrum $\{E_{\mathbf{n}}(\hbar); \mathbf{n} \in \mathbf{Z}_+^d\}$. To apply the unfolding of the *second* kind, we first note that for each $\mathbf{x} \in \mathbf{R}_+^d \setminus \{\mathbf{0}\}$ and $E > 0$, the equation $H(\hbar\mathbf{x}) = E$ can be solved for $\hbar$. If we denote this solution by $\hbar_E(\mathbf{x})$, then the unfolded energy level is given by $x_{\mathbf{n}}(E) = U_E(\mathbf{n})$, where we have defined

$$U_E(\mathbf{x}) = \nu(E)\hbar_E(\mathbf{x})^{-d} \ , \quad \mathbf{x} \in \mathbf{R}_+^d \ . \tag{133}$$

Since $\beta\hbar_E(\beta\mathbf{x}) = \hbar_E(\mathbf{x})$ for any $\beta > 0$, we see that $U_E(\mathbf{x})$ is a homogeneous function of degree d, namely

$$U_E(\beta\mathbf{x}) = \beta^d U_E(\mathbf{x}) \ , \tag{134}$$

for any $\beta > 0$. (See [3].)

To perform the level statistics of §4 for this sequence $\{U_E(\mathbf{n})\}_{\mathbf{n}}$, which is asymptotically uniformly distributed, we need to consider for $I = (a, b]$

$$N_L(t + I) = \sum_{\mathbf{n}\in\mathbf{Z}_+^d} 1_{(t+a,t+b]}(U_E(\mathbf{n})) \ . \tag{135}$$

If we define

$$\theta(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \ , \quad \mathbf{x} \in \mathbf{R}_+^d \setminus \{\mathbf{0}\} \tag{136}$$

and

$$\Pi_I(t) := \{\mathbf{x} \in \mathbf{R}_+^d |\ (t + a)^{1/d} U_E(\theta(\mathbf{x}))^{-1/d} < |\mathbf{x}| \leq (t + b)^{1/d} U_E(\theta(\mathbf{x}))^{-1/d}\} \ , \tag{137}$$

then

$$N_L(t + I) = \sum_{\mathbf{n}\in\mathbf{Z}_+^d} 1_{\Pi_I(t)}(\mathbf{n}) \ , \tag{138}$$

which is the number of lattice points contained in the d-dimensional domain $\Pi_I(t)$. It is then easy to see $|\Pi_I(t)| = b - a = |I|$, so that as t gets large, the domain $\Pi_I(t)$ expands in the space $\mathbf{R}_+^d$, but at the same time gets thinner and thinner to keep its volume constant. Hence provided the boundary of $\Pi_I(t)$ is not too regular, e.g. flat or spherical, then each point in $\mathbf{Z}_+^d$ would randomly belong to $\Pi_I(t)$ if t were chosen at random from a long interval $[0, L]$, so that the total number of lattice points in $\Pi_I(t)$, namely $N_L(t + I)$, would obey Poisson distribution with mean $|I|$. Namely we expect that

$$\pi_k(I) := \lim_{L\to\infty} \pi_k(L; I) = e^{-|I|}\frac{|I|^k}{k!} \ , \quad k = 0, 1, 2, \ldots \tag{139}$$

holds. This is conceivable if we recall how Poisson's law of small numbers is proved in elementary probability theory. Actually, it is a difficult problem to rigorously justify this intuition. In fact, no example of classically integrable Hamiltonian is known for which (139) is verified.

Let us now apply the unfolding of the *first* kind to our regular spectrum. This gives

$$x_{\mathbf{n}}(L) := L \cdot \nu(E_{\mathbf{n}}(L^{-1/d})) = L \cdot (\nu \circ H)(L^{-1/d}\mathbf{n}) \ , \quad L > 0, \ \mathbf{n} \in \mathbf{Z}_+^d \ ,$$

as unfolded levels. To proceed further, we assume that our Hamiltonian $H(\mathbf{x})$ satisfies the following

Condition 3 *For any* $\mathbf{x}, \mathbf{z} \in \mathbf{R}_+^d$ *and* $\beta > 0$, $H(\beta\mathbf{z}) \le H(\beta\mathbf{x})$ *implies* $H(\mathbf{z}) \le H(\mathbf{x})$.

