A CRITERION FOR HILL OPERATORS TO BE SPECTRAL OPERATORS OF SCALAR TYPE

FRITZ GESZTESY AND VADIM TKACHENKO

Abstract. We derive necessary and sufficient conditions for a Hill operator (i.e., a one-dimensional periodic Schrödinger operator) $H = -d^2/dx^2 + V$ to be a spectral operator of scalar type. The conditions show the remarkable fact that the property of a Hill operator being a spectral operator is independent of smoothness (or even analyticity) properties of the potential V. In the course of our analysis we also establish a functional model for periodic Schrödinger operators that are spectral operators of scalar type and develop the corresponding eigenfunction expansion.

The problem of deciding which Hill operators are spectral operators of scalar type appears to have been open for about 40 years.

1. Introduction

The principal aim of this paper is to establish a criterion for deciding when a Hill operator (i.e., a one-dimensional periodic Schrödinger operator)

$$H = -\frac{d^2}{dx^2} + V(x) \tag{1.1}$$

in $L^2(\mathbb{R})$ with $V(x)$ periodic, is a spectral operator of scalar type in the sense of Dunford. To describe this longstanding open problem in more detail requires a bit of preparation and so we present a brief historical introduction of topics closely related to the material in this paper.

For the notion of spectral operators we refer to the classical monograph of Dunford and Schwartz [15] (cf. Appendix A for a very brief summary). Generally speaking, a spectral operator T in a Hilbert space $\mathcal{H}$ possesses a spectral measure $E_T(\cdot)$ (i.e., a homomorphism from the σ-algebra of Borel subsets of $\mathbb{C}$ into the Boolean algebra of projection operators on $\mathcal{H}$ with E_T bounded and $E_T(\mathbb{C}) = I_{\mathcal{H}}$) such that $E_T(\omega)$ commutes with T and $\sigma\big(T|_{E_T(\omega)\mathcal{H}}\big) \subseteq \overline{\omega}$ for all Borel subsets ω of $\mathbb{C}$ (cf. Definition A.7). Here $\sigma(\cdot)$ denotes the spectrum. Spectral operators of scalar type S are then of the form $S = \text{s-}\lim_{n\uparrow\infty} \int_{|\lambda|\leq n} \lambda\, dE_S(\lambda)$ (cf. Definition A.9).

The literature on self-adjoint Hill operators of the form $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R})$ with V real-valued and periodic, is too enormous to be reviewed here, so we just note that the basic spectral properties of such H, namely, a countable set of closed intervals which may degenerate into finitely many closed intervals and a half-line, and an explicit formula for the spectral expansions generated by H,

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have been established since 1950 (see, e.g., [16, Ch. 6], [53, Sect. XIII.16], [64], [65, Ch. XXI]). In particular, since all self-adjoint operators are spectral operators in the sense of [15], we immediately turn to the case of periodic but non-self-adjoint operators H with complex-valued and periodic potentials V. In this case the spectrum shows much more complexity compared to the self-adjoint situation. Indeed, as shown by Serov [59] (see also [54], [66], [41]) in the first half of the 1960's, the spectrum now consists of a countable system of piecewise analytic arcs which may in fact exhibit crossings (again this can degenerate into a finite system of piecewise analytic arcs and a semi-infinite arc). However, necessary and sufficient conditions (in fact, even just sufficient conditions) for the existence of a uniformly bounded family of spectral projections of a (non-self-adjoint) Hill operator H, and especially, the property of H being a spectral operator of scalar type, remained elusive since the mid 1960's. Following up on successful applications of the formalism of spectral operators of scalar type to Schrödinger operators on the half-line $(0,\infty)$ with sufficiently fast decaying potentials as $|x| \uparrow \infty$ (cf. the discussions in [15, Sect. XX.1], [47], and [48, Appendix II]), McGarvey [40]–[43] started a systematic study of periodic differential operators (including higher-order differential operators) in a series of papers in 1962–1966. Using a combination of direct integral decompositions and perturbation techniques, McGarvey [42] proved in 1965 that certain nth-order differential operators in $L^2(\mathbb{R})$ with periodic coefficients and certain second-order periodic differential operators of the form

$$-\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x), \quad x \in \mathbb{R}, \tag{1.2}$$

with π-periodic functions p and q under the restriction

$$\operatorname{Im}\bigg(\int_0^\pi dx\, p(x)\bigg) \neq 0, \tag{1.3}$$

are spectral operators at infinity. The spectra of operators (1.2) outside a sufficiently large disc are composed of some separated ovals, permitting McGarvey to prove that these operators are in some sense *asymptotically* spectral operators [42]. Such results ignore the existence of local spectral singularities and, at any rate, are not applicable to non-self-adjoint Hill operators $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R})$.

In spite of a flurry of activities in connection with spectral theory for non-self-adjoint Hill operators since the early eighties, many of which were inspired by connections to the Korteweg–deVries hierarchy of evolution equations (we refer, e.g., to [2]–[6], [9]–[11], [20], [21], [25]–[28], [31], [50], [51], [55]–[57], [60]–[62], [67]–[71], [77], [78]), to the best of our knowledge, no progress on the question of which Hill operators are spectral operators of scalar type was made since McGarvey's investigations in the early 1960's. In Section 8 we will further discuss results from [17], [44], [46], [72]–[76], [79] on spectral expansions associated with a non-self-adjoint Hill operator H.

At first sight, a natural expectation with respect to such non-self-adjoint operators would be the following: The "better" the properties of its potential V and the "smaller" its imaginary part, the better should be its chance to be a scalar spectral operator. From this (admittedly, perhaps a bit naive) point of view, operators with potentials V analytic in a half-plane $\operatorname{Im}(z) > a$, $a \leq 0$, should be the best possible candidates. However, it follows from results obtained by Gasymov [20], that actually no such operator with non-constant potential is spectral. In other words, no smoothness or analyticity conditions imposed on V can guarantee that H is a spectral operator of scalar type.

In this paper we prove two versions of a criterion for a Hill operator to be a spectral operator of scalar type, one analytic and one geometric. The analytic version is stated in terms of Hill's discriminant $\Delta_+(z) = [\theta(z,\pi) + \phi'(z,\pi)]/2$ and the functions $\Delta_-(z) = [\theta(z,\pi) - \phi'(z,\pi)]/2$ and $\phi(z,\pi)$, where $\{\theta(z,x), \phi(z,x)\}$ is a fundamental system of distributional solutions of $H\psi(z,x) = z\psi(z,x)$, satisfying canonical boundary conditions at $x = 0$ recorded in (2.4). The triple $\Delta_+(z)$, $\Delta_-(z)$, and $\phi(z,\pi)$ was introduced in [55], [67] as a complete system of independent parameters which uniquely determines the potential V. The geometric version of the criterion uses algebraic and geometric properties of spectra of periodic/antiperiodic and Dirichlet boundary value problems generated by H in the space $L^2([0,\pi])$.

Finally we briefly describe the content of each section. Section 2 contains basic facts on Floquet theory and a summary of the spectral results of (non-self-adjoint) Hill operators (see [59], [54], [66], [41]) and some comments on direct integral decompositions of H and its Green's function. Section 3 then summarizes our principal new results. We provide three theorems (Theorems 3.5–3.7 announced in [23]) which each provide a criterion for H to be a spectral operator of scalar type. Section 4 provides a detailed discussion of the reduced operators $H(t)$, $t \in [0,2\pi]$, in $L^2([0,\pi])$, their spectra, and their spectral expansion. Here H is the direct integral over $H(t)$, $H = (2\pi)^{-1} \int_{[0,2\pi]}^{\oplus} dt\, H(t)$. Section 5 proves some auxiliary results and Section 6 proves the necessity of our conditions for H to be a spectral operator of scalar type. The proof of sufficiency of our conditions for H to be a spectral operator of scalar type is then presented in Section 7. Our final Section 8 presents a series of concluding remarks which put our results in proper perspective and underscores the subtleties involved when trying to determine whether or not a Hill operator is a spectral operator of scalar type.

2. Preliminaries on Hill Operators and Floquet Theory

In this section we briefly recall some standard results on (not necessarily self-adjoint) Hill operators and their associated Floquet theory.

Throughout this paper the $L^2(\Omega)$ and $L^2_{\mathrm{loc}}(\Omega)$ spaces without specifying the corresponding measure on the set $\Omega \subseteq \mathbb{R}$ refer to Lebesgue measure on Ω. The scalar product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)_{L^2(\Omega)}$ (it is assumed to be linear in the second factor), the corresponding norm is denoted by $\|\cdot\|_{L^2(\Omega)}$. For simplicity, the identity operators in $L^2(\Omega)$ will be denoted by I. We use the symbol $'$ to denote x-derivatives and $\bullet$ to denote z-derivatives (and occasionally, ζ derivatives, where $z = \zeta^2$). Moreover, we denote by $\sigma(\cdot)$, $\sigma_{\mathrm{p}}(\cdot)$, $\sigma_{\mathrm{r}}(\cdot)$, $\sigma_{\mathrm{c}}(\cdot)$, and $\rho(\cdot)$ the spectrum, the point spectrum (i.e., the set of eigenvalues), the residual spectrum, the continuous spectrum, and resolvent set of a densely defined, closed, linear operator in a Hilbert space. We also use the abbreviation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For the remainder of this paper we assume the following hypothesis.

Hypothesis 2.1. *Suppose*

$$V \in L^2_{\mathrm{loc}}(\mathbb{R}), \quad V(x+\pi) = V(x) \ \text{for a.e. } x \in \mathbb{R}. \tag{2.1}$$

Without loss of generality we chose the period of V to be π for subsequent notational convenience.

Given Hypothesis 2.1, one introduces the differential expression

$$L = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R} \text{ (or } x \in [0,\pi]) \tag{2.2}$$

and defines the corresponding Schrödinger operator H in $L^2(\mathbb{R})$ by

$$\begin{aligned} &(Hf)(x) = (Lf)(x), \quad x \in \mathbb{R}, \\ &f \in \mathrm{dom}(H) = \{g \in L^2(\mathbb{R}) \,|\, g, g' \in AC_{\mathrm{loc}}(\mathbb{R});\, Lg \in L^2(\mathbb{R})\}. \end{aligned} \tag{2.3}$$

Then H is known to be a densely defined, closed, linear operator in $L^2(\mathbb{R})$. It is self-adjoint if and only if V is real-valued.

Associated with H one introduces the fundamental system of distributional solutions $\theta(z,\cdot)$ and $\phi(z,\cdot)$ of $L\psi = z\psi$ satisfying

$$\theta(z,0) = \phi'(z,0) = 1, \quad \theta'(z,0) = \phi(z,0) = 0, \quad z \in \mathbb{C}. \tag{2.4}$$

For each $x \in \mathbb{R}$, $\theta(z,x)$ and $\phi(z,x)$ are entire with respect to z. The monodromy matrix $\mathcal{M}(z)$ is then given by

$$\mathcal{M}(z) = \begin{pmatrix} \theta(z,\pi) & \phi(z,\pi) \\ \theta'(z,\pi) & \phi'(z,\pi) \end{pmatrix}, \quad z \in \mathbb{C} \tag{2.5}$$

and its eigenvalues $\rho_\pm(z)$, the Floquet multipliers, satisfy

$$\rho_+(z)\rho_-(z) = 1 \tag{2.6}$$

since

$$\det(\mathcal{M}(z)) = \theta(z,\pi)\phi'(z,\pi) - \theta'(z,\pi)\phi(z,\pi) = 1. \tag{2.7}$$

The Floquet discriminant $\Delta_+(\cdot)$ is then defined by

$$\Delta_+(z) = \mathrm{tr}(\mathcal{M}(z))/2 = [\theta(z,\pi) + \phi'(z,\pi)]/2, \quad z \in \mathbb{C}, \tag{2.8}$$

and one obtains

$$\rho_\pm(z) = \Delta_+(z) \pm i\sqrt{1 - \Delta_+(z)^2} \tag{2.9}$$

with an appropriate choice of the square root branches. We also note that

$$|\rho_\pm(z)| = 1 \text{ if and only if } \Delta_+(z) \in [-1,1]. \tag{2.10}$$

The following theorem describes the well-known fundamental properties of the spectrum of (non-self-adjoint) Hill operators in (2.3) (cf. [59], [54], [66], [41], [14, p. 1486–98]).

Theorem 2.2. *Assume Hypothesis* 2.1.
(*i*) *The point spectrum and residual spectrum of H are empty and hence the spectrum of H is purely continuous,*

$$\sigma_{\mathrm{p}}(H) = \sigma_{\mathrm{r}}(H) = \emptyset, \tag{2.11}$$

$$\sigma(H) = \sigma_{\mathrm{c}}(H). \tag{2.12}$$

(*ii*) *$\sigma(H)$ is given by*

$$\sigma(H) = \big\{\lambda \in \mathbb{C} \,\big|\, -1 \le \Delta_+(\lambda) \le 1\big\} \tag{2.13}$$

$$= \{\lambda \in \mathbb{C} \,|\, \textit{there exists at least one non-trivial bounded distributional solution } \psi \in L^\infty(\mathbb{R}; dx) \textit{ of } H\psi = \lambda\psi\}. \tag{2.14}$$

The latter set equals the conditional stability set of H. In addition, $\sigma(H)$ contains no isolated points.
(iii) $\sigma(H)$ is contained in the semi-strip

$$\sigma(H) \subset \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \in [M_1, M_2],\, \mathrm{Re}(z) \geq M_3\}, \tag{2.15}$$

where

$$M_1 = \inf_{x\in\mathbb{R}}[\mathrm{Im}(V(x))], \quad M_2 = \sup_{x\in\mathbb{R}}[\mathrm{Im}(V(x))], \quad M_3 = \inf_{x\in\mathbb{R}}[\mathrm{Re}(V(x))] \tag{2.16}$$

(iv) Qualitatively, $\sigma(H)$ consists of countably many, simple, analytic arcs which may degenerate into finitely many analytic arcs and one simple semi-infinite analytic arc. Asymptotically, these analytic arcs approach the half-line

$$L_{\langle V\rangle} = \{z \in \mathbb{C} \,|\, z = \langle V\rangle + x,\, x \geq 0\}. \tag{2.17}$$

Moreover, crossings of arcs (subject to certain restrictions, see, e.g., item (v)) are permitted.
(v) The resolvent set $\mathbb{C}\backslash\sigma(H)$ of H is path-connected.

Here the mean value $\langle h\rangle$ of a periodic function $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ of period $\pi > 0$ is given by

$$\langle h\rangle = \frac{1}{\pi}\int_{x_0}^{x_0+\pi} dx\, h(x), \tag{2.18}$$

independent of the choice of $x_0 \in \mathbb{R}$.

Remark 2.3. A set $\sigma \subset \mathbb{C}$ is called an *arc* if there exists a parameterization $\gamma \in C([0,1])$ such that $\sigma = \{\gamma(t) \in \mathbb{C} \,|\, t \in [0,1]\}$. The arc σ is called *simple* if there exists a parameterization γ such that $\gamma\colon [0,1] \to \mathbb{C}$ is injective. The arc σ is called *analytic* if there is a parameterization γ that is analytic at each $t \in (0,1)$. Finally, σ_∞ is called a *semi-infinite* arc if there exists a parameterization $\gamma \in C([0,\infty))$ such that $\sigma_\infty = \{\gamma(t) \,|\, t \in [0,\infty)\}$ and σ_∞ is an unbounded subset of $\mathbb{C}$. Analytic semi-infinite arcs are defined analogously, and by a simple semi-infinite arc we mean one that is without self-intersection (i.e., corresponds to a injective parameterization) with the additional restriction that the unbounded part of σ_∞ consists of precisely one branch tending to infinity.

Next, we take a closer look at spectral arcs of H. Let Δ_+ be defined as in (2.8), and let $\lambda_0 \in \sigma(H)$. Then there exists $t_0 \in [0,\pi]$ such that $\Delta_+(\lambda_0) = \cos(t_0)$. If $\lambda_0 \in \sigma(H)$ with $\Delta_+^\bullet(\lambda_0) \neq 0$, then there exist closed intervals $[\alpha,\beta] \subset [0,\pi]$ and a function $\lambda(\cdot)$ continuous on $[\alpha,\beta]$, analytic in an open neighborhood of $[\alpha,\beta]$, such that $\lambda(t_0) = \lambda_0,\ t_0 \in [\alpha,\beta]$ and

$$\begin{aligned} &\Delta_+(\lambda(t)) = \cos(t),\ t \in [\alpha,\beta], \\ &\Delta_+^\bullet(\lambda(t)) \neq 0, \quad \lambda'(t) \neq 0,\ t \in (\alpha,\beta). \end{aligned} \tag{2.19}$$

Definition 2.4. A closed spectral arc of H

$$\sigma = \{z \in \mathbb{C} \,|\, z = \lambda(t),\, t \in [\alpha,\beta]\} \tag{2.20}$$

with $\Delta_+^\bullet(z) \neq 0$ for all $z \in \sigma$ will be called a *regular spectral arc* of H, and the points $\lambda(\alpha)$ and $\lambda(\beta)$ will be called its *endpoints*.

It follows from this definition that all regular spectral arcs of H are compact subsets of $\mathbb{C}$.

Finally, we assume that the orientation of a regular spectral arc σ of H is induced by the function $\lambda(t)$ as t varies from 0 to π (cf. (2.19)). In the special self-adjoint case, this has the effect that for every odd-numbered spectral band (we index them by $k \in \mathbb{N}$, see also (2.40), (2.41), and Lemma 4.1), $\lambda(0) < \lambda(\pi)$, whereas for every even-numbered spectral band $\lambda(0) > \lambda(\pi)$. This is consistent with integrals such as (3.1) for $P(\sigma)$ since we assume that

$$\sqrt{1 - \Delta_+(\lambda)^2} \geq 0, \quad \lambda \in \sigma(H). \tag{2.21}$$

Next, we denote by

$$\{\mu_k\}_{k\in\mathbb{N}} = \{z \in \mathbb{C} \,|\, \phi(z, \pi) = 0\} \tag{2.22}$$

the set of zeros of the entire function $\phi(\cdot, \pi)$. The set $\{\mu_k\}_{k\in\mathbb{N}}$ represents the Dirichlet spectrum associated with the restriction of L to the interval $[0, \pi]$. More precisely, define the densely defined, closed, linear operator H^D in $L^2([0, \pi])$ with Dirichlet boundary conditions by

$$\begin{aligned} &(H^D f)(x) = (Lf)(x), \quad x \in [0, \pi], \\ &f \in \mathrm{dom}(H^D) = \{g \in L^2([0, \pi]) \,|\, g, g' \in AC([0, \pi]);\ Lg \in L^2([0, \pi]); \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad g(0) = g(\pi) = 0\} \end{aligned} \tag{2.23}$$

(H^D is self-adjoint if and only if V is real-valued), then

$$\sigma(H^D) = \{\mu_k\}_{k\in\mathbb{N}}. \tag{2.24}$$

In a similar fashion one defines Schrödinger operator H^N in $L^2([0, \pi])$ with the Dirichlet boundary condition in (2.23) replaced by the Neumann boundary condition

$$g'(0) = g'(\pi) = 0. \tag{2.25}$$

Moreover, we also mention the family of Schrödinger operators H^α, $\alpha \in \mathbb{R}$, in $L^2([0, \pi])$, replacing the boundary condition in (2.23) by

$$g'(0) + \alpha g(0) = g'(\pi) + \alpha g(\pi) = 0, \quad \alpha \in \mathbb{R}. \tag{2.26}$$

Of course, $H^N = H^0$ and formally, $H^D = H^\infty$.

For future purposes it is convenient to introduce

$$\Delta_-(z) = [\theta(z, \pi) - \phi'(z, \pi)]/2, \quad z \in \mathbb{C}. \tag{2.27}$$

Floquet solutions $\psi_\pm(z, \cdot)$ of $L\psi = z\psi$ normalized at $x = 0$ associated with H are then given by ($z \in \mathbb{C}\backslash\{\mu_j\}_{j\in\mathbb{N}}$)

$$\begin{aligned} \psi_\pm(z, x) &= \theta(z, x) + [\rho_\pm(z) - \theta(z, \pi)]\phi(z, \pi)^{-1}\phi(z, x) \\ &= \theta(z, x) + m_\pm(z)\phi(z, x), \quad m_\pm(z) = \frac{-\Delta_-(z) \pm i\sqrt{1 - \Delta_+(z)^2}}{\phi(z, \pi)}, \\ \psi_\pm(z, 0) &= 1. \end{aligned} \tag{2.28}$$

One then verifies (for $z \in \mathbb{C}\backslash\{\mu_j\}_{j\in\mathbb{N}}$, $x \in \mathbb{R}$),

$$\begin{aligned}\psi_\pm(z, x+\pi) &= \rho_\pm(z)\psi_\pm(z,x), \\ &= e^{\pm it}\psi_\pm(z,x) \ \text{ with } \Delta_+(z) = \cos(t), \end{aligned} \tag{2.29}$$

$$W(\psi_+(z,\cdot), \psi_-(z,\cdot)) = m_-(z) - m_+(z) = -\frac{2i\sqrt{1-\Delta_+(z)^2}}{\phi(z,\pi)}, \tag{2.30}$$

$$m_+(z) + m_-(z) = -2\Delta_-(z)/\phi(z,\pi), \tag{2.31}$$

$$m_+(z)m_-(z) = -\theta'(z,\pi)/\phi(z,\pi), \tag{2.32}$$

where $W(f,g) = fg' - f'g$ denotes the Wronskian of f and g. Moreover, applying Lagrange's formula one computes

$$\Delta_+^\bullet(z) = -\phi(z,\pi)\frac{1}{2}\int_0^\pi dx\, \psi_+(z,x)\psi_-(z,x), \quad z \in \mathbb{C}. \tag{2.33}$$

To describe the Green's function of H (i.e., the integral kernel of the resolvent of H) we assume that the square root in (2.9) is chosen such that

$$|\rho_+(z)| < 1, \quad |\rho_-(z)| > 1, \quad z \in \mathbb{C}\backslash\sigma(H). \tag{2.34}$$

Then

$$\psi_\pm(z,\cdot) \in L^2([x_0, \pm\infty)) \ \text{ for all } \ x_0 \in \mathbb{R}, \ z \in \mathbb{C}\backslash(\sigma(H) \cup \{\mu_j\}_{j\in\mathbb{N}}) \tag{2.35}$$

and the Green's function of H is of the form

$$\begin{aligned} G(z,x,y) &= (H - zI)^{-1}(x,y) \\ &= \frac{1}{W(\psi_+(z), \psi_-(z))}\begin{cases}\psi_-(z,x)\psi_+(z,y), & x \le y, \\ \psi_-(z,y)\psi_+(z,x), & x \ge y,\end{cases} \\ &= -\frac{\phi(z,\pi)}{2i\sqrt{1-\Delta_+(z)^2}}\begin{cases}\psi_-(z,x)\psi_+(z,y), & x \le y, \\ \psi_-(z,y)\psi_+(z,x), & x \ge y,\end{cases} \quad z \in \mathbb{C}\backslash\sigma(H). \end{aligned} \tag{2.36}$$

By (2.7) and (2.32) one infers that the singularities of $\psi_\pm(z,x)$ at $z = \mu_j$, $j \in \mathbb{N}$, cancel in (2.36) and hence the latter indeed extends to all $z \in \mathbb{C}\backslash\sigma(H)$.

