RENORMALIZATION AND CENTRAL LIMIT THEOREM FOR CRITICAL DYNAMICAL SYSTEMS WITH WEAK EXTERNAL NOISE

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Abstract. We study of the effect of weak noise on critical one dimensional maps; that is, maps with a renormalization theory.

We establish a one dimensional central limit theorem for weak noises and obtain Berry–Esseen estimates for the rate of this convergence.

We analyze in detail maps at the accumulation of period doubling and critical circle maps with golden mean rotation number. Using renormalization group methods, we derive scaling relations for several features of the effective noise after long times. We use these scaling relations to show that the central limit theorem for weak noise holds in both examples.

We note that, for the results presented here, it is essential that the maps have parabolic behavior. They are false for hyperbolic orbits.

To the memory of D. Khmelev

1. Introduction

The goal of this paper is to develop a rigorous renormalization theory for weak noise superimposed to one dimensional systems whose orbits have some self-similar structure.

Some examples we consider in detail are period doubling and critical circle maps with golden mean rotation number.

To be more precise, we consider systems of the form

$$x_{n+1} = f(x_n) + \sigma\xi_{n+1} \tag{1.1}$$

where f is a map of a one dimensional space ($\mathbb{R}$, $\mathbb{T}^1$ or $I = [-1, 1]$) into itself, (ξ_n) is a sequence of real valued independent mean zero random variables of comparable sizes, and $\sigma > 0$ is a small parameter –called noise level– that controls the size of the noise.

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We will study the scaling limit of the effective noise of (1.1) for small noise level, and large number of iterations of the system. For f either a map at the accumulation of period doubling or a critical circle with golden mean rotation and the noises satisfying some mild conditions (existence of moments and the like), we will show that the scaling limits of the noise resemble a Gaussian in an appropriate sense. That is, if we fix a small value of the noise level σ and then, look at the effective noise after a long time, the distribution of effective noise, normalized to have variance one, will have a distribution close to a Gaussian. See Theorems 2.1 and 2.2 for precise statements. We will also show that statistical properties of the noise, namely Wick-ordered moments or *cumulants* satisfy some scaling properties. Observe that Theorems 2.1 and 2.2 are similar to the classical central limit and the Berry-Esseen theorems, even if the powers which appear as normalization factors are different from those in the classical theorems.

The papers of [CNR81, SWM81] considered heuristically a renormalization theory for weak Gaussian noise perturbing one dimensional maps at the accumulation of period doubling. The main result in those papers was that after appropriately rescaling space and time, the effective noise of this renormalized system satisfies some scaling relations. The paper [VSK84] developed a rigorous thermodynamic formalism for critical maps with period doubling. Among many other results, [VSK84], study the effect of noise on the ergodic theory of these maps and showed that for systems at the accumulation of period doubling with weak noise, there is a stationary measure depending on the magnitude of noise that converges to the invariant measure in the attractor. A very different rigorous renormalization theory for systems with noise is obtained in [CL89].

The results we present here can be considered as a rigorous version of the theory of [CNR81, SWM81]. The theory developed here also applies to noise of arbitrary shape and shows that the scaling limit is Gaussian. The main idea is that one can also renormalize other statistical properties of the map (cumulants) and, by analyzing the different rates of convergence of all these different renormalization operators (these are what we call the convexity properties of the spectral radii (See Theorem 4.5), we obtain that the effective noise is approximately Gaussian. The argument we present uses relatively little properties of the renormalization operator. Basically we just need that there is some convergence to a scaling limit. We also apply similar ideas to the case of circle maps. The results there are very similar.

Besides the results for the renormalization, we show that the previous results about the renormalization group give information about the behavior of the noise along a whole orbit.

There is some overlap between the results in this paper and some of the results in [VSK84]. The main emphasis on the paper [VSK84] is on statistical properties for a fixed level of noise. In this paper, we emphasize the behavior on single orbits for longer times but with weaker noises. The paper [VSK84] uses mainly the thermodynamic formalism and this paper relies mainly on transfer operators. Nevertheless, there are some relations between the two points of view. Of course, the relation between thermodynamics and transfer operators goes back to the beginning of thermodynamic formalism. See [May80]. In [VSK84, p. 31] the authors introduce one of the cumulant operators we use and find a relation between its spectral radius and thermodynamic properties. We think it would be possible and interesting to develop thermodynamic formalism analogues of the convexity properties of transfer operators obtained in Theorem 4.5. It would also be interesting to develop thermodynamic formalism analogues of the arguments developed in Sections 4.6, 5.6 which allow to study the behavior along a whole orbit from the study of renormalization operators.

This paper will be organized as follows. In Section 2 we state a general central limit theorem (Theorems 2.1) and a result on Berry–Esseen estimates (Theorem 2.2)for the convergence in the central limit theorem for one dimensional dynamical systems with weak noise. The main hypothesis of these results is that some combinations of derivatives grow at certain rate (see (2.9)). This condition is reminiscent of the classical Lindeberg–Lyapunov central limit theorem sums of independent random variables.

One important class of systems that satisfy the condition (2.9) of Theorem 2.1 is that of fixed points of renormalization operators. Specific results for systems at the accumulation of period doubling (Theorem 2.3) and for critical circle maps with golden mean rotation number(Theorem 2.3) will be stated in sections 2.2 and 2.3 respectively.

The rest of the paper is devoted to providing proofs of the results above. In Section 3, Theorems 2.1 and 2.2 are proved in detail. The method of the proof is to use the Lindeberg–Lyapunov central limit theorem for a linearized approximation of the effective noise, and then to control the error terms. Some examples will also be discussed in Section 3.6.

In Section 4 we study in detail unimodal maps at the accumulation of period–doubling [Fei77, CE80, CEL80], and prove Theorem 2.3. In section 5 we study critical maps of the circle with golden mean rotation number and prove Theorem 2.4.

The techniques used in Sections 4 and 5 consists on introducing some auxiliary linear operators – which we called Lindeberg–Lyapunov operators (see Sections 4.2.1 and 5.3) – that describe the statistical properties of the renormalized noise. The crucial part of the argument is to show that the spectral radii of these operators satisfy some convexity properties (Theorem 4.5), so that in the scaling limit, the properties of the noise are that of a Gaussian. This implies that the sufficient conditions of Theorem 2.1 hold for a sequence of times. A separate argument developed in Section 4.6 shows that, from the knowledge at these exponentially separated times, one can obtain information of the orbit starting at zero for all times. From that, one can obtain information for orbits starting at any iterate of the critical point.

2. Statement of results

Throughout this paper, we will make the following assumptions

A1. $f : M \mapsto M$ is a C^2 map where $M = \mathbb{R}$, $I \equiv [-1, 1]$ or $\mathbb{T}^1$.

A2. If $M = I$, we will further assume that there is a number $a > 0$ such that $f \in C^2([-1 - a, 1 + a])$ and $f([-1 - a, 1 + a]) \subset I$.

For any function $f \in C(M)$, we will denote by $\|f\|_{C_0} = \sup_{x\in M} |f(x)|$.

Let (ξ_n) be a sequence of independent random variables (defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$), with $p > 2$ finite moments. We will assume that

A3(p). $\mathbb{E}[\xi_n] = 0$ for all n, and that

$$c \leq \|\xi_n\|_2 \leq \|\xi_n\|_p \leq C$$

for some constants c, C. Here $\|\xi_n\|_s = (\mathbb{E}[|\xi_n|^s])^{1/s}$, for any $s > 0$.

A direct consequence of A3(p) and independence is that

$$c \leq \left\| \max_{1\leq j\leq n} |\xi_j| \right\|_p \leq n^{1/p}\, C$$

This inequality will be useful to give sufficient conditions on the noise level to obtain a central limit theorem (see Theorem 2.1 below).

2.1. General results for one–dimensional dynamical systems with external weak noise.

Let $x \in M$ and $\sigma > 0$ be fixed. We consider the system

$$x_m = f(x_{m-1}) + \sigma\xi_m, \quad x_0 = x \tag{2.1}$$

σ will be referred to as the *"noise level"*.

For $M = \mathbb{R}$ or $M = \mathbb{T}^1$, define the process $x_n(x, \sigma)$ as the value at time n of (2.1).

For $M = I$, we define the process $x_n(x, \sigma)$ by (2.1) provided that

$$\{f(x_{j-1}) + \sigma\xi_j\}_{j=1}^n \subset [-1 - a, 1 + a]$$

Observe that

$$\bigcup_{j=1}^n \{|f(x_{j-1}) + \sigma\xi_j| > 1 + a\} \subset \left\{\sigma \max_{1\le j\le n} |\xi_j| > a\right\} := C_n,$$

hence, by Chebyshev's inequality we get

$$\mathbb{P}[C_n] \le \sigma\frac{\mathbb{E}[\max_{1\le j\le n} |\xi_j|]}{a}$$

Hence, for a small value of σ, the event C_n occurs with low probability. Therefore, in order to define $x_n(x, \sigma)$, it suffices to condition on the event

$$\Omega \setminus C_n = \left\{\sigma \max_{1\le j\le n} |\xi_j| \le a\right\}$$

In the scaling limit, we will consider a sequence of noise levels $\{\sigma_n\}$ converging to 0 so that $\Omega \setminus C_n$ is close to the whole space Ω. Notice that if $(\xi_j)_{j\in\mathbb{N}}$ is supported on a compact interval, then by taking σ_n small enough, the events C_n will be empty sets.

2.1.1. *General central limit theorem for one dimensional dynamical systems with random weak noise.* Our first result is a general central limit theorem for one–dimensional dynamical systems with weak random noise (Theorem 2.1). This result is based upon the classical Lindeberg–Lyapunov central limit theorem for sums of random variables [Bil68, p. 44]. This is reflected in the sufficient condition (2.9) in the statement of Theorem 2.1.

We introduce the following notation

Definition 2.1. Let f be a map on M satisfying A1, A2, and let (ξ_n) be a sequence of random variables with $p > 2$ finite moments, that satisfies A3(p).

1. The Lyapunov functions Λ_s ($s \ge 0$) and $\widehat{\Lambda}(x, n)$ are defined by

$$\Lambda_s(x, n) = \sum_{j=1}^n \left|\left(f^{n-j}\right)' \circ f^j(x)\right|^s \tag{2.2}$$

$$\widehat{\Lambda}(x, n) = \max_{0\le i\le n} \sum_{j=0}^i \left|\left(f^{i-j}\right)' \circ f^j(x)\right| \tag{2.3}$$

When needed, we will use the notation $\widehat{\Lambda}^f$ and Λ_s^f to emphasize the dependence on f.

2. Let $x \in M$ and $\sigma > 0$ be fixed. The linearized effective noise is defined as

$$L_n(x) = \sum_{j=1}^{n} \left(f^{n-j}\right)' \circ f^j(x)\xi_j \tag{2.4}$$

Remark 2.1. It is very important to observe that for each $s \geq 0$, the Lyapunov function $\Lambda_s(x, n)$ satisfies

$$\Lambda_s(x, n+m) = |(f^m)' \circ f^n(x)|^s \Lambda_s(x, n) + \Lambda_s(f^n(x), m) \tag{2.5}$$

We will use (2.5) in the study of central limit theorems for systems near the accumulation of period doubling, Section 4, and for critical maps of the circle, Section 5.

Remark 2.2. The Lyapunov functions (2.2) are used to estimate the sums of the moments or order s of the terms in (2.4). In particular, notice that

$$\text{var}[L_n(x)] = \sum_{j=1}^{n} \left(\left(f^{n-j}\right)' \circ f^j(x)\right)^2 \mathbb{E}[\xi_j^2] \tag{2.6}$$

The assumption A3(p) and Hölder's inequality imply that there are constants c, C such that for any $0 < s \leq p$,

$$c\Lambda_s(x, n) \leq \sum_{j=1}^{n} \left|\left(f^{n-j}\right)' \circ f^j(x)\right|^s \mathbb{E}[|\xi_j|^s] \leq C\Lambda_s(x, n) \tag{2.7}$$

In particular, if A3($p = 2$) implies

$$c\Lambda_2(x, n) \leq \text{var}[L_n(x)] \leq C\Lambda_2(x, n) \tag{2.8}$$

The main result that we obtain for orbits of a dynamical system is that if a deterministic condition on the orbit (expressed in terms of Lyapunov functions) holds, then, the noise perturbing this orbit satisfies a central limit theorem.

Theorem 2.1. *Let f, M be a function satisfying A1 and A2, and let (ξ_n) be a sequence of independent random variables, with $p > 2$ finite moments, that satisfies A3(p). Suppose that for some $x \in M$ there is an increasing sequence of positive integers n_k such that*

$$\lim_{k \to \infty} \frac{\Lambda_p(x, n_k)}{\left(\Lambda_2(x, n_k)\right)^{p/2}} = 0 \tag{2.9}$$

Let $(\sigma_k)_k$ be a sequence of positive numbers. Assume furthermore either of the two conditions

- *[H1]*
  The noise satisfies A3(p) with $p > 2$.
  And the sequence (σ_k) satisfies:

$$\lim_{k\to\infty} \frac{\|f''\|_{C_0} \| \max_{1\le j\le n_k} |\xi_j| \|_p^2 (\widehat{\Lambda}(x,n_k))^6 \sigma_k}{\sqrt{\Lambda_2(x,n_k)}} = 0 \tag{2.10}$$

- *[H2]*
  The noise satisfies A3(p) with $p \ge 4$.
  And the sequence (σ_k) satisfies:

$$\lim_{k\to\infty} \frac{\|f''\|_{C_0} \| \max_{1\le j\le n_k} |\xi_j| \|_p^2 (\widehat{\Lambda}(x,n_k))^3 \sigma_k}{\sqrt{\Lambda_2(x,n_k)}} = 0 \tag{2.11}$$

Then, there exists a sequence of events $B_k \in \mathcal{F}$ such that
M1. $\lim_{k\to\infty} \mathbb{P}[B_k] = 1$
M2. *The processes defined by*

$$w_{n_k}(x,\sigma_k) = \frac{(x_{n_k}(x,\sigma_k) - f^{n_k}(x))}{\sigma_k \sqrt{\mathrm{var}[L_{n_k}(x)]}} \tag{2.12}$$

$$\widetilde{w}_{n_k}(x,\sigma_k) = \frac{(x_{n_k}(x,\sigma_k) - f^{n_k}(x))\mathbf{1}_{B_k}}{\sqrt{\mathrm{var}[(x_{n_k}(x,\sigma_k) - f^{n_k}(x))\mathbf{1}_{B_k}]}} \tag{2.13}$$

converge in distribution to a standard Gaussian as $k \to \infty$.

Furthermore, if the sequence ξ_n is supported on a compact set then, we can choose $B_k = \Omega$ for all k.

The sets $\Omega \setminus B_k$, which we call *outliers*, are events where large fluctuations of noise occur. We will refer to

$$(x_{n_k}(x,\sigma_k) - f^{n_k}(x))\mathbf{1}_{B_k} \tag{2.14}$$

as the *effective noise.*

Condition (2.9) in Theorem 2.1 is closely related to the Lyapunov condition of the classical limit theorem for sums of independent random variables applied to the linearized effective noise $L_{n_k}(x)$ defined by (2.4).

We will show in Section 3.3 that if the sequence of noise levels (σ_k) satisfy (2.10) then, the events where large fluctuations of noise occur (outliers) have low probability. As a consequence, we will have that the linearized effective noise (2.4) is, with large probability, a very good approximation to the effective noise defined by (2.14).

Remark 2.3. Note that there are two variants of the results (2.12) and (2.13) in Theorem 2.1 as well as two variants on the hypothesis.

The difference in the conclusions is that the effective noise is normalized in two – in principle different – ways. The version (2.12) normalizes the effective noise by its variance and the version normalizes by the variance of the linear approximation (see Remark 2.2.

It could, in principle happen that the difference between the linear approximation and true process converged to zero in probability but that had a significance contribution to the variance. This pathology can be excluded by assuming that the noise has sufficient moments and that the noise is weak enough. These two hypothesis can be traded off. In the first version of the hypothesis [H1], we use only $p > 2$ moments and a somewhat stronger smallness conditions in the size of noise (2.10). In the second version of the hypothesis [H2], we assume $p \geq 4$ moments, but the smallness conditions in the noise are weaker.

The subsequent Theorems 2.2 will be true under either of the hypothesis.

Remark 2.4. As we will see later, in the proof, the conditions on σ_k are just upper bounds. If we consider two sequences $\tilde{\sigma}_k \leq \sigma_k$ and σ_k satisfies either of (2.10) or (2.11), then $\tilde{\sigma}_k$ satisfies the same conditions.

Furthermore, the bounds that we obtain for the proximity $w_{n_k}(x, \tilde{\sigma}_k)$ to the standard Gaussian are smaller than the bounds that we obtain on the proximity of $w_{n_k}(x, \sigma_k)$ to the standard Gaussian.

2.1.2. *Berry–Esseen estimates.* The relevance of the outliers will become more evident in our second result, Theorem 2.2 below, which provides the rate of convergence to Gaussian in Theorem 2.1.

We will use $\Phi(z)$ to denote the distribution function of the standard Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, that is

$$\Phi(z) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

Theorem 2.2. *Let f, $(\xi_n)_n$ be as in Theorem 2.1 and let $s = \min(p, 3)$. Assume that condition* (2.9) *holds at some $x \in M$ If σ_k is a sequence of positive numbers such that*

$$\frac{(\widehat{\Lambda}(x, n_k))^3}{\sqrt{\mathrm{var}[L_{n_k}(x)]}} \|f''\|_{C_0} \left\| \max_{1 \leq j \leq n_k} |\xi_j| \right\|_s^2 \sigma_k \leq \left(\frac{\Lambda_s(x, n_k)}{\left(\Lambda_2(x, n_k)\right)^{s/2}} \right)^2 \quad (2.15)$$

then, we have that

$$\sup_{z \in \mathbb{R}} |\mathbb{P}[w_{n_k}(x) \mathbf{1}_{B_k} \leq z] - \Phi(z)| \leq A \frac{\Lambda_s(x, n_k)}{(\Lambda_2(x, n_k))^{s/2}} \quad (2.16)$$

where $A > 0$ depends only on x.

Remark 2.5. When $M = \mathbb{R}$, or $\mathbb{T}^1$ and $f(x) = x + c$ for some constant c, Theorems 2.1 and 2.2 coincide with the classical central limit and Berry–Esseen theorems for sums of independent random variables. Since $f'' \equiv 0$, the result holds regardless of what values σ_k takes, and the outliers are empty sets.

Remark 2.6. In section 3 we will consider the map $f(x) = 2x$. This system does not satisfy (2.9), and indeed the conclusion of Theorem 2.1 fails. Systems with enough hyperbolicity satisfy other types of central limit theorems for weak noise [GK97], or even in the absence noise [Liv96], [FMNT05]. Those results are very different from the ones we consider in this paper.

2.1.3. *Sketch of the proof of Theorems 2.1 and 2.2.* The proof of these results is obtained in Section 3 by showing that:

1) The linear approximation to the process satisfies a central limit theorem (or a Berry-Esseen theorem)
2) Under smallness conditions on the noise level (see (2.10)), the linear approximation is much larger than the Taylor reminder, so that we can transfer the Gaussian behavior from one to the other.

The main source of difficulties in the proof are situations when the noise is much larger than expected from the statistical properties of the linear approximation (outliers). These events, of course have small probability and, therefore, do not affect the convergence in probability. However, it could happen in principle that they change the variance. We will see in Section 3.3.3 that the variance of the effective noise and that of its linear approximation are asymptotically equal. As a consequence, we will have the effective noise normalized by its variance converges in distribution to the standard Gaussian.

The existence of moments of high order will provide an improvement on the choice of the noise level. In particular, we will see that if the noise has compact support, large fluctuations of the effective noise never occur; that is, the outliers are empty sets.

The procedure of cutting off outliers in Theorem 2.1 is similar to the process of elimination of *"Large fields"* that occurs in the rigorous study of renormalization group in [GK85, GKK87].

In this paper, we consider two examples of maps that have a renormalization theory, and for which Theorems 2.1 and 2.2 apply. Namely, systems at the accumulation of period doubling and critical circle maps with golden mean rotation number.

2.2. **Results for systems at the accumulation of period doubling.** In section 4.1 we consider systems of the form (1.1) where f is a $2k$–order analytic unimodal map of the interval I onto itself. That is $f(0) = 1$, $f^{(j)}(0) = 0$, for $1 \le 2k-1$, $f^{(2k)}(0) \neq 0$, and $xf'(x) < 0$ for $x \neq 0$. Here, $f^{(j)}$ denotes the j–th order derivative of f.

The period doubling renormalization group operator T acting on the space of unimodal maps is defined by

$$Tf(x) = f^2(\lambda_f x)/\lambda_f$$

where $\lambda_f = f(1)$. We refer to Section (4.1) for a precise definition of unimodal maps and the period-doubling renormalization operator.

For each k, there is a set of analytic functions $\mathcal{W}_s(g_k)$, such for maps $f \in \mathcal{W}_s(g_k)$, we have that $T^n f$ converges to a universal function g_k which is a fixed point of T (see [Fei77, Lan82, TC78, Eps86, Sul92, Mar98, dMvS93, JŚ02]).

We will show in Section 4 that

Theorem 2.3. *Let $f \in \mathcal{W}_s(g_k)$ then,*

F1. *For any $x = f^l(0)$, $l \in \mathbb{N}$ and any $p > 2$,*

$$\lim_{n\to\infty} \frac{\Lambda_p(x,n)}{\{\Lambda_2(x,n)\}^{p/2}} = 0$$

Let (ξ_n) be a sequence of independent random variables which have $p > 2$ finite moments, and that satisfies A3(p).

F2. *Let $w_n(x,\sigma)$ be*

$$w_n(x,\sigma) = \frac{(x_n(x,\sigma_n) - f^n(x))}{\sigma\sqrt{\sum_{j=1}^n \left((f^{n-j})' \circ f^j(x)\right)^2 \mathbb{E}[\xi_j^2]}}$$

Then, there is a constant $\gamma > 0$, such that if

$$\lim_n \sigma_n n^{\gamma+1} = 0$$

then, for each $x = f^l(0)$, $l \in \mathbb{N}$ there are events $\{B_n(x)\}_n \subset \mathcal{F}$ with

$$\lim_n \mathbb{P}[B_n(x)] = 1$$

such that $w_n(x,\sigma_n)\mathbf{1}_{B_n(x)}$ and hence $w_n(x,\sigma_n)$ converge in distribution to the standard Gaussian.

F3. *Furthermore, there are constants $\alpha > 0$ and $\theta > 0$ depending on p such that if $\sigma_n \le n^{-\theta}$, then*

$$\sup_{z\in\mathbb{R}} |\mathbb{P}[w_n(x,\sigma_n)\mathbf{1}_{B_n(x)} \le z] - \Phi(z)| \le C_x n^{-\alpha} \tag{2.17}$$

Some explicit values for γ, α, and θ that follow from the arguments of the proof are given in Section 4.6.

In [DEdlL06], empirical values for γ are obtained for the quadratic Feigenbaum fixed point. The values obtained there suggest that the sequence of level of noise σ_n decays as a power γ_* of the number of iterations, that is not very different from the one we obtain in Section 4.6.

2.3. **Results for critical circle maps.** In Section 5 we study critical maps of the circle with golden mean rotation number [Lan84, Eps89, dFdM99]. That is, we consider strictly increasing analytic maps f such that $f(x+1) = f(x)+1$, $f^{(j)}(0) = 0$ for $j = 1\ldots, 2k$, $f^{(2k+1)}(x) \neq 0$, and $\lim_n (f^n(x) - x)/n = (\sqrt{5}-1)/2 \equiv \beta$.

2.3.1. *Central limit theorem for Fibonacci times.* Recall that the sequence of Fibonacci numbers (Q_n), is defined by $Q_0 = 1 = Q_1$, $Q_{n+1} = Q_n + Q_{n-1}$. Any integer n admits a unique Fibonacci decomposition

$$n = Q_{m_0} + \ldots + Q_{m_{r_n}}$$

where $m_0 > \ldots > m_{r_n} > 0$ are non-consecutive integers, (i. e. $m_i \geq m_{i-1} + 2$). Notice that $r_n + 1$ is the number of terms in the Fibonacci representation of n, and that $r_n \leq m_0 \leq [\log_{\beta^{-1}} n]+$ (Here [ ] stands for the integer part function).

Theorem 2.4. *Let f be a critical circle map. If $\{n_k\}$ is a increasing sequence of integers such that*

$$\lim_{k\to\infty} \frac{r_{n_k}}{\log_{\beta^{-1}} n_k} = 0 \tag{2.18}$$

then,

C1. *For all $x = f^l(0)$, $l \in \mathbb{N}$, and any $p > 2$,*

$$\lim_{k\to\infty} \frac{\Lambda_p(x, n_k)}{\{\Lambda_2(x, n_k)\}^{p/2}} = 0$$

Let (ξ_n) be a sequence of random independent variables that have $p > 2$ moments, and that satisfies A3(p).

C2. *Let $w_n(x, \sigma)$ be the process defined by*

$$w_n(x, n) = \frac{(x_n(x, \sigma) - f^n(x))}{\sigma\sqrt{\mathrm{var}[L_n(x)]}}$$

Then, there is a constant $\gamma > 0$ such that if

$$\lim_k \sigma_k n_k^{\gamma+1} = 0$$

there are events $\{B_k(x)\} \subset \mathcal{F}$ *with*

$$\lim_k \mathbb{P}[B_k(x)] = 1$$

such that $w_{n_k}(x, \sigma_k)\mathbf{1}_{B_k(x)}$ *and hence* $w_{n_k}(x, \sigma_k)$ *converge in distribution to the standard Gaussian.*

C3. *Furthermore, there are constants* $\tau > 0$ *and* $\upsilon > 0$ *depending on* p *such that if* $\sigma_k \leq {n_k}^{-\tau}$, *then*

$$\sup_{z\in\mathbb{R}} |\mathbb{P}[w_{n_k}(x, \sigma_k)\mathbf{1}_{B_k(x)} \leq z] - \Phi(z)| \leq D_x\, n_k^{-\upsilon}$$

2.3.2. *Central limit theorem along the whole sequence of times.* In Section 5.6.3 it is shown that the Lyapunov condition (2.9), for orbits starting in $\{f^l(0) : l \in \mathbb{N}\}$, holds along the whole sequence of times provided that some numerical condition (5.44) is satisfied (see Proposition 5.6 and Theorem 5.7). The technical condition (5.44) is some relation between properties of fixed points of a renormalization operators and the spectral radii of some auxiliary operators. It could be verified by some finite computation or implied by some monotonicity properties of the fixed point theorem,

Under the technical condition (5.44), the central limit theorem holds along the whole sequence of times. The proof presented gives results that do not depend on the condition (5.44). Namely, we show that there as a central limit theorem along sequences of numbers which can be expressed as sum of sufficiently "few" Fibonacci numbers in terms of their size. The hypothesis (5.44) implies that all the numbers satisfy this property.

