ALGEBRO-GEOMETRIC FINITE-BAND SOLUTIONS OF THE ABLOWITZ–LADIK HIERARCHY

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Abstract. We provide a detailed derivation of all complex-valued algebro-geometric finite-band solutions of the Ablowitz–Ladik hierarchy. In addition, we survey a recursive construction of the Ablowitz–Ladik hierarchy and its zero-curvature and Lax formalism.

1. Introduction

In the mid seventies, Ablowitz and Ladik, in a series of papers [3]–[6] (see also [1], [2, Sect. 3.2.2], [7, Ch. 3], [20]), used inverse scattering methods to analyze certain integrable differential-difference systems. One of their integrable variants of such systems included a discretization of the celebrated AKNS-ZS system, the pair of coupled nonlinear differential difference equations,

$$\begin{aligned} -i\alpha_t - (\alpha^+ - 2\alpha + \alpha^-) + \alpha\beta(\alpha^+ + \alpha^-) = 0, \\ -i\beta_t + (\beta^+ - 2\beta + \beta^-) - \alpha\beta(\beta^+ + \beta^-) = 0 \end{aligned} \tag{1.1}$$

with $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Here we used the notation $f^\pm(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, for complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}}$. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where $\beta = -\overline{\alpha}$, and in the defocusing case, where $\beta = \overline{\alpha}$, (1.1) yields the discrete analog of the nonlinear Schrödinger equation

$$-i\alpha_t + 2\alpha - (1 \pm |\alpha|^2)(\alpha^+ + \alpha^-) = 0. \tag{1.2}$$

We will refer to (1.1) as the Ablowitz–Ladik system. The principal theme of this paper will be to derive the algebro-geometric finite-band solutions of the Ablowitz–Ladik (AL) hierarchy, a completely integrable sequence of systems of nonlinear evolution equations on the lattice $\mathbb{Z}$ whose first nonlinear member is the Ablowitz–Ladik system (1.1).

Since the mid seventies there has been an enormous amount of activity in the area of integrable differential-difference equations. Two principal directions of research are responsible for this development: Originally, the development was driven by the theory of completely integrable systems and its applications to fields such as nonlinear optics, and more recently, it gained additional momentum due to its intimate connections with the theory of orthogonal polynomials. In the following

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we first briefly recall the development in connection with integrable systems and subsequently turn to the one influenced by research on orthogonal polynomials.

The first systematic discussion of the Ablowitz–Ladik (AL) hierarchy appears to be due to Schilling [47] (cf. also [54], [58], [61]), infinitely many conservation laws are derived, for instance, by Ding, Sun, and Xu [24], the bi-Hamiltonian structure of the AL hierarchy is considered by Ercolani and Lozano [26], connections between the AL hierarchy and the motion of a piecewise linear curve have been established by Doliwa and Santini [25], Bäcklund and Darboux transformations were studied by Geng [29] and Vekslerchik [59], and the Hirota bilinear formalism, AL τ-functions, etc., were considered by Vekslerchik [58]. The initial value problem for half-infinite AL systems was discussed by Common [22], for an application of the inverse scattering method to (1.2) we refer to Vekslerchik and Konotop [60]. This just scratches on the surface of these developments and the interested reader will find much more material in the references cited in these papers and the ones discussed below.

Algebro-geometric (and periodic) solutions of the AL system (1.1) have briefly been studied by Ahmad and Chowdhury [8], [9], Bogolyubov, Prikarpatskii, and Samoilenko [16], Bogolyubov and Prikarpatskii [17], Chow, Conte, and Xu [21], Geng, Dai, and Cao [30], Vaninsky [56]. In an effort to analyze models describing oscillations in non-linear dispersive wave systems, Miller, Ercolani, Krichever, and Levermore [43] (see also [42]) gave a detailed analysis of algebro-geometric solutions of the AL system (1.1). Introducing

$$U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}, \quad V(z) = i \begin{pmatrix} z - 1 - \alpha\beta^- & \alpha - \alpha^- z^{-1} \\ \beta^- z - \beta & 1 + \alpha^-\beta - z^{-1} \end{pmatrix} \tag{1.3}$$

with $z \in \mathbb{C}$ a spectral parameter, the authors in [43] relied on the fact that the Ablowitz–Ladik system (1.1) is equivalent to the zero-curvature equations

$$U_t + UV - V^+U = 0, \tag{1.4}$$

the latter being the compatibility relation for the spatial and temporal linear problems

$$\Phi = U\Phi^-, \quad \Phi_t^- = V\Phi^-. \tag{1.5}$$

Here we extended the notation $f^\pm(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, to $\mathbb{C}^2$-valued and 2×2-matrix-valued sequences with complex-valued entries.

Miller, Ercolani, Krichever, and Levermore [43] then performed a thorough analysis of the solutions $\Phi = \Phi(z, n, t)$ associated with the pair (U, V) and derived the theta function representations of α, β satisfying the AL system (1.1). In the particular focusing and defocusing cases they also discussed periodic and quasi-periodic solutions α with respect to $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Vekslerchik [57] also studied finite-genus solutions for the AL hierarchy by establishing connections with Fay's identity for theta functions.

The connection between the Ablowitz–Ladik system (1.1) and orthogonal polynomials comes about as follows: Let $\{\alpha(n)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers subject to the condition $|\alpha(n)| < 1$, $n \in \mathbb{N}$, and define the transfer matrix

$$T(z) = \begin{pmatrix} z & \alpha \\ \overline{\alpha} z & 1 \end{pmatrix}, \quad z \in \mathbb{T}, \tag{1.6}$$

with spectral parameter z on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Consider the system of difference equations

$$\Phi(z, n) = T(z, n)\Phi(z, n - 1), \quad (z, n) \in \mathbb{T} \times \mathbb{N}, \tag{1.7}$$

with initial condition $\Phi(z,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where

$$\Phi(z,n) = \begin{pmatrix} \varphi(z,n) \\ z^n \overline{\varphi}(1/z,n) \end{pmatrix}, \quad (z,n) \in \mathbb{T} \times \mathbb{N}_0. \tag{1.8}$$

(Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.) Then $\varphi(\,\cdot\,,n)$ are monic polynomials of degree n first introduced by Szegő in the 1920's in his seminal work on the asymptotic distribution of eigenvalues of sections of Toeplitz forms [51], [52] (see also [53, Ch. XI]). Szegő's point of departure was the trigonometric moment problem and hence the theory of orthogonal polynomials on the unit circle. Indeed, given a probability measure $d\sigma$ supported on an infinite set on the unit circle, one is interested in finding monic polynomials $\chi(\,\cdot\,,n)$ of degree $n \in \mathbb{N}_0$ in $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, such that

$$\int_0^{2\pi} d\sigma(e^{i\theta})\, \overline{\chi(e^{i\theta},m)} \chi(e^{i\theta},n) = \gamma(n)^{-2} \delta_{m,n}, \quad m,n \in \mathbb{N}_0, \tag{1.9}$$

where $\gamma(0)^2 = 1$, $\gamma(n)^2 = \prod_{j=1}^n \big(1 - |\alpha(j)|^2\big)^{-1}$, $n \in \mathbb{N}$. Szegő showed that the corresponding polynomials (1.8) with φ replaced by χ satisfy the recurrence formula (1.7). Early work in this area includes important contributions by Akhiezer, Geronimus, Krein, Tomčuk, Verblunsky, Widom, and others, and is summarized in the books by Akhiezer [10], Geronimus [35], Szegő [53], and especially in the recent two-volume treatise by Simon [48].

Unaware of the paper [43], Geronimo and Johnson [33] studied (1.7) in the case where the coefficients α are random variables. Under appropriate ergodicity assumptions on α and the hypothesis of a vanishing Lyapunov exponent on prescribed spectral arcs on the unit circle $\mathbb{T}$, Geronimo and Johnson [33] (cf. also [32], [34]) developed the corresponding spectral theory associated with (1.7) and the unitary operator it generates in $\ell^2(\mathbb{Z})$. In this sense the discussion in [33] is a purely stationary one and connections with a zero-curvature formalism, theta function representations, and integrable hierarchies are not made in [33] (but in this context we refer to the discussion concerning [31] in the next paragraph). More recently, the defocusing case with periodic and quasi-periodic coefficients was studied in great detail by Deift [23], Golinskii and Nevai [39], Killip and Nenciu [40], Li [41], Nenciu [45], [46], Simon [48, Ch. 11], [49], and [50].

An important extension of (1.7) was developed by Baxter in a series of papers on Toeplitz forms [11]– [14] in 1960–63. In these papers the transfer matrix T in (1.6) is replaced by the more general (complexified) transfer matrix $U(z) = \left(\begin{smallmatrix} z & \alpha \\ \beta z & 1 \end{smallmatrix}\right)$ in (1.5), that is, precisely the matrix U responsible for the spatial part in the Ablowitz–Ladik system in its zero-curvature formulation (1.3)–(1.5). Here $\alpha = \{\alpha(n)\}_{n\in\mathbb{N}}$, $\beta = \{\beta(n)\}_{n\in\mathbb{N}}$ are subject to the condition $\alpha(n)\beta(n) \neq 1$, $n \in \mathbb{N}$. Studying the following extension of (1.7),

$$\Psi(z,n) = U(z,n)\Psi(z,n-1), \quad (z,n) \in \mathbb{T} \times \mathbb{N}, \tag{1.10}$$

Baxter was led to biorthogonal polynomials on the unit circle with respect to a complex-valued measure on $\mathbb{T}$. Baxter's U matrix in (1.5) led to a new hierarchy of nonlinear difference equations, called the Szegő–Baxter (SB) hierarchy in [31], in honor of these two pioneers of orthogonal polynomials on the unit circle. The latter reference also contains an in-depth study of algebro-geometric solutions of (1.7).

In addition to these recent developments on the AL system and the AL hierarchy, we offer a variety of results in this paper apparently not covered before. These include:

• An elementary, yet effective recursive construction of the AL hierarchy using Laurent polynomials.

• Explicit formulas for Lax pairs for the AL hierarchy.

• The detailed connection between the AL hierarchy and a "complexified" version of CMV matrices first introduced by Baxter.

• A unified treatment of stationary algebro-geometric finite-band solutions and their theta function representations of the entire AL hierarchy.

• A unified treatment of algebro-geometric solutions and their theta function representations of the time-dependent AL hierarchy by solving the rth AL flow with initial data given by stationary algebro-geometric finite-band solutions.

The structure of this paper is as follows: In Section 2 we describe our zero-curvature formalism for the Ablowitz–Ladik (AL) hierarchy. Extending a recursive polynomial approach discussed in great detail in [36] in the continuous case and in [18], [37, Ch. 4], [55, Chs. 6, 12] in the discrete context to the case of Laurent polynomials with respect to the spectral parameter, we derive the AL hierarchy of systems of nonlinear evolution equations whose first nonlinear member is the Ablowitz–Ladik system (1.1). Section 3 is devoted to a detailed study of the stationary AL hierarchy. We employ the recursive Laurent polynomial formalism of Section 2 to describe nonnegative divisors of degree $2p-1$ on a hyperelliptic curve $\mathcal{K}_{2p-1}$ of genus $2p-1$ associated with the pth equation in the stationary AL hierarchy. By means of a fundamental meromorphic function ϕ on $\mathcal{K}_{2p-1}$ (an analog of the Weyl–Titchmarsh function for the system (1.7)) we then proceed to derive the theta function representations of the associated Baker–Akhiezer vector and all stationary algebro-geometric finite-band solutions of the AL hierarchy. The corresponding time-dependent results for the AL hierarchy are presented in detail in Section 4. Appendix A collects relevant material on hyperelliptic curves and their theta functions and introduces the terminology freely used in Sections 3 and 4. Appendix B is of a technical nature and summarizes expansions of various key quantities related to the Laurent polynomial recursion formalism as the spectral parameter tends to zero and to infinity.

2. The Ablowitz–Ladik Hierarchy, Recursion Relations, Zero-Curvature Pairs, and Hyperelliptic Curves

In this section we summarize the construction of the Ablowitz–Ladik hierarchy employing a Laurent polynomial recursion formalism and derive the associated sequence of Ablowitz–Ladik zero-curvature pairs (we also hint at Lax pairs). Moreover, we discuss the Burchnall–Chaundy Laurent polynomial in connection with the stationary Ablowitz–Ladik hierarchy and the underlying hyperelliptic curve. For a detailed treatment of this material we refer to [37], [38].

We denote by $\mathbb{C}^{\mathbb{Z}}$ the set of complex-valued sequences indexed by $\mathbb{Z}$.

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that α, β satisfy*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \neq 1, \ n \in \mathbb{Z}. \tag{2.1}$$

In the time-dependent case we assume that α, β satisfy

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,), \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \neq 1,\ (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{2.2}$$

We denote by $S^{\pm}$ the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ according to

$$(S^{\pm} f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}. \tag{2.3}$$

Moreover, we will frequently use the notation

$$f^{\pm} = S^{\pm} f, \quad f \in \mathbb{C}^{\mathbb{Z}}. \tag{2.4}$$

Consider the 2×2 matrix

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \tag{2.5}$$

a transfer matrix associated with the difference expression L in (2.28). Here $z \in \mathbb{C}$ plays the role of a spectral parameter for L. To construct the Ablowitz–Ladik hierarchy one needs a second 2×2 matrix, denoted by $V_p(z)$, $p \in \mathbb{N}_0$, defined recursively in the following. We take the shortest route to the construction of V_p and hence to that of the Ablowitz–Ladik hierarchy by starting from the recursion relation (2.6)–(2.13) below.