For subsequent purposes, we denote the set of critical points of Δ_+ by

$$\{\delta_k\}_{k\in\mathbb{N}} = \{z \in \mathbb{C} \,|\, \Delta_+^\bullet(z) = 0\} \tag{2.37}$$

and the set of critical values of Δ_+ by

$$\{\gamma_k = \Delta_+(\delta_k)\}_{k\in\mathbb{N}}. \tag{2.38}$$

Next, we introduce one more family of densely defined, closed, linear operators $H(t)$, $t \in [0,2\pi]$ in $L^2([0,\pi])$ by

$$\begin{aligned} &(H(t)f)(x) = (Lf)(x), \quad t \in [0,2\pi], \ x \in [0,\pi], \\ &f \in \mathrm{dom}(H(t)) = \{g \in L^2([0,\pi]) \,|\, g, g' \in AC([0,\pi]);\ Lg \in L^2([0,\pi]); \\ &\qquad\qquad g(\pi) = e^{it}g(0),\ g'(\pi) = e^{it}g'(0)\}. \end{aligned} \tag{2.39}$$

Again, $H(t)$, $t \in [0,2\pi]$, is self-adjoint if and only if V is real-valued. The spectrum of $H(t)$ is then given by

$$\sigma(H(t)) = \{E_k(t)\}_{k\in\mathbb{N}_0} = \{z \in \mathbb{C} \,|\, \Delta_+(z) = \cos(t)\}, \quad t \in [0,2\pi], \tag{2.40}$$

and the spectrum of H is given by

$$\sigma(H) = \bigcup_{0\le t\le\pi} \sigma(H(t)). \tag{2.41}$$

Finally, we establish the connection between the original Schrödinger operator H in $L^2(\mathbb{R})$ and the family of operators $\{H(t)\}_{t\in[0,2\pi]}$ in $L^2([0,\pi])$ using the notion of direct integrals and the Gel'fand transform [22]. To this end we consider the direct integral of Hilbert spaces

$$\mathcal{K} = \frac{1}{2\pi}\int_{[0,2\pi]}^{\oplus} dt\, L^2([0,2\pi]) \tag{2.42}$$

with constant fibers $L^2([0,2\pi])$. Elements $F\in\mathcal{K}$ are represented by

$$\begin{aligned} &F = \big\{F(\cdot,t)\in L^2([0,\pi];dx)\big\}_{t\in[0,2\pi]}, \\ &\|F\|_{\mathcal{K}}^2 = \frac{1}{2\pi}\int_{[0,2\pi]} dt\, \|F(\cdot,t)\|_{L^2([0,\pi])}^2 = \frac{1}{2\pi}\int_{[0,2\pi]} dt \int_{[0,\pi]} dx\, |F(x,t)|^2 \end{aligned} \tag{2.43}$$

with scalar product in $\mathcal{K}$ defined by

$$\begin{aligned} (F,G)_{\mathcal{K}} &= \frac{1}{2\pi}\int_{[0,2\pi]} dt\, (F(\cdot,t),G(\cdot,t))_{L^2([0,\pi])} \\ &= \frac{1}{2\pi}\int_{[0,2\pi]} dt \int_{[0,\pi]} dx\, \overline{F(x,t)}G(x,t), \quad F,G\in\mathcal{K}. \end{aligned} \tag{2.44}$$

Frequently, $\mathcal{K}$ is written as the vector-valued Hilbert space

$$\mathcal{K} = L^2\big([0,2\pi];dt/(2\pi);L^2([0,\pi];dx)\big). \tag{2.45}$$

The Gel'fand transform [22] is then defined by

$$\mathcal{G}\colon \begin{cases} L^2(\mathbb{R})\to\mathcal{K} \\ \qquad f\mapsto (\mathcal{G}f)(x,t) = F(x,t) = \text{l.i.m.}_{N\uparrow\infty}\sum_{n=-N}^{N} f(x+n\pi)e^{-int}, \end{cases} \tag{2.46}$$

where l.i.m. denotes the limit in $\mathcal{K}$. By inspection, $\mathcal{G}$ is a unitary operator. The inverse transform is given by

$$\mathcal{G}^{-1} = \begin{cases} \mathcal{K}\to L^2(\mathbb{R}) \\ F\mapsto (\mathcal{G}^{-1}F)(x+n\pi) = \frac{1}{2\pi}\int_{[0,2\pi]} dt\, F(x,t)e^{int}, \quad n\in\mathbb{Z}. \end{cases} \tag{2.47}$$

One then infers that

$$\mathcal{G}\varphi(H)\mathcal{G}^{-1} = \frac{1}{2\pi}\int_{[0,2\pi]}^{\oplus} dt\, \varphi(H(t)) \tag{2.48}$$

for a large class of functions φ on $\mathbb{C}$ (including polynomials and bounded continuous functions on $\mathbb{C}$). In particular, (2.48) applies to the resolvent $(H-zI)^{-1}$, $z\in\rho(H)$, of H and one obtains

$$\begin{aligned} G(z,x+m\pi,y+n\pi) &= \frac{1}{2\pi}\int_{[0,2\pi]} dt\, G(t,z,x,y)e^{it(m-n)}, \\ &\qquad z\in\rho(H), \quad x,y\in[0,\pi], \quad m,n\in\mathbb{Z}, \end{aligned} \tag{2.49}$$

where

$$G(t,z,x,y) = (H(t)-zI)^{-1}(x,y), \quad z\in\rho(H(t)),\ t\in[0,2\pi],\ \ x,y\in[0,\pi]. \tag{2.50}$$

As discussed by McGarvey [40], [41], [42], a bounded decomposable operator B commuting with translations by π is a spectral operator of scalar type if, roughly speaking, the associated family $\{B(t)\}_{t\in[0,2\pi]}$ consists of spectral operators which are uniformly spectral with respect to $t\in[0,2\pi]$. Since one can show that H is a spectral operator of scalar type if and only if $(H-zI)^{-1}$, $z\in\rho(H)$, is one whose spectral resolution satisfies $E_{(H-zI)^{-1}}(\{0\})=0$, this seems to offer a reasonable strategy to characterize those Hill operators which are spectral operators of scalar type. However, to the best of our knowledge, this strategy has never been pursued successfully. In this paper we proceed somewhat differently and focus our attention directly on analyzing spectral projections $P(\sigma)$ of H in terms of the corresponding spectral projections of $H(t)$, $t\in[0,2\pi]$.

3. Principal Results

For every compactly supported element $g\in L^2(\mathbb{R})$ and every regular spectral arc $\sigma\subset\sigma(H)$ of H we set, following [66],

$$\begin{aligned}(P(\sigma)g)(x)&\\ &=\frac{1}{2\pi}\int_\sigma\frac{d\lambda}{\sqrt{1-\Delta_+(\lambda)^2}}\Big[\big[\phi(\lambda,\pi)\theta(\lambda,x)-\Delta_-(\lambda)\phi(\lambda,x)\big]F_\theta(\lambda;g)\\ &\qquad-\big[\theta'(\lambda,\pi)\phi(\lambda,x)+\Delta_-(\lambda)\theta(\lambda,x)\big]F_\phi(\lambda;g)\Big],\end{aligned}\tag{3.1}$$

where

$$\begin{gathered}F_\theta(\lambda;g)=\int_{\mathbb{R}}dy\,\theta(\lambda,y)g(y),\quad F_\phi(\lambda;g)=\int_{\mathbb{R}}dy\,\phi(\lambda,y)g(y),\\ \lambda\in\sigma(H),\quad g\in L^2(\mathbb{R}),\ \operatorname{supp}(g)\text{ compact},\end{gathered}\tag{3.2}$$

and the square root in (3.1) is understood to have a positive value on $\sigma(H)$.

Using the Floquet solutions (2.28), we can write (3.1) in the form

$$(P(\sigma)g)(x)=\frac{1}{4\pi}\int_\sigma d\lambda\frac{\phi(\lambda,\pi)}{\sqrt{1-\Delta_+(\lambda)^2}}\big[\psi_+(\lambda,x)F_-(\lambda;g)+\psi_-(\lambda,x)F_+(\lambda;g)\big],\tag{3.3}$$

where

$$F_\pm(\lambda;g)=\int_{\mathbb{R}}dy\,\psi_\pm(\lambda,y)g(y),\quad\lambda\in\sigma(H),\quad g\in L^2(\mathbb{R}),\ \operatorname{supp}(g)\text{ compact}.\tag{3.4}$$

Theorem 3.1. *If σ is a regular spectral arc of H, then there exists a constant $C(\sigma)>0$ such that*

$$\int_\sigma|d\lambda|\frac{|\phi(\lambda,\pi)|}{\sqrt{1-\Delta_+(\lambda)^2}}\bigg(\frac{w_+(\lambda)}{w_-(\lambda)}|F_-(\lambda;g)|^2+\frac{w_-(\lambda)}{w_+(\lambda)}|F_+(\lambda;g)|^2\bigg)\le C(\sigma)^2\|g\|^2_{L^2(\mathbb{R})},\tag{3.5}$$

where $|d\lambda|$ denotes the arc length measure along σ and

$$w_\pm(\lambda)=\|\psi_\pm(\lambda,\cdot)\|_{L^2([0,\pi])},\quad\lambda\in\sigma(H).\tag{3.6}$$

The operator $P(\sigma)$ extends to a bounded linear projection operator in $L^2(\mathbb{R})$ satisfying

$$P(\sigma_1)P(\sigma_2)=P(\sigma_1\cap\sigma_2)\ \text{ for all regular spectral arcs }\sigma_k,\ k=1,2,\text{ of }H.\tag{3.7}$$

The subspace $\operatorname{ran}(P(\sigma))$ *is invariant with respect to H and*

$$\sigma\big(H|_{\operatorname{ran}(P(\sigma))}\big)=\sigma.\tag{3.8}$$

Finally, if for some $g \in L^2(\mathbb{R})$, $P(\sigma)g = 0$ *for all regular spectral arcs* σ *of* H, *then* $g = 0$.

Later we will prove that if H is a spectral operator of scalar type, then the family of projections $P(\sigma)$, σ a regular spectral arc of H, extends to a spectral resolution $E_H(\cdot)$ of H such that

$$\|E_H(\sigma)\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq C, \quad \sigma \subseteq \sigma(H) \text{ measurable} \tag{3.9}$$

for some finite positive constant C independent of $\sigma \subseteq \sigma(H)$. (We denote by $\mathcal{B}(\mathcal{H})$ the Banach space of bounded, linear operators on the complex, separable Hilbert space $\mathcal{H}$.)

Given the notion of projections $P(\sigma)$ for regular spectral arcs σ of H as defined in (3.1), we can now define the notion of spectral singularities for general Hill operators H.

Definition 3.2. The point $\lambda_0 \in \sigma(H)$ is called a *spectral singularity of* H if for all sufficiently small $\delta > 0$ the relation

$$\sup_{\sigma \in \Sigma(\lambda_0, \delta)} \|P(\sigma)\|_{\mathcal{B}(L^2(\mathbb{R}))} = \infty \tag{3.10}$$

holds, where $\Sigma(\lambda_0, \delta)$ is the set of all regular spectral arcs of H located in the set

$$\sigma(H) \cap \{z \in \mathbb{C} \,|\, 0 < |z - \lambda_0| < \delta\}. \tag{3.11}$$

According to Theorem 3.1 (cf. also Definition 2.4), all spectral singularities of H are contained in the discrete set

$$\mathcal{S}_+ = \{\lambda \in \sigma(H) \,|\, \Delta_+^{\bullet}(\lambda) = 0\}. \tag{3.12}$$

We note that the Riesz projection associated with a contour surrounding a spectral singularity λ_0 is not necessarily unbounded as cancellations may occur when several spectral arcs of H end at (or cross through) λ_0. The crucial point in Definition 3.2 is the blowup of the norm of the projection $P(\sigma)$ for some arc σ that converges to λ_0.

Lemma 3.3. *If the Hill operator* H *in* (2.3) *is a spectral operator of scalar type,* E_H *is its spectral resolution, and* σ *is a regular spectral arc of* H, *then* $E_H(\sigma) = P(\sigma)$.

Next, we define the Paley–Wiener class $\mathbb{PW}_\pi$ as the set of all entire functions of exponential type not exceeding π satisfying

$$\|f\|^2_{\mathbb{PW}_\pi} = \int_{\mathbb{R}} dx\, |f(x)|^2 < \infty. \tag{3.13}$$

We note that

$$|f(\zeta)| \leq C\|f\|_{L^2(\mathbb{R})} e^{\pi|\mathrm{Im}(\zeta)|}, \quad \zeta \in \mathbb{C},\ f \in \mathbb{PW}_\pi, \tag{3.14}$$

for some $C > 0$ independent of $f \in \mathbb{PW}_\pi$.

The following theorem in [55] (see also [56], [57]) provides a spectral parametrization of the operator H in (2.3).

Theorem 3.4 ([55], [57])**.** *A triple of entire functions* $\{s = s(\zeta), u_+ = u_+(\zeta), u_- = u_-(\zeta)\}$, *with* $\zeta \in \mathbb{C}$, *coincides with a triple* $\{\phi(\zeta^2, \pi), \Delta_+(\zeta^2), \Delta_-(\zeta^2)\}$ *of some Hill* *operator if and only if the following conditions are satisfied:*

$$(i) \quad s(\zeta) = \frac{\sin(\pi\zeta)}{\zeta} - \pi\langle V\rangle\frac{\cos(\pi\zeta)}{2\zeta^2} + \frac{g(\zeta)}{\zeta^2}, \quad \zeta \in \mathbb{C},\ g \in \mathbb{PW}_\pi. \tag{3.15}$$

$$(ii) \quad u_+(\zeta) = \cos(\pi\zeta) + \pi\langle V\rangle\frac{\sin(\pi\zeta)}{2\zeta} - \pi^2\langle V\rangle^2\frac{\cos(\pi\zeta)}{8\zeta^2} + \frac{f(\zeta)}{\zeta^2}, \quad \zeta \in \mathbb{C},\ f \in \mathbb{PW}_\pi, \tag{3.16}$$

$$\text{with } \{f(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}). \tag{3.17}$$

$$(iii) \quad \frac{u_+(\zeta)^2 - 1 - u_-(\zeta)^2}{s(\zeta)} \text{ is an entire function in } \zeta. \tag{3.18}$$

$$(iv) \quad \zeta u_-(\cdot) \in \mathbb{PW}_\pi. \tag{3.19}$$

(v) *The Gel'fand–Levitan equation for the transformation kernel* $K(x,y)$*,*

$$K(x,y) + F(x,y) + \int_0^x ds\, K(x,s)F(s,y) = 0, \quad 0 \le y \le x \le \pi, \tag{3.20}$$

$$F(x,y) = [\Phi(x-y) + \Phi(x+y)]/2, \quad 0 \le y \le x \le \pi, \tag{3.21}$$

$$\Phi(x) = \sum_{\ell\in\mathbb{Z}} \left[\mathrm{res}_{\zeta=m_\ell}\left(\frac{u_-(\zeta)+u_+(\zeta)}{\zeta s(\zeta)}\cos(\zeta x)\right) - \frac{1}{\pi}\cos(\ell x)\right], \quad x \in [0,2\pi], \tag{3.22}$$

is uniquely solvable in $L^2([0,x])$ *for all* $x \in [0,\pi]$ *with*

$$V(x) = 2\frac{d}{dx}K(x,x) \text{ for a.e. } x \in [0,\pi]. \tag{3.23}$$

Here, $\langle V\rangle$ denotes the mean value of V

$$\langle V\rangle = \frac{1}{\pi}\int_0^\pi dx\, V(x) \tag{3.24}$$

and $\{m_\ell\}_{\ell\in\mathbb{Z}}$ denotes all zeros of $\zeta s(\zeta)$.

In the following theorems (and in the remainder of this paper), C denotes a finite positive constant whose value varies from place to place.

Next we turn to the principal results of this paper. The fundamental question to be answered is the following: When is H a spectral operator of scalar type (cf. [15, p. 1938 and 2242] and Appendix A)? The answer, in terms of intrinsically Floquet theoretic terms participating in the spectral parametrization of H in Theorem 3.4, reads as follows:

Theorem 3.5. *A Hill operator H is a spectral operator of scalar type if and only if the following conditions (i) and (ii) are satisfied:*
(i) *The function*

$$\frac{\Delta_+(z)^2 - 1 - \Delta_-(z)^2}{\phi(z,\pi)\Delta_+^\bullet(z)} \tag{3.25}$$

is analytic in an open neighborhood of $\sigma(H)$*.*
(ii) *The inequalities*

$$\left|\frac{\phi(\lambda,\pi)}{\Delta_+^\bullet(\lambda)}\right| \le C, \quad \left|\frac{\Delta_-(\lambda)}{(\sqrt{|\lambda|}+1)\Delta_+^\bullet(\lambda)}\right| \le C, \quad \lambda \in \sigma(H), \tag{3.26}$$

are satisfied with C a finite positive constant independent of $\lambda \in \sigma(H)$*.*

If both conditions (3.25) and (3.26) are satisfied, and a point $\lambda_0 \in \sigma(H)$ is such that $\Delta_+^\bullet(\lambda_0) = 0$, then

$$\Delta_+(\lambda_0)^2 - 1 = \Delta_-(\lambda_0) = \Delta_+^\bullet(\lambda_0) = 0, \quad \Delta_+^{\bullet\bullet}(\lambda_0) \neq 0, \tag{3.27}$$

implying that the spectrum of a Hill operator, which is a spectral operator of scalar type, is formed by a system of countably many, simple, nonintersecting, analytic arcs. (The latter may degenerate into finitely many simple analytic arcs and a simple analytic semi-infinite arc, all of which are nonintersecting).

According to Theorem 3.4, the operator H and, in particular, the function $\theta'(z,\pi)$ is uniquely determined by the triple $\{\phi(z,\pi),\ \Delta_+(z),\ \Delta_-(z)\}$. If all points $\{\mu_k\}_{k\in\mathbb{N}}$ of the Dirichlet spectrum are simple zeros of $\phi(\cdot,\pi)$, then the function $\Delta_-(z)$ may be uniquely recovered from its values $\Delta_-(\mu_k) = \pm\sqrt{\Delta_+(\mu_k)^2 - 1}$, and such a connection restricts the freedom in fixing $\Delta_-(z)$ to a choice of a sign for every k. Next, we state an alternative version of Theorem 3.5 in terms of the functions appearing in the definition (3.1) of the spectral projections $P(\sigma)$.

Theorem 3.6. *A Hill operator H is a spectral operator of scalar type if and only if the estimates*

$$\left|\frac{\phi(\lambda,\pi)}{\Delta_+^\bullet(\lambda)}\right| \leq C, \quad \left|\frac{\theta'(\lambda,\pi)}{(|\lambda|+1)\Delta_+^\bullet(\lambda)}\right| \leq C, \quad \left|\frac{\Delta_-(\lambda)}{(\sqrt{|\lambda|}+1)\Delta_+^\bullet(\lambda)}\right| \leq C \tag{3.28}$$

hold for all $\lambda \in \sigma(H)$, with C a finite positive constant independent of $\lambda \in \sigma(H)$.

If the conditions (3.28) *are satisfied, then the functions*

$$\frac{\phi(z,\pi)}{\Delta_+^\bullet(z)}, \quad \frac{\theta'(z,\pi)}{\Delta_+^\bullet(z)}, \quad \frac{\Delta_-(z)}{\Delta_+^\bullet(z)} \tag{3.29}$$

are analytic in an open neighborhood of $\sigma(H)$.

The following criterion involves the spectrum $\sigma(H)$ of H, the Dirichlet spectrum $\{\mu_k\}_{k\in\mathbb{N}}$, the periodic spectrum $\{E_k(0)\}_{k\in\mathbb{N}_0}$, and the antiperiodic spectrum $\{E_k(\pi)\}_{k\in\mathbb{N}_0}$, and is connected with the algebraic and geometric multiplicities of the eigenvalues in the sets $\sigma(H(t))$. It follows from (3.16), see [38, Sect. 1.5] and also Lemma 4.1 below, that the union of the periodic and antiperiodic spectra is formed by the numbers

$$\lambda_k^\pm = \left(k + \frac{\langle V\rangle}{2k} + \frac{f_k^\pm}{k}\right)^2, \ k \in \mathbb{N}, \quad \sum_{k\in\mathbb{N}} |f_k^\pm|^2 < \infty, \tag{3.30}$$

$$\{E_k(0), E_k(\pi)\}_{k\in\mathbb{N}_0} = \{\lambda_0^+, \lambda_k^\pm\}_{k\in\mathbb{N}}. \tag{3.31}$$

Theorem 3.7. *A Hill operator H is a spectral operator of scalar type if and only if the following conditions (i)–(iii) are satisfied:*

(i) *Every multiple point of either the periodic or antiperiodic spectrum of H is a point of its Dirichlet spectrum.*

(ii) *For all $t \in [0,2\pi]$ and all $E_k(t) \in \sigma(H(t))$, each root function of the operator $H(t)$ associated with $E_k(t)$ is an eigenfunction of $H(t)$. In particular, the geometric and algebraic multiplicity of each eigenvalue $E_k(t)$ of $H(t)$ coincide.*

(iii) *Let*

$$\mathcal{Q} = \{k \in \mathbb{N} \,|\, d_k = \operatorname{dist}(\delta_k, \sigma(H)) > 0\}, \tag{3.32}$$

then

$$\sup_{k\in\mathcal{Q}} \frac{|\lambda_k^+ - \lambda_k^-|}{\operatorname{dist}(\delta_k, \sigma(H))} < \infty, \quad \sup_{k\in\mathcal{Q}} \frac{|\mu_k - \lambda_k^-|}{\operatorname{dist}(\delta_k, \sigma(H))} < \infty, \quad \sup_{k\in\mathcal{Q}} \frac{|\mu_k - \lambda_k^+|}{\operatorname{dist}(\delta_k, \sigma(H))} < \infty. \tag{3.33}$$

Here a root function of $H(t)$ associated with the eigenvalue $E_k(t)$ denotes any element f satisfying $(H(t) - E_k(t))^m f = 0$ for some $m \in \mathbb{N}$ (i.e., any element in the algebraic eigenspace of $H(t)$ corresponding to $E_k(t)$).

Remark 3.8. It is well-known (see, e.g., [7, Sect. 8.3], [16, Ch. 6], [29, Sect. 10.8], [38, Sect. 3.4], [53, Sect. XIII.16], [64], [65, Ch. XXI]) that if $V \in L^2_{\text{loc}}(\mathbb{R})$ is real-valued, then H and $H(t)$, $t \in [0, 2\pi]$, are self-adjoint operators, all relevant spectra are real, and the interlacing conditions

$$\lambda_0^+ \le E_0(t) \le \lambda_1^- \le \mu_1 \le \lambda_1^+ \le E_1(t) \le \lambda_2^- \le \mu_2 \le \lambda_2^+ \le E_2(t) \le \lambda_3^- \le \mu_3 \le \cdots \tag{3.34}$$

are satisfied. The spectrum $\sigma(H)$ of H is then formed by the system of bands

$$\sigma(H) = \bigcup_{k=0}^{\infty} [\lambda_k^+, \lambda_{k+1}^-] \tag{3.35}$$

separated by spectral gaps

$$(\lambda_k^-, \lambda_k^+), \quad k \in \mathbb{N}. \tag{3.36}$$

For each $k \in \mathbb{N}$, the closure of the spectral gap (3.36), $[\lambda_k^-, \lambda_k^+]$, contains exactly one point μ_k of the Dirichlet spectrum and one non-degenerate critical point δ_k of Δ_+. The conditions of Theorem 3.7 in the self-adjoint setting are quite transparent: Indeed, if λ_0 is a multiple point of the periodic/antiperiodic spectra, then for some $k \ge 1$, the interval $[\lambda_k^-, \lambda_k^+]$ collapses to the point $\lambda_0 = \lambda_k^- = \mu_k = \lambda_k^+$, the point μ_k of the Dirichlet spectrum is trapped in the collapsed gap and condition (i) is satisfied.

Being self-adjoint, no operator $H(t)$ has a root function which is not its eigenfunction, implying property (ii).

The interlacing conditions yield

$$|\mu_k - \lambda_k^\pm| \le \lambda_k^+ - \lambda_k^- \underset{k\uparrow\infty}{=} \pi(\gamma_k^2 - 1)^{1/2}[1 + o(1)], \tag{3.37}$$

$$\operatorname{dist}(\delta_k, \sigma(H)) = \min\{\lambda_k^+ - \delta_k, \delta_k - \lambda_k^-\} \underset{k\uparrow\infty}{=} \frac{\pi}{2}(\gamma_k^2 - 1)^{1/2}[1 + o(1)] \tag{3.38}$$

which proves (3.33) and hence property (iii).

We also note that Marchenko and Ostrovskii [39] gave a complete description of all sequences $\{\lambda_0^+, \lambda_k^\pm\}_{k\in\mathbb{N}}$ which are spectra of self-adjoint Hill operators and of all their Hill discriminants Δ_+. Such a description for non-self-adjoint Hill operators was given in [67] and [69].

4. Spectral Properties of the Operators $H(t)$

In this section we take a closer look at the spectral properties of $H(t)$, $t \in [0, \pi]$ defined in (2.39). For simplicity, we assume

$$\langle V \rangle = 0 \tag{4.1}$$

for the remainder of this paper. For notational convenience we will also identify $E_0^\pm(t) = E_0(t)$ in the following.

Lemma 4.1. *The spectrum $\sigma(H(t)) = \{E_k(t)\}_{k\in\mathbb{N}_0}$ of $H(t)$, $t \in [0,\pi]$, can be represented in the form*

$$\sigma(H(t)) = \{E_0(t), E_n^{\pm}(t)\}_{n\in\mathbb{N}}, \tag{4.2}$$

whose asymptotic expansion (since $\langle V\rangle = 0$) is of the type

$$E_n^{\pm}(t) \underset{n\uparrow\infty}{=} (p_n^{\pm}(t))^2, \quad p_n^{\pm}(t) \underset{n\uparrow\infty}{=} 2n \pm \frac{t}{\pi} + \frac{g_n^{\pm}(t)}{n}, \tag{4.3}$$

where

$$\sum_{n=1}^{\infty} |g_n^{\pm}(t)|^2 \leq C < \infty, \tag{4.4}$$

and $C > 0$ is independent of $t \in [0,\pi]$.