Theorem 2.5. *Let* f *be a critical circle map. Under the numerical condition* (5.44) *(see Section 5.6.3)*

C4. *If* $x = f^l(0)$, $l \in \mathbb{N}$ *and* $p > 2$, *then*

$$\lim_{n\to\infty} \frac{\Lambda_p(x, n)}{\{\Lambda_2(x, n)\}^{p/2}} = 0$$

Let (ξ_n) *be a sequence of random variables which have* $p > 2$ *moments, and that satisfies A3. Then,*

C5. *There is a constant* $\delta > 0$ *such that if*

$$\lim_k \sigma_n n^{\gamma+1} = 0$$

there are events $\{B_n(x)\} \subset \mathcal{F}$ *with*

$$\lim_n \mathbb{P}[B_n(x)] = 1$$

such that $w_n(x, \sigma_k)\mathbf{1}_{B_n(x)}$ *and hence* $w_n(x, \sigma_n)$ *converge in distribution to the standard Gaussian.*

C3. *Furthermore, there are constants $\tau > 0$ and $\upsilon > 0$ depending on p such that if $\sigma_n \le n^{-\tau}$, then*

$$\sup_{z\in\mathbb{R}} |\mathbb{P}[w_n(x,\sigma_n)\mathbf{1}_{B_n(x)} \le z] - \Phi(z)| \le D_x\, n^{-\upsilon}$$

2.3.3. *Discussion of the results for critical maps (Theorems 2.3, 2.4 and 2.5).* The proof of Theorems 2.3 and 2.4 are obtained using renormalization methods.

Roughly, renormalization gives us control of the effects of noise on small scales around the critical point for a fixed increasing sequence of times (powers of 2 in the case of period doubling and Fibonacci numbers in the case of circle maps with golden mean rotation). This gives us, rather straightforwardly, a central limit theorem when the orbit of zero is observed along these sequences of times.

To obtain a central limit theorem along the sequence of all times, we use the fact that an arbitrary number can be written as sum of these good numbers. We argue by approximation. We observe that the sequence of times accessible to renormalization is also the sequence of times at which the orbit of zero comes close to the origin. Hence, we can write the orbit of zero as a sum of approximate Gaussians.

The argument we present has some delicate steps. We need to balance how close is the approximation to the Gaussian (how fast is the convergence to the CLT) with how fast is the recurrence at the indicated times.

In our approach, to prove a CLT along all times, we have to compute and compare the two effects. This comparison depends on quantitative properties of the fixed point of the renormalization group and some of the auxiliary operators.

In the period doubling case, the properties required by our approach can be established and proved by conceptual methods (convexity and the like) from the properties of the fixed point.

In the case of circle maps however, our methods require a property (see (5.44)) which seems to be true numerically, but which we do not know how to verify using only analytical methods.

The analysis presented above raises the possibility that, for some systems, the weak noise limit could have a CLT along some sequences but not along other ones. Of course, it is possible that there are other methods of proof that do not require such comparisons. We think that it would be interesting either to develop a proof that does not require these conditions or to present an example of a system whose weak noise limit converges to Gaussian along a sequence of times but not others.

3. Proof of Theorem 2.1 and Theorem 2.2

In this section, we prove the two general theorems about one dimensional dynamical systems with weak random noise, namely Theorem 2.1 (a central limit theorem) and Theorem 2.2 (a Berry–Esseen theorem).

First, we consider a linear approximation of the system (1.1) and show that it satisfies a central limit theorem and a Berry–Esseen theorem, see section 3.1. Then, in section 3.2 we make a comparison between the linear approximation process and $x_n(x, \sigma)$, see Lemma 3.2. The proofs of Theorems 2.1 and 2.2 are given in Sections 3.3 and 3.4 respectively. We show that for σ small enough, see (2.10), the linear approximation process is a good approximation to $x_n(x, \sigma)$, except perhaps in sets of decreasing probability, which we call outliers. Since these sets have probability going to zero, they do not affect the convergence in probability. Showing that these outliers do not affect the variance requires some extra arguments, which we present later.

3.1. **Linear approximation of the effective noise.** For $x \in M$ and $\sigma > 0$ fixed, using Taylor expansion, we decompose the process $x_n(x, \sigma)$ as

$$x_n(x, \sigma) = f^n(x) + \sigma L_n(x) + \sigma^2 Q_n(x, \sigma), \tag{3.1}$$

where the linear term L_n is the sum of independent random variables defined by (2.4). The linear approximation process $y_k(x, \sigma)$

$$y_n(x, \sigma) = f^n(x) + \sigma L_n(x) \tag{3.2}$$

satisfies the following central limit theorem.

Lemma 3.1. *Let f be a function satisfying A1 and A2, and (ξ_n) be a sequence of independent random variables with $p > 2$ moments, satisfying A3. If condition* (2.9) *holds for some point $x \in M$ then,*

$$l_{n_k}(x) \equiv \frac{L_{n_k}(x)}{\sqrt{\mathrm{var}[L_n(x)]}} \tag{3.3}$$

converges in distribution to the standard Gaussian as $k \to \infty$. Moreover, there is a universal constant C such that

$$\sup_{z \in R} |\mathbb{P}[l_{n_k}(x) \leq z] - \Phi(z)| \leq C \frac{\Lambda_{\min(p,3)}(x, n_k)}{(\Lambda_2(x, n_k))^{\min(p,3)/2}}, \tag{3.4}$$

Proof. From (2.7) and (2.8), there is $c > 0$ such that

$$c^{-1} \frac{\Lambda_p(x, n_k)}{\sqrt{\Lambda_2(x, n_k)}} \leq \frac{\sum_{j=1}^{n_k} \left|(f^{n_k - j})' \circ f^j(x)\right|^p}{\sqrt{\mathrm{var}[L_{n_k}(x)]}} \leq c \frac{\Lambda_p(x, n_k)}{\sqrt{\Lambda_2(x, n_k)}}$$

Hence, the convergence of l_{n_k} to the standard Gaussian follows from the classical Lindeberg–Lyapunov central limit theorem [Bil68, p. 44], since condition (2.9) is equivalent in this case to the Lyapunov condition for sums of independent random variables.

The second assertion, (3.4), follows from Berry–Esseen's Theorem for sums of independent random variables [Pet75, p. 115] □

Remark 3.1. Notice that the linear dependence of $y_n(x,\sigma)$ on the random variables ξ_n implies that the convergence in Lemma 3.1 is independent of the noise level σ. The size of the noise σ will be important in the control of the non-linear term $\sigma^2 Q_n(x,\sigma)$.

3.2. **Nonlinear theory.** In this section, we prove a result (Lemma 3.2) that will help us make a comparison between $x_n(x,\sigma)$ and $y_n(x,\sigma)$. This result is analog to a well known result on variational equations for ODE's [Har82]. The method we use in the proof of Lemma 3.2 is similar to the proof of the Shadowing Lemma in [Shu87, Kat72, Szű81].

In the rest of this section, we will use the norm $\|\mathbf{x}\| := \max_{1\le j\le m} |x_j|$ for vectors $\mathbf{x}$ in Euclidean space $\mathbb{R}^m$, and the corresponding induced norms for linear and bilinear operators.

Lemma 3.2. *Suppose $f \in C^2(M)$, $x_0 \in M$, and let $\mathbf{\Delta} \in \mathbb{R}^{\mathbf{N+1}}$ with $\Delta_0 = 0$. Consider the sequences $\bar{\mathbf{x}}$, $\mathbf{x}$ and $\mathbf{c}$ in $\mathbb{R}^{N+1}$ defined by*

a) $\bar{x}_0 = x_0$, $\bar{x}_{i+1} = f(\bar{x}_i)$,
b) $x_{i+1} = f(x_i) + \Delta_{i+1}$,
c) $c_{i+1} = f'(\bar{x}_i)c_i + \Delta_{i+1}$ with $c_0 = 0$,

for $i = 0,\dots,N-1$.
Assume that

$$\|\mathbf{\Delta}\|\|f''\|_{C_0}\{\widehat{\Lambda}(x_0,N)\}^2 \le \frac{1}{4} \tag{3.5}$$

Then,

$$\begin{aligned}\|\bar{\mathbf{x}}+\mathbf{c}-\mathbf{x}\| &\le \|\mathbf{c}\|^2\widehat{\Lambda}(x_0,N)\|f''\|_{C_0}\\ &\le \|\mathbf{\Delta}\|^2(\widehat{\Lambda}(x_0,N))^3\|f''\|_{C_0}\end{aligned} \tag{3.6}$$

Proof. Fixing a parameterization on M, we can assume without loss of generality that $M = \mathbb{R}$. We will prove (3.6) by showing that $\mathbf{x}$ is a fixed point of the contractive function ψ defined in (3.16).

Notice that

$$c_i = \sum_{j=0}^{i} \left(f^{i-j}\right)'(\bar{x}_j)\Delta_j$$

If $\mathbf{A} = (a_{ij})$ is the lower triangular matrix defined by

$$a_{ij} = \begin{cases} \left(f^{i-j}\right)'(\bar{x}_j) & \text{for} \quad 0 \le j \le i \le N \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$\mathbf{c} = \mathbf{A}\boldsymbol{\Delta} \tag{3.7}$$

$$\|\mathbf{A}\| = \widehat{\Lambda}(x_0, N)$$

Let us define a function $\tau : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}$ by

$$\tau(\mathbf{z}, \alpha) = (x_0, f(z_0) + \alpha_1, \ldots, f(z_{N-1}) + \alpha_N)^\top$$

Observe that

a) $\tau(\bar{\mathbf{x}}, 0) = \bar{\mathbf{x}}$
b) $\tau(\mathbf{z}, \boldsymbol{\Delta}) = \mathbf{z}$ if and only if $\mathbf{z} = \mathbf{x}$
c) For all $(\mathbf{z}, \alpha) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$

$$\begin{aligned} D\tau(\mathbf{z}, \alpha) &= D\tau(\mathbf{z}, 0) \\ D_{11}\tau(\mathbf{z}, \alpha) &= D_{11}\tau(z, 0) \end{aligned}$$

The following identities will be useful

$$D_1\tau(\bar{\mathbf{x}}, 0)\mathbf{c} + D_2\tau(\bar{\mathbf{x}}, 0)\boldsymbol{\Delta} = \mathbf{c} \tag{3.8}$$

$$(D_1(\tau(\bar{\mathbf{x}}, 0)) - I)^{-1} = \mathbf{A} \tag{3.9}$$

A direct computation shows that for any point $(\mathbf{z}, \alpha)$ and vectors $[h, k]$, $[\tilde{h}, \tilde{k}] \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$

$$D^2\tau(\mathbf{z}, \alpha)([h, k], [\tilde{h}, \tilde{k}]) = (0, \tilde{h}_0 f''(z_0)h_0, \ldots, \tilde{h}_{N-1} f''(z_{N-1})h_{N-1})^\top$$

Therefore

$$\sup_{(\mathbf{z},\alpha)} \|D^2\tau(\mathbf{z}, \alpha)\| = \sup_{\mathbf{z}} \|D_{11}\tau(\mathbf{z}, 0)\| \le \|f''\|_{C_0} \tag{3.10}$$

We define an auxiliary function $\mathcal{N} : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}$ by

$$\mathcal{N}(\mathbf{z}, \alpha) = -\mathbf{A}(\tau(\mathbf{z}, \alpha) - \mathbf{z}) + \mathbf{z}$$

Using (a)–(c) we have

d) $\mathcal{N}(\mathbf{z}, 0) = z$ if and only if $\mathbf{z} = \bar{\mathbf{x}}$
e) $\mathcal{N}(\mathbf{z}, \boldsymbol{\Delta}) = z$ if and only if $\mathbf{z} = \mathbf{x}$
f) $D_1\mathcal{N}(\mathbf{z}, \alpha) = D_1\mathcal{N}(\mathbf{z}, 0)$ for all $(\mathbf{z}, \alpha)$.

It follows from (3.9)

$$\mathcal{N}(\mathbf{z}, \alpha) - \mathcal{N}(\mathbf{z}, 0) = -\mathbf{A}[\tau(\mathbf{z}, \alpha) - \tau(\mathbf{z}, 0)] \tag{3.11}$$

$$\begin{aligned} D_1\mathcal{N}(\mathbf{z}, \alpha) &= -\mathbf{A}(D_1\tau(\mathbf{z}, 0) - I) + I \\ &= \mathbf{A}[D_1\tau(\bar{\mathbf{x}}, 0) - D_1\tau(\mathbf{z}, 0)] \end{aligned} \tag{3.12}$$

$$D_2\mathcal{N}(\mathbf{z},\alpha)k \;=\; -\mathbf{A}[0,k_1,\ldots,k_N] \tag{3.13}$$

From (3.8) we have

$$\tau(\bar{\mathbf{x}}+\mathbf{c},\boldsymbol{\Delta}) = \bar{\mathbf{x}}+\mathbf{c}+\frac{1}{2}D_{11}\tau(\tilde{\xi},\tilde{\boldsymbol{\Delta}})(\mathbf{c},\mathbf{c}), \tag{3.14}$$

where $(\tilde{\xi},\tilde{\boldsymbol{\Delta}})$ is a point on the line segment between $(\bar{\mathbf{x}}+\mathbf{c},\boldsymbol{\Delta})$ and $(\bar{\mathbf{x}},0)$. Thus, by (3.7), (3.10), and (3.14) we have

$$\|\mathcal{N}(\bar{\mathbf{x}}+\mathbf{c},\boldsymbol{\Delta})-(\bar{\mathbf{x}}+\mathbf{c})\| \le \frac{1}{2}\|\mathbf{A}\|^3\|f''\|_{C_0}\|\boldsymbol{\Delta}\|^2$$

Let $\bar{B}(\bar{\mathbf{x}};r)$ be the closed ball in $\mathbb{R}^{N+1}$ centered at $\bar{\mathbf{x}}$ with radius $r = 2\|\boldsymbol{\Delta}\|\|\mathbf{A}\|$. By (3.12), for all $(h,\alpha)\in\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$ and all $\mathbf{z}\in\bar{B}(\bar{\mathbf{x}};r)$ $\eta\in\bar{B}(\bar{\mathbf{x}};r)$

$$D_1\mathcal{N}(\mathbf{z},\alpha)h = -\mathbf{A}D_{11}\tau(\eta,0)(h,\mathbf{z}-\bar{\mathbf{x}})$$

for some η in the line segment between $\mathbf{z}$ and $\bar{x}$. Therefore, if condition (3.5) holds, then

$$\|D_1\mathcal{N}(\mathbf{z},\alpha)\| \le 2\|\mathbf{A}\|^2\|\boldsymbol{\Delta}\|\|f''\|_{C_0} \le \frac{1}{2} \tag{3.15}$$

for all $\mathbf{z}\in\bar{B}(\bar{\mathbf{x}};r)$.

On the other hand (3.11) and (3.15) imply that

$$\begin{aligned}\|\mathcal{N}(\mathbf{z},\boldsymbol{\Delta})-\bar{\mathbf{x}}\| &\le \|\mathcal{N}(\mathbf{z},\boldsymbol{\Delta})-\mathcal{N}(\bar{\mathbf{x}},\boldsymbol{\Delta})\|+\|\mathcal{N}(\bar{\mathbf{x}},\boldsymbol{\Delta})-\mathcal{N}(\bar{\mathbf{x}},0)\| \\ &\le \frac{1}{2}\|z-\bar{\mathbf{x}}\|+\|\mathbf{A}\|\|\boldsymbol{\Delta}\|\le r\end{aligned}$$

It follows that the function

$$\psi(\mathbf{z}) = \mathcal{N}(\mathbf{z},\boldsymbol{\Delta}) \tag{3.16}$$

is a contraction by a factor 1/2 of the closed ball $\bar{B}(\bar{\mathbf{x}};r)$ into itself. Moreover, $\mathbf{x}$ is the unique fixed point.

Inequality (3.6) follows from

$$\begin{aligned}\|\bar{\mathbf{x}}+\mathbf{c}-\mathbf{x}\| &\le \|\bar{\mathbf{x}}+\mathbf{c}-\mathcal{N}(\bar{\mathbf{x}}+\mathbf{c},\boldsymbol{\Delta})\|+\|\mathcal{N}(\mathbf{x},\boldsymbol{\Delta})-\mathcal{N}(\bar{\mathbf{x}}+\mathbf{c},\boldsymbol{\Delta})\| \\ &\le \tfrac{1}{2}\|\bar{\mathbf{x}}+\mathbf{c}-\mathbf{x}\|+\tfrac{1}{2}\|\mathbf{A}\|^3\|\boldsymbol{\Delta}\|^2\|f''\|_{C_0}\end{aligned}$$

□

3.3. **Proof of central limit theorem (Theorem 2.1).** The conclusion of Theorem 2.1 will follow by combining Lemmas 3.1 and 3.2. The key argument is to define events (outliers) where the linear approximation process is very different from $x_n(x,\sigma)$, and show that they have small probability.

Notice that if $\|f''\|_{C_0} = 0$, then Theorem 2.1 coincides with the Lindeberg–Lyapunov central limit theorem. Therefore, we will assume that $\|f''\|_{C_0} > 0$.

3.3.1. *Outliers.* For each k and $j = 1, \dots, n_k$, let $\Delta_j = \sigma_k \xi_j$, and let $\mathbf{\Delta}$, $\bar{\mathbf{x}}$, $\mathbf{x}$ and $\mathbf{c}$ be as in Lemma 3.2. Then, by (2.5), (3.2) and the definition of the linear approximation process, we have that for $j = 1, \dots, n_k$

$$\begin{aligned} f^j(x) &= \bar{x}_j \\ y_j(x,\sigma_k) &= \bar{x}_j + c_j \\ \sigma_k L_j(x) &= c_j \\ x_j(x,\sigma_k) &= x_j \end{aligned}$$

For any sequence (σ_k) of noise levels decreasing to 0, Lemma 3.2 implies that in the event

$$\bar{B}_k = \left[\|f''\|_{C_0} \sigma_k (\widehat{\Lambda}(x,n_k))^2 \max_{1 \le j \le n_k} |\xi_j| \le \frac{1}{4} \right], \tag{3.17}$$

the linear approximation process $y_{n_k}(x,\sigma_k)$ (3.2) is close to $x_{n_k}(x,\sigma_k)$. Thus, we will restrict the process $x_{n_k}(x,\sigma_k)$ to events $B_k \in \mathcal{F}$ such that

$$B_k = \left[\max_{1 \le j \le n_k} |\xi_j| \le a_k \right] \subset \bar{B}_k \tag{3.18}$$

for some appropriate sequence a_k. We will choose (a_k) so that

$$\lim_{k \to \infty} \mathbb{P}[\Omega \setminus B_k] \to 0 \tag{3.19}$$

For this purpose, it will be enough to define (a_k) by

$$a_k = \frac{1}{4\|f''\|_{C_0}} (\widehat{\Lambda}(x,n_k))^{-\beta} \sigma_k^{-\alpha}, \tag{3.20}$$

where β and α are chosen so that (3.18) and (3.18) hold. This means that

$$\sigma_k (\widehat{\Lambda}(x,n_k))^2 \le \sigma_k^{\alpha} (\widehat{\Lambda}(x,n_k))^{\beta} \tag{3.21}$$

$$\lim_{k \to \infty} \left\| \max_{1 \le j \le n_k} |\xi_j| \right\|_p^{1/\alpha} (\widehat{\Lambda}(x,n_k))^{\beta/\alpha} \sigma_k = 0 \tag{3.22}$$

Remark 3.2. Recall that if $M = I$, we define the process $x_{n_k}(x,\sigma_k)$ conditioned on the event

$$\Omega \setminus C_{n_k} = \left\{ \sigma_k \max_{1 \le j \le n_k} |\xi_j| \le a \right\}$$

Chebyshev's inequality implies that

$$\mathbb{P}[C_{n_k}] \le \left(\frac{\sigma_k \, \|\max_{1 \le j \le n_k} |\xi_j|\|_p}{a} \right)^p$$

Notice that (3.22), and the fact that $\widehat{\Lambda}(x, n_k) > 1$ imply that $\mathbb{P}[C_{n_k}] \to 0$ as $k \to \infty$.

We refer to $\{\Omega \setminus B_k\}_k$ ($\{\Omega \setminus (B_k \cap C_k\}_k$ if $M = I$) as the sequence of outliers.

3.3.2. *Estimates on the effective noise.* In this section, we will show that the outliers have small probability. This means that linearized effective noise (2.4) is a good approximation of the effective noise (2.14) with high probability. As a consequence, we will have that the effective noise scaled by the standard deviation of the linearized effective noise approaches a Gaussian.

Recall from (3.1) that the effective noise is decomposed as

$$(x_{n_k}(x, \sigma_k) - f^{n_k}(x))\mathbf{1}_{B_k} = \sigma_k L_{n_k}(x)\mathbf{1}_{B_k} + \sigma_k^2 Q_{n_k}(x, \sigma_k)\mathbf{1}_{B_k}$$

To control the effect of nonlinear terms in the effective noise, it will be enough to require that the variance of the noise of $\sigma_k^2 Q_{n_k}(x, \sigma_k)$ is small compared to the variance of the linearized effective noise $\sigma_k L_{n_k}(x)$. In terms of the scaled processes $w_{n_k}(x, \sigma_k)$ and $l_{n_k}(x)$ defined by (2.12) and (3.3), this requirement is equivalent to

$$\lim_{k\to\infty} \|(w_{n_k}(x, \sigma_k) - l_{n_k}(x))\mathbf{1}_{B_k}\|_2 = 0 \tag{3.23}$$

By (3.18) and Lemma 3.2 we have that

$$|Q_{n_k}(x, \sigma_k)|\mathbf{1}_{B_k} \quad \leq \quad \|f''\|_{C_0}(\widehat{\Lambda}(x, n_k))^3 \max_{1\leq j\leq n_k} |\xi_j|^2 \mathbf{1}_{B_k}$$

Then, by (3.20) we get

$$\|(w_{n_k}(x, \sigma_k) - l_{n_k}(x))\mathbf{1}_{B_k}\|_2 \leq C \frac{(\widehat{\Lambda}(x, n_k))^{3-2\beta}\sigma_k^{1-2\alpha}}{\sqrt{\Lambda_2(x, n_k)}} \tag{3.24}$$

for some $C > 0$. To obtain (3.23), it suffices to require that

$$\lim_{k\to\infty} \frac{(\widehat{\Lambda}(x, n_k))^{3-2\beta}\sigma_k^{1-2\alpha}}{\sqrt{\Lambda_2(x, n_k)}} = 0 \tag{3.25}$$

We have the following Theorem

Theorem 3.3. *Let f and (ξ_n) be as in Theorem 2.1. Assume that the Lyapunov condition* (2.9) *holds and that (σ_k) satisfies* (2.10)*. If $\alpha^{-1} = 2 = \beta$, then $w_{n_k}(x, \sigma_k)$ converges in distribution to the standard Gaussian.*

Proof. Notice that the Lyapunov condition (2.9) implies that

$$\lim_{k\to\infty} \Lambda_2(x, n_k) \to \infty$$

Hence, by (3.21) and (3.25), it will be enough to consider $0 < \alpha \leq 1/2$ and $\beta \geq 2$.

In particular, if $\alpha^{-1} = 2 = \beta$ and (σ_k) satisfies (2.10), then $\mathbf{1}_{B_k}$ and $(w_{n_k}(x, \sigma_k) - l_{n_k}(x))\mathbf{1}_{B_k}$ converge in $L_{p/2}(\mathbb{P})$ and hence in probability to 1 and 0 respectively. By Lemma 3.3 and the *"converging together"* Lemma [Dur05, p. 89], it follows that $w_{n_k}(x, \sigma_k)$ converges in distribution to the standard Gaussian. $\square$

The result of Theorem 3.3 is improved by by considering the weaker condition (2.11) for (σ_k) and smaller outliers $\Omega \setminus \bar{B}_k$.

Notice that by Chebyshev's inequality

$$\mathbb{P}[\Omega \setminus \bar{B}_k] \leq \left(4\|f\|_{C_0}(\widehat{\Lambda}(x, n_k))^2 \left\|\max_{1\leq j\leq n_k} |\xi_n|\right\|_p \sigma_k\right)^p \tag{3.26}$$

On the other hand, we have that

$$\left\| \frac{\sigma_k Q_{n_k}(x, \sigma_k)}{\sqrt{\mathrm{var}[L_{n_k}(x)]}} \mathbf{1}_{B_k} \right\|_{p/2} \leq C \frac{\| \max_{1\leq j\leq n_k} |\xi_j| \|_p^2 (\widehat{\Lambda}(x, n_k))^3 \sigma_k}{\sqrt{\Lambda_2(x, n_k)}} \tag{3.27}$$

The following result is a direct consequence of the "*convergence together*" Lemma, (3.26), and (3.27).

Corollary 3.4. *Let f and (ξ_n) be as in Theorem 2.1. Assume that the Lyapunov condition* (2.9) *holds and that (σ_k) is a sequence of positive numbers that satisfy* (2.11). *Then, the process $w_{n_k}(x, \sigma_k)\mathbf{1}_{\bar{B}_k}$ and hence $w_{n_k}(x, \sigma_k)$ converge in distribution to the standard Gaussian.*

3.3.3. *Comparison between the variance of the effective noise and the variance of the linear approximation.* Observed that the scaling given in (3.24) uses the variance of a random variable $L_{n_k}(x)$ which is defined in the whole space Ω. Since the outliers occur with very low probability, it is more natural to restrict all the quantities to the complement of the outliers, that is to the events B_k.

If (σ_k) satisfies (2.10), we will show that the variance of the effective noise and that if its linear approximation are asymptotically equal. By the converging together Lemma, we will have as a consequence a central limit theorem for the effective noise scaled by its variance.