Define sequences $\{f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ recursively by

$$g_{0,+} = \tfrac{1}{2} c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^+, \quad h_{0,+} = c_{0,+}\beta, \tag{2.6}$$

$$g_{\ell+1,+} - g_{\ell+1,+}^- = \alpha h_{\ell,+}^- + \beta f_{\ell,+}, \quad \ell \in \mathbb{N}_0, \tag{2.7}$$

$$f_{\ell+1,+}^- = f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \tag{2.8}$$

$$h_{\ell+1,+} = h_{\ell,+}^- + \beta(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \tag{2.9}$$

and

$$g_{0,-} = \tfrac{1}{2} c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^+, \tag{2.10}$$

$$g_{\ell+1,-} - g_{\ell+1,-}^- = \alpha h_{\ell,-} + \beta f_{\ell,-}^-, \quad \ell \in \mathbb{N}_0, \tag{2.11}$$

$$f_{\ell+1,-} = f_{\ell,-}^- + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0, \tag{2.12}$$

$$h_{\ell+1,-}^- = h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0. \tag{2.13}$$

Here $c_{0,\pm} \in \mathbb{C}$ are given constants. For later use we also introduce

$$f_{-1,\pm} = h_{-1,\pm} = 0. \tag{2.14}$$

Remark 2.2. *The sequences $\{f_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,+}\}_{\ell\in\mathbb{N}_0}$ can be computed recursively as follows: Assume that $f_{\ell,+}$, $g_{\ell,+}$, and $h_{\ell,+}$ are known. Equation* (2.7) *is a first-order difference equation in $g_{\ell+1,+}$ that can be solved directly and yields a local lattice function that is determined up to a new constant denoted by $c_{\ell+1,+} \in \mathbb{C}$. Relations* (2.8) *and* (2.9) *then determine $f_{\ell+1,+}$ and $h_{\ell+1,+}$, etc. The sequences $\{f_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,-}\}_{\ell\in\mathbb{N}_0}$ are determined similarly.*

Upon setting

$$\gamma = 1 - \alpha\beta, \tag{2.15}$$

one explicitly obtains

$$\begin{aligned}
&f_{0,+} = c_{0,+}(-\alpha^+), \quad f_{1,+} = c_{0,+}\big(-\gamma^+\alpha^{++} + (\alpha^+)^2\beta\big) + c_{1,+}(-\alpha^+),\\
&g_{0,+} = \tfrac{1}{2}c_{0,+}, \quad g_{1,+} = c_{0,+}(-\alpha^+\beta) + \tfrac{1}{2}c_{1,+},\\
&h_{0,+} = c_{0,+}\beta, \quad h_{1,+} = c_{0,+}\big(\gamma\beta^- - \alpha^+\beta^2\big) + c_{1,+}\beta,\\
&f_{0,-} = c_{0,-}\alpha, \quad f_{1,-} = c_{0,-}\big(\gamma\alpha^- - \alpha^2\beta^+\big) + c_{1,-}\alpha,\\
&g_{0,-} = \tfrac{1}{2}c_{0,-}, \quad g_{1,-} = c_{0,-}(-\alpha\beta^+) + \tfrac{1}{2}c_{1,-},\\
&h_{0,-} = c_{0,-}(-\beta^+), \quad h_{1,-} = c_{0,-}\big(-\gamma^+\beta^{++} + \alpha(\beta^+)^2\big) + c_{1,-}(-\beta^+), \text{ etc.}
\end{aligned} \tag{2.16}$$

Here $\{c_{\ell,\pm}\}_{\ell\in\mathbb{N}}$ denote summation constants which naturally arise when solving the difference equations for $g_{\ell,\pm}$ in (2.7), (2.11). Subsequently, it will also be useful to work with the corresponding homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, defined by the vanishing of all summation constants $c_{k,\pm}$ for $k = 1,\dots,\ell$, and choosing $c_{0,\pm} = 1$,

$$\hat{f}_{0,+} = -\alpha^+, \quad \hat{f}_{0,-} = \alpha, \quad \hat{f}_{\ell,\pm} = f_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}, \tag{2.17}$$

$$\hat{g}_{0,\pm} = \tfrac{1}{2}, \quad \hat{g}_{\ell,\pm} = g_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}, \tag{2.18}$$

$$\hat{h}_{0,+} = \beta, \quad \hat{h}_{0,-} = -\beta^+, \quad \hat{h}_{\ell,\pm} = h_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}. \tag{2.19}$$

By induction one infers that

$$f_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{f}_{k,\pm}, \quad g_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{g}_{k,\pm}, \quad h_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{h}_{k,\pm}. \tag{2.20}$$

In a slight abuse of notation we will occasionally stress the dependence of $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ on α,β by writing $f_{\ell,\pm}(\alpha,\beta)$, $g_{\ell,\pm}(\alpha,\beta)$, and $h_{\ell,\pm}(\alpha,\beta)$.

One can show (cf. [38]) that all homogeneous elements $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, $\ell\in\mathbb{N}_0$, are polynomials in α,β, and some of their shifts.

Remark 2.3. *As an efficient tool to later distinguish between homogeneous and nonhomogeneous quantities $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, and $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, $\hat{h}_{\ell,\pm}$, respectively, we now introduce the notion of degree as follows. Denote*

$$f^{(r)} = S^{(r)}f, \quad f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad S^{(r)} = \begin{cases} (S^+)^r, & r\geq 0,\\ (S^-)^{-r}, & r<0, \end{cases} \quad r\in\mathbb{Z}, \tag{2.21}$$

and define

$$\deg\big(\alpha^{(r)}\big) = r, \quad \deg\big(\beta^{(r)}\big) = -r, \quad r\in\mathbb{Z}. \tag{2.22}$$

This then results in

$$\begin{aligned}
&\deg\big(\hat{f}^{(r)}_{\ell,+}\big) = \ell+1+r, \quad \deg\big(\hat{f}^{(r)}_{\ell,-}\big) = -\ell+r, \quad \deg\big(\hat{g}^{(r)}_{\ell,\pm}\big) = \pm\ell,\\
&\deg\big(\hat{h}^{(r)}_{\ell,+}\big) = \ell-r, \quad \deg\big(\hat{h}^{(r)}_{\ell,-}\big) = -\ell-1-r, \quad \ell\in\mathbb{N}_0,\ r\in\mathbb{Z}.
\end{aligned} \tag{2.23}$$

We also note the following useful result (cf. [38]): Assume (2.1), then,

$$\begin{aligned}
&g_{\ell,+} - g^-_{\ell,+} = \alpha h_{\ell,+} + \beta f^-_{\ell,+}, \quad \ell\in\mathbb{N}_0,\\
&g_{\ell,-} - g^-_{\ell,-} = \alpha h^-_{\ell,-} + \beta f_{\ell,-}, \quad \ell\in\mathbb{N}_0.
\end{aligned} \tag{2.24}$$

Moreover, we note the following symmetries,

$$\hat{f}_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat{h}_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \hat{g}_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat{g}_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \ell\in\mathbb{N}_0. \tag{2.25}$$

In the special case $c_{0,+} = 2$, $c_{0,-} = 0$ (describing the stationary Baxter–Szegő hierarchy, cf. Remark 2.5 (ii)), the precise structure of the sequences $f_{\ell,+}, g_{\ell,+}, h_{\ell,+}$ is described in [31, Lemma 2.2].

Next we relate the homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$ to certain matrix elements of L, where L will later be identified as the Lax difference expression associated with the Ablowitz–Ladik hierarchy. For this purpose it is useful to introduce the standard basis $\{\delta_m\}_{m\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ by

$$\delta_m = \{\delta_{m,n}\}_{n\in\mathbb{Z}},\ m \in \mathbb{Z}, \quad \delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases} \tag{2.26}$$

The scalar product in $\ell^2(\mathbb{Z})$, denoted by $(\,\cdot\,,\,\cdot\,)_{\ell^2(\mathbb{Z})}$, is defined by

$$(f, g)_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \overline{f(n)}g(n), \quad f, g \in \ell^2(\mathbb{Z}). \tag{2.27}$$

In the basis just introduced, we introduce the difference expression L by

$$L = \left(\begin{array}{cccccccc}
\ddots & \ddots & \ddots & \ddots & \ddots & & & \\
0 & -\alpha(0)\rho(-1) & -\beta(-1)\alpha(0) & -\alpha(1)\rho(0) & \rho(0)\rho(1) & & \mbox{\Large 0} & \\
 & \rho(-1)\rho(0) & \beta(-1)\rho(0) & -\beta(0)\alpha(1) & \beta(0)\rho(1) & 0 & & \\
 & & 0 & -\alpha(2)\rho(1) & -\beta(1)\alpha(2) & -\alpha(3)\rho(2) & \rho(2)\rho(3) & \\
 & & & \rho(1)\rho(2) & \beta(1)\rho(2) & -\beta(2)\alpha(3) & \beta(2)\rho(3) & 0 \\
 & \mbox{\Large 0} & & & & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right), \tag{2.28}$$

where terms of the form $-\beta(n)\alpha(n+1)$ represent the diagonal (n,n)-entries, $n \in \mathbb{Z}$, in the infinite matrix (2.28). Here we used the abbreviation

$$\rho = \gamma^{1/2} = (1 - \alpha\beta)^{1/2}. \tag{2.29}$$

The half-lattice (i.e., semi-infinite) version of L was recently rediscovered by Cantero, Moral, and Velázquez [19] in the special case where $\beta = \overline{\alpha}$ (see also Simon [48]–[50] who coined the term CMV matrix in this context).

The next result details the connections between L and the recursion coefficients $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$:

Lemma 2.4. *Assume* (2.1) *and let* $n \in \mathbb{Z}$. *Then the homogeneous coefficients* $\{\hat{f}_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{\hat{g}_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, *and* $\{\hat{h}_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ *satisfy the following relations:*

$$\begin{aligned}
&\hat{f}_{\ell,+}(n) = \alpha(n)(\delta_n, L^{\ell+1}\delta_n)_{\ell^2(\mathbb{Z})} + \rho(n) \begin{cases} (\delta_{n-1}, L^{\ell+1}\delta_n)_{\ell^2(\mathbb{Z})}, & n \text{ even}, \\ (\delta_n, L^{\ell+1}\delta_{n-1})_{\ell^2(\mathbb{Z})}, & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0, \\
&\hat{f}_{\ell,-}(n) = \alpha(n)(\delta_n, L^{-\ell}\delta_n)_{\ell^2(\mathbb{Z})} + \rho(n) \begin{cases} (\delta_{n-1}, L^{-\ell}\delta_n)_{\ell^2(\mathbb{Z})}, & n \text{ even}, \\ (\delta_n, L^{-\ell}\delta_{n-1})_{\ell^2(\mathbb{Z})}, & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0, \\
&\quad \hat{g}_{0,\pm} = 1/2, \quad \hat{g}_{\ell,\pm}(n) = (\delta_n, L^{\pm\ell}\delta_n)_{\ell^2(\mathbb{Z})}, \quad \ell \in \mathbb{N}, \\
&\hat{h}_{\ell,+}(n) = \beta(n)(\delta_n, L^{\ell}\delta_n)_{\ell^2(\mathbb{Z})} + \rho(n) \begin{cases} (\delta_n, L^{\ell}\delta_{n-1})_{\ell^2(\mathbb{Z})}, & n \text{ even}, \\ (\delta_{n-1}, L^{\ell}\delta_n)_{\ell^2(\mathbb{Z})}, & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0, \\
&\hat{h}_{\ell,-}(n) = \beta(n)(\delta_n, L^{-\ell-1}\delta_n)_{\ell^2(\mathbb{Z})} + \rho(n) \begin{cases} (\delta_n, L^{-\ell-1}\delta_{n-1})_{\ell^2(\mathbb{Z})}, & n \text{ even}, \\ (\delta_{n-1}, L^{-\ell-1}\delta_n)_{\ell^2(\mathbb{Z})}, & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0.
\end{aligned} \tag{2.30}$$

For the proof of Lemma 2.4 and some of its applications in connection with conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy we refer to [38].