Proof. We set $z = \zeta^2$, $u_+(\zeta) = \Delta_+(z)$ and represent the equation

$$u_+(\zeta) = \cos(t), \quad t \in [0,\pi], \tag{4.5}$$

in the form

$$2\sin\left(\frac{\pi\zeta - t}{2}\right)\sin\left(\frac{\pi\zeta + t}{2}\right) = \frac{f(\zeta)}{\zeta^2}. \tag{4.6}$$

For every $n \in \mathbb{N}$, $n > 1$, let $\delta_n = n^{-1}$ and $0 < \delta_n \leq t \leq \pi - \delta_n$. Setting $\pi\zeta = 2\pi n + t + \xi$ and assuming $|\xi| \leq n^{-1}$, one obtains

$$\operatorname{dist}(t + (\xi/2), \{\pi n\}_{n\in\mathbb{N}}) \geq (2n)^{-1}, \quad |\sin(t + (\xi/2))| \geq Cn^{-1} \tag{4.7}$$

for some constant $C > 0$ and represents (4.6) in the form

$$\xi - \frac{f_n^+(\xi,t)}{n} = 0 \tag{4.8}$$

with f_n^+ given by

$$f_n^+(\xi,t) = n\,\frac{f(2n + t/\pi + \xi/\pi)}{(2n + t/\pi + \xi/\pi)^2}\,\frac{\xi}{2\sin(\xi/2)\sin(t + (\xi/2))}. \tag{4.9}$$

To estimate f_n^+ we now use well-known subharmonic function arguments (cf., [32, Lecture 20, p. 150]). Indeed, for $n \in \mathbb{Z}$ and $\tau \in \mathbb{C}$ such that $|\tau| \leq 2$, let

$$Q_n = \{z \in \mathbb{C} \,|\, z = 2n + x + iy, |x| \leq 4, |y| \leq 4\}, \tag{4.10}$$

$$D_n(\tau) = \{z \in \mathbb{C} \,|\, |z - 2n - \tau| \leq 2\}. \tag{4.11}$$

For every $q \in \mathbb{N}$ the function $|f(\cdot)|^q$ is subharmonic and hence the inequalities

$$\begin{aligned}(4\pi)^{-1}\iint_{Q_n} dx\,dy\,|f(x+iy)|^q &\geq (4\pi)^{-1}\iint_{D_n(\tau)} dx\,dy\,|f(x+iy)|^q \\ = (4\pi)^{-1}\int_0^2 r\,dr\int_0^{2\pi} d\theta\,|f(2n+\tau+re^{i\theta})|^q &\geq |f(2n+\tau)|^q \end{aligned}\tag{4.12}$$

hold. Next, we set $q = 2$ in (4.12), and since $f \in \mathbb{PW}_\pi$, we obtain

$$\sum_{n\in\mathbb{Z}} |f(2n+\tau)|^2 \leq (2\pi)^{-1}\int_{-\infty}^{\infty} dx \int_{|y|\leq 4} dy\,|f(x+iy)|^2 < \infty, \tag{4.13}$$

where the right-hand side of the latter inequality is independent of the variable τ appearing in the left-hand side of (4.13). As result, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, $N > 1$, such that

$$\sum_{n=N}^{\infty} |f(2n+\tau)|^2 \le \varepsilon^2. \tag{4.14}$$

It follows now that if we set

$$r_n^+ = \max_{|\xi|\le n^{-1},\, \delta_n \le t \le \pi-\delta_n} |f_n^+(\xi,t)|, \tag{4.15}$$

then

$$\sum_{n=N}^{\infty} (r_n^+)^2 \le \varepsilon^2. \tag{4.16}$$

Now we apply Rouché's theorem to the function $\xi - f_n^+(\xi,t)/n$ in the disc $\{\xi \in \mathbb{C} \,|\, |\xi| \le 2r_n^+/n\}$ and find that there exists a solution $p_n^+(t) = 2n + (t + \xi_n^+(t))\pi^{-1}$ of (4.6) such that

$$|\xi_n^+(t)| \le 2\frac{r_n^+}{n}, \quad \delta_n \le t \le \pi - \delta_n, \quad \sum_{n=1}^{\infty} (r_n^+)^2 < \infty. \tag{4.17}$$

Similar arguments prove that the substitution $\pi\zeta = 2\pi n - t + \xi$ produces another solution $p_n^-(t) = 2n - (t + \xi_n^-(t))\pi^{-1}$ of (4.6) such that

$$|\xi_n^-(t)| \le 2\frac{r_n^-}{n}, \quad \delta_n \le t \le \pi - \delta_n, \quad \sum_{n=1}^{\infty} (r_n^-)^2 < \infty, \tag{4.18}$$

which proves (4.3) with $g_n^{\pm}(t) = \pi^{-1} n\, \xi^{\pm}(t)$.

Next, let $0 \le t \le \delta_n$. If $\pi\zeta = 2\pi n + \xi$, then (4.6) takes on the form

$$\xi^2 - t^2 = \frac{f_n(\xi,t)}{n^2}, \tag{4.19}$$

where

$$f_n(\xi,t) = n^2 \frac{f(2n+\xi/\pi)}{(2n+\xi/\pi)^2}\, \frac{\xi - t}{2\sin((\xi-t)/2)}\, \frac{\xi+t}{\sin((\xi+t)/2)}. \tag{4.20}$$

We note that since $f' \in \mathbb{PW}_\pi$, the estimate (4.14) is satisfied with f' instead of f. In addition, according to condition (3.17), we have $\{f(2n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. We use (4.12) with $q = 1$ and the representation

$$f(\zeta) = f(2n) + \int_{2n}^{\zeta} d\tau\, f'(\tau) \tag{4.21}$$

to arrive at the estimate

$$\sum_{n=N}^{\infty} |f(2n+\tau)| \le \varepsilon, \quad |\tau| \le 2. \tag{4.22}$$

Therefore, the sequence

$$r_n = \max_{|\xi|\le n^{-1},\, 0 \le t \le \delta_n} |f_n(\xi,t)| \tag{4.23}$$

is such that

$$\sum_{n=N}^{\infty} r_n \le \varepsilon. \tag{4.24}$$

If $|\xi| = 2\delta_n$, then $|\xi^2 - t^2| \geq |\xi^2| - |t^2| \geq 3\delta_n^2$, and by Rouché's theorem, for all $n \geq N$, every disc $\{\xi \in \mathbb{C} \,|\, |\xi| \leq 2\delta_n\}$ contains two solutions $\xi_n^{\pm}(t)$ of (4.6). If for a given $n \geq N$ we have $t \leq 2n^{-1}\sqrt{r_n}$, then (4.19) implies $|\xi_n^{\pm}(t)|^2 \leq 5n^{-2}r_n$, which means that there exist two solutions

$$p_n^{\pm}(t) = 2n \pm \frac{t}{\pi} + \frac{g_n^{\pm}(t)}{n}, \quad g_n^{\pm}(t) = \frac{\mp t + \xi_n^{\pm}(t)}{\pi}, \quad |g_n^{\pm}(t)| \leq Cn^{-1}\sqrt{r_n} \tag{4.25}$$

of (4.6), and hence (4.4) is satisfied. On the other hand, if $2n^{-1}\sqrt{r_n} \leq t \leq n^{-1}$, then

$$\xi = \pm t\left(1 + \frac{f_n(\xi,t)}{n^2t^2}\right)^{1/2} = \pm t + \frac{g_n^{\pm}(\xi,t)}{n}, \tag{4.26}$$

where $|g_n^{\pm}(\xi,t)| \leq C\sqrt{|f_n(\xi,t)|} \leq C\sqrt{r_n}$. The same arguments apply to the case $\pi - \delta_n \leq t \leq \pi$ and we again obtain (4.25), completing the proof of Lemma 4.1 for all $t \in [0,\pi]$. □

Corollary 4.2. *There exists a finite positive constant C such that for every sequence $\{c_n^{\pm}\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$ and every $t \in [0,\pi]$, the series*

$$\Theta^{\pm}(x,t) = \sum_{n=0}^{\infty} c_n^{\pm}\theta(E_n^{\pm}(t),x), \quad \Phi^{\pm}(x,t) = \sum_{n=0}^{\infty} c_n^{\pm}\sqrt{|E_n^{\pm}(t)|+1}\phi(E_n^{\pm}(t),x) \tag{4.27}$$

converge in $L^2([0,\pi])$ and their sums satisfy the estimate

$$\left\|\Theta^{\pm}(\cdot,t)\right\|^2_{L^2([0,\pi])} + \left\|\Phi^{\pm}(\cdot,t)\right\|^2_{L^2([0,\pi])} \leq C\sum_{n=0}^{\infty} |c_n^{\pm}|^2. \tag{4.28}$$

Corollary 4.3. *There exists a finite positive constant C such that for every element $f \in L^2([0,\pi])$ and every $t \in [0,\pi]$, the estimate*

$$\sum_{n=0}^{\infty}\left|\int_0^{\pi} dx\, f(x)\theta(E_n^{\pm}(t),x)\right|^2 + \sum_{n=0}^{\infty}\left|\int_0^{\pi} dx\, f(x)\sqrt{|E_n^{\pm}(t)|+1}\phi(E_n^{\pm}(t),x)\right|^2 \\ \leq C\|f\|^2_{L^2([0,\pi])} \tag{4.29}$$

holds.

Proof. Indeed, using the transformation operators [38, Ch. 1] for L and Lemma 4.1 one obtains

$$\theta(E_n^{\pm}(t),x) = \cos\left(\left(2n \pm \frac{t}{\pi}\right)x\right) + \frac{\theta_n^{\pm}(x,t)}{n}, \tag{4.30}$$

$$\sqrt{E_n^{\pm}(t)}\phi(E_n^{\pm}(t),x) = \sin\left(\left(2n \pm \frac{t}{\pi}\right)x\right) + \frac{\phi_n^{\pm}(x,t)}{n}, \quad n \geq 1, \tag{4.31}$$

where

$$\sum_{n=0}^{\infty}\left(\|\theta_n^{\pm}(\cdot,t)\|^2_{L^2([0,\pi])} + \|\phi_n^{\pm}(\cdot,t)\|^2_{L^2([0,\pi]}\right) \leq C, \quad 0 \leq t \leq \pi, \tag{4.32}$$

with C independent of t. The statements of both Corollaries follow from the corresponding properties of the trigonometric system $\{\cos(2nx), \sin(2nx)\}_{n\in\mathbb{N}_0}$ in the space $L^2([0,\pi])$. □

A standard application of the Lagrange formula yields the following result (cf. (2.30)).

Lemma 4.4. *If $\lambda(t) \in \sigma(H(t))$, $0 \leq t \leq \pi$, and $\Delta_+^\bullet(\lambda(t)) \neq 0$, $\phi(\lambda(t),\pi) \neq 0$, then $\lambda(t)$ is a simple eigenvalue of $H(t)$ and $H(2\pi - t)$, the corresponding eigenvectors are given by $\psi_+(\lambda(t),\cdot)$ and $\psi_-(\lambda(t),\cdot)$, and the identity*

$$\int_0^\pi dx\, \psi_+(\lambda(t),x)\psi_-(\lambda(t),x) = -2\,\frac{\Delta_+^\bullet(\lambda(t))}{\phi(\lambda(t),\pi)} = 2\,\frac{\sqrt{1-\Delta_+(\lambda(t))^2}}{\lambda'(t)\phi(\lambda(t),\pi)} \tag{4.33}$$

holds with the positive value of the square root understood.

The last equality in (4.33) follows from differentiating $\Delta_+(\lambda(t)) = \cos(t)$ with respect to t.

For $t \neq 0, \pi$ the boundary conditions in (2.39) are regular (cf. [48, Ch. V]) and if all zeros $E_k(t)$ of the function $\Delta_+(z) - \cos(t)$ are simple, then the spectral resolution for the corresponding operator $H(t)$ has the form

$$\begin{aligned}
f(x) = &-\frac{1}{2}\sum_{k\in\mathbb{N}} \frac{1}{\Delta_+^\bullet(E_k(t))} \\
&\times \left(\theta(E_k(t),x) \begin{vmatrix} -\Delta_-(E_k(t)) - i\sqrt{1-\Delta_+(E_k(t))^2} & -\phi(E_k(t),\pi) \\ \widetilde F_\theta(E_k(t);f) & \widetilde F_\phi(E_k(t);f) \end{vmatrix} \right. \\
&\left. - \phi(E_k(t),x) \begin{vmatrix} \theta'(E_k(t),\pi) & -\Delta_-(E_k(t)) + i\sqrt{1-\Delta_+(E_k(t))^2} \\ \widetilde F_\theta(E_k(t);f) & \widetilde F_\phi(E_k(t);f) \end{vmatrix} \right), \\
&\qquad f \in L^2([0,\pi]),
\end{aligned} \tag{4.34}$$

where $\widetilde F_\theta(E_k(t);f)$ and $\widetilde F_\phi(E_k(t);f)$ are defined by

$$\begin{aligned}
&\widetilde F_\theta(\lambda(t);f) = \int_0^\pi dy\, \theta(\lambda(t),y)f(y), \quad \widetilde F_\phi(\lambda(t);f) = \int_0^\pi dy\, \phi(\lambda(t),y)f(y), \\
&\qquad \lambda(t) \in \sigma(H(t)),\ t \in [0,2\pi], \quad f \in L^2([0,\pi]).
\end{aligned} \tag{4.35}$$

The Floquet form of the same expansion, if $\phi(E_k(t),\pi) \neq 0$, is then of the following type

$$\begin{aligned}
&f(x) = -\frac{1}{2}\sum_{k\in\mathbb{N}} \frac{\phi(E_k(t),\pi)}{\Delta_+^\bullet(E_k(t))}\psi_+(E_k(t),x)\int_0^\pi dy\, \psi_-(E_k(t),y)f(y), \\
&\qquad t \in [0,\pi], \quad f \in L^2([0,\pi]).
\end{aligned} \tag{4.36}$$

Here $\{\psi_+(E_k(t),\cdot), \psi_-(E_k(t),\cdot)\}_{k\in\mathbb{N}}$ are a biorthogonal system of eigenfunctions of $H(t)$ and $H(t)^*$, that is,

$$\begin{aligned}
&\big(\overline{\psi_-(E_k(t),\cdot)}, \psi_+(E_\ell(t),\cdot)\big)_{L^2([0,\pi])} = \int_0^\pi dx\, \psi_-(E_k(t),x), \psi_+(E_\ell(t),x) \\
&\quad = -\delta_{k,\ell}\frac{2\Delta_+^\bullet(E_k(t))}{\phi(E_k(t),\pi)}, \quad k,\ell \in \mathbb{N}, \quad t \in [0,2\pi].
\end{aligned} \tag{4.37}$$

5. Proofs of Theorem 3.1 and Lemma 3.3

We start with some preliminary considerations.

For a given regular spectral arc $\sigma \subset \sigma(H)$ of H, there exists an interval $I = [\alpha,\beta] \subseteq [0,\pi]$ and an analytic function $\lambda(\cdot)$ such that

$$\sigma = \{z \in \mathbb{C} \,|\, z = \lambda(t),\ \Delta_+(\lambda(t)) = \cos(t),\ t \in [\alpha,\beta]\}. \tag{5.1}$$

In addition, let $I^* = [2\pi - \beta, 2\pi - \alpha]$.

Next, we use the functions $w_\pm(\lambda)$ defined by (3.6) and introduce the 2×2 matrix-valued functions (cf. (3.6))

$$W(\lambda) = \begin{pmatrix} [w_-(\lambda)/w_+(\lambda)]^{1/2} & 0 \\ 0 & [w_+(\lambda)/w_-(\lambda)]^{1/2} \end{pmatrix}, \quad \lambda \in \sigma(H), \tag{5.2}$$

$$\Omega(z) = \begin{pmatrix} 1 & m_+(z) \\ 1 & m_-(z) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\{\mu_k\}_{k\in\mathbb{N}}, \tag{5.3}$$

and the $\mathbb{C}^2$-vector functions

$$\mathbf{Y}(z,x) = \begin{pmatrix} \theta(z,x) \\ \phi(z,x) \end{pmatrix}, \ z \in \mathbb{C}, \quad \mathbf{\Psi}(z,x) = \begin{pmatrix} \psi_+(z,x) \\ \psi_-(z,x) \end{pmatrix}, \ z \in \mathbb{C}\backslash\{\mu_k\}_{k\in\mathbb{N}}, \tag{5.4}$$

$$\mathbf{F}(\lambda; g) = \begin{pmatrix} F_\theta(\lambda; g) \\ F_\phi(\lambda; g) \end{pmatrix}, \ \lambda \in \sigma(H), \tag{5.5}$$

where $g \in L^2(\mathbb{R})$ is a compactly supported function and $F_\theta(\cdot; g)$ and $F_\phi(\cdot; g)$ are defined by (3.2).

We use the standard scalar product in $\mathbb{C}^2$,

$$(\mathbf{Y}, \mathbf{Z})_{\mathbb{C}^2} = \overline{y_1} z_1 + \overline{y_2} z_2, \quad \mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ \mathbf{Z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2, \tag{5.6}$$

and set

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.7}$$

Given a measurable set $\sigma \subseteq \sigma(H)$, we define the Hilbert space $L^2(\sigma)^2$ of measurable $\mathbb{C}^2$-vector elements

$$\mathbf{F}(\lambda) = \begin{pmatrix} F_1(\lambda) \\ F_2(\lambda) \end{pmatrix}, \quad \lambda \in \sigma, \tag{5.8}$$

with the finite norm

$$\|\mathbf{F}\|_{L^2(\sigma)^2} = \bigg(\int_\sigma |d\lambda| \frac{|\phi(\lambda,\pi)|}{\sqrt{1-(\Delta_+(\lambda))^2}} \|W(\lambda)\Omega(\lambda)\mathbf{F}(\lambda)\|^2_{\mathbb{C}^2} \bigg)^{1/2} \tag{5.9}$$

and scalar product

$$(\mathbf{F}, \mathbf{G})_{L^2(\sigma)^2} = \int_\sigma |d\lambda| \frac{|\phi(\lambda,\pi)|}{\sqrt{1-(\Delta_+(\lambda))^2}} (W(\lambda)\Omega(\lambda)\mathbf{F}(\lambda), W(\lambda)\Omega(\lambda)\mathbf{G}(\lambda))_{\mathbb{C}^2}. \tag{5.10}$$

We note that $|d\lambda|$ abbreviates the arc length measure along the arc σ and hence the integrals in (5.9) and (5.10) are independent of the orientation of the spectral arc σ.

Lemma 5.1. *The relation*

$$B_\sigma(\mathbf{F}, \mathbf{H}) = \int_\sigma d\lambda \frac{\phi(\lambda,\pi)}{\sqrt{1-(\Delta_+(\lambda))^2}} \big(\overline{\Omega(\lambda)\mathbf{F}(\lambda)}, J\Omega(\lambda)\mathbf{H}(\lambda)\big)_{\mathbb{C}^2} \tag{5.11}$$

defines a bounded bilinear form in $L^2(\sigma)^2$ and the estimate

$$|B_\sigma(\mathbf{F}, \mathbf{H})| \le \|\mathbf{F}\|_{L^2(\sigma)^2} \|\mathbf{H}\|_{L^2(\sigma)^2} \tag{5.12}$$

holds.

Proof. It suffices to note that

$$W(\lambda)^{-1}J = JW(\lambda), \tag{5.13}$$

$$\big(\overline{\Omega(\lambda)\mathbf{F}(\lambda)}, J\Omega(\lambda)\mathbf{H}(\lambda)\big)_{\mathbb{C}^2} = \big(\overline{W(\lambda)\Omega(\lambda)\mathbf{F}(\lambda)}, JW(\lambda)\Omega(\lambda)\mathbf{H}(\lambda)\big)_{\mathbb{C}^2}, \tag{5.14}$$

and then to apply the Schwarz inequality. □

Lemma 5.2. *Let σ be a regular spectral arc of H and introduce*

$$K_\sigma(H)^2 = 2\pi \max_{\lambda\in\sigma,\, x\in[0,\pi]} \big[|\theta(\lambda,x)|^2 + (|\lambda|+1)|\phi(\lambda,x)|^2\big]. \tag{5.15}$$

Then for every compactly supported function $g \in L^2(\mathbb{R})$ with associated Gel'fand transform $G(x,t)$, the estimate

$$\|\mathbf{F}(\cdot;g)\|^2_{L^2(\sigma)^2} \leq C(\sigma)^2 \int_{I\cup I^*} dt\, \big\|M(\lambda(t))\widetilde{\mathbf{F}}(\lambda(t);G(.,t))\big\|^2_{\mathbb{C}^2} \tag{5.16}$$

holds. Here $I=[\alpha,\beta]\subseteq[0,\pi]$, $I^*=[2\pi-\beta,2\pi-\alpha]$,

$$\begin{aligned} C(\sigma)^2 &= 4K_\sigma(H)^2 \\ &\times \sup_{\lambda\in\sigma}\left[\left|\frac{\phi(\lambda,\pi)}{\Delta^\bullet_+(\lambda)}\right|^2 + \left|\frac{\theta'(\lambda,\pi)}{(|\lambda|+1)\Delta^\bullet_+(\lambda)}\right|^2 + 2\frac{[1-\Delta_+(\lambda)^2+|\Delta_-(\lambda)|^2]}{(|\lambda|+1)|\Delta^\bullet_+(\lambda)|^2}\right], \end{aligned} \tag{5.17}$$

the matrix function $M(\cdot)$ is defined by

$$M(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & (|\lambda|+1)^{1/2}\end{pmatrix}, \quad \lambda\in\sigma(H), \tag{5.18}$$

and the vector function $\widetilde{\mathbf{F}}(\lambda;f)$ is of the form

$$\widetilde{\mathbf{F}}(\lambda;f) = \int_0^\pi dy\, \mathbf{Y}(\lambda,y)f(y) = \begin{pmatrix} \widetilde{F}_\theta(\lambda;f) \\ \widetilde{F}_\phi(\lambda;f)\end{pmatrix}, \quad \lambda\in\sigma(H),\ f\in L^2([0,\pi]) \tag{5.19}$$

with $\widetilde{F}_\theta$ and $\widetilde{F}_\phi$ defined in (4.35).

Proof. For most of this proof we agree to suppress the explicit t-dependence of $\lambda=\lambda(t)$, $t\in I$, and for notational simplicity replace it by λ. Since $\mathbf{\Psi}(z,x) = \Omega(z)\mathbf{Y}(z,x)$, one obtains (we recall that $\Delta_+(\lambda(t)) = \cos(t)$, $t\in I$)

$$\begin{aligned} \Omega(\lambda)\mathbf{F}(\lambda;g) &= \int_{\mathbb{R}} dy\, \mathbf{\Psi}(\lambda,y)g(y) = \sum_{n\in\mathbb{Z}}\int_0^\pi dy \begin{pmatrix} e^{int}\psi_+(\lambda,y) \\ e^{-int}\psi_-(\lambda,y)\end{pmatrix} g(y+n\pi) \\ &= P_1\Omega(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,2\pi-t)) + P_2\Omega(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,t)). \end{aligned} \tag{5.20}$$

Thus, one obtains,

$$\phi(\lambda,y)\Big)$$

$$\times\, G(y,2\pi-t)\Big|^2$$

$$+\, w_+(\lambda)^2\Big|\textstyle\int_0^\pi dy\,\Big(\theta(\lambda,y) + \frac{e^{-it}-\theta(\lambda,\pi)}{(|\lambda|+1)^{1/2}\phi(\lambda,\pi)}(|\lambda|+1)^{1/2}$$

$$\phi(\lambda,y)\Big)G(y,t)\Big|^2\Bigg\}$$

$$= \tfrac{1}{w_-(\lambda)w_+(\lambda)}\Bigg\{w_-(\lambda)^2\Big|\widetilde{F}_\theta(\lambda,G(\cdot,2\pi-t)) + \tfrac{e^{it}-\theta(\lambda,\pi)}{(|\lambda|+1)^{1/2}\phi(\lambda,\pi)}(|\lambda|+1)^{1/2}$$

$$\widetilde{F}_\phi(\lambda,G(\cdot,2\pi-t)))\Big|^2$$

$$+ w_+(\lambda)^2\Big|\widetilde{F}_\theta(\lambda,G(\cdot,t)) + \tfrac{e^{-it}-\theta(\lambda,\pi)}{(|\lambda|+1)^{1/2}\phi(\lambda,\pi)}(|\lambda|+1)^{1/2}$$

$$\widetilde{F}_\phi(\lambda,G(\cdot,t)))\Big|^2\Bigg\}$$

$$\leq \tfrac{2}{w_-(\lambda)w_+(\lambda)}$$

$$\times\Bigg\{w_-(\lambda)^2\bigg(1+\tfrac{|e^{it}-\theta(\lambda,\pi)|}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2$$

$$\|$$

$$\mathrm{M}(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,2\pi-t))\big\|^2_{\mathbb{C}^2}$$

$$+ w_+(\lambda)^2\bigg(1+\tfrac{|e^{-it}-\theta(\lambda,\pi)|}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2$$

$$\|$$

$$\mathrm{M}(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,t))\big\|^2_{\mathbb{C}^2}\Bigg\}$$

$$\leq \tfrac{2K_\sigma(H)^2}{w_-(\lambda)w_+(\lambda)}\bigg(1+\tfrac{|e^{it}-\theta(\lambda,\pi)|}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2$$

$$\bigg($$

$$1 + \text{—}\mathrm{e}^{-it} - \theta(\lambda,\pi)|\frac{}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2$$

$$\times\Big[\big\|M(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,2\pi-t))\big\|^2_{\mathbb{C}^2} + \big\|M(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,t))\big\|^2_{\mathbb{C}^2}\Big], \quad \lambda\in\sigma,$$

where we used

$$\phi(\lambda,y)\Big|^2$$

$$\leq 2\int_0^\pi dy\left[|\theta(\lambda,y)|^2 + \left|\tfrac{e^{\pm it}-\theta(\lambda,\pi)}{(|\lambda|+1)^{1/2}\phi(\lambda,\pi)}\right|^2(|\lambda|+1)|\phi(\lambda,y)|^2\right]$$

$$\leq 2\pi\max_{\lambda\in\sigma,\,x\in[0,\pi]}\big[|\theta(\lambda,x)|^2+(|\lambda|+1)|\phi(\lambda,x)|^2\big]$$

$$\times\bigg(1+\tfrac{|e^{\pm it}-\theta(\lambda,\pi)|}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2$$

$$= K_\sigma(H)^2\bigg(1+\tfrac{|e^{\pm it}-\theta(\lambda,\pi)|}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2,$$

$\lambda \in \sigma,$

to arrive at the last inequality in (5).