Lemma 3.5. *Let f and (ξ_n) be as in Theorem 2.1. Assume that the Lyapunov condition* (2.9) *holds. For any sequence (d_k) of positive numbers such that*

$$\lim_{k\to\infty} d_k^{-2(p-1)} \frac{(\widehat{\Lambda}(x, n_k))^2}{\Lambda_2(x, n_k)} = 0, \tag{3.28}$$

define the event $D_k = [\max_{1\le j\le n_k} |\xi_j| < d_k]$. *Then,*

$$\lim_{k\to\infty} \frac{\mathrm{var}[L_{n_k}(x)\mathbf{1}_{D_k}]}{\mathrm{var}[L_{n_k}(x)]} = 1 \tag{3.29}$$

Proof. Notice that

$$\mathrm{var}[L_{n_k}(x)\mathbf{1}_{D_k}] = \sum_{j,i=1}^{n_k} \left(f^{n_k-j}\right)'(f^j(x)) \left(f^{n_k-i}\right)'(f^i(x)) \,\mathrm{cov}[\xi_j\mathbf{1}_{D_k}, \xi_i\mathbf{1}_{D_k}]$$

where

$$\mathrm{cov}[\xi_j\mathbf{1}_{D_k}, \xi_i\mathbf{1}_{D_k}] = \mathbb{E}[\xi_j\xi_i\mathbf{1}_{D_k}] - \mathbb{E}[\xi_j\mathbf{1}_{D_k}]\mathbb{E}[\xi_i\mathbf{1}_{D_k}]$$

First, we estimate $\mathrm{cov}[\xi_j\mathbf{1}_{D_k}, \xi_i\mathbf{1}_{D_k}]$ when $j \neq i$.

Since $\mathbb{E}[\xi_j] = 0$ for all j, from the independence of (ξ_n) we get

$$\begin{aligned} |\mathbb{E}[\xi_j\mathbf{1}_{D_k}]| &= |\mathbb{E}[\xi_j\{|\xi| \le d_k\}]| \prod_{m\neq j} \mathbb{P}[|\xi_m| \le d_k] \\ &= |\mathbb{E}[\xi_j\{|\xi| > d_k\}]| \prod_{m\neq j} \mathbb{P}[|\xi_m| \le d_k] \end{aligned}$$

By Hölder's and Chebyshev's inequality we get

$$|\mathbb{E}[\xi_j\mathbf{1}_{D_k}]| \le \|\xi_j\|_p^p d_k^{1-p} \prod_{m\neq j} \mathbb{P}[|\xi_m| \le d_k] \tag{3.30}$$

Notice that (3.28) implies that $\lim_k d_k\infty$ and that $\lim_k \mathbb{P}[D_k] \to 1$. Consequently, the product of probabilities on the right hand side of (3.30) is close to (and smaller than) 1. Furthermore, A(p) implies that (ξ_n) is bounded in $L_p(\mathbb{P})$. Hence,

$$|\mathbb{E}[\xi_j\mathbf{1}_{D_k}]| \le C d_k^{1-p} \tag{3.31}$$

Similarly, the independence of (ξ_n) implies

$$\begin{aligned} |\mathbb{E}[\xi_j\xi_i\mathbf{1}_{D_k}]| &= |\mathbb{E}[\xi_j\{|\xi_j| \ge d_k\}]\mathbb{E}[\xi_i\{|\xi_i| \ge d_k\}]| \prod_{m\neq j,i} \mathbb{P}[|\xi_m| \le d_k] \\ &= |\mathbb{E}[\xi_j\{|\xi_j| > d_k\}]\mathbb{E}[\xi_i\{|\xi_i| > d_k\}]| \prod_{m\neq j,i} \mathbb{P}[|\xi_m| \le d_k] \end{aligned}$$

Using Hölder's and Chebyshev's inequalities we get

$$|\mathbb{E}[\xi_j\xi_i\mathbf{1}_{D_k}]| \le \|\xi_j\|_p^p \|\xi_i\|_p^p d_k^{-2(p-1)} \prod_{m\neq j,i} \mathbb{P}[|\xi_m| \le d_k] \tag{3.32}$$

The product of probabilities on the right hand side of (3.32) is close to (and smaller than) 1. Therefore, using A3(p), we have:

$$|\mathbb{E}[\xi_j\xi_i\mathbf{1}_{D_k}]| \le C d_k^{-2(p-1)} \tag{3.33}$$

Combining (3.31) and (3.33) we have, for $j \neq i$, that

$$|\operatorname{cov}[\xi_j \mathbf{1}_{D_k}, \xi_i \mathbf{1}_{D_k}]| \leq C d_k^{-2(p-1)} \tag{3.34}$$

When $j = i$, we obtain

$$\begin{aligned} |\operatorname{var}[\xi_j \mathbf{1}_{D_k}] - \operatorname{var}[\xi]| &= \mathbb{E}[\xi_j^2 \mathbf{1}_{B_k^c}] + |\mathbb{E}[\xi_j \mathbf{1}_{D_k}]|^2 \\ &\leq C(\mathbb{P}[D_k^c]^{(p-2)/p} + d_k^{-2(p-1)}) \end{aligned} \tag{3.35}$$

Recall that $\Lambda_2(x, n_k)$ and $\operatorname{var}[L_{n_k}(x)]$ are of the same order (see (2.7)). Since $p > 2$, by combining (3.34) and (3.35), we get

$$\left|1 - \frac{\operatorname{var}[L_{n_k}(x)\mathbf{1}_{D_k}]}{\operatorname{var}[L_{n_k}(x)]}\right| \leq C\left(\frac{d_k^{-2(p-1)}(\Lambda_1(x, n_k))^2}{\Lambda_2(x, n_k)} + \mathbb{P}[D_k^c]^{(p-2)/p}\right)$$

Since $\Lambda_1(x, n_k) \leq \widehat{\Lambda}(x, n_k)$, the conclusion of Lemma 3.5 follows from (3.28) by passing to the limit. □

The following result shows that the variance of effective noise and that of the its linear approximation are asymptotically equal

Lemma 3.6. *Let f and (ξ_n) be as in Theorem 2.1. Assume that the Lyapunov condition* (2.9) *holds and that (σ_k) is a sequence of positive numbers that satisfy* (2.10). *Then,*

$$\lim_{k\to\infty} \frac{\operatorname{var}[(x_{n_k}(x, \sigma_k) - f^{n_k}(x))\mathbf{1}_{B_k}]}{\sigma_k^2 \operatorname{var}[L_{n_k}(x)]} = 1, \tag{3.36}$$

where $\{B_k\}$ are the events defined in Section 3.3.1.

Proof. If (σ_k) satisfies (2.10), we know from Section 3.3.1 that

$$\lim_{k\to\infty} \|(w_{n_k}(x, \sigma_k) - l_{n_k}(x))\mathbf{1}_{B_k}\|_2 = 0$$

Therefore, by Lemma 3.5 it is enough to show that (3.28) holds for $d_k = a_k$, where a_k is defined by (3.20).

Notice that

$$a_k^{-2(p-1)} \frac{(\widehat{\Lambda}(x, n_k))^2}{\Lambda_2(x, n_k)} \leq C\left(\frac{\|\max_{1\leq j\leq n_k}|\xi_j|\|_p^2 (\widehat{\Lambda}(x, n_k))^6 \sigma_k}{\sqrt{\Lambda_2(x, n_k)}}\right)^{4(p-1)}$$

Thus, the conclusion of Lemma 3.6 follows by passing to the limit. □

Theorem 3.7. *Let f and (ξ_n) be as in Theorem 2.1. Assume that the Lyapunov condition* (2.9) *holds and that (σ_k) is a sequence of positive numbers that satisfy* (2.10). *Then, the process $\widetilde{w}_{n_k}(x, \sigma_k)$ defined by* (2.13) *converges in distribution to the standard Gaussian.*

Proof. We know from Theorem 3.3 that $w_{n_k}(x,\sigma_k)$ converges in distribution to the standard Gaussian. Notice that

$$\widetilde{w}_{n_k}(x,\sigma_k) = \sqrt{\frac{\operatorname{var}[L_{n_k}(x)]}{\operatorname{var}[(x_{n_k}(x,\sigma_k) - f^{n_k}(x))\mathbf{1}_{B_k}]}}\, w_{n_k}(x,\sigma_k)\mathbf{1}_{B_k}$$

The conclusion of Theorem 3.7 follows from Lemma 3.6 and the "*converging together*" Lemma. □

3.3.4. *Noise with finite moments of order $p \geq 4$.* Now, we will prove Theorem 2.1 under hypothesis H2.

We will show that if the sequence (ξ_n) has moments of order $p \geq 4$, then the choice of noise level (σ_k) (and hence the choice of outliers) is improved.

Theorem 3.8. *Let f be as in Theorem 2.1, assume that the Lyapunov condition* (2.9) *holds. Let (ξ_n) be a sequence of independent random variables with $p \geq 4$ finite moments, and that it satisfies A3(p).*

For any sequence (σ_k) that satisfies (2.11)*, let $\{B_k\}$ be the events defined by* (3.17)*. Then, the process*

$$\widehat{w}_{n_k} = \frac{(x_n(x,\sigma_k) - f^{n_k}(x))\mathbf{1}_{\bar{B}_k}]}{\sigma_k^2 \operatorname{var}[L_{n_k}(x)]} \tag{3.37}$$

converges to the standard Gaussian.

Proof. It suffices to verify that

$$d_k = \frac{1}{4\|f''\|_{C_0}} (\widehat{\Lambda}(x,n_k))^{-2}\sigma_k^{-1}$$

satisfy condition (3.28) in Lemma 3.5.

Since $\sqrt{\Lambda_2(x,n_k)} \leq \Lambda_1(x,n_k) \leq \widehat{\Lambda}(x,n_k)$, we have that

$$d_k^{-2(p-1)} \frac{(\widehat{\Lambda}(x,n_k))^2}{\Lambda_2(x,n_k)} \leq C \left(\frac{\|\max_{1\leq j\leq n_k} |\xi_j|\|_p^2 (\widehat{\Lambda}(x,n_k))^3 \sigma_k}{\sqrt{\Lambda_2(x,n_k)}} \right)^{2(p-1)}$$

The conclusion of Theorem 3.8 follows from (2.11) □

Remark 3.3. Notice from the definition of the outliers, that if the sequence of noises (ξ_n) is supported on an interval $[-b,b]$, then for k large enough $B_k = \Omega$, that is, the outliers are empty sets.

3.4. **Proof of Berry–Esseen estimates (Theorem 2.2).** In this section, we present a proof of Theorem 2.2. First, we will prove two results (Lemmas 3.9 and 3.10) that show how much the distribution of a random variable changes when we add a small perturbation.

Lemma 3.9. *Let ξ be a real random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let h be a bounded Borel measurable function in $\mathbb{R}$, and $B \in \mathcal{F}$. Then*

$$|\mathbb{E}\left[h\left(\xi \mathbf{1}_B\right)\right] - \mu h| \leq 2\|h\|_\infty \mathbb{P}[B^c] + |\mathbb{E}\left[h(\xi)\right] - \mu h| \tag{3.38}$$

where $\mu h = \int h(x)\mu(dx)$.

Proof. From the identity

$$h\left(\xi \mathbf{1}_B\right) = h(\xi)\mathbf{1}_B + h(0)\mathbf{1}_{B^c}$$

we have

$$|\mathbb{E}[h(\xi)] - \mathbb{E}\left[h\left(\xi \mathbf{1}_B\right)\right]| \leq 2\|h\|_\infty \mathbb{P}[B^c]$$

We obtain (3.38) by the triangle inequality □

Lemma 3.10. *Let ξ and η be real random variables defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that for some Lipschitz function G on $\mathbb{R}$ and some constant $0 < a < 1$ we have*

1) $\sup_{z\in\mathbb{R}} |\mathbb{P}[\eta \leq z] - G(z)| \leq a$
2) $\mathbb{E}[|\xi - \eta|] \leq a^2$

Then

$$\sup_{z\in\mathbb{R}} |\mathbb{P}[\xi \leq z] - G(z)| \leq (2 + L)a$$

where $L = L(G)$ is the Lipschitz constant of G.

Proof. Define the event $A = [|\xi - \eta| < a]$. Then by Chebyshev's inequality, we have $\mathbb{P}[A^c] \leq a$.

Partitioning the probability space Ω, we have

$$\begin{aligned}\mathbb{P}[\xi \leq z] &= \mathbb{P}[\{\xi \leq z\} \cap A] + \mathbb{P}[\{\xi \leq z\} \cap A^c] \\ &\leq \mathbb{P}[\eta \leq z + a] + a\end{aligned}$$

Adding and subtracting $G(z + a) - G(z)$ we obtain

$$\sup\left\{\mathbb{P}[\xi \leq z] - G(z)\right\} \leq (2 + L)a \tag{3.39}$$

Similarly, by partitioning the space we have

$$\begin{aligned}\mathbb{P}[\eta \leq z - a] &= \mathbb{P}[\{\eta \leq z - a\} \cap A] + \mathbb{P}[\{\eta \leq z - a\} \cap A^c] \\ &\leq \mathbb{P}[\xi \leq z] + a\end{aligned}$$

Adding and subtracting $G(z) - G(z - a)$ we obtain

$$-a(2 + L) \leq \inf_{z\in\mathbb{R}} \left\{\mathbb{P}[\xi \leq z] - G(z)\right\} \tag{3.40}$$

We finish the proof by combining (3.39) and (3.40) □

Notice that from Hölder's inequality, if the Lyapunov condition (2.9) holds at a point $x \in M$ for some number $2 < p$, then it also holds for $s = \min(3, p)$. By the same token, if (ξ_n) is a sequence of independent random variables with p finite moments that satisfies A3(p), then it satisfies A3(s).

If we denote by

$$a_k = \frac{\Lambda_s(x, n_k)}{(\Lambda_2(x, n_k))^{s/2}},$$

and let b_k be the left–hand side of (2.15). The assumption (2.15) is expressed as

$$b_k \leq a_k^2$$

Recall the processes w_{n_k} and l_{n_k} defined in (2.12) and (3.3) respectively. By Theorem 2.1, we know that there are events B_k such that

$$\max\{\mathbb{P}[B_k^c], \mathbb{E}[(w_{n_k} - l_{n_k})\mathbf{1}_{B_k}]\} \leq C a_k$$

Using Lemma 3.9 with $h = \mathbf{1}_{(-\infty, z]}$ and $\mu(dt) = \Phi'(t)dt$, we get

$$\sup_{z\in\mathbb{R}} |\mathbb{P}\left[l_k \mathbf{1}_{B_k} \leq z\right] - \Phi(z)| \leq 2b_k + a_k < \tfrac{3}{2}a_k$$

for k large enough. Then, the conclusion of Theorem 2.2 follows from Lemma 3.10 by letting $\xi = w_k\mathbf{1}_{B_k}$, $\epsilon = l_{n_k}\mathbf{1}_{B_k}$, $G(z) = \Phi(z)$ and $a = 3a_k/2$. □

Remark 3.4. Notice that the sequence of sizes of noises $\{\sigma_k\}_k$ and the sequence of outliers $\{B_k\}_k$ depend on the initial condition x_0 of the process x_n. Therefore, the central limit theorem in Theorem 2.1 and the Berry–Esseen estimates in Theorem 2.2 are point wise results. In section 3.6, we will give an example where the convergence to central limit theorem holds uniformly with respect all initial conditions.

3.5. **Cumulants.** The Lyapunov functions Λ_s used in Theorem 2.1 have a simple interpretation in terms of rather well known statistical quantities called cumulants in the statistical literature or Wick-ordered moments in statistical mechanics.

The cumulants, for a given random variable ξ, are formally defined as the coefficients K_p in the power series

$$\log \mathbb{E}\left[e^{it\xi}\right] = \sum_{p=0}^{\infty} \frac{K_p}{p!}(it)^p \tag{3.41}$$

Notice that $K_2[\xi] = \mathrm{var}[\xi]$. It can be seen [Pet75, Ch. IV] that $K_p[\xi]$ are homogeneous polynomials in moments, of degree p. Furthermore, for any $a \in \mathbb{R}$, and any pair of independent random variables ξ, η we have that

$$\begin{aligned} K_p[a\xi] &= a^p K_p[\xi] \\ K_p[\xi+\eta] &= K_p[\xi] + K_p[\eta] \end{aligned} \tag{3.42}$$

The cumulants $K_3[\xi]/(K_2[\xi])^{3/2}$ and $K_4[\xi]/(K_2[\xi])^2$ of $\bar{\xi} = \xi/\sqrt{K_2[\xi]}$, called *skewness* and *kurtosis* respectively, are empirical measures of resemblance to Gaussian: the closer to zero the closer $\bar{\xi}$ is to a Gaussian [MGB74, p. 78]. Observe that these measures are scale invariant.

Definition 3.1. For each positive integer p, we consider the cumulant functional $\widetilde{\Lambda}_p$ defined by

$$\widetilde{\Lambda}_p(x,n) = \sum_{j=1}^{n} \Big(\big(f^{n-j}\big)' \circ f^j(x) \Big)^p \tag{3.43}$$

Observe that $|\widetilde{\Lambda}_p(x,n)| \leq \Lambda_p(x,n)$ and that if p even, $\widetilde{\Lambda}_p = \Lambda_p$. Furthermore, the cumulant functionals $\widetilde{\Lambda}_p(x,n)$ satisfy the analog to (2.5) for the Lyapunov functionals, namely

$$\widetilde{\Lambda}_p(x,n+m) = ((f^m)' \circ f^n(x))^p \, \widetilde{\Lambda}_p(x,n) + \widetilde{\Lambda}_p(f^n(x),m) \tag{3.44}$$

In section 4.5 we will use (3.44) to study the asymptotic behavior of the cumulants in the case of systems near the accumulation of period doubling.

Notice that when $(\xi_n)_n$ is an i.i.d sequence with $p \in \mathbb{N}$ finite moments, we have that

$$K_s[L_n(x)] = K_s[\xi_1]\widetilde{\Lambda}_s(x,n) \tag{3.45}$$

for all $0 \leq s \leq p$.

If $(\xi_n)_n$ is a sequence of random variables that satisfies A3(p) for an integer $p > 2$, then from (2.4) and (3.42)

$$\left| K_p\left[\frac{L_n(x)}{\sqrt{K_2[L_n(x)]}} \right] \right| \leq c_p \left| \frac{\widetilde{\Lambda}_p(x,n)}{(\Lambda_2(x,n))^{p/2}} \right| \leq c_p \frac{\Lambda_p(x,n)}{(\Lambda_2(x,n))^{p/2}} \tag{3.46}$$

for some constant $c_p > 0$. Therefore, condition (2.9) in Theorem 2.1 can be interpreted as a measure of closeness to Gaussian.

Using renormalization theory, we will obtain sharp asymptotics for (3.46) for maps near the accumulation of period doubling and critical circle maps with golden mean rotation number.

3.6. **Examples.** In this section, we consider a nontrivial example of a system to which Theorem 2.1 applies. We also show that systems with enough hyperbolicity do not have a central limit theorem in the direction of Theorem 2.1.

3.6.1. *Smooth diffeomorphisms of the circle.* We start this section by showing a central limit theorem for conjugate maps. We will assume that $M = I$ or $\mathbb{T}^1$.

Lemma 3.11. *Let $f, g \in C^2(M)$, and let (ξ_n) be as in 2.1. Assume*

$$f = h^{-1} \circ g \circ h$$

for some map h such that $h, h^{-1} \in C^1(M)$. The Lyapunov condition (2.9) holds for f at some point x if and only if it holds for g at $h(x)$.

Proof. Notice that $f^j = h^{-1} \circ g^j \circ h$. Since h, $h^{-1} \in C^1(M)$ and M compact, we have that

$$c \left|(g^j)' \circ h(x)\right| \leq \left|(f^j)'(x)\right| \leq C \left|(g^j)' \circ h(x)\right|$$

for some constants $C, c > 0$. If we denote $y = f(x)$, we have

$$c^s \Lambda_s^g(y, n) \leq \Lambda_s^f(x, n) \leq C^s \Lambda_s^g(y, n)$$

for all $0 < s \leq q$. Now, Lemma 3.11 follows immediately. □

Recall that every orientation–preserving homeomorphism f of the circle $\mathbb{T}^1$ is given by $f(x) = F(x) \mod 1$, where F is a strictly increasing continuous function such that $F(x) = f(x) + 1$ for all $x \in \mathbb{R}$. For any $\alpha \in \mathbb{R}$, we denote by $R_\alpha(x) = x + \alpha \mod 1$ the rotation of the circle with rotation number α.

We will use Lemma 3.11 to show that Theorem 2.1 applies for smooth diffeomorphisms of the circle with Diophantine rotation number. This will be a consequence of the following Herman type theorem:

Theorem 3.12. *Let $k > 2$ and assume that $f \in C^k(\mathbb{T}^1)$ is an orientation preserving diffeomorphism, whose rotation number α is a Diophantine number with exponent β. That is*

$$\left|\alpha - \tfrac{p}{q}\right| > \tfrac{C}{q^{2+\beta}}$$

for all $\frac{p}{q} \in \mathbb{Q}$, where $C > 0$ is a fixed constant. If $\beta < k-2$, then there is a map $h \in C^{k-1-\beta}$ such that

$$f = h^{-1} \circ R_\alpha \circ h$$

Theorem 3.12, due to [SK89], is the most comprehensive global result on the problem of smoothness of conjugacies of diffeomorphisms of the circle with rotations. The first global result on this problem is due to [Her79] for $k \geq 3$, and generalized later by [Yoc84]. The first mayor improvement ($k > 2$) in the solution of this problem was obtained by [KS87, KO89, SK89]. A brief historical account of the problem and a detail proof of this and related results can be found in [SK89]. The technique developed in [SK89] constitutes one of the most important applications of the renormalization group method in dynamical systems.

Example 1. Notice that Theorem 2.1 applied to R_α is the classical central limit theorem. Thus, we can conclude that any Diophantine diffeomorphism of the circle has a Gaussian scaling limit for weak noises along the whole sequence of integers. Moreover, if the sequence of random variables ξ_n have finite moments of order $p = 3$ and we let $\sigma_n = n^{-3/2}$, we can define the outliers by

$$B_n^c = \left[\|f''\|_{C_0} \max_{i \le l \le n} |\xi_l| \ge \sqrt{n} \right]$$

Notice that the outliers are independent of the initial point of the orbit. Furthermore, by Theorem 2.2 we have

$$\sup_{(x,z) \in \mathbb{T}^1 \times \mathbb{R}} |\mathbb{P}[w_n(x) \le z] - \Phi(z)| \le \frac{A}{\sqrt{n}}$$

for some constant $A > 0$. □

3.6.2. *Systems with positive Lyapunov exponents.* In this section, we give an example of a system that does not satisfy the Lyapunov condition (2.9), and for which the conclusion of Theorem 2.1 fails.

Example 2. Consider the map on $\mathbb{R}$ (or $\mathbb{T}^1$) given by

$$f(x) = 2x$$

Condition (2.9) is not satisfied, and as we will see the conclusion of Theorem 2.1 fails. We will show that there is a limit for the scaled noise, not necessarily Gaussian, which depends strongly on the distribution of the original noise ξ_n. Notice that

$$\begin{aligned} w_n &= \frac{x_n(x_0, \sigma_n) - 2^n x_0}{\sigma_n \operatorname{var}\left(\sum_{j=1}^n 2^{n-j} \epsilon_j \right)} \\ &= \frac{3\sqrt{2}}{2\sqrt{1 - 4^{-n}}} \sum_{j=1}^n 2^{-j} \xi_j \end{aligned}$$

If (ξ_n) is an i.i.d. sequence with uniform distribution $U[-1, 1]$ then, w_n will converge in law to a compactly supported random variable ξ with characteristic function

$$\phi(z) = \prod_{k=2}^{\infty} \frac{2\sqrt{2} \sin(2^{-k}\sqrt{2}3z)}{2^{-k}3z}$$

On the other hand, if (ξ_n) is an i.i.d sequence with standard normal distribution, then w_n has standard normal distribution for all times.

Similar results hold for hyperbolic orbits and the reason is that derivatives at hyperbolic points grow (or decay) exponentially, that is $|(f^{n-j})' \circ f^j(x)| \approx a^{n-j}$ for some number $a > 0$. If (ξ_n) is an i.i.d sequence for instance, then the cumulants K_p of order $p > 2$ of the normalized variable L_n do not decay to zero. In fact we have

$$K_p\left[\frac{L_n(x)}{\sqrt{\mathrm{var}[L_n(x)]}}\right] \approx K_p[\xi_1]\sqrt{|a^2-1|^{p-2}}$$

where K_p is the cumulant of order p of ξ_1. Therefore, for hyperbolic orbits the scaling limit depends strongly on the distribution of the sequence ξ_n. □

In view of the example above, we can see that the scaling version of the central limit theorem we have proved does not apply for maps which are hyperbolic. In that respect, it is curious to mention that a celebrated result [GŚ96, Lyu97, GŚ97] shows that the maps in the quadratic family are hyperbolic for a dense set of parameters.

4. Central limit theorem for maps the accumulation of period doubling

In this section, we will show that the orbits (1.1) at the accumulation of period–doubling satisfy a central limit theorem (Theorem 2.3). We consider some statistical characteristics of the effective noise, i.e. Lindeberg–Lyapunov sums, and show that they satisfy some scaling relations (Corollary 4.8). We use this scaling relations to show that the Lyapunov condition 2.9 holds for certain class of initial conditions (Theorem 4.15).

4.1. **The period doubling renormalization group operator.** We will consider analytic even maps f of the interval $I = [-1, 1]$ into itself such that

- P1. $f(0) = 1$
- P2. $xf'(x) < 0$ for $x \neq 0$
- P3. $Sf(x) < 0$ for all $x \neq 0$ where S is the Schwartzian derivative $(Sf(x) = (f'''(x)/f'(x)) - (3/2)(f''(x)/f'(x))^2)$

The set of functions that satisfy conditions P1–P3 is called the set of unimodal maps.

Let us denote by $\lambda_f = f(1)$ and $b_f = f(\lambda_f)$. We will further assume that

- P4. $0 < |\lambda_f| < b_f$

P5. $0 < f(b_f) < |\lambda_f|$

The set $\mathcal{P}$ of functions that satisfy P1–P5 is called the space of period–doubling renormalizable functions.

It follows from P4 that the intervals $I_0 = [-|\lambda_f|, |\lambda_f|]$ and $I_1 = [b_f, 1]$ do not overlap. If $f \in \mathcal{P}$, then $f \circ f|_{I_0}$ has a single critical point, which is a minimum. The change of variables $x \mapsto \lambda_f x$ replaces I_0 by I and the minimum by a maximum.

