TANGENTIAL HOMOCLINIC CROSSINGS OF INVARIANT MANIFOLDS, HAVING CO-DIMENSIONS HIGHER THAN ONE.

VICTORIA RAYSKIN

Abstract. Let $f : M \to M$ denote a diffeomorphism of a smooth manifold M with a hyperbolic fixed point and a degenerate homoclinic crossing at a point $B \in M$. Suppose that at the point B the order of contact of stable and unstable manifolds is finite. It is shown that, subject to certain conditions on the invariant manifolds and on the eigenvalues of the linear part of f, a transverse homoclinic crossing will arise arbitrarily close to B. This proves the existence of a horseshoe structure arbitrarily close to B as well as analytic non-integrability.

Present result extends our earlier result [16], restricted to the case when one of the invariant manifolds has codimension one. We also improve another result of [16], which is the key part of the proof: the Tangential Inclination lemma. The Tangential Inclination lemma, proved in this paper does not have a non-resonance assumption.

1. Introduction

The well-known Theorem of Birkhoff and Smale (see, for instance, [17]) is one of the basic results describing chaotic dynamics. It provides a method for rigorously concluding the existence of a horseshoe structure. On of the main assumptions of this theorem is the presence of a transverse homoclinic point.

Theorem 1.1 (Birkhoff-Smale). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with a hyperbolic fixed point p. If $q \neq p$ is a transverse homoclinic point, then there exist a hyperbolic invariant set Λ, a set of bi-infinite sequences Σ and a shift map σ such that for some m the following diagram commutes:*

$$\begin{array}{ccc} \Lambda & \stackrel{f^m}{\rightarrow} & \Lambda \\ h\downarrow & & \downarrow h \\ \Sigma & \stackrel{\sigma}{\rightarrow} & \Sigma \end{array}$$

Here $\Sigma = \{...A_{-2}A_{-1}.A_0A_1A_2...\}$, $A_i \in \{0, ..., N\}$, $\sigma\{...A_{-2}A_{-1}.A_0A_1A_2...\} = \{...A_{-2}A_{-1}A_0.A_1A_2...\}$ and h is a homeomorphism mapping Λ onto Σ.

Although, there is a variety of methods that help to conclude that the manifolds intersect at an isolated point, it is much harder to prove that the intersection is transverse. There were many attempts to remove this assumption.

The papers of K. Burns and H. Weiss [3], A. J. Homburg and H. Weiss [10], M. Hirsch [9], R. Churchill and D. Rod [4], Gavrilov and Shilnikov [5], [6], Gonchenko

---

1991 *Mathematics Subject Classification.* 37B10, 37C05, 37C15, 37D10.

Key words and phrases. Analytic non-integrability, Homoclinic tangency, invariant manifolds, λ-Lemma, order of contact, horseshoe structure.

and Shilnikov [7] address the question of existence of horseshoes arbitrarily close to a homoclinic point in *planar* dynamics.

The object of the papers of Hirsch [9] and Churchill and Rod [4] was to show that, in the planar case, the manifolds W_S and W_U cross transversely at some other point, arbitrarily close to the original crossing, if the original crossing is two-sided and of finite contact.

Churchill and Rod [4] imposed the additional assumptions that f is *analytic* and *area-preserving*. In that case, by Moser's Theorem [13], this diffeomorphism can be assumed to be in the Birkhoff normal form, and any branch of the stable (unstable) manifold can be re-parameterized so that $f^n(W_U) \cap W_S$ contains a positive orbit $q', fq', f^2q', ..., f^kq'$ that lies on a hyperbola. Then, direct calculations imply that q' is a transverse homoclinic point. But area preservation is a strong restriction. Furthermore, analyticity allows one to consider the case of a finite order of contact only.

Hirsch [9], replaced the above assumptions with the smooth linearizability assumption (near p). If we substitute L, the linear part of f, for f, the order of contact will not be changed, and we can calculate the "slope" between two manifolds at the point q', where $L^n(W_U)$ crosses W_S. If the crossing at q' is transverse, then $f^n(W_U) = \Phi L^n \Phi^{-1}(W_U)$ also crosses W_S non-degenerately, where Φ is the C^1 linearizing map.

Homburg and Weiss [10] considered a surface diffeomorphism and its homoclinic *one-sided* tangency of arbitrary high (possibly infinite) order of contact. They proved that some power of this diffeomorphism has the full shift on two symbols as a topological factor. Then, Katok's theorem [11], [12] implies that the map possesses a horseshoe arbitrarily close to homoclinic tangency.

Important cases of homoclinic tangencies were also described in works of Shilnikov, Gonchenko and Gavrilov (see [7], [5], [6]). They considered *one-sided* planar homoclinic tangencies.

Gavrilov and Shilnikov [5], [6] established a topological conjugacy (on a closed invariant set) between a surface diffeomorphism having a dissipative hyperbolic periodic point with certain types of quadratic (*one-sided*) tangencies and the full shift on two symbols.

Gonchenko and Shilnikov considered higher orders of contact. Homburg and Weiss write: "Gonchenko claims that a proof of this result for finite order tangencies appeared in his unpublished (Russian) thesis in 1984. The result for finite order tangencies was announced by Gonchenko and Shilnikov [7] in 1986, but we are unable to find any proof in the literature." We were unable to find such a proof in the existing literature either.

Homburg and Weiss, in their paper [10], state an Open Problem, asking whether their results could be extended to dimensions higher than two. In our earlier paper [16] we give a partial answer to this question, for the case, when the codimension of one of the invariant manifolds is one. We also have some additional assumptions such as graph portion, non-Newhose type of homoclinic point and linearizability. In the same paper, as one of the main proof's ingredients, we also prove an analog of the Inclination lemma of Palis [15] for the case of tangential homoclinic point. In this lemma we make an unfortunate resonance assumption on the eigenvalues of the linear part of the diffeomorphism.

In the present paper we remove this assumption on the eigenvalues in the Tangential Inclination lemma. In the main theorem, we consider a diffeomorphism f of a smooth manifold M (of an arbitrary dimension) with a hyperbolic fixed point. We no longer restrict our proof to the case when one of the invariant manifolds of f is a curve. We assume that the order of contact of invariant manifolds is finite, and that the invariant manifolds locally have a special geometric form (one manifold is covering and another one has a graph portion). We prove (Theorem 3.1) that, subject to certain conditions on the eigenvalues of f (which is not necessarily a non-resonance assumption), a transverse homoclinic crossing will arise arbitrarily close to the tangential crossing. This implies the existence of a horseshoe arbitrarily close to the tangential crossing.

