The boundary control approach to the Titchmarsh-Weyl $m-$function. I. The response operator and the $A-$amplitude

Sergei Avdonin, Victor Mikhaylov and Alexei Rybkin

Dedicated with great pleasure to B.S. Pavlov on the occasion of his 70th birthday

Abstract. We link the Boundary Control Theory and the Titchmarsh-Weyl Theory. This provides a natural interpretation of the $A-$amplitude due to Simon and yields a new efficient method to evaluate the Titchmarsh-Weyl $m-$function associated with the Schrödinger operator $H = -\partial_x^2 + q(x)$ on $L_2(0,\infty)$ with Dirichlet boundary condition at $x = 0$.

1. Introduction

Consider the Schrödinger operator

$$H = -\partial_x^2 + q(x) \tag{1.1}$$

on $L_2(\mathbb{R}_+), \mathbb{R}_+ := [0,\infty)$, with a real-valued locally integrable potential q. We assume that (1.1) is limit point case at ∞, that is, for each $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ the equation

$$-u'' + q(x)u = zu \tag{1.2}$$

has a unique, up to a multiplicative constant, solution u_+ which is in L_2 at ∞ :

$$\int_{\mathbb{R}_+} |u_+(x,z)|^2 \, dx < \infty, z \in \mathbb{C}_+. \tag{1.3}$$

Such solution u_+ is called a *Weyl solution* and its existence for a very broad class of real potentials q is the central point of the Titchmarsh-Weyl theory.

The (principal or Dirichlet) *Titchmarsh-Weyl m-function*, $m(z)$, is defined for $z \in \mathbb{C}_+$ as

$$m(z) := \frac{u_+'(0,z)}{u_+(0,z)}. \tag{1.4}$$

Function $m(z)$ is analytic in $\mathbb{C}_+$ and satisfies the Herglotz property:

$$m : \mathbb{C}_+ \to \mathbb{C}_+, \tag{1.5}$$

1991 *Mathematics Subject Classification.* 34B20, 34E05, 34L25, 34E40, 47B20, 81Q10.

Key words and phrases. Schrödinger operator, Boundary control, Titchmarsh-Weyl $m-$function.

so m satisfies a Herglotz representation theorem,

$$m(z) = c + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t), \tag{1.6}$$

where $c = \operatorname{Re} m(i)$ and μ is a positive measure subject to

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty, \tag{1.7}$$

$$d\mu(t) = \text{w-}\lim_{\varepsilon \to +0} \frac{1}{\pi} \operatorname{Im} m(t + i\varepsilon)\, dt. \tag{1.8}$$

It is a fundamental fact of the spectral theory of ordinary differential operators that the measure μ is the spectral measure of the Schrödinger operator (1.1) with a Dirichlet boundary condition at $x = 0$. Another fundamental fact is the *Borg-Marchenko uniqueness theorem* stating

$$m_1 = m_2 \Longrightarrow q_1 = q_2. \tag{1.9}$$

There is no explicit formula realizing (1.9) but there are some Gelfand-Levitan-Marchenko type procedures to recover the potential q by given $m-$function (see, e.g., Freiling-Yurko [**7**]).

The Titchmarsh-Weyl $m-$function is a central object of the spectral theory of linear ordinary differential operators but its actual computation is problematic. In fact, (1.6) is suitable if the spectral measure μ of (1.1) with Dirichlet boundary condition at 0 is available, which is not usually the case. Instead, (1.6) is used to find μ by (1.8) but not the other way around.

The definition (1.4) is not always practical either since finding $m(z)$ by (1.4) is essentially equivalent to solving[1]

$$\begin{cases} -u'' + q(x)\, u = zu \\ u(0) = 1, \int_0^\infty |u|^2 < \infty \end{cases} \tag{1.10}$$

for all $z \in \mathbb{C}_+$. The analysis of the asymptotic behavior of $m(z)$ for large $|z|$ has received enormous attention and the picture is now quite clear (see, e.g. [**6**], [**19**] and the literature cited therein). Loosely speaking,

$$m(z) = i\sqrt{z} + \frac{q(0)}{2i\sqrt{z}} + o\left(\frac{1}{\sqrt{z}}\right),\ z \to \infty,\ \varepsilon \le \arg z \le \pi - \varepsilon,\ \varepsilon > 0, \tag{1.11}$$

which means that the $m-$functions for all q coinciding on $[0, a]$ with arbitrarily small $a > 0$ have the same asymptotic behavior. Due to (1.9), it is therefore $m(z)$ for finite z that is of particular interest, which requires a very accurate control of the solution to (1.10) at $x \to \infty$. In other words, the main issue here is the asymptotic behavior of $u(x, z)$ as $x \to \infty$ for finite z. Typically, such asymptotics are derived by transforming (1.10) to a suitable linear Volterra type integral equation. This can efficiently be done when, e.g., q decays at ∞ fast enough ($q \in L_1(\mathbb{R}_+)$ is sufficient). Equation (1.10) can then be transformed to

$$y(x, z) = 1 + \int_x^\infty K(x, s, z)\, y(s, z)\, ds,\ y(x, z) := e^{-i\sqrt{z}x} u(x, z),$$

[1] In the sequel, unless it leads to a confusion, $\int_a^b f = \int_a^b f(x)\, dx$.

where

$$K(x,s,z) := \frac{e^{-2i\sqrt{z}(s-x)}-1}{2i\sqrt{z}}q(x),$$

which can be solved by iteration.