Under this condition, the function $G := \nu \circ H$ is homogeneous with degree d. In fact for any $\beta > 0$,

$$G(\beta\mathbf{x}) = \int \cdots \int_{\mathbf{R}_+^d} 1_{\{H(I) \le H(\beta\mathbf{x})\}} dI_1 \cdots dI_d = \beta^d \int \cdots \int_{\mathbf{R}_+^d} 1_{\{H(\beta\mathbf{z}) \le H(\beta\mathbf{x})\}} dz_1 \cdots dz_d = \beta^d G(\mathbf{x}) \ .$$

Thus we have $x_{\mathbf{n}}(L) = G(\mathbf{n})$, and hence replacing U_E by $G = \nu \circ H$, we are lead to the same lattice point problem as before.

If in particular, the Hamiltonian $H(\mathbf{x})$ itself is a homogeneous function with degree $\gamma > 0$, say, then Condition 3 is automatically satisfied and for each $E > 0$, we have $\nu(E) = c_H E^{d/\gamma}$ with

$$c_H = \int \cdots \int_{\mathbf{R}_+^d} 1_{\{H(\mathbf{x}) \le 1\}} dx_1 \cdots dx_d \ ,$$

and $\hbar_E(\mathbf{x}) = E^{1/\gamma} H(\mathbf{x})^{-1/\gamma}$. Hence

$$U_E(\mathbf{x})\nu(E)\hbar_E(\mathbf{x})^{-d} = c_H H(\mathbf{x})^{d/\gamma} = \nu(H(\mathbf{x})) = G(\mathbf{x}) \ .$$

Thus the unfolded levels $x_{\mathbf{n}}(E) = U_E(\mathbf{n})$ of the second kind does not depend on E, and coincide with the unfolded levels $x_{\mathbf{n}}(L) = G(\mathbf{n})$ of the first kind.

At this point, it is appropriate to discuss the limit theorem of Sinai [38] and Major [15] (see also [23]). They consider the case $d = 2$ in which the curve

$$r = f(\theta) \ , \quad 0 \le \theta \le \pi/2$$

defining the boundary of the domain

$$\Pi_I(t) := \{\mathbf{x} \in \mathbf{R}_+^2 | \ \sqrt{t+a} f(\theta) < r \le \sqrt{t+b} f(\theta)\}$$

is very random. It was then proved by Major ([15]) that (139) actually holds for almost all realization of this random curve, and by the present author ([23]) that

$$\mu_k(I) := \lim_{L \to \infty} \mu_k(L; I) = |I|^k \ , \quad k = 1, 2, \ldots, \tag{140}$$

also holds almost surely, which is a slightly stronger assertion than (139). Since the randomness assumed in these works is so strong that the typical realizations of the curve is non-smooth, the direct connection of these limit theorems to the question of level statistics is lost. But the analysis in [38], [15], and [23] suggests us that it would be possible to prove that $\mu_k(I) = |I|^k$, $k = 1, 2, \ldots, K$, should hold almost surely with some finite K if the curve has only a finite dimensional randomness. This was actually done by VanderKam ([39]), the significance of which we shall discuss again in the next section.

5.4.4 Schrödinger operator with white noise potential in the thermodynamic limit

Let $\{B_\omega(t);\ t \ge 0\}$ be the standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and consider for each $L > 0$ the Schrödinger operator

$$H_\omega^L := -\frac{d^2}{dt^2} \ + \ B_\omega'(t) \ , \quad 0 \le t \le L \tag{141}$$

under Dirichlet boundary conditions at $x = 0, L$. Here $B'_\omega(t)$ is the formal derivative of the sample function $B_\omega(t)$. Although $B'_\omega(t)$ is singular, H^L_ω is well defined as a random self-adjoint operator acting in $L^2([0, L]; dt)$, which has a purely discrete spectrum. Compared to previous definitions (e.g. [8], [20]) of the operators of the form

$$H = -\frac{d^2}{dt^2} \ + \ Q'(t) \tag{142}$$

with a non-smooth $Q(t)$, the recent method by Savchuk and Shkalikov ([35]) seems to be the most ingeneous in that it yields Sturm's oscillation theorem through almost identical argument as in the classical theory ([41], [29]).