Definition 4.1. The period doubling renormalization operator T on $\mathcal{P}$ is defined by

$$Tf(x) = \frac{1}{\lambda_f} f \circ f(\lambda_f x)$$

If $f, Tf, \ldots, T^{(n-1)}f \in \mathcal{P}$, we denote by $\Gamma_n = f^{2^n}(0)$ (or Γ_n^f if we need to emphasize the dependence on f). We have that

$$\begin{aligned} T^n f(x) &= \frac{f^{2^n}(\Gamma_n x)}{\Gamma_n} \\ \lambda_{T^n f} &= \frac{\Gamma_{n+1}}{\Gamma_n} \end{aligned}$$

The renormalization group operator T has been well studied by many authors, [Fei77, TC78, CE80, CEL80, Lan82, VSK84, EW85, Eps86, CL89, Sul92, Mar98, dMvS93, JŚ02]. The following results, whose proofs can be found in the reference above, will be very useful for our purposes.

R1. For each $k \in \mathbb{N}$, there is an analytic function h_k such that $g_k(x) = h_k(x^{2k})$ is analytic on some open bounded domain $D_k \subset \mathbb{C}$ containing the interval $[-1, 1]$, g_k restricted to I is concave, and for all $x \in I$

$$Tg_k(x) = g_k(x) \tag{4.1}$$

Furthermore, since $Sf(x) < 0$ for all $x \neq 0$, it follows that $g''(x) < 0$ for all $x \neq 0$.

Notice that (4.1) implies that

$$g_k'(x) = g_k'(g_k(\lambda_k x))g_k'(\lambda_k x)$$

Then, by taking the limit $x \to 0$ we get

$$g_k'(1) = \lim_{x \to 0} \frac{g_k'(x)}{g_k'(\lambda_k x)} = \lambda_k^{1-2k}$$

R2. The domain D_k can be chosen so that ∂D_k is smooth, $\overline{\lambda_k D_k} \subset D_k$ and $\overline{g_k(\lambda_k D_k)} \subset D_k$.

This condition is closely related to the fact that g_k is in the domain of the period doubling renormalization operator.

For each k, denote by $H(D_k)$ the space of analytic functions on D_k, and let $\mathcal{G}_k \subset H(D_k)$ be the Banach space of analytic functions

$$\mathcal{G}_k = \left\{ f:\ f(z) = z^{2k}\hat{f}(z),\ \hat{f}(z) \in H(D_k),\ \hat{f} \in C_0(\overline{D_k}) \right\}$$

endowed with the sup norm. Notice that if $f \in \mathcal{G}_k$, then $f^{(j)}(0) = 0$ for all derivatives of order $0 \le j < 2k$.

- R3. There is a neighborhood $\mathcal{V}$ of g_k in $(\mathcal{G}_k + 1) \cap \mathcal{P}$ where T is differentiable.
- R4. A consequence of R2 is that if $f \in \mathcal{V}$, the derivative $DT(f)$ is a compact operator on $\mathcal{G}^k$.
- R5. T admits an invariant stable manifold $\mathcal{W}_s(g_k)$, such that if $f \in \mathcal{W}_s(g_k)$ then,

$$\lim_{n\to\infty} \|T^n f - g_k\|_{C(\bar{D}_k)} = O(\omega_k^n) \tag{4.2}$$

where $0 < \omega_k < 1$ is a universal constant.

The set $\mathcal{W}_s(g_k)$ is called the universality class of g_k.

If we denote by $\lambda_{g_k} = \lambda_k$ then, for each $n \in \mathbb{N}$ we have that $\Gamma_n^{g_k} = (\lambda_k)^n$. Moreover, the exponential convergence (R4) implies that for any $f \in \mathcal{W}_s(g_k)$, λ_{T^n} converges to λ_k exponentially fast.

4.2. **Renormalization theory of the noise.** In this section, we develop a theory of renormalization for the noise. Since the renormalization group operator T gives information for times 2^n at small scales, we will study first the effective noise at times 2^n for orbits x_n starting at zero.

Let $f \in \mathcal{W}_s(g)$ be fixed, and for each $p \ge 0$ denote by $k_p^{(n)}(x) = \Lambda_p^f(x, 2^n)$. From (2.5) we have

$$k_p^{(n)}(x) = \left| \left(f^{2^{n-1}} \right)' \circ f^{2^{n-1}}(x) \right|^p k_p^{(n-1)}(x) + k_p^{(n-1)} \circ f^{2^{n-1}}(x) \tag{4.3}$$

Since $f \in \mathcal{P}$, then $-f'(x) > 1 > 0$ for all $x \in [f(\lambda_f), 1]$. Therefore, by taking a determination of logarithm defined on

$$\mathbb{C} \setminus \{x + iy : x \le 0,\, y = 0\}$$

we can analytically define the map $z \mapsto (-f' \circ f(\lambda_f z))^p$ on D_k. In particular, notice that if $z \in D_k \cap \mathbb{R}$, then

$$(-f' \circ f(\lambda_f z))^p = |f' \circ f(\lambda_f z)|^p \tag{4.4}$$

4.2.1. *Lindeberg–Lyapunov operators.* The following family of operators will be very useful in the study of the propagation of noise.

Definition 4.2. For each $f \in \mathcal{W}_s(g)$, the family of Lindeberg–Lyapunov operators $\{\mathcal{K}_{f,p} : p \geq 0\}$ acting on the space $H^r(\bar{D}_k)$ of real and bounded analytic functions on D_k which are continuous in $\bar{D}_k$, is defined by

$$\mathcal{K}_{f,p}h(z) = \frac{1}{(-\lambda_f)^p}\left[(-f' \circ f(\lambda_f z))^p\, h(\lambda_f z) + h(f(\lambda_f z))\right] \tag{4.5}$$

where $h \in H^r.(\bar{D}_k)$.

The operators defined by (4.5) belong to a class of operators called transfer operators, which have been extensively studied [Bal00, May80].

It follows from (4.3) and (4.5) that if

$$x = \Gamma^f_n z, \qquad \widetilde{k}^{(n)}_p(z) = \left|\Gamma^f_n\right|^{-p} k^{(n)}_p(\Gamma^f_n z),$$

then, defining the function $\mathbf{1}$ by $\mathbf{1}(z) = 1$

$$\widetilde{k}^{(n)}_p(z) = (\mathcal{K}_{T^{n-1}f,p} \cdots \mathcal{K}_{f,p}\mathbf{1})(z) \tag{4.6}$$

Equation (4.6) provides a relation between the Lindeberg–Lyapunov operators defined by (4.5) and the effective noise. Indeed, recall that the sequence of noises $(\xi_n)_n$ satisfies $c^{-1} < \inf_n \|\xi\|_2 \leq \sup_n \|\xi\|_p < c$. Then, for $s = 2$, p we have that

$$c^{-1}\widetilde{k}^{(n)}_s(z) \leq |\lambda_k|^{n\,s} \sum_{j=1}^{2^n} \left|\left(f^{2^n-j}\right)' \circ f^j(\lambda_k^n z)\right|^s \|\xi\|_s^s \leq c\,\widetilde{k}^{(n)}_s(z)$$

This means that (4.6) estimates the growth of the linearized propagation of noise at times 2^n with initial condition at the origin.

4.2.2. *Exponential convergence of the Lindeberg–Lyapunov operators.* Recall that for a map $f \in \mathcal{W}_s(g_k)$, we have that $T^n f(z)$ converges exponentially fast to g_k. Our next result implies that the sequence of Lindeberg–Lyapunov operators $\mathcal{K}_{T^n f,p}$ also converges exponentially to $\mathcal{K}_{g_k,p}$. This implication will be important in Section 4.3, Proposition 4.7, where we analyze the asymptotic behavior of the product of operators in (4.6).

Lemma 4.1. *Let $f \in \mathcal{W}_s(g_k)$ close enough to g_k. For all $p \geq 0$, $\mathcal{K}_{f,p}$ is a compact operator on $H^r(\overline{D}_k)$.*

Furthermore, there is a constant C_p such that

$$\|\mathcal{K}_{f,p} - \mathcal{K}_{g_k,p}\| < C_p\|f - g_k\|_{C(\bar{D}_k)} \tag{4.7}$$

Proof. For any set $D \subset \mathbb{C}$, let $D^\epsilon = \{z : d(x, D) \le \epsilon\}$. Recall that

$$\begin{aligned} \overline{\lambda_k D_k} &\subset D_k \\ \overline{g_k(\lambda_k D_k)} &\subset D_k \end{aligned}$$

Since for all $z \in \bar{D}_k$

$$|\lambda_f\, z - \lambda_k\, z| \le \|f - g_k\|_{\bar{D}_k} \operatorname{diam}(D_k),$$

we can choose a neighborhood $\mathcal{U}$ of g_k such that for all $f \in \mathcal{W}_s(g_k) \cap \mathcal{U}$,

$$\overline{\lambda_f D_k} \subset \overline{\lambda_k D_k}^\epsilon \subset D_k \tag{4.8}$$

where $\epsilon > 0$ is small enough. Let us denote by $V_k = \overline{\lambda_k D_k}^\epsilon$.

Notice that for all $z \in D_k$,

$$\begin{aligned} |g_k(\lambda_k\, z) - g_k(\lambda_f\, z)| &\le \|g_k'\|_{V_k} \operatorname{diam}(D_k) \|f - g_k\|_{D_k} \\ |f(\lambda_f\, z) - g_k(\lambda_f\, z)| &\le \|f - g_k\|_{V_k} \end{aligned}$$

Then, for all $z \in D_k$ we have that

$$|f(\lambda_f\, z) - g_k(\lambda_k\, z)| \le C \|f - g_k\|_{D_k} \tag{4.9}$$

for some constant C. Therefore, shrinking $\mathcal{U}$ if necessary, we can assume that

$$\overline{f(\lambda_f D_k)} \subset \overline{g_k(\lambda_k D_k)}^\epsilon \subset D_k \tag{4.10}$$

Let us denote by $W_k = \overline{g_k(\lambda_k D_k)}^\epsilon$.

For all $z \in D_k$ we have that

$$\begin{aligned} |f'(f(\lambda_f\, z)) - g_k'(g_k(\lambda_k\, z))\| \le |g_k'(g_k(\lambda_k\, z)) - g_k'(f(\lambda_k\, z))| \\ + |g_k'(f(\lambda_f\, z)) - f'(f(\lambda_f\, z))| \end{aligned}$$

Using Cauchy estimates and the following inequalities

$$\begin{aligned} |g_k'(g_k(\lambda_k\, z)) - g_k'(f(\lambda_k\, z))| &\le \|g_k''\|_{W_k} \|g_k - f\|_{V_k} \\ |g_k'(f(\lambda_f\, z)) - f'(f(\lambda_f\, z))| &\le \|g_k' - f'\|_{W_k}, \end{aligned}$$

we obtain that

$$\|f' \circ f \circ \lambda_f - g_k' \circ g_k \circ \lambda_k\|_{\bar{D}_k} \le \tilde{C} \|g_k - f\|_{\bar{D}_k} \tag{4.11}$$

Let $p > 0$ be fixed. Notice that the Lindeberg–Lyapunov operator $\mathcal{K}_{f,p}$ has the form

$$\mathcal{K}_{f,p} h = U_f h + R_f h$$

where $U : h \mapsto (\psi \circ f' \circ f \circ \lambda_f)(h \circ \lambda_f)$ and $R : h \mapsto h \circ f \circ \lambda_f$, where $\psi(z) = z^p$ is defined on $O = \mathbb{C} \setminus \{x + i\, y : x < 0,\, y = 0\}$. By shrinking $\mathcal{U}$ if necessary, we can assume by (4.11) that if $f \in \mathcal{W}_s(g_k) \cap \mathcal{U}$ then,

$$\overline{f'(f(\lambda_f D_k))} \subset \overline{g_k'(g_k(\lambda_k D_k))}^\epsilon \subset O$$

Let us denote by $Y_k = \overline{g_k'(g_k(\lambda_k D_k))}^{\epsilon}$ From (4.9), we have that

$$\begin{aligned} \|(R_f - R_{g_k})h\|_{D_k} &\leq C\|h'\|_{W_k}\|f-g_k\|_{\bar{D}_k} \\ &\leq \hat{C}\|h\|_{\bar{D}_k}\|f-g_k\|_{\bar{D}_k} \end{aligned} \tag{4.12}$$

From (4.11), we have that

$$\begin{aligned} \|(U_f - U_{g_k})h\| \leq \hat{C}\|\psi\|_{Y_k}\|h\|_{\bar{D}_k}\|f-g_k\|_{\bar{D}_k} \\ + \tilde{C}\|\psi'\|_{Y_k}\|h\|_{V_k}\|f-g_k\|_{\bar{D}_k} \end{aligned} \tag{4.13}$$

The exponential convergence follows from (4.12) and (4.13).

Notice that the operators $\mathcal{K}_{f,p}$ send bounded sets in $H^r(\overline{D}_k)$ into bounded sets in $H^r(\overline{D}_k)$.

Furthermore, for all z_1, $z_2 \in D_k$ we have from (4.8) and (4.10) that

$$\begin{aligned} |R_f h(z_1) - R_f h(z_2)| &\leq \|(h\circ f)'\|_{V_k}|z_1 - z_2| \\ |U_f h(z_1) - U_f h(z_2)| &\leq \left|((\psi\circ f)h)'\right|_{V_k}|z_1 - z_2| \end{aligned}$$

The compactness of the operators $\mathcal{K}_{f,p}$ follows from Cauchy estimates and the Arzela-Ascoli theorem. □

We will study the spectral properties of the Lindeberg–Lyapunov operators to show that (2.9) holds, and then by Theorem 2.1 we will obtain the Gaussian scaling limit for systems at the accumulation of period–doubling.

4.3. **Spectral analysis of the Lindeberg–Lyapunov operators.** Let $\mathcal{C}$ be the cone of functions on $H^r(\bar{D}_k)$ which are non-negative when restricted to $\bar{D}_k \cap \mathbb{R}$. Note that int$\mathcal{C}$, the interior of $\mathcal{C}$ consists of the functions in $H^r(\bar{D}_k)$ which are strictly possible when restricted to $\bar{D}_k \cap \mathbb{R}$.

Let $f \in \mathcal{W}_s(g)$ and $p \geq 0$ fixed, and denote $\mathcal{K}_p = \mathcal{K}_{f,p}$.

We have that

(i) $\mathcal{K}_p(\mathcal{C}\setminus\{0\}) \subset \mathcal{C}\setminus\{0\}$
(ii) $\mathcal{K}_p(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$, and
(iii) for each $h \in \mathcal{C}\setminus\{0\}$, there is an integer $n = n(h)$ such that $\mathcal{K}_p^n h \in \mathrm{int}(\mathcal{C})$.

Properties (i), (ii) are quite obvious. Property (iii) follows from the observation that if a function is non-negative and strictly positive in an interval I^* – the set of zeros has to be isolated – then, by the definition of $\mathcal{K}_p$ (4.5) it is strictly positive in the image of I^* under the inverse of the maps $\lambda_f\cdot$, $f(\lambda_f\cdot))$. Since the functions λ_f, $f(\lambda_f\cdot)$ are have derivatives strictly smaller than 1, it follows that a finite number of iterations of the inverse functions covers the whole interval.

By Kreĭn–Rutman's Theorem [Sch71, p. 265], [Tak94], we have that

Proposition 4.2. *Let* $f \in \mathcal{W}_s(g)$ *and* $p \geq 0$*, and denote by* $\mathcal{K}_p = \mathcal{K}_{f,p}$*. Then,*

K1. *The spectral radius* ρ_p *of* $\mathcal{K}_p$ *is a positive simple eigenvalue of* $\mathcal{K}_p$*.*
K2. *An eigenvector* $\psi_p \in H^r(\bar{D}_k) \setminus \{0\}$ *associated with* ρ_p *can be chosen in int*$(\mathcal{C})$*.*
K3. *If* μ *is in the spectrum of* $\mathcal{K}_p$*,* $0 \neq \mu \neq \rho_p$*, then* μ *is an eigenvalue of* $\mathcal{K}_{f,p}$ *and* $|\mu| < \rho_p$*.*
K4. *If* $h \in \mathcal{C} \setminus \{0\}$ *is an eigenvector of* $\mathcal{K}_p$*, then the corresponding eigenvalue is* ρ_p*.*

Remark 4.1. We emphasize that Krein-Rutman's theorem only needs to assume that the operators are compact and that preserve a cone. Hence, the only properties of the operators $\mathcal{K}_p$ that are used are precisely $i), ii), iii)$ above. Later in Section 5 we will see another application of the abstract set up. In the case of Section 5 the space will be a space whose elements are pairs of functions. It is important to note that Corollary 4.3, Proposition 4.4 and Proposition 4.6 remain valid in the case of several components.

We will say that two sequences of functions $\{a_n(z)\}_n$ and $\{b_n(z)\}_n$ of function in $H^r(\bar{D}_k)$ are asymptotically similar (denoted by $a_n \asymp b_n$) if

$$\lim_{n\to\infty} \|a_n/b_n\|_{H^r(\bar{D}_k)} = 1$$

As standard, we use the notation $\psi > \phi$ to denote $\psi - \phi \in \text{int}\mathcal{C}$. In particular $\phi > 0$ means $\phi \in \text{int}\mathcal{C}$. For function in $H^r(\bar{D}_k)$, $\phi > \psi$ is just means that $\phi(z) > \psi(z)$ for all $z \in \bar{D}_k \cap \mathbb{R}$.

One property of the cones in the spaces we are considering in this Section and in Section 5 is that:

property 4.1. Let $\phi, \psi \in \text{int}\mathcal{C}$ Then, there exists $\delta > 0$ such that $\psi > \delta\phi$.

A consequence of Proposition 4.2 is:

Corollary 4.3. $h \in \mathcal{C} \setminus \{0\}$ *then, there is a constant* $c_p = c_p(h) > 0$ *such that*

$$||\mathcal{K}_p^n h(z) - c_p \rho_p^n \psi_p(z)|| \leq C(\rho_p - \delta)^n \tag{4.14}$$

for some $C > 0, \delta > 0$*.*

In the one-dimensional case, this implies.

$$\mathcal{K}_p^n h(z) \asymp c_p \rho_p^n \psi_p(z) \tag{4.15}$$

To prove (4.14) we note that we can define spectral projections corresponding to the spectral radius and to the complement of the spectrum. If $h = c_p\psi_p + h^<$ with $h^<$ in the space corresponding to the spectrum away from the spectral radius.

By the spectral radius formula, we have $||\mathcal{K}_p^n h^<|| \leq C(\rho_p - \delta)^n$ for some $\delta > 0$. On the other hand, we have that for some $\delta > 0$, $h > \delta\psi_p$ over the reals. Hence, $\mathcal{K}_p^n h > \delta\rho_p^n\psi_p$ over the reals. By comparing with $\mathcal{K}_p^n h = \delta\rho_p^n\psi_p + \mathcal{K}_p^n h^<$, we obtain that $c_p > 0$.

The conclusion (4.15) follows from the fact that ψ_p is bounded away from zero.

Another important consequence of Proposition 4.2 and the analyticity improving is that the spectrum and the eigenvalues are largely independent of the domain of the functions we are considering. This justifies that we have not included it in the notation.

Proposition 4.4. *Assume that D_k, $\tilde{D}_k$ are domains satisfying the assumptions R_2.*

Assume that applying a finite number of times the inverse functions of either $\lambda_k\cdot$ or $g(\lambda_k\cdot)$ D_k we obtain a domain which contains $\tilde{D}_k$. Then, the spectrum of $\mathcal{K}_{f,p}$ acting on $H^r(D_k)$ is contained in the spectrum of $\mathcal{K}_{f,p}$ acting on $H^r(\tilde{D}_k)$.

We know that the spectrum is just eigenvalues (and zero). If we use the functional equation satisfied by an eigenvalue of $\mathcal{K}_{f,p}$, we see that the eigenfunction extends to the images of D_k under the inverse images of $\lambda_k\cdot$ or $g(\lambda_k\cdot)$

Hence, by the assumption, we obtain that the eigenvalues in $H(D_k)$ are eigenvalues in $H(\tilde{D}_k)$. □

Of course, if the hypothesis of Proposition 4.4 is satisfied also when we exchange D_k and $\tilde{D}_k$, we conclude that the spectrum is the same.

Note that in our case, since the inverse maps are expansive, Proposition 4.4 shows that the spectrum of the operator is independent of the domain, provided that the domain is not too far away from the unit interval.

For the case of the quadratic fixed point there is a very detailed study of the maximal domains in [EL81]. The results of [EL81] imply that there is a natural boundary for the eigenfunctions of the operators. Of course, if we consider domains larger than that, the results are different. Nevertheless, for domains inside this natural boundary, the spectrum is independent of the domain.

4.3.1. *Properties of the spectral radius of the Lindeberg–Lyapunov operators.* For $f \in \mathcal{W}_s(g)$, we have that the spectral radii of the operators $\mathcal{K}_{f,p}$ satisfy some convexity properties.

Theorem 4.5. *Let f be a map in $\mathcal{W}_s(g_k)$ close enough to g_k. For each $p \geq 0$, denote by $\mathcal{K}_p = \mathcal{K}_{f,p}$, and the spectral radius of $\mathcal{K}_p$ by ρ_p. Then*

S1.
$$(\lambda_f^{-1} f'(1))^p < \rho_p < (\lambda_f^{-1} f'(1))^p + (-\lambda_f)^{-p} \tag{4.16}$$
In particular, if $f = g_k$ then
$$1 < \lambda_k^{2kp} \rho_p < 1 + (-\lambda_k)^{(2k-1)p} \tag{4.17}$$

S2. *For $p \geq 0$. The map $p \mapsto \rho_p$ is strictly increasing and strictly log–convex.*

S3. *The map $p \mapsto \log[\rho_p]/p$ is strictly decreasing.*

For future purposes in this paper, [S3.] is the most important of the consequences claimed in Theorem 4.5.

Proof. S1: For fixed $p \geq 0$, let ψ_p be a positive eigenfunction of $\mathcal{K}_p$. Denote by $b = (-\lambda_f)^{-p}$, and notice that $b > 1$. Let U_p, R and S_p be the nonnegative compact operators defined by

$$\begin{aligned} U_p h(z) &= (-f' \circ f(\lambda_f z))^p \, h(\lambda_f z) \\ R h(z) &= h \circ f(\lambda_f z) \\ S_p h(z) &= (-f'(1))^p h(\lambda_f z) \end{aligned}$$

Since $\mathcal{K}_p \, h = b \, (U_p + R) \, h$, for any $h \in \mathcal{C} \setminus \{0\}$ and any $n \in \mathbb{N}$ we have that, by the binomial theorem

$$\mathcal{K}_p^n h > b^n (U_p^n + R^n) h > b^n \, U_p^n h. \tag{4.18}$$

Notice that

$$U_p^n h(z) = \left\{ \prod_{j=1}^{n} \left(-f' \circ f(\lambda_f^j z) \right) \right\}^p h(\lambda_f^n z)$$

If we take $h(z) = \psi_p(z)$ and then let $z = 0$, the left–hand side of (4.16) follows.

Since $1 < -f'(z) < -f'(1)$ for all $z \in [f(\lambda_f), 1]$, we have that

$$\mathcal{K}_p^n h < b^n (S_p + R)^n h$$

positive function h. Notice that the spectral radius of the positive operator $S_p + R$ is $(-f'(1))^p + 1$. Then, the right–hand side of (4.16) follows.

The claim in [S1.] for the special case $f = g_k$ follows from

$$g_k'(1) = \lambda_{g_k}^{1-2k} = \lambda_k^{1-2k}$$

To prove [S2.], [S3.] we will use the following Proposition

Proposition 4.6. *In the conditions of Proposition 4.2. Assume that ϕ is a positive function λ is a positive number such that*

$$\mathcal{K}_p \phi > \lambda \phi$$

Then,

$$\rho_p > \lambda$$

Note that because the functions are strictly positive, we have $\mathcal{K}_p\phi \geq (\lambda + \delta)\phi$ for some $\delta > 0$. Then, applying this repeatedly, we obtain $\mathcal{K}_p^n \phi \geq (\lambda+\delta)^n\phi$. By Corollary 4.3, we have $\mathcal{K}_p^n\phi = c_p\rho_p^n\psi_p + o((\rho-\delta)^n)$. Since the function ψ_p is strictly positive, we have the desired result. □

Now, we continue with the proof of [S2.], [S3.]

We recall that the operator $\mathcal{K}_p$ has the structure:

$$\mathcal{K}_p h = A^p h \circ f_1 + h \circ f_2 \tag{4.19}$$

where $A(z) = -f' \circ f(\lambda_f z)$, $f_1(z) = \lambda_f z$, $f_2(z) = f(\lambda_f z)$. In particular, $A > 1$.

The strict monotonicity of ρ_p follows from the fact that if $q > p$

$$\begin{aligned} \rho_q \psi_q &= A^q \psi_q \circ f_1 + \psi_q \circ f_2 \\ &> A^p \psi_q \circ f_1 + \psi_q \circ f_2 \\ &= \mathcal{K}_p \psi_q \end{aligned}$$

To prove that $\log(\rho_p)/p$ is strictly decreasing, we will show that for any $0 < p$ and any $\alpha > 1$ we have $\rho_{p\alpha} < \rho_p^\alpha$, which is equivalent to the desired result.

If we raise the eigenvalue equation for $\mathcal{K}_p$ to the α power, we obtain.

$$\begin{aligned} \rho_p^\alpha \psi_p^\alpha &= (A^p \psi_p \circ f_1 + \psi_p \circ f_2)^\alpha \\ &> A^{p\alpha} \Psi_p^\alpha + \psi_p^\alpha \circ f_2 \\ &= \mathcal{K}_{p\alpha} \psi_p^\alpha \end{aligned} \tag{4.20}$$

The inequality above is a consequence of the binomial theorem for fractional powers. Applying Proposition 4.6 we obtain the desired result $\rho_{p\alpha} < \rho_p^\alpha$.

To prove the strict log-convexity of ρ_p, we argue similarly. We will show that for $0 < p < q$ and for $\rho_{(p+q)/2} < \rho_p^{1/2}\rho_q^{1/2}$, which implies strict convexity.

Multiplying the eigenvalue equations satisfied by the spectral radius for $\mathcal{K}_p$ and for $\mathcal{K}_q$ and raising them to the 1/2, we have:

$$\begin{aligned}(\rho_p\rho_q)^{1/2}\psi_p^{1/2}\psi_q^{1/2} &= \big[(A^p\psi_p\circ f_1+\psi_p\circ f_2)\cdot(A^q\psi_q\circ f_1+\psi_q\circ f_2)\big]^{1/2}\\ &> \big(A^{p+q}(\psi_p\psi_q)\circ f_1+(\psi_p\psi_q)\circ f_1\big)^{1/2}\\ &> A^{(p+q)/2}(\psi_p\psi_q)^{1/2}\circ f_1+(\psi_p\psi_q)^{1/2}\circ f_1)^{1/2}\\ &= \rho_{(p+q)/2}(\psi_p\psi_q)^{1/2}\end{aligned}$$

The first inequality comes by expanding the product and ignoring the cross terms, and the second inequality is a consequence of $(1+x)^{1/2} < 1+x^{1/2}$ for $x>0$.