Next we define the 2×2 matrix

$$V_p(z) = i \begin{pmatrix} G_p^-(z) & -F_p^-(z) \\ H_p^-(z) & -K_p^-(z) \end{pmatrix}, \quad p \in \mathbb{N}_0, \tag{2.31}$$

for appropriate Laurent polynomials $F_p(z)$, $G_p(z)$, $H_p(z)$, and $K_p(z)$ in the spectral parameter $z \in \mathbb{C} \setminus \{0\}$ to be determined shortly. By postulating the stationary zero-curvature relation

$$0 = UV_p - V_p^+ U, \tag{2.32}$$

one concludes that (2.32) is equivalent with the following relations

$$z(G_p^- - G_p) + z\beta F_p + \alpha H_p^- = 0, \tag{2.33}$$

$$z\beta F_p^- + \alpha H_p - K_p + K_p^- = 0, \tag{2.34}$$

$$-F_p + zF_p^- + \alpha(G_p + K_p^-) = 0, \tag{2.35}$$

$$z\beta(G_p^- + K_p) - zH_p + H_p^- = 0. \tag{2.36}$$

In order to make the connection between the zero-curvature formalism and the recursion relations (2.6)–(2.13), we now define Laurent polynomials F_p, G_p, H_p, and K_p by

$$F_p(z) = \sum_{\ell=1}^{p} f_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+} z^{\ell}, \tag{2.37}$$

$$G_p(z) = \sum_{\ell=1}^{p} g_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+} z^{\ell}, \tag{2.38}$$

$$H_p(z) = \sum_{\ell=0}^{p-1} h_{p-1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+} z^{\ell}, \tag{2.39}$$

$$K_p(z) = \sum_{\ell=0}^{p} g_{p-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} g_{p-\ell,+} z^{\ell} = G_p(z) + g_{p,-} - g_{p,+}. \tag{2.40}$$

The corresponding homogeneous quantities are defined by ($\ell \in \mathbb{N}_0$)

$$\begin{aligned}
&\widehat{F}_0(z) = 0, \quad \widehat{F}_\ell(z) = \sum_{k=1}^{\ell} \hat{f}_{\ell-k,-} z^{-k} + \sum_{k=0}^{\ell-1} \hat{f}_{\ell-1-k,+} z^{k}, \\
&\widehat{G}_0(z) = \frac{1}{2}, \quad \widehat{G}_\ell(z) = \sum_{k=1}^{\ell} \hat{g}_{\ell-k,-} z^{-k} + \sum_{k=0}^{\ell} \hat{g}_{\ell-k,+} z^{k}, \\
&\widehat{H}_0(z) = 0, \quad \widehat{H}_\ell(z) = \sum_{k=0}^{\ell-1} \hat{h}_{\ell-1-k,-} z^{-k} + \sum_{k=1}^{\ell} \hat{h}_{\ell-k,+} z^{k}, \\
&\widehat{K}_0(z) = \frac{1}{2}, \quad \widehat{K}_\ell(z) = \sum_{k=0}^{\ell} \hat{g}_{\ell-k,-} z^{-k} + \sum_{k=1}^{\ell} \hat{g}_{\ell-k,+} z^{k} = \widehat{G}_\ell(z) + \hat{g}_{\ell,-} - \hat{g}_{\ell,+}.
\end{aligned} \tag{2.41}$$

The stationary zero-curvature relation (2.32), $0 = UV_p - V_p^+ U$, is then equivalent to

$$-\alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- = 0, \tag{2.42}$$

$$\beta(g_{p,+}^- + g_{p,-}) + h_{p-1,+}^- - h_{p-1,-} = 0. \tag{2.43}$$

Thus, varying $p \in \mathbb{N}_0$, equations (2.42) and (2.43) give rise to the stationary Ablowitz–Ladik (AL) hierarchy which we introduce as follows

$$\text{s-AL}_p(\alpha,\beta) = \begin{pmatrix} -\alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ \beta(g_{p,+}^- + g_{p,-}) + h_{p-1,+}^- - h_{p-1,-} \end{pmatrix} = 0, \quad p \in \mathbb{N}_0. \tag{2.44}$$

Explicitly (recalling $\gamma = 1 - \alpha\beta$),

$$\text{s-AL}_0(\alpha,\beta) = \begin{pmatrix} -c_0\alpha \\ c_0\beta \end{pmatrix} = 0,$$

$$\text{s-AL}_1(\alpha,\beta) = \begin{pmatrix} -\gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_1\alpha \\ \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_1\beta \end{pmatrix} = 0,$$

$$\text{s-AL}_2(\alpha,\beta) = \begin{pmatrix} -\gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix}$$

$$+ \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_2\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_2\beta \end{pmatrix} = 0, \text{ etc.}, \tag{2.45}$$

represent the first few equations of the stationary Ablowitz–Ladik hierarchy. Here we introduced

$$c_p = (c_{p,+} + c_{p,-})/2, \quad p \in \mathbb{N}_0. \tag{2.46}$$

By definition, the set of solutions of (2.44), with p ranging in $\mathbb{N}_0$ and $c_{\ell,\pm} \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, represents the class of algebro-geometric Ablowitz–Ladik solutions.

In the following we will frequently assume that α, β satisfy the pth stationary Ablowitz–Ladik equations. By this we mean it satisfies one of the pth stationary Ablowitz–Ladik equations after a particular choice of summation constants $c_{\ell,\pm} \in \mathbb{C}$, $\ell = 0, \dots, p$, $p \in \mathbb{N}$, has been made.

In accordance with our notation introduced in (2.17)–(2.19) and (2.41), the corresponding homogeneous stationary Ablowitz–Ladik equations are defined by

$$\text{s-}\widehat{\text{AL}}_p(\alpha,\beta) = \text{s-}\widehat{\text{AL}}_p(\alpha,\beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p}, \quad p \in \mathbb{N}_0. \tag{2.47}$$

If one assumes in addition to (2.1) that

$$\gamma(n) = 1 - \alpha(n)\beta(n) \neq 0, \quad n \in \mathbb{Z}, \tag{2.48}$$

one can show (cf. [38]) that $g_{p,+} = g_{p,-}$ up to a lattice constant which can be set equal to zero without loss of generality. Thus, we will henceforth assume that

$$g_{p,+} = g_{p,-}, \tag{2.49}$$

which in turn implies that

$$K_p = G_p \tag{2.50}$$

and hence renders V_p in (2.31) traceless in the stationary context. (We note that equations (2.49) and (2.50) cease to be valid in the time-dependent context, though.)

Remark 2.5. (*i*) *The particular choice* $c_{0,+} = c_{0,-} = 1$ *yields the stationary Ablowitz–Ladik equation. Scaling* $c_{0,\pm}$ *with the same constant then amounts to scaling* V_p *with this constant which drops out in the stationary zero-curvature equation* (2.32).
(*ii*) *Different ratios between* $c_{0,+}$ *and* $c_{0,-}$ *will lead to different stationary hierarchies. In particular, the choice* $c_{0,+} = 2$, $c_{0,-} = 0$ *yields the stationary Baxter–Szegő hierarchy considered in detail in* [31]. *However, in this case some parts from the recursion relation for the negative coefficients still remain. In fact,* (2.13) *reduces to* $g_{p,-} - g_{p,-}^- = \alpha h_{p-1,-}$, $h_{p-1,-} = 0$ *and thus requires* $g_{p,-}$ *to be a constant in addition to* (2.61) *below. This renders the system* (2.61) *overdetermined except in the stationary case.*

Next, still assuming (2.1) and (2.48), and taking into account (2.50), one infers that the expression R_{2p}, defined as

$$R_{2p} = G_p^2 - F_p H_p, \tag{2.51}$$

is a lattice constant, that is, $R_{2p} - R_{2p}^- = 0$, by taking determinants in the stationary zero-curvature equation (2.32). Hence, $R_{2p}(z)$ only depends on z, and assuming in addition to (2.1) and (2.48) that

$$c_{0,\pm} \in \mathbb{C} \setminus \{0\}, \tag{2.52}$$

one may write R_{2p} as[1]

$$R_{2p}(z) = (c_{0,+}^2/4) z^{-2p} \prod_{m=0}^{4p-1} (z - E_m), \quad \{E_m\}_{m=0}^{4p-1} \subset \mathbb{C} \setminus \{0\}. \tag{2.53}$$

Moreover, multiplying (2.51) by z^{2p} and taking $z \to 0$ yields

$$\prod_{m=0}^{4p-1} E_m = \frac{c_{0,-}^2}{c_{0,+}^2}. \tag{2.54}$$

Relation (2.51) allows one to introduce a hyperelliptic curve $\mathcal{K}_{2p-1}$ of (arithmetic) genus $2p-1$ (possibly with a singular affine part), where

$$\mathcal{K}_{2p-1} \colon \mathcal{F}_{2p-1}(z,y) = y^2 - 4c_{0,+}^{-2} z^{2p} R_{2p}(z) = 0. \tag{2.55}$$

Next we turn to the time-dependent Ablowitz–Ladik hierarchy. For that purpose the coefficients α and β are now considered as functions of both the lattice point and time. For each equation in the hierarchy, that is, for each p, we introduce a deformation (time) parameter $t_p \in \mathbb{R}$ in α, β, replacing $\alpha(n), \beta(n)$ by $\alpha(n, t_p), \beta(n, t_p)$. Moreover, the definitions (2.5), (2.31), and (2.37)–(2.40) of U, V_p, and F_p, G_p, H_p, K_p, respectively, still apply. Imposing the zero-curvature relation

$$U_{t_p} + U V_p - V_p^+ U = 0, \quad p \in \mathbb{N}_0, \tag{2.56}$$

then results in the equations

$$\alpha_{t_p} = i\big(z F_p^- + \alpha(G_p + K_p^-) - F_p\big), \tag{2.57}$$

$$\beta_{t_p} = -i\big(\beta(G_p^- + K_p) - H_p + z^{-1} H_p^-\big), \tag{2.58}$$

$$0 = z(G_p^- - G_p) + z\beta F_p + \alpha H_p^-, \tag{2.59}$$

$$0 = z\beta F_p^- + \alpha H_p + K_p^- - K_p. \tag{2.60}$$

[1]We use the convention that a product over an empty index set is to be interpreted equal to 1.

Varying $p \in \mathbb{N}_0$, the collection of evolution equations

$$\mathrm{AL}_p(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_p} - \alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ -i\beta_{t_p} + \beta(g_{p,+}^- + g_{p,-}) - h_{p-1,-} + h_{p-1,+}^- \end{pmatrix} = 0, \quad (n,t_p) \in \mathbb{Z}\times\mathbb{R},\ p \in \mathbb{N}_0, \tag{2.61}$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly,

$$\begin{aligned}
\mathrm{AL}_0(\alpha,\beta) &= \begin{pmatrix} -i\alpha_{t_0} - c_0\alpha \\ -i\beta_{t_0} + c_0\beta \end{pmatrix} = 0, \\
\mathrm{AL}_1(\alpha,\beta) &= \begin{pmatrix} -i\alpha_{t_1} - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_1\alpha \\ -i\beta_{t_1} + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_1\beta \end{pmatrix} = 0, \\
\mathrm{AL}_2(\alpha,\beta) &= \begin{pmatrix} -i\alpha_{t_2} - \gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ -i\beta_{t_2} + \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix} \\
&\quad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_2\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_2\beta \end{pmatrix} = 0, \text{ etc.},
\end{aligned} \tag{2.62}$$

represent the first few equations of the time-dependent Ablowitz–Ladik hierarchy.

By (2.61), (2.7), and (2.11), the time derivative of $\gamma = 1 - \alpha\beta$ is given by

$$\gamma_{t_p} = i\gamma\big((g_{p,+} - g_{p,+}^-) - (g_{p,-} - g_{p,-}^-)\big). \tag{2.63}$$

Remark 2.6. *From* (2.33)–(2.36) *and the explicit computations of the coefficients* $f_{\ell,\pm}$*,* $g_{\ell,\pm}$*, and* $h_{\ell,\pm}$*, one concludes that the zero-curvature equation* (2.56) *and hence the Ablowitz–Ladik hierarchy is invariant under the scaling transformation*

$$\alpha \to \alpha_c = \{c\,\alpha(n)\}_{n\in\mathbb{Z}}, \quad \beta \to \beta_c = \{\beta(n)/c\}_{n\in\mathbb{Z}}, \quad c \in \mathbb{C}\setminus\{0\}. \tag{2.64}$$

Moreover, $R_{2p} = G_p^2 - H_pF_p$ *and hence* $\{E_m\}_{m=0}^{4p-1}$ *are invariant under this transformation. Furthermore, choosing* $c = \mathrm{e}^{ic_pt}$*, one verifies that it is no restriction to assume* $c_p = 0$*. This also indicates that stationary solutions* α,β *can only be constructed up to a constant* (*compare Theorem* 3.7*, in particular,* (3.63)*,* (3.64))*.*

Finally, we briefly hint at explicit expressions of the Lax pair for the Ablowitz–Ladik hierarchy. More details will be presented in [38]. First we need some notation. Let T be a bounded operator in $\ell^2(\mathbb{Z})$. Given the standard basis (2.26) in $\ell^2(\mathbb{Z})$, we represent T by

$$T = \big(T(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T(m,n) = (\delta_m, T\,\delta_n)_{\ell^2(\mathbb{Z})}, \quad (m,n)\in\mathbb{Z}^2. \tag{2.65}$$

Moreover, we introduce the upper and lower triangular parts $T_\pm$ of T by

$$T_\pm = \big(T_\pm(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T_\pm(m,n) = \begin{cases} T(m,n), & \pm(n-m) > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.66}$$

Next, consider the finite difference expression P_p defined by

$$P_p = \frac{i}{2}\sum_{j=1}^{p}\Big(c_{p-j,+}\big((L^j)_+ - (L^j)_-\big) + c_{p-j,-}\big((L^{-j})_+ - (L^{-j})_-\big)\Big) + \frac{i}{2}c_p\,Q_d, \quad p \in \mathbb{N}_0, \tag{2.67}$$

with L given by (2.28) and Q_d denoting the doubly infinite diagonal matrix

$$Q_d = \big((-1)^k \delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}}. \tag{2.68}$$

Then one can show that (L, P_p) represents the Lax pair for the Ablowitz–Ladik equations (2.61) for $p \in \mathbb{N}$. In particular, the hierarchy of nonlinear Ablowitz–Ladik evolution equations (2.61) then can alternatively be derived by imposing the Lax commutator equations

$$L_{t_p}(t_p) - [P_p(t_p), L(t_p)] = 0, \quad t_p \in \mathbb{R},\ p \in \mathbb{N}. \tag{2.69}$$

For additional representations of P_p in terms of L and a particular factorization of L we refer to [38]. The Lax operator in the special defocusing case, where $\beta = \overline{\alpha}$, in the finite-dimensional context was recently derived by Nenciu [46].

In the special stationary case, where P_p and L commute, $[P_p, L] = 0$, they satisfy an algebraic relationship of the type

$$\begin{aligned} &\mathcal{F}_p(L, P_p) = P_p^2 + R_{2p}(L) = P_p^2 + (c_{0,+}^2/4)L^{-2p}\prod_{m=0}^{4p-1}(L - E_m) = 0, \\ &R_{2p}(z) = (c_{0,+}^2/4)z^{-2p}\prod_{m=0}^{4p-1}(z - E_m), \quad z \in \mathbb{C}. \end{aligned} \tag{2.70}$$

The expression $\mathcal{F}_p(L, P_p)$ in (2.70) then represents the Burchnall–Chaundy Laurent polynomial of the pair (L, P_p).