One of the important applications of our result is the question of analytic integrability. The proof of our result is similar to our earlier proof [16], but more technical. We consider (Section 3) tangent subspaces of stable and unstable manifolds at a certain sequence of their intersection points. We show that, at some intersection points, a pair of the tangent subspaces spans the entire $\mathbb{R}^n$. In Section 2 we present basic definitions and lemmas that are used in Section 3. In particular, we prove tangential λ-Lemma 2.12, essential for the proof of the main result. Also, in Section 2, we present a new definition of the order of contact of two manifolds. This definition does not require high differentiability and plays a key role in our approach.

2. Definitions and Preliminaries

In this section we give basic definitions and state several preliminary lemmas to be used in the proof of the main result. The references for this Section are V. I. Arnold, S. M. Guseĭn-Zade [1] and A. N. Varchenko, K. Burns and H. Weiss [3], M. Hirsch [9], A. J. Homburg and H. Weiss [10], S. Newhouseand J. Palis [14], J. Palis [15], V. Rayskin [16], R. Uribe-Vargas [18].

Throughout this section, we will be considering two submanifolds in a certain ambient smooth manifold (of arbitrary dimension). Suppose two such submanifolds meet at an isolated point A. We will discuss the contact properties of these submanifolds at point A. Since these properties are local, we may assume without loss of generality that the ambient manifold is just $\mathbb{R}^n$.

First, assume that each manifold is a curve. We repeat our definition, which was originally introduces in [16], and which is different form the classical definition defined and studied in works of Arnold, Guseĭn-Zade [1] and A. N. Varchenko, Burns and Weiss [3], Hirsch [9], Homburg and Weiss [10], Uribe-Vargas [18]. Their definitions of the l-th order of contact require high differentiability; i.e. l is the maximal integer, such that the first $l-1$ derivatives of the two curves coincide at the point of their contact. Our definition below does not require high differentiability and thus, allows us to apply C^1-linearization in our main Theorem 3.1.

Definition 2.1. Let γ_i $(i = 1, 2)$ denote two immersed C^1-curves in $\mathbb{R}^n$. Suppose the two curves meet at an isolated point A. Then we will say that a number l is the *order of contact* of the curve γ_1 with the curve γ_2 at the point A if there exist positive real numbers m and M such that for all points x on γ_1 sufficiently close to A

$$m \leq \frac{d(x, \gamma_2)}{\|x - A\|^l} \leq M\ .$$

Proposition 2.2. *If the curves are C^1-smooth, then the definition of the order of contact is symmetric with respect to γ_1 and γ_2.*

Proof. Since the curves are C^1-smooth, we can assume that γ_2 is the X-axis (in some coordinate system), and γ_1 is a graph of a C^1 function f in this system. Let the order of contact of γ_1 with γ_2 be l. By Definition 2.1 it means that

$$m \leq \frac{\|f(x)\|}{(x^2 + \|f(x)\|^2)^{l/2}} \leq M \ ,$$

so that there exists a constant C such that for all sufficiently small x

$$m \leq \frac{\|f(x)\|}{|x|^l} \leq C \ ,$$

which means that γ_1 is sandwiched between two order l paraboloids $\mathcal{P}_m$ and $\mathcal{P}_C$, respectively.

Let us now look at the order of contact of γ_2 with γ_1. By the above, for any point $x \in X$

$$d(x, \mathcal{P}_m) \leq d(x, \gamma_1) \leq d(x, \mathcal{P}_C) \ .$$

Since $\mathcal{P}_m$ and $\mathcal{P}_C$ are order l paraboloids, the distances to them are uniformly equivalent to $|x|^l$, whence the claim □

Remark 2.3. Our notion of the order of contact is more general than the classical one. Obviously, if the order of contact of two curves is l in the classical sense, then it is also l in our sense. The converse, however, need not be true.

Remark 2.4. Without the C^1 assumption Proposition 2.2 may fail. For instance, let us consider the graphs of the functions $\gamma_1(x) = 0$ and $\gamma_2(x) = x^2 + (1+\sin(1/x))x$ on the plane. γ_2 oscillates between x^2 and x with the distance between consecutive crests of order x^2. Then, the order of contact of γ_1 with γ_2 in the sense of our Definition 2.1 is 2, whereas the order of contact of γ_2 with γ_1 does not exists.

Naturally, the l-th order of contact (in the sense of [1], [3], [9], [10], [18]) between two C^l-curves is preserved under a C^1-diffeomorphism. Here, we state the Lemma 2.5, (proved in [16]) which shows that the order of contact in the sense of our Definition 2.1 is preserved by C^1-diffeomorphisms. In particular, our order of contact from Definition 2.1 does not depend on the choice of a coordinate chart in the ambient manifold, so that it is well-defined in the general manifold setup as well.

Lemma 2.5. *Given two C^1-curves in $\mathbb{R}^n$ intersecting at an isolated point, any diffeomorphism of a neighborhood of this point preserves the order of contact of these curves.*

Let us now pass to a discussion of the order of contact for higher dimensional intersecting manifolds.

Definition 2.6. Let S_1 and S_2 denote two immersed C^1-manifolds in $\mathbb{R}^n$. Suppose the two manifolds meet at an isolated point A. Their *order of contact* l at A is the supremum of the orders of contact of curves $\gamma_1 \subset S_1$ and $\gamma_2 \subset S_2$ passing through point A.

As it follows from Lemma 2.5, the order of contact is preserved under C^1-diffeomorphisms.

The well known λ-Lemma of Palis [15] gives an important description of chaotic dynamics. The basic assumption of this theorem is the presence of a transverse homoclinic point, which means that the manifolds have contact of order 1.

Theorem 2.7 (λ-Lemma, Palis)**.** *Let f be a diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at 0 and m- and p-dimensional stable and unstable manifolds W_S and W_U ($m+p=n$). Let D be a p-disk in W_U, and w be another p-disk in W_U meeting W_S at some point A* transversely. *Then $\bigcup_{n\geq 0} f^n(w)$ contains p-disks arbitrarily C^1-close to D.*

Later in this section we will prove an analog of this lemma (Lemma 2.12) for non-transversal homoclinic intersections.

For the estimates in the proofs of the Singular λ-Lemma (i.e., in the case when the order of contact is greater than 1) and the Main Theorem we need the following definitions. The first one, the definition of a graph portion, deals with the shape of a global invariant manifold in the vicinity of a homoclinic point. This homoclinic point must be chosen close to a hyperbolic reference point. Then, the local properties of the invariant manifold are checked.