Applying Proposition 4.6, we obtain the desired result. □

□

4.4. **Asymptotic properties of the renormalization.** From Lemma (4.1), we know that $\|T^nf-g_k\|_{D_k}=O(\omega_k^n)$ implies

$$\|\mathcal{K}_{T^nf,p}-\mathcal{K}_{g_k,p}\|=O(\omega_k^n) \tag{4.21}$$

for all $p>0$. Therefore, the asymptotic properties of $\mathcal{K}_{T^nf,p}$ become similar to those of $\mathcal{K}_{g_k,p}$ for large n. In Proposition 4.7 below, we show that the dominant eigenspaces of the Lindeberg–Lyapunov operators, after a large number of renormalizations, align to the dominant eigenspace of the cumulant operator at the Feigenbaum fixed point g_k.

Proposition 4.7. *Let ρ_p be the spectral radius of $\mathcal{K}_p=\mathcal{K}_{g_k,p}$. There exists a neighborhood $\mathcal{U}$ of g_k such that if $f\in\mathcal{W}_s(g_k)\cap\mathcal{U}$ and h is a positive function $h\in H^r$ then,*

$$c^{-1}\rho_p^n\le\mathcal{K}_{T^{n-1}f,p}\cdots\mathcal{K}_{f,p}h(z)\le c\rho_p^n \tag{4.22}$$

for some constant c.

The proof of this Proposition will be done after stating a consequences. A Corollary of (4.22) is the following

Corollary 4.8. *For any $p\ge 0$, let ρ_p be the spectral radius of $\mathcal{K}_p=\mathcal{K}_{g_k,p}$, then:*

1. *There are constants $0<c<C$ such that*
$$c\le\inf_{z\in I}\frac{\Lambda_p^f(\Gamma_n^fz,2^n)}{\{|\lambda_k|^p\rho_p\}^n}\le\sup_{z\in I}\frac{\Lambda_p^f(\Gamma_n^fz,2^n)}{\{|\lambda_k|^p\rho_p\}^n}\le C \tag{4.23}$$
2. *If $f\in\mathcal{W}_s(g_k)$ is close to g_k then, for any $p>2$*
$$\lim_{n\to\infty}\sup_{z\in I}\frac{\Lambda_p^f(\Gamma_n^fz,2^n)}{\{\Lambda_2^f(\Gamma_n^fz,2^n)\}^{p/2}}=0$$

Proof. (1.) From (4.6) and Proposition 4.7 we have that

$$c\rho_p^n|\Gamma_n^f|^p \leq \Lambda_p(\Gamma_n^f z, 2^n) \leq C\rho_p^n|\Gamma_n^f|^p$$

for constants $0 < c < C$. Part (1) follows from the exponential convergence since (4.2) implies that

$$\hat{c}|\lambda_k|^n \leq |\Gamma_n^f| \leq \hat{C}|\lambda_k|^n$$

for some positive constants $\hat{c}$ and $\hat{C}$.

Parts (2) follows from part (1) and Proposition 4.5 since for all $p > 2$, we have $\rho_p < \sqrt{\rho_2^p}$. □

The proof of Proposition 4.7 is based on the following general result (Proposition (4.9)) on convergent sequences of operators on a Banach space. The method that we will use is a standard technique in hyperbolic theory [HP70], where it is used to establish the stability of hyperbolic splittings

Lemma 4.9. *Let X be a Banach space, $\{\mathcal{K}_n\}_{n=1}^\infty$ be a convergent sequence of operators on X such that $\lim_n \|\mathcal{K} - \mathcal{K}_n\| = 0$. Assume that there is a decomposition of $X = E^< \oplus E^>$, and constants $0 < \lambda_- < \lambda_+$ such that $E^{<,>}$ are closed subspaces and*

$$\begin{aligned}
\mathcal{K}\left(E^>\right) &= E^> \\
\mathcal{K}\left(E^<\right) &\subset E^< \\
\operatorname{Spec}\left(\mathcal{K}|_{E^<}\right) &\subset \{z \in \mathbb{C} : |z| < \lambda_-\} \\
\operatorname{Spec}\left(\mathcal{K}^{-1}|_{E^>}\right) &\subset \left\{z \in \mathbb{C} : |z| < \frac{1}{\lambda_+}\right\}.
\end{aligned}$$

There exist $\epsilon > 0$ such that if $\sup_n \|\mathcal{K}_n - \mathcal{K}\| < \epsilon$, then there are sequences of linear operators $\{A_n\}_{n=0}^\infty \subset \mathcal{L}(E^<, E^>)$ and $\{B_n\}_{n=0}^\infty \subset \mathcal{L}(E^>, E^<)$ with $\lim_n \|A_n\| = 0$ and $\lim_n \|B_n\| = 0$, such that the spaces

$$E_n^> = \{x \oplus A_n x : \quad x \in E^<\} \tag{4.24}$$

$$E_n^< = \{x \oplus B_n x : \quad x \in E^<\} \tag{4.25}$$

satisfy $X = E_n^> \oplus E_n^<$ and

$$E_n^> = \mathcal{K}_n \cdots \mathcal{K}_1(E_0^>) \tag{4.26}$$

$$E_n^< = \mathcal{K}_n \cdots \mathcal{K}_1(E_0^<) \tag{4.27}$$

Proof. Since X is the direct sum of invariant closed subspaces with respect to $\mathcal{K}$, using adapted norms [CFdlL03, Appen. A] we can assume without loss of generality that

$$\|\mathcal{K}|_{E^<}\| \leq \lambda_- \tag{4.28}$$

$$\left\|\mathcal{K}^{-1}|_{E^>}\right\| \leq \frac{1}{\lambda_+} \tag{4.29}$$

and that if $x = x^< + x^>$ with $x^< \in E^<$ and $x^> \in E^>$, then $\|x\| = \|x^<\| + \|x^>\|$.

The operators $\mathcal{K}$ and $\mathcal{K}_n$ are represented by the following matrices of operators with respect to the fixed decomposition $X = E^< \oplus E^>$

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}^< & 0 \\ 0 & \mathcal{K}^> \end{pmatrix} \tag{4.30}$$

$$\mathcal{K}_n = \begin{pmatrix} \mathcal{K}^< + \delta_n^{<<} & \delta_n^{<>} \\ \delta_n^{><} & \mathcal{K}^> + \delta_n^{>>} \end{pmatrix} \tag{4.31}$$

where $\|\delta_n^{<<}\| + \|\delta_n^{<>}\| + \|\delta_n^{><}\| + \|\delta_n^{>>}\| \to 0$.

I. We show first the existence of the sequence of transformations $\{A_n\}$ in (4.26).

If use (4.30) to compute the image of a point $x + A_n x$ we obtain that it has its $E^>, E^<$ components are, respectively:

$$(K^> + \delta^{>>})A_n x + \delta^< > x \tag{4.32}$$

$$(K^< + \delta^{<<})x + A_n\delta^> < x \tag{4.33}$$

The invariance condition that $K_n(x + A_n x)$ is in the graph of A_{n+1} is equivalent to saying that applying A_{n+1} to the second row of (4.32), we obtain the first.

Therefore, the invariance condition is equivalent to

$$\begin{aligned} &A_{n+1}[(K^< + \delta^{<<}) + A_n\delta^> <] \\ &= (K^> + \delta^{>>})A_n + \delta^< > \end{aligned}$$

Isolating the first A_n in the RHS, we obtain that the invariance (4.26) for the spaces is equivalent to the following invariance equation for $\{A_n\}$

$$A_n = (\mathcal{K}^> + \delta_n^{>>})^{-1}\left\{A_{n+1}\left(\mathcal{K}^< + \delta_n^{<<}\right) + A_{n+1}\delta_n^{<>}A_n - \delta_n^{><}\right\} \tag{4.34}$$

Notice that if $\{A_n\} \in \mathcal{A}$ satisfies (4.34), then (4.26) holds.

We will show that there is a unique solution $\{A_n\}$ of the invariance equation (4.34) in an appropriate space of sequences. To be more precise, let $\mathcal{A}$ be the space of sequences of operators in $\mathcal{L}(E^<, E^>)$ that converge to zero. Observe that $\|A\| = \sup_n \|A_n\|$ defines a norm on $\mathcal{A}$, which makes it into a Banach space.

Let τ be the operator on $\mathcal{A}$ defined by the right–hand side of equation (4.34). For A and $\widetilde{A}$ in $\mathcal{A}$ we have that

$$\begin{aligned}|\tau(A)_n - \tau(\widetilde{A})_n| \leq |(\mathcal{K}^{>} + \delta_n^{>>})^{-1}(A_{n+1} - \widetilde{A}_{n+1})(\mathcal{K}^{<} + \delta_n^{<<})| + \\ |(\mathcal{K}^{>} + \delta_n^{>>})^{-1}||(A_{n+1} - \widetilde{A}_{n+1})\delta_n^{<>}A_n| + \\ |(\mathcal{K}^{>} + \delta_n^{>>})^{-1}||\widetilde{A}_{n+1}\delta_n^{<>}(A_n - \widetilde{A}_n)|\end{aligned}$$

Let $R \geq \max(\|A\|, \|\widetilde{A}\|)$. Since $\delta_n \to 0$, we can assume with no loss of generality that for a given $\epsilon > 0$, $\sup_n \|\delta_n\| < \epsilon$.

$$\|\tau(A) - \tau(\widetilde{A})\| \leq (\lambda_+^{-1}\lambda_- + \epsilon)\|A - \widetilde{A}\| + 2(\lambda_+^{-1} + \epsilon)\epsilon\|A - \widetilde{A}\|R$$

Since $\lambda_+^{-1}\lambda_- < 1$, we can choose $\epsilon > 0$ small enough so that

$$\begin{aligned}\epsilon + 2R(\lambda_+^{-1}\epsilon)\epsilon &< \tfrac{1-\lambda_+^{-1}\lambda_-}{2} \\ (\lambda_+^{-1}\lambda_- + \epsilon)R + (\lambda_+^{-1}\epsilon)(\epsilon R^2 + \epsilon) &< R\end{aligned}$$

we get that τ is a contraction on the ball $\bar{B}(0; R) \subset \mathcal{A}$. Therefore, there exists a unique solution, A, to the invariance equation (4.34).

II. An argument very similar to the one above shows that (4.27) holds if and only if $\{B_n\} \subset \mathcal{L}(E^{>}, E^{<})$ satisfies the following invariance equation

$$B_{n+1} = \{(\mathcal{K}^{<} + \delta_n^{<<})B_n + \delta_n^{<<} - B_{n+1}\delta_n^{><}B_n\}(\mathcal{K}^{>} + \delta_n^{>>})^{-1} \quad (4.35)$$

Let $\mathcal{B}$ be the space of sequences in $\mathcal{L}(E^{>}, E^{<})$ that converge to zero with norm $\|B\| = \sup_n \|B_n\|$, and let $\tilde{\tau}$ be the operator on $\mathcal{B}$ defined by the right hand side of (4.35). An idea similar to the one used in part (I), shows that $\tilde{\tau}$ is a contraction on an appropriate closed ball. Therefore, there is a unique solution to the invariance equation (4.35). □

Remark 4.2. $\|A_n\|$ and $\|B_n\|$ are measures of distance between the spaces $E_n^{<,>}$ and $E^{<,>}$. Lemma 4.9 states that these respective spaces align in the limit.

The next Corollary gives control on the growth of the products of operators $\mathcal{K}_n$.

Corollary 4.10. *Under the hypothesis of Proposition 4.9, let us assume that dim*$(E^{>}) = 1$. *Then:*

(1) *For any* $v \in X$

$$\prod_{j=1}^{n}(\lambda_+ - \epsilon_j)\,\|v\| \leq \|\mathcal{K}_n \cdots \mathcal{K}_0 v\| \leq \prod_{j=1}^{n}(\lambda_+ + \epsilon_j)\,\|v\| \quad (4.36)$$

where $\epsilon_n = \epsilon_n(\|\mathcal{K}_j - \mathcal{K}\|)$, *is such that* $\lim_{t\to 0}\epsilon_n(t) = 0$.

(2) *If* $\|\mathcal{K}_n - \mathcal{K}\| = O(\omega^n)$ *with* $0 < \omega < 1$ *, then we have that*

$$c^{-1}\lambda_+^n\|v\| \leq \|\mathcal{K}_n \cdots \mathcal{K}_0 v\| \leq c\lambda_+^n\,\|v\| \tag{4.37}$$

for some constant $c > 0$*.*

Proof. Using adapted norms, we can assume that (4.28), (4.29) hold, and that if $x = x^{<}+x^{>}$ with $x^{<} \in E^{<}$ and $x^{>} \in E^{>}$, then $\|x\| = \|x^{<}\|+\|x^{>}\|$. For $s =$"$<$", "$>$", let $E_0^s = E^s$ and define $E_{n+1}^s = \mathcal{K}_n(E_n^s)$.
(1) Let $v \in E_n^{<}$, then by Lemma 4.9 there exists a unique $w \in E^{<}$ such that

$$v = w + A_n w$$

Since $\|A_n\|$, $\|B_n\| \to 0$, we may further assume that

$$\begin{aligned} \sup_n \|A_n\| &\leq \frac{\delta}{2\|\mathcal{K}\|} \\ \sup_n \|B_n\| &\leq \frac{\delta}{2\|\mathcal{K}\|} \\ \|\mathcal{K}_n - \mathcal{K}\| &\leq \frac{\delta}{2} \end{aligned}$$

for $\delta > 0$ small. Since $\|\mathcal{K}w\| < \lambda_-\|w\| \leq \lambda_+\|v\|$, from triangle inequality we have

$$\begin{aligned} \|\mathcal{K}_n v\| &\leq \|(\mathcal{K}_n - \mathcal{K})v\| + \|\mathcal{K}v\| \\ &\leq \frac{\delta}{2}\|v\| + \|\mathcal{K}(w + A_n w)\| \\ &\leq (\lambda_+ + \delta)\|v\|. \end{aligned}$$

Let $v \in E_n^{>}$ then, by Lemma 4.9 there is unique $z \in E^{>}$ such that

$$v = B_n z + z.$$

Therefore, $\|v\| \leq \frac{\delta}{2\lambda_+}\|z\| + \|z\|$, which is equivalent to

$$\|z\| \geq \frac{\|v\|}{1+\delta}$$

From the triangle inequality, we have that

$$\begin{aligned} \|\mathcal{K}_n v\| &\geq \|\mathcal{K}v\| - \|(\mathcal{K}_n - \mathcal{K})v\| \\ &> \|\mathcal{K}(B_n z + z)\| - \frac{\delta}{2}\|v\| \\ &> \|\mathcal{K}z\| - \frac{\delta}{2}\|v\| \\ &> (\lambda_+ - \delta)\|v\| \end{aligned}$$

Combining these results, we obtain inequality (4.36).

(2) In the case of exponential convergence, (4.37) follows from an elementary result on sums and products, namely: if $\lim_n \sup_j |c_{nj}| = 0$, and $\sup_n \sum_{j=1}^{n} |c_{n,j}| < \infty$, then

$$\prod_j (1 + c_{nj}) \to e^c \quad \text{if} \quad \sum_j c_{nj} \to c$$

See [Dur05, p. 78]. □

Proof of Proposition 4.7. By Proposition 4.2 we have that that the spectral radius ρ_p of $\mathcal{K}_p$ is a simple eigenvalue. Then, (4.22) follows from Corollary 4.10 and the exponential convergence, see (4.21), of the Lindeberg–Lyapunov operators. □

Remark 4.3. The estimates (4.23) for $p = 2$ are obtained in [VSK84] using the Thermodynamic formalism.

4.5. **Cumulant operators.** In Section 3.5 we introduced the concept of cumulants, see (3.41). One important fact is the cumulants of random variables, normalized to have variance one, provide an empirical measures of resemblance to Gaussian. In this section, we make a connection between the spectral properties of the Lindeberg–Lyapunov operators studied in Section 4.3 and the cumulants of the propagation of noise.

Recall that the functions $f \in \mathcal{W}_s(g_k)$ are defined in a domain D_k, see R2 in section 4.1.

Definition 4.3. For each $f \in \mathcal{W}_s(g)$, the family of cumulant operators $\{\tilde{\mathcal{K}}_{f,p} : p \in \mathbb{N}\}$ acting on the space $H^r(\bar{D}_k)$ are defined by

$$\tilde{\mathcal{K}}_{f,p} h(z) = \frac{1}{(\lambda_f)^p} \left[(f' \circ f(\lambda_f z))^p \, h(\lambda_f z) + h(f(\lambda_f z)) \right] \tag{4.38}$$

By an argument identical to that given in Lemma 4.1, it follows from the exponential convergence of $T^n f$ that

Lemma 4.11. *Let $f \in \mathcal{W}_s(g_k)$ close to g_k. For each nonnegative integer p, $\widetilde{\mathcal{K}}_{f,p}$ is a compact operator on $H^r(\overline{D}_k)$. Furthermore, and $\widetilde{\mathcal{K}}_{T^{n-1}f,p}$ converges to $\widetilde{\mathcal{K}}_{g_k,p}$ exponentially fast.*

Notice from Definition 4.3 that

$$\begin{aligned} \widetilde{\mathcal{K}}_{f,p} &= \mathcal{K}_{f,p} \qquad p \text{ even} \\ \|\widetilde{\mathcal{K}}_{f,p}\| &\leq \|\mathcal{K}_{f,p}\| \qquad p \text{ odd} \end{aligned} \tag{4.39}$$

Denote the spectral radius of $\widetilde{\mathcal{K}}_{f,p}$ and $\mathcal{K}_{f,p}$ by $\tilde{\rho}_{f,p}$ and $\rho_{f,p}$ respectively. Then, from (4.39) we have that $\tilde{\rho}_{f,p} = \rho_{f,p}$ for all p even, and for all integers $p > 2$

$$\tilde{\rho}_{f,p} \le \rho_{f,p} \le (\rho_{f,2})^{p/2} \tag{4.40}$$

Let $f \in \mathcal{W}_s(g_k)$ be fixed, and recall from Section 3.5 the cumulant functional $\widetilde{\Lambda}$. For $x \in I$, let

$$\tilde{k}_{n,p}(x) = \Gamma_n^{-p}\widetilde{\Lambda}_p(\Gamma_n x, 2^n)$$

Then, from (3.44), we have that

$$\tilde{k}_{n,p}(s)(x) = \widetilde{\mathcal{K}}_{T^{n-1}f,p} \cdots \widetilde{\mathcal{K}}_{f,p}\mathbf{1}(x) \tag{4.41}$$

We have that from Corollary 4.10 that for very $h \in H^r(\overline{D}_k)$ there is a constant c such that

$$c^{-1}\tilde{\rho}_p^n \le \|\widetilde{\mathcal{K}}_{T^{n-1}f,p} \cdots \widetilde{\mathcal{K}}_{f,p}h\| \le c\tilde{\rho}_p^n$$

Recalling the notation $K_p[\xi]$ for the p–th order cumulant of a random variable ξ, we have from (4.40) that

$$K_p\left[\frac{L_{2^n}(\Gamma_n^f x)}{\sqrt{\operatorname{var}[L_{2^n}(\Gamma_n^f x)]}}\right] \asymp \left(\frac{\tilde{\rho}_p}{\tilde{\rho}_2^{p/2}}\right)^n \frac{h_{g_k,p}(x)}{(h_{g_k,2}(x))^{p/2}} \to 0 \tag{4.42}$$

as $n \to \infty$, where $h_{g_k,p}$ is a positive eigenfunction of $\mathcal{K}_{g_k,p}$. From the compactness of the cumulant and Lindeberg–Lyapunov operators, the asymptotic expansions (4.42) are improved by adding more terms that involve eigenfunctions of smaller eigenvalues [FdlL06]. That is, we start from the standard asymptotic expansion of powers of a linear compact operator:

$$\mathcal{K}_{g_k,p}^n u(z) = \rho_p^n h_{g_k,p} c(u) + \sum_{j=1}^{N} \bar{\mu}_j^n c_j(u)\Psi_j(z) + O(\mu_{N+1}^n) \tag{4.43}$$

where $\bar{\mu}_j$ are the Jordan blocks associated to the eigenvalues μ_j (in decreasing order of size), $c_j(u)$ are the projections on this space and $\Psi_j(z)$ is a basis for the eigenspace.

For the nonlinear renormalization operator, the theory of [FdlL06] shows that there are very similar expansions for the asymptotic noise. The expansion involves not only powers of the eigenvalues of $\mathcal{K}$ but also powers of products of eigenvalues.

These asymptotic approximations are very reminiscent to the higher order asymptotic expansions in the central limit theorem, such as Edgeworth expansions [Fel71, p. 535]. (The asymptotic expansions (4.43) corresponds to Edgeworth expansions of 2^n terms). It seems possible that one could also prove Edgeworth expansions for the asymptotics of the scaling limits but it looks like the powers appearing in the expansions will be very different form the standard semi-integer powers and will be related to the spectrum of some of the operators giving the renormalization of the cumulants.

4.5.1. *Numerical conjectures on spectral properties of the cumulant operators and the Lindeberg–Lyapunov operators.* If p is even, we know from Proposition 4.2 that $\rho_{f,p}$ is a that single eigenvalue of $\widetilde{K}_{f,p} = \mathcal{K}_{f,p}$.

If p is odd, we have by compactness that $\widetilde{\mathcal{K}}_{f,p}$ has an eigenvalue $\tilde{\mu}_{f,p} \in \{z \in \mathbb{C} : |z| = \tilde{\rho}_{f,p}\}$.

In the quadratic case ($k = 1$), numerical computations [DEdlL06] suggest that

- C1. If $\mathrm{Spec}(\mathcal{K}_{g_1,p}) \setminus \{0\} = \{\mu_{n,p}\}_n$ with $|\mu_{n,p}| \leq |\mu_{n-1,p}|$, then $\mu_{n,p} \sim \lambda_1^n$ (Here, $u_n \sim v_n$ means that $\lim_n u_n/v_n = 1$).
- C2. $\mathrm{Spec}(\mathcal{K}_{g_1,p}) \setminus \{0\}$ is asymptotically real, that is: $\mu_n \in \mathbb{R}$ for all n large enough.

For all p odd

- C3. $\tilde{\rho}_{g_1,p}$ is an eigenvalue of $\widetilde{\mathcal{K}}_{g_1,p}$. The rest of $\mathrm{Spec}(\widetilde{\mathcal{K}}_{g_1,p})$ is contained in the interior of the ball of radius $\tilde{\rho}_{g_1,p}$.
- C4. $\lambda_1^{-2p} \sim \rho_{g_1,p} \sim \rho_{g_1,p} \quad \text{as } p \to \infty$.
- C5. $\mathrm{Spec}(\widetilde{\mathcal{K}}_{g_1,p}) \setminus \{0\} = \{\tilde{\mu}_{n,p}\}$, where $|\tilde{\mu}_{n,p}| \leq |\nu_{n-1,p}|$ is asymptotically real.
- C6. $\tilde{\mu}_{n,p} \sim \mu_{n,p} \sim \lambda_1^n$.

Conjectures C1–C6 are related to conjectures on behavior of the Perron-Frobenius operators of real analytic expanding maps, see [Bal00, May80, Rug94]. The case $p = 1$ corresponds to the problem of reality of the spectrum of the linearized period doubling operator. It is observed in [CCR90] that the spectrum in this special case appears to be real, and that all the eigenvalues behave as λ_1^n.

4.6. **Proof of the central limit theorem for systems in a domain of universality (Theorem 2.3).**

4.6.1. *Introduction.* In this section, we prove the central limit theorem (Theorem 2.3) for maps f in the domain of universality $\mathcal{W}_s(g_k)$.

The first step is to prove a central limit theorem and Berry–Esseen estimates for $x_n(x, n)$ with initial condition x in the orbit of 0.

We will verify the Lyapunov condition (2.9) along the whole sequence of positive integers. Then, we apply the result of Theorems 2.1 and 2.2.

The main observation is that the renormalization theory developed in Section 4.2 gives control on $\Lambda_s(x, 2^n)$ for x sufficiently small (roughly

$|x| \leq |\Gamma_{n+1}|$), see Corollary 4.8. Then, the main tool to obtain control of the effect of the noise on a segment of orbit, will be to decompose the segment into pieces whose lengths are powers of 2 and chosen so that the renormalization theory applies.

This decomposition will be accomplished in Section 4.6.2. The main conclusion of this section will be that this decomposition is possible and that, the final effect over the whole interval is the sum of the binary blocks – which are understandable by renormalization affected by weights. See (4.45). The weights measure how the effect of the dyadic block propagates to the end of the interval.

The effect of the dyadic blocks in the final result depends on somewhat delicate arguments. This follows from the the following heuristic argument.

Note that when an orbit passes though zero, the derivatives become very small (hence the effect of the noise up to this time is erased). Hence, when an orbit includes extremely close passages through zero, the next iterations tend to behave as if they were short orbits (and, hence the noise is far from Gaussian). On the other hand, the close passage to zero are unavoidable (and they are the basis of the renormalization).

Hence, our final result will be some quantitative estimates that measure the effect of the returns. The main ingredients is how close the returns are and how long do we need to wait for the following close return (this is closely related to the gaps in the binary decomposition). The combination of these properties depends on quantitative properties of the fixed point. See Theorem 4.15.

The fact that the gaps in the binary decomposition play a role is clearly illustrated in the numerical calculations in [DEdlL06].

4.6.2. *The binary decomposition.* Given a number $n \in \mathbb{N}$, we will use the binary expansion

$$n = 2^{m_0} + \cdots + 2^{m_{r_n}} \tag{4.44}$$

where $m_0 > m_1 > \ldots > m_{r_n} \geq 0$.

If necessary, we will use $m_j(n)$ to emphasize the dependence on n of the power m_j in (4.44).

We will denote the integer part of any number u by $[u]$. Notice that that $m_0(n) = [\log_2(n)]$, and $r_n \leq m_0$.