3. The Stationary Ablowitz–Ladik Formalism

This section is devoted to a detailed study of the stationary Ablowitz–Ladik hierarchy and its algebro-geometric solutions. Our principal tools are derived from combining the Laurent polynomial recursion formalism introduced in Section 2 and a fundamental meromorphic function ϕ on a hyperelliptic curve $\mathcal{K}_{2p-1}$. With the help of ϕ we study the Baker–Akhiezer vector Ψ, Dubrovin-type equations governing the motion of auxiliary divisors on $\mathcal{K}_{2p-1}$, trace formulas, and theta function representations of ϕ, Ψ, α, and β. For proofs of the elementary results of the stationary formalism we refer to [37], [38].

Unless explicitly stated otherwise, we suppose throughout this section that

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0, 1\},\ n \in \mathbb{Z}, \tag{3.1}$$

and assume (2.5)–(2.14), (2.31)–(2.32), (2.37)–(2.40), (2.44), (2.50), (2.51), (2.53), keeping $p \in \mathbb{N}_0$ fixed.

We recall the hyperelliptic curve

$$\begin{aligned} &\mathcal{K}_{2p-1} \colon \mathcal{F}_{2p-1}(z, y) = y^2 - 4c_{0,+}^{-2} z^{2p} R_{2p}(z) = 0, \\ &R_{2p}(z) = \left(\frac{c_{0,+}}{2z^p}\right)^2 \prod_{m=0}^{4p-1}(z - E_m), \quad \{E_m\}_{m=0}^{4p-1} \subset \mathbb{C} \setminus \{0\}, \end{aligned} \tag{3.2}$$

as introduced in (2.55). Throughout this section we assume the affine part of $\mathcal{K}_{2p-1}$ to be nonsingular, that is, we suppose that

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 4p-1. \tag{3.3}$$

$\mathcal{K}_{2p-1}$ is compactified by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_{2p-1}$. Points P on $\mathcal{K}_{2p-1} \setminus$

$\{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_{2p-1}$ satisfying $\mathcal{F}_{2p-1}(z, y) = 0$. The complex structure on $\mathcal{K}_{2p-1}$ is then defined in the usual way, see Appendix A. Hence, $\mathcal{K}_{2p-1}$ becomes a two-sheeted hyperelliptic Riemann surface of genus $2p-1$ in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_{2p-1}$ (i.e., by fixing $E_0, \dots, E_{4p-1}$), the summation constants $c_{1,\pm}, \dots, c_{p,\pm}$ in $f_{p,\pm}$, $g_{p,\pm}$, and $h_{p,\pm}$ (and hence in the corresponding stationary s-AL$_p$ equations) are uniquely determined as is clear from (B.11) which establishes the summation constants $c_{\ell,\pm}$ as symmetric functions of $E_0^{\pm 1}, \dots, E_{4p-1}^{\pm 1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$.

We denote by $\{\mu_j(n)\}_{j=1,\dots,2p-1}$ and $\{\nu_j(n)\}_{j=1,\dots,2p-1}$ the zeros of $(\,\cdot\,)^p F_p(\,\cdot\,, n)$ and $(\,\cdot\,)^{p-1} H_p(\,\cdot\,, n)$, respectively; they will play a special role in the following. Thus, we may write

$$F_p(z) = -c_{0,+}\alpha^+ z^{-p} \prod_{j=1}^{2p-1} (z - \mu_j), \tag{3.4}$$

$$H_p(z) = c_{0,+}\beta z^{-p+1} \prod_{j=1}^{2p-1} (z - \nu_j), \tag{3.5}$$

and we recall that (cf. (2.51))

$$R_{2p} - G_p^2 = -F_p H_p. \tag{3.6}$$

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_j of F_p and H_p from the complex plane $\mathbb{C}$ to the curve $\mathcal{K}_{2p-1}$. From (3.6) one infers that

$$R_{2p}(z) - G_p(z)^2 = 0, \quad z \in \{\mu_j, \nu_k\}_{j,k=1,\dots,2p-1}. \tag{3.7}$$

We now introduce $\{\hat{\mu}_j\}_{j=1,\dots,2p-1} \subset \mathcal{K}_{2p-1}$ and $\{\hat{\nu}_j\}_{j=1,\dots,2p-1} \subset \mathcal{K}_{2p-1}$ by

$$\hat{\mu}_j(n) = (\mu_j(n), (2/c_{0,+})\mu_j(n)^p G_p(\mu_j(n), n)), \quad j = 1, \dots, 2p-1, \; n \in \mathbb{Z}, \tag{3.8}$$

and

$$\hat{\nu}_j(n) = (\nu_j(n), -(2/c_{0,+})\nu_j(n)^p G_p(\nu_j(n), n)), \quad j = 1, \dots, 2p-1, \; n \in \mathbb{Z}. \tag{3.9}$$

We also introduce the points $P_{0,\pm}$ by

$$P_{0,\pm} = (0, \pm(c_{0,-}/c_{0,+})) \in \mathcal{K}_{2p-1}, \quad \frac{c_{0,-}^2}{c_{0,+}^2} = \prod_{m=0}^{4p-1} E_m. \tag{3.10}$$

We emphasize that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_{2p-1}$.

Next we introduce the fundamental meromorphic function on $\mathcal{K}_{2p-1}$ by

$$\begin{aligned} \phi(P, n) &= \frac{(c_{0,+}/2)z^{-p}y + G_p(P, n)}{F_p(P, n)} && (3.11)\\ &= \frac{-H_p(P, n)}{(c_{0,+}/2)z^{-p}y - G_p(P, n)}, && (3.12)\\ &\qquad P = (z, y) \in \mathcal{K}_{2p-1}, \; n \in \mathbb{Z}, \end{aligned}$$

with divisor $(\phi(\,\cdot\,, n))$ of $\phi(\,\cdot\,, n)$ given by

$$(\phi(\,\cdot\,, n)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}, \tag{3.13}$$

using (3.4) and (3.5). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_{2p-1}\}, \ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_{2p-1}\} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}). \tag{3.14}$$

Given $\phi(\,\cdot\,, n)$, the meromorphic stationary Baker–Akhiezer vector $\Psi(\,\cdot\,, n, n_0)$ on $\mathcal{K}_{2p-1}$ is then defined by

$$\Psi(P,n,n_0) = \begin{pmatrix} \psi_1(P,n,n_0) \\ \psi_2(P,n,n_0) \end{pmatrix},$$

$$\psi_1(P,n,n_0) = \begin{cases} \prod_{m=n_0+1}^{n} (z+\alpha(m)\phi^-(P,m)), & n \geq n_0+1, \\ 1, & n = n_0, \\ \prod_{m=n+1}^{n_0} (z+\alpha(m)\phi^-(P,m))^{-1}, & n \leq n_0-1, \end{cases} \tag{3.15}$$

$$\psi_2(P,n,n_0) = \phi(P,n_0) \begin{cases} \prod_{m=n_0+1}^{n} (z\beta(m)\phi(z,m-1)^{-1}+1), & n \geq n_0+1, \\ 1, & n = n_0, \\ \prod_{m=n+1}^{n_0} (z\beta(m)\phi(z,m-1)^{-1}+1)^{-1}, & n \leq n_0-1. \end{cases} \tag{3.16}$$

Basic properties of ϕ and Ψ are summarized in the following result.

Lemma 3.1 ([38])**.** *Suppose that α, β satisfy (3.1) and the pth stationary Ablowitz–Ladik system (2.44). Moreover, assume (3.2) and (3.3) and let $P = (z,y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n,n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{3.17}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_p(z)}{F_p(z)}, \tag{3.18}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_p(z)}{F_p(z)}, \tag{3.19}$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p}\frac{y(P)}{F_p(z)}. \tag{3.20}$$

The vector Ψ satisfies

$$U(z,n)\Psi^-(P,n,n_0) = \Psi(P,n,n_0), \tag{3.21}$$

$$V_p(z,n)\Psi^-(P,n,n_0) = -(i/2)c_{0,+}z^{-p}y\Psi^-(P,n,n_0), \tag{3.22}$$

$$\psi_2(P,n,n_0) = \phi(P,n)\psi_1(P,n,n_0), \tag{3.23}$$

$$\psi_1(P,n,n_0)\psi_1(P^*,n,n_0) = z^{n-n_0}\frac{F_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0), \tag{3.24}$$

$$\psi_2(P,n,n_0)\psi_2(P^*,n,n_0) = z^{n-n_0}\frac{H_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0), \tag{3.25}$$

$$\begin{aligned} &\psi_1(P,n,n_0)\psi_2(P^*,n,n_0) + \psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \\ &\quad = 2z^{n-n_0}\frac{G_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0), \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\psi_1(P,n,n_0)\psi_2(P^*,n,n_0) - \psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \\ &\quad = -c_{0,+}z^{n-n_0-p}\frac{y}{F_p(z,n_0)}\Gamma(n,n_0), \end{aligned} \tag{3.27}$$

where we used the abbreviation

$$\Gamma(n,n_0) = \begin{cases} \prod_{m=n_0+1}^{n} \gamma(m) & n \geq n_0+1, \\ 1 & n = n_0, \\ \prod_{m=n+1}^{n_0} \gamma(m)^{-1} & n \leq n_0 - 1. \end{cases} \tag{3.28}$$

Combining the Laurent polynomial recursion approach of Section 2 with (3.4) and (3.5) readily yields trace formulas for $f_{\ell,\pm}$ and $h_{\ell,\pm}$ in terms of symmetric functions of the zeros μ_j and ν_k of F_p and H_p, respectively. For simplicity we just record the simplest cases.

Lemma 3.2 ([38])**.** *Suppose that α, β satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.44)*. Then,*

$$\frac{\alpha}{\alpha^+} = \prod_{j=1}^{2p-1} \mu_j \bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{-1/2}, \tag{3.29}$$

$$\frac{\beta^+}{\beta} = \prod_{j=1}^{2p-1} \nu_j \bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{-1/2}, \tag{3.30}$$

$$\sum_{j=1}^{2p-1} \mu_j = \alpha^+\beta - \gamma^+\frac{\alpha^{++}}{\alpha^+} - \frac{c_{1,+}}{c_{0,+}}, \tag{3.31}$$

$$\sum_{j=1}^{2p-1} \nu_j = \alpha^+\beta - \gamma\frac{\beta^{-}}{\beta} - \frac{c_{1,+}}{c_{0,+}}. \tag{3.32}$$

Next we turn to asymptotic properties of ϕ and Ψ in a neighborhood of $P_{\infty_\pm}$ and $P_{0,\pm}$.

Lemma 3.3 ([38])**.** *Suppose that α, β satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.44)*. Moreover, let $P = (z,y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n,n_0) \in \mathbb{Z}^2$. Then ϕ has the asymptotic behavior*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \beta + \beta^-\gamma\zeta + O(\zeta^2), & P \to P_{\infty_+}, \\ -(\alpha^+)^{-1}\zeta^{-1} + (\alpha^+)^{-2}\alpha^{++}\gamma^+ + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.33}$$

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \alpha^{-1} - \alpha^{-2}\alpha^-\gamma\zeta + O(\zeta^2), & P \to P_{0,+}, \\ -\beta^+\zeta - \beta^{++}\gamma^+\zeta^2 + O(\zeta^3), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.34}$$

The components of the Baker–Akhiezer vector Ψ have the asymptotic behavior

$$\psi_1(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{n_0-n}(1+O(\zeta)), & P \to P_{\infty_+}, \\ \frac{\alpha^+(n)}{\alpha^+(n_0)}\Gamma(n,n_0) + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.35}$$

$$\psi_1(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \frac{\alpha(n)}{\alpha(n_0)} + O(\zeta), & P \to P_{0,+}, \\ \zeta^{n-n_0}\Gamma(n,n_0)(1+O(\zeta)), & P \to P_{0,-}, \end{cases} \quad \zeta = z, \tag{3.36}$$

$$\psi_2(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \beta(n)\zeta^{n_0-n}(1+O(\zeta)), & P \to P_{\infty_+}, \\ -\frac{1}{\alpha^+(n_0)}\Gamma(n,n_0)\zeta^{-1}(1+O(\zeta)), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.37}$$

$$\psi_2(P,n,n_0) \underset{\zeta\to 0}{=} \begin{cases} \frac{1}{\alpha(n_0)} + O(\zeta), & P \to P_{0,+}, \\ -\beta^+(n)\Gamma(n,n_0)\zeta^{n+1-n_0}(1+O(\zeta)), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.38}$$

The divisors (ψ_j) of ψ_j, $j=1,2$, are given by

$$(\psi_1(\,\cdot\,,n,n_0)) = \mathcal{D}_{\underline{\hat{\mu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n-n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}), \tag{3.39}$$

$$(\psi_2(\,\cdot\,,n,n_0)) = \mathcal{D}_{\underline{\hat{\nu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n-n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}) + \mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_-}}. \tag{3.40}$$

Since nonspecial divisors play a fundamental role in this section and the next, we now take a closer look at them.