Definition 2.8. Let f be a diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at the origin and $U \in \mathbb{R}^n$ be some small neighborhood of 0. Denote by W_S (resp., W_U) the associated stable (resp., unstable) manifold, and by m (resp., p) its dimension ($m+p=n$). Let A be a homoclinic point of W_S and W_U. Denote by $\mathcal{V}$ a small p-neighborhood in W_U around the origin. Define a local coordinate system E_1 at 0, which spans $\mathcal{V}$. Similarly, define a local coordinate system E_2 which spans W_S near 0. Let $E = E_1 + E_2$, $E \subset U$. A p-neighborhood $\Lambda \subset W_U$ is a *graph portion* in U, associated with the homoclinic point A, if for some $n > 0$ $f^n(A) \in \Lambda$, $\Lambda \subset (U \cap W_U)$, and Λ is a graph in E-coordinates of some C^1-function defined on $\mathcal{V}$.

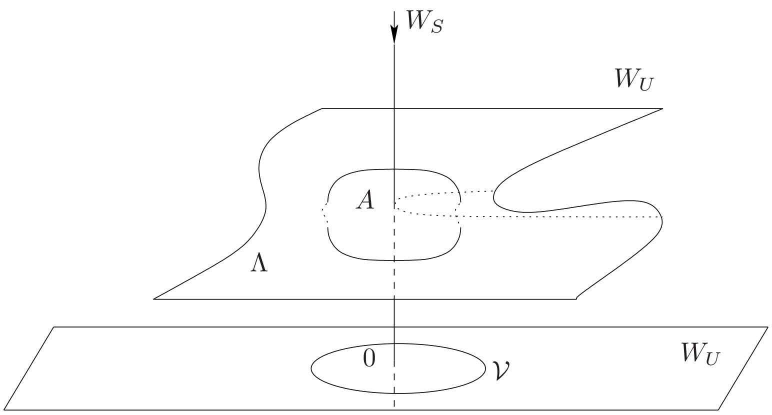

FIGURE 1. In this figure, Λ is not a graph portion of the manifold W_U, because, for any n, an iteration of Λ with diffeomorphism f^n is never a graph in the given coordinates.

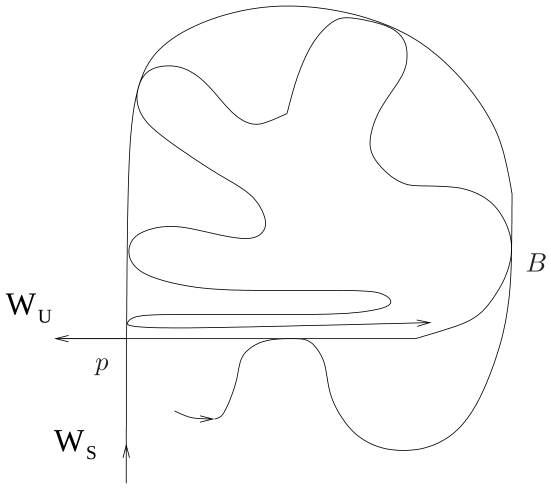

FIGURE 2. Newhouse example. The invariant manifolds do not have a graph portion. There is no open neighborhood in the unstable (stable, if we replace f with f^{-1}) branch, which would represent Λ.

In order to simplify the proofs, we shell consider the class of diffeomorphisms that can be C^1-conjugated to the diffeomorphisms, containing only mixed second order terms and their linear parts. In our context "mixed" means the following: the second order term consists of the product of expanding and contracting coordinates. The next lemma (Lemma 2.9) allows us to ascertain belonging to this class.

Lemma 2.9. *A C^∞ diffeomorphism f with a hyperbolic fixed point 0 can be C^1-conjugated to the diffeomorphism consisting of the sum of the linear part and mixed second order terms.*

Remark 2.10. The C^∞ assumption in the above Lemma 2.9 can be replaced with a C^k assumption, where k depends on the spectrum of the linear part, like in Bronstein and Kopanskii [2] (Theorem 11.9). As this dependence is a bit complicated we assume C^∞.

Remark 2.11. This is the only place where we use the fact that the diffeomorphism is C^∞. All further proofs are simplified by the assumption that higher order terms and non-mixed second order terms can be dropped. The assumption that the diffeomorphism $f \in C^\infty$ is of the technical nature.

Proof. Let $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ $(p + m = n)$ and $f(x, y) : \mathbb{R}^n \to \mathbb{R}^n$ have the linear part

$$((\beta x)_1, \ldots, (\beta x)_p, (\alpha y)_1, \ldots, (\alpha y)_m),$$

with $\|\beta^{-1}\|$, $\|\alpha\| < \lambda < 1$.

We will establish conjugation between f and the following normal form:

$$\beta x + \left(\sum_{\substack{i=1,\ldots,p;\\ j=1,\ldots,m}} a^1_{ij} x_i y_j, \ldots, \sum_{\substack{i=1,\ldots,p;\\ j=1,\ldots,m}} a^p_{ij} x_i y_j \right),$$

$$\alpha y + \left(\sum_{\substack{i=1,\ldots,p;\\ j=1,\ldots,m}} b^1_{ij} x_i y_j, \ldots, \sum_{\substack{i=1,\ldots,p;\\ j=1,\ldots,m}} b^m_{ij} x_i y_j \right).$$

Conjugation of f with this quadratic polynomial can be constructed C^1-smooth, because any term of degree higher than 2 in the polynomial expansion, of a C^∞ diffeomorphism with a hyperbolic fixed point satisfies the $S(1)$ condition defined by Bronstein and Kopanskii [2] (Definition 7.4, page 110).

Suppose that the polynomial expansion of f contains a term $cx^k y^l$ and at least one of the two inequalities holds: $|k| > 1$, $|l| > 1$. For definiteness let us assume that $|k| > 1$ (where $|k|$ denotes the length of the multiindex k). Also, we can assume that the eigenvalues are ordered: $\beta_1 \le \cdots \le \beta_p$. Let $k = (k_1, \ldots, k_p)$ and $j = \max\{i : k_i \neq 0\}$. Then,

$$\beta_1^{k_1} \cdots \beta_j^{k_j} > \beta_j,$$

i.e. $cx^k y^l$ satisfies the $S(1)$ condition.

Remark 7.6 in Bronstein and Kopanskii [2] (page 111) asserts that $S(1) \implies \beta(1)$. Then, Samovol's theorem ([2], Theorem 10.1, page 179) implies C^1-conjugation of f with the quadratic polynomial.

