GROUND STATES AND CRITICAL POINTS FOR GENERALIZED FRENKEL-KONTOROVA MODELS

RAFAEL DE LA LLAVE AND ENRICO VALDINOCI

Abstract. We consider a multidimensional model of Frenkel-Kontorova type but we allow non-nearest neighbor interactions.

For every possible frequancy vector, we show that there are quasi-periodic ground states which enjoy further geometric properties. The gound states we produce are either bigger or smaller than the state. They are are at bounded distance from the plane wave with the given frequency.

The comparison property above implies that the ground states and the translations are organized into laminations. If these leave a gap, we show that there are critical points inside the gap which also satisfy the comparizon properties.

In particular, given any frequency, we show that either there is a continuous parameter of ground states or there is a ground state and another critical point which is not a ground state.

Introduction

This paper deals with existence and multiplicity results for periodic and quasiperiodic equilibria of generalized Frenkel-Kontorova models.

The equilibrium solutions we construct have further geometric properties, such as satisfying comparison properties and lying at a bounded distance from a linear function. The results presented here generalize some well known results in Aubry-Mather theory. We allow the lattices to be of *any dimension* and we also allow interactions which are *not nearest neighbor*. We also allow *Degenerate twists*.

Our Theorem 1 constructing minimizers generalizes the main result in [A83, AD83, M82], whereas our Theorem 2 constructing solutions which are not minimizers is an extension of the main result of [Ma86]. Our results and techniques may be also applied to more general lattices and group actions (see [LlV06a, LlV06b] for a more abstract framework). Arguments along the same line as the ones presented here apply to PDEs too but, then, one has to pay attention to regularity, existence etc., which are easier issues in our case (see [LlV05]).

The Aubry-Mather theory had its origin in [A83, AD83, M82] and important precedents may be considered the works of [M24, H32, P74, P79]. A deep connection with the theory of viscosity solutions in PDEs has been developed too (see, e.g., [F97]). The techniques we use here will be a combination of the dynamical system and the PDE approaches.

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The situation we consider deals with a lattice $\mathbb{Z}^d$. At every site, we have a particle whose state can be described by one real parameter.

A configuration is an asignation of a state to each of the sites. That is, a configuration is $u_i \in \mathbb{R}$, with $i \in \mathbb{Z}^d$.

Given a configuration, we consider the formal variational principle

$$S(u) := \sum_{i,j\in\mathbb{Z}^d} a_{ij}|u_i - u_j - c_{ij}|^{p_{ij}} + \sum_{i\in\mathbb{Z}^d} V(u_i)\,.$$

We suppose that $p_{ij} = \bar{p}(|i-j|)$, $c_{ij} = \bar{c}(|i-j|)$, $a_{ij} := \bar{a}(|i-j|)$, with $2 \le \bar{p} \le \Lambda$, $-\Lambda \le \bar{c} \le \Lambda$, $0 \le \bar{a} \le \Lambda$ and $\bar{a}(r) = 0$ if $r \in \{0\} \cup [R, +\infty)$, for suitable Λ, $R > 0$. The last assumption means that the interaction is finite range. A discussion of how one can extend our results by considering infinite range interactions as the limit of finite range ones is in [LlV06a].

We also suppose that V is smooth and that $V(r+1) = V(r)$ for any $r \in \mathbb{R}$.

From the physical point of view, such a model may be seen as a system of (possibly nonlinear and nonhomogeneus) interacting springs in a background periodic potential. Another possible physical interpretation is spin waves. It is a generalization of the so called Frenkel-Kontorova model. Notice that we allow lattices of higher dimensions and the interactions to be nonnearest neighbor, which is very natural for spin-waves. For simplicity, we require only finite range. The form of the interaction – chosen here to be (possibly nonlinear) harmonic oscillators – is not too relevant. The only properties that matter are some convexity properties (configurations tend to be aligned), and the fact that the interaction is coercive (it penalizes close particles for becoming very different). See [LlV06a].

From the mathematical viewpoint, the formal functional S may be seen as a discretization of analogous ones arising in the calculus of variations in relation with degenerate elliptic PDEs. The degeneration, which corresponds to the case $p_{ij} > 2$, then reflects into the flatness of the twist condition in the dynamical system framework.

The following definitions are standard in the calculus of variations.

We say that a configuration u_i is a *ground state* if

$$\begin{aligned}(1)\qquad &\sum_{\substack{i,j\in\mathbb{Z}^d\\ \min\{|i|,|j|\}\le K}} a_{ij}|u_i - u_j - c_{ij}|^{p_{ij}} + \sum_{\substack{i\in\mathbb{Z}^d\\ |i|\le K}} V(u_i)\\ &\qquad \le \sum_{\substack{i,j\in\mathbb{Z}^d\\ \min\{|i|,|j|\}\le K}} a_{ij}|v_i - v_j - c_{ij}|^{p_{ij}} + \sum_{\substack{i\in\mathbb{Z}^d\\ |i|\le K}} V(v_i)\,,\end{aligned}$$

for any configuration v_i such that $v_j = u_j$ as long as $|j| > K$, for any given $K > 0$.

We say that u_i is a *critical point* if

$$\begin{aligned}(2)\qquad &\sum_{j\in\mathbb{Z}^d} a_{ij}p_{ij}(u_i - u_j)\Big[|u_i - u_j - c_{ij}|^{p_{ij}-2} + |u_j - u_i - c_{ij}|^{p_{ij}-2}\Big] + V'(u_i)\\ &= \sum_{j\in\mathbb{Z}^d} a_{ij}p_{ij}c_{ij}\Big[|u_i - u_j - c_{ij}|^{p_{ij}-2} - |u_j - u_i - c_{ij}|^{p_{ij}-2}\Big]\end{aligned}$$

for any $i \in \mathbb{Z}^d$ (of course, the case of ideal springs $p_{ij} := 2$ and $c_{ij} := 0$ is the easiest to visualize).

Note that the series in (1) and (2) are just finite sums. Roughly, (1) says that, for a ground state, any change of a finite number of sites increases the energy (note that we sum the change of energy in the terms which are affected). Equation (2) is just the Euler-Lagrange equation of the functional S. In general, the energy functional S is only a formal sum and may well not converge, but the definitions of ground state and critical point make sense since they involve only finite sums.