Lemma 3.4 ([38])**.** *Suppose that α,β satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.44)*. Moreover, assume* (3.2) *and* (3.3) *and let $n\in\mathbb{Z}$. Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_{2p-1}\}$, and $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\hat{\nu}_1,\dots,\hat{\nu}_{2p-1}\}$, be the pole and zero divisors of degree $2p-1$, respectively, associated with α, β, and ϕ defined according to* (3.8) *and* (3.9)*, that is,*

$$\begin{aligned} \hat{\mu}_j(n) &= (\mu_j(n), (2/c_{0,+})\mu_j(n)^p G_p(\mu_j(n),n)), \quad j=1,\dots,2p-1,\ n\in\mathbb{Z}, \\ \hat{\nu}_j(n) &= (\nu_j(n), -(2/c_{0,+})\nu_j(n)^p G_p(\nu_j(n),n)), \quad j=1,\dots,2p-1,\ n\in\mathbb{Z}. \end{aligned} \tag{3.41}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial for all $n\in\mathbb{Z}$.

Next, we shall provide an explicit representation of ϕ, Ψ, α, and β in terms of the Riemann theta function associated with $\mathcal{K}_{2p-1}$. We freely employ the notation established in Appendix A. In order to avoid the trivial case $p=0$ we assume $p\in\mathbb{N}$ for the remainder of this argument.

Let θ denote the Riemann theta function associated with $\mathcal{K}_{2p-1}$ and introduce a fixed homology basis $\{a_j,b_j\}_{j=1,\dots,2p-1}$ on $\mathcal{K}_{2p-1}$. Choosing as a convenient fixed base point one of the branch points, $Q_0=(E_{m_0},0)$, the Abel maps $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ are defined by (A.20) and (A.21) and the Riemann vector $\underline{\Xi}_{Q_0}$ is given by (A.31). Let $\omega^{(3)}_{P_+,P_-}$ be the normal differential of the third kind holomorphic on $\mathcal{K}_{2p-1}\backslash\{P_+,P_-\}$ with simple poles at $P_\pm$ and residues ± 1, respectively. In particular, one obtains for $\omega^{(3)}_{P_{0,-},P_{\infty_\pm}}$,

$$\omega^{(3)}_{P_{0,-},P_{\infty_\pm}} = \bigg(\frac{y+y_{0,-}}{z} \mp \prod_{j=1}^{2p-1}(z-\lambda_{\pm,j})\bigg)\frac{dz}{2y}, \quad P_{0,-}=(0,y_{0,-}), \tag{3.42}$$

where the constants $\{\lambda_{\pm,j}\}_{j=1}^{2p-1}\subset\mathbb{C}$ are uniquely determined by employing the normalization

$$\int_{a_j}\omega^{(3)}_{P_{0,-},P_{\infty_\pm}} = 0, \quad j=1,\dots,2p-1. \tag{3.43}$$

The explicit formula (3.42) then implies the following asymptotic expansions (using the local coordinate $\zeta=z$ near $P_{0,\pm}$ and $\zeta=1/z$ near $P_{\infty_\pm}$),

$$\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ \ln(\zeta) \end{Bmatrix} + \omega_0^{0,\pm}(P_{0,-},P_{\infty_-}) + O(\zeta) \text{ as } P\to P_{0,\pm}, \tag{3.44}$$

$$\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ -\ln(\zeta) \end{Bmatrix} + \omega_0^{\infty_\pm}(P_{0,-},P_{\infty_-}) + O(\zeta) \text{ as } P\to P_{\infty_\pm}, \tag{3.45}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_+}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ \ln(\zeta) \end{Bmatrix} + \omega_0^{0,\pm}(P_{0,-},P_{\infty_+}) + O(\zeta) \text{ as } P \to P_{0,\pm}, \tag{3.46}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_+}} \underset{\zeta\to 0}{=} \begin{Bmatrix} -\ln(\zeta) \\ 0 \end{Bmatrix} + \omega_0^{\infty_\pm}(P_{0,-},P_{\infty_+}) + O(\zeta) \text{ as } P \to P_{\infty_\pm}. \tag{3.47}$$

Lemma 3.5. *With $\omega_0^{\infty_\sigma}(P_{0,-},P_{\infty_\pm})$ and $\omega_0^{0,\sigma'}(P_{0,-},P_{\infty_\pm})$, $\sigma,\sigma'\in\{+,-\}$, defined as in* (3.44)–(3.47) *one has*

$$\begin{aligned} &\exp\big(\omega_0^{0,-}(P_{0,-},P_{\infty_\pm}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_\pm}) \\ &\quad - \omega_0^{\infty_-}(P_{0,-},P_{\infty_\pm}) + \omega_0^{0,+}(P_{0,-},P_{\infty_\pm})\big) = 1. \end{aligned} \tag{3.48}$$

Proof. Pick $Q_{1,\pm} = (z_1,\pm y_1) \in \mathcal{K}_{2p-1}\setminus\{P_{\infty_\pm}\}$ in a neighborhood of $P_{\infty_\pm}$ and $Q_{2,\pm} = (z_2,\pm y_2) \in \mathcal{K}_{2p-1}\setminus\{P_{0,\pm}\}$ in a neighborhood of $P_{0,\pm}$. Without loss of generality one may assume that P_{∞_+} and $P_{0,+}$ lie on the same sheet. Then by (3.42),

$$\begin{aligned} &\int_{Q_0}^{Q_{2,-}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} - \int_{Q_0}^{Q_{1,+}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} - \int_{Q_0}^{Q_{1,-}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} + \int_{Q_0}^{Q_{2,+}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \\ &\quad = \int_{Q_0}^{Q_{2,+}} \frac{dz}{z} - \int_{Q_0}^{Q_{1,+}} \frac{dz}{z} = \ln(z_2) - \ln(z_1) + 2\pi i k, \end{aligned} \tag{3.49}$$

for some $k\in\mathbb{Z}$. On the other hand, by (3.44)–(3.47) one obtains

$$\begin{aligned} &\int_{Q_0}^{Q_{2,-}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} - \int_{Q_0}^{Q_{1,+}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} - \int_{Q_0}^{Q_{1,-}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} + \int_{Q_0}^{Q_{2,+}} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \\ &\quad = \ln(z_2) + \ln(1/z_1) + \omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-}) \\ &\qquad - \omega_0^{\infty_-}(P_{0,-},P_{\infty_-}) + \omega_0^{0,+}(P_{0,-},P_{\infty_-}) + O(z_2) + O(1/z_1), \end{aligned} \tag{3.50}$$

and hence the part of (3.48) concerning $\omega^{(3)}_{P_{0,-},P_{\infty_-}}$ follows. The corresponding result for $\omega^{(3)}_{P_{0,-},P_{\infty_+}}$ is proved analogously. ☐

In the following it will be convenient to use the abbreviation

$$\begin{aligned} &\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ &P\in\mathcal{K}_{2p-1},\ \underline{Q} = \{Q_1,\dots,Q_{2p-1}\} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}). \end{aligned} \tag{3.51}$$

We note that $\underline{z}(\,\cdot\,,\underline{Q})$ is independent of the choice of base point Q_0.

For later use we state the following result.

Lemma 3.6. *The following relations hold:*

$$\underline{z}(P_{\infty_+},\underline{\hat{\mu}}^+) = \underline{z}(P_{\infty_-},\underline{\hat{\nu}}) = \underline{z}(P_{0,-},\underline{\hat{\mu}}) = \underline{z}(P_{0,+},\underline{\hat{\nu}}^+), \tag{3.52}$$

$$\underline{z}(P_{\infty_+},\underline{\hat{\nu}}^+) = \underline{z}(P_{0,-},\underline{\hat{\nu}}), \quad \underline{z}(P_{0,+},\underline{\hat{\mu}}^+) = \underline{z}(P_{\infty_-},\underline{\hat{\mu}}). \tag{3.53}$$

Proof. We indicate the proof of some of the relations to be used in (3.71) and (3.72). Let $\underline{\hat{\lambda}}$ denote either $\underline{\hat{\mu}}$ or $\underline{\hat{\nu}}$. Then,

$$\begin{aligned} \underline{z}(P_{0,+},\underline{\hat{\lambda}}^+) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+}) \\ &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}) \\ &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) \end{aligned}$$

$$= \underline{z}(P_{\infty_-}, \underline{\hat{\lambda}}), \tag{3.54}$$

$$\begin{aligned}
\underline{z}(P_{\infty_+}, \underline{\hat{\lambda}}^+) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+}) \\
&= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}) \\
&= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) \\
&= \underline{z}(P_{0,-}, \underline{\hat{\lambda}}), \text{ etc.}
\end{aligned} \tag{3.55}$$

Here we used $\underline{A}_{Q_0}(P^*) = -\underline{A}_{Q_0}(P)$, $P \in \mathcal{K}_{2p-1}$, since Q_0 is a branch point of $\mathcal{K}_{2p-1}$, and $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+})$. The latter equality immediately follows from (3.39) in the case $\underline{\hat{\lambda}} = \underline{\hat{\mu}}$ and from combining (3.13) and (3.40) in the case $\underline{\hat{\lambda}} = \underline{\hat{\nu}}$. □

Given these preparations, the theta function representations of ϕ, ψ_1, ψ_2, α, and β then read as follows.

Theorem 3.7. *Suppose that α, β satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.44). *Moreover, assume* (3.2) *and* (3.3) *and let $P \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ and $(n, n_0) \in \mathbb{Z}^2$. Then for each $n \in \mathbb{Z}$, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial. Moreover,*

$$\phi(P, n) = C(n) \frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n)))} \exp\left(\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-}, P_{\infty_-}} \right), \tag{3.56}$$

$$\psi_1(P, n, n_0) = C(n, n_0) \frac{\theta(\underline{z}(P, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n_0)))} \exp\left((n - n_0) \int_{Q_0}^{P} \omega^{(3)}_{P_{0,-}, P_{\infty_+}} \right), \tag{3.57}$$

$$\begin{aligned}
\psi_2(P, n, n_0) &= C(n) C(n, n_0) \frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n_0)))} \\
&\quad \times \exp\left(\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-}, P_{\infty_-}} + (n - n_0) \int_{Q_0}^{P} \omega^{(3)}_{P_{0,-}, P_{\infty_+}} \right),
\end{aligned} \tag{3.58}$$

where

$$\begin{aligned}
C(n) &= (-1)^{n-n_0} \exp\big((n - n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-})) \big) \\
&\quad \times \frac{1}{\alpha(n_0)} \exp\big(-\omega_0^{0,+}(P_{0,-}, P_{\infty_-}) \big) \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n_0)))},
\end{aligned} \tag{3.59}$$

$$C(n, n_0) = \exp\big(-(n - n_0) \omega_0^{\infty_+}(P_{0,-}, P_{\infty_+}) \big) \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n)))}. \tag{3.60}$$

The Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n - n_0) \underline{A}_{P_{0,-}}(P_{\infty_+}), \tag{3.61}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) + (n - n_0) \underline{A}_{P_{0,-}}(P_{\infty_+}). \tag{3.62}$$

Finally, α, β are of the form

$$\begin{aligned}
\alpha(n) &= \alpha(n_0)(-1)^{n-n_0} \exp\big(-(n - n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-})) \big) \\
&\quad \times \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n_0))) \theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0))) \theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n)))},
\end{aligned} \tag{3.63}$$

$$\beta(n) = \beta(n_0)(-1)^{n-n_0} \exp\big((n - n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_-})) \big)$$

$$
\times \frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))}, \tag{3.64}
$$

$$
\begin{aligned}
\alpha(n)\beta(n) &= \exp\big(\omega_0^{\infty_+}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big) \\
&\quad \times \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))},
\end{aligned} \tag{3.65}
$$

$$
\begin{aligned}
\Gamma(n,n_0) &= \exp\big((n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_+}))\big) \\
&\quad \times \frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))}.
\end{aligned} \tag{3.66}
$$

Proof. Applying Abel's theorem (cf. Theorem A.1, (A.29)) to (3.13) yields

$$
\begin{aligned}
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) - \underline{A}_{Q_0}(P_{0,-}) + \underline{A}_{Q_0}(P_{\infty_-}) \\
&= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) + \underline{A}_{P_{0,-}}(P_{\infty_-})
\end{aligned} \tag{3.67}
$$

and applying it to (3.39), (3.40) results in

$$
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \tag{3.68}
$$

$$
\begin{aligned}
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + \underline{A}_{P_{0,-}}(P_{\infty_-}) \\
&= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}).
\end{aligned} \tag{3.69}
$$

By Lemma 3.4, $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ are nonspecial. By equation (3.13) and Theorem A.3, $\phi(P,n)\exp\big(-\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\big)$ must be of the type

$$
\phi(P,n)\exp\bigg(-\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\bigg) = C(n)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))} \tag{3.70}
$$

for some constant $C(n)$. A comparison of (3.70) and the asymptotic relations (3.33) then yields, with the help of (3.44), (3.45) and (3.52), (3.53), the following expressions for α and β:

$$
\begin{aligned}
(\alpha^+)^{-1} &= C^+ e^{\omega_0^{0,+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}^+))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}^+))} \\
&= C^+ e^{\omega_0^{0,+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}))} \\
&= -C e^{\omega_0^{\infty_-}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}))}.
\end{aligned} \tag{3.71}
$$

Similarly one obtains

$$
\begin{aligned}
\beta^+ &= C^+ e^{\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}^+))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}^+))} \\
&= C^+ e^{\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}))} \\
&= -C e^{\omega_0^{0,-}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}))}.
\end{aligned} \tag{3.72}
$$

Here we used (3.61) and (3.62), more precisely,

$$
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}), \quad \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}). \tag{3.73}
$$

Thus, one concludes

$$C(n+1) = -\exp\big[\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-})\big]C(n), \quad n\in\mathbb{Z}, \tag{3.74}$$

and

$$C(n+1) = -\exp\big[\omega_0^{\infty_-}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big]C(n), \quad n\in\mathbb{Z}, \tag{3.75}$$

which is consistent with (3.48). The first-order difference equation (3.74) then implies

$$C(n) = (-1)^{(n-n_0)}\exp\big[(n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-}))\big]C(n_0), \quad n,n_0\in\mathbb{Z}. \tag{3.76}$$

Thus one infers (3.63) and (3.64). Moreover, (3.76) and taking $n=n_0$ in the first line in (3.71) yield (3.59). Dividing the first line in (3.72) by the first line in (3.71) then proves (3.65).