Next, we recall that

$$m_{\pm}(\lambda) = \frac{e^{\pm it} - \theta(\lambda,\pi)}{\phi(\lambda,\pi)}, \quad t \in (0,\pi),\ \lambda \in \sigma(H), \tag{5.21}$$

and hence (2.32) implies

$$\begin{aligned}
&\bigg(1 + \frac{|e^{it} - \theta(\lambda,\pi)|}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2 \bigg(1 + \frac{|e^{-it} - \theta(\lambda,\pi)|}{(|\lambda|+1)^{1/2}|\phi(\lambda,\pi)|}\bigg)^2 \\
&\quad \leq \frac{4}{|\phi(\lambda,\pi)|^2}\bigg[|\phi(\lambda,\pi)|^2 + \bigg(\frac{|e^{it} - \theta(\lambda,\pi)|^2 + |e^{-it} - \theta(\lambda,\pi)|^2}{|\lambda|+1}\bigg) + \frac{|\theta'(\lambda,\pi)|^2}{(|\lambda|+1)^2}\bigg], \\
&\hspace{25em} \lambda \in \sigma(H).
\end{aligned} \tag{5.22}$$

Insertion of (5.22) into (5) then yields

$$\begin{aligned}
&\|W(\lambda)\Omega(\lambda)\mathbf{F}(\lambda;g)\|_{\mathbb{C}^2}^2 \\
&\quad \leq \frac{8K_\sigma(H)^2}{w_-(\lambda)w_+(\lambda)|\phi(\lambda,\pi)|^2} \\
&\qquad \times \bigg[|\phi(\lambda,\pi)|^2 + \bigg(\frac{|e^{it} - \theta(\lambda,\pi)|^2 + |e^{-it} - \theta(\lambda,\pi)|^2}{|\lambda|+1}\bigg) + \frac{|\theta'(\lambda,\pi)|^2}{(|\lambda|+1)^2}\bigg] \\
&\qquad \times \Big[\big\|M(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,2\pi-t))\big\|_{\mathbb{C}^2}^2 + \big\|M(\lambda)\widetilde{\mathbf{F}}(\lambda;G(\cdot,t))\big\|_{\mathbb{C}^2}^2\Big], \quad \lambda \in \sigma,
\end{aligned} \tag{5.23}$$

Applying the Schwarz inequality to (4.33) implies

$$\frac{1}{w_+(\lambda)w_-(\lambda)} \leq \frac{|\phi(\lambda,\pi)|}{2|\Delta^\bullet(\lambda)|}, \quad \lambda \in \sigma. \tag{5.24}$$

In addition, we note that by (2.28) and (5.21) one computes

$$|e^{it} - \theta(\lambda,\pi)|^2 + |e^{-it} - \theta(\lambda,\pi)|^2 = 2\big[1 - \Delta_+(\lambda)^2 + |\Delta_-(\lambda)|^2\big], \quad \lambda \in \sigma. \tag{5.25}$$

Moreover, since by (4.33)

$$|d\lambda|\frac{|\phi(\lambda,\pi||}{\sqrt{1-\Delta_+(\lambda)^2}} = dt\,\frac{|\phi(\lambda(t),\pi)|}{|\Delta_+^\bullet(\lambda(t))|}, \quad t \in I, \tag{5.26}$$

(5.23)–(5.26) imply

$$\begin{aligned}\int_\sigma |d\lambda| &\frac{|\phi(\lambda,\pi||}{\sqrt{1-\Delta_+(\lambda)^2}} \|W(\lambda)\Omega(\lambda)\mathbf{F}(\lambda;g)\|^2_{\mathbb{C}^2} \\ &\leq 4K_\sigma(H)^2 \sup_{\lambda\in\sigma} \bigg[\frac{\phi(\lambda,\pi)|^2}{|\Delta_+^\bullet(\lambda)|^2} + \frac{|e^{it}-\theta(\lambda,\pi)|^2+|e^{-it}-\theta(\lambda,\pi)|^2}{(|\lambda|+1)|\Delta_+^\bullet(\lambda)|^2} \\ &\qquad\qquad + \frac{|\theta'(\lambda,\pi)|^2}{(|\lambda|+1)^2|\Delta_+^\bullet(\lambda)|^2}\bigg] \\ &\quad \times \int_{I\cup I^*} dt\, \big\|M(\lambda)\widetilde{\mathbf{F}}(\lambda,G(\cdot,t))\big\|^2_{\mathbb{C}^2} \\ &= 4K_\sigma(H)^2 \sup_{\lambda\in\sigma} \bigg[\frac{\phi(\lambda,\pi)|^2}{|\Delta_+^\bullet(\lambda)|^2} + 2\frac{[1-\Delta_+(\lambda)^2+|\Delta_-(\lambda)|^2]}{(|\lambda|+1)|\Delta_+^\bullet(\lambda)|^2} \\ &\qquad\qquad + \frac{|\theta'(\lambda,\pi)|^2}{(|\lambda|+1)^2|\Delta_+^\bullet(\lambda)|^2}\bigg] \\ &\quad \times \int_{I\cup I^*} dt\, \big\|M(\lambda)\widetilde{\mathbf{F}}(\lambda,G(\cdot,t))\big\|^2_{\mathbb{C}^2}, \quad \lambda\in\sigma, \end{aligned} \tag{5.27}$$

implying (5.16). □

Using analogous arguments one finds that if σ is a regular spectral arc of H, then

$$\|\mathbf{Y}(\cdot,x)\|^2_{L^2(\sigma)^2} \leq C(\sigma)^2 \sup_{\mu\in\sigma}\Big(|\theta(\mu,x)|^2+(|\mu|+1)|\phi(\mu,x)|^2\Big) < \infty \tag{5.28}$$

with $C(\sigma)$ independent of $x\in\mathbb{R}$.

Proof of Theorem 3.1. First we note that according to (5.16),

$$\|\mathbf{F}(\cdot;g)\|_{L^2(\sigma)^2} \leq 2\pi C(\sigma)K(H)\|g\|_{L^2(\mathbb{R})} \tag{5.29}$$

holds. Hence (3.5) is satisfied and the mapping

$$\mathbf{T}\colon \begin{cases} L^2(\mathbb{R})\to L^2(\sigma)^2 \\ \qquad g\mapsto \mathbf{F}(\lambda;g)\end{cases} \tag{5.30}$$

defines a bounded linear operator.

To show that the range of $\mathbf{T}$ coincides with $L^2(\sigma)^2$, we assume that $\mathbf{H}\in L^2(\sigma)^2$ is an arbitrary element and set

$$\begin{aligned} G_{\mathbf{H}}(x,t) = -\frac{\phi(\lambda,\pi)}{2\Delta^\bullet(\lambda)}\Big[&\chi(t)\big(\overline{P_2\Omega(\lambda)\mathbf{H}(\lambda)}, J\Omega(\lambda)\mathbf{Y}(\lambda,x)\big)_{\mathbb{C}^2} \\ &+ \chi^*(t)\big(\overline{P_1\Omega(\lambda)\mathbf{H}(\lambda)}, J\Omega(\lambda)\mathbf{Y}(\lambda,x)\big)_{\mathbb{C}^2}\Big], \end{aligned} \tag{5.31}$$

where $\lambda=\lambda(t)=\lambda(2\pi-t)\in\sigma$, and $\chi(t)$ and $\chi^*(t)$ are the characteristic functions of the intervals I and I^*, respectively. Then

$$\begin{aligned} |G_{\mathbf{H}}(x,t)| \leq &\left|\frac{\phi(\lambda,\pi)}{2\Delta^\bullet(\lambda)}\right| \|W(\lambda)\Omega(\lambda)\mathbf{H}(\lambda)\|_{\mathbb{C}^2} \\ &\times \left[\chi(t)\left(\frac{w_-(\lambda)}{w_+(\lambda)}\right)^{1/2}|\psi_+(\lambda,x)| + \chi^*(t)\left(\frac{w_+(\lambda)}{w_-(\lambda)}\right)^{1/2}|\psi_-(\lambda,x)|\right], \end{aligned} \tag{5.32}$$

and similar to the proof of (5.16) one obtains

$$\|G_{\mathbf{H}}\|_{\mathcal{K}} \leq C(\sigma)\|\mathbf{H}\|_{L^2(\sigma)^2}, \tag{5.33}$$

where $C(\sigma)$ is a parameter independent of $\mathbf{H} \in L^2(\sigma)^2$. We denote by $h \in L^2(\mathbb{R})$ the inverse Gel'fand transform of $G_{\mathbf{H}}$.

Next, we abbreviate $e_1 = (1\ 0)^\top$ and $e_2 = (0\ 1)^\top$, where $\top$ denotes transposition. Then,

$$P_2 J\Omega(\lambda)\mathbf{Y}(\lambda, x) = \psi_+(\lambda, x)e_2, \quad P_2\mathbf{\Psi}(\lambda, x) = \psi_-(\lambda, x)e_2, \tag{5.34}$$

and for $t \in I$ we have

$$\begin{aligned} P_2\Omega(\lambda)\widetilde{\mathbf{F}}(\lambda; G_{\mathbf{H}}(\cdot, t)) &= \int_0^\pi dx\, G_{\mathbf{H}}(x,t)(P_2\mathbf{\Psi})(\lambda, x) \\ &= -\frac{\phi(\lambda,\pi)}{2\Delta^\bullet(\lambda)}\int_0^\pi dx\ \psi_+(\lambda,x)\psi_-(\lambda,x)\big(\overline{\Omega(\lambda)\mathbf{H}(\lambda)}, e_2\big)_{\mathbb{C}^2}e_2 = \big(\overline{\Omega(\lambda)\mathbf{H}(\lambda)}, e_2\big)_{\mathbb{C}^2}e_2. \end{aligned} \tag{5.35}$$

Similarly,

$$P_1\Omega(\lambda)\widetilde{\mathbf{F}}(\lambda; G_{\mathbf{H}}(\cdot, 2\pi - t)) = \big(\overline{\Omega(\lambda)\mathbf{H}(\lambda)}, e_1\big)_{\mathbb{C}^2}e_1. \tag{5.36}$$

Using (5.20) one finds that $\Omega(\lambda)\mathbf{F}(\lambda; h) = \Omega(\lambda)\mathbf{H}(\lambda)$ and $\mathbf{F}(\lambda; h) = \mathbf{H}(\lambda)$.

A comparison of (3.3) and (5.11) then shows that the operator $P(\sigma)$, initially defined on the subspace of compactly supported functions $g \in L^2(\mathbb{R})$, can be extended to all $g \in L^2(\mathbb{R})$ by the relation

$$(P(\sigma)g)(x) = \frac{1}{4\pi}B_\sigma(\mathbf{F}(\cdot; g), \mathbf{Y}(\cdot, x)), \quad g \in L^2(\mathbb{R}). \tag{5.37}$$

If $h \in L^2(\mathbb{R})$, then

$$|(\overline{h}, P(\sigma)g)_{L^2(\mathbb{R})}| = \frac{1}{4\pi}|B_\sigma(\mathbf{F}(\cdot; g), \mathbf{F}(\cdot; h))| \leq C(\sigma)\|g\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}. \tag{5.38}$$

Thus, $P(\sigma)$ is a bounded operator on $L^2(\mathbb{R})$.

To prove (3.7) we assume that σ_1 and σ_2 are two regular spectral arcs of H:

$$\sigma_j = \{z \in \mathbb{C} \,|\, z = \lambda_j(t),\ \Delta_+(\lambda_j(t)) = \cos(t),\ t \in [\alpha_j, \beta_j] \subseteq [0, \pi]\},\ j = 1, 2. \tag{5.39}$$

Then for $g \in L^2(\mathbb{R})$ with compact support,

$$F_+(\lambda_1(t); P(\sigma_2)g) = \Phi_+(t) + \Psi_+(t), \quad t \in [\alpha_1, \beta_1], \tag{5.40}$$

where

$$\Phi_+(t) = \frac{1}{4\pi}\int_{\mathbb{R}} dy\, \psi_+(\lambda_1(t), y)\int_{\sigma_2}\frac{d\lambda\, \phi(\lambda,\pi)}{\sqrt{1-\Delta_+(\lambda)^2}}\, \psi_+(\lambda, y)F_-(\lambda; g) \tag{5.41}$$

and

$$\Psi_+(t) = \frac{1}{4\pi}\int_{\mathbb{R}} dy\, \psi_+(\lambda_1(t), y)\int_{\sigma_2}\frac{d\lambda\, \phi(\lambda,\pi)}{\sqrt{1-\Delta_+(\lambda)^2}}\, \psi_-(\lambda, y)F_+(\lambda; g). \tag{5.42}$$

Substituting $\lambda = \lambda_2(s)$ in the inner integrals, using the distributional relations

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{i(t-s)n} = \delta(t-s), \quad \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{i(t+s)n} = 0, \quad t, s \in (0, \pi), \tag{5.43}$$

and the identity (2.29), we obtain

$$\begin{aligned}\Phi_+(t) = \int_{\alpha_2}^{\beta_2} &\frac{ds\, \phi(\lambda_2(s),\pi)\lambda_2'(s)}{\sqrt{1-\Delta_+(\lambda_2(s))^2}} F_-(\lambda_2(s);g) \\ &\times \frac{1}{4\pi}\sum_{n\in\mathbb{Z}} e^{i(t+s)n} \int_0^\pi dx\, \psi_+(\lambda_1(t),x)\psi_+(\lambda_2(s),x) = 0,\end{aligned} \tag{5.44}$$

and

$$\begin{aligned}\Psi_+(t) = \int_{\alpha_2}^{\beta_2} &\frac{ds\, \phi(\lambda_2(s),\pi)\lambda_2'(s)}{\sqrt{1-\Delta_+(\lambda_2(s))^2}} F_+(\lambda_2(s);g) \\ &\times \frac{1}{4\pi}\sum_{n\in\mathbb{Z}} e^{i(t-s)n} \int_0^\pi dx\, \psi_+(\lambda_1(t),x)\psi_-(\lambda_2(s),x) \\ = \frac{1}{2}&\chi_{\sigma_2}(\lambda_2(t)) \frac{\phi(\lambda_2(t),\pi)\lambda_2'(t)}{\sqrt{1-\Delta_+(\lambda_2(t))^2}} F_+(\lambda_2(t);g)) \int_0^\pi dx\, \psi_+(\lambda_1(t),x)\psi_-(\lambda_2(t),x) \\ = \frac{1}{2}&\chi_{\sigma_1\cap\sigma_2}(\lambda_1(t)) F_+(\lambda_1(t);g)).\end{aligned} \tag{5.45}$$

In the last step we used the biorthogonality of $\{\psi_+(E_k(t),\cdot),\psi_-(E_k(t),\cdot)\}_{k\in\mathbb{N}}$ in (5.23). Thus, by (4.33),

$$F_+(\lambda_1(t);P(\sigma_2)g) = \chi_{\sigma_1\cap\sigma_2}(\lambda_1(t))F_+(\lambda_1(t);g)), \tag{5.46}$$

and in exactly the same way one obtains

$$F_-(\lambda_1(t);P(\sigma_2)g) = \chi_{\sigma_1\cap\sigma_2}(\lambda_1(t))F_-(\lambda_1(t);g)). \tag{5.47}$$

Hence,

$$\begin{aligned}(P(\sigma_1)P(\sigma_2)g)(x) &= \frac{1}{4\pi}\int_{\sigma_1} \frac{d\lambda\, \phi(\lambda,\pi)}{\sqrt{1-\Delta_+(\lambda)^2}} \big[\psi_+(\lambda,x)F_-(\lambda;P(\sigma_2)g) \\ &\qquad\qquad\qquad\qquad + \psi_-(\lambda,y)F_+(\lambda;P(\sigma_2)g))\big] \\ &= \frac{1}{4\pi}\int_{\sigma_1} \frac{d\lambda\, \phi(\lambda,\pi)}{\sqrt{1-\Delta_+(\lambda)^2}} \chi_{\sigma_2}(\lambda)\big[\psi_+(\lambda,x)F_-(\lambda;g) + \psi_-(\lambda,y)F_+(\lambda;g))\big] \\ &= (P(\sigma_1\cap\sigma_2)g)(x), \quad x\in\mathbb{R}.\end{aligned} \tag{5.48}$$

It is evident that the space $L^2(\sigma)^2$ is invariant with respect to the operator Λ of multiplication by the independent variable. This implies that the space $\mathrm{ran}(P(\sigma))$ is contained in the domain of H and hence it is invariant with respect to H. Thus, (3.8) holds.

To prove the last statement of Theorem 3.1 we first note that the sequence of roots of the function $\Delta_+^\bullet(\lambda)$ is asymptotic to $\{k^2\}_{k=1}^\infty$ (we recall the assumption (4.1), $\langle V\rangle = 0$), and therefore, the set

$$\mathcal{T} = \{t\in[0,\pi] \,|\, \text{there exists } \lambda\in\sigma \text{ such that } \Delta_+(\lambda)=\cos(t),\ \Delta_+^\bullet(\lambda)=0\} \tag{5.49}$$

is either finite or countable, with the only possible accumulation points at 0 and π.

Let I be a closed interval in the set $[0,\pi]\backslash\mathcal{T}$ and let $t\in I$. Then the spectrum of each operator $H(t)$ consists of simple eigenvalues $E_k(t)$, $k=1,2,\dots$, and each set $\sigma_k = \{\lambda\in\mathbb{C}\,|\,\lambda=E_k(t),\ t\in I\}$ is a regular spectral arc of H. If $P(\sigma_k)g=0$ for all such arcs, then $B_{\sigma_k}(\mathbf{F}(\cdot;g),\mathbf{Y}(\cdot,x))=0$, $x\in\mathbb{R}$. Applying the operator H^m to the latter identity one obtains $B_{\sigma_k}(\Lambda^m\mathbf{F}(\cdot;g),\mathbf{Y}(\cdot,x))=0$, $x\in\mathbb{R}$, and since the system $\{\lambda^m\}_{m=0}^{\infty}$ is complete in the Hilbert space $L^2(\sigma_k)^2$, one has $\Omega(\lambda)\mathbf{F}(\lambda;g) = 0,\ \lambda \in \sigma_k$. Therefore,

$$F_-(E_k(t);g) = \widetilde{F}_-(E_k(t);G(\cdot,t)) = F_+(E_k(t);g) = \widetilde{F}_+(E_k(t);G(\cdot,2\pi - t)) = 0, \quad t \in I, \tag{5.50}$$

where

$$\begin{gathered}\widetilde{F}_\pm(\lambda(t);f) = \int_0^\pi dy\, \psi_\pm(\lambda(t),y)f(y), \\ \lambda(t) \in \sigma(H(t)),\ t \in [0,2\pi], \quad f \in L^2([0,\pi]).\end{gathered} \tag{5.51}$$

But then the completeness of the eigensystems of the operators $H(t)$ and $H(2\pi - t)$ implies $G(x,t) = G(x,2\pi - t) = 0$ for almost all $(x,t) \in [0,\pi] \times I$, and hence $g = 0$. □

Proof of Lemma 3.3. Let σ be a regular spectral arc of H given by (2.20). For every $g \in L^2(\mathbb{R})$ we set

$$(Q(z,\sigma)g)(x) = \frac{1}{4\pi}\int_\sigma \frac{d\lambda\, \phi(\lambda,\pi)}{\sqrt{1-\Delta_+(\lambda)^2}}\big[\psi_+(\lambda,x)F_-(\lambda;g) + \psi_-(\lambda,x)F_+(\lambda;g)\big]\frac{1}{\lambda - z} \tag{5.52}$$

and check that $Q(z,\sigma)$ is an analytic operator-valued function in $z \in \mathbb{C}\backslash\sigma$. A straightforward computation shows that it coincides with the resolvent

$$R(z) = (H - zI)^{-1}, \quad z \in \rho(H), \tag{5.53}$$

on the subspace $\operatorname{ran}(P(\sigma))$ and hence $R(z)P(\sigma) = Q(z,\sigma)$. In addition, according to (5.46) and (5.47), one has

$$F_+(\lambda;P(\sigma)g) = F_+(\lambda;g), \quad F_-(\lambda;P(\sigma)g) = F_-(\lambda;g), \quad \lambda \in \sigma, \tag{5.54}$$

and the identity

$$Q(z,\sigma)P(\sigma) = Q(z,\sigma) \tag{5.55}$$

holds.

To describe the analytic behavior of $R(z)$ with z approaching the spectral arc σ, we use the representation (2.49). If z is sufficiently close to σ, then

$$\begin{aligned} G(z,x+m\pi,y+n\pi) &= \frac{1}{2\pi}\int_{[\alpha,\beta]\cup[2\pi-\beta,2\pi-\alpha]} dt \\ &\times \left(\frac{\psi_+(\lambda(t),x)\psi_-(\lambda(t),y)}{\lambda(t) - z}\ \frac{\phi(\lambda(t),\pi)}{2\Delta_+^\bullet(\lambda(t))} + \Gamma(z,x,y,t)\right)e^{it(m-n)}, \end{aligned} \tag{5.56}$$

where the remainder $\Gamma(z,x,y,t)$ is analytic at interior points $z \in \sigma$ for all $x,y \in [0,\pi]$ and $t \in [\alpha,\beta]\cup[2\pi-\beta,2\pi-\alpha]$. Equations (5.20) then show that the difference $R(z) - Q(z,\sigma)$ is also analytic at the same points. Using (5.55) we obtain

$$R(z)(I - P(\sigma)) = R(z) - R(z)P(\sigma)^2 = R(z) - Q(z,\sigma), \tag{5.57}$$

proving that the operator-valued function $R(z)(I - P(\sigma))$ is analytic at interior points of σ.

Next, assume that H is a spectral operator of scalar type and E_H is its spectral resolution. Since $R(z)$ and $E_H(\sigma)$ commute for all regular arcs σ, the operator-valued functions $R(z)E_H(\sigma)$ and $R(z)(I - E_H(\sigma))$ are analytic at interior points of

$\sigma(H)\backslash\sigma$ and σ, respectively. Therefore, the equations

$$\Gamma_1(z) = R(z)E_H(\sigma)(I - P(\sigma)) = E_H(\sigma)R(z)(I - P(\sigma)), \tag{5.58}$$

$$\Gamma_2(z) = R(z)(I - E_H(\sigma))P(\sigma) = (I - E_H(\sigma))R(z)P(\sigma) \tag{5.59}$$

define operator-valued functions analytic in $\mathbb{C}\backslash\{\lambda(\alpha), \lambda(\beta)\}$. It follows from (2.28) and (2.49) that

$$\|\Gamma_j(z)\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq C(|z - \lambda(\alpha)||z - \lambda(\beta)|)^{-\gamma} \tag{5.60}$$

in some neighborhoods of $\lambda(\alpha)$ and $\lambda(\beta)$ with $\gamma > 0,\ C > 0$ independent of z. In addition,

$$\lim_{M\uparrow\infty}\left[\sup_{|\mathrm{Im} z|\geq M}\|\Gamma_j(z)\|_{\mathcal{B}(L^2(\mathbb{R}))}\right] = 0, \tag{5.61}$$

and hence Γ_j, $j = 1, 2$, are rational operator-valued functions with possible poles at $\lambda(\alpha)$ and $\lambda(\beta)$. If one of these points is a pole of Γ_j, then it is an eigenvalue of H which contradicts Theorem 2.2. Hence Γ_j are entire functions and by the Liouville theorem they are both equal to the zero operator implying

$$E_H(\sigma)(I - P(\sigma)) = (I - E_H(\sigma))P(\sigma) = 0. \tag{5.62}$$

Finally,

$$E_H(\sigma) - P(\sigma) = E_H(\sigma)(I - P(\sigma)) - (I - E_H(\sigma))P(\sigma) = 0, \tag{5.63}$$

completing the proof of Lemma 3.3. □

6. Necessary Conditions for a Hill Operator to be a Spectral Operator of Scalar Type

Let H be a spectral operator of scalar type and let E_H be the corresponding resolution of the identity. Then, according to Lemma 3.3, for every regular spectral arc σ of H we have $E_H(\sigma) = P(\sigma)$ and hence there exists a constant $C > 0$ such that

$$\sup_{\sigma\in\sigma(H)}\|P(\sigma)\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq C. \tag{6.1}$$

The proof of the necessity of the conditions of Theorem 3.6 is contained in the following two lemmas.