Now let

$$E^\omega_1(L) < E^\omega_2(L) < \cdots < E^\omega_j(L) < \cdots$$

be the eigenvalues of H^L_ω. As a special case of an ergodic Schrödinger operator (see Example 11), there exists a non- decreasing function $\mathcal{N}(E)$ such that for almost all $\omega \in \Omega$,

$$\sharp\{j \geq 1;\ E^\omega_j(L) \leq E\} \sim \mathcal{N}(E)L \ , \quad L \to \infty$$

holds for every $E \in \mathbf{R}$. Actually, it is known ([8]) that

$$\mathcal{N}(E) = \left\{ \sqrt{2\pi} \int_0^\infty \exp\left(-\frac{1}{6}u^3 - 2Eu \right) \frac{du}{\sqrt{u}} \right\}^{-1} \ . \tag{143}$$

Hence for almost all $\omega \in \Omega$, the spectrum of H^L_ω is unfoldable with $h = L^{-1}$, $\alpha = 1$, and $\nu(E) = \mathcal{N}(E)$, the latter being continuous in E. The first kind unfolding is carried through by letting

$$x^\omega_j(L) := L\mathcal{N}(E^\omega_j(L)) \ . \tag{144}$$

If we consider the statistics $\pi_k(L; I)$ defined by (80) for this family of sequences $\{x^\omega_j(L)\}_j$, then in view of the results by Molchanov ([27]) and by McKean ([18]) (see the remark below), we would expect that

$$\lim_{L\to\infty} \pi_k(L; (0, c]) = e^{-c}\frac{c^k}{k!}$$

holds almost surely for each $c > 0$, or at least that $\{x^\omega_j(L); j \geq 1\}$ converges in law to the Poisson point process on $(0, \infty)$ with mean density 1. But for the moment, this is only a conjecture.

Remark 9 *McKean actually proved that as $L \to \infty$, the distribution of $x^\omega_1(L) = L\mathcal{N}(E^\omega_1(L))$ converges weakly to the exponential distribution $1_{(0,\infty)}(x)e^{-x}dx$. See [10] for related topics.*

Let $\psi^\omega_E(t)$ be the solution of the equation $H_\omega\psi = E\psi$ satisfying the initial condition $\psi(0) = 0$, $\psi'(0) = 1$. Then by the Sturm's oscillation theorem, $E = E^\omega_j(L)$ if and only if $t = L$ is the j-th zero of the function $\psi^\omega_E(t)$. Moreover $E^\omega_j(L)$ is increasing in $h := L^{-1}$, and hence we can apply the unfolding of the second kind. For this purpose, fix an energy value $E \in \mathbf{R}$ and let $L^\omega_j(E)$ be defined by the equation $E^\omega_j(L) = E$. Then $L^\omega_j(E)$ has the following meaning. If the stochastic processes $\{r_E(t)\}$ and $\{\theta_E(t)\}$ are introduced through the Prüfer's transformation

$$\psi_E(t) = r_E(t)\sin\theta_E(t) \ ; \quad \psi'_E(t) = r_E(t)\cos\theta_E(t) \ , \tag{145}$$

then $\{\theta_E(t)\}_t$ is a diffusion process on the circle $\mathbf{R}/\pi\mathbf{Z}$ ([18], [8]) with $\theta_E(0) = 0 \pmod{\pi}$, and $L_j(E)$ $(j \geq 1)$ are the j-th hitting times to the origin 0 of this process. Namely letting $L_0(E) := 0$, we have

$$L_j(E) := \inf\{t > L_{j-1}(E);\ \theta_E(t) = 0 \pmod{\pi}\} \tag{146}$$

for $j \geq 1$. By the strong Markov property of $\{\theta_E(t)\}_t$, $\tau_j(E) := L_j(E) - L_{j-1}(E)$, $j = 1, 2, \ldots$, are independent and identically distributed random variables, and $L^\omega_j(E) = \sum_{k=1}^j \tau_k(E)$. Moreover one has $\mathbf{E}[\tau_1(E)] = \mathcal{N}(E)^{-1}$ ([8]). Hence the unfolded levels of the second kind, given by