Ground states are critical points, as it is easily checked.

The first result we present deals with the existence of ground states which lie at a bounded distance from any given linear function. These ground states have an interesting geometric property (known in the literature under the name of Birkhoff or selfconforming see (4) below). We will interpret geometrically this property by saying that the ground states and their translations are organized in a lamination.

Theorem 1. *Given any vector $\sigma \in \mathbb{R}^n$, there exists a ground state u_i such that*

$$|u_i - \sigma \cdot i| \leq 2\,, \tag{3}$$

for any $i \in \mathbb{Z}^d$.

Moreover,

$$\begin{aligned} &\text{if } j \in \mathbb{Z}^d \text{ and } s \in \mathbb{Z} \text{ are such that } \sigma \cdot j \geq s, \\ &\text{we have that } u_{i+j} \geq u_i + s \text{ for any } i \in \mathbb{Z}^d. \end{aligned} \tag{4}$$

We observe that (4) implies that $u_{i+j} = u_i$ for any $j \in \mathbb{Z}^d \cap \sigma^\perp$, and thus, for $\sigma \in \mathbb{Q}^d$, it is just a periodicity condition. When $\sigma \in \mathbb{R}^d \setminus \mathbb{Q}^d$, we obtain that the solutions are quasiperiodic since they are the limit of periodic solutions. Here we follow the notation common in variational method of calling quasiperiodic the limits in whatever sense of periodic solutions. In other areas of dynamics, one uses a different name depending on the different meanings of limit (e.g. pointwise, uniform etc.) So, our solutions could be called quasiautomorphic in the language of topological dynamics.

The property in (4) says that the integer translation of the graph are all comparable functions. Furthermore, it is easy to see that the pointwise limit of these functions is also a ground state. We will call *lamination* a set of comparable functions whose graphs in $\mathbb{Z}^d \times (\mathbb{R}/\mathbb{Z})$ are a closed set under pointwise limits. These definition is very similar to the usual definition of laminations in topology. The graphs of each of the functions is a sheet. Nevertheless, in our case, we will not assume that different sheets are disjoint. We allow that they can touch, even if they cannot cross. As we will see later, with an extra assumption on the interaction we will prove a maximum principle that shows that the graphs of comparable equilibria have to be disjoint.

The ground states satisfying (4) are organized in laminations, hence it is natural to formulate the results for laminations. It can happen that the lamination covers $\mathbb{Z}^d \times (\mathbb{R}/\mathbb{Z})$ – this is analogue of the case of existence of invariant circles for twist maps – or that it contains gaps, which is an analogue of the minimizers being a Cantor set.

Our next result shows that, in the case that a lamination of minimizers contains gaps, we can find a further critical point in the gap.

Theorem 2. *Assume that the graphs of $u_i^{(0)}$ and $u_i^{(1)}$ belong to a lamination of ordered ground states as produced in Theorem 1. Suppose that $u_i^{(1)} \geq u_i^{(0)}$ for any $i \in \mathbb{Z}^d$ and that $u_{i_o}^{(1)} > u_{i_o}^{(0)}$ for some $i_o \in \mathbb{Z}^d$.*

Then, there exists a critical point u_i satisfying (4), different from $u_i^{(0)}$ and $u_i^{(1)}$, such that $u_i^{(1)} \geq u_i \geq u_i^{(0)}$ for any $i \in \mathbb{Z}^d$.

Under a further assumption, the result of Theorem 2 may be sharpened by obtaining a strict order property. For this, we say that the interaction is *transitive* if given any $i,\, j \in \mathbb{Z}^d$ there exist $i_1, \dots, i_\ell \in \mathbb{Z}^d$ in such a way that

$$a_{ii_1}\,,\, a_{i_1 i_2}\,,\, \dots, a_{i_\ell j} \neq 0\,.$$

Physically, this means that two sites are always connected by a certain system of springs. Then, Theorem 2 takes the following stronger form:

Corollary 1. *Suppose that the interaction is transitive. Assume that the graphs of $u_i^{(0)}$ and $u_i^{(1)}$ belong to a lamination of ordered ground states as produced in Theorem 1. Suppose that $u_i^{(1)} \geq u_i^{(0)}$ for any $i \in \mathbb{Z}^d$ and that $u_{i_o}^{(1)} > u_{i_o}^{(0)}$ for some $i_o \in \mathbb{Z}^d$.*

Then, there exists a critical point u_i satisfying (4) *and such that $u_i^{(1)} > u_i > u_i^{(0)}$ for any $i \in \mathbb{Z}^d$.*

One of the tool of the proofs of Theorem 2 and Corollary 1 will be a comparison property (see Lemmata 1 and 2 below).

The proofs of Theorems 1 and 2 that we present here are remarkably simple and they depend on very few elements of the problem. Thus, the ideas we expose may be applied to quite general variational problems which satisfy some sort of "ferromagnetism", "coercivity" and "invariance under translations" (see [LlV06a, LlV06b]).

Comparison Principles

We will deal with the following differential equation:

$$(5)\qquad \begin{cases} -\partial_t \Phi^t(u)_i = \displaystyle\sum_{j\in\mathbb{Z}^d} a_{ij} p_{ij} (\Phi^t(u)_i - \Phi^t(u)_j) \Big[|\Phi^t(u)_i - \Phi^t(u)_j - c_{ij}|^{p_{ij}-2} \\ \qquad\qquad + |\Phi^t(u)_j - \Phi^t(u)_i - c_{ij}|^{p_{ij}-2} \Big] + V'(\Phi^t(u)_i) \\ \qquad\qquad + \displaystyle\sum_{j\in\mathbb{Z}^d} a_{ij} p_{ij} c_{ij} \Big[|\Phi^t(u)_j - \Phi^t(u)_i - c_{ij}|^{p_{ij}-2} \\ \qquad\qquad - |\Phi^t(u)_i - \Phi^t(u)_j - c_{ij}|^{p_{ij}-2} \Big] \qquad \text{for } t > 0, \\ \\ \qquad\qquad \Phi^0(u)_i = u_i\,. \end{cases}$$

Here, we will take $t \in [0, T^\star]$, for some $T^\star > 0$. The above Cauchy problem is the gradient flow of S.