□

In [16] we proved Singular λ-Lemma (an analog of the λ-Lemma of Palis for the case of tangential homoclinic crossing). In that lemma, there is an assumption that the diffeomorphism is in the special normal form. A diffeomorphism can be conjugated to this norm with a C^1 transformation, if there is a non-resonance assumption in all contracting (or expanding) directions. Below we prove a stronger result. This new Singular λ-Lemma does not have the assumptions on resonances.

We will now prove this improved version of the Singular λ-Lemma in $\mathbb{R}^n$.

Lemma 2.12 (Singular λ-Lemma)**.** *Let f be a C^∞ diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at 0 and with m- and p-dimensional stable and unstable manifolds W_S and W_U $(m + p = n)$, respectively. Assume that A is an isolated homoclinic point, and that the order of contact at A, l, is finite. Also assume that there exists $\Lambda \subset W_U$, a graph portion in a small neighborhood of 0, associated with the homoclinic point A (cf. Definition 2.8).*

Then, for any $\rho > 0$, for an arbitrarily small ϵ-neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of the origin and for the graph portion Λ, the set $(\bigcup_{n\ge 0} f^n(\Lambda)) \setminus \mathcal{U}$ contains disks ρ-C^1 close to $\mathcal{V} \setminus \mathcal{U}$ (here $\mathcal{V}$ is a small neighborhood of 0 in the local unstable manifold).

Proof. of Lemma 2.12 Since Λ is a graph portion that is C^1-isolated from W_S at point A, we can assume that locally Λ is the graph of a function

$$\Lambda(x) = A + r(x) : \mathbb{R}^p \to \mathbb{R}^n, \quad r(0) = 0,$$

and for any sufficiently small $\sigma > 0$

$$|r(x)| \le \text{const} \cdot |x|^l \quad \text{and} \quad \left| \frac{\partial}{\partial x_i} r(x) \right| \le \text{const} \cdot |x_i|^{l-1}$$

for all $|x| < \sigma$, $i = 1, \ldots, p$. Let $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ $(p + m = n)$ and $f(x, y) : \mathbb{R}^n \to \mathbb{R}^n$ have the linear part

$$((\beta x)_1, \ldots, (\beta x)_p, (\alpha y)_1, \ldots, (\alpha y)_m).$$

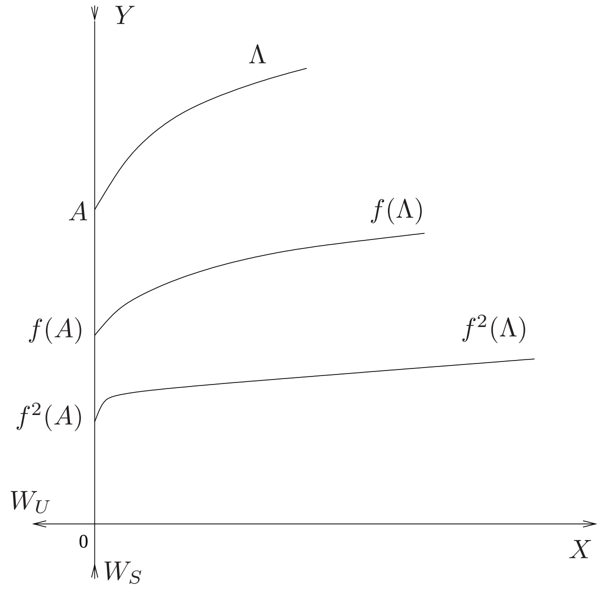

FIGURE 3. Iterations of the graph portion Λ with the diffeomorphism f.

Assume that $\|\beta^{-1}\|$, $\|\alpha\| < \lambda < 1$. Choose an arbitrarily small Δ. We know that locally f can be written in the form $f(x,y) = (S_1(x,y), S_2(x,y))$, where $S_2(x,y)$ is linear, $S_2(x,y) = \alpha y$. Since f is C^2, the existence of invariant manifolds implies that, in appropriate coordinates and in a sufficiently small neighborhood of the origin, f can be written as

$$S_1(x,y) = \beta x + \left(\sum_{i=1,\dots,p} x_i U_i^1(x,y), \dots, \sum_{i=1,\dots,p} x_i U_i^p(x,y) \right),$$

$$S_2(x,y) = \alpha y + \left(\sum_{i=1,\dots,p} x_i V_i^1(x,y), \dots, \sum_{i=1,\dots,p} x_i V_i^p(x,y) \right),$$

with $U(0) = V(0) = 0$, $\|U\|_{C^0} \le \Delta$, $\|V\|_{C^0} \le \Delta$ (for an arbitrary small Δ), and $\|U\|_{C^1}$, $\|V\|_{C^1}$ bounded.

Consider $f(x, \Lambda(x)) = (T_1^\Lambda(x), T_2^\Lambda(x))$. We will work with $(x, T_2^\Lambda \circ (T_1^\Lambda)^{-1}(x))$ and deduce that $f^n(x, \Lambda(x))$ is C^1-small for n big enough, $|x| \in (\epsilon, \sigma)$, and $\sigma > 0$ sufficiently small. First we will show that, in C^1-topology, $(T_1^\Lambda)^{-1}$ is Δ-close to β^{-1}. For simplicity, we will denote T_1^Λ by T_1 and T_2^Λ by T_2. Then

$$T_1(x) = \beta x + \left(\sum_{i=1,\dots,p} x_i U_i^1(x, \Lambda(x)), \dots, \sum_{i=1,\dots,p} x_i U_i^p(x, \Lambda(x)) \right).$$

Through the proof of this Theorem, $O(1)$ will denote the set of functions

$$O(1) = \big\{ \gamma : \mathbb{R} \mapsto \mathbb{R}, \text{ such that there exists a positive constant } c \text{ with } |\gamma(\zeta)| \le c \text{ for all sufficiently small } \zeta \big\}$$

Claim 2.13.

$$\left\|T_1^{-n}\right\|_{C^1} \leq \left(\left\|\beta^{-1}\right\|_{C^1} + O(1)\cdot\Delta\right)^n$$

for $|x| < \sigma$ ($\sigma > 0$ sufficiently small).