$$x_j = x_j(E) := \mathcal{N}(E)L^\omega_j(E) \ , \quad j = 1, 2, \ldots \tag{147}$$

constitute a renewal process on $(0,\infty)$ with mean density 1, so that the corresponding level statistics is rather trivial. For example, one has

$$\lim_{L\to\infty}\rho(L;c) = \lim_{L\to\infty}\frac{1}{L}\sharp\{j\geq 1;\ x_j\leq L,\ x_{j+1}-x_j>c\} = \mathbf{P}\left(\frac{\tau_1(E)}{\mathbf{E}[\tau_1(E)]}>c\right) \tag{148}$$

with probability one. In this case, the result of unfolding of the second kind does depend on the fixed energy value E, the distribution of $\tau_1(E)/\mathbf{E}[\tau_1(E)]$ being dependent on E. As we let $E\to-\infty$, this distribution tends to the exponential distribution $1_{(0,\infty)}(x)e^{-x}dx$, as was proved by McKean ([18]). Accordingly the renewal process $\{x_j(E)\}_j$ converges weakly to the Poisson point process on $(0,\infty)$ with mean density 1.

6 Clustering and repulsion of levels

As we noted in §5.4.3, no concrete example of regular spectrum is known for which Berry-Tabor conjecture is literally true in the sense that

$$\rho(c) := \lim_{L\to\infty}\rho(L;c) = e^{-c} \qquad c>0 \tag{149}$$

holds for the unfolded spectrum. The assertion (149) on the level spacing distribution function $\rho(c)$ is what Berry and Tabor called the "level clustering". We have also seen in §5.4.2 that there is even an example of quantum Hamiltonian for which level statistics is carried out in full details, yet yielding a level spacing distribution function $\rho(c)$ which is close to, but different from e^{-c}. This situation suggests us that we should give some wider, or looser definition to the term "level clustering" than the strict assertion like (149). To formulate this problem, suppose we are given a sequence $\{x_j(L)\}_j$ depending on a parameter $L>0$ which satisfies Condition 1 with $\lambda=1$, and which we regard as an unfolded quantal spectrum. For this sequence $\{x_j(L)\}_j$, let us consider statistics defined by (80)–(83) with $I=(0,c]$. Except that we always have $\mu_1(c) := \lim_{L\to\infty}\mu_1(L;(0,c]) = c$, none of these is guaranteed to converge to some limit as $L\to\infty$. Let us then define

$$\underline{\pi}_k(c) := \liminf_{L\to\infty}\pi_k(L;(0,c])\ ,\quad \overline{\pi}_k(c) := \limsup_{L\to\infty}\pi_k(L;(0,c])\ ; \tag{150}$$

$$\underline{\mu}_k(c) := \liminf_{L\to\infty}\mu_k(L;(0,c])\ ,\quad \overline{\mu}_k(c) := \limsup_{L\to\infty}\mu_k(L;(0,c])\ ; \tag{151}$$

$$\underline{\rho}(c) := \liminf_{L\to\infty}\rho(L;c)\ ,\quad \overline{\rho}(c) := \limsup_{L\to\infty}\rho(L;c)\ ; \tag{152}$$

and

$$\underline{R}_k(c) := \liminf_{L\to\infty}R_k(L;c)\ ,\quad \overline{R}_k(c) := \limsup_{L\to\infty}R_k(L;c)\ ; \tag{153}$$

When the corresponding upper and lower limits coincide, we shall denote their common value by $\pi_k(c)$, $\mu_k(c)$, $\rho(c)$, or $R_k(c)$ as before.

Now one of the characteristic of the exponential distribution is that its density is strictly positive at the origin $c=0$. Hence if (149) were true, we would have relatively many closely adjacent pairs of levels. This situation would be well described in the following way.