By (3.39) and Theorem A.3, $\psi_1(P,n,n_0)$ must be of the type (3.57). A comparison of (3.15), (3.33), and (3.57) as $P\to P_{\infty_+}$ (with local coordinate $\zeta=1/z$) then yields

$$\psi_1(P,n,n_0) \underset{\zeta\to 0}{=} \zeta^{n_0-n}(1+O(\zeta)) \tag{3.77}$$

and

$$\begin{aligned}\psi_1(P,n,n_0) \underset{\zeta\to 0}{=}\, & C(n,n_0)\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0)))} \\ & \times\exp\big[(n-n_0)\omega_0^{\infty_+}(P_{0,-},P_{\infty_+})\big]\zeta^{n_0-n}(1+O(\zeta))\end{aligned} \tag{3.78}$$

proving (3.60). Equation (3.58) is clear from (3.23), (3.56), and (3.57).

Finally, a comparison of (3.36) and (3.57) as $P\to P_{0,-}$ (with local coordinate $\zeta=z$) yields

$$\psi_1(P,n,n_0) \underset{\zeta\to 0}{=} \Gamma(n,n_0)\zeta^{n-n_0}(1+O(\zeta)) \tag{3.79}$$

$$\begin{aligned}\underset{\zeta\to 0}{=}\, & C(n,n_0)\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n_0)))}\exp\big((n-n_0)\omega_0^{0,-}(P_{0,-},P_{\infty_+})\big) \\ & \times\zeta^{n-n_0}(1+O(\zeta))\end{aligned} \tag{3.80}$$

and hence

$$\begin{aligned}\Gamma(n,n_0) &= C(n,n_0)\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n_0)))}\exp\big((n-n_0)\omega_0^{0,-}(P_{0,-},P_{\infty_+})\big) \\ &= \exp\big((n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_+}))\big) \\ &\quad\times\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))}\end{aligned} \tag{3.81}$$

using (3.60). ☐

The algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data, that is, the construction of α and β by starting from a set of initial data (nonspecial divisors) of full measure, will be presented in [38].

4. The Time-Dependent Ablowitz–Ladik Formalism

In this section we extend the algebro-geometric analysis of Section 3 to the time-dependent Ablowitz–Ladik hierarchy. For proofs of the elementary results of the time-dependent formalism we refer to [37], [38].

For most of this section we assume the following hypothesis.

Hypothesis 4.1. (i) *Suppose that* α, β *satisfy*

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,),\ \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{4.1}$$

(ii) *Assume that the hyperelliptic curve* $\mathcal{K}_{2p-1}$ *satisfies* (3.2) *and* (3.3).

The basic problem in the analysis of algebro-geometric solutions of the Ablowitz–Ladik hierarchy consists of solving the time-dependent rth Ablowitz–Ladik flow with initial data a stationary solution of the pth equation in the hierarchy. More precisely, given $p \in \mathbb{N}_0$ we consider a solution $\alpha^{(0)}$, $\beta^{(0)}$ of the pth stationary Ablowitz–Ladik equation s-AL$_p(\alpha^{(0)}, \beta^{(0)}) = 0$, associated with the hyperelliptic curve $\mathcal{K}_{2p-1}$ and a corresponding set of summation constants $\{c_{\ell,\pm}\}_{\ell=1,\dots,p} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution α, β of the rth Ablowitz–Ladik flow AL$_r(\alpha,\beta) = 0$ with $\alpha(t_{0,r}) = \alpha^{(0)}$, $\beta(t_{0,r}) = \beta^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the summation constants in the definitions of the stationary and the time-dependent Ablowitz–Ladik equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde V_r$, $\widetilde F_r$, $\widetilde G_r$, $\widetilde H_r$, $\tilde f_{s,\pm}$, $\tilde g_{s,\pm}$, $\tilde h_{s,\pm}$, $\tilde c_{s,\pm}$, in order to distinguish them from V_p, F_p, G_p, H_p, $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, $c_{\ell,\pm}$, in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth Ablowitz–Ladik flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are looking for solutions α, β of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} &\widetilde{\mathrm{AL}}_r(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_r} - \alpha(\tilde g_{r,+} + \tilde g_{r,-}^-) + \tilde f_{r-1,+} - \tilde f_{r-1,-}^- \\ -i\beta_{t_r} + \beta(\tilde g_{r,+}^- + \tilde g_{r,-}) - \tilde h_{r-1,-} + \tilde h_{r-1,+}^- \end{pmatrix} = 0, \\ &(\alpha,\beta)\big|_{t=t_{0,r}} = \big(\alpha^{(0)}, \beta^{(0)}\big), \end{aligned} \tag{4.2}$$

$$\text{s-AL}_p\left(\alpha^{(0)},\beta^{(0)}\right) = \begin{pmatrix} -\alpha^{(0)}(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ \beta^{(0)}(g_{p,+}^- + g_{p,-}) - h_{p-1,-} + h_{p-1,+}^- \end{pmatrix} = 0 \tag{4.3}$$

for some $t_{0,r} \in \mathbb{R}$, where $\alpha = \alpha(n,t_r)$, $\beta = \beta(n,t_r)$ satisfy (4.1) and a fixed curve $\mathcal{K}_{2p-1}$ is associated with the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ in (4.3). In terms of the zero-curvature formulation this amounts to solving

$$U_{t_r}(z,t_r) + U(z,t_r)\widetilde V_r(z,t_r) - \widetilde V_r^+(z,t_r)U(z,t_r) = 0, \tag{4.4}$$

$$U(z,t_{0,r})V_p(z,t_{0,r}) - V_p^+(z,t_{0,r})U(z,t_{0,r}) = 0. \tag{4.5}$$

One can show (cf. [38]) that the stationary Ablowitz–Ladik system (4.5) is actually satisfied for all times $t_r \in \mathbb{R}$. Thus, we impose

$$U_{t_r}(z,t_r) + U(z,t_r)\widetilde V_r(z,t_r) - \widetilde V_r^+(z,t_r)U(z,t_r) = 0, \tag{4.6}$$

$$U(z,t_r)V_p(z,t_r) - V_p^+(z,t_r)U(z,t_r) = 0, \tag{4.7}$$

instead of (4.4) and (4.5). For further reference, we recall the relevant quantities here (cf. (2.5), (2.31), (2.37)–(2.39), (2.50)):

$$\begin{aligned} &U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \\ &V_p(z) = i\begin{pmatrix} G_p^-(z) & -F_p^-(z) \\ H_p^-(z) & -G_p^-(z) \end{pmatrix}, \quad \widetilde V_r(z) = i\begin{pmatrix} \widetilde G_r^-(z) & -\widetilde F_r^-(z) \\ \widetilde H_r^-(z) & -\widetilde K_r^-(z) \end{pmatrix}, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} F_p(z) &= \sum_{\ell=1}^p f_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+} z^\ell = -c_{0,+}\alpha^+ z^{-p} \prod_{j=1}^{2p-1} (z-\mu_j), \\ G_p(z) &= \sum_{\ell=1}^p g_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+} z^\ell, \\ H_p(z) &= \sum_{\ell=0}^{p-1} h_{p-1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+} z^\ell = c_{0,+}\beta z^{-p+1} \prod_{j=1}^{2p-1} (z-\nu_j), \\ \widetilde F_r(z) &= \sum_{s=1}^r \tilde f_{r-s,-} z^{-s} + \sum_{s=0}^{r-1} \tilde f_{r-1-s,+} z^s, \\ \widetilde G_r(z) &= \sum_{s=1}^r \tilde g_{r-s,-} z^{-s} + \sum_{s=0}^{r} \tilde g_{r-s,+} z^s, \\ \widetilde H_r(z) &= \sum_{s=0}^{r-1} \tilde h_{r-1-s,-} z^{-s} + \sum_{s=1}^{r} \tilde h_{r-s,+} z^s, \\ \widetilde K_r(z) &= \sum_{s=0}^r \tilde g_{r-s,-} z^{-s} + \sum_{s=1}^{r} \tilde g_{r-s,+} z^s = \widetilde G_r(z) + \tilde g_{r,-} - \tilde g_{r,+} \end{aligned} \tag{4.9}$$

for fixed $p, r \in \mathbb{N}_0$. Here $f_{\ell,\pm}$, $\tilde f_{s,\pm}$, $g_{\ell,\pm}$, $\tilde g_{s,\pm}$, $h_{\ell,\pm}$, and $\tilde h_{s,\pm}$ are defined as in (2.6)–(2.13) with appropriate sets of summation constants $c_{\ell,\pm}$, $\ell \in \mathbb{N}$, and $\tilde c_{k,\pm}$, $k \in \mathbb{N}$. Explicitly, (4.6) and (4.7) are equivalent to (cf. (2.33)–(2.36), (2.57)–(2.60)),

$$\alpha_{t_r} = i\big(z\widetilde F_r^- + \alpha(\widetilde G_r + \tilde K_r^-) - \widetilde F_r\big), \tag{4.10}$$
$$\beta_{t_r} = -i\big(\beta(\widetilde G_r^- + \tilde K_r) - \widetilde H_r + z^{-1}\widetilde H_r^-\big), \tag{4.11}$$
$$0 = z(\widetilde G_r^- - \widetilde G_r) + z\beta\widetilde F_r + \alpha\widetilde H_r^-, \tag{4.12}$$
$$0 = z\beta\widetilde F_r^- + \alpha\widetilde H_r + \tilde K_r^- - \tilde K_r, \tag{4.13}$$
$$0 = z(G_p^- - G_p) + z\beta F_p + \alpha H_p^-, \tag{4.14}$$
$$0 = z\beta F_p^- + \alpha H_p - G_p + G_p^-, \tag{4.15}$$
$$0 = -F_p + zF_p^- + \alpha(G_p + G_p^-), \tag{4.16}$$
$$0 = z\beta(G_p + G_p^-) - zH_p + H_p^-, \tag{4.17}$$

respectively. In particular, (2.51) holds in the present t_r-dependent setting, that is,

$$G_p^2 - F_p H_p = R_{2p}. \tag{4.18}$$

As in the stationary context (3.8), (3.9) we introduce

$$\begin{aligned}\hat{\mu}_j(n,t_r) = (\mu_j(n,t_r), (2/c_{0,+})\mu_j(n,t_r)^p G_p(\mu_j(n,t_r),n,t_r)) \in \mathcal{K}_{2p-1},\\ j=1,\dots,2p-1,\ (n,t_r)\in\mathbb{Z}\times\mathbb{R},\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\hat{\nu}_j(n,t_r) = (\nu_j(n,t_r), -(2/c_{0,+})\nu_j(n,t_r)^p G_p(\nu_j(n,t_r),n,t_r)) \in \mathcal{K}_{2p-1},\\ j=1,\dots,2p-1,\ (n,t_r)\in\mathbb{Z}\times\mathbb{R},\end{aligned} \tag{4.20}$$

and note that the regularity assumptions (4.1) on α,β imply continuity of μ_j and ν_k with respect to $t_r\in\mathbb{R}$ (away from collisions of these zeros, μ_j and ν_k are of course C^∞).

In analogy to (3.11), (3.12), one defines the following meromorphic function $\phi(\,\cdot\,,n,t_r)$ on $\mathcal{K}_{2p-1}$,

$$\phi(P,n,t_r) = \frac{(c_{0,+}/2)z^{-p}y + G_p(z,n,t_r)}{F_p(z,n,t_r)} \tag{4.21}$$

$$= \frac{-H_p(z,n,t_r)}{(c_{0,+}/2)z^{-p}y - G_p(z,n,t_r)}, \tag{4.22}$$

$$P=(z,y)\in\mathcal{K}_{2p-1},\ (n,t_r)\in\mathbb{Z}\times\mathbb{R},$$

with divisor $(\phi(\,\cdot\,,n,t_r))$ of $\phi(\,\cdot\,,n,t_r)$ given by

$$(\phi(\,\cdot\,,n,t_r)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n,t_r)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_r)}. \tag{4.23}$$

The time-dependent Baker–Akhiezer vector is then defined in terms of ϕ by

$$\Psi(P,n,n_0,t_r,t_{0,r}) = \begin{pmatrix}\psi_1(P,n,n_0,t_r,t_{0,r})\\ \psi_2(P,n,n_0,t_r,t_{0,r})\end{pmatrix}, \tag{4.24}$$

$$\psi_1(P,n,n_0,t_r,t_{0,r}) = \exp\Big(i\int_{t_{0,r}}^{t_r} ds\big(\widetilde{G}_r(z,n_0,s) - \widetilde{F}_r(z,n_0,s)\phi(P,n_0,s)\big)\Big) \tag{4.25}$$

$$\times\begin{cases}\prod_{m=n_0+1}^{n}(z+\alpha(m,t_r)\phi^-(P,m,t_r)), & n\geq n_0+1,\\ 1, & n=n_0,\\ \prod_{m=n+1}^{n_0}(z+\alpha(m,t_r)\phi^-(P,m,t_r))^{-1}, & n\leq n_0-1,\end{cases}$$

$$\psi_2(P,n,n_0,t_r,t_{0,r}) = \exp\Big(i\int_{t_{0,r}}^{t_r} ds\big(\widetilde{G}_r(z,n_0,s) - \widetilde{F}_r(z,n_0,s)\phi(P,n_0,s)\big)\Big) \tag{4.26}$$

$$\times\,\phi(P,n_0,t_r)\begin{cases}\prod_{m=n_0+1}^{n}\Big(\frac{z\beta(m,t_r)}{\phi^-(P,m,t_r)}+1\Big), & n\geq n_0+1,\\ 1, & n=n_0,\\ \prod_{m=n+1}^{n_0}\Big(\frac{z\beta(m,t_r)}{\phi^-(P,m,t_r)}+1\Big)^{-1}, & n\leq n_0-1,\end{cases}$$

$$P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,t_r)\in\mathbb{Z}\times\mathbb{R}.$$

One observes that

$$\begin{aligned}&\psi_1(P,n,n_0,t_r,\tilde{t}_r) = \psi_1(P,n_0,n_0,t_r,\tilde{t}_r)\psi_1(P,n,n_0,t_r,t_r),\\ &P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,n_0,t_r,\tilde{t}_r)\in\mathbb{Z}^2\times\mathbb{R}^2.\end{aligned} \tag{4.27}$$

The following lemma records basic properties of ϕ, ψ, and Ψ in analogy to the stationary case discussed in Lemma 3.1.