Proof. of Claim 2.13 Fix some $t \in \{1, \ldots, p\}$. Recall that $\Lambda(x) = A + r(x)$.

$$\begin{aligned}\left|x_i\frac{\partial}{\partial x_t}\Lambda_j(x)\right| &\leq |x_i|\cdot\left|\frac{\partial}{\partial x_t}\Lambda_j(x)\right| \\ &\leq |x|\cdot O(1)|x|^{l-1} \\ &\leq O(1)|x|^l\end{aligned}$$

Also,

$$\begin{aligned}|x|\cdot\left|\frac{\partial}{\partial x_t}U_i^t(x,\Lambda(x))\right| &= |x|\cdot\left|\frac{\partial}{\partial x_t}U_i^t(x,y) + \sum_{k=1}^m \frac{\partial}{\partial y_k}U_i^t(x,y)\cdot\frac{\partial}{\partial x_t}\Lambda_k(x)\right| \\ &= O(1)|x|^\alpha,\end{aligned}$$

and

$$\left\|U(x,\Lambda(x))\right\|_{C^0} \leq \Delta.$$

Therefore,

$$\left\|\left(\sum_{i=1,\ldots,p} x_iU_i^1(x,\Lambda(x)),\ldots,\sum_{i=1,\ldots,p} x_iU_i^p(x,\Lambda(x))\right)\right\|_{C^1} \leq \Delta\cdot O(1),$$

if σ is sufficiently small and $|x| < \sigma$. (An arbitrarily small Δ was chosen earlier). Then,

$$\left\|T_1^{-n}\right\|_{C^1} \leq \left(\left\|\beta^{-1}\right\|_{C^1} + O(1)\cdot\Delta\right)^n$$

□

Now, we continue the proof of Lemma 2.12 and show that $\|T_2^n \circ T_1^{-n}\|_{C^1}$ is ρ-small.

$$\begin{aligned}\left\|T_2\circ T_1^{-n}\right\|_{C^1} &= \left\|\alpha\cdot\Lambda(T_1^{-n})\right\|_{C^1} + \left\|T_1^{-n}\cdot V(T_1^{-n},\Lambda(T_1^{-n}))\right\|_{C^1} \\ &\leq \left(\|\alpha\|\cdot\|(T_1^{-n})^{l-1}\|_{C^0} + O(1)\cdot\Delta\right)\cdot\|T_1^{-n}\|_{C^1} \\ &\leq \|\alpha\|\cdot\|(T_1^{-n})^l\|_{C^1}\cdot\|x\|^{l-1} + O(1)\cdot\Delta\cdot\|T_1^{-n}\|_{C^1}\end{aligned}$$

If $\epsilon < \left\|x\right\| < \delta$ and n is big enough, then

$$\|T_2^n\circ T_1^{-n}\|_{C^1} \leq \left(\|\alpha\|\cdot\|(T_1^{-n})^l\|_{C^1}\cdot\|x\|^{l-1} + O(1)\cdot\Delta\cdot\|T_1^{-n}\|_{C^1}\right)^n$$

is small.

□

Remark 2.14. Clearly, if W_U does not contain any graph portion Λ, the conclusion of the λ-Lemma 2.12 is not true. Figure 4 illustrates this situation.

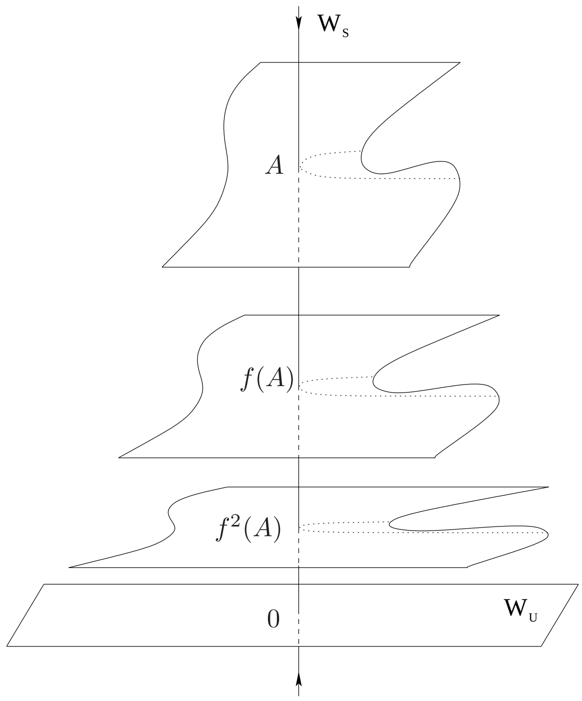

FIGURE 4. The iterated manifold is not a graph portion. The iterations of this manifold do not come C^1-close to the neighborhood of 0 in W_U.

3. The Main Theorem

In this section, we extend the results of Hirsch [9], Homburg and Weiss [10], and Gavrilov and Shilnikov [5], [6] to certain types of non-planar homoclinic tangencies with finite orders of contact. Our goal is to establish a transversal crossing arbitrarily close to the non-transversal one. Then, it would follow that there exist horseshoes, located arbitrarily close to the degenerate crossing.

Theorem 3.1. *Let M be a smooth manifold, and let a diffeomorphism $f : M \to M$ satisfies the assumptions of the Singular λ-Lemma 2.12. Assume that some part of the unstable manifold of f is a covering manifold (see definition 3.2 below). Also assume that one of the following three conditions holds:*

Case 1: *The diffeomorphism f is non-resonant.*

Case 2: *Suppose that the linear part of f has contracting eigenvalues $\alpha_1, ..., \alpha_p$ and expanding eigenvalues $\beta_1, ..., \beta_m$. Then, the following inequality holds: $\beta_j \cdot \alpha_k^{\frac{l-1}{l}} > 1$ for all pairs of j, k ($j = 1, ..., p; k = 1, ..., m$).*

Case 3: *For the same set of contracting and expanding eigenvalues, $\beta_j^{-\frac{l-1}{l}} \cdot \alpha_k^{-1} > 1$*

Then the invariant manifolds have a point of transverse intersection arbitrarily close to the point B. Therefore, f has a horseshoe (arbitrarily close to B) according to the Birkhoff-Smale Theorem.

Proof. of Theorem 3.1 Consider a C^1 transformation $\phi : N' \mapsto N$ from a neighborhood $N' \subset M$ of p onto a neighborhood $N \subset \mathbb{R}^n$ of the origin, which linearizes f in the subspace, where the linear part of is expanding. Such linearization is possible, according to the Hartman's theorem for contracting hyperbolic time-one map (see [8]). We choose the linearization in such a way that f has a normal Jordan form in the new coordinates. We may assume that ϕ takes the local stable manifold at p to a neighborhood in the Y-axis, and the local unstable manifold to a neighborhood in the X-hyperplane.