Definition 3 *We shall say that the unfolded spectrum $\{x_j(L)\}_j$ exhibits level clustering [resp. repulsion] in the wide sense if the condition*

$$\liminf_{c\searrow 0}\frac{1-\overline{\rho}(c)}{c}>0 \qquad \left[\textit{resp.}\ \limsup_{c\searrow 0}\frac{1-\underline{\rho}(c)}{c}=0\right] \tag{154}$$

holds.

We can give sufficient conditions for repulsion or clustering in the wide sense in terms of $\mu_k(L;(0,c])$, $k=2,3$, as follows.

Proposition 20 *(i) If $\overline{\mu}_2(c) = o(c^2)$ as $c \searrow 0$, then the spectrum exhibits level repulsion in the wide sense. (ii) If for some $a > 0$, $\underline{\mu}_2(c) \geq ac^2$ for all $c > 0$ and if $\overline{\mu}_3(c) = o(c^2)$ as $c \searrow 0$, then the spectrum exhibits level clustering in the wide sense.*

Proof. Applying Lemma 2.2 of [6] to the probability distribution $\{\pi_k(L;(0,c])\}_{k\geq 0}$, we obtain the following equality which is valid for $n \geq 1$ and $k \geq 0$:

$$P_n(L;c) := \sum_{j\geq n} \pi_j(L;(0,c]) = S_{n,k}(c) + (-1)^{k+1}T \ , \tag{155}$$

where

$$S_{n,k}(c) = \sum_{j=n}^{n+k} (-1)^{j-n} \binom{j-1}{n-1} \frac{1}{j!} \mu_j(L;(0,c]) \ , \tag{156}$$

and $T = T_{n,k} \geq 0$ is given by

$$T_{n,k} = \sum_{j\geq n+k+1} \binom{j-1}{n-1} \binom{j-n-1}{k} P_j(L;c) \ , \tag{157}$$

though this explicit expression for T will not be used in the following argument. When $n = 1$, formula (155) reduces to

$$1 - \pi_0(L;(0,c]) = \sum_{j=1}^{k+1} (-1)^{j-1} \frac{1}{j!} \mu_j(L;(0,c]) + (-1)^{k+1}T \ , \tag{158}$$

and hence

$$\pi_0(L;(0,c]) \leq 1 - \mu_1(L;(0,c]) + \frac{1}{2!}\mu_2(L;(0,c]) + \cdots + (-1)^k \frac{1}{k!}\mu_k(L;(0,c]) \tag{159}$$

when k is even, and

$$\pi_0(L;(0,c]) \geq 1 - \mu_1(L;(0,c]) + \frac{1}{2!}\mu_2(L;(0,c]) + \cdots + (-1)^k \frac{1}{k!}\mu_k(L;(0,c]) \tag{160}$$

when k is odd. (These are special cases of Theorem 2.3 of [6].)

Given $L > 0$, define N by $x_N(L) \leq L < x_{N+1}(L)$, namely by

$$N := n(L) = \sharp\{j \geq 1; x_j(L) \leq L\} \ , \tag{161}$$

and put $x_0(L) = 0$. Then we have

$$\begin{aligned}
L\pi_0(L;(0,c]) &= \sum_{j=0}^{N-1} (x_{j+1}(L) - x_j(L) - c)_+ + \{(x_{N+1}(L) - x_N(L) - c)_+ \wedge (L - x_N(L))\} \\
&= \sum_{j=0}^{N-1} (x_{j+1}(L) - x_j(L) - c) + \sum_{j=0}^{N-1} (c - (x_{j+1}(L) - x_j(L)))_+ \\
&\quad + \{(x_{N+1}(L) - x_N(L) - c)_+ \wedge (L - x_N(L))\} \\
&\leq x_N(L) - cN + c \cdot \sharp\{j \geq 0;\ x_{j+1}(L) \leq L,\ x_{j+1}(L) - x_j(L) \leq c\} \\
&\quad + (L - x_N(L)) \\
&= L - c \cdot \sharp\{j \geq 0;\ x_{j+1}(L) \leq L,\ x_{j+1}(L) - x_j(L) > c\} \ .
\end{aligned} \tag{162}$$