Lemma 4.2 ([38]). *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. In addition, let* $P = (z, y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}\}$, $(n, n_0, t_r, t_{0,r}) \in \mathbb{Z}^2 \times \mathbb{R}^2$, *and* $r \in \mathbb{N}_0$. *Then* ϕ *satisfies*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{4.28}$$

$$\phi_{t_r}(P) = i\widetilde{F}_r\phi^2(P) - i(\widetilde{G}_r(z) + \widetilde{K}_r(z))\phi(P) + i\widetilde{H}_r(z), \tag{4.29}$$

$$\phi(P)\phi(P^*) = \frac{H_p(z)}{F_p(z)}, \tag{4.30}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_p(z)}{F_p(z)}, \tag{4.31}$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p}\frac{y(P)}{F_p(z)}. \tag{4.32}$$

Moreover, assuming $P = (z, y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, *then* Ψ *satisfies*

$$\psi_2(P, n, n_0, t_r, t_{0,r}) = \phi(P, n, t_r)\psi_1(P, n, n_0, t_r, t_{0,r}), \tag{4.33}$$

$$U(z, n, t_r)\Psi^-(P, n, n_0, t_r, t_{0,r}) = \Psi(P, n, n_0, t_r, t_{0,r}), \tag{4.34}$$

$$V_p(z, n, t_r)\Psi^-(P, n, n_0, t_r, t_{0,r}) = -(i/2)c_{0,+}z^{-p}y\Psi^-(P, n, n_0, t_r, t_{0,r}), \tag{4.35}$$

$$\Psi_{t_r}(P, n, n_0, t_r, t_{0,r}) = \widetilde{V}_r^+(n, t_r)\Psi(P, n, n_0, t_r, t_{0,r}), \tag{4.36}$$

$$\psi_1(P, n, n_0, t_r, t_{0,r})\psi_1(P^*, n, n_0, t_r, t_{0,r}) = z^{n-n_0}\frac{F_p(z, n, t_r)}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r), \tag{4.37}$$

$$\psi_2(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) = z^{n-n_0}\frac{H_p(z, n, t_r)}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r), \tag{4.38}$$

$$\begin{aligned}&\psi_1(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) + \psi_1(P^*, n, n_0, t_r, t_{0,r})\psi_2(P, n, n_0, t_r, t_{0,r})\\ &\quad = 2z^{n-n_0}\frac{G_p(z, n, t_r)}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r),\end{aligned} \tag{4.39}$$

$$\begin{aligned}&\psi_1(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) - \psi_1(P^*, n, n_0, t_r, t_{0,r})\psi_2(P, n, n_0, t_r, t_{0,r})\\ &\quad = -c_{0,+}z^{n-n_0-p}\frac{y}{F_p(z, n_0, t_{0,r})}\Gamma(n, n_0, t_r).\end{aligned} \tag{4.40}$$

In addition, as long as the zeros $\mu_j(n_0, s)$ *of* $F_p(\,\cdot\,, n_0, s)$ *are all simple and distinct from zero for* $s \in \mathcal{I}_\mu$, $\mathcal{I}_\mu \subseteq \mathbb{R}$ *an open interval,* $\Psi(\,\cdot\,, n, n_0, t_r, t_{0,r})$ *is meromorphic on* $\mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ *for* $(n, t_r, t_{0,r}) \in \mathbb{Z} \times \mathcal{I}_\mu^2$.

Next we consider the t_r-dependence of F_p, G_p, and H_p.

Lemma 4.3 ([38]). *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. In addition, let* $(z, n, t_r) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{R}$. *Then,*

$$F_{p,t_r} = -2iG_p\widetilde{F}_r + i(\widetilde{G}_r + \widetilde{K}_r)F_p, \tag{4.41}$$

$$G_{p,t_r} = iF_p\widetilde{H}_r - iH_p\widetilde{F}_r, \tag{4.42}$$

$$H_{p,t_r} = 2iG_p\widetilde{H}_r - i(\widetilde{G}_r + \widetilde{K}_r)H_p. \tag{4.43}$$

In particular, (4.41)–(4.43) *are equivalent to*

$$V_{p,t_r} = \big[\widetilde{V}_r, V_p\big]. \tag{4.44}$$

Next we turn to the Dubrovin equations for the time variation of the zeros μ_j of F_p and ν_j of H_p governed by the AL_r flow (indexed by the parameter t_r).

Lemma 4.4 ([38]). *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold on* $\mathbb{Z}\times I_\mu$ *with* $I_\mu\subseteq\mathbb{R}$ *an open interval. In addition, assume that the zeros* μ_j, $j=1,\dots,2p-1$, *of* $F_p(\,\cdot\,)$ *remain distinct and nonzero on* $\mathbb{Z}\times I_\mu$. *Then* $\{\hat\mu_j\}_{j=1,\dots,2p-1}$, *defined in* (4.19), *satisfies the following first-order system of differential equations on* $\mathbb{Z}\times I_\mu$,

$$\mu_{j,t_r}=-iy(\hat\mu_j)\widetilde F_r(\mu_j)(\alpha^+)^{-1}\prod_{\substack{k=1\\k\neq j}}^{2p-1}(\mu_j-\mu_k)^{-1},\quad j=1,\dots,2p-1. \tag{4.45}$$

For the zeros ν_j, $j=1,\dots,2p-1$, *of* $H_p(\,\cdot\,)$, *identical statements hold with* μ_j *and* I_μ *replaced by* ν_j *and* I_ν, *etc.* (*with* $I_\nu\subseteq\mathbb{R}$ *an open interval*). *In particular,* $\{\hat\nu_j\}_{j=1,\dots,2p-1}$, *defined in* (4.20), *satisfies the first-order system on* $\mathbb{Z}\times I_\nu$,

$$\nu_{j,t_r}=iy(\hat\nu_j)\widetilde H_r(\nu_j)(\beta\nu_j)^{-1}\prod_{\substack{k=1\\k\neq j}}^{2p-1}(\nu_j-\nu_k)^{-1},\quad j=1,\dots,2p-1. \tag{4.46}$$

When attempting to solve the Dubrovin systems (4.45) and (4.46), they must be augmented with appropriate divisors $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}\in\operatorname{Sym}^{2p-1}\mathcal{K}_{2p-1}$, $t_{0,r}\in I_\mu$, and $\mathcal{D}_{\underline{\hat\nu}(n_0,t_{0,r})}\in\operatorname{Sym}^{2p-1}\mathcal{K}_{2p-1}$, $t_{0,r}\in I_\nu$, as initial conditions.

Next, we turn to the asymptotic expansions of ϕ and Ψ in a neighborhood of $P_{\infty_\pm}$ and $P_{0,\pm}$. Since this is a bit more involved we provide some details.

Lemma 4.5. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. Moreover, let* $P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$, $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$. *Then* ϕ *has the asymptotic behavior*

$$\phi(P)\underset{\zeta\to0}{=}\begin{cases}\beta+\beta^-\gamma\zeta+O(\zeta^2), & P\to P_{\infty_+},\\ -(\alpha^+)^{-1}\zeta^{-1}+(\alpha^+)^{-2}\alpha^{++}\gamma^++O(\zeta), & P\to P_{\infty_-},\end{cases}\quad \zeta=1/z, \tag{4.47}$$

$$\phi(P)\underset{\zeta\to0}{=}\begin{cases}\alpha^{-1}-\alpha^{-2}\alpha^-\gamma\zeta+O(\zeta^2), & P\to P_{0,+},\\ -\beta^+\zeta-\beta^{++}\gamma^+\zeta^2+O(\zeta^3), & P\to P_{0,-},\end{cases}\quad \zeta=z. \tag{4.48}$$

The component ψ_1 *of the Baker–Akhiezer vector* Ψ *has the asymptotic behavior*

$$\begin{aligned}\psi_1(P,n,n_0,t_r,t_{0,r})&\underset{\zeta\to0}{=}\exp\bigg(\pm\frac{i}{2}(t_r-t_{0,r})\sum_{s=0}^r\tilde c_{r-s,+}\zeta^{-s}\bigg)(1+O(\zeta))\\ &\quad\times\begin{cases}\zeta^{n_0-n}, & P\to P_{\infty_+},\\ \Gamma(n,n_0,t_r)\frac{\alpha^+(n,t_r)}{\alpha^+(n_0,t_{0,r})}\\ \times\exp\Big(i\int_{t_{0,r}}^{t_r}ds\big(\tilde g_{r,+}(n_0,s)-\tilde g_{r,-}(n_0,s)\big)\Big), & P\to P_{\infty_-},\end{cases}\quad\zeta=1/z,\end{aligned} \tag{4.49}$$

$$\begin{aligned}\psi_1(P,n,n_0,t_r,t_{0,r})&\underset{\zeta\to0}{=}\exp\bigg(\pm\frac{i}{2}(t_r-t_{0,r})\sum_{s=0}^r\tilde c_{r-s,-}\zeta^{-s}\bigg)(1+O(\zeta))\\ &\quad\times\begin{cases}\frac{\alpha(n,t_r)}{\alpha(n_0,t_{0,r})}, & P\to P_{0,+},\\ \Gamma(n,n_0,t_r)\zeta^{n-n_0}\\ \times\exp\Big(i\int_{t_{0,r}}^{t_r}ds\big(\tilde g_{r,+}(n_0,s)-\tilde g_{r,-}(n_0,s)\big)\Big), & P\to P_{0,-},\end{cases}\quad\zeta=z.\end{aligned} \tag{4.50}$$

The divisors (ψ_j) *of* ψ_j, $j=1,2$, *are given by*

$$(\psi_1(\,\cdot\,,n,n_0,t_r,t_{0,r}))=\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}-\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_r)}+(n-n_0)(\mathcal{D}_{P_{0,-}}-\mathcal{D}_{P_{\infty_+}}), \tag{4.51}$$

$$(\psi_2(\,\cdot\,,n,n_0,t_r,t_{0,r}))=\mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n,t_r)}-\mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n_0,t_{0,r})}+(n-n_0)(\mathcal{D}_{P_{0,-}}-\mathcal{D}_{P_{\infty_+}}). \tag{4.52}$$

Proof. Since by the definition of ϕ in (4.21) the time parameter t_r can be viewed as an additional but fixed parameter, the asymptotic behavior of ϕ remains the same as in Lemma 3.3. Similarly, also the asymptotic behavior of $\psi_1(P,n,n_0,t_r,t_r)$ is derived in an identical fashion to that in Lemma 3.3. This proves (4.49) and (4.50) for $t_{0,r}=t_r$, that is,

$$\psi_1(P,n,n_0,t_r,t_r)\underset{\zeta\to 0}{=}\begin{cases}\zeta^{n_0-n}(1+O(\zeta)), & P\to P_{\infty_+},\\ \Gamma(n,n_0,t_r)\frac{\alpha^+(n,t_r)}{\alpha^+(n_0,t_r)}+O(\zeta), & P\to P_{\infty_-},\end{cases}\qquad \zeta=1/z, \tag{4.53}$$

$$\psi_1(P,n,n_0,t_r,t_r)\underset{\zeta\to 0}{=}\begin{cases}\frac{\alpha(n,t_r)}{\alpha(n_0,t_r)}+O(\zeta), & P\to P_{0,+},\\ \Gamma(n,n_0,t_r)\zeta^{n-n_0}(1+O(\zeta)), & P\to P_{0,-},\end{cases}\qquad \zeta=z. \tag{4.54}$$

It remains to investigate

$$\psi_1(P,n_0,n_0,t_r,t_{0,r})=\exp\bigg(i\int_{t_{0,r}}^{t_r}dt\big(\widetilde{G}_r(z,n_0,t)-\widetilde{F}_r(z,n_0,t)\phi(P,n_0,t)\big)\bigg). \tag{4.55}$$

The asymptotic expansion of the integrand is derived using Theorem B.1. Focusing on the homogeneous coefficients first, one computes as $P\to P_{\infty_\pm}$,

$$\begin{aligned}\widehat{G}_s-\widehat{F}_s\phi&=\widehat{G}_s-\widehat{F}_s\frac{G_p+(c_{0,+}/2)z^{-p}y}{F_p}=\widehat{G}_s-\widehat{F}_s\bigg(\frac{2z^p}{c_{0,+}}\frac{G_p}{y}+1\bigg)\bigg(\frac{2z^p}{c_{0,+}}\frac{F_p}{y}\bigg)^{-1}\\ &\underset{\zeta\to 0}{=}\pm\frac{1}{2}\zeta^{-s}+\frac{\hat{g}_{0,+}\mp\frac{1}{2}}{\hat{f}_{0,+}}(\hat{f}_{s,+}-\hat{f}_{s-1,-})+O(\zeta),\quad P\to P_{\infty_\pm},\ \zeta=1/z.\end{aligned} \tag{4.56}$$

Since

$$\widetilde{F}_r\underset{\zeta\to 0}{=}\sum_{s=0}^{r}\tilde{c}_{r-s,+}\widehat{F}_s+O(\zeta),\quad \widetilde{G}_r\underset{\zeta\to 0}{=}\sum_{s=0}^{r}\tilde{c}_{r-s,+}\widehat{G}_s+O(\zeta), \tag{4.57}$$

one infers from (4.47)

$$\widetilde{G}_r-\widetilde{F}_r\phi\underset{\zeta\to 0}{=}\frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,+}\zeta^{-s}+O(\zeta),\quad P\to P_{\infty_+},\ \zeta=1/z, \tag{4.58}$$

Insertion of (4.58) into (4.55) then proves (4.49) as $P\to P_{\infty_+}$.