There are images of $\phi(B)$ in N, under forward and backward iterates of f, on both the local stable and local unstable manifolds of 0. There is a compact curve, $\Lambda \subset N$ (see Figure 5), with the following properties:

- Γ is an image under f of the stable manifold;
- Γ has topological intersection with X-hyperplane and meets with Λ (without loss of generality – in the first quadrant);
- A is arbitrarily close to the origin.

This C^1-manifold Γ, immersed in $\mathbb{R}^n$, which has homoclinic intersection with the X-hyperplane at the point $A = (A_1, ..., A_p) \neq 0$, locally can be parameterized as:

$$\Gamma = \begin{cases} u^1 & = & a_1(\tau_1, ..., \tau_m) + A_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^p & = & a_p(\tau_1, ..., \tau_m) + A_p \\ v^1 & = & \psi_1(\tau_1, ..., \tau_m) \\ v^2 & = & \psi_2(\tau_1, ..., \tau_m) \\ . & . & . \\ . & . & . \\ . & . & . \\ v^m & = & \psi_m(\tau_1, ..., \tau_m), \end{cases}$$

for non-negative parameters τ_k, for some constants A_j, and C^1 functions a_j, such that $a_j(0) = 0$. Also, ψ_i $(i = 1, ..., m)$ here are C^1 functions, $\psi(0) = 0$.

Then, one can can introduce the following definition:

Definition 3.2. *We will say that the manifold Γ is covering, if there exists a parameterization with the properties, described above, such that the determinant $|D\psi| \neq 0$ for all sufficiently small $\tau \neq 0$.*

Since Λ is a graph portion, it can be parameterized as:

$$\Lambda = \begin{cases} x^1 & = & t_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ x^p & = & t_p \\ y^1 & = & \phi_1(t_1, ..., t_p) + B_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ y^m & = & \phi_m(t_1, ..., t_p) + B_m, \end{cases}$$

for some constant $B \neq 0$ and C^1 functions ϕ_i, $\phi_i(0) = 0$ $(i = 1, ..., m)$.

Now it suffices to refer to the following lemma, in order to have the proof of this theorem completed.

Lemma 3.3. *Suppose*

$$f : (x, y) \to (\beta_1 x^1 + \sigma_2 x^2, \beta_2 x^2 + \sigma_3 x^3, ..., \beta_{p-1} x^{p-1} + \sigma_p x^p, \beta_p x^p, f_1(y), ..., f_m(y)),$$

$$1 < \beta_j \quad (j = 1, ..., p)$$

and σ_k are 0 or 1 – according to the Jordan normal form of f.

Then there exists a natural number k_* such that $f^k\Lambda$ crosses Γ transversely if $k > k_*$.

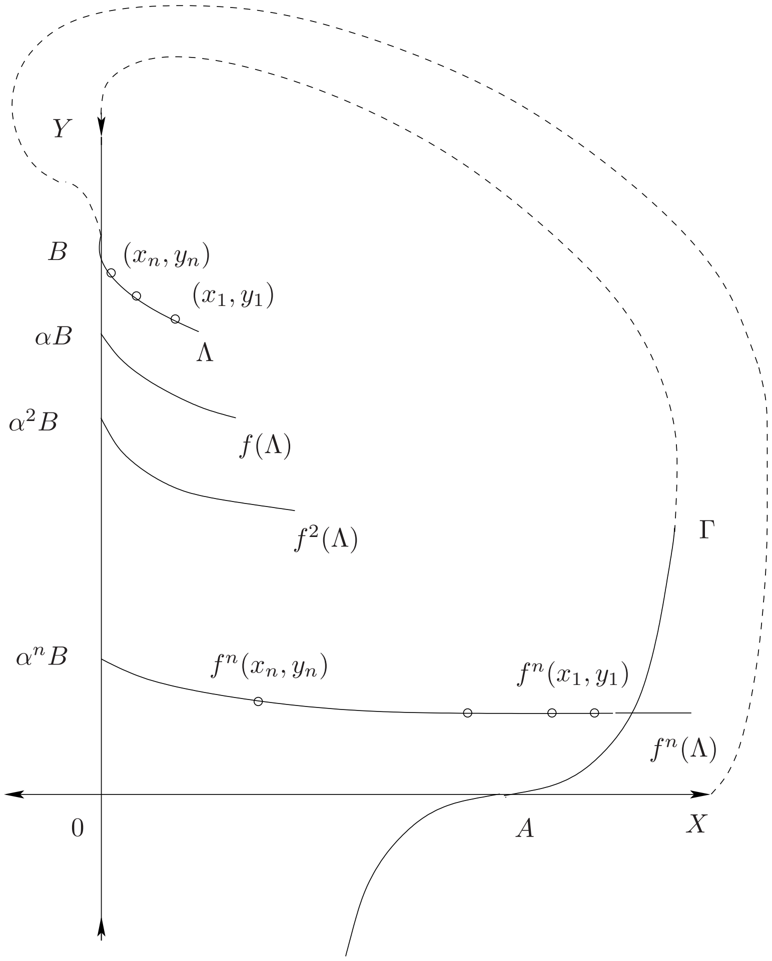

Figure 5. Degenerate homoclinic crossing near the fixed point 0. Illustration of the proof.

Proof. of Lemma 3.3 To simplify notations, first assume that f has a diagonal Jordan normal form, i.e., all σ_k are 0. Consider the iterations of Λ by f:

$$f^{k_i}\Lambda = \begin{cases} u^1 & = & \beta_1^{k_i} t_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^p & = & \beta_p^{k_i} t_p \\ v^1 & = & f_1^{k_i}\left(\phi_1(t_1,...,t_p) + B_1\right) \\ . & . & . \\ . & . & . \\ . & . & . \\ v^m & = & f_m^{k_i}\left(\phi_m(t_1,...,t_p) + B_m\right) \end{cases}$$

Since Λ is a graph portion, it has topological intersection with Y-hyperplane. Γ also has topological intersection with X-hyperplane. We can find sequences $(x_i, y_i) \in \Lambda$ and $k_i \to \infty$ such that $(x_i, y_i) \to (0, B)$ and $f^{k_i}(x_i, y_i) \to (A, 0)$. This implies that $f^{k_i}\Lambda$ and Γ will cross at some point other than $(0, B)$ and $(A, 0)$ (see Figure 5). Thus, we only need to show that this intersection is transversal.

Consider the matrix, constructed in the following way:

- The first p rows represent the linear space of the tangent hyperplane to the p-manifold $f^{k_i}\Lambda$.
- The next m rows represents the linear space of the tangent hyperplane to the m-manifold Γ.