Using (2.5) several times we have for all $p \geq 0$ and for any $x \in [-1, 1]$ that

$$\Lambda_p(x,n) = \sum_{j=0}^{r_n} |\Psi_{j,n}(x)|^p \, \Lambda_p\left(f^{n-2^{m_j}-\dots-2^{m_r}}(x), 2^{m_j}\right) \tag{4.45}$$

$$\Psi_{j,n}(x) \quad = \quad \left(f^{n-2^{m_0}-\dots-2^{m_j}}\right)' \circ \left(f^{2^{m_j}+\dots 2^{m_0}}\right)(x) \tag{4.46}$$

When the initial value $x_0 = x$ of $x_n = f(x_{n-1})$ satisfies $|x| \le |\Gamma_{m_0+1}|$, the points

$$\upsilon_j = f^{2^{m_j}}(\upsilon_{j-1}), \quad \upsilon_{-1} = x \tag{4.47}$$

where $m_0 > \dots m_{r_n} \ge 0$, are return points to a small neighborhood of 0 in the domain of renormalization.

The decomposition in dyadic scales (4.45) will be our main tool for the analysis of the growth of (4.45).

We will show in Lemma 4.13 that the size of the weights $\Psi_{j,n}(x)$ are controlled using the renormalization. Then, using Corollary 4.8 we will get control over the size of the $\Lambda_p(\upsilon_{j-1}, 2^{m_j})$. Combining these results, the Lyapunov condition 2.9 will follow for the orbit of zero, thus establishing the central limit theorem.

4.6.3. *Estimating the weights.* The main tasks of this section will be to estimate the weights $\Psi_{n,j}$ in (4.45) and to show that they do not affect too much the conclusions of renormalization.

For $f \in \mathcal{W}_s(g_k)$ fixed, recall the notation $\lambda_f = f(1)$, and for each nonnegative integer m, $\Gamma_m = f^{2^m}(0)$. Then

$$\lambda_{T^n f} = \frac{\Gamma_{n+1}}{\Gamma_n}$$

If necessary, we will use Γ^f_n to emphasize the role of f. For the Feigenbaum fixed point g_k of order $2k$, we use $\lambda_k = \lambda_{g_k}$. We have that

$$\Gamma^{g_k}_n = \lambda^n_k$$

Recall that the map g_k has a series expansion of the form

$$g_k(x) = 1 + \sum_{l=1}^{\infty} b_l x^{2kl}$$

and that $g'_k(1) = \lambda_k^{1-2k}$.

Lemma 4.12. *Let h_k be the function over the interval $[-1, 1]$ defined by*

$$h_k(x) = \frac{g'_k(x)}{x^{2k-1}} \tag{4.48}$$

for $x \neq 0$ and $h_k(0) = 2k\, b_1$. Then h is a negative even function, increasing on $[0,1]$.

Proof. From the fixed point equation

$$g_k(x) = \frac{1}{\lambda_k} g_k(g_k(\lambda_k x))$$

we have that

$$h_k(x) = \frac{g_k'(g_k(\lambda_k x))}{g_k'(1)} h(\lambda_k x) \tag{4.49}$$

Repeated applications on (4.49) gives

$$h_k(x) = h_k(\lambda_k^{n+1} x) \prod_{j=1}^{n} \frac{g_k'(g_k(\lambda_k^j x))}{g_k'(1)} \tag{4.50}$$

Notice that $h_k(\lambda_k^{n+1}x) \to h_k(0)$ uniformly on $[-1,1]$ as $n \to \infty$. Furthermore, the factors in the product (4.50) converge exponentially to 1. Therefore, we can pass to the limit as $n \to \infty$ in (4.50) and get

$$h_k(x) = h_k(0) \prod_{n=1}^{\infty} \frac{g_k'(g_k(\lambda_k^n x))}{g_k'(1)}$$

Each term in the product is a nonnegative decreasing function on $[0,1]$ therefore, h_k is increasing on $[0,1]$. □

In the rest of this section, we assume that the domain of universality $\mathcal{W}_s(g_k)$ is fixed. From the exponential convergence of $T^m f$ to g_k, we can choose a small neighborhood $\mathcal{U}$ of g_k so that of

$$T(\mathcal{W}_s(g_k) \cap \mathcal{U}) \subset \mathcal{W}_s(g_k) \cap \mathcal{U} \tag{4.51}$$

The following result gives control of derivatives for points close to the origin. Some of the bounds for the derivatives in Lemma 4.13 are done in [VSK84] using the thermodynamic formalism.

Lemma 4.13. *Assume that $f \in \mathcal{W}_s(g_k) \cap \mathcal{U}$ and as before, let $h_k(x) = g_k'(x)x^{1-2k}$. For each finite decreasing sequence of nonnegative integers $\{m_j\}_{j=0}^r$, and $|x| \leq |\Gamma_{m_0+1}|$, let $\{\upsilon_j\}_{j=-1}^r$ with $\upsilon_{-1} = x$ be as in 4.47. There exists $0 \leq \epsilon \ll 1$, depending only on $\mathcal{U}$, such that if*

$$\begin{aligned} G &= g_k(\lambda_k) - \epsilon & (4.52)\\ c &= |h(\lambda_k)| - \epsilon & (4.53)\\ d &= |h_k(0)| + \epsilon & (4.54) \end{aligned}$$

then $1 < c\, G^{2k-1} < d$, and

(1) *For each* $0 \le j \le r$,

$$G|\Gamma_{m_j}| \le |\upsilon_j| \le |\Gamma_{m_j}| \tag{4.55}$$

(2) *For* $1 \le j \le r$

$$c\, G^{2k-1}\left|\frac{\Gamma_{m_{j-1}}}{\Gamma_{m_j}}\right|^{2k-1} \le \left|\left(f^{2^{m_j}}\right)'(\upsilon_{j-1})\right| \le d\left|\frac{\Gamma_{m_{j-1}}}{\Gamma_{m_j}}\right|^{2k-1} \tag{4.56}$$

Proof. By (4.51), we can assume without loss of generality that

$$\sup_{m\in\mathbb{N}} |\lambda_{T^m f} - \lambda_k| \le \epsilon \tag{4.57}$$

$$|T^m f(\lambda_k) - g_k(\lambda_k)| \le \epsilon \tag{4.58}$$

for some $0 \le \epsilon \ll 1$. Since $g_k(\lambda_k) > |\lambda_k|$ and

$$|g_k'(\lambda_k)| = \left|\frac{g_k'(1)}{g_k'(g_k(\lambda_k))}\right| > 1,$$

by choosing $0 \le \epsilon \ll 1$ smaller if necessary, we get $1 < c\, G^{2k-1} < d$.

The exponential convergence implies that there is $0 \le \delta \ll 1$ such that

$$\sup_{m\in\mathbb{N}} \left|\Gamma_{m+1}\Gamma_m^{-1}\right| \le |\lambda_{T^m f}| < |\lambda_k| + \delta \tag{4.59}$$

We know that or all $m \in \mathbb{N}$, $T^m f$ is even, decreasing in $[0,1]$, and $\sup_{\upsilon\in I} |T^m f(\upsilon)| \le 1$. Shrinking $\mathcal{U}$ if necessary, from (4.58), we can assume without loss of generality that

$$\{g_k(\lambda_k) - \epsilon\} \le |T^m f(x)| \le 1 \tag{4.60}$$

for all $x \in [\lambda_k - \delta, |\lambda_k| + \delta]$.
(1) For $j = 0$ we have that

$$\upsilon_0 = f^{2^{m_0}}(\upsilon_{-1}) = \Gamma_{m_0} T^{m_0} f(\Gamma_{m_0}^{-1}\upsilon_{-1})$$

It follows from (4.57) that

$$|\Gamma_{m_0}^{-1}\upsilon_{-1}| < |\lambda_k| + \delta$$

Then, by (4.60) we get

$$\{g_k(\lambda_k) - \epsilon\}|\Gamma_{m_0}| \le |\upsilon_0| \le |\Gamma_{m_0}|$$

We continue by induction on j. Assume (4.55) holds for all numbers $0 \le i \le j$, where $j < r$. For $j+1$ we have that

$$\upsilon_{j+1} = f^{2^{m_{j+1}}}(\upsilon_j) = \Gamma_{m_{j+1}} T^{m_{j+1}} f\left(\Gamma_{m_{j+1}}^{-1}\upsilon_j\right)$$

From (4.57) and (4.59), it follows that

$$\{g_k(\lambda_k) - \epsilon\}|\Gamma_{m_{j+1}}| \le |\upsilon_{j+1}| \le |\Gamma_{m_{j+1}}|$$

(2) Shrinking $\mathcal{U}$ if necessary, we can assume that

$$\sup_{x\in I}\left|\frac{f'(x)}{x^{2k-1}} - h_k(x)\right| < \epsilon \tag{4.61}$$

for all $f \in \mathcal{W}_s(g_k)\cap\mathcal{U}$.

We know from Lemma 4.12 that

$$|h_k(\lambda_k)||x|^{2k-1} \le |g_k'(x)| \le |h_k(0)|$$

for all $|x| \le |\lambda_k|$. Hence, for each $m \in \mathbb{N}$, (4.61) and the exponential convergence of $T^m f$ imply that

$$(|h(\lambda_k)| - \epsilon)\,|x|^{2k-1} \le \left|(T^m f)'(x)\right| \le (|h_k(0)| + \epsilon)\,|x|^{2k-1} \tag{4.62}$$

for $x \in [f(\lambda_f)|\Gamma_m|, |\Gamma_m|]$.

Therefore, from part (1) of the present Lemma, (4.62), and the identity

$$\left(f^{2^{m_{j+1}}}\right)'(\upsilon_j) = (T^{m_{j+1}}f)'\left(\Gamma^{-1}_{m_{j+1}}\upsilon_j\right)$$

we have that

$$\begin{aligned}\left|\left(f^{2^{m_{j+1}}}\right)'(\upsilon_j)\right| &\ge\ c|\Gamma^{-1}_{m_{j+1}}\upsilon_j|^{2k-1} \ge\ c\,G^{2k-1}\left|\frac{\Gamma_{m_j}}{\Gamma_{m_{j+1}}}\right|^{2k-1}\\ \left|\left(f^{2^{m_{j+1}}}\right)'(\upsilon_j)\right| &\le\ d|\Gamma^{-1}_{m_{j+1}}\upsilon_j|^{2k-1} \le\ d\left|\frac{\Gamma_{m_j}}{\Gamma_{m_{j+1}}}\right|^{2k-1}\end{aligned}$$

for all $0 \le j \le r-1$. □

4.6.4. *Lyapunov condition for all times for functions in the domain of universality.* In this section, we show now that the Lyapunov condition 2.9 holds along the whole sequence of positive integers for orbits starting at any point of the form $x = f^l(0)$, $l \in \mathbb{N}$, with $f \in \mathcal{W}_s(g_k)\cap\mathcal{U}$.

Proposition 4.14. *Let $f \in \mathcal{W}_s(g_k)\cap\mathcal{U}$ be fixed, and for each $n \in \mathbb{N}$, let $m_0(n) = [\log_2(n)]$, r_n+1 be the number of ones in the binary expansion of n, and $I_{m_0(n)}$ be the interval $\left[-|\Gamma_{m_0(n)+1}|, |\Gamma_{m_0(n)+1}|\right]$.*

(1) *Let c, G, and d be as in Lemma 4.13. For any real number $p > 0$, there are constants a_p, b_p such that*

$$a_p\left(\frac{(c\,G^{2k-1})^{r_n}}{\lambda_k^{(2k-1)m_{r_n}}}\right)^p \le \frac{\Lambda_p(x,n)}{(\lambda_k^{2kp}\rho_p)^{m_0(n)}} \le b_p\left(\frac{d^{r_n}}{\lambda_k^{(2k-1)m_{r_n}}}\right)^p \tag{4.63}$$

for all $x \in I_{m_0(n)}$.

(2) *For any real number $p > 2$, we have that*

$$\lim_{n\to\infty}\sup_{x\in I_{m_0(n)}}\frac{\Lambda_p(x,n)}{\{\Lambda_2(x,n)\}^{p/2}} = 0 \tag{4.64}$$

Proof. For each $m \in \mathbb{N}$, let $I_m = [-|\Gamma_{m+1}|, |\Gamma_{m+1}|]$. Let n be fixed for the moment, and consider the sequence of powers, $\{m_j\}_{j=0}^{r_n}$, in the binary expansion of n, see (4.44).

Let $x \in I_{m_0}$ be fixed, and consider the sequence of returns $\{\upsilon_j\}_{j=-1}^{r_n}$ with $\upsilon_{-1} = x$ as in (4.47).

From (4.45), we have for each $p \geq 0$ that

$$\Lambda_p(x,n) = \sum_{j=0}^{r_n} |\Psi_{j,n}(x)|^p \Lambda_p(\upsilon_{j-1}, 2^{m_j}), \tag{4.65}$$

(1) From Lemma 4.13, we have that

$$\left(c\,G^{2k-1}\right)^{r_n-j+1} \leq |\Psi_{j,n}(x)| \left|\frac{\Gamma_{m_{r_n}}}{\Gamma_{m_j}}\right|^{2k-1} \leq d^{r_n-j+1}$$

where $1 < c\,G^{2k-1} < d$. Since

$$|\Gamma_{m+1}\Gamma_m^{-1} - \lambda_k| \leq C\omega_k^m$$

with $0 < \omega_k < 1$, there is a constant a such that

$$a^{-1}(c\,G^{2k-1})^{r_n-j+1} \leq |\Psi_{j,n}(x)||\lambda_k|^{(m_{r_n}-m_j)(2k-1)} \leq ad^{r_n-j+1} \tag{4.66}$$

for all $x \in I_{m_0(n)}$. By Corollary 4.8, there are positive constants $\hat{c}_p$, $\hat{C}_p$ such that

$$\hat{c}_p \leq \Lambda_p(\nu_{j-1}, 2^{m_j})(|\lambda_k|^p \rho_p)^{-m_j} \leq \hat{C}_p \tag{4.67}$$

Using (4.66) and (4.67), we obtain estimates for the size of each term in (4.65). Since $\lambda_k^{2kp}\rho_p > 1$, the estimate (4.65) follows by noticing that $m_0 - m_j \geq j$.

(2) Since $0 \leq r_n \leq m_0 = [\log_2(n)]$, (4.63) implies that there are constants $0 < C_p < D_p$ such that for all n large enough

$$C_p\{\lambda_k^{2kp}\rho_p\}^{m_0} \leq \Lambda_p(x,n) \leq D_p\{d^p\rho_p\}^{m_0} \tag{4.68}$$

From Proposition 4.5, we know that $\lambda_k^{2kp}\rho_p > 1$ for all $p > 0$, therefore

$$\lim_{n\to\infty} \inf_{x\in I_{m_0}(n)} \Lambda_p(x,n) = \infty \tag{4.69}$$

Assume that $p > 2$. By Corollary 4.8, we have that for any given $\epsilon > 0$, there is an integer M_ϵ such that if $m \geq M_\epsilon$, then

$$\Lambda_p(x, 2^m) < \epsilon(\Lambda_2(x, 2^m))^{p/2}$$

for all $|x| \leq |\Gamma_m|$.

Denote by $\mathcal{S}_{r_n} = \{0, \ldots, r_n\}$ and let

$$\mathcal{A}_n^1 = \left\{ j \in \mathcal{S}_{r_n} : \sup_{u \in I_{m_j}} \frac{\Lambda_p\left(u, 2^{m_j}\right)}{\left(\Lambda_2\left(u, 2^{m_j}\right)\right)^{p/2}} \leq \epsilon \right\}$$

and $\mathcal{A}_n^2 = \mathcal{S}_{r_n} \setminus \mathcal{A}_n^1$. For each $x \in I_{m_0}$ and $s = 2, p$, we split $\Lambda_s(x, n)$ as

$$\Lambda_s(x, n) = H_s(x, \mathcal{A}_n^1) + H_s(x, \mathcal{A}_n^2),$$

where $H_s(x, \mathcal{A}_n^i)$, $i = 1, 2$ is defined by

$$H_s(x, \mathcal{A}_n^i) = \sum_{j \in \mathcal{A}_n^i} |\Psi_{j,n}(x)|^s \Lambda_s(\upsilon_{j-1}, 2^{m_j})$$

Notice that $H_p(x, \mathcal{A}_n^2)$ is the sum of bounded number of terms (at most M_ϵ) all of which are bounded. Hence,

$$H_p(x, \mathcal{A}_n^2) \leq C_\epsilon \tag{4.70}$$

for some constant $C_\epsilon > 0$.

In addition, notice that

$$H_p(x, \mathcal{A}_n^1) \leq \epsilon (H_2(x, \mathcal{A}_n^1))^{p/2} \leq \epsilon (\Lambda_2(x, n))^{p/2} \tag{4.71}$$

Combining (4.70), (4.71), we get

$$\begin{aligned} \frac{\Lambda_p(x, n)}{(\Lambda_2(x, n))^{p/2}} &\leq \frac{H_p(x, \mathcal{A}_n^1) + H_p(x, \mathcal{A}_n^2)}{(\Lambda_2(x, n))^{p/2}} \\ &\leq \epsilon + \frac{C_\epsilon}{(\Lambda_2(x, n))^{p/2}} \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain from (4.69), that

$$\limsup_{n \to \infty} \sup_{x \in I_{m_0(n)}} \frac{\Lambda_p(x, n)}{\{\Lambda_2(x, n)\}^{p/2}} \leq \epsilon.$$

By letting $\epsilon \to 0$, we get (4.64). □

For any $l \in \mathbb{N}$ fixed and for all $m > [\log_2(l)]$, denote by $I_m = [-|\Gamma_{m+1}|, |\Gamma_{m+1}|]$. We will show that the Lyapunov condition 2.9 holds for orbits starting at any point on $\{f^l(0)\}_{l \in \mathbb{N}}$.

Theorem 4.15. *Let $f \in \mathcal{W}_s(g_k) \cap \mathcal{U}$ be fixed, and $l \in \mathbb{N}$. For all $p > 2$,*

$$\lim_{n \to \infty} \sup_{x \in I_{m_0(n+l)}} \frac{\Lambda_p(f^l(x), n)}{\left(\Lambda_2(f^l(x), n)\right)^{p/2}} = 0 \tag{4.72}$$

The limit (4.72) is not uniform in l.

Proof. The idea of the proof is to use the dyadic decomposition (4.45) and show that for any fixed integer $l \geq 1$, the Lyapunov functionals $\Lambda_p(f^l(x), n)$ and $\Lambda_p(x, n+l)$ for $x \in I_{m_0(n+l)}$ are very similar for large values of n (see (4.77)).

Let $n \in \mathbb{N}$ be a large integer and decompose $n+l$ in its binary decomposition:

$$n + l = 2^{m_0} + \cdots 2^{m_r}$$

where $m_0 = m_0(n+l) = [\log_2(n+l)]$. Consider $x \in I_{m_0(n+l)+1}$, and observe that

$$\Lambda_p(f^l(x), n) = \sum_{j=l+1}^{n+l} \left|\left(f^{n+l-j}\right) \circ f^j(x)\right|^p$$

Using $\{m_j\}$, consider the sequence of returns $\{\upsilon\}_j$ with $\upsilon_{-1} = x$ (see (4.47)). It follows that

$$\Lambda_p(f^l(x), n) = |\Psi_{n+l,0}(x)|^p A_p(x, 2^{m_0}) + \sum_{i=1}^{r} |\Psi_{n+l,i}(x)\|^p \Lambda_p(\upsilon_{i-1}, 2^{m_i}) \tag{4.73}$$

where

$$\Lambda_p(x, 2^{m_0}) - A_p(x, 2^{m_0}) = \sum_{j=1}^{l} \left|\left(f^{2^{m_0}-j}\right) \circ f^j(x)\right|^p \tag{4.74}$$

We will show that the difference (4.74) is small compared to $\Lambda_p(x, 2^{m_0})$. Indeed, since

$$\left(f^{2^{m_0}-j}\right)' \circ f^j(x) = \frac{\left(f^{2^{m_0}}\right)'(x)}{\left(f^j\right)'(x)},$$

it follows from Lemma 4.13 that

$$\frac{c_j}{|\lambda_k|^{(2k-1)m_0}} \leq \left|\left(f^{2^{m_0}-j}\right)' \circ f^j(x)\right| \leq \frac{d_j}{|\lambda_k|^{(2k-1)m_0}} \tag{4.75}$$

where c_j and d_j depend only on j. Since(4.74) contains l terms, (4.17) and Corollary 4.8 imply that

$$\frac{\Lambda_p(x, 2^{m_0}) - A_p(x, 2^{m_0})}{\Lambda_p(x, 2^{m_0})} \leq C_{l,p} \frac{1}{(\lambda_k^{2kp} \rho_p)^{m_0}} \to 0 \tag{4.76}$$

as $m_0 \to \infty$. Therefore

$$\lim_{n \to} \sup_{x \in I_{m_0(n+l)}} \frac{\Lambda_p(f^l(x), n)}{\Lambda_p(x, n+l)} = 1 \tag{4.77}$$

The limit (4.72) follows from Proposition 4.14 and (4.77). □

We will use the result of Theorem 4.15 together with the results of Theorems 2.1 and 2.2 to prove the central limit theorem for systems in the domain of universality $\mathcal{W}_s(g_k)$.

4.6.5. *Proof of Theorem 2.3.* By Theorem 2.1 and Theorem 4.15, it suffices to choose σ_n that satisfies (2.10).

Recall that for any integer n, $m_0(n) = [\log_2(n)]$ and $r_n + 1$ equals the number of ones in the binary expansion of n.

For $l \in \mathbb{N}$ fixed, denote by $y_l = f^l(0)$. Let c, G, and d be as in Lemma 4.13. From (4.77) and Proposition 4.14, for each $p > 0$ there there are constants $C_{p,l}$ and $D_{p,l}$ such that for all for all $x \in I_{m_0(n+l)}$,

$$C_{p,l}\left(\frac{(c\,G^{2k-1})^{r_{n+l}}}{\lambda_k^{(2k-1)m_{r_{n+l}}}}\right)^p \leq \frac{\Lambda_p(f^l(x),n)}{(\lambda_k^{2kp}\rho_p)^{m_0(n+l)}} \leq D_{p,l}\left(\frac{d^{r_{n+l}}}{\lambda_k^{(2k-1)m_{r_{n+l}}}}\right)^p$$

Therefore, we have that

$$1 \leq \frac{(\widehat{\Lambda}(f^l(x),n))^3}{\sqrt{\Lambda_2(f^l(x),n)}} \leq C\left(\frac{d^3\lambda_k^{4k}}{c\,G^{2k-1}}\right)^{r_{n+l}}\left(\frac{\rho_1^3}{\sqrt{\rho_2}}\right)^{m_0(n+l)} \tag{4.78}$$

for some constant $C > 0$.

Hence, if

$$\left|\frac{d^3\lambda_k^{4k}}{c\,G^{2k-1}}\right| \leq 1 \tag{4.79}$$

then, it suffices to consider

$$\sigma_m = \frac{1}{n^{\gamma+1}}$$

with $\gamma > \log_2(\rho_1^3/\sqrt{\rho_2})$.

In the case

$$\left|\frac{d^3\lambda_k^{4k}}{c\,G^{2k-1}}\right| > 1, \tag{4.80}$$

it is enough to consider

$$\sigma_n = \frac{1}{n^{\gamma^*+1}}$$

with $\gamma^* > \log_2(d^3\lambda_k^{4k}\rho_1^3) + \log_2(c\,G^{2k-1}/\sqrt{\rho_2})$.

To obtain the Berry–Esseen estimates of Theorem 2.2, we choose σ_n that satisfies (2.15)

If (4.79) holds, then it suffices to consider

$$\sigma_n = \frac{1}{n^{\theta}}$$

with $\theta > \log_2(\rho_1^3/\sqrt{\rho_2}) + \log_2(\rho_2^3/\rho_3) + 1$.

In the case (4.80), it suffices to consider

$$\sigma_n = \frac{1}{n^{\theta^*}}$$

with $\theta^* > \log_2(d^3 \lambda_k^{4k} \rho_1^3) + \log_2(c\, G^{2k-1}/\sqrt{\rho_2}) + \log_2(\rho_2^3/\rho_3) + 1$ □

Remark 4.4. If we consider an increasing sequence of iterations $\{n_k\}_k$ which have lacunar binary expansions, that is

$$\frac{r_{n_k}}{\log_2(n_k)} \to 0$$

as $k \to \infty$ then, from (4.78), it suffices to consider $\sigma_{n_k} = n_k^{-(\gamma+1)}$ to have a central limit holds along $\{n_k\}$.

Remark 4.5. In the quadratic ($k = 1$) case, the power which arises from the argument in the central limit theorem presented here computed in [DEdlL06]. The result is $\gamma = 3.8836\ldots$ The method used in [DEdlL06] follows a numerical scheme similar to [Lan82] to compute g_1 and the spectral radii of the operators $\mathcal{K}_{g_1,1}$ and $\mathcal{K}_{g_1,2}$.

The numerics in [DEdlL06] give a confirmation that the dependence of the optimal σ_n on the number of iterations is a power of exponent γ_* which is not too far from the the γ coming from the arguments in this paper.

4.6.6. *Conjectures for orbits of points in the basin of attraction.* The topological theory of interval maps [CE80, CEL80] implies that for each $f \in \mathcal{V} \cap \mathcal{W}_s(g_k)$, the set set $\mathcal{C}_f = \overline{\{f^n(0)\}_{n\in\mathbb{N}}}$ attracts all points in $[-1, 1]$, except for periodic points of period 2^n – which are unstable - and their preimages. More formally, for any point $x \in I$, either

(a) $\lim_{n\to\infty} d(f^n(x), \mathcal{C}_f) = 0$, or
(b) $f^{2^n}(f^l(x)) = f^l(x)$ for some nonnegative integers l and n,

In the second alternative, we have $\left|(f^{2^n})'(f^l(x))\right| > 1$.

The set $\mathcal{C}_f$ is commonly referred to as the Feigenbaum attractor since it attracts a set of full measure (not open, however). It is also known that $f|_{\mathcal{C}_f}$ is uniquely ergodic since it is a substitution system [CEL80]. The set of points $\mathcal{B}_f$ that are attracted to $\mathcal{C}_f$ is called the basin of attraction of f.

Numerical simulations [DEdlL06] for the quadratic Feigenbaum fixed point and the quartic map suggest that the effective noise of orbits starting in the basin of attraction $\mathcal{B}_f$ and affected by weak noise approaches a Gaussian. We conjecture that the is indeed the case.

Of course, the convergence to Gaussian is not expected to be uniform in the point in the basin. We have already shown that the convergence to Gaussian is false for the preimages of unstable periodic orbits which are accumulation points for the basin.