Another well-known transformation of (1.2) is the Green-Liouville transformation (see, e.g. [**21**])

$$y''+y+\left[\frac{1}{4}\frac{q''(x)}{\{z-q(x)\}^2}+\frac{5}{16}\frac{q'^2(x)}{\{z-q(x)\}^3}\right]y=0, \tag{1.12}$$

where $y(x,z)=\{z-q(x)\}^{1/4}u(x,z)$. Equation (1.12) is a crucial ingredient in the WKB-analysis and can be reduced to a linear Volterra integral equation for a wide range of potentials (even growing at infinity) but requires that q be twice differentiable. Even for smooth potentials like $q(x)=x^{-\alpha}\sin x^{\beta}, 0<\alpha\le\beta\le 1$, the transformation (1.12) is not of much help since q' and q'' unboundedly oscillate at ∞. Note, that, as it was shown by Buslaev-Matveev [**4**], the Green-Liouville transformation (1.12) works well for slowly decaying potentials subject to

$$\left|q^{(l)}(x)\right|\le Cx^{-\alpha-l},\ \alpha>0,\ l=0,1,2. \tag{1.13}$$

One of the authors [**18**] has recently put forward yet another transformation that allows one to obtain and analyze the asymptotics for the solution to (1.10) for general non-smooth potentials q with a very mild decay at ∞. Namely, if a potential q is such that the sequence $\left\{\int_j^{j+1}|q|\right\}_{j=1}^{\infty}$ is from $l_p, p=2^n$ with some $j\in\mathbb{N}$ then (1.10) can be transformed to

$$y(x,z)=1+\int_x^{\infty}K_n(x,s,z)\,y(s,z)\,ds, \tag{1.14}$$

where $y(x,\lambda):=\Theta_n^{-1}(x,\lambda)\,u(x,\lambda)$ and

$$\begin{aligned}\Theta_n(s,z)\ &:\ =\Theta_n(0,s,z),\Theta_n(x,s,z):=\exp\left\{i\sqrt{z}s+\int_x^{s+x}\sum_{m=1}^{n}q_m(t,z)\,dt\right\},\\ K_n(x,s,\lambda)\ &:\ =(q_n\Theta_n)^2(s,\lambda)\int_x^s\Theta_n^{-2}(t,\lambda)\,dt.\end{aligned}$$

The functions q_m are, in turn, defined by the following recursion formulas:

$$\begin{aligned}q_1(x,z)\ &:\ =-\int_0^{\infty}e^{2i\sqrt{z}s}q(s+x)\,ds,\\ q_{m+1}(x,z)\ &:\ =\int_0^{\infty}\Theta_m^2(x,s,z)\,q_m^2(s+x,z)\,ds,\ m\in\mathbb{N}.\end{aligned} \tag{1.15}$$

Formulas (1.15) can be viewed as "energy dependent" transformations of the original potential q improving its rate of decay at infinity. For $n\ge 2$ these transformations are highly nonlinear and were previously considered by many authors (see, e.g. [**9**], [**10**], [**12**]) in connection with a variety of improvements of asymptotics (1.11). The main feature of the transformations (1.14) – (1.15) is that they yield higher order WKB type asymptotics of the Weyl solution $u_+(x,z)$ as $x\to\infty$ at fixed finite z (see. [**18**]).

Our list of transformations of the original equation (1.2) is of course incomplete and given here to demonstrate how drastically computational complexity of solving (1.10) (and hence the $m-$function) tends to increase when one relaxes decay conditions. It should be particularly emphasized that in order to get $m(z)$ one has to solve the integral equations for each z separately. In addition, q's with no decay at ∞ should be considered on ad hog basis.

In the present note we put forward a different approach to evaluate the Titchmarsh-Weyl $m-$function which is based on the Boundary Control Theory. The main idea is that to study the (dynamic) Dirichlet-to-Neumann map $u(0,t) \mapsto u_x(0,t)$ for the wave equation associated with (1.2):

$$u_{tt} - u_{xx} + q(x)\,u = 0,\ x > 0,\ t > 0,$$

with zero initial conditions. The Dirichlet-to-Neumann map defined this way turns out to be the so-called response operator, an important object of the Boundary Control method in inverse problems [**3**], [**2**]. In the frequency domain the latter becomes the operator of multiplication by the Titchmarsh-Weyl m-function[2] associated with the operator $-\partial_x^2 + q(x)$ with Dirichlet boundary condition at $x = 0$ (Pavlov [**15**] has noticed that m-function can be interpreted as a one-dimensional (spectral) Dirichlet-to-Neumann map). This approach allows one to employ powerful techniques developed for the wave equation to the study of the Titchmarsh-Weyl $m-$function. In this paper we concentrate on the direct problem only. That is, given potential q, we evaluate the $m-$function in terms of the response operator (response function, to be precise) which is exactly Simon's representation of the $m-$function via his $A-$amplitude (see [**20**] and [**8**]). Our approach however provides a clear physical interpretation of the $A-$amplitude and gives a new procedure to compute it. The latter can potentially be used for numerical analysis of the $m-$function.