In order to prove that $f^{k_i}\Lambda$ crosses Γ transversely at some points of the form $(u^1(t_{1,k_i},...,t_{p,k_i}),...,u^p(t_{1,k_i},...,t_{p,k_i}),v^1(t_{1,k_i},...,t_{p,k_i}),...,v^m(t_{1,k_i},...,t_{p,k_i}))$ $= (u^1(\bar{t}_{k_i}),...,u^p(\bar{t}_{k_i}),v^1(\bar{t}_{k_i}),...,v^m(\bar{t}_{k_i}))$, we show that the determinant of this matrix is never 0 for k_i sufficiently large. To simplify the notations, we will write subscript i for k_i, and consider the following determinant D_i, which corresponds to the k_i-th iteration of f:

$$D_i = \begin{vmatrix} \beta_1^i & & 0 & \\ & \ddots & & D\Big(f^i\phi(t_i)\Big) \\ 0 & & \beta_p^i & \\ & D\Big(a(\tau_i)\Big) & & D\Big(\psi(\tau_i)\Big) \end{vmatrix}$$

We calculate the determinant D_i at the point of intersection of Γ and $f^i(\Lambda)$. For our future calculations, we note that the condition of the intersection can be written as

$$a_j(\tau_i) + A_j = \beta_j^i t_i, \qquad j = 1,...,p$$

and

$$\psi_r(\tau_i) = f_r^i(\phi(t_i) + B), \qquad r = 1,...,m,$$

and therefore

$$\frac{\partial \psi_s(\tau_i)}{\partial \tau^r} = O(1) \cdot (f_r^i(\phi_r(\hat{t^i}) + B_r))^{\frac{l-1}{l}}$$

Here $\hat{t^i}$ is a sequence of intersection points of $(\tau_i^r)^l$ (an approximation of $\psi_r(\tau_i)$) and $(f_r^i(\phi_r(\hat{t^i}) + B_r)$. Of cause, $\hat{t^i}$ is close to t^i.

Now we begin to estimate the determinant at the sequence t_i and τ_i of intersection points of the invariant manifolds.

$$|D_i| = \beta_1^i ... \beta_p^i \left| D\left(\psi(\tau_i)\right) + \sum_{j=1}^{p} \sum_{k,r=1}^{m} O(\beta_j^{-i}) \cdot \frac{\partial \left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right|,$$

where $D_r^k\left(\psi(\tau_i)\right)$ stays for the r,k-th minor of the bloc $\psi(\tau_i)$, and $D\left(\psi(\tau_i)\right)$ is non-zero. Then,

$$\begin{aligned}
|D_i| \geq \beta_1^i ... \beta_p^i & \left(\sum_s \prod_{r=1,\ldots,m} \left| \frac{\partial \left(\psi_{s_r}(\tau_i)\right)}{\partial \tau^r} \right| \right. \\
& \left. - \sum_{j=1}^{p} \sum_{k,r=1}^{m} \left| O(\beta_j^{-i}) \cdot \frac{\partial \left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right| \right) \\
&= \beta_1^i ... \beta_p^i \left(\sum_s \prod_{r=1}^{m} O(1) \left(\tau_i^r\right)^{l-1} \right. \\
& \left. - \sum_{j=1}^{p} \sum_{k,r=1}^{m} \left| O(\beta_j^{-i}) \cdot \frac{\partial \left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right| \right) \\
&= \beta_1^i ... \beta_p^i \left(\sum_s \prod_{r=1}^{m} O(1) \left((\tau_i^r)^l\right)^{\frac{l-1}{l}} \right. \\
& \left. - \sum_{j=1}^{p} \sum_{k,r=1}^{m} \left| O(\beta_j^{-i}) \cdot \frac{\partial \left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right| \right) \\
&= \beta_1^i ... \beta_p^i \left(\sum_s \prod_{r=1}^{m} O(1) \left(f_r^i\left(\phi_r(\hat{t}_i) + B_r\right)\right)^{\frac{l-1}{l}} \right. \\
& \left. - \sum_{j=1}^{p} \sum_{k,r=1}^{m} \left| O(\beta_j^{-i}) \cdot \frac{\partial \left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right| \right)
\end{aligned}$$

We can assume that for sufficiently high i-iterations $f_r^i\left(\phi(\tau_i)+B\right) > f_r^i(B)/2$. Then,

$$\begin{aligned}
|D_i| \geq \beta_1^i ... \beta_p^i & \left(\sum_s \prod_{r=1}^m O(1) \left(f_r^i\left(B_r\right)/2\right)^{\frac{l-1}{l}} \right. \\
& \left. - \sum_{j=1}^p \sum_{k,r=1}^m \left| O(\beta_j^{-i}) \cdot \frac{\partial\left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right| \right) \\
= \beta_1^i ... \beta_p^i & \left(\prod_{r=1}^m O(1) \left(\alpha_r^i \cdot B_r/2\right)^{\frac{l-1}{l}} \right. \\
& \left. - \sum_{j=1}^p \sum_{k,r=1}^m \left| O(\beta_j^{-i}) \cdot \frac{\partial\left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right| \right) \\
= \beta_1^i ... \beta_p^i & \left(\prod_{r=1}^m O(1) \alpha_r^{i\frac{l-1}{l}} \right. \\
& \left. - \sum_{j=1}^p \sum_{k,r=1}^m \left| O(\beta_j^{-i}) \cdot \frac{\partial\left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot D_r^k\left(\psi(\tau_i)\right) \right| \right)
\end{aligned}$$

In a similar way one can simplify the second term. After factoring out $\left(\alpha_1^i ... \alpha_m^i\right)^{\frac{l-1}{l}}$, we obtain

$$|D_i| \geq \left(\alpha_1^i ... \alpha_m^i\right)^{\frac{l-1}{l}} \beta_1^i ... \beta_p^i \left(O(1) - \sum_{j=1}^p \sum_{k,r=1}^m O(\beta_j^{-i}) \cdot \frac{\partial\left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^j} \cdot \alpha_r^{-i\frac{l-1}{l}} \right).$$

Case 1. Suppose f is non-resonant. Then, it is locally C^1-linearizable and

$$\frac{\partial\left(f_k^i\left(\phi_k(t_i)\right)\right)}{\partial t^k} = \alpha_k^i \cdot \frac{\partial\left(\phi_k(t_i)\right)}{\partial t^k},$$

where $\frac{\partial(\phi_k(t_i))}{\partial t^k} \to \frac{\partial(\phi_k(A))}{\partial t^k} < \infty$ as $i \to \infty$. Then,

$$|D_i| \geq \left(\alpha_1^i ... \alpha_m^i\right)^{\frac{l-1}{l}} \beta_1^i ... \beta_p^i \left(O(1) - \sum_{j=1}^p \sum_{k=1}^m O(\beta_j^{-i} \cdot \alpha_k^{i\frac{1}{l}}) \right) > 0$$

for big i.