If we fix a frequency ω, we can consider the RHS defined on the affine space consisting of functions u such $|u_i - \omega \cdot i|$ is bounded uniformly in i. We denote this space as X_ω. It is an affine space modelled on $\ell^\infty(\mathbb{Z}^d)$. We note that when $p \geq 2$, the RHS is a C^1 mapping from X_ω to ℓ^∞. Furtehrmore, the Lipschitz constant is uniform on bounded sets. Therefore, applying the standard theory of existence and regularity of solutions of equations on Banach spaces (or Banach manifolds) (see e.g. [H80]) we can conclude that, when the initial data u range over a bounded set, the flow is uniquely defined for a time $T > 0$ and, moreover that the flow depends C^1 on the initial conditions. Furthermore, we can compute the derivative of the flow integrating the variational equations.

Our next goal is to use the variational equations to show that there are Comparison Principles for the equation. We will prove a weak one, Lemma 1 and a strong one, Lemma 2 which requires an extra hypothesis. As we will see later, as a consequence of the Comparison Principles, we will obtain that, in the situations we are interested, the flow can be extended for all times.

The Comparison Principles we obtain are the following:

Lemma 1 (Weak Comparison Principle)**.** *Let* (5) *holds. Suppose that* $u_i \geq v_i$ *for all* $i \in \mathbb{Z}^d$*. Then* $\Phi^t(u)_i \geq \Phi^t(v)_i$ *for any* $i \in \mathbb{Z}^d$ *and any* $t \in [0, T^\star]$.

Lemma 2 (Strong Comparison Principle)**.** *Let the interaction be transitive. Let the assumptions of Lemma 1 hold. If also* $u \not\equiv v$*, then* $\Phi^t(u)_i > \Phi^t(v)_i$ *for any* $i \in \mathbb{Z}^d$ *and any* $t \in [0, T^\star]$.

As we will now see in the course of the proof, Lemma 1 uses only that the nondiagonal elements of the variational equation are nonnegative, while Lemma 2 exploits the fact that the variational equations have a positive kernel.

In order to prove Lemma 1 we argue as follows. For $\lambda \in [0,1]$, let $u_i^\lambda := \lambda u_i + (1-\lambda)v_i$ and differentiate (5) to obtain that

$$-\partial_t(\partial_\lambda \Phi^t(u^\lambda)_i) \;=\; \delta_i^\lambda(t)\,\partial_\lambda\Phi^t(u^\lambda)_i - \sum_{\substack{j\in\mathbb{Z}^d\\ 0<|i-j|\leq R}} \mu_{ij}^\lambda(t)\,\partial_\lambda\Phi^t(u^\lambda)_j\,, \tag{6}$$

for suitable $\delta_i^\lambda(t)$ and $\mu_{ij}^\lambda(t)$. The main property we use is that

$$\begin{aligned}\mu_{ij}^\lambda(t) \;&=\; a_{ij}p_{ij}(p_{ij}-1)\Big(|\Phi^t(u^\lambda)_i - \Phi^t(u^\lambda)_j - c_{ij}|^{p_{ij}-2} \\ &\qquad + |\Phi^t(u^\lambda)_j - \Phi^t(u^\lambda)_i - c_{ij}|^{p_{ij}-2}\Big) \geq 0\,.\end{aligned} \tag{7}$$

Let now

$$C_i^\lambda(t) := e^{\int_0^t \delta_i^\lambda(s)\,ds}\,\partial_\lambda\Phi^t(u^\lambda)_i\,.$$

Then, from (6),

$$\partial_t C_i^\lambda(t) \;=\; \sum_{\substack{j\in\mathbb{Z}^d\\ 0<|i-j|\leq R}} \nu_{ij}^\lambda(t)\,C_j^\lambda(t)\,, \tag{8}$$

with

$$\nu_{ij}^\lambda(t) \;:=\; e^{\int_0^t \delta_i^\lambda(s)\,ds} e^{-\int_0^t \delta_j^\lambda(s)\,ds}\mu_{ij}^\lambda(t) \;\geq\; 0$$

and

$$C_i^\lambda(0) = u_i - v_i \geq 0\,. \tag{9}$$

Therefore, $C_i^\lambda(t) \geq 0$ and so $\partial_\lambda\Phi^t(u^\lambda)_i \geq 0$ (as can be proved, for small $t > 0$, via Picard approximation and, for any $t \in (0, T^\star]$, by iterating the ODE flow). Accordingly,

$$\Phi^t(u)_i = \Phi^t(u^1)_i \geq \Phi^t(u^\lambda)_i \geq \Phi^t(u^0)_i = \Phi^t(v)_i\,, \tag{10}$$

for any $\lambda \in [0,1]$, which completes the proof of Lemma 1.

Let us now deal with the proof of Lemma 2. If

$$\Phi^{t_o}(u)_{i_o} = \Phi^{t_o}(v)_{i_o} \text{ for some } i_o \in \mathbb{Z}^d \text{ and some } t_o > 0, \tag{11}$$

we gather from (10) that $\partial_\lambda\Phi^{t_o}(u^\lambda)_{i_o} = 0$, thence

$$C_{i_o}^\lambda(t_o) = 0 \tag{12}$$

for any $\lambda \in (0,1)$.