Lemma 6.1. *If H is a spectral operator of scalar type and $\Delta_+^\bullet(\lambda_0) = 0,\ \lambda_0 \in \sigma(H)$, then λ_0 is a simple root of $\Delta_+^\bullet$, that is, $\Delta_+^{\bullet\bullet}(\lambda_0) \neq 0$. Moreover,*

$$\phi(\lambda_0, \pi) = \theta'(\lambda_0, \pi) = \Delta_-(\lambda_0) = 0, \tag{6.2}$$

and

$$\Delta_+(\lambda_0)^2 = 1. \tag{6.3}$$

Proof. Let σ be a spectral arc of H in $\sigma(H)$ with λ_0 one of its endpoints, let $[a, c] \subseteq [0, \pi]$ be a closed interval and let $\lambda(\cdot)$ be the function on $[a, c]$ such that

$$\sigma = \{z \in \mathbb{C}\,|\, z = \lambda(t),\ \Delta_+(\lambda(t)) = \cos(t),\ t \in [a, c]\}. \tag{6.4}$$

We start with some preparatory constructions. To this end we first assume that $\lambda_0 = \lambda(a)$. Since $\phi(\lambda_0, x) \not\equiv 0$, we can choose an element $Q \in L^2([0, \pi])$ such that

$$\int_0^\pi dx\, \phi(\lambda_0, x)Q(x) \neq 0, \tag{6.5}$$

and find $b \in [a, c]$ such that for every interval $I = [\alpha, \beta] \subseteq [a, b]$, the condition

$$\int_0^\pi dx\, \phi(\lambda(t), x) Q(x) \neq 0, \quad t \in I, \tag{6.6}$$

is satisfied. Furthermore, we set $I^* = [2\pi - \beta, 2\pi - \alpha]$, and for every $F_0 \in L^2([0, \pi])$ satisfying

$$\int_0^\pi dx\, \phi(\lambda_0, x) F_0(x) = 0, \tag{6.7}$$

we define

$$F(x, t) = \begin{cases} F_0(x) + \rho(t) Q(x), & t \in I \cup I^*, \\ 0, & \text{otherwise} \end{cases} \tag{6.8}$$

with

$$\rho(t) = -\frac{\int_0^\pi dx\, \phi(\lambda(t), x) F_0(x)}{\int_0^\pi dx\, \phi(\lambda(t), x) Q(x)}, \quad t \in I \cup I^*. \tag{6.9}$$

Then $\rho(a + 0) = 0$,

$$\int_0^\pi dx\, \phi(\lambda(t), x) F(x, t) = 0, \quad t \in [0, 2\pi], \tag{6.10}$$

the function

$$\Theta(t) = \int_0^\pi dx\, \theta(\lambda(t), x) F(x, t), \quad t \in [0, 2\pi], \tag{6.11}$$

is continuous in $I \cup I^*$, and there exists the finite limit

$$\Theta_0 = \lim_{\alpha \downarrow a} \Theta(\alpha) = \int_0^\pi dx\, \theta(\lambda_0, x) F_0(x). \tag{6.12}$$

In addition,

$$\int_0^\pi dx \int_0^{2\pi} dt\, |F(x, t)|^2 \leq C|\beta - \alpha| \tag{6.13}$$

with $C > 0$ independent of α and β. If f is the inverse Gel'fand transform of $F(x, t)$, then

$$\|f\|^2_{L^2(\mathbb{R})} \leq C|\beta - \alpha|, \quad |(\overline{f}, P(\sigma_{\alpha,\beta}) f)_{L^2(\mathbb{R})}| \leq C|\beta - \alpha|, \tag{6.14}$$

where $\sigma_{\alpha,\beta} \subseteq \sigma$ is the spectral arc of H in σ with endpoints $\lambda(\alpha)$ and $\lambda(\beta)$, and $\overline{f}$ denotes the complex conjugate of f. According to (6.10) and (3.1) we have

$$F_\pm(\lambda(t); f) = \int_0^\pi dy\, \psi_\pm(\lambda(t), y) F(y, t) = \Theta(t). \tag{6.15}$$

Thus,

$$(\overline{f}, P(\sigma_{\alpha,\beta}) f)_{L^2(\mathbb{R})} = -\frac{1}{2\pi} \int_\alpha^\beta dt\, \frac{\phi(\lambda(t), \pi)}{\Delta_+^\bullet(\lambda(t))}\, \Theta(t)^2. \tag{6.16}$$

Similarly, we can choose an element $S \in L^2([0, \pi])$ such that

$$\int_0^\pi dx\, \theta(\lambda_0, x) S(x) \neq 0, \tag{6.17}$$

and find $b \in [a, c]$ such that for every interval $I = [\alpha, \beta] \subseteq [a, b]$, the condition

$$\int_0^\pi dx\, \theta(\lambda(t), x) S(x) \neq 0, \quad t \in I, \tag{6.18}$$

is satisfied. Furthermore, for every element $G_0 \in L^2([0,\pi])$ satisfying

$$\int_0^\pi dx\, \theta(\lambda_0, x) G_0(x) = 0, \tag{6.19}$$

we set

$$G(x,t) = \begin{cases} G_0(x) + \widetilde{\rho}(t) S(x), & t \in I \cup I^*, \\ 0, & \text{otherwise} \end{cases} \tag{6.20}$$

with

$$\widetilde{\rho}(t) = -\frac{\int_0^\pi dx\, \theta(\lambda(t), x) G_0(x)}{\int_0^\pi dx\, \theta(\lambda(t), x) S(x)}, \quad t \in I \cup I^*. \tag{6.21}$$

Then $\tilde{\rho}(a+0) = 0$,

$$\int_0^\pi dx\, \theta(\lambda(t), x) G(x,t) = 0, \quad t \in [0, 2\pi], \tag{6.22}$$

and the function

$$\Phi(t) = \int_0^\pi dx\, \phi(\lambda(t), x) G(x,t), \quad t \in [0, 2\pi], \tag{6.23}$$

is continuous in $I \cup I^*$. In addition,

$$\int_0^\pi dx \int_0^{2\pi} dt\, |G(x,t)|^2 \leq C|\beta - \alpha| \tag{6.24}$$

with $C > 0$ independent of α and β. If g is the inverse Gel'fand transform of $G(x,t)$ then

$$\|g\|^2_{L^2(\mathbb{R})} \leq C|\beta - \alpha|, \quad |(\overline{g}, P(\sigma_{\alpha,\beta}) g)_{L^2(\mathbb{R})}| \leq C|\beta - \alpha|. \tag{6.25}$$

According to (6.22) and (2.28) we infer

$$F_\pm(\lambda(t); g) = \int_0^\pi dx\, \psi_\pm(\lambda(t), x) G(x,t) = m_\pm(\lambda(t)) \Phi(t). \tag{6.26}$$

Thus,

$$(\overline{g}, P(\sigma_{\alpha,\beta}) g)_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_\alpha^\beta dt\, \frac{\theta'(\lambda(t)), \pi)}{\Delta_+^\bullet(\lambda(t))}\, \Phi(t)^2. \tag{6.27}$$

Since $\Delta_+^\bullet(\lambda_0) = 0$, we have

$$\Delta_+(\lambda) = \Delta_+(\lambda_0) + c_0(\lambda - \lambda_0)^k [1 + o(1)] \tag{6.28}$$

with $c_0 \neq 0$ and $k \geq 2$.

After these preparations we now turn to the proof of Lemma 6.1.

If we assume that $\Delta_+(\lambda_0)^2 \neq 1$, then $a \neq 0$, $\sin(a) \neq 0$. Together with the expansion

$$\cos(t) = \cos(a) - (t - a)\sin(a)[1 + o(1)], \tag{6.29}$$

valid for t sufficiently close to a, this yields

$$\lambda(t) = \lambda_0 + c_1(t-a)^{1/k}[1 + o(1)], \quad \Delta_+^\bullet(\lambda(t)) = c_2(t-a)^{1-(1/k)}[1 + o(1)] \tag{6.30}$$

with some non-vanishing constants $c_1, c_2 \in \mathbb{C}$.

In addition to $\Delta_+(\lambda_0)^2 \neq 1$, we now assume $\phi(\pi, \lambda_0) \neq 0$. For all $F_0 \in L^2([0,\pi])$ satisfying (6.7), we define $F(x,t)$ by (6.8) and note that if for all such F_0 the constant Θ_0 defined by (6.12) is zero, then the Hahn–Banach theorem implies $\theta(\lambda_0, x) = C\phi(\lambda_0, x)$, which represents a contradiction. Hence we can assume F_0

to be such that $\Theta_0 \neq 0$. The inverse Gel'fand transform f of $F(x,t)$ then satisfies (6.14). Using (6.16) we find

$$\left| \int_\alpha^\beta dt\, (\phi(\lambda_0,\pi)\Theta_0^2 + \varepsilon(t))(t-a)^{-1+(1/k)} \right| \leq C|\beta-\alpha|, \tag{6.31}$$

where $\varepsilon(t) \downarrow 0$ uniformly with respect to $t \in [\alpha,\beta]$ as $\beta \downarrow a$. Since $\phi(\lambda_0,\pi) \neq 0$, we obtain

$$k((\beta-a)^{1/k} - (\alpha-a)^{1/k}) = \left| \int_\alpha^\beta dt\, (t-a)^{-1+(1/k)} \right| \leq C|\beta-\alpha|. \tag{6.32}$$

Since $k \geq 2$ and β are close to a, this is impossible. Hence, $\phi(\lambda_0,\pi) = 0$.

If, as before, $\Delta_+(\lambda_0)^2 \neq 1$ and, in addition, $\theta'(\lambda_0,\pi) \neq 0$, then (6.25) yields

$$\left| \int_\alpha^\beta dt\, (\theta'(\lambda_0,\pi)\Phi_0^2 + o(1))(t-a)^{-1+(1/k)} \right| \leq C|\beta-\alpha|, \tag{6.33}$$

where

$$\Phi_0 = \lim_{\alpha \downarrow a} \Phi(\alpha) = \int_0^\pi dx\, \phi(\lambda_0,x) G_0(x) \tag{6.34}$$

for every element $G_0 \in L^2([0,\pi])$ satisfying (6.19). Using again the Hahn–Banach arguments we conclude that $\Phi_0 \neq 0$, which again leads to a contradiction. Therefore, $\theta'(\lambda_0,\pi) = 0$. Thus,

$$\frac{\phi(\lambda(t),\pi)}{\Delta_+^\bullet(\lambda(t))} = O\big(|t-a|^{-1+(2/k)}\big), \quad \frac{\theta'(\lambda(t),\pi)}{\Delta_+^\bullet(\lambda(t))} = O\big(|t-a|^{-1+(2/k)}\big). \tag{6.35}$$

Let $R \in L^2([0,\pi])$ such that

$$X(t) = \int_0^\pi dx\, \theta(\lambda(t),x)R(x) \neq 0, \quad \Psi(t) = \int_0^\pi dx\, \phi(\lambda(t),x)R(x) \neq 0, \tag{6.36}$$

for all $t \in [a,\beta]$ with β sufficiently close to a, and let

$$R(x,t) = \begin{cases} R(x), & t \in I \cup I^*, \\ 0, & \text{otherwise.} \end{cases} \tag{6.37}$$

If r is the inverse Gel'fand transform of $R(x,t)$, then

$$F_\pm(\lambda(t);r) = \begin{cases} X(t) + m_\pm(\lambda(t))\Psi(t), & t \in I \cup I^*, \\ 0, & \text{otherwise,} \end{cases} \tag{6.38}$$

and

$$\begin{aligned} (\overline{r}, P(\sigma_{\alpha,\beta})r)_{L^2(\mathbb{R})} = \frac{1}{2\pi} & \int_\alpha^\beta dt \left[\frac{\phi(\lambda(t),\pi)}{\Delta_+^\bullet(\lambda(t))} X(t)^2 - \frac{\theta'(\lambda(t),\pi)}{\Delta_+^\bullet(\lambda(t))} \Psi(t)^2 \right] \\ & - \frac{1}{\pi} \int_\alpha^\beta dt\, \frac{\Delta_-(\lambda(t))}{\Delta_+^\bullet(\lambda(t))} X(t)\Psi(t). \end{aligned} \tag{6.39}$$

Because of (6.35), the absolute value of the former integral is not greater than $C((\beta-a)^{2/k} - (\alpha-a)^{2/k})$. On the other hand, $\|r\|_{L^2(\mathbb{R})}^2 = \|R\|_{\mathcal{K}}^2 \leq C|\beta-\alpha|$, and hence,

$$\left| \int_\alpha^\beta dt\, \frac{\Delta_-(\lambda(t))}{\Delta_+^\bullet(\lambda(t))} X(t)\Psi(t) \right| \leq C((\beta-a)^{2/k} - (\alpha-a)^{2/k}). \tag{6.40}$$

Since $X(a)\Psi(a) \neq 0$ and (6.30) is valid, we conclude that $\Delta_-(\lambda_0) = 0$. Together with $\theta'(\pi, \lambda_0) = \phi(\pi, \lambda_0) = 0$ this implies $\Delta_+(\lambda_0)^2 = 1$ contradicting the assumption $\Delta_+(\lambda_0)^2 \neq 1$.

Repeating the above arguments for the case $\lambda_0 = \lambda(c)$ we find that the condition $\Delta_+^\bullet(\lambda) = 0$ for $\lambda \in \sigma(H)$ is only satisfied for

$$\Delta_+(\lambda_0)^2 = 1. \tag{6.41}$$

Next, we will show that if this is the case, then $k = 2$ in (6.28). For simplicity we restrict ourselves to the case $a = 0$, $\lambda_0 = \lambda(0)$ and $\Delta_+(\lambda_0) = 1$.

First we note that instead of (6.30) we have

$$\lambda(t) = \lambda_0 + c_1 t^{2/k}[1 + o(1)], \quad \Delta_+^\bullet(\lambda(t)) = c_2 t^{2-(2/k)}[1 + o(1)] \tag{6.42}$$

for some $c_1, c_2 \in \mathbb{C}\backslash\{0\}$. Assume for a moment that $k \geq 3$. Then if $\phi(\lambda_0, \pi) \neq 0$ then according to (6.14) and (6.16) we now obtain, in place of (6.32),

$$\left|\phi(\lambda_0, \pi)\, \Theta_0^2 \int_\alpha^\beta dt\, t^{-2+(2/k)}\right| \leq C|\beta - \alpha| \tag{6.43}$$

for some constant $\Theta_0 \neq 0$. Hence,

$$\big|\phi(\lambda_0, \pi)\big(\beta^{-1+(2/k)} - \alpha^{-1+(2/k)}\big)\big| \leq C|\beta - \alpha|, \tag{6.44}$$

which is impossible for $k \geq 3$. Therefore, as before, $\phi(\lambda_0, \pi) = 0$ and $\phi(\lambda, \pi) = A(\lambda - \lambda_0)^p[1 + o(1)]$ for some integer $p \in \mathbb{N}$ and $A \neq 0$. In the same manner we obtain $\theta'(\lambda, \pi) = B(\lambda - \lambda_0)^q[1 + o(1)]$ and $\Delta_-(\lambda) = C(\lambda - \lambda_0)^r[1 + o(1)]$ with integers $q \geq 1$, $r \geq 1$. Using again (6.14), (6.16), (6.25), (6.27) and(6.40) we deduce $p \geq k, q \geq k, r \geq k$. The identity

$$\Delta_+(\lambda)^2 - 1 - \Delta_-(\lambda)^2 = -\phi(\lambda, \pi)\theta'(\lambda, \pi) \tag{6.45}$$

then leads to the relation

$$D(\lambda - \lambda_0)^k = C^2(\lambda - \lambda_0)^{2r}[1 + o(1)] - AB(\lambda - \lambda_0)^{p+q}[1 + o(1)] \tag{6.46}$$

for some $D \neq 0$, which is impossible for $k \geq 3$. Thus, $k = 2$ in (6.28).

Assuming $\phi(\lambda_0, \pi) \neq 0$ and inserting $k = 2$ in (6.43) yields

$$|\phi(\lambda_0, \pi)\, \Theta_0^2 \ln(\beta/\alpha)| = \left|\phi(\lambda_0, \pi)\, \Theta_0^2 \ln\left(1 + \frac{\beta - \alpha}{\alpha}\right)\right| \leq C|\beta - \alpha|. \tag{6.47}$$

Since $\Theta_0 \neq 0$ and α can be chosen arbitrarily close to zero, (6.47) immediately leads to a contradiction. Therefore,

$$\phi(\lambda_0, \pi) = 0. \tag{6.48}$$

Analogous considerations yield

$$\theta'(\lambda_0, \pi) = 0 \text{ and hence } \Delta_-(\lambda_0) = 0, \tag{6.49}$$

completing the proof of Lemma 6.1. ☐

Remark 6.2. If for some $\lambda_0 \in \mathbb{C}$,

$$\phi(\lambda_0, \pi) = \theta'(\lambda_0, \pi) = \Delta_-(\lambda_0) = 0, \tag{6.50}$$

then the Dirichlet and Neumann operators H^D and H^N (cf. (2.23), (2.25)) and, in fact, all operators H^α, $\alpha \in \mathbb{R}$ (cf. (2.26)), have the eigenvalue λ_0. This is the familiar situation in the self-adjoint case when $\Delta^\bullet(\lambda_0) = 0$ and $\Delta(\lambda_0)^2 = 1$. In this case a spectral gap closes at λ_0 and an eigenvalue for all separated self-adjoint boundary conditions associated with L restricted to $[0, \pi]$ is trapped at the position

λ_0. Consequently, no self-adjoint Hill operator can have spectral singularities. Of course, this is clear from the general fact that all self-adjoint (respectively, normal) operators in a complex separable Hilbert space are of course spectral operators of scalar type.

Lemma 6.3. *If H is a spectral operator of scalar type then the functions*

$$\frac{\phi(z,\pi)}{\Delta_+^\bullet(z)}, \quad \frac{\theta'(z,\pi)}{\Delta_+^\bullet(z)}, \quad \frac{\Delta_-(z)}{\Delta_+^\bullet(z)} \tag{6.51}$$

are analytic in an open neighborhood of $\sigma(H)$ and the functions

$$\frac{\phi(\lambda,\pi)}{\Delta_+^\bullet(\lambda)}, \quad \frac{\theta'(\lambda,\pi)}{\lambda\Delta_+^\bullet(\lambda)}, \quad \frac{\Delta_-(\lambda)}{\sqrt{\lambda}\Delta_+^\bullet(\lambda)} \tag{6.52}$$

are bounded on the set

$$\sigma_{\mathrm{ext},R}(H) = \sigma(H) \cap \{z \in \mathbb{C} \,|\, |z| \geq R\} \tag{6.53}$$

with $R > 0$ sufficiently large.

Proof. The analyticity of the functions (6.51) in an open neighborhood of $\sigma(H)$ is an immediate consequence of Lemma 6.1. To prove the remaining part of Lemma 6.3 we now fix a constant $R > 0$ sufficiently large for all expressions of the form $1 + o(1)$ to be close to 1 uniformly with respect to the variables t, n, x on which they depend.

Let $\sigma \subset \sigma(H)$ be a closed spectral arc of H. Without loss of generality we can assume that the latter is a part of a spectral arc

$$\sigma_n^+ = \{z \in \mathbb{C} \,|\, z = E_n^+(t),\ \Delta_+(E_n^+(t)) = \cos(t),\ t \in [0,\pi]\},\ n \in \mathbb{N}, \tag{6.54}$$

where $n \geq N_0$ is sufficiently large and

$$E_n^+(t) = (p_n^+(t))^2, \quad p_n^+(t) = 2n + \frac{t}{\pi} + \frac{g_n^+(t)}{n}, \quad |g_n^+(t)| \leq C \tag{6.55}$$

with $C > 0$ independent of $t \in [0,\pi]$ and $n \geq N_0$.

We denote by $T = T(\sigma) \subseteq [0,\pi]$ the pre-image of σ with respect to the function $E_n^+(t)$ and set $T^* = \{t^* \in [\pi, 2\pi] \,|\, t^* = 2\pi - t, t \in T\} \subseteq [\pi, 2\pi]$. Since

$$\theta(z,x) = \cos(\sqrt{z}x) + \int_0^x dy\, K_\theta(x,y)\cos(\sqrt{z}y), \tag{6.56}$$

$$\sqrt{z}\phi(z,x) = \sin(\sqrt{z}x) + \int_0^x dy\, K_\phi(x,y)\sin(\sqrt{z}y), \tag{6.57}$$

where $K_\theta(x,y)$ and $K_\phi(x,y)$ are integral kernels of transformation operators, (cf. [38, Ch. 1]), we obtain

$$\theta(E_n^+(t),x) = \cos\left(\left(2n + \frac{t}{\pi}\right)x\right) + \frac{\varepsilon_{\theta,n}^+(t)}{n}, \tag{6.58}$$

$$\sqrt{E_n^+(t)}\phi(E_n^+(t),x) = \sin\left(\left(2n + \frac{t}{\pi}\right)x\right) + \frac{\varepsilon_{\phi,n}^+(t)}{n}, \tag{6.59}$$

with

$$|\varepsilon_{\theta,n}^+(t)| + |\varepsilon_{\phi,n}^+(t)| \leq C \tag{6.60}$$

and $C > 0$ independent of $t \in [0,\pi]$ and $n \geq N_0$.

Next, set

$$\alpha_n(t)=\begin{cases}-\dfrac{\big(\overline{\theta(E_n^+(t),\cdot)},\phi(E_n^+(t),\cdot)\big)_{L^2([0,\pi])}}{\sqrt{E_n^+(t)}\big(\overline{\phi(E_n^+(t),\cdot)},\phi(E_n^+(t),\cdot)\big)_{L^2([0,\pi])}}, & t\in T\cup T^*,\\ 0, & \text{otherwise.}\end{cases} \tag{6.61}$$

The above asymptotic representations show that an inequality $|\alpha_n(t)|\leq C$ is satisfied with $C>0$ independent of $t\in[0,\pi]$ and $n\geq N_0$. For the function

$$F_n(x,t)=\begin{cases}\theta(E_n^+(t),x)+\alpha_n(t)\sqrt{E_n^+(t)}\phi(E_n^+(t),x), & t\in T\cup T^*,\\ 0, & \text{otherwise,}\end{cases} \tag{6.62}$$

we obtain

$$\int_0^\pi dx\,F_n(x,t)\phi(E_n^+(t),x)=0, \quad t\in[0,2\pi], \tag{6.63}$$

and

$$\int_0^\pi\int_0^{2\pi}dx\,dt\,|F_n(x,t)|^2\leq C|T|, \tag{6.64}$$

where $|\cdot|$ abbreviates Lebesgue measure.

If $f_n\in L^2(\mathbb{R})$ is the inverse Gel'fand transform of $F_n(x,t)$, then $\|f_n\|^2_{L^2(\mathbb{R})}\leq C|T|$ and

$$(\overline{f_n},P(\sigma)f_n)_{L^2(\mathbb{R})}=-\frac{1}{2\pi}\int_T dt\,\frac{\phi(E_n^+(t),2\pi)}{\Delta_+^\bullet(E_n^+(t))}\,\bigg(\int_0^\pi dx\,\theta(E_n^+(t),x)^2+o(1)\bigg)^2. \tag{6.65}$$

According to (6.1), there exists a finite positive constant C such that for every set $\sigma\subset\sigma(H)$ and every element $g\in L^2(\mathbb{R})$ the inequality

$$|(\overline{g},P(\sigma)g)_{L^2(\mathbb{R})}|\leq C\|g\|^2_{L^2(\mathbb{R})} \tag{6.66}$$

holds. Thus,

$$\bigg|\int_T dt\,\frac{\phi(E_n^+(t),\pi)}{\Delta_+^\bullet(E_n^+(t))}[1+o(1)]\bigg|\leq C\|f_n\|^2_{L^2(\mathbb{R})}\leq C|T|. \tag{6.67}$$

Since σ is an arbitrary closed part of the spectral arc σ_n of H, we find

$$\bigg|\frac{\phi(\lambda,\pi)}{\Delta_+^\bullet(\lambda)}\bigg|\leq C, \quad \lambda\in\sigma_{\text{ext},R}(H) \tag{6.68}$$

proving that the first function in (6.52) is bounded on the set $\sigma(H)$.