We emphasize that all the ingredients we use in the present paper are already known in different inverse problems communities ([**8**], [**13**], [**11**], [**5**], [**7**] to name just five) but it is the new way to combine them that makes our main contribution to this well developed area. However, we do not utilize here the full power of the Boundary Control approach, which is in inverse methods. We plan to address this important issue in our sequel on this topic.

The paper is organized as follows. In Section 2 we introduce the main ingredient of our approach, the response operator R, and give its connection with the Titchmarsh-Weyl $m-$function. We also show that its kernel is closely related to the $A-$amplitude.

In Section 3 we derive a linear Volterra type integral equation for a function $A(x,y)$ which diagonal value is the $A-$amplitude (Theorem 1).

Section 4 is devoted to the analysis of the integral equation for the kernel $A(x,y)$ producing an important bound for the $A-$amplitude (Theorem 2) which answers an open question by Gesztesy-Simon [**8**].

In short Section 5 we present our algorithm of practical evaluation of the $m-$function and make some concluding remarks.

[2] Usually referred to as the Dirichlet (or principal) Titchmarsh-Weyl $m-$function

2. The response operator and the A-amplitude

Let us associate with the Schrödinger equation (1.1) the axillary wave equation

$$\left\{\begin{array}{c} u_{tt}(x,t)-u_{xx}(x,t)+q(x)u(x,t)=0, \quad x>0,\ t>0, \\ u(x,0)=u_t(x,0)=0,\ u(0,t)=f(t), \end{array}\right. \tag{2.1}$$

where f is an arbitrary $L_2(\mathbb{R}_+)$ function referred to as a *boundary control.* It can be verified by a direct computation that the weak solution $u^f(x,t)$ to the initial-boundary value problem (2.1) admits the representation

$$u^f(x,t)=\left\{\begin{array}{c} f(t-x)+\int_x^t w(x,s)f(t-s)\,ds,\ x\le t, \\ 0, \quad x>t, \end{array}\right. \tag{2.2}$$

in terms of the solution $w(x,s)$ to the Goursat problem:

$$\left\{\begin{array}{c} w_{ss}(x,s)-w_{xx}(x,s)+q(x)w(x,s)=0, \quad 0<x<s, \\ w(x,0)=0, w(x,x)=-\frac{1}{2}\int_0^x q. \end{array}\right. \tag{2.3}$$

We introduce now the *response operator* R:

$$(Rf)(t)=u_x(0,t), \tag{2.4}$$

so it transforms $u(0,t)\mapsto u_x(0,t)$. By this reason it can also be called the (dynamic) Dirichlet-to-Neumann map. From (2.2) we easily get the representation

$$(Rf)(t)=-\frac{d}{dt}f(t)+\int_0^t r(t-s)f(s)\,ds \tag{2.5}$$

$$r(\cdot):=w_x(0,\cdot). \tag{2.6}$$

In other words, the response operator is the operator of differentiation plus the convolution. The kernel r of the convolution part of (2.5) is called the *response function* which plays an important role in the Boundary Control method.

In fact, besides the Boundary Control method some close analogs of the response function have independently been discovered in the half-line short-range scattering [**16**] and more recently in the connection with inverse spectral problem for the half-line Schrodinger operator [**20**], [**8**], [**17**].

We now demonstrate the connection between the response function $r(s)$ and the (Dirichlet) Titchmarsh-Weyl m-function. An interplay between spectral and time-domain data is widely used in inverse problems, see, e.g., [**11**] where the equivalence of several types of boundary inverse problems is discussed for smooth coefficients; notice, however, that we consider the case of not smooth but just L^1_{loc} potentials.

Let $f\in C_0^\infty(0,\infty)$ and

$$\widehat{f}(k):=\int_0^\infty f(t)\,e^{-kt}\,dt$$

be its Laplace transform. Function $\widehat{f}(k)$ is well defined for $k\in\mathbb{C}$ and, if $\operatorname{Re}k>0$,

$$|\widehat{f}(k)|\le C_\alpha(1+|k|)^{-\alpha} \tag{2.7}$$

for any $\alpha>0$. Going in (2.1) and (2.4) over to the Laplace transforms, one has

$$-\widehat{u}_{xx}(x,k)+q(x)\widehat{u}(x,k) = -k^2\widehat{u}(x,k), \tag{2.8}$$

$$\widehat{u}(0,k) = \widehat{f}(k), \tag{2.9}$$

and

$$\widehat{(Rf)}(k) = \widehat{u}_x(0,k)\,, \tag{2.10}$$

respectively.