Since $C^\lambda_{i_o}(0) \geq 0$ and the map $t \mapsto C^\lambda_{i_o}(t)$ is nondecreasing due to (8), we conclude that

$$C^\lambda_{i_o}(t) = 0 \text{ for any } t \in [0, t_o] \text{ and any } \lambda \in (0,1). \tag{13}$$

In particular,

$$0 = C^\lambda_{i_o}(0) = u_{i_o} - v_{i_o}\,. \tag{14}$$

Now, we prove that

$$\begin{array}{c}\text{if (11) holds,}\\ \text{then } \phi^\tau(u)_j = \phi^\tau(v)_j \text{ for any } \tau \in [0,t_o] \text{ and any } j \in \mathbb{Z}^d \text{ such that } a_{i_o j} \neq 0.\end{array} \tag{15}$$

To confirm this, we first note that, up to replacing u_i and v_i with $\Phi^\tau(u)_i$ and $\Phi^\tau(v)_i$, we may and do assume that $\tau = 0$ in (15). We also observe that, given j in such a way that $u_j \neq v_j$, there exist $\lambda_j \in (0,1)$ in such a way that

$$u^{\lambda_j}_{i_o} \neq u^{\lambda_j}_j\,. \tag{16}$$

Indeed, if $u^\lambda_{i_o} = u^\lambda_j$ for any λ, we would have that

$$0 = \partial_\lambda(u^\lambda_{i_o} - u^\lambda_j) = (u_{i_o} - v_{i_o}) - (u_j - v_j) = 0 - u_j + v_j\,,$$

due to (14), which is a contradiction.

Moreover, from (13) and (8), $\nu^\lambda_{i_o j}(t) C^\lambda_j(t) = 0$ for any $j \in \mathbb{Z}^d$, any $t \in [0,t_o]$ and any $\lambda \in (0,1)$. In particular, if, by contradiction, $u_j \neq v_j$ for some j with $a_{i_o j} \neq 0$, we would have that

$$0 = \nu^{\lambda_j}_{i_o j}(0)\, C^{\lambda_j}_j(0) = \mu^{\lambda_j}_{i_o j}(0)\,(u_j - v_j)$$

and so $\mu^{\lambda_j}_{i_o j}(0) = 0$. By (7), this would give that $u^{\lambda_j}_j = u^{\lambda_j}_{i_0}$, in contradiction with (16). This says that $u_j = v_j$ for any j as long as $a_{i_o j} \neq 0$, yielding (15).

The proof of Lemma 2 is now ended by recursion, thanks to the fact that the interaction is transitive. For this, let t_o and i_o be as in (11), take any $j \in \mathbb{Z}^d$ and suppose that

$$a_{i_o i_1},\, a_{i_1 i_2},\, \ldots, a_{i_\ell j} \neq 0\,,$$

for suitable $i_1, \ldots, i_\ell$. Then, by (11) and (15), we have that $\Phi^\tau(u)_{i_1} = \Phi^\tau(v)_{i_1}$ for any $\tau \in [0,t_o]$. Then, by applying (15) again, we get that $\Phi^\tau(u)_{i_2} = \Phi^t(v)_{i_2}$ for any $\tau \in [0,t_o]$. By repeatedly applying (15), we thus conclude that $\Phi^\tau(u)_{i_k} = \Phi^\tau(v)_{i_k}$ for $k = 1, \ldots, \ell$ and so $\Phi^\tau(u)_j = \Phi^\tau(v)_j$ for any $\tau \in [0,t_o]$. In particular, $u_j = v_j$.

The proof of Lemma 2 is thus concluded. □

As a straightforward consequence of Lemma 2, we obtain:

Corollary 2. *Let the interaction be transitive. Let u_i and v_i be two critical points in such a way that $u_i \geq v_i$ for any $i \in \mathbb{Z}^d$ and $u_{i_o} > v_{i_o}$ for some $i_o \in \mathbb{Z}^d$.*

Then, $u_i > v_i$ for any $i \in \mathbb{Z}^d$.

Therefore, when the interaction is transitive, we see that when we consider a lamination of critical points, the sheets are disjoint, in more agreement with the common usage in topology. If the interaction is not transitive, the conclusions of Corollary 2 could be false. For example, if we consider a layered material in which different layers do not interact, it is easy to have minimizers so that they agree on one layer but are different – but comparable – on the others.

Proof of Theorem 1

The proof we present uses the rearrangement method of [CLl00] and it is based the construction of a "minimal minimizer" which possesses additional geometric properties. Further details and generalizations are given in [LlV06a].

We will first consider the case $\sigma \in \mathbb{Q}^d$ and prove good uniform estimates. The case $\sigma \in \mathbb{R}^d \setminus \mathbb{Q}^d$ is then obtained by a limit process.

We fix some finite index subgroup $\mathcal{K}$ of

$$\mathcal{K}(\sigma) := \{k \in \mathbb{Z}^d \text{ s.t. } \sigma \cdot k \in \mathbb{Z}\} \tag{17}$$

and we consider the **periodic configurations** in

$$\mathcal{C}_{\mathcal{K}} := \Big\{ u \text{ s.t. } \sup_{i \in \mathbb{Z}^d} |u_i - \sigma \cdot i| \le 10 \text{ and } u_{i+j} = u_i + \sigma \cdot j \text{ for any } j \in \mathcal{K} \Big\}. \tag{18}$$

In what follows, we identify, with a slight abuse of notation, the quotient $\mathbb{Z}^d/\mathcal{K}$ with an integer polytope centered at the origin.

We can assume without loss of generality that the sides are much larger than the range of the interaction.

We now **quotient our variational problem** with respect to $\mathcal{K}$. That is, given $i \in \mathbb{Q}^d$, we consider the equivalence class

$$[i] := \{i' \in \mathbb{Q}^d \text{ s.t. } i - i' \in \mathcal{K}\}.$$

Analogously, given i, $j \in \mathbb{Q}^d$, we set

$$[(i,j)] := \{(i',j') \in \mathbb{Q}^d \times \mathbb{Q}^d \text{ s.t. } i - i' = j - j' \in \mathcal{K}\}.$$

Given any $i \in \mathbb{Q}^d$ (resp., i, $j \in \mathbb{Q}^d$), we then choose one element $i_{\mathcal{K}} \in [i]$ (resp., $(i_{\mathcal{K}}, j_{\mathcal{K}}) \in [(i,j)]$).