Next, we replace $F_n(x,t)$ by the function

$$G_n(x,t)=\begin{cases}\beta_n(t)\theta(E_n^+(t),x)+\sqrt{E_n^+(t)}\phi(E_n^+(t),x), & t\in T\cup T^*,\\ 0, & \text{otherwise}\end{cases} \tag{6.69}$$

with

$$\beta_n(t)=-\frac{\sqrt{E_n^+(t)}\big(\overline{\theta(E_n^+(t),\cdot)},\phi(E_n^+(t),\cdot)\big)_{L^2([0,\pi])}}{\big(\overline{\theta(E_n^+(t),\cdot)},\theta(E_n^+(t),\cdot)\big)_{L^2([0,\pi])}}, \quad t\in[0,2\pi]. \tag{6.70}$$

Then

$$\int_0^\pi dx\,G_n(x,t)\theta(E_n^+(t),x)=0 \tag{6.71}$$

and

$$\int_0^{\pi}\int_0^{2\pi} dxdt\,|G_n(x,t)|^2 \le C|T|. \tag{6.72}$$

Using (3.3) we obtain

$$\begin{aligned}&(\overline{g_n}, P(\sigma)g_n)_{L^2(\mathbb{R})}\\ &\quad = \frac{1}{2\pi}\int_T dt\,\frac{\theta'(E_n^+(t),\pi)}{E_n^+(t)\Delta_+^\bullet(E_n^+(t))}\left(E_n^+(t)\int_0^{\pi} dx\,\phi(E_n^+(t),x)^2 + o(1)\right)^2,\end{aligned} \tag{6.73}$$

where $g_n \in L^2(\mathbb{R})$ is the inverse Gel'fand transform of $G_n(x,t)$. This implies the estimate

$$\left|\frac{\theta'(\lambda,\pi)}{\lambda\Delta_+^\bullet(\lambda)}\right| \le C, \quad \lambda \in \sigma_{\mathrm{ext},R}(H). \tag{6.74}$$

Finally, set

$$R_n(x,t) = \begin{cases}\theta(E_n^+(t),x) + \sqrt{E_n^+(t)}\phi(E_n^+(t),x), & t \in T\cup T^*,\\ 0, & \text{otherwise},\end{cases} \tag{6.75}$$

and denote by $r_n(x)$ the inverse Gel'fand transform of $R_n(x,t)$. Then

$$\begin{aligned}(\overline{r_n}, P(\sigma)r_n)_{L^2(\mathbb{R})} &= \frac{1}{2\pi}\int_T dt\,\frac{\phi(E_n^+(t),\pi)}{\Delta_+^\bullet(E_n^+(t))}\big(\overline{\theta(E_n^+(t),\cdot)}, R_n(\cdot,t)\big)^2_{L^2([0,\pi])}\\ &- \frac{1}{2\pi}\int_T dt\,\frac{\theta'(E_n^+(t),\pi)}{\Delta_+^\bullet(E_n^+(t))}\big(\overline{\phi(E_n^+(t),\cdot)}, R_n(\cdot,t)\big)^2_{L^2([0,\pi])}\\ &- \frac{1}{\pi}\int_T dt\,\frac{\Delta_-(E_n^+(t))}{\Delta_+^\bullet(E_n^+(t))}\big(\overline{\theta(E_n^+(t),\cdot)}, R_n(\cdot,t)\big)_{L^2([0,\pi])}\big(\overline{\phi(E_n^+(t),\cdot)}, R_n(\cdot,t)\big)_{L^2([0,\pi])}.\end{aligned} \tag{6.76}$$

Since

$$\big(\overline{\theta(E_n^+(t),\cdot)}, R_n(\cdot,t)\big)_{L^2([0,\pi])} = \frac{\pi}{2} + o(1), \tag{6.77}$$

$$\sqrt{E_n^+(t)}\big(\overline{\phi(E_n^+(t),\cdot)}, R_n(\cdot,t)\big)_{L^2([0,\pi])} = \frac{\pi}{2} + o(1), \tag{6.78}$$

and the estimates (6.68) and (6.74) are already proved, we have

$$\left|\int_T dt\,\frac{\Delta_-(E_n^+(t))}{\sqrt{E_n^+(t)}\Delta_+^\bullet(E_n^+(t))}[1+o(1)]\right| \le C|T|, \tag{6.79}$$

and

$$\left|\frac{\Delta_-(\lambda)}{\sqrt{\lambda}\Delta_+^\bullet(\lambda)}\right| \le C < \infty, \quad \lambda \in \sigma_{\mathrm{ext},R}(H), \tag{6.80}$$

completing the proof of Lemma 6.3. □

The necessity of the conditions in Theorem 3.5 now follows from the necessity of the conditions in Theorem 3.6 and the identity (6.45).

Proof of necessity of the conditions in Theorem 3.7. To prove necessity of condition (i) of Theorem 3.7 we note that if λ_0 is a multiple point of the periodic or antiperiodic spectra then $\lambda_0 \in \sigma(H)$ and $\Delta_+^\bullet(\lambda_0) = 0$ and the claim follows from the analyticity of the first fraction in (3.29).

Now, let λ_0 be an eigenvalue of an operator $H(t)$, $t \in [0,\pi]$. Then $\Delta_+(\lambda_0) = \cos(t) + c(\lambda-\lambda_0)^m[1+o(1)]$ with some $c \neq 0$ and an integer $m \geq 1$. Let

$$\mathcal{E}(\lambda_0,t) = \{f \in \mathrm{dom}(H(t)^m) \,|\, (H(t)-\lambda_0 I)^m f = 0\} \tag{6.81}$$

and

$$\ker(H(t)-\lambda_0 I) = \{f \in \mathrm{dom}(H(t)) \,|\, (H(t)-\lambda_0 I) f = 0\} \tag{6.82}$$

the corresponding algebraic (root) and geometric eigenspaces, respectively. (Cf. [24] for a detailed discussion of algebraic multiplicities in terms of the behavior of the discriminant Δ_+.) If $m = 1$ then both subspaces coincide. In the case $m \geq 2$ we have $\Delta_+^\bullet(\lambda_0) = 0$ and, according to (3.27), either $t = 0$ or $t = \pi$, and in both cases $m = 2$, $\mathcal{M} - \Delta_+(\lambda_0)I_2 = 0$ and $2 = \dim(\ker(H(t)-\lambda_0)) \leq \dim(\mathcal{E}(\lambda_0,t))$=2. Hence, $\ker(H(t)-\lambda_0) = \mathcal{E}(\lambda_0,t)$, proving claim *(ii)* of Theorem 3.7.

To prove that *(iii)* is also nessesary, we first note that for every $\delta > 0$ there exists a constant $C = C(\delta) > 0$ such that

$$C^{-1} \leq |\sin \pi\zeta| e^{-\pi|\mathrm{Im}(\zeta)|} \leq C, \quad \zeta \notin \bigcup_{k\in\mathbb{Z}} D_k, \quad D_k = \{\zeta' \in \mathbb{C} \,|\, |\zeta' - k| \leq \delta\}. \tag{6.83}$$

In the following we use the abbreviations,

$$\ell_k^\pm = \sqrt{\lambda_k^\pm}, \quad m_k = \sqrt{\mu_k}, \quad d_k = \sqrt{\delta_k}, \tag{6.84}$$

where, as before, $\{\lambda_k^\pm\}_{k\in\mathbb{N}}$ are the periodic/antiperiodic spectra, $\{\mu_k^\pm\}_{k\in\mathbb{N}}$ denotes the Dirichlet spectrum, and $\{\delta_k^\pm\}_{k\in\mathbb{N}}$ is the set of critical points of $\Delta_+(\mu)$. Moreover, we use the notation (cf. Theorem 3.4)

$$u_+(\zeta) = \Delta_+(\zeta^2), \quad u_-(\zeta) = \Delta_-(\zeta^2), \quad s(\zeta) = \phi(\zeta^2,\pi). \tag{6.85}$$

Next, we fix a small δ and use representations (3.15)-(3.18) to arrive at the estimates

$$C^{-1} \leq \left|\frac{\zeta s(\zeta)}{\zeta - m_k}\right| \leq C, \quad C^{-1} \leq \left|\frac{u_+^\bullet(\zeta)}{\zeta - d_k}\right| \leq C, \quad \zeta \notin D_k, \tag{6.86}$$

for all $k \geq k_0$ with sufficiently large k_0. In addition, we will use the expansions

$$u_+(\zeta) = \gamma_k + \frac{1}{2} u_+^{\bullet\bullet}(d_k)(\zeta - d_k)^2[1 + \varepsilon_k(\zeta)], \quad \zeta \in D_k, \tag{6.87}$$

where

$$\lim_{k\uparrow\infty} \max_{\zeta \in D_k} |\varepsilon_k(\zeta)| = 0. \tag{6.88}$$

Substituting $\zeta = \ell_k^\pm$ we obtain

$$\ell_k^\pm - d_k = \pm\frac{1}{\pi}(\gamma_k^2 - 1)^{1/2}[1 + o(1)], \quad k \geq k_0 \tag{6.89}$$

and

$$\ell_k^+ - \ell_k^- = \frac{2}{\pi}(\gamma_k^2 - 1)^{1/2}[1 + o(1)], \quad k \geq k_0. \tag{6.90}$$

Therefore, there exists a constant $C > 0$ such that

$$C^{-1}|\ell_k^+ - \ell_k^-| \leq |\ell_k^\pm - d_k| \leq C|\ell_k^+ - \ell_k^-|. \tag{6.91}$$

Since

$$\begin{aligned} u_-(m_k)^2 &= u_+(m_k)^2 - 1 \\ &= \gamma_k u_+^{\bullet\bullet}(d_k)\big\{(m_k - d_k)^2[1 + o(1)] + (\ell_k^\pm - d_k)^2[\pi^2 + o(1)]\big\}, \end{aligned} \tag{6.92}$$

we have

$$|u_-(m_k)|^2 \leq C(|m_k - d_k|^2 + |\ell_k^\pm - d_k|^2). \tag{6.93}$$

Next, let $\lambda = \zeta^2, \zeta \in D_k \cap \sigma'(H)$ with

$$\sigma'(H) = \{\zeta \in \mathbb{C} \,|\, u_+(\zeta) = \cos(t),\, 0 \le t \le \pi\}. \tag{6.94}$$

Then for every $k \in \mathcal{Q}$ defined in (3.32), the inequalities (3.26) and (6.86) imply

$$\left|\frac{\zeta - m_k}{\zeta - d_k}\right| = \left|\frac{\zeta - m_k}{2\zeta\phi(\zeta^2)}\right| \left|\frac{u_+^\bullet(\zeta)}{\zeta - d_k}\right| \left|\frac{\phi(\lambda,\pi)}{\Delta_+^\bullet(\lambda)}\right| \le C, \quad \zeta \in D_k \cap \sigma'(H), \tag{6.95}$$

proving an estimate

$$\left|\frac{d_k - m_k}{\zeta - d_k}\right| \le C, \quad \zeta \in D_k \cap \sigma'(H). \tag{6.96}$$

On the other hand, $|u_-(\zeta) - u_-(m_k)| \le C|\zeta - m_k|$, and hence

$$\left|\frac{u_-(m_k)}{\zeta - d_k}\right| \le \left|\frac{\Delta_-(\lambda)}{2\lambda^{1/2}\Delta_+^\bullet(\lambda)}\frac{u_+^\bullet(\zeta)}{\zeta - d_k}\right| + C\left|\frac{\zeta - m_k}{\zeta - d_k}\right|. \tag{6.97}$$

Combining the latter inequality with (3.26) we obtain

$$\left|\frac{u_-(m_k)}{\zeta - d_k}\right| \le C, \quad \zeta \in D_k \cap \sigma'(H). \tag{6.98}$$

Therefore, (6.92) and (6.96) yield

$$\left|\frac{\ell_k^\pm - d_k}{\zeta - d_k}\right| + \left|\frac{(\gamma_k^2 - 1)^{1/2}}{\zeta - d_k}\right| \le C, \quad \zeta \in D_k \cap \sigma'(H). \tag{6.99}$$

Inequalities (3.33) follow from (6.96) and (6.90). □

7. Sufficient Conditions for a Hill Operator to be a Spectral Operator of Scalar Type

Similar to the proof of Theorem 3.1, we define the Hilbert space $L^2(\sigma)^2$, $\sigma \subseteq \sigma(H)$, of measurable $\mathbb{C}^2$-vector elements with the finite norm (5.9). The following expansion theorem contains an exact description of the Fourier–Floquet transform generated by H and will turn out to be a cornerstone in the proof of sufficiency of the conditions in Theorem 3.6.

Theorem 7.1. *Suppose the conditions* (3.28) *of Theorem* 3.6 *are satisfied. Then for every element $g \in L^2(\mathbb{R})$ there exists the $L^2(\sigma(H))^2$-limit*

$$\mathbf{F}(\lambda; g) = \text{s-lim}_{R\uparrow\infty} \int_{-R}^{R} dy\, \mathbf{Y}(\lambda, x) g(y) \tag{7.1}$$

and the $L^2(\mathbb{R})$-representation

$$\begin{aligned} g(x) = \frac{1}{2\pi}\text{s-lim}_{R\uparrow\infty} &\int_{\sigma(H)\cap\{\lambda\in\mathbb{C}\,|\,|\lambda|\le R\}} \frac{d\lambda}{\sqrt{1 - \Delta_+(\lambda)^2}} \\ &\times \Big\{\big[\phi(\lambda,\pi)\theta(\lambda,x) - \Delta_-(\lambda)\phi(\lambda,x)\big] F_\theta(\lambda; g) \\ &\quad - \big[\theta'(\lambda,\pi)\phi(\lambda,x) + \Delta_-(\lambda)\theta(\lambda,x)\big] F_\phi(\lambda; g)\Big\} \end{aligned} \tag{7.2}$$

is valid.

Proof. Assume that (3.28) is valid and $\Delta_+^\bullet(\lambda_0) = 0$ for some $\lambda_0 \in \sigma(H)$. Then (6.2) follows, (6.45) implies (6.3), and we find that the function

$$\omega(\lambda) = \frac{\Delta_+(\lambda)^2 - 1}{\Delta_+^\bullet(\lambda)^2} \tag{7.3}$$

is analytic in a neighborhood of λ_0. Moreover, for $\lambda \to \lambda_0$ we have

$$\Delta_+(\lambda) = \pm 1 + c(\lambda - \lambda_0)^p [1 + o(1)], \quad \Delta_+^\bullet(\lambda) = cp(\lambda - \lambda_0)^{p-1}[1 + o(1)] \tag{7.4}$$

with some integer $p \geq 2$ and $c \neq 0$, and $\omega(\lambda) = \pm 2c^{-1}p^{-2}(\lambda - \lambda_0)^{-p+2}[1+o(1)]$. The analyticity of $\omega(\lambda)$ at λ_0 is possible only if $p = 2$, which implies that the point λ_0 is a simple zero of $\Delta_+^\bullet(\lambda)$. Therefore, no spectral arcs of H cross at interior points, at most two spectral arcs of H can meet at a point $\lambda_0 \in \mathbb{C}$ with $\Delta_+(\lambda_0)^2 = 1$, and the spectrum of H is of the form

$$\sigma(H) = \bigcup_{n=1}^\infty \Lambda_n \tag{7.5}$$

with the spectral arcs Λ_n of H being given by

$$\begin{aligned} \Lambda_n = \{z \in \mathbb{C} \,|\, z = E_{n-1}(t),\, \Delta_+(E_{n-1}(t)) = \cos(t),\, t \in [0, \pi], \\ \Delta_+^\bullet(E_{n-1}(t)) \neq 0,\, t \in (0, \pi)\}. \end{aligned} \tag{7.6}$$

In addition, all functions in (3.29) are analytic in a neighborhood of $\sigma(H)$ and the function

$$\frac{\Delta_+(\lambda)^2 - 1}{(|\lambda| + 1)\Delta_+^\bullet(\lambda)^2} \tag{7.7}$$

is bounded on $\sigma(H)$.

If $\Delta_+(\lambda)^2 - 1 = \Delta_+^\bullet(\lambda) = 0$ for some $\lambda \in \sigma(H)$, then the spectral arcs meeting at the point λ are not regular. Nevertheless, according to conditions (3.28), the constant $C = C(\sigma)$ in (5.16) is finite and is independent of the spectral arc $\sigma = \Lambda_n \subseteq \sigma(H)$.

If $g \in L^2(\mathbb{R})$, then using Lemma 5.2 one obtains $\mathbf{F}(\cdot; g) \in L^2(\Lambda_n)^2$ for all $n \in \mathbb{N}_0$, and

$$\begin{aligned} \|\mathbf{F}(\lambda; g)\|^2_{L^2(\sigma(H))^2} &= \sum_{n=1}^\infty \|\mathbf{F}(\lambda; g)\|^2_{L^2(\Lambda_n)^2} \\ &\leq C^2 \int_0^{2\pi} dt \sum_{n=0}^\infty \Big[\big|\widetilde{F}_\theta(E_n(t); G(.,t))\big|^2 + \big|\sqrt{|E_n(t)| + 1}\widetilde{F}_\phi(E_n(t); G(.,t))\big|^2 \Big]. \end{aligned} \tag{7.8}$$

Corollary 4.3 yields the inequality

$$\|\mathbf{F}(\lambda; g)\|^2_{L^2(\sigma(H))^2} \leq C^2 \int_0^{2\pi} dt \|G(\cdot, t)\|^2_{L^2([0,\pi])} = C^2 \|g\|^2_{L^2(\mathbb{R})}, \tag{7.9}$$

proving the existence of the limit in (7.1). Therefore, the mapping $\mathbf{T}g = \mathbf{F}(\lambda; g)$ defines a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\sigma(H))^2$.

To prove that $\mathbf{T}$ is a surjective map, we now assume that $\mathbf{F} \in L^2(\sigma(H))^2$ and set $\mathbf{F}_k(\lambda) = \chi_k(\lambda)\mathbf{F}(\lambda)$, $k \in \mathbb{N}$, where χ_k is the characteristic function of Λ_k. We have seen in the proof of Theorem 3.1 that there exists a sequence of functions $\{v_k(x)\}_{k=1}^\infty \subset L^2(\mathbb{R})$ such that

$$P(\Lambda_k)v_k = v_k, \quad \mathbf{F}_k(\lambda; v_k) = \mathbf{F}_k(\lambda),\ \lambda \in \Lambda_k. \tag{7.10}$$

Next, we set

$$w_n = \sum_{k=1}^{n} v_k = \sum_{k=1}^{n} P(\Lambda_k)v_k \tag{7.11}$$

and let $h \in L^2(\mathbb{R})$ be an arbitrary element. Then, as we have already proved, $\mathbf{F}(\cdot; h) \in L^2(\sigma(H))^2$ and

$$\begin{aligned} |(\overline{h}, w_n)_{L^2(\mathbb{R})}| &\leq \sum_{k=1}^{n} \frac{1}{4\pi} B_{\Lambda_k} |(\mathbf{F}_k, \mathbf{F}(\cdot; h))| \\ &\leq C\bigg(\sum_{k=1}^{n} \|\mathbf{F}_k\|^2_{L^2(\Lambda_k)^2}\bigg)^{1/2} \|h\|_{L^2(\mathbb{R})}. \end{aligned} \tag{7.12}$$

Therefore,

$$\|w_n\|_{L^2(\mathbb{R})} \leq C\bigg(\sum_{k=1}^{n} \|\mathbf{F}_k\|^2_{L^2(\Lambda_k)^2}\bigg)^{1/2} = C\|\mathbf{F}\|_{L^2(\sigma(H))^2}, \tag{7.13}$$

and there exists the $L^2(\mathbb{R})$-limit $w = \text{s-lim}_{n\uparrow\infty} w_n$. For $m \in \mathbb{N}$ one has in the norm of $L^2(\sigma(H))^2$,

$$\chi_m \mathbf{F}(\cdot; w) = \text{s-lim}_{n\uparrow\infty} \sum_{k=1}^{n} \chi_m \mathbf{F}(\cdot; v_k) = \text{s-lim}_{n\uparrow\infty} \sum_{k=1}^{n} \chi_m \mathbf{F}_k = \mathbf{F}_m. \tag{7.14}$$

Thus, $\mathbf{F}(\cdot; w) = \mathbf{F}$ and $\operatorname{ran}(\mathbf{T}) = L^2(\sigma(H))^2$. Moreover, for every $\mathbf{F} \in L^2(\sigma(H))^2$ there exists the $L^2(\mathbb{R})$-limit

$$w(x) = \text{s-lim}_{n\uparrow\infty} \frac{1}{4\pi} \sum_{k=1}^{n} B_{\Lambda_k}(\mathbf{F}, \mathbf{Y}(\cdot, x)) \tag{7.15}$$

such that $\mathbf{F}(\cdot; w) = \mathbf{F}$ and the estimate

$$\|w\|_{L^2(\mathbb{R})} \leq C\|\mathbf{F}\|_{L^2(\sigma(H))^2} \tag{7.16}$$

holds. In particular, these arguments are applicable to $\mathbf{F} = \mathbf{F}(\cdot; g)$ with an arbitrary element $g \in L^2(\mathbb{R})$ and one concludes that there exists the $L^2(\mathbb{R})$-limit

$$(P(\sigma(H))g)(x) = \text{s-lim}_{n\uparrow\infty} \frac{1}{4\pi} \sum_{k=1}^{n} B_{\Lambda_k}(\mathbf{F}(\cdot; g), \mathbf{Y}(\cdot, x)). \tag{7.17}$$

Combining (7.9) and (7.16) one finds that $P(\sigma(H))$ is a bounded linear operator on $L^2(\mathbb{R})$.

For an element $g \in L^2(\mathbb{R})$ we set $h = g - P(\sigma(H))g$. If $\sigma \subset \sigma(H)$ is an arbitrary regular spectral arc of H, then

$$P(\sigma)P(\sigma(H))g = \text{s-lim}_{n\uparrow\infty} \sum_{k=1}^{n} P(\sigma)P(\Lambda_k)g = P(\sigma)g, \tag{7.18}$$

implying $P(\sigma)h = 0$. According to the last statement of Theorem 3.1 this implies $h = 0$, $g = P(\sigma(H))g$ and $P(\sigma(H)) = I$, which proves assertion (7.2). $\square$

Proof of sufficiency of the conditions in Theorem 3.6. Theorem 7.1 contains a description of the functional model for the operator H: The latter is equivalent to the operator of multiplication by the independent variable in the space $L^2(\sigma(H))^2$.

At this point we can extend (3.3) to an arbitrary Borel set $\sigma \subseteq \sigma(H)$. Indeed, for every such set and an arbitrary element $g \in L^2(\mathbb{R})$, we define

$$(P(\sigma)g)(x) = \text{s-lim}_{n\uparrow\infty} \frac{1}{4\pi} \sum_{k=1}^{n} B_{\Lambda_k}(\chi_\sigma(\cdot)\mathbf{F}(\cdot; g), \mathbf{Y}(\cdot, x)) \tag{7.19}$$

and find that the family $\{E_H(\omega) = P(\omega \cap \sigma(H))\}_{\omega\in\mathcal{B}_\mathbb{C}}$ is a spectral resolution for the operator H. (Here $\mathcal{B}_\mathbb{C}$ denotes the collection of Borel subsets of $\mathbb{C}$.) □

Proof of sufficiency of the conditions in Theorem 3.5. If the function in (3.25) is analytic in an open neighborhood of $\sigma(H)$ and if $\Delta_+^\bullet(\lambda_0) = 0$, then $\Delta_+(\lambda_0)^2 - 1 = \phi(\lambda_0, \pi) = \Delta_-(\lambda_0) = 0$, the identity (6.46) implies that λ_0 is a simple zero of $\Delta_+^\bullet(\lambda_0) = 0$, and the functions (3.29) are locally bounded on $\sigma(H)$.

Next, we will prove that (3.26) implies the second inequality in (3.28).

In addition to employing the notation (6.85), we also introduce

$$c_1(\zeta) = \theta'(\zeta^2, \pi), \tag{7.20}$$

and recall the abbreviations $m_k = \sqrt{\mu_k}$, $d_k = \sqrt{\delta_k}$.