Estimate (2.7) implies that $|\widehat{u}(x,k)|$ decreases rapidly when $|k| \to \infty$, $\operatorname{Re} k \geq \epsilon > 0$. The values of the function $\widehat{u}(0,k)$ and its first derivative at the origin, $\widehat{u}_x(0,k)$, are related through the Titchmarsh-Weyl m-function

$$\widehat{u}_x(0,k) = m(-k^2)\widehat{f}(k)\,. \tag{2.11}$$

Therefore,

$$\widehat{(Rf)}(k) = m(-k^2)\widehat{f}(k)\,, \tag{2.12}$$

and thus the spectral and dynamic Dirichlet-to-Neumann maps are in one-to-one correspondence.

Taking the Laplace transform of (2.5) we get

$$\widehat{(Rf)}(k) = -k\widehat{f}(k) + \widehat{r}(k)\widehat{f}(k). \tag{2.13}$$

In Section 4 we show that, under some mild conditions on the potential q, (2.12) and (2.13) imply

$$m(-k^2) = -k + \int_0^\infty e^{-k\alpha} r(\alpha)\, d\alpha\,, \tag{2.14}$$

where the integral is absolutely convergent in a proper domain of k.

Representation (2.14) is not new. In the form

$$m(-k^2) = -k - \int_0^\infty A(\alpha)e^{-2\alpha k}\, d\alpha \tag{2.15}$$

(with the absolutely convergent integral) it was proven for $q \in L^1(\mathbb{R}_+)$ and $q \in L^\infty(\mathbb{R}_+)$ by Gesztesy-Simon [**8**] who call the function A in (2.15) the $A-$*amplitude*. Clearly, one has

$$A(\alpha) = -2r(2\alpha). \tag{2.16}$$

Remark 1. *The fundamental role of representation* (2.15) *and the $A-$amplitude was emphasized in* [**20**] *and* [**8**]. *However no direct interpretation of* (2.15) *and A is given in* [**20**], [**8**]. *On the other hand,* (2.14) *says that the Titchmarsh-Weyl $m-$function is the Laplace transform of the kernel of the response operator R (see* (2.5)*). Or, equivalently, the matrix (one by one in our case) of the response operator R in the spectral representation of $H_0 = -\partial_x^2, u(0) = 0$, coincides with the Titchmarsh-Weyl $m-$function associated with $H = H_0 + q$. The response operator R, in turn, describes the reaction of the system. In particular, for the string the operator R connects the displacement and tension at the endpoint $x = 0$. For electric circuits it relates the current and voltage (see, e.g.,* [**15**], [**11**] *and references therein for additional information about the physical meaning of the Dirichlet-to-Neumann map). In the theory of linear dynamical systems the response operator is the input-output map and the Laplace transform of its kernel is the transfer function of a system.*

REMARK 2. *In fact,* (2.12) *can be regarded as a definition of the Titchmarsh-Weyl $m-$function which could be effortlessly extended to matrix valued and complex potentials since the Boundary Control method is readily available in these situations (see,* [**3, 2**]*). We hope to return to this important point elsewhere. Also, since the Dirichlet-to-Neumann map can be viewed as a $3D$ analog of the $m-$function,* (2.12) *could hopefully yield a canonical way to define (operator valued) $m-$functions for certain partial differential operators. It is worth mentioning that Amrein-Pearson* [**1**] *have recently generalized (using quite different methods) the theory of the Weyl-Titchmarsh $m-$function for second-order ordinary differential operators to partial differential operators of the form $-\Delta + q(x)$ acting in three space dimensions.*

Despite the clear physical interpretation of the response function r some formulas in Section 3 look slightly prettier in terms of the $A-$amplitude. Since our interest to the topic was originally influenced by [**20**], [**8**] we therefore are going to deal with A related to r by (2.16). In Section 4 we prove the absolute convergence of the integral in (2.15) (and, therefore, of the integral in (2.14)) for

$$q \in l^{\infty}\left(L^{1}\left(\mathbb{R}_{+}\right)\right) := \left\{ q : \int_{n}^{n+1} |q| \in l^{\infty} \right\}.$$

3. An integral equation for the A-amplitude

In this section we derive a linear Volterra type integral equation closely related to the $A-$amplitude.

THEOREM 1. *Let $q \in L^{1}_{\text{loc}}\left(\mathbb{R}_{+}\right)$. Then for a. e. $\alpha > 0$*

$$A(\alpha) = A(\alpha, \alpha), \tag{3.1}$$

where $A(x, y)$ is the solution to the integral equation

$$A(x, y) = q(x) - \int_{0}^{y} \left(\int_{v}^{x} A(u, v)\, du \right) q(x - v)\, dv\,;\ \ x, y > 0\,. \tag{3.2}$$