We also consider the quotient energy functional

$$S_{\mathcal{K}}(u) := \sum_{(i_{\mathcal{K}}, j_{\mathcal{K}})} a_{i_{\mathcal{K}} j_{\mathcal{K}}} |u_{i_{\mathcal{K}}} - u_{j_{\mathcal{K}}} - c_{ij}|^{p_{ij}} + \sum_{i_{\mathcal{K}}} V(u_{i_{\mathcal{K}}}). \tag{19}$$

The functional $S_{\mathcal{K}}$ is well defined for $u \in \mathcal{C}_{\mathcal{K}}$ and it does not depend on the choice of $i_{\mathcal{K}}$, $j_{\mathcal{K}}$. We also note that the series in (19) are just finite sums. Indeed, the set $\mathbb{Z}^d/\mathcal{K}$ is finite, since $\sigma \in \mathbb{Q}^d$ and $\mathcal{K}$ has finite index in $\mathcal{K}(\sigma)$, thus we may always think that the contribution of the above series is given by some $i_{\mathcal{K}}$ in such a finite set and by $j_{\mathcal{K}}$ satisfying $|i_{\mathcal{K}} - j_{\mathcal{K}}| < R$, which, again, is a finite set. Note that the value of $S_{\mathcal{K}}$ on $\mathcal{C}_{\mathcal{K}} \ni u$ is determined once we know u_i for $i \in \mathbb{Z}^d/\mathcal{K}$, which is a finite set. Therefore, minimizing $S_{\mathcal{K}}$ on $\mathcal{C}_{\mathcal{K}}$ is a finite dimensional problem. It is easy to check that if the distance between two sites goes to infinity, the energy goes to infinity. The problem is also invariant by addition of an integer to the state. Consequently, the set $\mathcal{M}_{\mathcal{K}}$ of **minimizers** of $S_{\mathcal{K}}$ on $\mathcal{C}_{\mathcal{K}}$ is non empty.

Moreover, if $u \in \mathcal{M}_{\mathcal{K}}$, then so does $v := u + k$ for any $k \in \mathbb{Z}$ and, if

$$k \ge \max_{i,\ell \in \mathbb{Z}^d/\mathcal{K}} |u_{i+\ell}| + |\sigma| \max_{j \in \mathbb{Z}^d/\mathcal{K}} |j|,$$

we have that, given any $i \in \mathbb{Z}^d/\mathcal{K}$ and $j \in \mathbb{Z}^d$, with $j = j_1 + j_2$, $j_1 \in \mathbb{Z}^d/\mathcal{K}$ and $j_2 \in \mathcal{K}$,

$$v_{i+j} = u_{i+j_1+j_2} + k = u_{i+j_1} + \sigma \cdot j_2 + k = u_{i+j_1} + \sigma \cdot j - \sigma \cdot j_1 + k \ge \sigma \cdot j.$$

Consequently the set

$$\mathcal{M}^{\star}_{\mathcal{K}} := \mathcal{M}_{\mathcal{K}} \cap \{u_{i+j} \ge \sigma \cdot j \text{ for any } i \in \mathbb{Z}^d/\mathcal{K} \text{ and } j \in \mathbb{Z}^d\}$$

is non empty.

Given two configurations u_i and v_i we now consider the configurations

$$\min\{u,v\}_i := \min\{u_i, v_i\} \quad \text{and} \quad \max\{u,v\}_i := \max\{u_i, v_i\}.$$

The following inequality will be fundamental for our argument:

$$S_{\mathcal{K}}(\min\{u,v\}) + S_{\mathcal{K}}(\max\{u,v\}) \le S_{\mathcal{K}}(u) + S_{\mathcal{K}}(v)\,. \tag{20}$$

To prove (20), we set $\alpha_i := \max\{u,v\}_i - u_i$ and $\beta_i := u_i - \min\{u,v\}_i$. Then $\alpha_i\beta_i \ge 0$, $\alpha_i\beta_i = 0$ and $v_i = u_i + \alpha_i - \beta_i$. Thus, if we consider the function

$$[-1,1]^2 \ni (s,t) \longmapsto F(s,t) := |(u + t\alpha + s\beta)_i - (u + t\alpha + s\beta)_j - c_{ij}|^{p_{ij}}\,,$$

we have that

$$\partial^2_{st} F = -\sum_{i,j} p_{ij}(p_{ij}-1)|(u+t\alpha+s\beta)_i - (u+t\alpha+s\beta)_j - c_{ij}|^{p_{ij}-2}(\alpha_i\beta_j + \alpha_j\beta_i) \le 0\,.$$

Accordingly,

$$\begin{aligned} 0 \quad &\ge \quad \int_0^1 \int_{-1}^0 \partial^2_{st} F \, ds\, dt \\ &= \quad |\max\{u,v\}_i - \max\{u,v\}_j - c_{ij}|^{p_{ij}} + |\min\{u,v\}_i - \min\{u,v\}_j - c_{ij}|^{p_{ij}} \\ &\quad\; -|u_i - u_j - c_{ij}|^{p_{ij}} - |v_i - v_j - c_{ij}|^{p_{ij}}\,, \end{aligned}$$

from which (20) plainly follows.

The above considerations imply that

$$u_i^{\mathcal{K}} := \min_{u \in \mathcal{M}^\star_{\mathcal{K}}} u_i \tag{21}$$

is also in $\mathcal{M}^\star_{\mathcal{K}}$ and it thus will be called the **minimal minimizer**.

One of the main feature of $u^{\mathcal{K}}$ is that it satisfies the **no symmetry breaking** condition, namely, that

$$u^{\mathcal{K}} = u^{\mathcal{K}(\sigma)} \tag{22}$$

for any finite index subgroup $\mathcal{K}$ of $\mathcal{K}(\sigma)$.

To confirm (22), we argue as follows. Given $k \in \mathcal{K}(\sigma)$, we observe that $\tilde u \in \mathcal{M}^\star_{\mathcal{K}}$, where $\tilde u_i := u^{\mathcal{K}}_{i+k} - \sigma \cdot k$, and therefore $u_i^{\mathcal{K}} \le \tilde u_i$ and so

$$u_i^{\mathcal{K}} \le \min_{k \in \mathcal{K}(\sigma)} u^{\mathcal{K}}_{i+k} - \sigma \cdot k\,.$$

Since the other inequality is obvious, we have that

$$u_i^{\mathcal{K}} = \min_{k \in \mathcal{K}(\sigma)} u^{\mathcal{K}}_{i+k} - \sigma \cdot k$$

and so that

$$u_i^{\mathcal{K}} = u^{\mathcal{K}}_{i+k} - \sigma \cdot k$$

for any $k \in \mathcal{K}(\sigma)$.