5. Central limit theorem for critical circle maps

In this section, we consider another example of dynamical systems with a non trivial renormalization theory, namely circle maps with a critical point and golden mean rotation number. The theory has been developed both heuristically and rigorously in [FKS82, ÖRSS83, Mes84, Lan84, SK87, dF99]

We will adapt the argument developed in section 4.1 to the case of critical circle maps with golden mean rotation number.

5.1. Critical circle maps with golden mean rotation number.

We will consider the following class of maps of the circle.

Definition 5.1. The space of critical circle maps is defined as the set of analytic functions f, that are strictly increasing in in $\mathbb{R}$ and satisfy

G1. $f(x+1) = f(x) + 1$

G2. f has rotation number $\beta = \frac{\sqrt{5}-1}{2}$. Recall that for a circle map f the rotation number $r(f)$ is defined by

$$r(f) = \lim_{n\to\infty} \frac{f^n(x) - x}{n}$$

G3. $f^{(j)}(0) = 0$ for all $0 \le j \le 2k$, and $f^{(2k+1)}(0) \neq 0$.

From the well known relation between the golden mean and the Fibonacci numbers, $Q_0 = 0$, $Q_1 = 1$ and $Q_{n+1} = Q_{n-1} + Q_n$, it follows that $Q_n\beta - Q_{n-1} = (-1)^{n-1}\beta^n$ is the rotation number of the map

$$f_{(n)}(x) = f^{Q_n}(x) - Q_{n-1} \tag{5.1}$$

The sequence of maps $f_{(n)}$ will be useful in section 5.3 where we analyze the cumulants of the noise.

5.2. Renormalization theory of circle maps.

There are different rigorous renormalization formalisms for circle maps. For our purposes, we will need very little about the renormalization group, so that we will use the very basic formalism of scaling limits.

Definition 5.2. Given a map f as in Definition 5, let $f_{(n)}$ be as in (5.1) and denote $\lambda_{(n)} = f_{(n)}(0)$. The n–th renormalization of f is defined by

$$f_n(x) = \frac{1}{\lambda_{(n-1)}} f_{(n)}(\lambda_{(n-1)}x) \tag{5.2}$$

Remark 5.1. Since the rotation number of $f_{(n)}(x)$ is $(-1)^{n-1}\beta^n$, we have that

$$(-1)^{n-1}\left(f_{(n)}(x) - x\right) > 0$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. In particular, for $x = 0$ we obtain that $(-1)^{n-1}\lambda_{(n)} > 0$. Therefore, it follows that each function $f_n(x)$, defined by (5.2), is increasing in x and satisfies $f_n(x) < x$

It is known [dFdM99, Yam01, Yam02, Yam03] that for every $k \in \mathbb{N}$ there is a universal constant $-1 < \lambda_k < 0$ and universal function η_k such that:

RC1. η_k is increasing, $\eta_k(x) \le x$, and $\eta_k(x) = H_k(x^{2k+1})$ for some analytic function H_k.

RC2. The sequence of ratios $\alpha_n = \lambda_{(n)}/\lambda_{(n-1)}$ converges to a limit $-1 < \lambda_k < 0$

RC3. For some $0 < \delta_k < 1$, $\|f_n(x) - \eta_k(x)\|_{C_0} = O(\delta_k^n)$ where the norm $\|\cdot\|_{C_0}$ is taken over an appropriate complex open domain D_k that contains the real line $\mathbb{R}$.

RC4. The function η_k is a solution of the functional equations

$$\eta_k(x) = \tfrac{1}{\lambda_k}\eta_k\left(\tfrac{1}{\lambda_k}\eta_k(\lambda_k^2 x)\right) \tag{5.3}$$

$$\eta_k(x) = \tfrac{1}{\lambda_k^2}\eta_k\left(\lambda_k\eta_k(\lambda_k x)\right) \tag{5.4}$$

RC5. The domain D_k can be taken so that ∂D_k is smooth and

$$\overline{\lambda_k D_k} \subset D_k \tag{5.5}$$

$$\lim_{n\to\infty} \sup\{|\operatorname{Im}(z)| : z \in D_k,\ |z| > n\} = 0 \tag{5.6}$$

$$\overline{\lambda_k^{-1}\eta_k(\lambda_k^2 D_k)} \subset D_k \tag{5.7}$$

$$\overline{\lambda_k\eta_k(\lambda_k D_k)} \subset D_k \tag{5.8}$$

A few useful relations can be obtain directly from (5.3) and (5.4). For instance, by letting $x = 0$ we get

$$\eta_k(1) = \lambda_k^2 \qquad \eta_k(\lambda_k^2) = \lambda_k^3$$

Taking derivatives on (5.3) and (5.4) we have that

$$\frac{\eta_k'(x)}{\eta_k'(\lambda_k^2 x)} = \eta_k'\left(\frac{1}{\lambda_k}\eta_k(\lambda_k^2 x)\right)$$

$$\frac{\eta_k'(x)}{\eta_k'(\lambda_k x)} = \eta_k'\left(\lambda_k\eta_k(\lambda_k x)\right)$$

Then letting $x \to 0$ we get

$$\eta'_k(1) = \frac{1}{\lambda_k^{4k}}$$
$$\eta'_k(\lambda_k^2) = \frac{1}{\lambda_k^{2k}}$$

Solutions of the equations (5.3), (5.4) are constructed in [Eps89] for all orders of tangency at the critical point.

In the case of cubic circle maps, [Mes84, LdlL84] using computer assisted proofs constructed a fixed point. The exponential convergence follows from the compactness of the derivative of the renormalization transformation at η_1, which is a consequence of the analyticity improving of the auxiliary functions. This is part of the conclusions of the computer assisted proofs.

5.3. **Renormalization theory of the noise.** In this section, we will develop in parallel two renormalization theories for the noise. For the purposes of these paper, either one will be enough so we will not show to which extent they are equivalent. We will assume that the order of tangency k is fixed.

The renormalization group scheme for critical maps with rotation number β gives information at small scales and at Fibonacci times. Observe that if f is a circle maps, then Lyapunov functions $\widehat{\Lambda}^f$ and Λ_p^f defined by (2.3) and (2.2) are periodic with period 1.

Denote by

$$k_n(x) = \Lambda_p^f(x, Q_n)$$

Since f is increasing, equation (2.5) for the Lyapunov functions Λ_p implies that

$$k_n(x) = \left(f'_{(n-1)} \circ f_{(n-2)}(x)\right)^p k_{n-2}(x) + k_{n-1}(f_{(n-2)}(x)) \quad (5.9)$$
$$k_n(x) = \left(f'_{(n-2)} \circ f_{(n-1)}(x)\right)^p k_{n-1}(x) + k_{n-2}(f_{(n-1)}(x)) \quad (5.10)$$

5.3.1. *Lindeberg–Lyapunov operators.* To study the propagation of noise at small scales, we will use the following definition

Definition 5.3. Let f be a critical circle map, and for each $p \geq 0$ and $n \in \mathbb{N}$, define the operators

$$U_{n,p}h(z) = \left[f'_{n-2}(\alpha_{n-2}f_{n-1}(\alpha_{n-1}z))\right]^p h(\alpha_{n-1}z)$$
$$T_{n,p}q(z) = \left[f'_{n-1}\left(\alpha_{n-2}^{-1}f_{n-2}(\alpha_{n-1}\alpha_{n-2}z)\right)\right]^p q(\alpha_{n-1}\alpha_{n-2}z)$$
$$R_nh(z) = h(\alpha_{n-2}^{-1}f_{n-2}(\alpha_{n-1}\alpha_{n-2}z))$$
$$P_nq(z) = q\left(\alpha_{n-2}f_{n-1}(\alpha_{n-1}z)\right)$$

acting on the space or real analytic functions in the domain D_k. For $p \geq 0$, and $n \in \mathbb{N}$, the *Lindeberg–Lyapunov* operators, $\mathcal{K}_{n,p}$ and $\widehat{\mathcal{K}}_{n,p}$, acting on pairs of real analytic functions in D_k are defined by the matrices of operators

$$\mathcal{K}_{n,p} = \begin{pmatrix} R_n & T_{n,p} \\ I & 0 \end{pmatrix} \quad \widehat{\mathcal{K}}_{n,p} = \begin{pmatrix} U_{n,p} & P_n \\ I & 0 \end{pmatrix} \tag{5.11}$$

where I is the identity map and 0 is the zero operator. Similarly, consider the operators acting on the space of real analytic functions on D_k defined by

$$\begin{aligned} U_p h(z) &= [\eta_k'(\lambda_k \eta_k \lambda_k z))]^p \, h(\lambda_k z) \\ T_p q(z) &= \left[\eta_k'\left(\lambda_k^{-1} \eta_k(\lambda_k^2 z)\right)\right]^p q(\lambda_k^2 z) \\ Rh(z) &= h(\lambda_k^{-1} \eta_k(\lambda_k^2 z)) \\ Pq(z) &= q\left(\lambda_k \eta_k(\lambda_k z)\right) \end{aligned}$$

The Lindeberg–Lyapunov operators $\mathcal{K}_p$ and $\widehat{\mathcal{K}}_p$ are defined by

$$\mathcal{K}_p = \begin{pmatrix} R & T_p \\ I & 0 \end{pmatrix} \quad \widehat{\mathcal{K}}_p = \begin{pmatrix} U_p & P \\ I & 0 \end{pmatrix} \tag{5.12}$$

Notice from (5.9) and (5.10) that the change variables

$$x = \lambda_{(n-1)} z, \qquad \tilde{k}_n(z) = k_n(\lambda_{(n-1)} z),$$

implies that

$$\begin{bmatrix} \tilde{k}_n \\ \tilde{k}_{n-1} \end{bmatrix} = \mathcal{K}_{n,p} \cdots \mathcal{K}_{3,p} \begin{bmatrix} \tilde{k}_2 \\ \tilde{k}_1 \end{bmatrix} \tag{5.13}$$

$$\begin{bmatrix} \tilde{k}_n \\ \tilde{k}_{n-1} \end{bmatrix} = \widehat{\mathcal{K}}_{n,p} \cdots \widehat{\mathcal{K}}_{3,p} \begin{bmatrix} \tilde{k}_2 \\ \tilde{k}_1 \end{bmatrix} \tag{5.14}$$

Remark 5.2. Notice that the derivatives involved in Definition 5.3 are positive. Therefore, the notions of cumulant operators and Lindeberg–Lyapunov operators coincide in this case.

Equations (5.13) and (5.14) are the analogs to (4.6) for period doubling. These equations measure the growth of the linearized propagation of noise at Fibonacci times for an orbit starting at the critical point 0.

5.3.2. *Exponential convergence of the Lindeberg–Lyapunov operators.* An important consequence of the exponential convergence of f_n to η_k is that the Lindeberg–Lyapunov operators $\mathcal{K}_{n,p}$ and $\widehat{\mathcal{K}}_{n,p}$ will converge exponentially fast to operators $\mathcal{K}_p$ and $\widehat{\mathcal{K}}_p$ respectively, as $n \to \infty$. Indeed, we have that

Lemma 5.1. *Let f be a circle map of order k satisfying G1–G3. The operators $\mathcal{K}_{n,p}$, $\mathcal{K}_p$, $\widehat{\mathcal{K}}_{n,p}$ and $\widehat{\mathcal{K}}_p$ acting on the space of pairs of bounded analytic functions defined on some compact set $B_k \subset D_k$ that has smooth boundary and contains $[-1,1]$ in its interior, are compact. Furthermore, for p fixed,*

$$\|\mathcal{K}_{n,p} - \mathcal{K}_p\| \leq c_p \left(\|f_n - \eta_k\|_{B_k} + \|f_{n-1} - \eta_k\|_{B_k}\right) \tag{5.15}$$

$$\|\widehat{\mathcal{K}}_{n,p} - \widehat{\mathcal{K}}_p\| \leq c_p \left(\|f_n - \eta_k\|_{B_k} + \|f_{n-1} - \eta_k\|_{B_k}\right) \tag{5.16}$$

Proof. We only show (5.16), since the proof of (5.15) is very similar.

For any $A \subset \mathbb{C}$, recall the notation $A^\epsilon = \{z \in \mathbb{C} : d(z, A) \leq \epsilon\}$. Since $|\alpha_{n-1} - \lambda_k| = O(\delta_k^n)$ for some $0 < \delta_k < 1$, we have that for any compact $B_k \subset D_k$ and for all n large enough

$$\overline{\alpha_{n-1}B_k} \subset \overline{\lambda_k B_k}^\epsilon \subset D_k$$

where $\epsilon > 0$ is small.

For $z \in D_k$, we have that

$$\begin{aligned}|\alpha_{n-2}f_{n-1}(\alpha_{n-1}z) - \lambda_k\eta_k(\lambda_k z)| &\leq |\alpha_{n-2}||f_{n-1}(\alpha_{n-1}z) - \eta_k(\alpha_{n-1}z)| \\ &+ |\alpha_{n-2} - \lambda_k||\eta_k(\alpha_{n-1}z)| + |\lambda_k||\eta_k(\alpha_{n-1}z) - \eta_k(\lambda_k z)|\end{aligned}$$

Therefore, using Cauchy estimates, we have that for any compact set $B_k \subset D_k$ containing 0

$$\begin{aligned}\|\alpha_{n-2}f_{n-1} \circ \alpha_{n-1} - \lambda_k\eta_k \circ \lambda_k\|_{B_k} \leq C\,(&\|f_{n-1} - \eta_k\|_{B_k} \\ &+ \|f_{n-2} - \eta_k\|_{B_k})\end{aligned} \tag{5.17}$$

Therefore, we by taking ϵ small enough, we have that for all n large enough

$$\overline{\alpha_{n-2}f_{n-1}(\alpha_{n-1}B_k)}^\epsilon \subset \overline{\lambda_k\eta_k(\lambda_k B_k)}^\epsilon \subset D_k \tag{5.18}$$

Consider B_k with smooth boundary and let $W_k = \overline{\lambda_k\eta_k(\lambda_k B_k)}^\epsilon$. For $z \in B_k$ we have

$$\begin{aligned}|f'_{n-2}(\alpha_{n-2}f_{n-1}(\alpha_{n-1}z)) - \eta'_k(\lambda_k\eta_k\lambda_k z))| \leq \big|&f'_{n-2}(\alpha_{n-2}f_{n-1}(\alpha_{n-1}z)) \\ &-\eta'_k(\alpha_{n-2}f_{n-1}(\alpha_{n-1}z))| \\ &+ |\eta'_k(\alpha_{n-2}f_{n-1}(\alpha_{n-1}z)) - \eta'_k(\lambda_k\eta_k\lambda_k z))|\end{aligned}$$

Then, from (5.18) and (5.17) we get

$$\begin{aligned}\|f'_{n-2}\circ(\alpha_{n-2}f_{n-1}\circ\alpha_{n-1}) - \eta'_k\circ(\lambda_k\eta\circ\lambda_k)\| \le |f'_{n-2}-\eta'_k|_{W_k}\\ + C\|\eta''_k\|_{W_k}\|\left(\|f_{n-1}-\eta_k\|_{B_k} + \|f_{n-2}-\eta_k|_{B_k}\right)\end{aligned}$$

Using Cauchy estimates we obtain

$$\begin{aligned}\|f'_{n-2}\circ(\alpha_{n-2}f_{n-1}\circ\alpha_{n-1}) - \eta'_k\circ(\lambda_k\eta\circ\lambda_k)\|_{B_k} \le c\left(\|f_{n-1}-\eta_k\|_{D_k}\right.\\ \left.+\|f_{n-2}-\eta_k|_{D_k}\right) \quad (5.19)\end{aligned}$$

Denote by $\mathcal{O} = \mathbb{C}\setminus\{x+iy:\ x\le 0,\ y=0\}$, and let $B_k\subset D_k$ be a bounded domain containing $[-1,1]$ with smooth boundary such that

$$\overline{\lambda_k\eta_k(\lambda_k B_k)}\subset\mathcal{O}$$

From (5.19) we can assume, by taking ϵ smaller if necessary, that

$$\overline{\alpha_{n-2}f_{n-1}(\alpha_{n-1}B_k)}\subset\overline{\lambda_k\eta_k(\lambda_k B_k)}^{\epsilon}\subset\mathcal{O}$$

Recall the operators $U_{n,p}$, U_p, $P_{n,p}$ and P_p defined in Definition 5.3. We restrict these operators to the space of bounded analytic functions on B_k. Denote by $Y_k = \overline{\lambda_k\eta_k(\lambda_k B_k)}^{\epsilon}$ and let $\psi(z)=z^p$ defined on Y_k. Hence, using the triangle inequality and Cauchy estimates, we get for any bounded analytic functions h, q on B_k that

$$\begin{aligned}\|(U_{n,p}-U_p)h\|_{B_k} \le \|\psi\|_{Y_k}\|h\|_{\lambda_k B_k}|\alpha_{n-1}-\lambda_k|\\ + \tilde{C}\|\psi'\|_{Y_k}\|h\|_{\lambda_k B_k}\left(\|f_{n-1}-\eta_k\|_{B_k}+\|f_{n-2}-\eta_k\|_{B_k}\right) \quad (5.20)\end{aligned}$$

$$\begin{aligned}\|(P_{n,p}-P_p)q\|_{B_k} \le C\|q'\|_{W_k}\left(\|f_{n-1}-\eta_k\|_{B_k}\right.\\ \left.+\|f_{n-2}-\eta_k\|_{B_k}\right. \quad (5.21)\end{aligned}$$

Combining (5.20) and (5.21) we obtain (5.16).

For z_1, $z_2\in B_k$ close enough we have that

$$\begin{aligned}|T_{n,p}h(z_1)-T_{n,p}h(z_2)| &\le \left\|\left(\psi\circ f'_{n-2}(\alpha_{n-2}f_{n-1}\circ\alpha_{n-1})\right)'\right\|_{B_k}|z_1-z_2|\\ |P_{n,p}h(z_1)-P_{n,p}h(z_2)| &\le \left\|\left(q\circ(\alpha_{n-2}f_{n-1}\circ\alpha_{n-1})\right)'\right\||z_1-z_2|\end{aligned}$$

Therefore, the compactness of the operators $\widehat{\mathcal{K}}_{n,p}$ follows from Cauchy estimates and the Arzela–Ascoli theorem. □

As in section 4.2, the spectral properties of the operators $\mathcal{K}_{n,p}$, $\mathcal{K}_p$, $\widehat{\mathcal{K}}_{n,p}$ and $\widehat{\mathcal{K}}_p$ will prove important for our analysis of the Gaussian properties of the scaling limit.

Notice that for $p\in\mathbb{N}$, since f_n and η_k are increasing, the cumulant functions defined by (3.43) and the Lyapunov functionals (2.2) coincide.

Hence, the study of the properties of cumulants is equivalent to that of the properties of the Lindeberg–Lyapunov functionals.

5.4. **Spectral analysis of Lindeberg–Lyapunov operators.** From Lemma 5.1, we know that the Lindeberg–Lyapunov operators $\mathcal{K}_p$, $\widehat{\mathcal{K}}_p$ when defined on in a space space of real valued analytic functions with domains in D that satisfy (5.5), (5.6), (5.7), (5.8) are such that their squares are compact. Again, the compactness follows from the fact that the operators $U_{n,p}, T_{n,p}, R_n, Q_n$ in Definition 5.3 are analyticity improving.

The compactness of the operators $\mathcal{K}_{n,p}$, $\widehat{\mathcal{K}}_{n,p}$, is slightly more subtle since they have a block (in the lower diagonal) that is the identity. Using the fact that this block appear in the lower diagonal, the operator can be made compact by considering that the second component is analytic in a slightly smaller domain. In this way, the identity becomes an immersion, which is compact. If the domain for the second component is chosen only slightly smaller, the analyticity improving properties of the other operators are still maintained.

Remark 5.3. An slightly different approach is to remark that the square of the operators consist only of compact operators. For our purposes, the study of the squares will be very similar.

Notice that the operators $\mathcal{K}_{n,p}, \widehat{\mathcal{K}}_{n,p}$ preserve the cone $\mathcal{C}$ of pairs of functions, both of whose components are strictly positive when restricted to the reals. Note that the interior of the cone $\mathcal{C}$ are the pairs of functions whose components are strictly positive when restricted to the reals.

It is also easy to verify the properties (i), (ii), (iii) in Section 4.3. Again, (i), (ii) are obvious and we note that if a pair is not identically zero, then, one of the components has to be strictly positive in an interval. The structure of the square of the operator implies that the component is strictly positive in an interval which is larger. This, in turn, implies that the second component becomes positive. Both components become strictly positive by repeating the operation a finite number of times.

Therefore, we can apply the the Kreĭn–Rutman theorem and obtain a result similar to Proposition 4.2.

Proposition 5.2. *Let $\mathcal{K}_{n,p}$ and $\widehat{\mathcal{K}}_{n,p}$ be the Lindeberg–Lyapunov operators defined by* (5.11)*. Denote by $\mathcal{K}_{\infty,p} = \mathcal{K}_p$ and $\widehat{\mathcal{K}}_{\infty,p} = \widehat{\mathcal{K}}_p$. For all $p \geq 0$ and $n \in \mathbb{N} \cup \{\infty\}$, let $\rho_{n,p}$ ($\widehat{\rho}_{n,p}$) be the spectral radius of $\mathcal{K}_{n,p}$ ($\widehat{\mathcal{K}}_{n,p}$). Then,* *a)* $\rho_{n,p}$ ($\widehat{\rho}_{n,p}$) *is a positive eigenvalue of* $\mathcal{K}_{n,p}$ ($\widehat{\mathcal{K}}_{n,p}$).

b) The rest of $\mathrm{Spec}(\mathcal{K}_{n,p})\setminus\{0\}$ ($\mathrm{Spec}(\widehat{\mathcal{K}}_{n,p})\setminus\{0\}$) *consists of eigenvalues* μ *with* $|\mu| < \rho_{n,p}$ ($|\mu| < \widehat{\rho}_{n,p}$).

c) A pair of positive functions $(\psi_{n,p}, \phi_{n,p})$ *is an eigenvector of* $\mathcal{K}_{n,p}$ ($\widehat{\mathcal{K}}_{n,p}$) *if and only if the corresponding eigenvalue is* $\rho_{n,p}$ ($\widehat{\rho}_{n,p}$).

5.4.1. *Properties of the spectral radius of the Lindeberg–Lyapunov operators.* The convexity properties of the Lyapunov functions Λ_p will also imply similar properties for the cumulants $\mathcal{K}_{n,p}$ and $\mathcal{K}_{n,p}$. In the next Theorem, we will only consider the the operators $\mathcal{K}_p$ and $\widehat{\mathcal{K}}_p$. We also remark that the argument in Proposition 4.4 applies in our case also, so that the spectrum is independent of the domain considered. Hence, we will not include the domain in the notation.

Theorem 5.3. *Let* ρ_p *be the spectral radius of* $\mathcal{K}_p$. *(Similarly for* $\widehat{\rho}_p$ *and* $\widehat{\mathcal{K}}_p$.*)*

K1. *For all* $p > 0$

$$\lambda_k^{2kp}\rho_p > 1$$

K2. *The map* $p \mapsto \rho_p$ *is strictly log–convex*

K3. *The map* $p \mapsto \log(\rho_p)/p$ *is strictly decreasing.*

A Similar result holds for $\widehat{\rho}_p$.

Proof. (K1) For any pair of positive functions $[h,\, \ell]$, we have that

$$\mathcal{K}_p^{2m}[h,\, \ell] > \left[T_p^m h, T_p^m \ell\right]$$

If $[h,\, \ell]$ is the dominant eigenvector of $\mathcal{K}_p$, then we have that

$$\begin{aligned} \rho_p^{2m}[h(z),\, \ell(z)] &> \left[T_p^m h, T_p^m \ell\right](z) \\ &= \prod_{j=1}^{m} \left(\eta'\left(\lambda_k^{-1}\eta_k\lambda_k^{2j}z\right)\right)^p \left[h(\lambda_k^{2mz}), \ell(\lambda_k^{2m}z)\right] \end{aligned}$$

Since $\eta_k'(1) = \lambda_k^{-4k}$, the conclusion of (M1) follows by letting $z = 0$. To prove [K2.] [K3.] we proceed in a similar way as in the proof of Theorem 4.5. We study the eigenvalue equation for the leading eigenvalue and use it to construct positive functions that satisfy eigenvalue equations with a positive remainder and then, we apply Proposition 4.6. We suppress the subindex n for typographical clarity. We will use the notations for operators introduced in Definition 5.3.

The eigenvalue equation for the leading eigenvalue of $\mathcal{K}_p$ is

$$\rho_p\ell_p = Rh_p + T_p\ell_p \tag{5.22}$$

$$\rho_p\ell_p = h_p \tag{5.23}$$

We raise the equations (5.22) to the power $\alpha > 1$ and use the inequality for the binomial theorem and the elementary identities $(T_p\ell_p)^\alpha = T_{p\alpha}\ell_p^\alpha$, $(Rh_p)^\alpha = Rh_p^\alpha$. Then, we have:

$$\begin{aligned} \rho_p^\alpha \ell_p^\alpha &= (Rh_p + T_p\ell_p)^\alpha && (5.24)\\ &> (Rh_p)^\alpha + (T_p\ell_p)^\alpha && (5.25)\\ &= Rh_p^\alpha + T_{p\alpha}\ell_p^\alpha && (5.26)\\ \rho_p^\alpha \ell_p^\alpha &= h_p^\alpha && (5.27) \end{aligned}$$

Hence, we obtain

$$\mathcal{K}_{\alpha p}[h^\alpha, \ell^\alpha] \geq \rho_p^\alpha [h^\alpha, \ell^\alpha]$$

and we have that the inequality is strict in the first component. Therefore

$$\mathcal{K}_{\alpha p}^2[h^\alpha, \ell^\alpha] > \rho_p^{2\alpha}[h^\alpha, \ell^\alpha]$$

Using that the leading eigenvalue of $\mathcal{K}_p^2$ is the square of the leading eigenvalue of $\mathcal{K}_p$, we obtain, applying Proposition 4.6 that

$$\rho_{p\alpha} < \rho_p^\alpha$$

which is property [K3.].