Case 2. Suppose $\beta_j \cdot \alpha_k^{\frac{l-1}{l}} > 1$ for all pairs of j and k ($j = 1, ..., p$, $k = 1, ..., m$). By Lemma 2.12 for any $\rho > 0$ there exists i_* such that for any $i > i_*$

$$\frac{\partial}{\partial t^j} f_k^i\left(\phi_k(t_i)\right) < \rho,$$

for all $j = 1, ...p$. Then

$$|D_i| \geq \beta_1^i ... \beta_p^i \cdot \left(\alpha_1^i ... \alpha_m^i\right)^{\frac{l-1}{l}} \left[O(1) - O(1) \sum_{j=1}^p \sum_{k=1}^m \beta_j^{-i} \cdot \alpha_k^{-i\frac{l-1}{l}} \cdot \rho \right] > 0$$

Case 3: $\beta_j^{-\frac{l-1}{l}} \cdot \alpha_k^{-1} > 1$ for all pairs of j and k ($j = 1, ..., p$, $k = 1, ..., m$). This case is similar to the Case 2. We just consider f^{-1} instead of f. Thus, in each of the three case we proved that for some k_*, the iterations f^{k_i} $(k_i > k_*)$ applied to the manifold Λ near the origin will produce a transverse crossing with Γ.

$\square$

Applying Lemma 3.3, we complete the proof of Theorem 3.1. $\square$

Remark 3.4. It is clear from the proof of Theorem 3.1 that, if W_U does not contain any graph portion, then W_U possibly has no transversal crossings with Γ. Figure 6 shows that $\Gamma \cap f^n(\Lambda)$ can be a 1-dimensional manifold.

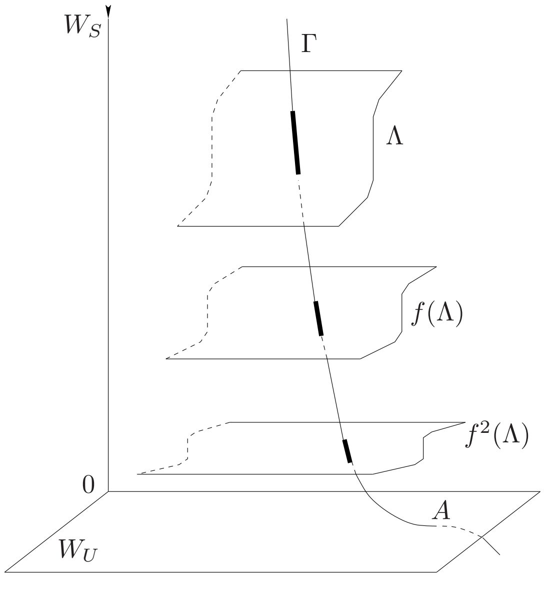

FIGURE 6. Λ is not a graph portion. Intersections of Γ with $f^n(\Lambda)$ are indicated by a thicker line. Γ and $f^n(\Lambda)$ have no transversal intersections.

References

[1] V. I. Arnold, S. M. Guseĭn-Zade, A. N. Varchenko, Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts, Birkhäuser, Boston, 1985.

[2] I. U. Bronstein, A. Ya. Kopanskii, Smooth Invariant Manifolds and Normal Forms, World Scientific, Singapore, 1994.

[3] K. Burns, H. Weiss, *A geometric criterion for positive topological entropy*, Comm. Math. Phys., 172 (1995), 95-118.

[4] R. Churchill and D. Rod, *Pathology in dynamical systems. III. Analytic Hamiltonians*, J. Diff. Equations, 37 (1980), 23-38.

[5] N. Gavrilov, L. Silnikov *On 3-dimensional dynamical systems close to systems with a structurally unstable homoclinic curve. I*, Math. USSR Sb., 88 (1972), 467-485.

[6] N. Gavrilov, L. Silnikov *On 3-dimensional dynamical systems close to systems with a structurally unstable homoclinic curve*, Math. USSR Sb., 90 (1973), 139-156.

[7] S. Gonchenko, L. Shilnikov *Dynamical systems with structurally unstable homoclinic curves*, Soviet Math. Dokl., 33 (1986), 234-238.

[8] P. Hartman, *On local homeomorphisms of Euclidean spaces*, Bol. Soc. Mat. Mexicana (2), 5 (1960), 220-241.

[9] M. Hirsch, *Degenerate homoclinic crossings in surface diffeomorphisms*, preprint (1993).

[10] A. J. Homburg, H. Weiss, *A geometric criterion for positive topological entropy. II: Homoclinic tangencies*, Comm. Math. Phys., 208 (1999), 267-273.

[11] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. IHES, 51 (1980), 137-173.

[12] A. Katok, *Nonuniform hyperbolicity and structure of smooth dynamical systems*, Proc. International Congress of Mathematicians, Warszawa, 2 (1983), 1245-1254.

[13] J. Moser, *The analytic invariants of an area-preserving mapping near a hyperbolic fixed point*, Comm. Pure Appl. Math., 9 (1956), 673-692.

[14] S. Newhouse and J. Palis, *Cycles and bifurcation theory*, Asterisque, 31, Soc. Math. France, Paris, 43-140 (1976).

[15] J. Palis, *On Morse-Smale dynamical systems*, Topology, 8 (1969), 385-405.

[16] V. Rayskin, *Homoclinic tangencies in* $\mathbb{R}^n$, Discrete Contin. Dyn. Syst., 12 (2005), 465-480.

[17] S. Smale, *Diffeomorphisms with many periodic points*, in:Differential and Combinatorial topology (edited by S. S. Cairnes), Princeton University Press, 63-80, (1965).

[18] R. Uribe-Vargas, *Four-vertex theorems in higher dimensional spaces for a larger class of curves than the convex ones*, C.R. Acad. Sci. Paris, Ser. I, 330 (2000), 1085-1090.

Victoria Rayskin

Department of Mathematics, PO Box 750561, International University Bremen, 28725 Bremen, Germany

E-mail address: v.rayskin@iu-bremen.de