In particular, $u^{\mathcal{K}}$ belongs to $\mathcal{C}_{\mathcal{K}(\sigma)}$ (and not only to $\mathcal{C}_{\mathcal{K}}$). This and the fact that

$$S_{\mathcal{K}} = \sharp(\mathcal{K}(\sigma)/\mathcal{K})\, S_{\mathcal{K}(\sigma)}$$

on $\mathcal{C}_{\mathcal{K}(\sigma)}$ yield (22).

Also, if j and s are as in (4), we have that $v_i := u^{\mathcal{K}(\sigma)}_{i+j} - s \in \mathcal{M}^\star_{\mathcal{K}}$ and therefore $v_i \ge u_i^{\mathcal{K}(\sigma)}$, yielding the **order property** in (4).

Let now $j \in \mathbb{Z}^d$. Take s such that $\sigma \cdot j \in [s, s+1]$. Then, the order property in (4) implies that

$$\sigma \cdot j - 1 \le s \le u^{\mathcal{K}(\sigma)}_{i+j} - u^{\mathcal{K}(\sigma)}_i \le s+1 \le \sigma \cdot j + 1$$

thence

$$\sup_{i,j\in\mathbb{Z}^d} |u^{\mathcal{K}(\sigma)}_{i+j} - u^{\mathcal{K}(\sigma)}_i - \sigma \cdot j| \le 1\,. \tag{23}$$

We also observe that there must be some $j_o \in \mathbb{Z}^d$ such that $u^{\mathcal{K}(\sigma)}_{j_o} - \sigma \cdot j_o \in [0,1]$. Indeed, if not, the fact that $u^{\mathcal{K}(\sigma)} \in \mathcal{M}^\star_{\mathcal{K}}$ would give that $u^{\mathcal{K}(\sigma)}_j - \sigma \cdot j \ge 1$ for any $j \in \mathbb{Z}^d$, and so $v_i := u^{\mathcal{K}(\sigma)}_i - 1$ would also lie in $\mathcal{M}^\star_{\mathcal{K}}$, in contradiction with the definition of minimal minimizer given in (21).

Then, by (23),

$$|u^{\mathcal{K}(\sigma)}_i - \sigma \cdot i| \le |u^{\mathcal{K}(\sigma)}_i - u^{\mathcal{K}(\sigma)}_{j_o} - \sigma \cdot (i - j_o)| + 1 \le 2\,,$$

for any $i \in \mathbb{Z}^d$, thus giving (3) and the fact that $u^{\mathcal{K}(\sigma)}$ does not touch the boundary of $\mathcal{C}_{\mathcal{K}}$ as defined in (18).

Also, $u^{\mathcal{K}(\sigma)}$ **is a ground state**. Indeed, let $K > 0$ and v_i be a configuration such that $v_i = u^{\mathcal{K}(\sigma)}_i$ for $|i| > K$. We take $\ell \in \mathbb{N}$ so large that the set

$$\{j \in \mathbb{Z}^d \text{ s.t. } |i-j| \le R \text{ for some } |i| \le K\}$$

is contained in the fundamental polytope $\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))$.

We now introduce the "periodic extension" of v. Namely, given $i \in \mathbb{Z}^d$ with $i = i_1 + i_2$, $i_1 \in \mathbb{Z}^d/(\ell\mathcal{K}(\sigma))$ and $i_2 \in \ell\mathcal{K}(\sigma)$, we define $w_i := v_{i_1} + \sigma \cdot i_2$. Then, $w \in \mathcal{C}_{\ell\mathcal{K}(\sigma)}$ and so

$$S_{\ell\mathcal{K}(\sigma)}(w) \ge S_{\ell\mathcal{K}(\sigma)}(u^{\ell\mathcal{K}(\sigma)}) = S_{\ell\mathcal{K}(\sigma)}(u^{\mathcal{K}(\sigma)})\,.$$

This says that

$$\begin{aligned} &\sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)),\, |j-i|\le R} a_{ij}|w_i - w_j - c_{ij}|^{p_{ij}} + \sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))} V(w_i) \\ \ge\ &\sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)),\, |j-i|\le R} a_{ij}|u^{\mathcal{K}(\sigma)}_i - u^{\mathcal{K}(\sigma)}_j - c_{ij}|^{p+ij} + \sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))} V(u^{\mathcal{K}(\sigma)}_i)\,. \end{aligned}$$

Note that

$$\sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))} V(w_i) = \sum_{|i|\le K} V(v_i) + \sum_{\substack{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)) \\ |i|>K}} V(u^{\mathcal{K}(\sigma)}_i)\,.$$

Also, it is easily seen that if $i \in \mathbb{Z}^d/\mathcal{K}(\sigma)$, $|i-j| \le R$ and $|j| > K$, we have that $|j+h| > K$ for any $h \in \mathcal{K}(\sigma)$ and so

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)),\,|j-i|\le R} a_{ij}|w_i - w_j - c_{ij}|^{p_{ij}} \\
= &\sum_{|i|\le K} a_{ij}|v_i - v_j - c|^{p_{ij}} + \sum_{\substack{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)),\,|j-i|\le R\\ |i|>K\ge|j|}} a_{ij}|v_i - v_j - c_{ij}|^{p_{ij}} \\
&+ \sum_{\substack{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)),\,|j-i|\le R\\ |i|,|j|>K}} a_{ij}|v_i - w_j - c_{ij}|^{p_{ij}} \\
= &\sum_{\min\{|i|,|j|\}\le K} a_{ij}|v_i - v_j - c_{ij}|^{p_{ij}} + \sum_{\substack{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))\\ |i|,|j|>K}} a_{ij}|u_i^{\mathcal{K}(\sigma)} - u_j^{\mathcal{K}(\sigma)} - c_{ij}|^{p_{ij}}\,.
\end{aligned}$$

The above considerations yields that $u^{\mathcal{K}(\sigma)}$ is a ground state.