PROOF. We go through a chain of standard transformations of the Goursat problem (2.3). By setting $u = s + x, v = s - x$ and

$$V(u, v) = w\left(\frac{u - v}{2}, \frac{u + v}{2}\right), \tag{3.3}$$

equation (2.3) reduces to

$$\begin{cases} V_{uv} + 4q\left(\dfrac{u-v}{2}\right) V = 0 \\ V(u, u) = 0 \\ V(u, 0) = -\frac{1}{2}\int_{0}^{u/2} q \end{cases}$$

which can be easily transformed into

$$V(u, v) = -\frac{1}{2}\int_{v/2}^{u/2} q - \frac{1}{4}\int_{0}^{v} dv_1 \int_{v}^{u} du_1 q\left(\frac{u_1 - v_1}{2}\right) V(u_1, v_1)\,. \tag{3.4}$$

Doubling the variables in (3.4) yields

$$V(2u, 2v) = -\frac{1}{2}\int_{v}^{u} q - \int_{0}^{v} dv_1 \int_{v}^{u} du_1 q\left(u_1 - v_1\right) V\left(2u_1, 2v_1\right). \tag{3.5}$$

Introduce a new function

$$U(x,y) := \int_0^y dvq(x-v)\,V(2x,2v)\,. \tag{3.6}$$

It follows from (3.5) that $U(x,y)$ satisfies the integral equation

$$U(x,y) = -\frac{1}{2}\int_0^y dv\,q(x-v)\int_v^x du\,q(u) - \int_0^y dv\,q(x-v)\int_v^x du\,U(u,v)\,.$$

The function

$$A(x,y) = q(x) + 2U(x,y) \tag{3.7}$$

then obeys equation (3.2).

It is left to show (3.1). By (2.16) and (2.6)

$$A(\alpha) = -2r(2\alpha) = -2w_x(0,2\alpha)\,. \tag{3.8}$$

But it follows from (3.3) that

$$w_x(x,s) = (V_u - V_v)(s+x, s-x)$$

and hence

$$w_x(0,2\alpha) = (V_u - V_v)(2\alpha,2\alpha)\,. \tag{3.9}$$

Differentiating (3.5) with respect to u and v and setting $u=v=2\alpha$, we have

$$\begin{aligned} V_u(2\alpha,2\alpha) &= -\frac{1}{4}q(\alpha) - \frac{1}{4}\int_0^{2\alpha} dv_1 q\left(\alpha - \frac{v_1}{2}\right)V(2\alpha,v_1)\,,\\ V_v(2\alpha,2\alpha) &= \frac{1}{4}q(\alpha) + \frac{1}{4}\int_0^{2\alpha} dv_1 q\left(\alpha - \frac{v_1}{2}\right)V(2\alpha,v_1)\,. \end{aligned}$$

Inserting these formulas into (3.9) we get

$$w_x(0,2\alpha) = -\frac{1}{2}q(\alpha) - \frac{1}{2}\int_0^{2\alpha} dv_1 q\left(\alpha - \frac{v_1}{2}\right)V(2\alpha,v_1)\,. \tag{3.10}$$

Setting in (3.10) $v_1 = 2v$ and plugging it then in (3.8), yields

$$A(\alpha) = q(\alpha) + 2\int_0^{\alpha} dv\,q(\alpha - v)\,V(2\alpha,2v)\,. \tag{3.11}$$

It is left to notice that by (3.6) the right hand side of (3.11) is $q(\alpha) + 2U(2\alpha,2\alpha)$ which by (3.7) is equal to $A(\alpha,\alpha)$ and (3.1) is proven. □

The kernel $A(x,y)$ in Theorem 1 is not related to $A(\alpha,x)$ appearing in [**20**], [**8**] where $A(\alpha,x)$ is the $A-$amplitude corresponding to the $m-$function associated with the interval (x,∞).

4. Analysis of iterations

In this section we demonstrate that integral equation (3.2) is quite easy to analyze. We need the following technical

Lemma 1. *Let $f(x)$ be a non-negative function and*

$$\|f\| := \sup_{x\geq 0} \int_x^{x+1} f < \infty. \tag{4.1}$$

Then for any $a, b \geq 0$ and natural n

$$\int_0^a (x+b)^n f(x)\, dx \leq \frac{(a+b+1)^{n+1}}{n+1} \|f\|. \tag{4.2}$$

Proof. We may assume $\|f\| = 1$. Integrating the left hand side of (4.2) by parts yields

$$\begin{aligned} \int_0^a (x+b)^n f(x)\, dx &= -\int_0^a (x+b)^n\, d\left(\int_x^a f\right) \\ &= b^n \int_0^a f + \int_0^a \left(\int_x^a f\right) d(x+b)^n . \end{aligned} \tag{4.3}$$

Due to the trivial inequality

$$\int_\alpha^\beta f < \beta - \alpha + 1,$$

(4.3) can be estimated above as follows

$$\begin{aligned} \int_0^a (x+b)^n f(x)\, dx &< b^n(a+1) + \int_0^a (a-x+1)\, d(x+b)^n \\ &= b^n(a+1) + (a+b)^n + \frac{(a+b)^{n+1}}{n+1} - b^n(a+1) - \frac{b^{n+1}}{n+1} \\ &\leq (a+b)^n + \frac{(a+b)^{n+1}}{n+1} \\ &= \frac{1}{n+1}\left\{(a+b)^{n+1} + (n+1)(a+b)^n\right\} \\ &\leq \frac{(a+b+1)^{n+1}}{n+1}. \end{aligned}$$

At the last step we used the obvious inequality $(x \geq 0)$

$$(x+1)^{n+1} \geq x^{n+1} + (n+1)\, x^n.$$

$\square$

The following theorem is the main result of this section.