Suppose that the condition of (ii) is satisfied. Then dividing bothe sides of (162) by L and letting $L \to \infty$, we obtain from (160)

$$\begin{aligned}
1 - c\overline{\rho}(c) &\geq \underline{\pi}_0(c) \geq 1 - \mu_1(c) + \frac{1}{2}\underline{\mu}_2(c) - \frac{1}{3!}\overline{\mu}_3(c) \\
&\geq 1 - c + \frac{a}{2}c^2 + o(c^2) \ ,
\end{aligned} \tag{163}$$

or

$$c(1-\overline{\rho}(c)) \geq \frac{ac^2}{2} + o(c^2) \ .$$

Hence we get

$$\liminf_{c\searrow 0} \frac{1-\overline{\rho}(c)}{c} \geq \frac{a}{2} > 0 \ . \tag{164}$$

Next suppose that the condition of (i) is satisfied. For each $0<\delta<1$, we have

$$\begin{aligned} L\pi_0(L;(0,c]) \ &\geq \ x_N(L) - cN + (L - x_N(L) - c) \\ &\quad + \sum_{j=0}^{N-1} 1_{(0,\delta c]}(x_{j+1}(L) - x_j(L))\{c - (x_{j+1}(L) - x_j(L))\} \\ &\geq \ (L-c) - cN \\ &+ \ (1-\delta)c\,[N - \sharp\{j\geq 0;\ x_{j+1}(L)\leq L,\ x_{j+1}(L) - x_j(L) > \delta c\}] \ . \end{aligned} \tag{165}$$

Again dividing by L and noting $N/L\to 1$ as $L\to\infty$, we obtain from (159)

$$\begin{aligned} 1 - c + (1-\delta)c[1-\underline{\rho}(\delta c)] \ &\leq \ \overline{\pi}_0(c) \\ &\leq \ 1 - \mu_1(c) + \frac{1}{2}\overline{\mu}_2(c) \\ &= \ 1 - c + o(c^2) \ , \end{aligned} \tag{166}$$

and hence

$$\limsup_{c\searrow 0} \frac{1-\underline{\rho}(c)}{c} = \limsup_{c\searrow 0} \frac{1-\underline{\rho}(\delta c)}{\delta c} = 0 \ , \tag{167}$$

as desired.

Corollary 1 *(i) If $\overline{R}_2(c) = o(c)$ as $c\searrow 0$, then the spectrum exhibits level repulsion in the wide sense.*
(ii) If for some $a>0$, $\underline{R}_2(c)\geq ac$ for all $c>0$ and if $\overline{R}_3(c) = o(c)$ as $c\searrow 0$, then the spectrum exhibits level clustering in the wide sense.

Proof. By the inequality (92) in the proof of Proposition 18, we see that

$$kc\frac{1}{\ell}\sum_{\ell=0}^{m} \underline{R}_k\left(c\left(1-\frac{m}{\ell}\right)\right) \leq \underline{\mu}_k(c) \leq \overline{\mu}_k(c) \leq kc\frac{1}{\ell}\sum_{\ell=0}^{m} \overline{R}_k\left(c\left(1-\frac{m}{\ell}\right)\right) \tag{168}$$

holds for all integer $\ell > 1$. Letting $\ell\to\infty$, we get

$$k\int_0^c \underline{R}_k(c')dc \leq \underline{\mu}_k(c) \leq \overline{\mu}_k(c) \leq k\int_0^c \overline{R}_k(c')dc' \ . \tag{169}$$

It is now obvious that the conditions of the corollary imply those of Proposition 20.

If we impose conditions on $R_k(c)$, we get more precise estimates on $\overline{\rho}(c)$ and $\underline{\rho}(c)$.

Proposition 21 *If the limits $R_j(c)$ exist for $j=2,\dots,k$, $k\geq 2$, then we have*

$$\overline{\rho}(c) \leq 1 + \sum_{j=1}^{k} (-1)^j \frac{1}{j!} R_{j+1}(c) \tag{170}$$

when k is even, and

$$\underline{\rho}(c) \geq 1 + \sum_{j=1}^{k} (-1)^j \frac{1}{j!} R_{j+1}(c) \tag{171}$$

when k is odd.