The lamination property is a consequence of (4).

This ends the proof of Theorem 1. $\square$

Proof of Theorem 2

The proof we provide is based on the gradient flow method of [A90,KLlR97, CLl98, LlV05]. Further details and generalizations are given in [LlV06b].

We fix $\sigma \in \mathbb{R}^d$ (not necessarily rational) and we consider the "resonant set" $\mathcal{K}(\sigma)$ given in (17). For $\ell \in \mathbb{N}$, with a slight abuse of notation, we identify the quotient spaces $\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))$ with a sequence of nested fundamental domains, that is, each $\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))$ is a polytope centered at the origin. Note that, when $\sigma \in \mathbb{R}^d \setminus \mathbb{Q}^d$, some sides of such a polytope may have infinite length and, in fact, if $\mathcal{K}(\sigma) = \{0\}$ then $\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)) = \mathbb{Z}^d$. In any case, $\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)) \subset \mathbb{Z}^d/((\ell+1)\mathcal{K}(\sigma)) \subset \dots$ is a sequence of polytopes invading the whole $\mathbb{Z}^d$. We also remark that, as a consequence of the periodicity and lamination properties,

$$\sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))} |u_i^{(1)} - u_i^{(0)}| \le \sharp\Big(\mathcal{K}(\sigma)/(\ell\mathcal{K}(\sigma))\Big)\,. \tag{24}$$

We define the space

$$\mathcal{C} := \{u_i \text{ satisfying (4) s.t. } u_i^{(0)} \le u \le u_i^{(1)} \text{ for any } i \in \mathbb{N}\}$$

and, for u, $v \in \mathcal{C}$, we consider the **renormalized energy**

$$\begin{aligned}
S_\ell(u,v) \;:=\; &\sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma)),j\in\mathbb{Z}^d} a_{ij}\Big(|u_i - u_j - c_{ij}|^{p_{ij}} - |v_i - v_j - c_{ij}|^{p_{ij}}\Big) \\
&+ \sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))} \Big(V(u_i) - V(v_i)\Big)\,.
\end{aligned} \tag{25}$$

Note that

$$\begin{aligned}
&u_i - u_j \pm c_{ij} \le u_i^{(1)} - u_j^{(0)} + \Lambda \\
&\le |u_i^{(1)} - \sigma\cdot i| + |\sigma\cdot j - u_j^{(0)}| + |\sigma|\,|i-j| + \Lambda \le 4 + |\sigma|\,R + \Lambda\,,
\end{aligned} \tag{26}$$

if $u \in \mathcal{C}$ and $|i-j| \le R$, because of (3). Then, the series in (25) are convergent, thanks to (24) and (26).

Notice also that if $v^{(n)} \in \mathcal{C}$ is so that

$$\lim_{n\to+\infty} v_i^{(n)} = v_i \quad \text{for any } i \in \mathbb{Z}^d/(\ell\mathcal{K}(\sigma))$$

it follows from (24) and the Dominated Convergence Theorem that

$$\lim_{n\to+\infty} \sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))} |v_i^{(n)} - v_i| \,=\, 0\,,$$

thence the pointwise and uniform convergence agree in $\mathcal{C}$.

We now show that **the renormalized energy is good in detecting the ground states**. Namely, we prove that, if $u \in \mathcal{C}$,

(27) $\quad S_\ell(u, u^{(0)}) \geq 0$ and equality holds if and only if u is a ground state.

The proof of such a statement will be accomplished in several steps. First, we show that

(28) $\quad$ if v is a ground state, $S_\ell(u, v) \geq 0$.

In the proof of (28), we may assume, without loss of generality, that

(29) $\quad u_i = v_i$ if $|\sigma \cdot i| > A|\sigma|$, for some $A > 0$.

Indeed, once (28) is proven under such additional assumption, we define

$$u_i^\star := \begin{cases} u_i & \text{if } |\sigma\cdot i| \leq A|\sigma|, \\ v_i & \text{if } |\sigma\cdot i| > A|\sigma|, \end{cases}$$

and we use (26) and (24) to obtain (28) for u from the same result for $u^\star$, by sending $A \to +\infty$.

Then, we focus on the proof of (28) under the additional assumption (29). To this extent, we denote here by C suitable positive constants which may depend on σ, A, ℓ and on the structural constants of the problem, and which may be different at different steps of the computation. We let $r := \dim(\mathcal{K}(\sigma))$. We suppose $r \geq 1$ (if $r = 0$, (28) follows by a cut-off argument and (24)). We also take n and K to be large integers in such a way that $K \in [n\ell - C,\, n\ell + C]$ and $\{j \in \mathbb{Z}^d$ s.t. $|i - j| \leq R$ for some $i \in \mathbb{Z}^d,\ |i| \leq K\} \subseteq \mathbb{Z}^d/(n\ell\mathcal{K}(\sigma))$. We define

$$w_i := \begin{cases} u_i & \text{if } |i| < K, \\ v_i & \text{if } |i| \geq K \end{cases}$$

and we deduce from the ground state property of v that

$$\sum_{\min\{|i|,|j|\}\leq K} a_{ij}\Big(|w_i - w_j - c_{ij}|^{p_{ij}} - |v_i - v_j - c_{ij}|^{p_{ij}}\Big) + \sum_{|i|\leq K}\Big(V(w_i) - V(v_i)\Big) \geq 0\,.$$

We use the latter inequality and (26) to gather that

$$\begin{aligned} \sharp\left(\frac{\ell\mathcal{K}(\sigma)}{n\ell\mathcal{K}(\sigma)}\right) S_\ell(u,v) \;&=\; S_{n\ell}(u,v) \\ &\geq\; -C \sum_{\substack{j\in\mathbb{Z}^d/(n\ell\mathcal{K}(\sigma)) \\ |\sigma\cdot j|\leq C|\sigma| \\ |j|\geq K-C}} |u_j - v_j| \\ &\geq\; -C\,\sharp\Big\{j\in\mathbb{Z}^d/(n\ell\mathcal{K}(\sigma))\,,\ |\sigma\cdot j|\leq C|\sigma|\,,\ |j|\geq K-C\Big\} \\ &\geq\; -C\,(n\ell)^{r-1}\,. \end{aligned}$$

We then divide both sides by $\sharp\left(\frac{\ell\mathcal{K}(\sigma)}{n\ell\mathcal{K}(\sigma)}\right) \sim (n\ell)^r$ and we take n arbitrarily large in order to obtain that $S_\ell(u,v) \geq 0$. This ends the proof of (28).