Theorem 2. *Let q be subject to*

$$\|q\| := \sup_{x\geq 0} \int_x^{x+1} |q| < \infty\,. \tag{4.4}$$

Then for $\alpha \geq 0$

$$|A(\alpha) - q(\alpha)| \leq \frac{1}{2}\left(\int_0^\alpha |q|\right)^2 \left\{\exp\left(2\sqrt{2}\sqrt{\|q\|\alpha}\right) + \frac{1}{\sqrt{2\pi}} \exp\left(2e\|q\|\alpha\right)\right\}. \tag{4.5}$$

Proof. Rewriting (3.2) as

$$A = q - KA,$$

where

$$(Kf)(x,y) := \int_0^y dv\, q(x-v) \int_v^x du\, f(u,v),$$

and formally solving it by iteration, we get

$$A(\alpha) = q(\alpha) + \sum_{n\geq 1} (-1)^n A_n(\alpha), \ A_n(\alpha) := (K^n q)(\alpha,\alpha),$$

and hence

$$|A(\alpha) - q(\alpha)| \leq \sum_{n\geq 1} |A_n(\alpha)| \leq \sum_{n\geq 1} I_n(\alpha) \ ,$$

where

$$I_n(\alpha) := (|K|^n |q|)(\alpha,\alpha) \tag{4.6}$$

with the agreement that $|K|$ is the integral operator K with $|q|$ in place of q. We now need a suitable estimate for $(|K|^n |q|)(x,y)$. For $n=1$,

$$\begin{aligned} (|K|\,|q|)(x,y) &= \int_0^y \left(\int_v^x |q(u)|\, du \right) |q(x-v)|\, dv \\ &\leq \int_{x-y}^x |q| \int_0^x |q| \leq \left(\int_0^x |q| \right)^2 =: Q^2(x). \end{aligned} \tag{4.7}$$

Similarly,

$$\begin{aligned} \left(|K|^2 |q|\right)(x,y) &= \int_0^y \left(\int_v^x (|K|\,|q|)(u,v)\, du \right) |q(x-v)|\, dv \\ &\leq \sup_{0\leq v\leq u\leq x} (|K|\,|q|)(u,v) \int_0^y \left(\int_v^x du \right) |q(x-v)|\, dv \\ &\leq Q^2(x) \int_0^y (x-v)\, |q(x-v)|\, dv \\ &\leq Q^2(x)\ x \int_0^y |q(x-v)|\, dv = Q^2(x)\ x \int_0^y |q(v+(x-y))|\, dv \\ &\leq Q^2(x)\ x\,(y+1)\ \|q\|. \end{aligned} \tag{4.8}$$

Here the supremum in (4.8) was estimated by (4.2). We are now able to make the induction assumption

$$(|K|^n |q|)(x,y) \leq Q^2(x) \frac{x^{n-1}}{(n-1)!} \frac{(y+n-1)^{n-1}}{(n-1)!} \|q\|^{n-1}. \tag{4.9}$$

Then

$$
\begin{aligned}
\left(|K|^{n+1}|q|\right)(x,y) &\leq \int_0^y \left(\int_v^x \left(|K|^n|q|\right)(u,v)\,du\right)|q(x-v)|\,dv \\
&\leq \int_0^y \left(\int_v^x Q^2(u)\frac{u^{n-1}}{(n-1)!}\frac{(v+n-1)^{n-1}}{(n-1)!}||q||^{n-1}du\right)|q(x-v)|\,dv \\
&\leq \sup_{0\leq u\leq x} Q^2(u)\left(\int_0^x \frac{u^{n-1}}{(n-1)!}du\right)\left(\int_0^y \frac{(v+n-1)^{n-1}}{(n-1)!}|q(x-v)|\,dv\right)||q||^{n-1} \\
&\leq Q^2(x)\frac{x^n}{n!}\frac{(y+n)^n}{n!}||q||^n.
\end{aligned}
$$

At the last step we estimated the second integral by Lemma 1. One now concludes that (4.9) holds for all natural n.