Proof. Let $n(L) := \sum_j 1_{(0,L]}(x_j(L))$ and define the probability distribution $\{\rho_k(L;c)\}_{k\geq 0}$ by

$$\rho_k(L;c) := \frac{1}{n(L)} \sum_j 1_{(0,L]}(x_j(L)) 1_{\{N_L(x_j(L),x_j(L)+c]=k\}} \ . \tag{172}$$

Then

$$\rho_0(L;c) = \frac{n(L)}{L} \rho(L;c) \ . \tag{173}$$

Note that $n(L) \sim L$, and hence the upper and lower limits of $\rho_0(L;c)$ as $L \to \infty$ coincide with $\overline{\rho}(c)$ and $\underline{\rho}(c)$ respectively. If we let

$$r_k(L;c) := \frac{L}{n(L)} R_k(L;c) \tag{174}$$

for $k \geq 2$, then it is not difficult to see

$$r_k(L;c) = \frac{1}{n(L)} \sum_j 1_{(0,L]}(x_j(L)) \left(N_L(x_j(L), x_j(L)+c]\right)^{[k-1]} = \sum_{i \geq k-1} i^{[k-1]} \rho_i(L;c) \ . \tag{175}$$

Namely $r_k(L;c)$ is the $(k-1)$-th factorial moment of the probability distribution $\{\rho_i(L;c)\}_{i\geq 0}$. Thus again by Lemma 2.2 of [6], we can write

$$Q_n(L;c) := \sum_{j \geq n} \rho_i(L;c) = \sigma_{n,k}(c) + (-1)^{k+1}\tau \ , \tag{176}$$

where $\tau \geq 0$ and

$$\sigma_{n,k}(c) = \sum_{j=n}^{n+k} (-1)^{j-n} \binom{j-1}{n-1} \frac{1}{j!} r_{j+1}(L;c) \ . \tag{177}$$

When $n = 1$, this reduces to

$$1 - \rho_0(L;c) = \sum_{j=1}^{k} (-1)^{j-1} \frac{1}{j!} r_{j+1}(L;c) + (-1)^k \tau \ , \tag{178}$$

which is valid for $k \geq 1$. The assertion of the proposition immediately follows from this equality.

Example 12 *In [39], VanderKam considered the level statistics for the Laplacian on a flat d-dimensional torus, which is the quantization of the uniform motion on that torus. The corresponding Hamiltonian, expressed in the action variables, is a positive definite quadratic form*

$$P(x_1, \ldots, x_d) = {}^t\mathbf{x}A\mathbf{x} \ , \tag{179}$$

where A is a $d \times d$ real symmetric, positive definite matrix, and the eigenvalues of the Laplacian are values taken by P on the lattice $\hbar\mathbf{Z}^d$. To eliminate the obvious degeneracy, $\mathbf{x}$ and $-\mathbf{x}$ are identified. Let us denote this equivalence relation by $\sim$. Then the spectrum $\{P(\hbar\mathbf{n}); \mathbf{n} \in \mathbf{Z}^d/\sim\}$ is unfoldable with $\alpha = d$ and $\nu(E) = c_P E^{d/2}$, where c_P is the volume of the domain $\{\mathbf{x} \in \mathbf{R}^d; P(\mathbf{x}) \leq 1\}/\sim$. Since $P(\mathbf{x})$ is a homogeneous function of degree 2, both kinds of unfoldings discussed in §5 give the same unfolded sequence $\{\xi_\mathbf{n}; \mathbf{n} \in \mathbf{Z}^d\}$, where

$$\xi_\mathbf{n} := c_P P(\mathbf{n})^{d/2} \ . \tag{180}$$

Then VanderKam proves that for almost all realization of the coefficient matrix A of $P(\mathbf{x})$, one has $R_k(c) = c^{k-1}$ for $k = 2, \ldots, [d/2]$, for the correlation functions of the unfolded spectrum (180). Hence if $d \geq 6$, Corollary 1 tells us that $\{\xi_\mathbf{n}\}$ exhibits level clustering in the wide sense. Moreover, by Proposition 20, we can assert that when d is large and $c > 0$ is small, $\overline{\rho}(c)$ and $\underline{\rho}(c)$ are very close to each other, and are approximated by e^{-c}.

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