It is sufficient to prove

$$\sup_{\zeta\in\sigma'(H)\cap S(\delta,k_0)} \left|\frac{c_1(\zeta)}{\zeta u_+^\bullet(\zeta)}\right| < \infty, \tag{7.21}$$

where $\sigma'(H)$ is defined by (6.94), and $S(\delta, k_0)$ is the union of discs $D_k = \{\zeta \in \mathbb{C} \,|\, |\zeta - d_k| \leq \delta, k \geq k_0\}$ for some small $\delta > 0$ and sufficiently large k_0.

Now we use the following expansions in D_k:

$$\zeta s(\zeta) = m_k s^\bullet(m_k)(\zeta - m_k)[1 + o(1)], \tag{7.22}$$

$$u_+(\zeta) = \gamma_k + \frac{1}{2} u_+^{\bullet\bullet}(d_k)(\zeta - d_k)^2[1 + o(1)], \tag{7.23}$$

$$u_+^\bullet(\zeta) = u_+^{\bullet\bullet}(d_k)(\zeta - d_k)[1 + o(1)], \tag{7.24}$$

$$c_1(\zeta) = -\zeta \sin(\pi\zeta)[1 + o(1)]. \tag{7.25}$$

Using relation (6.45) we find

$$\frac{c_1(m_k)}{m_k} = 2\frac{u_+(m_k)u_+^\bullet(m_k) - u_-(m_k)u_-^\bullet(m_k)}{m_k s^\bullet(m_k)} \tag{7.26}$$

and thus,

$$\left|\frac{c_1(\zeta)}{\zeta}\right| \leq C(|u_+^\bullet(m_k)| + |u_-(m_k)| + |\zeta - m_k|), \quad \zeta \in D_k \cap \sigma'(H). \tag{7.27}$$

As we have seen in the proof of Theorem 3.7, conditions (3.26) imply (6.95)–(6.98), which in turn lead to (7.21). □

Proof of sufficiency of the conditions in Theorem 3.7. To this end we assume that $\Delta_+^\bullet(\lambda_0) = 0$ for some $\sigma(H(t))$, $t \in [0, \pi]$. Then

$$\Delta_+(\lambda) = \cos(t) + c(\lambda - \lambda_0)^p[1 + o(1)], \quad \Delta_+^\bullet(\lambda) = c\,p(\lambda - \lambda_0)^{p-1}[1 + o(1)] \tag{7.28}$$

with $c \neq 0$ and some integer $p \geq 2$. Since according to condition (ii), $p = \dim(\mathcal{E}(\lambda_0, t)) = \dim(\ker(H(t) - \lambda_0 I)) \leq 2$ (cf. (6.81) and (6.82)), we conclude that $p = 2$. Thus, the boundary value problem

$$(H(t) - \lambda_0 I)y = 0, \quad y(\pi) = e^{it}y(0),\ y'(\pi) = e^{it}y'(0), \tag{7.29}$$

has two linearly independent solutions y_1 and y_2. In this case the $\mathbb{C}^2$-vectors

$$\mathbf{Y}_1 = \begin{pmatrix} y_1(0) \\ y_1'(0) \end{pmatrix}, \quad \mathbf{Y}_2 = \begin{pmatrix} y_2(0) \\ y_2'(0) \end{pmatrix} \tag{7.30}$$

are linearly independent solutions of the equation

$$\big(\mathcal{M}(t) - e^{it} I_2\big)\, \mathbf{Y} = 0 \tag{7.31}$$

and $\rho = e^{it}$ is a double zero of the characteristic equation $\rho^2 - 2\Delta_+(\lambda_0)\rho + 1 = 0$. This is possible only if $\Delta_+(\lambda_0)^2 - 1 = 0$. The upshot is that either $t = 0$, or $t = \pi$ and hence λ_0 is a double zero of $\Delta_+(\lambda)^2 - 1$ and a simple zero of $\Delta_+^\bullet(\lambda)$. We use (6.46) once more and obtain $\phi(\lambda_0, \pi) = \Delta_-(\lambda_0) = \theta'(\lambda_0, \pi) = 0$. Therefore, all functions in (3.29) are locally bounded on $\sigma(H)$. It remains to prove (3.26) for $\lambda \in \sigma_R(H)$ with a sufficiently large $R > 0$. Moreover, because of representations (3.15)–(3.18), it is sufficient to prove (3.26) for $\lambda = \zeta^2$, $\zeta \in D_k \cap \sigma'(H)$, for all sufficiently large $k \geq k_0$.

If $\delta_k \in \sigma(H)$ for such k, then the functions

$$\frac{\zeta \phi(\zeta^2)}{u_+^\bullet(\zeta)}, \quad \frac{u_-(\zeta)}{u_+^\bullet(\zeta)} \tag{7.32}$$

are analytic on the open interior of the disc D_k and, according to (6.86), are bounded on the boundary ∂D_k by a constant independent of k. It follows from the maximum principle that they are bounded by the same constant inside D_k.

If $\delta_k \notin \sigma(H)$ for $k \geq k_0$, then similar to (6.95) (cf. also the abbreviations in (6.84)),

$$\begin{aligned} \left| \frac{\phi(\lambda, \pi)}{\Delta_+^\bullet(\lambda)} \right| &= \left| \frac{2\zeta\phi(\zeta^2)}{\zeta - m_k} \right| \left| \frac{\zeta - d_k}{u_+^\bullet(\zeta)} \right| \left| \frac{\zeta - m_k}{\zeta - d_k} \right| \\ &\leq C \left(1 + \frac{|m_k - \ell_k^\pm| + |\ell_k^+ - \ell_k^-|}{|\zeta - d_k|} \right), \quad \zeta \in D_k \cap \sigma'(H). \end{aligned} \tag{7.33}$$

At last, we apply (6.93) and obtain

$$\begin{aligned} \left| \frac{\Delta_-(\lambda)}{\sqrt{\lambda}\Delta_+^\bullet(\lambda)} \right| &= \left| \frac{u_-(\zeta)}{u_+^\bullet(\zeta)} \right| \leq C \frac{|u_-(m_k)| + |\zeta - m_k|}{|u_+^\bullet(\zeta)|} \\ &\leq C \left(1 + \frac{|m_k - \ell_k^\pm| + |\ell_k^+ - \ell_k^-|}{|\zeta - d_k|} \right), \quad \zeta \in D_k \cap \sigma'(H). \end{aligned} \tag{7.34}$$

According to the assumptions (3.33), the inequalities (3.26) are satisfied, and by virtue of Theorem 3.5, H is a spectral operator of scalar type. $\square$

8. Concluding Remarks

In our final section we further illustrate the principal results of this paper in a series of remarks.

Remark 8.1. We emphasize again some of the main points and consequences of Theorems 3.5–3.7.

If the conditions (3.28) are satisfied, then the following assertions are valid:

(α) H has no spectral singularities and $\{P(\sigma)\}_{\sigma \subseteq \sigma(H)}$ defines a spectral resolution for H in the sense that $E_H(\omega) = P(\omega \cap \sigma(H))$ for all Borel sets $\omega \in \mathbb{C}$.

(β) The spectrum of H consists of a system of countably many, simple, nonintersecting, analytic arcs (the latter may degenerate into finitely many simple analytic arcs and a simple analytic semi-infinite arc, all of which are nonintersecting).

(γ) The functions

$$\frac{\phi(z,\pi)}{\Delta_+^\bullet(z)}, \quad \frac{\Delta_-(z)}{\Delta_+^\bullet(z)}, \quad \frac{\theta'(z,\pi)}{\Delta_+^\bullet(z)}, \quad \frac{\Delta_+(z)^2-1}{\Delta_+^\bullet(z)^2}, \quad \frac{\Delta_+(z)^2-1-\Delta_-(z)^2}{\phi(z,\pi)\Delta_+^\bullet(z)} \tag{8.1}$$

are analytic in an open neighborhood of $\sigma(H)$.

Remark 8.2. Theorem 7.1 shows that conditions (3.28) are sufficient for establishing a spectral decomposition of non-self-adjoint Hill operators with complex-valued potentials in complete analogy to the well-known spectral decomposition of self-adjoint Hill operators (cf. [64], [65, Ch. XXI]). In this sense, Hill operators which are spectral operators of scalar type, demonstrate their great similarity to the case of self-adjoint Hill operators. We also note that equation (7.2) can be rewritten in the form

$$g(x) = \text{s-lim}_{R\uparrow\infty} \int_{\sigma(H)\cap\{\lambda\in\mathbb{C}\,|\,|\lambda|\leq R\}} d\lambda\, (\mathcal{S}(\lambda)\mathbf{F}(\lambda;g), \mathbf{Y}(\lambda,x))_{\mathbb{C}^2}, \tag{8.2}$$

where

$$\mathcal{S}(\lambda) = \frac{1}{2\pi\sqrt{1-\Delta_+(\lambda)^2}} \begin{pmatrix} \phi(\lambda,\pi) & -\Delta_-(\lambda) \\ -\Delta_-(\lambda) & -\theta'(\lambda,\pi) \end{pmatrix}, \quad \lambda\in\sigma(H), \tag{8.3}$$

represents the analog of the 2×2 spectral matrix of H familiar from the self-adjoint context (cf. [14, Sect. XIII.5], [48, Ch. VI], [65, Ch. III]).

Remark 8.3. Next, we will show that the two inequalities in (3.26) are independent. We start by constructing a Hill operator for which the first inequality in (3.26) holds but the second does not. To this end, let $\gamma_{2k-1} = -1$, $\gamma_{2k} = 1-\pi^2(\rho_{|k|}^2 + i\omega_{|k|}^2)/2$, $k=0,\pm 1,\dots$, with sufficiently small and rapidly vanishing numbers $\rho_k>0$ and $\omega_k>0$ such that $\omega_k\rho_k^{-1} \underset{k\uparrow\infty}{\to} 0$. According to [69] there exists a function $u_+(\zeta)$ satisfying property (ii) in Theorem 3.4 such that $\{\gamma_k\}_{k\in\mathbb{Z}}$ is its set of critical values. The corresponding critical points are of the form $d_k \underset{k\uparrow\infty}{=} k+o(1)$ and if $r>0$ is a sufficiently small fixed constant and $D_k = \{\zeta\in\mathbb{C}\,|\,|\zeta-d_k|\leq r\}$, then

$$u_+(\zeta) = \gamma_k - \pi^2(-1)^k(\zeta-d_k)^2(1+o(1))/2, \quad \zeta\in D_k. \tag{8.4}$$

An elementary analysis show that the set

$$\sigma_{2k}' = \{\zeta\in\mathbb{C}\,|\,u_+(\zeta)=\cos(t),\ t\in[0,\pi]\}\cap D_{2k} \tag{8.5}$$

is formed by two analytic arcs which are close to the sets

$$c_{2k}^\pm = \Big\{\zeta\in\mathbb{C}\,|\,\zeta = d_{2k}\pm i\sqrt{\rho_{|k|}^2 + i\omega_{|k|}^2 - t^2/(\pi)^2}\Big\} \tag{8.6}$$

and therefore, there exists a point $\nu_{2k}\in\sigma_{2k}'$ such that $|\nu_{2k}-d_{2k}|\leq 2\omega_k$. We note that the condition $\omega_k>0$ eliminates the intersection of spectral gaps.

Furthermore, we set $s(\zeta) = -u^\bullet(z)/(\pi\zeta)$ and define a function $u_-(\zeta)$ using the interpolation data $u_-(d_k)^2 = u_+(d_k)^2-1$. According to Theorem 3.4, with $\rho_k>0$ and $\omega_k>0$ being sufficiently small, there exists a Hill operator H such that $\Delta_+(\lambda) = u_+(\sqrt{\lambda})$, $\phi(\lambda,\pi) = s(\sqrt{\lambda})$, $\Delta_-(\lambda) = u_-(\sqrt{\lambda})$. It is evident that the first condition in (3.26) is satisfied for every such operator. On the other hand, for all sufficiently large $|k| \in \mathbb{N}$, we have $|u_-(d_{2k})| = |\gamma_{2k}^2 - 1|^{1/2} \geq 2^{-1}\rho_{|k|}$, and if $\lambda_k = \nu_{2k}^2$, then the estimates from below

$$\left|\frac{\Delta_-(\lambda_k)}{2\lambda_k^{1/2}\Delta_+^\bullet(\lambda_k)}\right| = \left|\frac{u_-(\nu_{2k})}{u_+^\bullet(\nu_{2k})}\right| \geq 10^{-1}\left(\left|\frac{u_-(d_{2k})}{d_{2k}-\nu_{2k}}\right| - \left|\frac{u_-(\nu_{2k}) - u_-(d_{2k})}{d_{2k}-\nu_{2k}}\right|\right)$$
$$\geq C\frac{\rho_{|k|}}{\omega_{|k|}} \tag{8.7}$$

hold with $C > 0$ independent of k. Thus, H is not a spectral operator of scalar type and the second condition in (3.26) cannot be omitted in Theorem 3.5.

Next, we consider the Hill operator H associated with the triple $\{\Delta_+(\lambda),\ \phi(\lambda) = s(\sqrt{\lambda}), 0\}$ with the same function $\Delta_+(\lambda)$ as above and with the entire function $s(\zeta)$ uniquely determined by the interpolation data $s(l_k^+) = 0$, $k \neq 0$, and by its exponential type π. With the same $\lambda_k = \nu_k^2$ as above, one obtains

$$\left|\frac{\phi(\lambda_k)}{\Delta_+^\bullet(\lambda_k)}\right| = \left|\frac{2\nu_{2k}s(\nu_{2k})}{u_+^\bullet(\nu_{2k})}\right| \geq 10^{-1}\left|\frac{l_{2k}^+ - \nu_{2k}}{d_{2k}-\nu_{2k}}\right| \geq C\frac{\rho_{|k|}}{\omega_{|k|}}. \tag{8.8}$$

Thus, the first inequality in (3.26) fails, while the second obviously holds since $\Delta_-(\lambda) \equiv 0$.

Remark 8.4. It is evident that the first inequality in (3.33) follows from two remaining ones and thus it could have been omitted. Nevertheless, we decided to keep it in the statement of Theorem 3.7 because of the following well-known interpretation of $|\lambda_k^+ - \lambda_k^-|$ in the self-adjoint situation: If V is real-valued, then λ_k^+ and λ_k^- are the end-points of spectral gaps, $|\lambda_k^+ - \lambda_k^-|$ are the lengths of spectral gaps, the points μ_k of the Dirichlet spectrum are trapped inside the gaps, and the second and third inequality in (3.33) become redundant. However, we will show next that, generally speaking, they are indispensable for complex-valued potentials V.

Indeed, if all critical values γ_k of $\Delta_+(z)$ are real and $|\gamma_k| \geq 1$, then the first condition in (3.33) is satisfied. We can choose γ_k exponentially close to $(-1)^k$ and afterwards fix real zeroes m_k of $s(\zeta) = \phi(\zeta^2, \pi)$ outside spectral gaps in such a way that $|m_k - d_k| = \varepsilon|k|^{-\text{any big integer}}$. The corresponding non-self-adjoint Hill operator exists according to Theorem 3.4, but neither the second nor the third inequality in (3.33) are satisfied.

The numerators of fractions in (3.33) contain distances between the closest points of periodic, antiperiodic, and Dirichlet boundary problems generated by the expression (2.2) in the space $L^2([0,\pi])$. Their denominator also has a spectral meaning: Multiplied by the factor 2, it is equivalent to the distance between adjacent spectral arcs and may be considered as the length of the gap between them. These lengths decay as k grows to infinity and equation (3.33), if valid, states that the distances between these three spectra vanish at least at the same rate.

It is interesting to note that the smoothness or analyticity properties of V are characterized in terms of ℓ^2-norms of the numerators in (3.33), with no connection with the relative rate of their decay (cf. [67], [70], [12]).

Remark 8.5. If λ_0 is a point of either the periodic or antiperiodic spectrum of the operator H and a simple Dirichlet eigenvalue of (2.23) such that $\Delta_+^\bullet(\lambda_0) = 0$, then the analyticity of the first two functions in (8.1) at λ_0 implies that λ_0 is a simple zero of $\Delta_+^\bullet(z)$ and the three remaining functions are analytic at λ_0 as well. We will next show that the situation is different if a Dirichlet eigenvalue λ_0 has algebraic multiplicity larger than one, that is, $\phi(\cdot,\pi)$ has a zero at λ_0 of order larger than one (cf. [24]).

According to [71], an arbitrary complex number λ_0 may be simultaneously a point of the Dirichlet spectrum of multiplicity $m > 2$ and a point of the periodic spectrum of multiplicity 2 of the operator H in (2.3). We fix $\lambda_0 \in \mathbb{C}$, set $m = 3$, and consider a corresponding Hill operator H. Following [55], [69], we can assume that the triple $\{\phi(z,\pi), \Delta_+(z), \Delta_-(z)\}$ parameterizing such an operator has, in addition to properties (i)–(v) in Theorem 3.4, the following properties (vi)–(ix):

(vi) If $z \to \lambda_0$, then

$$\Delta_+(z) = 1 + a(z-\lambda_0)^2[1+o(1)], \quad \phi(z,\pi) = b(z-\lambda_0)^3[1+o(1)] \tag{8.9}$$

with $ab \neq 0$.

(vii) There exists $R > 0$ such that if $\lambda \in \sigma_{\mathrm{ext},R}(H)$, $\Delta_+^{\bullet}(\lambda) = 0$, then $\Delta_+^{\bullet\bullet}(\lambda) \neq 0$ and

$$\Delta_+^{\bullet}(\lambda)^2 - 1 = \phi(\lambda,\pi) = \Delta_-(\lambda) = \lim_{z\to\lambda} \frac{\Delta_+(z)^2 - 1 - \Delta_-(z)^2}{\phi(z,\pi)} = 0. \tag{8.10}$$

$(viii)$ All critical points of $\Delta_+(\cdot)$ in the disc $\{z \in \mathbb{C} \,|\, |z| \leq R\}$, with the exception of λ_0, do not belong to the set $\sigma(H)$.

(ix) All zeros of $\phi(\cdot,\pi)$, except λ_0, are simple.

All conditions of Theorems 3.5 and 3.6 are satisfied for such an operator H, except perhaps, the analyticity of the function

$$\frac{\Delta_+(z)^2 - 1 - \Delta_-(z)^2}{\phi(z,\pi)\Delta_+^{\bullet}(z)} \tag{8.11}$$

at λ_0. For $z \to \lambda_0$ we have

$$\begin{aligned} \Delta_+(z)^2 - 1 - \Delta_-(z)^2 &= \big[\Delta_+^{\bullet\bullet}(\lambda_0) - \Delta_-^{\bullet}(\lambda_0)^2\big](z-\lambda_0)^2 \\ &\quad + \big[3^{-1}\Delta_+^{\bullet\bullet\bullet}(\lambda_0) - \Delta_-^{\bullet}(\lambda_0)\Delta_-^{\bullet\bullet}(\lambda_0)\big](z-\lambda_0)^3 + o((z-\lambda_0)^3) = O((z-\lambda_0)^3), \end{aligned} \tag{8.12}$$

and hence $\Delta_-^{\bullet}(\lambda_0) = \pm\sqrt{\Delta_+^{\bullet\bullet}(\lambda_0)} \neq 0$. In addition to the latter relation, the analyticity of the function (8.11) leads to another relation, $\Delta_+^{\bullet\bullet\bullet}(\lambda_0) - 3\Delta_-^{\bullet}(\lambda_0)\Delta_-^{\bullet\bullet}(\lambda_0) = 0$. In any case, whether the latter relation is satisfied or not, we construct a function $v(\lambda)$ using the interpolation data

$$\begin{aligned} &v(\mu_k) = \Delta_-(\mu_k),\ \mu_k \neq \lambda_0,\ v(\lambda_0) = 0,\ v^{\bullet}(\lambda_0) = \Delta_-^{\bullet}(\lambda_0), \\ &v^{\bullet\bullet}(\lambda_0) \neq \Delta_+^{\bullet\bullet\bullet}(\lambda_0)/[3v^{\bullet}(\lambda_0)] \end{aligned} \tag{8.13}$$

such that $\|\sqrt{z}[v(z) - \Delta_-(z)]\|_{\mathbb{PW}_\pi}$ is sufficiently small and find the Hill operator H (cf. (2.3)) corresponding to the triple $\{\phi(z,\pi), \Delta_+(z), v(z)\}$. Conditions (ii) of Theorem 3.5 are satisfied for such an operator, but analyticity of the function (8.11) at λ_0 fails. In other words, the analyticity of the function (3.25) does not follow from (3.26) and condition (i) in Theorem 3.5 is indispensable.

Remark 8.6. Gasymov showed in [20] (cf. also [21], [60]) that if

$$V(x) = \sum_{n=1}^{\infty} c_n e^{2inx}, \quad \{c_n\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}),\ x \in \mathbb{R}, \tag{8.14}$$

then $\Delta_+(z) = \cos(\pi\sqrt{z})$. Thus, in this case $\sigma(H) = [0,\infty)$ and the function

$$\frac{\phi(z,\pi)}{\Delta^\bullet_+(z)} = -\frac{2\sqrt{z}\phi(z,\pi)}{\pi\sin(\pi\sqrt{z})} \tag{8.15}$$

is analytic in an open neighborhood of $\sigma(H)$ if and only if $\phi(z,\pi) = \sin(\pi\sqrt{z})/\sqrt{z}$. In the latter case $\Delta_-(z) \equiv 0$ and $V(x) = 0$ for a.e. $x \in \mathbb{R}$. This means that no smoothness or analyticity conditions imposed on V can guarantee that the Hill operator (2.3) is a spectral operator of scalar type. Nevertheless, every Hill operator with a complex-valued locally square-integrable potential is an operator with a separable spectrum as defined by Lyubich and Matsaev [34], [35].

For every operator H with a nontrivial potential (8.14) there exists at least one integer $n_0 \in \mathbb{N}$ such that $\phi(n_0^2,\pi) \neq 0$ and the point n_0^2 is then a spectral singularity. For a generic element $f \in L^2(\mathbb{R})$, norm-convergent expansions of the form (7.2) are impossible, but Gasymov derived a regularized spectral expansion containing, in particular, terms of the type

$$\bigg(\int_{\mathbb{R}} dy\,\psi_+(n_0^2,y)f(y)\bigg)\psi_+(n_0^2,\cdot) \tag{8.16}$$

not belonging to $L^2(\mathbb{R})$, and converging for sufficiently smooth and sufficiently fast decaying functions f.

Regularized expansions of a similar type for arbitrary potentials V were also proposed by Veliev in [72] (cf. also [73]–[75]). However, from our point of view the proofs given in papers [72]–[75] are not satisfactory since they completely ignore the possible growth of functions in (3.28), leaving disputable the convergence of the series even for compactly supported continuous functions f.

Remark 8.7. Explicit examples of crossings of spectral arcs $\lambda(t)$ of H at interior points (i.e., for values of t different from 0 and π) were first constructed by Pastur and Tkachenko [52]. An explicit example of two crossing arcs in terms of the Weierstass $\wp$-function was found by Gesztesy and Weikard [24]. These crossings represent spectral singularities of the underlying Hill operator H. As shown by Tkachenko [68], these crossings, and hence the existence of such spectral singularities, is unstable with respect to arbitrarily small perturbations of V in $L^2([0,\pi])$. More precisely, for any $\varepsilon > 0$ there exists a periodic potential $V_\varepsilon \in L^2_{\text{loc}}(\mathbb{R})$ of period π with $\|V - V_\varepsilon\|_{L^2([0,\pi])} < \varepsilon$, such that the spectrum of $H_\varepsilon = -d^2/dx^2 + V_\varepsilon$ is a system of nonintersecting regular analytic arcs. Moreover, each spectral arc of H is mapped in a one-to-one manner on the interval $[-1,1]$ by the Floquet discriminant Δ_+. Thus, generically (in the $L^2([0,\pi])$-sense just described), the spectrum of a Hill operator H is formed by a system of nonintersecting analytic arcs. In addition, an approximating potential may be chosen in such a way that all conditions of Theorems 3.5–3.7 are satisfied. Put differently, generically, Hill operators H are spectral operators of scalar type.

Remark 8.8. There are two obstacles for the operator H not to be a spectral operator of scalar type, and both of them may be elucidated using the direct integral representation

$$P(\sigma) = \frac{1}{2\pi}\int^{\oplus}_{\{t\in[0,2\pi]\,|\,\Delta_+(\lambda)=\cos(t),\ \lambda\in\sigma\}} dt\,P(t;\sigma), \tag{8.17}$$

where $P(t;\sigma)$ is the spectral projection generated by $H(t)$.

The first obstacle is the presence of critical points of the Hill discriminant in the spectrum, and to explain their influence we assume that $t_0 \in (0,\pi)$ and $\lambda_0 \in \sigma(H(t_0))$ are such that $\Delta_+(\lambda_0)^2 - \cos(t_0) = \Delta_+^{\bullet}(\lambda_0) = \cdots = \Delta_+^{(m)}(\lambda_0) = 0$, $\Delta_+^{(m+1)}(\lambda_0) \neq 0$. Then the root subspace (i.e., the algebraic eigenspace) of $H(t_0)$ corresponding to the eigenvalue λ_0 is of dimension $m+1 \geq 2$. On the other hand, since $\Delta_+(\lambda_0)^2 - 1 = ((\theta(\lambda_0,\pi) - \phi'(\lambda_0,\pi)))/2)^2 - \theta'(\lambda_0,\pi)\phi(\lambda_0,\pi)) \neq 0$, at least one entry of the matrix $\mathcal{M}(\lambda_0) - e^{it_0}I$ does not vanish and the corresponding geometric eigenspace is one-dimensional.