Combining (4.6) and (4.9), we get

$$
I_n(\alpha) = \left(|K|^n|q|\right)(\alpha,\alpha) \leq Q^2(\alpha)\frac{\alpha^{n-1}}{(n-1)!}\frac{(\alpha+n-1)^{n-1}}{(n-1)!}||q||^{n-1}
$$

and hence for (4) we have

$$
|A(\alpha)-q(\alpha)| \leq Q^2(\alpha)\sum_{n\geq 0}\frac{\alpha^n}{n!}\frac{(\alpha+n)^n}{n!}||q||^n. \tag{4.10}
$$

By the inequality $(a+b)^n \leq 2^{n-1}(a^n+b^n)$, estimate (4.10) continues

$$
\begin{aligned}
|A(\alpha)-q(\alpha)| &\leq Q^2(\alpha)\left\{\sum_{n\geq 0}2^{n-1}\left(\frac{\alpha^n}{n!}\right)^2||q||^n + \sum_{n\geq 1}2^{n-1}\frac{\alpha^n}{n!}\frac{n^n}{n!}||q||^n\right\} \\
&= \frac{1}{2}Q^2(\alpha)\left\{\sum_{n\geq 0}\left(\frac{\sqrt{2||q||}\alpha}{n!}\right)^{2n} + \sum_{n\geq 1}\frac{(2\alpha)^n}{n!}\frac{n^n}{n!}||q||^n.\right\} \qquad (4.11)
\end{aligned}
$$

For the first series on the right hand side of (4.11) one has

$$
\sum_{n\geq 0}\left(\frac{\sqrt{2||q||}\alpha}{n!}\right)^{2n} \leq \exp^2\left(\sqrt{2||q||}\alpha\right) = \exp\left(2\sqrt{2||q||}\alpha\right). \tag{4.12}
$$

Evaluate the other one. It follows from the Stirling formula that

$$
n! \geq \sqrt{2\pi}\sqrt{n}\left(\frac{n}{e}\right)^n
$$

and hence

$$
\begin{aligned}
\sum_{n\geq 1}\frac{(2\alpha)^n}{n!}\frac{n^n}{n!}||q||^n &\leq \frac{1}{\sqrt{2\pi}}\sum_{n\geq 1}\frac{(2\alpha)^n}{n!}\frac{e^n}{\sqrt{n}}||q||^n \\
&= \frac{1}{\sqrt{2\pi}}\sum_{n\geq 1}\frac{(2e||q||\alpha)^n}{n!} < \frac{1}{\sqrt{2\pi}}\exp(2e||q||\alpha). \qquad (4.13)
\end{aligned}
$$

It follows now from (4.11) – (4.13) that

$$
|A(\alpha)-q(\alpha)| < \frac{1}{2}Q^2(\alpha)\left\{\exp\left(2\sqrt{2||q||}\alpha\right) + \frac{1}{\sqrt{2\pi}}\exp(2e||q||\alpha)\right\}
$$

and (4.5) is proven. □

Remark 3. *The exponential bounds* (4.5) *on* $A(\alpha)$ *in Theorem 2 can be easily improved for the cases* $q \in L^1(0,\infty)$ *and* $q \in L^\infty(0,\infty)$*:*

$$|A(\alpha) - q(\alpha)| \leqslant Q^2(\alpha)e^{\alpha Q(\alpha)}, Q(\alpha) = \int_0^\alpha |q|, \tag{4.14}$$

$$|A(\alpha) - q(\alpha)| \leqslant ||q||_\infty \sum_{n\geq 1} \frac{||q||_\infty^n \alpha^{2n}}{n!\,(n+1)!}, ||q||_\infty := \sup_{0\leq x\leq\infty} |q(x)|, \tag{4.15}$$

respectively. Bounds (4.14) *and* (4.15) *were found in* [**8**]. *In* [**8**] *(Section 10) Gesztesy-Simon also conjectured that A has exponential bound for potentials obeying* (4.4). *Theorem 2 gives an affirmative answer to their conjecture.*

5. A new procedure for evaluating the Titchmarsh-Weyl $m-$function

In this short section we present an algorithm to evaluate the $m-$function which should already be quite transparent to the reader.

Algorithm 1. *Given real valued potential* q *subject to* $||q|| = \sup \int_x^{x+1} |q| < \infty$, *the* $m-$*function can be computed as follows:*

1. Solve integral equation (3.2) *for* $A(x,y)$ *and evaluate* $A(\alpha)$ *by* (3.1).

2. Evaluate $m(z)$ *by* (2.15). *The integral in* (2.15) *is absolute convergent for* $z = -k^2$ *where* $\operatorname{Re} k > 2\max\{\sqrt{2\,||q||}, e\,||q||\}$.

Remark 4. *Our algorithm yields an absolutely convergent series representation of the* $m-$*function,*

$$m(-k^2) = -k - \sum_{n\geq 0} (-1)^n \int_0^\infty A_n(\alpha)e^{-2\alpha k}\, d\alpha,\ A_0 := q. \tag{5.1}$$

Under weaker conditions on q *representation* (5.1) *was obtained in* [**8**] *by completely different methods which do not imply the linear Volterra integral equation* (3.2). *Some other series representations can be found in* [**9**], [**10**], [**12**], [**19**], [**18**]. *It should be pointed out though that those series are quite unwieldy and, in addition, should be computed for each* z *separately. Our procedure has the advantage that once* $A(\alpha)$ *is found one only needs to compute its Laplace transform for different* k.