By exchanging the rôles of u and v, we also deduce from (28) that

$$\text{if } u \text{ and } v \text{ are ground states, } S_\ell(u,v) = 0. \tag{30}$$

We now show that

$$\text{if } v \text{ is a ground state and } S_\ell(u,v) = 0 \text{ then } u \text{ is a ground state too.} \tag{31}$$

To prove this, let w be such that $w_i = u_i$ if $|i| > K$. Let n_o so large that

$$\{j \in \mathbb{Z}^d \text{ s.t. } |i-j| \leq R \text{ for some } i \in \mathbb{Z}^d, |i| \leq K\} \subseteq \mathbb{Z}^d/(n_o\ell\mathcal{K}(\sigma)) \,. \tag{32}$$

We then use the fact that $S_\ell(u,v) = 0$ and (28) to conclude that

$$S_{n_o\ell}(u,w) = S_{n_o\ell}(u,v) + S_{n_o\ell}(v,w) = 0 - S_{n_o\ell}(w,v) \leq 0 \,,$$

which implies that u is a ground state, thanks to (32).

By collecting the results of (28), (30) and (31), the claim in (27) follows.

We also introduce the following **gradient flow**. Given $u_i \in \mathcal{C}$, we consider the configuration $\Phi^t(u)_i$, with $i \in \mathbb{N}$ and $t \geq 0$ to be a solution of the Cauchy Problem (5), which turns out to be the gradient flow of (25).

The right hand side of (5) is locally Lipschitz in ℓ^∞ so the flow satisfies existence and uniqueness for short times. The Comparison Principle in Lemma 1 implies that if $u_i \geq v_i$ for all $i \in \mathbb{Z}^d$, then $\Phi^t(u)_i \geq \Phi^t(v)_i$ for all $i \in \mathbb{Z}^d$. Hence we conclude that $\Phi^t(\mathcal{C}) \subseteq \mathcal{C}$.

Moreover, since, for functions in $\mathcal{C}$ the generator of the flow is uniformly Lipschitz, we conclude that Φ^t extends for all time for initial data in $\mathcal{C}$, due to the Prolongation Lemma (see [H80]).

Furthermore,

$$-\partial_t S_\ell(\Phi^t(u), u^{(0)}) = \sum_{i\in\mathbb{Z}^d/(\ell\mathcal{K}(\sigma))} \Big(\partial_t \Phi^t(u)_i\Big)^2 \geq 0 \,. \tag{33}$$

Note that the vanishing of the quantity in (33) is the same as being a critical point in (2).

We now look at the **evolution of the linear interpolation** between $u^{(0)}$ and $u^{(1)}$, that is, for $s \in [0,1]$, we define

$$u^{(s)} := su^{(1)} + (1-s)u^{(0)}$$

and we conclude by the above construction and Tychonoff Theorem that for any $s \in [0,1]$ there is a sequence of times $t_n(s)$ and a critical point $u^{(s,\star)}$ such that

$$\lim_{n\to+\infty} t_n(s) = +\infty \text{ and } \lim_{n\to+\infty} \Phi^{t_n(s)}(u^{(s)})_i = u_i^{(s,\star)} \,.$$

The proof of Theorem 2 will be completed once we are able to show that there exists $s \in [0,1]$ in such a way that $u^{(s,\star)}$ does not coincide with $u^{(0)}$ nor with $u^{(1)}$.

For this, given $r \geq 0$, we consider the **energy ball**

$$B_r^\star := \{u \in \mathcal{C} \text{ s.t. } S_\ell(u,u^{(0)}) \leq r\} \,.$$

Then, in the light of (27), we may suppose that

$$\mathcal{C} \cap B_0^\star = \{u^{(0)}, u^{(1)}\} \,, \tag{34}$$

otherwise another ground state (and so, a fortiori a critical point) would exist, thus ending the proof.

We now argue **by contradiction** and assume that no other critical point but $u^{(0)}$ and $u^{(1)}$ lies in $\mathcal{C}$. It thus follows from (27) and (34) that, for small $r > 0$, the energy ball $B_r^\star$ is disconnected, i.e., it has a connected component $C^{(0)}$ near $u^{(0)}$ and another connected component $C^{(1)}$ near $u^{(1)}$. This gives that once $\Phi^t(u^{(s)})$ has entered $C^{(0)}$ (resp., $C^{(1)}$) it cannot leave it and it thus approaches $u^{(0)}$ (resp., $u^{(1)}$).

This considerations allow us to define the **basin of attraction** $\beta^{(0)}$ (resp., $\beta^{(1)}$) as all the $s \in [0,1]$ in such a way $\Phi^t(u^{(s)})$ approaches $u^{(0)}$ (resp., $u^{(1)}$) as $t \to +\infty$. The continuous dependence of Φ^t on initial data implies that $\beta^{(0)}$ and $\beta^{(1)}$ are open. But then, $\beta^{(0)}$ and $\beta^{(1)}$ are open and disjoint and they cover $[0,1]$, which is a contradiction.

This completes the proof of Theorem 2. $\square$

Proof of Corollary 1

The desired result plainly follows from Theorem 2 and Corollary 2. $\square$

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The preprints [LlV05], [LlV06a] and [LlV06b] are available on-line at

`http://www.ma.utexas.edu/mp_arc/`

Department of Mathematics, University of Texas at Austin, Austin TX 78712-0257
E-mail address: llave@math.utexas.edu

Università di Roma Tor Vergata Dipartimento di Matematica Roma, Italia
E-mail address: valdinoc@mat.uniroma2.it