If $t \neq t_0$ is close to t_0, then the spectrum of $H(t)$ in a neighborhood of λ_0 is formed by $m+1$ simple eigenvalues, and the corresponding eigenvectors span an $(m+1)$-dimensional invariant subspace. The angle between any pair of linearly independent eigenvectors tends to zero as t tends to t_0 and the projection onto one of them along the other has its norm increasing to infinity. A similar situation arises if $t_0 = 0 \,(\mathrm{mod}\,\pi)$ and $m \geq 3$. In either case, the family $\{P(\sigma)\}$ with regular spectral arcs σ close to λ_0 is not bounded and we have a blowup at this point.

Another obstacle for H to be a spectral operator of scalar type is connected to the behavior of functions in (3.29) at infinity. For every $t \neq 0 \,(\mathrm{mod}\,\pi)$ the boundary conditions (2.39) are regular [48] and the corresponding eigenvalues are asymptotically separated (i.e., asymptotically they are simple, and consecutive eigenvalues are a fixed minimal, possibly t-dependent distance apart from each other). This has been discussed in [30], [45], and [15, Ch. XIX] for more general boundary conditions than the ones in (2.39). In addition, the normalized root system of $H(t)$ is a basis in the space $L^2([0,\pi])$ in this case. Moreover, as proved in [75], it is a uniform Riesz basis for $t \in [\varepsilon, \pi - \varepsilon]$, which means (in the absence of root functions) that the inequalities

$$C_\varepsilon^{-1} \|f\|^2_{L^2([0,\pi])} \leq \sum_{k\in\mathbb{N}_0} |(f, \hat{\psi}_\pm(E_k(t),\cdot))|^2 \leq C_\varepsilon \|f\|^2_{L^2([0,\pi])} \tag{8.18}$$

are valid for all such t's with some finite constant $C_\varepsilon > 0$. As a result, the family of projections $\{P(\sigma)\}$ with $\sigma \subseteq \{\lambda \in \sigma(H) \,|\, \Delta_+(\lambda) = \cos(t),\ \varepsilon \leq t \leq \pi - \varepsilon\}$ is bounded, but the possible dependence of C_ε on ε does not permit us to claim that H is a spectral operator of scalar type.

If, however, $t = 0 \,(\mathrm{mod}\,\pi)$, then the boundary conditions in (2.39) remain regular, but the eigenvalues of the operators $H(t)$ with such t's are grouped in converging pairs $(\lambda_k^+(t), \lambda_k^-(t))$ which may amalgamate as k tends to infinity. In the special case where H is self-adjoint, so is every operator $H(t)$, and because of the orthogonality of spectral projections, even in the worst case scenario of coincidence, $\lambda_k^+(t) = \lambda_k^-(t)$, the norm of the corresponding spectral projections $P(\sigma,t)$ remains equal to 1.

The situation is quite different for complex-valued potentials V. For t close to $0 \,(\mathrm{mod}\,\pi)$, the angle between eigenspaces of $H(t)$ corresponding to $\lambda_k^-(t)$ and $\lambda_k^+(t)$ may approach zero as $k \uparrow \infty$ with different k's for different t's and we can find a sequence of regular spectral arcs σ_m accumulating at infinity such that $\lim_{m\to\infty} \|P(\sigma_m)\| = \infty$. This may be considered a blowup at the point infinity, and the operator H in Remark 8.3 yields an example of such a behavior. The conditions described in Theorems 3.5–3.7 prevent both blowup phenomena from happening.

Remark 8.9. The problem we are discussing in this remark is similar to the following well-known question: When is the exponential system $\{e^{i\mu_k x}\}_{k\in\mathbb{Z}}$ a Riesz basis in $L^2([-\pi,\pi])$? The investigation of such problems for non-orthogonal systems of exponentials $\{\exp(i\mu_k x)\}_{k\in\mathbb{N}}$, $x\in[-\pi,\pi]$, which is similar to those presently discussed, has a long history. A criterion for the latter system to be a Riesz basis in $L^2([-\pi,\pi])$ was found by Pavlov [49]. Most recently, Minkin [46] made essential progress in studying the same property for systems of eigenfunctions of two-point boundary problems for higher-order differential operators, and Makin [36] found sufficient conditions for the root system of a Schrödinger operator on the interval $[0,1]$ associated with periodic and antiperiodic boundary conditions to be (or not to be) a Riesz basis. Using well-known asymptotic formulas (see [38]) for the periodic/antiperiodic and Dirichlet spectra of the corresponding operator H restricted to the interval $[0,1]$, it is easy to check that under the assumptions made on V in [38], the conditions of Theorem 3.6 are either satisfied (or not satisfied), with the corresponding conclusions about the system of eigenfunctions and its property of forming a Riesz basis. Recently, Djakov and Mityagin [12] constructed a series of potentials in some weighted spaces of periodic functions for which the corresponding eigensystems are not Riesz bases in $L^2([-\pi,\pi])$.

Returning to the eigensystems of the operators $H(t)$, we will next show that our conditions (3.28) are sufficient for the eigensystem of every such operator to form a Riesz basis in the space $L^2([0,\pi])$ with a constant C_ε in (8.18) independent of ε.

First of all, we fix $t\in[0,\pi]$ with $t\neq 0\,(\mathrm{mod}\,\pi)$, and note that for every $k\in\mathbb{N}_0$ such that $\phi(E_k(t),\pi)\neq 0$, the numbers $w_\pm(E_k(t))$, with $w_\pm(\lambda)$ being defined by (3.6), are finite, do not vanish, and the functions

$$\hat\psi_\pm(E_k(t),x)=\frac{\psi_\pm(E_k(t),x)}{w_\pm(E_k(t))},\quad t\in[0,\pi],\ t\neq 0\,(\mathrm{mod}\,\pi), \tag{8.19}$$

are normalized Floquet solutions of $L\psi=z\psi$. Second, for the function

$$m(\lambda)=-\frac{\phi(\lambda,\pi)w_+(\lambda)w_-(\lambda)}{2\Delta_+^\bullet(\lambda)}, \tag{8.20}$$

equation (4.33) yields $|m(E_k(t))|\geq 1$ while, similar to (5.16), conditions (3.28) imply $|m(E_k(t))|\leq C$, with $C>0$ independent of $t\in[0,\pi]$.

Furthermore, for each $k\in\mathbb{N}_0$ such that $\phi(E_k(t),\pi)=0$, we define the normalized Floquet solutions of $L\psi=z\psi$ and the numbers $m(E_k(t))$ by the relations

$$\hat\psi_\pm(E_k(t),x))=\lim_{t\to s}\hat\psi_\pm(E_k(s),x),\qquad m(E_k(t))=\lim_{t\to s}m(E_k(s)),$$

and obtain the estimates $1\leq|m(E_k(t))|\leq C$ with the same constant C for all $t\in(0,\pi)$ and $k\in\mathbb{N}_0$.

At last, we write the expansion (4.36) for every $t\in(0,\pi)$ in the form

$$f(x)=\sum_{k\in\mathbb{N}_0}m(E_k(t))\,(\overline{f},\hat\psi_-(E_k(t),\cdot))\hat\psi_+(E_k(t),x), \tag{8.21}$$

and using Corollaries 4.2 and 4.3 prove the inequalities (8.18) for $t\in(0,\pi)$ with C_ε actually independent of ε.

For $t=0\,(\mathrm{mod}\,\pi)$ the spectrum of $H(t)$ degenerates if $\Delta_+^\bullet(E_k(t))=0$ and the series in (8.18) must be modified. Assume, for instance, that $t=0$ and set $\mathcal{K}=\{k\in\mathbb{N}_0\,|\,|\Delta_+^\bullet(E_k(0))=0\}$. Then $E_k(0)=E_{k+1}(0)=\lambda_k$ for $k\in\mathcal{K}$ and there exists a small neighborhood U_0 of $t=0$, the same for all sufficiently large $k\in\mathcal{K}$, with solutions $E_k^+(t)$ and $E_k^-(t)$ of the equation $\Delta_+(\lambda)=\cos(t)$, $t\in U_0$, given by (4.3). Moreover, for all sufficiently large k these solutions are analytic in U_0. Since the constant C in (4.4) is independent of t, we can differentiate the representations (4.3) and obtain

$$\frac{dE_k^{\pm}(0)}{dt} = \pm\frac{4k}{\pi} + \frac{g_k^{\pm}}{k}, \quad \sum_{k\in\mathcal{K}} |g_k^{\pm}|^2 < \infty. \tag{8.22}$$

According to Theorem 3.6, the functions (3.29) are analytic in a neighborhood of λ_k, one has $\phi(\lambda_k,\pi) = \theta(\lambda_k,\pi) - 1 = 0$, and hence there exist the limits

$$\begin{aligned}\psi_k^{\pm}(x) &= \lim_{t\to 0} \psi_+(E_k^{\pm}(t), x) \\ &= \theta(\lambda_k, x) + \frac{i\left(\frac{dE_k^{\pm}(0)}{dt}\right)^{-1} + \theta^{\bullet}(\lambda_k,\pi)}{\phi^{\bullet}(\lambda_k,\pi)}\,\phi(\lambda_k,x), \quad k\in\mathcal{K}.\end{aligned}$$

These functions are two linearly independent Floquet solutions forming a basis in the eigenspace of $H(0)$ corresponding to its eigenvalue λ_k. Taking into account (4.30), (4.32), and (4.3), we obtain the representation

$$\psi_k^{\pm}(x) = e^{\pm 2\pi i x} + G_k(x), \quad \sum_{k\in\mathcal{K}} \|G_k\|^2_{L^2([0,\pi])} \le C. \tag{8.23}$$

After normalizing the system

$$\{\psi_k^{\pm}\}_{k\in\mathcal{K}} \cup \{\psi_+(E_k^{\pm}(0),\cdot)\}_{k\in\mathbb{N}_0\setminus\mathcal{K}}, \tag{8.24}$$

it forms a Riesz basis in $L^2([0,\pi])$.

The same arguments are valid for $t=\pi$ and hence inequalities (8.18) are satisfied for all $t\in[0,2\pi]$. Thus, if the conditions of at least one of the Theorems 3.5–3.7 are satisfied, then the system of eigenfunctions of every operator $H(t)$, $t\in[0,2\pi]$, forms a Riesz basis in the space $L^2([0,\pi])$.

Arguments similar to those used in the proof of Theorem 3.6 show that the system of eigenfunctions of $H(t)$ with $t = 0 \,(\mathrm{mod}\,\pi)$ is a Riesz basis if and only if

$$\left|\frac{\phi(\lambda_k(t),\pi)}{\Delta_+^{\bullet}(\lambda_k(t))}\right| + \left|\frac{\theta'(\lambda_k(t),\pi)}{(|\lambda_k(t)|+1)\Delta_+^{\bullet}(\lambda_k(t))}\right| + \left|\frac{\Delta_-(\lambda_k(t))}{(\sqrt{|\lambda_k(t)|}+1)\Delta_+^{\bullet}(\lambda_k(t))}\right| \le C \tag{8.25}$$

for such t's and all $k\ge 1$. Of course, condition (8.25) is weaker than (3.28), and, as the operator H constructed in Remark 8.3 demonstrates, the former condition may be satisfied, while the latter may not. As a result, the eigensystems of $H(t)$ are Riesz bases for all $t\in[0,2\pi]$, while H is not a spectral operator of scalar type, in complete agreement with the unboundness of the family $\{C_\varepsilon\}_{\varepsilon>0}$ in (8.18).

Remark 8.10. Finally, we briefly describe some historical developments related to spectral theory of non-self-adjoint Hill operators: The notion of spectral projections associated with regular spectral arcs (cf. Definition 2.4) was introduced by Tkachenko [66] in 1964. Spectral projections were also discussed by Veliev [72]–[74] who defined spectral singularities as described in 3.2. He was first to note that $\lambda = \lambda_k(t)$ is a point of spectral singularity of H if and only if the root subspace of $H(t)$ corresponding to λ contains a root function which is not an actual eigenfunction of $H(t)$. However, since he erroneously concluded that Hill operators lead to non-intersecting analytic arcs (cf. [73, Theorem 5]), the counter examples of Hill operators with crossing spectral arcs in the interior of these arcs found in [52] (and subsequently in [24]) invalidate some of his results concerning spectral projections and spectral singularities. A bit earlier spectral expansions for non-self-adjoint Hill operators were also briefly touched upon by Meiman [44] (but no proofs of his claims were offered). Moreover, he noted on page 846 that zeros of $\Delta_+^\bullet(\lambda)$ are integrable singularities for the functions in (3.29) with $\lambda = \lambda(t)$, but this is incorrect for $t_0 \in \{0, \pi, 2\pi\}$. Indeed, for potentials of the type (8.14) and $t_0 \in \{0, \pi, 2\pi\}$, one has $\Delta_+^\bullet(\lambda(t)) = (c+o(1))(t-t_0)$, $c \neq 0$, and if $\phi(\lambda(t_0), \pi) \neq 0$, the singularity at t_0 is not integrable. Another claim of that paper on page 846, suitably paraphrased, is the following: "The crude asymptotic estimates for $\phi(\lambda, \pi)$, $\Delta_+^\bullet(\lambda)$ and the Floquet solutions of H remain valid in the complex case. From this[1] it follows that the Fourier integral theory remains essentially valid also for the expansion in eigenfunctions of a non-self-adjoint Schrödinger operator with a complex-valued periodic potential." However, such a vague statement, without explanations as to what type of convergence and what type of function space is meant, is unsatisfactory, especially, taking into account Theorems 3.5–3.7.

For spectral expansion theorems in connection with finite interval problems and limit circle-type situations we refer to [17], [46], [75], and the literature cited therein. Spectral resolutions for a special class of non-self-adjoint operators were also considered by Volk [76]. Exponentially decaying perturbations of non-self-adjoint Hill operators were studied by Zheludev [79]. A rather different route was chosen by Marchenko [37], [38, Ch. 2] (see also [19]), who introduced the notion of spectral distributions for non-self-adjoint problems. For the notion of generalized spectral operators we refer, for instance, to [8], [33] and the references therein.

Appendix A. Spectral Operators in a Nutshell

In this appendix we recall a selection of basic facts on spectral operators as discussed in great detail in volume 3 of Dunford and Schwartz [15]. Unless explicitly stated otherwise, the material presented is taken from [15] (see also [1], [13], [18], [58]) and the reader is referred to this monograph for proofs and pertinent references on this subject. For simplicity we will restrict our discussion to operators in a separable complex Hilbert space. We note, however, that the material below is developed in a general Banach space context in [15].

To set the stage, we assume the following conventions for the rest of this section: $\mathcal{H}$ denotes a separable, complex Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ (linear in the second factor), norm $\|\cdot\|_{\mathcal{H}}$, and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. The Banach space of bounded linear operators on $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$ with norm $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$, and the set of densely defined, closed linear operators in $\mathcal{H}$ will be denoted by $\mathcal{C}(\mathcal{H})$. The domain and range of a linear operator S are denoted by $\mathrm{dom}(S)$ and $\mathrm{ran}(S)$.

Definition A.1.
(i) A *spectral measure* E in $\mathcal{H}$ is a homomorphic map of a σ-algebra $\mathcal{A}$ of sets into a Boolean algebra of projection operators in $\mathcal{H}$ such that the unit of $\mathcal{A}$ is mapped to $I_{\mathcal{H}}$. The spectral measure E is called *bounded* if for some $C > 0$, $\|E(\omega)\|_{\mathcal{B}(\mathcal{H})} \leq C$ for all $\omega \in \mathcal{A}$.
(ii) If $T \in \mathcal{C}(\mathcal{H})$, then $\sigma \subseteq \sigma(T)$ is called a *spectral set* if σ is both open and closed in the topology of $\sigma(T)$.

In the concrete applications we have in mind in the bulk of this paper, $\Omega = \mathbb{C}$ and Σ typically equals the σ-algebra $\mathcal{B}_{\mathbb{C}}$ of Borel subsets of $\mathbb{C}$; thus we confine ourselves to this case for the rest of this appendix.

Next we turn to the special case of bounded spectral operators on $\mathcal{H}$.

[1] and some formulas in Section 6 of [44]

Definition A.2. Let $T \in \mathcal{B}(\mathcal{H})$.
(i) A projection-valued spectral measure E on $\mathcal{B}_{\mathbb{C}}$ is called a *resolution of the identity* (or a *spectral resolution*) for T if

$$E(\omega)T = TE(\omega), \quad \sigma\big(T\big|_{E(\omega)\mathcal{H}}\big) \subseteq \overline{\omega}, \quad \omega \in \mathcal{B}_{\mathbb{C}}. \tag{A.1}$$

(ii) A projection-valued spectral measure E in $\mathcal{H}$ defined on $\mathcal{B}_{\mathbb{C}}$ is called *countably additive* if for all $f, g \in \mathcal{H}$, $(f, E(\cdot)g)_{\mathcal{H}}$ is countably additive on $\mathcal{B}_{\mathbb{C}}$.
(iii) T is called a *spectral operator* if it has a countably additive resolution of the identity defined on $\mathcal{B}_{\mathbb{C}}$.

Lemma A.3.
(i) *Any countably additive projection-valued spectral measure E on $\mathcal{B}_{\mathbb{C}}$ is countably additive in the strong operator topology and bounded.*
(ii) *Let $T \in \mathcal{B}(\mathcal{H})$ be a spectral operator, then $E(\sigma(T)) = I_{\mathcal{H}}$.*
(iii) *Every bounded spectral operator has a uniquely defined countably additive resolution of the identity defined on $\mathcal{B}_{\mathbb{C}}$.*

The uniquely defined spectral resolution for T will frequently be denoted by E_T in the following.

The important special case of bounded scalar spectral operators is introduced next.

Definition A.4.
(i) Let $S \in \mathcal{B}(\mathcal{H})$ be a spectral operator with spectral resolution E_S defined on $\mathcal{B}_{\mathbb{C}}$. Then S is said to be of *scalar type* (or a *scalar spectral operator*) if

$$S = \int_{\mathbb{C}} \lambda \, dE_S(\lambda). \tag{A.2}$$

(ii) $N \in \mathcal{B}(\mathcal{H})$ is called *quasi-nilpotent* if $\lim_{n\uparrow\infty} \|N^n\|_{\mathcal{B}(\mathcal{H})}^{1/n} = 0$.

Lemma A.5.
(i) *If E is a countably additive projection-valued spectral measure on $\mathcal{B}_{\mathbb{C}}$ which vanishes outside a compact subset of $\mathbb{C}$, then*

$$S = \int_{\operatorname{supp}(dE)} \lambda \, dE(\lambda) \tag{A.3}$$

is a bounded spectral operator of scalar type whose spectral resolution is E.
(ii) *$N \in \mathcal{B}(\mathcal{H})$ is quasi-nilpotent if and only if $\sigma(N) = \{0\}$.*

The following is a principal result on bounded spectral operators.

Theorem A.6 (The canonical reduction of bounded spectral operators)**.**
Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a spectral operator if and only if $T = S + N$, where $S \in \mathcal{B}(\mathcal{H})$ is a bounded spectral operator of scalar type and N is a quasi-nilpotent operator commuting with S. This decomposition is unique and

$$\sigma(T) = \sigma(S). \tag{A.4}$$

Moreover, T and S have the same resolution of the identity.

Next we turn to unbounded spectral operators.

Definition A.7. Let $T \in \mathcal{C}(\mathcal{H})$. Then T is called a *spectral operator* if there exists a regular, countably additive projection-valued spectral measure E (with respect to the strong operator topology) defined on $\mathcal{B}_{\mathbb{C}}$ such that the following conditions hold:

(α) $\mathrm{dom}(T) \supseteq E(\omega)\mathcal{H}$ for $\omega \in \mathcal{B}_{\mathbb{C}}$ bounded.

(β) $E(\omega)\mathrm{dom}(T) \subseteq \mathrm{dom}(T)$,
$TE(\omega)f = E(\omega)Tf, \quad f \in \mathrm{dom}(T),\ \omega \in \mathcal{B}_{\mathbb{C}}$.

(γ) Let $\omega \in \mathcal{B}_{\mathbb{C}}$. Then $T\big|_{E(\omega)\mathcal{H}}$ on $\mathrm{dom}\big(T\big|_{E(\omega)\mathcal{H}}\big) = \mathrm{dom}(T) \cap E(\omega)\mathcal{H}$ has spectrum contained in $\overline{\omega}$, $\sigma\big(T\big|_{E(\omega)\mathcal{H}}\big) \subseteq \overline{\omega}$.

E is called a *resolution of the identity* (or a *spectral resolution*) for T.

Lemma A.8.
(i) *Every spectral operator $T \in \mathcal{C}(\mathcal{H})$ has a uniquely defined, regular, countably additive resolution of the identity defined on $\mathcal{B}_{\mathbb{C}}$. (It will frequently be denoted by E_T in the following.)*
(ii) *Let $T \in \mathcal{C}(\mathcal{H})$ be a spectral operator with spectral resolution E_T and $\omega \in \mathcal{B}_{\mathbb{C}}$. Then, $T\big|_{E(\omega)\mathcal{H}}$ is a spectral operator with spectral resolution*

$$E_{T|_{E(\omega)\mathcal{H}}}(\sigma) = E_T(\sigma)\big|_{E(\omega)\mathcal{H}}, \quad \sigma \in \mathcal{B}_{\mathbb{C}}. \tag{A.5}$$

If ω is bounded, $T\big|_{E(\omega)\mathcal{H}}$ is a bounded spectral operator.
(iii) *Let $T \in \mathcal{C}(\mathcal{H})$ be a spectral operator, E_T its spectral resolution, and $\omega \in \mathcal{B}_{\mathbb{C}}$ open. Then,*

$$\sigma(T) \cap \omega \subseteq \sigma\big(T\big|_{E_T(\omega)}\big) \subseteq \sigma(T) \cap \overline{\omega}. \tag{A.6}$$

(iv) *If $P = P^2$, $P\mathrm{dom}(T) \subseteq \mathrm{dom}(T)$, $PTf = TPf$, $f \in \mathrm{dom}(T)$, then,*

$$\sigma(T) \supseteq \sigma\big(T\big|_{P\mathcal{H}}\big). \tag{A.7}$$

(v) *Let $T \in \mathcal{C}(\mathcal{H})$ be a spectral operator with spectral resolution E_T. Then,*

$$\sigma(T) = \bigcap_{\{\omega \in \mathcal{B}_{\mathbb{C}} \,|\, E_T(\omega) = I_{\mathcal{H}}\}} \overline{\omega}. \tag{A.8}$$

(vi) *Let $T \in \mathcal{C}(\mathcal{H})$ be a spectral operator with spectral resolution E_T satisfying $E_T(\sigma(T)) = I_{\mathcal{H}}$. If $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathbb{C}}$ is an increasing sequence of bounded Borel sets with $E_T\big(\bigcup_{n\in\mathbb{N}} \omega_n\big) = I_{\mathcal{H}}$, then*

$$\sigma(T) = \overline{\bigcup_{n\in\mathbb{N}} \sigma\big(T\big|_{E_T(\omega_n)\mathcal{H}}\big)}. \tag{A.9}$$

Definition A.9. Let $S \in \mathcal{C}(\mathcal{H})$. Suppose there exists a regular, countably additive projection-valued spectral measure E (with respect to the strong operator topology) defined on $\mathcal{B}_{\mathbb{C}}$ such that

$$\begin{aligned} &\mathrm{dom}(S) = \bigg\{ g \in \mathcal{H} \,\bigg|\, \mathrm{l.i.m.}_{n\uparrow\infty} \int_{\{\lambda\in\mathbb{C} \,|\, |\lambda|\leq n\}} \lambda\, d(E(\lambda)\, g) \text{ exists in } \mathcal{H} \bigg\}, \\ &Sf = \mathrm{l.i.m.}_{n\uparrow\infty} \int_{\{\lambda\in\mathbb{C} \,|\, |\lambda|\leq n\}} \lambda\, d(E(\lambda)\, f), \quad f \in \mathrm{dom}(S). \end{aligned} \tag{A.10}$$

Then S is called a *spectral operator of scalar type* and the projection-valued measure E is called the *resolution of the identity* for S.

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: fritz@math.missouri.edu
URL: http://www.math.missouri.edu/personnel/faculty/gesztesyf.html

Department of Mathematics, Ben Gurion University of the Negev, Beer–Sheva 84105, Israel
E-mail address: tkachenk@math.bgu.ac.il