Remark 5. *It can be easily seen that if* $q(x) \geq 0$ *then* $A_n(\alpha) \geq 0$ *which improves the rate of convergence of* $A(\alpha) = \sum_{n\geq 0} (-1)^n A_n(\alpha)$ *making our algorithm more efficient.*

6. Acknowledgments

Research of Sergei Avdonin was partially supported by the National Science Foundation, grant # OPP-0414128. Victor Mikhaylov was partially supported by the University of Alaska Fairbanks Graduate Fellowship.

References

[1] W.O. Amrein and D.B. Pearson, *M operators: a generalization of Weyl-Titchmarsh theory* J. Comput. Appl. Math. 171 (2004), no. 1-2, 1–26.

[2] S.A. Avdonin and M.I. Belishev, *Boundary control and dynamic inverse problem for nonselfadjoint Sturm–Liouville operator*, Control and Cybernetics, **25** (1996), 429–440.

[3] S.A. Avdonin, M.I. Belishev, and S. A. Ivanov, *Matrix inverse problem for the equation* $u_{tt} - u_{xx} + Q(x)u = 0$, Math. USSR Sbornik, **7** (1992), 287–310.

[4] V.S. Buslaev and V.B. Matveev, *Wave operators for the Schrödinger equation with slowly decreasing potential*, Theoret. and Math. Phys. 2 (1970), no. 3, 266–274.

[5] K. Chadan and P.C. Sabatier, *Inverse problems in quantum scattering theory. Second edition.* Texts and Monographs in Physics. Springer-Verlag, New York, 1989. xxxii+499 pp.

[6] S. Clark and F. Gesztesy, *Weyl-Titchmarsh M-function asymptotics, local uniqueness results, trace formulas, and Borg-type theorems for Dirac operators*, Trans. Amer. Math. Soc. 354 (2002), no. 9, 3475–3534

[7] G. Freiling, V. Yurko, *Inverse Sturm-Liouville problems and their applications.* Nova Science Publishers, Inc., Huntington, NY, 2001. x+356 pp.

[8] F. Gesztesy and B. Simon, *A new approach to inverse spectral theory. II. General real potentials and the connection to the spectral measure,* Ann. of Math. (2) 152 (2000), no. 2, 593–643.

[9] B. J. Harris, *An exact method for the calculation of certain Titchmarsh-Weyl m-functions.* Proc. Roy. Soc. Edinburgh Sect. A 106 (1987), no. 1-2, 137–142.

[10] D.B. Hinton, M. Klaus, and J.K. Shaw, *Series representation and asymptotics for Titchmarsh-Weyl* m-functions.*,* Differential Integral Equations 2 (1989), no. 4, 419–429.

[11] A. Kachalov, Y. Kurylev, M. Lassas and N. Mandache, *Equivalence of time-domain inverse problems and boundary spectral problems*, Inverse Problems, 20 (2004), 491–436.

[12] H.G. Kaper and M.K. Kwong, *Asymptotics of the Titchmarsh-Weyl m-coefficient for integrable potentials. II.* Differential equations and mathematical physics (Birmingham, Ala., 1986), 222–229, Lecture Notes in Math., 1285, Springer, Berlin, 1987.

[13] B.M. Levitan, I.S. Sargsjan, *Introduction to spectral theory: selfadjoint ordinary differential operators.* Translations of Mathematical Monographs, Vol. 39. American Mathematical Society, Providence, R.I., 1975. xi+525 pp.

[14] V.B. Matveev and M.M. Skriganov, *Scattering problem for radial Schrödinger equation with slowly decreasing potential,* Teor. Mat. Fiz. 10 (1972), no. 2, 238–248.

[15] B. Pavlov, *S-Matrix and Dirichlet-to-Neumann Operators*, In: Encyclopedia of Scattering, ed. R. Pike, P. Sabatier, Academic Press, Harcourt Science and Tech. Company (2001) pp 1678-1688.

[16] A.G. Ramm, *Recovery of the potential from $I-$function* C. R. Math. Rep. Acad. Sci. Canada 9 (1987), no. 4, 177–182.

[17] A. Ramm and B. Simon, *A new approach to inverse spectral theory. III. Short-range potentials* J. Anal. Math. 80 (2000), 319–334.

[18] A. Rybkin, *On a transformation of the Sturm-Liouville equation with slowly decaying potentials and the Titchmarsh-Weyl $m-$function* Spectral methods for operators of mathematical physics, 185–201, Oper. Theory Adv. Appl., 154, Birkhäuser, Basel, 2004.

[19] A. Rybkin, *Some new and old asymptotic representations of the Jost solution and the Weyl m-function for Schrödinger operators on the line.* Bull. London Math. Soc. 34 (2002), no. 1, 61–72.

[20] B. Simon, *A new approach to inverse spectral theory. I. Fundamental formalism,* Ann. of Math. 150 (1999), no. 2, 1029-1057.

[21] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations. Part I,* Second Edition, Clarendon Press, Oxford 1962, 203 pp.

Department of Mathematics and Statistics, University of Alaska Fairbanks, PO Box 756660, Fairbanks, AK 99775

E-mail address: ffsaa@uaf.edu, ftvsm@uaf,edu, ffavr@uaf.edu