To prove property [K2], we multiply the eigenvalue equation for leading eigenvectors of $\mathcal{K}_p$, $\widehat{\mathcal{K}}_p$ and raise to the 1/2 power. We use the identities $(Rh_p)(Rh_q) = R(h_p h_q)$, $[R(h_p h_q)]^{1/2} = R\left[(h_p h_q)^{1/2}\right]$ as well as similar identities for the operator T.

$$\begin{aligned} (\rho_p \rho_q)^{1/2}(h_p h_q)^{1/2} &= [(Rh_p + T_p\ell_p)\cdot(Rh_q + T_q\ell_q)]\\ &> (Rh_pRh_p + T_p\ell_pT_q\ell_q)^{1/2}\\ &> R(h_ph_q)^{1/2} + T_{(p+q)/2}(\ell_p\ell_q)^{1/2} \end{aligned}$$

Multiplying the second components we have

$$(h_ph_q)^{1/2} = (\rho_p\rho_q)^{1/2}(\ell_p\ell_q)^{1/2}$$

In other words, we have

$$\mathcal{K}_{(p+q)/2}[(h_ph_q)^{1/2}, (\ell_p\ell_q)^{1/2}] \leq (\rho_p\rho_q)^{1/2}[(h_ph_q)^{1/2}, (\ell_p\ell_q)^{1/2}]$$

with the inequality being strict in the first component. Proceeding as before, we obtain

$$\rho_{(p+q)/2} < (\rho_p\rho_q)^{1/2}$$

which is equivalent to strict log-convexity.

□

5.5. **Asymptotic properties of the renormalization.** A consequence of Lemma 5.1 is that the asymptotic properties of the operators $\mathcal{K}_{n,p}$ and $\widehat{\mathcal{K}}_{n,p}$ become similar to those of $\mathcal{K}_p$ and $\widehat{\mathcal{K}}_p$ respectively.

The following result, Corollary 5.4, is obtained as an application of Proposition 4.7. Since the proof is very similar to that of Corollary 4.8 for period doubling, we will omit the details.

Corollary 5.4. *Let $\mathcal{K}_{n,p}$, $\mathcal{K}_p$, and ρ_p be as in Proposition 4.2. Then*

1. *There is a constant $c_p > 0$ such that for all positive analytic pairs of functions $[h, q]$ in the domain $\mathcal{D}$ of the Lindeberg–Lyapunov operators*
$$c_p^{-1}\rho_p^n \leq \mathcal{K}_{n,p}\cdots\mathcal{K}_{1,p}[h(z), q(z)] \leq c_p\rho_p^n \tag{5.28}$$
2. *For $p > 2$, we have $\rho_p < (\rho_2)^{p/2}$*
3. *For any $p > 2$*
$$\lim_{n\to\infty}\sup_{x\in I_n}\frac{\Lambda_p^f(x, Q_n)}{\{\Lambda_2^f(x, Q_n)\}^{p/2}} = 0$$
where $I_m = [-|\lambda_{(n)}|, |\lambda_{(n)}|]$.

5.6. **Proof of the central limit theorem for circle maps (Theorems 2.4 and 2.5).** In this section, we will prove a central limit for critical circle maps (Theorem 2.4). The method of the proof is similar to the method developed in Section 4.6 for period doubling. That is, we will verify the Lyapunov condition (2.9) along the whole sequence of integers and then use Theorems 2.1 and 2.2.

The renormalization theory developed in Section 5.4 gives control on $\Lambda_s(x, Q_m)$ for all x sufficiently small ($|x| \leq |\lambda_{(m+2)}|$), see Corollary 5.4. The main tool to obtain control of the noise on a segment of the orbit will be to decompose the segment into pieces where renormalization applies, that is segments of Fibonacci length.

This decomposition is develop in Section 5.6.1. The main conclusion of this section is that the effect of the noise over a period of time n equals the sum of Fibonacci blocks – where renormalization applies – with some weights. See (5.30). This weights measure how the effect of Fibonacci blocks propagates to the end of the interval.

The effect of weights on the Fibonacci blocks is measure in Section 5.6.2.

5.6.1. *Fibonacci decomposition.* Given $n \in \mathbb{N}$, it admits a unique decomposition
$$n = Q_{m_0} + \cdots + Q_{m_{r_n}} \tag{5.29}$$

where $m_0 > \ldots > m_{r_n} > 0$ are non–consecutive integers and Q_{m_j} is the m_j–th Fibonacci number. If necessary, we will use $m_j(n)$ to emphasize the dependence on n. Notice that $r_n \leq m_0(n) \leq [\log_{\beta^{-1}} n + 1]$.

We have the following Fibonacci decomposition for $\Lambda_p(x, n)$

$$\Lambda_p(x,n) = \sum_{j=0}^{r_n} |\Psi_{j,n}(x)|^p \, \Lambda_p\left(f^{n-Q_{m_j}-\cdots-Q_{m_r}}(x), Q_{m_j}\right) \tag{5.30}$$

where

$$\Psi_{j,n}(x) \;\; = \;\; \left(f^{n-Q_{m_0}-\cdots-Q_{m_j}}\right)' \circ \left(f^{Q_{m_j}+\cdots Q_{m_0}}\right)(x) \tag{5.31}$$

Since $(f^m)'(x)$ is periodic of period 1 for any $m \in \mathbb{N}$, $\Lambda_p(x,n)$ can be expressed in terms of the functions $f_{(n)}$ defined by (5.1). Indeed, let

$$\upsilon_j = f_{(m_j)}(\upsilon_{j-1}) \qquad \upsilon_{-1} = x \tag{5.32}$$

then,

$$\Psi_{n,j} = \prod_{l=j+1}^{r_n} (f_{(m_l)})'(\upsilon_{l-1})$$

and

$$\Lambda_p(x,n) = \sum_{j=0}^{r_n} (\Psi_{n,j}(x))^p \Lambda_p(\upsilon_{j-1}, Q_{m_j}) \tag{5.33}$$

Let us fix a critical map of the circle of order k, and let $0 < \epsilon \ll 1$. For all n large enough we have that

$$\|f_n - \eta_k\|_{C(D_k)} < \epsilon \tag{5.34}$$

Using renormalization, we will control the size of the size of the weights $\Psi_{j,n}(x)$ for $|x| \leq |\lambda_{(m_0+2)}|$.

5.6.2. *Estimation of the weights.* In this section we estimate the weights $\Psi_{n,j}$ defined in (5.33). The method used is very similar to to one develop for period doubling.

Recall that $\eta_k(x) = \lambda_k + \sum_{j=1}^{\infty} b_j x^{(2k-1)j}$, with $b_1 > 0$. Consider the function H_k defined by

$$H_k(x) = \frac{\eta_k'(x)}{x^{2k}}$$

and define

$$s_k = \inf_{\{x:|x|\leq \lambda_k^2\}} H_k(x) \qquad u_k = \sup_{\{x:|x|\leq\lambda_k^2\}} H_k(x) \tag{5.35}$$

It follows that

$$cx^{2k} \leq \eta_k'(x) \leq dx^{2k} \tag{5.36}$$

In the rest of this section, we will assume with no loss of generality that all n are large enough so that

$$\sup_{\{x:|x|\le 1\}} |f_n(x) - \eta_k(x)| < \epsilon \tag{5.37}$$

with $0 < \epsilon \ll 1$.

Lemma 5.5. *Let $\{m_j\}_{j=0}^r$ be a decreasing sequence of non–consecutive positive integers. For $|x| \le \lambda_{(m_0+2)}$, let $\{\upsilon_j\}_{j=-1}^r$ be defined by (5.32).*

(1) *For all $0 \le j \le r$ for which m_j is large enough*

$$|\lambda_k^3||\lambda_{(m_j-1)}| \le |\upsilon_j| \le |\eta_k(-\lambda_k^2)||\lambda_{(m_j-1)}| \tag{5.38}$$

(2) *Let $c = s_k - \epsilon$ and $d = u_k + \epsilon$. Then,*

$$c\lambda^{6k}\left(\frac{\lambda_{(m_{j-1}-1)}}{\lambda^{(m_j-1)}}\right)^{2k} \le (f_{(m_j)})'(\upsilon_{j-1}) \le d\left(\frac{\lambda_{(m_{j-1}-1)}}{\lambda_{(m_j-1)}}\right)^{2k} \tag{5.39}$$

Proof. (1) For $j = 0$, we have that

$$\upsilon_0 = \lambda_{(m_0-1)} f_{m_0}\left(\frac{\upsilon_{-1}}{\lambda_{(m_0-1)}}\right)$$

The exponential convergence of f_m to η_k implies that

$$\left|\frac{\upsilon_{-1}}{\lambda_{(m_0-1)}}\right| \le \left|\frac{\lambda_{(m_0+2)}}{\lambda_{(m_0-1)}}\right| \le |\lambda_k|^3 + \epsilon$$

for some $0 < \epsilon \ll 1$. Hence,

$$(|\lambda_k|^3 - \epsilon)|\lambda_{(m_0-1)}| \le |\upsilon_0| \le |\eta_k(-\lambda_k^2)||\lambda_{(m_0-1)}|$$

By induction, assume that

$$|\upsilon_{j-1}| < |\eta_k(-\lambda_k^2)|\left|\lambda_{(m_{j-1}-1)}\right|$$

Notice that

$$\upsilon_j = f_{(m_j)}(\upsilon_{j-1}) = \lambda_{(m_j-1)} f_{m_j}\left(\frac{\upsilon_{j-1}}{\lambda_{(m_j-1)}}\right)$$

Since $m_{j-1} - m_j \ge 2$, we have that

$$\left|\frac{\upsilon_{j-1}}{\lambda_{(m_j-1)}}\right| \le |\eta_k(-\lambda_k^2)|(\lambda_k^2 + \epsilon) \le \lambda_k^2 \tag{5.40}$$

since $\epsilon \ll 1$. Therefore, we have that

$$|\lambda_k|^3\left|\lambda_{(m_j-1)}\right| \le |\upsilon_j| \le |\eta_k(-\lambda_k^2)|\left|\lambda_{(m_j-1)}\right|$$

(2) Taking the neighborhood of the fixed point smaller if necessary, assume that for all m large enough

$$\sup_{|x|\le\lambda_k^2}\left|\frac{(f_m)'(x)}{x^{2k}}-\frac{\eta_k'(x)}{x^{2k}}\right|<\epsilon$$

Notice that

$$(f_{(m_j)})'(\upsilon_{j-1})=(f_{m_j})'\left(\frac{\upsilon_{j-1}}{\lambda_{(m_j-1)}}\right)$$

If m_j is large enough, by the exponential convergence of the renormalized functions to the fixed point, we have that

$$\left|\frac{\upsilon_{j-1}}{\lambda_{(m_j-1)}}\right|\le\lambda_k^2+\epsilon$$

Therefore, from (5.36) we get

$$c\lambda_k^{6k}\left(\frac{\lambda_{(m_{j-1}-1)}}{\lambda_{(m_j-1)}}\right)^{2k}\le\left|(f_{(m_j)})'(\upsilon_{j-1})\right|\le d\left(\frac{\lambda_{(m_{j-1}-1)}}{\lambda_{(m_j-1)}}\right)^{2k}$$

□

5.6.3. *Lyapunov condition for critical circle maps.* In this section we show that the Lyapunov condition 2.9 holds for subsequences of the orbits starting at points of the form $x=f^l(0)$, $l\in\mathbb{N}$. The main result of this section is Theorem 5.7 which proves that the Lyapunov condition holds always if we choose appropriately subsequences (they are subsequences of numbers with very few terms in the Fibonnaci expansion. The precise conditions depend on numerical properties of the fixed point. We show that if the fixed point satisfies some conditions, we can obtain the limit along the full sequence.

First, we use Lemma 5.5 to estimate the growth of the Lyapunov functions $\Lambda_p(0,n)$ at zero at all times.

Proposition 5.6. *Let c and d as in Proposition 5.5. For each integer n, let $Q_{m_0(n)}$ be the largest Fibonacci number in the Fibonacci expansion of n and let r_n be the number of terms in expansion* (5.29) *of n. Denote by $I_{m_0}=[-|\lambda_{(m_0+2)}|,|\lambda_{(m_0+2)}|]$.*

1. *For each $p>0$, there are constants a_p and b_p such that*

$$a_p\left(\frac{c\,\lambda_k^{6k}}{\lambda_k^{2km_{r_n}}}\right)^p\le\frac{\Lambda_p(x,n)}{(\lambda_k^{2kp}\rho_p)^{m_0(n)}}\le b_p\left(\frac{d^{r_n}}{\lambda_k^{2km_{r_n}}}\right)^p \tag{5.41}$$

for all $x\in I_{m_0(n)}$.

2. *If $\{n_i\}$ is an increasing sequence of integers such that*

$$\lim_{i\to\infty} \frac{r_{n_i}}{m_0(n_i)} = 0 \tag{5.42}$$

then

$$\lim_{j\to\infty} \sup_{x\in I_{m_0(n_j)}} \frac{\Lambda_p(x,n_j)}{(\Lambda_2(x,n_j))^{p/2}} = 0 \tag{5.43}$$

3. *If the following condition*

$$\left(s_k\lambda_k^{6k}\right)^p \lambda_k^{2kp}\rho_p > 1, \tag{5.44}$$

holds (s_k defined as in (5.35)*) then, for each $p > 2$*

$$\lim_{n\to\infty} \sup_{x\in I_{m_0(n)}} \frac{\Lambda_p(x,n)}{(\Lambda_2(x,n))^{p/2}} = 0 \tag{5.45}$$

Proof. For a given $n \in \mathbb{N}$, let $\{m_j\}_{j=0}^{r_n}$ be the decreasing sequence of numbers that appear in the Fibonacci expansion (5.29) of n.
(1) Recall the weights $\Psi_{n,i}$ defined in (5.33). By (5.39) in Lemma 5.5 and (5.37), we can assume without loss of generality that all m_j are large enough, so that

$$(c\lambda_k^{6k})^{r-j} \le \Psi_{n,j}(x)\left(\frac{\lambda_{(m_{r_n}-1)}}{\lambda_{(m_j-1)}}\right)^{2k} \le d^{r-i}$$

for all $x \in I_{m_0(n)}$.

The exponential convergence of f_n to η_k, implies that there is a constant $C > 0$ such that

$$C^{-1}(c\lambda_k^{6k})^{r_n-j+1} \le \lambda_k^{2k(m_{r_n}-m_j)}\Psi_{n,j}(x) \le C\, d^{r_n-j+1} \tag{5.46}$$

By Corollary 5.4, there is a constant $D > 0$ such that

$$D^{-1} \le \rho_p^{-m_j}\Lambda_p(\upsilon_{j-1}, Q_{m_j}) < D \tag{5.47}$$

for all $j = 0, \dots r_n$.

Since each term in the decomposition (5.33) of $\Lambda_p(0,n)$ is positive, for some constants a_p and b_p we have that

$$a_p(c\lambda_k^{6k})^{p\,r_n}\left(\frac{\lambda_k^{m_0}}{\lambda_k^{m_{r_n}}}\right)^{2kp}\rho_p^{m_0} \le \Lambda_p(x,n) \le b_p\sum_{j=0}^{r_n} d^{p(r_n-j)}\left(\frac{\lambda_k^{m_j}}{\lambda_k^{m_{r_n}}}\right)^{2kp}\rho_p^{m_j}$$

for all $x \in I_{m_0(n)}$. We obtain (5.41) from (5.46), (5.47) by noticing that $m_0 - m_j \ge 2j$.

Notice that if $\{n_i\}$ is a sequence of integers that satisfy (5.42) then

$$\lim_{i\to\infty} \inf_{x\in I_{m_0(n_i)}} \Lambda_p(x,n_i) = \infty \tag{5.48}$$

On the other hand, if the condition (5.44) holds, then $(c\lambda_k^{6k})^p\lambda_k^{2kp}\rho_p > 1$. Therefore,

$$\lim_{n\to\infty}\inf_{x\in I_{m_0(n)}}\Lambda_p(x,n) = \infty \tag{5.49}$$

(2) Corollary 5.4 implies that for any $\epsilon > 0$, there is an integer M_ϵ such that if $m \geq M_\epsilon$, then

$$\Lambda_p(x,Q_m) < \epsilon\,(\Lambda_2(x,Q_m))^{p/2}$$

for all $x \in [-|\lambda_{(m)}|, |\lambda_{(m)}|]$. Let $\mathcal{S}_{r_n} = \{0,\dots,r_n\}$ and define

$$\mathcal{A}_n^1 = \left\{ j \in S_{r_n} : \sup_{x\in I_{m_j}} \frac{\Lambda_p\left(x,Q_{m_j}\right)}{\left(\Lambda_2\left(x,Q_{m_j}\right)\right)^{p/2}} \leq \epsilon \right\}$$

and $\mathcal{A}_n^2 = \mathcal{S}_{r_n}\setminus\mathcal{A}_n^1$. For each $x \in I_{m_0}$ and $s = 2,p$, we split $\Lambda_s(x,n)$ as

$$\Lambda_s(x,n) = H_s(x,\mathcal{A}_n^1) + H_s(x,\mathcal{A}_n^2),$$

where $H_s(x,\mathcal{A}_n^i)$, $i = 1,2$ is defined by

$$H_s(x,\mathcal{A}_n^i) = \sum_{j\in\mathcal{A}_n^i}|\Psi_{j,n}(x)|^s\Lambda_s(\upsilon_{j-1},Q_{m_j})$$

Notice that $H_p(x,\mathcal{A}_n^2)$ is the sum of at most M_ϵ bounded terms. Therefore, for some constant $C_\epsilon > 0$

$$H_p(x,\mathcal{A}_n^2) \leq C_\epsilon \tag{5.50}$$

For the term $H_p(x,\mathcal{A}_n^1)$ we have

$$H_p(x,\mathcal{A}_n^1) \leq \epsilon(H_2(x,\mathcal{A}_n^1))^{p/2} \leq \epsilon(\Lambda_2(x,n))^{p/2} \tag{5.51}$$

Therefore, combining (5.50), (5.51) we get

$$\frac{\Lambda_p(x,n)}{(\Lambda_2(x,n))^{p/2}} \leq \frac{C_\epsilon}{(\Lambda_2(x,n))^{p/2}} + \epsilon$$

Limit (5.43) follows from (5.48).

(3) If condition (5.44) holds, then limit (5.45) follows from (5.49). □

Remark 5.4. One natural hypothesis that implies condition (5.44) holds is that $s_k = K_k(\lambda_k^2)$. In this case, $H(\lambda_k^2)\lambda_k^{6k} = 1$. Then, (5.44) follows from Proposition 5.3.

Since

$$Q_m = \frac{\beta^{-m} - (-1)^m \beta^m}{\sqrt{5}}$$

for all m, it follows that $m_0(n) \sim [\log_{\beta^{-1}}(n)] + 1$ as $n \to \infty$.

In the following result, we show that the Lyapunov condition 2.9 holds for all orbits starting at points of the form $x = f^l(0)$, $l \in \mathbb{N}$.

Theorem 5.7. 1. *If* $\{n_i\}$ *is an increasing sequence of integers that satisfy* (5.42), *then*

$$\lim_{i\to\infty} \sup_{x \in I_{m_0(n_i+l)}} \frac{\Lambda_p(f^l(x), n_i)}{(\Lambda_2(f^l(x), n_i))^{p/2}} = 0 \tag{5.52}$$

for all $p > 2$.

2. *Under condition* (5.44)

$$\lim_{i\to\infty} \sup_{x \in I_{m_0(n_i+l)}} \frac{\Lambda_p(f^l(x), n_i)}{(\Lambda_2(f^l(x), n_i))^{p/2}} = 0 \tag{5.53}$$

for all $p > 2$.

Proof. We will use a method similar to the one used in Theorem 4.15. since the proof of (1) and (2) are very similar, we will only prove (2).

Let $l \in \mathbb{N}$ be fixed. For any $n \in \mathbb{N}$, consider

$$n + l = Q_{m_0} + \cdots Q_{m_r}$$

where $\{m_j\}$ is as in (5.29). For $x \in I_{m_0(n+l)}$, define the sequence of returns $\{\upsilon_j\}_j$ with $\upsilon_{-1} = x$ as in (5.32). Notice that

$$\Lambda_p(f^l(x), n) = |\Psi_{n+l,0}(x)|^p A_p(x, Q_{m_0}) + \sum_{i=1}^{r} |\Psi_{n+l,i}(x)\|^p \Lambda_p(\upsilon_{i-1}, Q_{m_i}) \tag{5.54}$$

where

$$\Lambda_p(x, Q_{m_0}) - A_p(x, Q_{m_0}) = \sum_{j=1}^{l} \left| \left(f^{Q_{m_0}-j}\right) \circ f^j(x) \right|^p \tag{5.55}$$

From the identity

$$\left(f^{Q_{m_0}-j}\right)' \circ f^j(x) = \frac{\left(f^{Q_{m_0}}\right)'(x)}{\left(f^j\right)'(x)}$$

and Lemma 5.5, it follows that

$$\frac{c_j}{|\lambda_k|^{2km_0}} \leq \left| \left(f^{Q_{m_0}-j}\right)' \circ f^j(x) \right| \leq \frac{d_j}{|\lambda_k|^{2km_0}} \tag{5.56}$$

where c_j and d_j depend only on j. Since(5.55) contains l terms, Theorem (5.3) and Corollary 5.4 imply that

$$\frac{\Lambda_p(x,Q_{m_0})-A_p(x,Q_{m_0})}{\Lambda_p(x,Q_{m_0})}\le C_{l,p}\frac{1}{(\lambda_k^{2kp}\rho_p)^{m_0}}\to 0$$

Therefore,

$$\lim_{n\to}\sup_{x\in I_{m_0(n+l)}}\frac{\Lambda_p(f^l(x),n)}{\Lambda_p(x,n+l)}=1 \tag{5.57}$$

and the limit (5.53) follows from Proposition (5.6). □

5.6.4. *Proof of Theorems 2.4 and 2.5.* Recall that for large integers n, $m_0(n)\sim[\log_{\beta^{-1}}(n)]+1$ and r_n+1 equals the number of terms in the Fibonacci expansion of n.

By Theorem 2.1 and Theorem 5.7, it suffices to choose σ_n that satisfies (2.10).

Let c and d be as in Lemma 5.5. From (5.57) and Proposition 5.6, for each $p>0$ there there are constants $C_{p,l}$ and $D_{p,l}$ such that for all for all $x\in I_{m_0(n+l)}$,

$$C_{p,l}\left(\frac{(c\,\lambda_k^{6k})^{r_{n+l}}}{\lambda_k^{(2k-1)m_{r_{n+l}}}}\right)^p\le\frac{\Lambda_p(f^l(x),n)}{(\lambda_k^{2kp}\rho_p)^{m_0(n+l)}}\le D_{p,l}\left(\frac{d^{r_{n+l}}}{\lambda_k^{(2k-1)m_{r_{n+l}}}}\right)^p$$

Therefore, we have that

$$1\le\frac{(\widehat{\Lambda}(f^l(x),n))^3}{\sqrt{\Lambda_2(f^l(x),n)}}\le C\left(\frac{d^3}{c\,\lambda_k^{2k}}\right)^{r_{n+l}}\left(\frac{\rho_1^3}{\sqrt{\rho_2}}\right)^{m_0(n+l)} \tag{5.58}$$

for some constant $C>0$.

If we consider an increasing sequence of iterations $\{n_k\}_k$ which have lacunar Fibonacci expansions, see (5.42). Then, from (5.58) it follows that it suffices to consider $\sigma_{n_k}=n_k^{-(\gamma+1)}$ to have a central limit holds along $\{n_k\}$.

Otherwise, under condition (5.44) we have the following cases:

Case (a) If

$$\left|\frac{d^3}{c\,\lambda_k^{2k}}\right|\le 1 \tag{5.59}$$

then, it suffices to consider

$$\sigma_m=\frac{1}{n^{\delta+1}}$$

with $\delta>\log_{\beta^{-1}}(\rho_1^3/\sqrt{\rho_2})$.

Case (b) If

$$\left| \frac{d^3}{c\,\lambda_k^{2k}} \right| > 1, \tag{5.60}$$

it is enough to consider

$$\sigma_n = \frac{1}{n^{\delta^*+1}}$$

with $\delta^* > \log_{\beta^{-1}}(d^3\rho_1^3) + \log_{\beta^{-1}}(c\,\lambda_k^{2k}/\sqrt{\rho_2})$.

The Berry–Esseen estimates follow by choosing σ_n that satisfies (2.15) in Theorem 2.2.

If (5.59) holds, then it suffices to consider

$$\sigma_n = \frac{1}{n^{\tau}}$$

with $\tau > \log_{\beta^{-1}}(\rho_1^3/\sqrt{\rho_2}) + \log_{\beta^{-1}}(\rho_2^3/\rho_3) + 1$.

In the case (5.60), it suffices to consider

$$\sigma_n = \frac{1}{n^{\tau^*}}$$

with $\tau^* > \log_{\beta^{-1}}(d^3\rho_1^3) + \log_2(c\,\lambda_k^{2k}/\sqrt{\rho_2}) + \log_2(\rho_2^3/\rho_3)\ +1$ □

Remark 5.5. We know from the theory of critical circle maps that $\{f^l(0) \mod 1 : l \in \mathbb{N}\}$ is dense in $\mathbb{T}^1$. Numerics in [DEdlL06] suggests that the Lyapunov condition holds for all points in the circle. We conjecture that this is indeed the case and that that the effective noise of orbits starting in the any arbitrary point of $\mathbb{T}^1$ approaches a Gaussian. We expect the the speed of convergence in the central limit theorem to be not uniform.

6. Possible extensions of the results

In this section, we suggest several extensions of the results in this paper that are presumably be accessible.

1. Assume that (ξ_n) is a sequence of independent random variables with mean zero p finite moments such that

$$A_- n^{\alpha_-} \le \|\xi_n\|_2 \le \|\xi_n\|_p \le A_+ n^{\alpha_+},$$

with some $\alpha_\pm$ in a small range.
2. Assume that the random variables (ξ_n) are weakly correlated (e.g. Martingale approximations).

These assumptions are natural in dynamical systems applications when the noise is generated by a discrete process. That is

$$\begin{aligned} x_{n+1} &= f(x_n) \quad + \quad \sigma\psi(y_n) \\ y_{n+1} &= h(y_n) \end{aligned}$$

and h is an expanding map or an Anosov system.

3. Related to the central limit theorem (even in the case independent random variables (ξ_n) of comparable sizes), it also would be desirable to obtain higher order asymptotic expansions in the convergence to Gaussian, namely Edgeworth expansions.
4. We note that the estimates for the asymptotic growth of the variance of the effective noise ((4.23) with $p = 2$) for systems at the accumulation of period doubling are obtained in [VSK84] using the Thermodynamic formalism. We think that it would be very interesting to develop analogues to the log–convexity properties of the Lindeberg–Lyapunov operators or the Edgeworth expansions with the thermodynamic formalism.

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