As $P\to P_{\infty_-}$, we need one additional term in the asymptotic expansion of $\widetilde{F}_r$, that is, we will use

$$\widetilde{F}_r\underset{\zeta\to 0}{=}\sum_{s=0}^{r}\tilde{c}_{r-s,+}\widehat{F}_s-\sum_{s=0}^{r}(\tilde{c}_{r-s,+}-\tilde{c}_{r-s,-})\hat{f}_{s-1,-}\zeta+O(\zeta^2). \tag{4.59}$$

This then yields

$$\widetilde{G}_r-\widetilde{F}_r\phi\underset{\zeta\to 0}{=}-\frac{1}{2}\sum_{s=0}^{r}\tilde{c}_{r-s,+}\zeta^{-s}-(\alpha^+)^{-1}(\tilde{f}_{r,+}-\tilde{f}_{r-1,-})+O(\zeta). \tag{4.60}$$

Invoking (2.8) and (4.2) one concludes that

$$\tilde{f}_{r-1,-} - \tilde{f}_{r,+} = -i\alpha^+_{t_r} + \alpha^+(\tilde{g}_{r,+} - \tilde{g}_{r,-}) \tag{4.61}$$

and hence

$$\widetilde{G}_r - \widetilde{F}_r\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r} \tilde{c}_{r-s,+}\zeta^{-s} - \frac{i\alpha^+_{t_r}}{\alpha^+} + \tilde{g}_{r,+} - \tilde{g}_{r,-} + O(\zeta). \tag{4.62}$$

Insertion of (4.62) into (4.55) then proves (4.49) as $P \to P_{\infty_-}$.

Using again Theorem B.1 one obtains in the same manner as $P \to P_{0,\pm}$,

$$\widehat{G}_s - \widehat{F}_s\phi \underset{\zeta\to 0}{=} \pm\frac{1}{2}\zeta^{-s} + \hat{g}_{s,+} - \hat{g}_{s,-} - \frac{\hat{g}_{0,-} \pm \frac{1}{2}}{\hat{f}_{0,-}}\big(\hat{f}_{s-1,+} - \hat{f}_{s,-}\big) + O(\zeta). \tag{4.63}$$

Since

$$\widetilde{F}_r \underset{\zeta\to 0}{=} \sum_{s=0}^{r} \tilde{c}_{r-s,-}\big(\widehat{F}_s - \hat{f}_{s-1,+}\big) + \tilde{f}_{r-1,+} + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z, \tag{4.64}$$

$$\widetilde{G}_r \underset{\zeta\to 0}{=} \sum_{s=0}^{r} \tilde{c}_{r-s,-}\big(\widehat{G}_s - \hat{g}_{s,+}\big) + \tilde{g}_{r,+} + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z, \tag{4.65}$$

(4.63)–(4.65) yield

$$\widetilde{G}_r - \widetilde{F}_r\phi \underset{\zeta\to 0}{=} \pm\frac{1}{2}\sum_{s=0}^{r} \tilde{c}_{r-s,-}\zeta^{-s} + \tilde{g}_{r,+} - \tilde{g}_{r,-} - \frac{\hat{g}_{0,-} \pm \frac{1}{2}}{\hat{f}_{0,-}}(\tilde{f}_{r-1,+} - \tilde{f}_{r,-}) + O(\zeta), \tag{4.66}$$

where we again used (4.48), (2.20), and (4.2). As $P \to P_{0,-}$, one thus obtains

$$\widetilde{G}_r - \widetilde{F}_r\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r} \tilde{c}_{r-s,-}\zeta^{-s} + \tilde{g}_{r,+} - \tilde{g}_{r,-}, \quad P \to P_{0,-},\ \zeta = z. \tag{4.67}$$

Insertion of (4.67) into (4.55) then proves (4.50) as $P \to P_{0,-}$.

As $P \to P_{0,+}$, one obtains

$$\begin{aligned}
\widetilde{G}_r - \widetilde{F}_r\phi &\underset{\zeta\to 0}{=} \frac{1}{2}\sum_{s=0}^{r} \tilde{c}_{r-s,-}\zeta^{-s} + \tilde{g}_{r,+} - \tilde{g}_{r,-} - \frac{1}{\alpha}(\tilde{f}_{r-1,+} - \tilde{f}_{r,-}) + O(\zeta), \\
&\underset{\zeta\to 0}{=} \frac{1}{2}\sum_{s=0}^{r} \tilde{c}_{r-s,-}\zeta^{-s} - \frac{i\alpha_{t_r}}{\alpha} + O(\zeta), \quad P \to P_{0,+},\ \zeta = z,
\end{aligned} \tag{4.68}$$

using $\tilde{f}_{r,-} = \tilde{f}^-_{r-1,-} + \alpha(\tilde{g}_{r,-} - \tilde{g}^-_{r,-})$ (cf. (2.12)) and (4.2). Insertion of (4.68) into (4.55) then proves (4.50) as $P \to P_{0,+}$. □

Next, we note that Lemma 3.4 on nonspecial divisors in the stationary context extends to the present time-dependent situation without a change since $t_r \in \mathbb{R}$ just plays the role of a parameter. We thus omit the details.

Finally, we turn to the principal result of this section, the representation of ϕ, Ψ, α, and β in terms of the Riemann theta function associated with $\mathcal{K}_{2p-1}$.

In addition to (3.42)–(3.48), let $\omega^{(2)}_{P_{\infty_\pm},q}$ and $\omega^{(2)}_{P_{0,\pm},q}$ be the normalized differentials of the second kind with a unique pole at $P_{\infty_\pm}$ and $P_{0,\pm}$, respectively, and principal parts

$$\omega^{(2)}_{P_{\infty_\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q} + O(1)\big)d\zeta, \quad P \to P_{\infty_\pm},\ \zeta = 1/z,\ q \in \mathbb{N}_0, \tag{4.69}$$

$$\omega^{(2)}_{P_{0,\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q}+O(1)\big)d\zeta, \quad P\to P_{0,\pm},\ \zeta=z,\ q\in\mathbb{N}_0, \tag{4.70}$$

with vanishing a-periods,

$$\int_{a_j}\omega^{(2)}_{P_{\infty_\pm},q} = \int_{a_j}\omega^{(2)}_{P_{0,\pm},q}=0, \quad j=1,\dots,2p-1. \tag{4.71}$$

Moreover, we define

$$\begin{aligned}\widetilde\Omega^{(2)}_r = \frac{i}{2}\bigg(&\sum_{s=1}^r s\tilde c_{r-s,-}\big(\omega^{(2)}_{P_{0,+},s-1}-\omega^{(2)}_{P_{0,-},s-1}\big)\\ &+\sum_{s=1}^r s\tilde c_{r-s,+}\big(\omega^{(2)}_{P_{\infty_+},s-1}-\omega^{(2)}_{P_{\infty_-},s-1}\big)\bigg),\end{aligned} \tag{4.72}$$

where $\tilde c_{\ell,\pm}$ are the summation constants in $\widetilde F_r$. The corresponding vector of b-periods of $\widetilde\Omega^{(2)}_r/(2\pi i)$ is then denoted by

$$\underline{\widetilde U}^{(2)}_r = \big(\widetilde U^{(2)}_{r,1},\dots,\widetilde U^{(2)}_{r,p}\big), \quad \widetilde U^{(2)}_{r,j}=\frac{1}{2\pi i}\int_{b_j}\widetilde\Omega^{(2)}_r, \quad j=1,\dots,2p-1. \tag{4.73}$$

Finally, we abbreviate

$$\widetilde\Omega^{\infty_\pm}_r = \lim_{P\to P_{\infty_\pm}}\bigg(\int_{Q_0}^P\widetilde\Omega^{(2)}_r \pm\frac{i}{2}\sum_{s=0}^r\tilde c_{r-s,+}\zeta^{-s}\bigg), \tag{4.74}$$

$$\widetilde\Omega^{0,\pm}_r = \lim_{P\to P_{0,\pm}}\bigg(\int_{Q_0}^P\widetilde\Omega^{(2)}_r \pm\frac{i}{2}\sum_{s=0}^r\tilde c_{r-s,-}\zeta^{-s}\bigg). \tag{4.75}$$

Theorem 4.6. *Assume Hypothesis* 4.1 *and suppose that* (4.6), (4.7) *hold. In addition, let $P\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$ and $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$. Then for each $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, $\mathcal{D}_{\underline{\hat\mu}(n,t_r)}$ and $\mathcal{D}_{\underline{\hat\nu}(n,t_r)}$ are nonspecial. Moreover,*

$$\phi(P,n,t_r)=C(n,t_r)\frac{\theta(\underline z(P,\underline{\hat\nu}(n,t_r)))}{\theta(\underline z(P,\underline{\hat\mu}(n,t_r)))}\exp\bigg(\int_{Q_0}^P\omega^{(3)}_{P_{0,-},P_{\infty_-}}\bigg), \tag{4.76}$$

$$\begin{aligned}\psi_1(P,n,n_0,t_r,t_{0,r}) &= C(n,n_0,t_r,t_{0,r})\frac{\theta(\underline z(P,\underline{\hat\mu}(n,t_r)))}{\theta(\underline z(P,\underline{\hat\mu}(n_0,t_{0,r})))}\\ &\quad\times\exp\bigg((n-n_0)\int_{Q_0}^P\omega^{(3)}_{P_{0,-},P_{\infty_+}}-(t_r-t_{0,r})\int_{Q_0}^P\widetilde\Omega^{(2)}_r\bigg),\end{aligned} \tag{4.77}$$

$$\begin{aligned}\psi_2(P,n,n_0,t_r,t_{0,r}) &= C(n,t_r)C(n,n_0,t_r,t_{0,r})\frac{\theta(\underline z(P,\underline{\hat\nu}(n,t_r)))}{\theta(\underline z(P,\underline{\hat\mu}(n_0,t_{0,r})))}\\ &\times\exp\bigg(\int_{Q_0}^P\omega^{(3)}_{P_{0,-},P_{\infty_-}}+(n-n_0)\int_{Q_0}^P\omega^{(3)}_{P_{0,-},P_{\infty_+}}-(t_r-t_{0,r})\int_{Q_0}^P\widetilde\Omega^{(2)}_r\bigg),\end{aligned} \tag{4.78}$$

where

$$\begin{aligned}C(n,t_r) &= \frac{(-1)^{n-n_0}}{\alpha(n_0,t_r)}\exp\big((n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_-})-\omega_0^{\infty_+}(P_{0,-},P_{\infty_-}))\big)\\ &\quad\times\exp\big(-\omega_0^{0,+}(P_{0,-},P_{\infty_-})\big)\frac{\theta(\underline z(P_{0,+},\underline{\hat\mu}(n_0,t_r)))}{\theta(\underline z(P_{0,+},\underline{\hat\nu}(n_0,t_r)))},\end{aligned} \tag{4.79}$$

$$C(n,n_0,t_r,t_{0,r}) = \frac{\theta(\underline z(P_{\infty_+},\underline{\hat\mu}(n_0,t_{0,r})))}{\theta(\underline z(P_{\infty_+},\underline{\hat\mu}(n,t_r)))} \tag{4.80}$$

$$\times \exp\big((t_r - t_{0,r})\widetilde{\Omega}_r^{\infty_+} - (n-n_0)\omega_0^{\infty_+}(P_{0,-}, P_{\infty_+})\big).$$

The Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(n,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n,t)}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_r - t_{0,r})\underline{\tilde{U}}_r^{(2)}, \quad (4.81)$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,r})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_r - t_{0,r})\underline{\tilde{U}}_r^{(2)}. \quad (4.82)$$

Finally, α, β are of the form

$$\begin{aligned}
\alpha(n,t_r) &= \alpha(n_0,t_{0,r}) \exp\big((n-n_0)(\omega_0^{0,+}(P_{0,-},P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_+}))\big) \quad (4.83)\\
&\quad \times \exp\big((t_r - t_{0,r})(\widetilde{\Omega}_r^{\infty_+} - \widetilde{\Omega}_r^{0,+})\big) \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n,t_r)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0,t_{0,r})))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n,t_r)))},\\
\beta(n,t_r) &= \frac{1}{\alpha(n_0,t_{0,r})} \exp\big((n-n_0)(\omega_0^{0,+}(P_{0,-},P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_+}))\big)\\
&\quad \times \exp\big(\omega_0^{\infty_+}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big) \quad (4.84)\\
&\quad \times \exp\big(-(t_r - t_{0,r})(\widetilde{\Omega}_r^{\infty_+} - \widetilde{\Omega}_r^{0,+})\big) \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0,t_{0,r})))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(n,t_r)))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n_0,t_{0,r})))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n,t_r)))},
\end{aligned}$$

and

$$\begin{aligned}
\alpha(n,t_r)\beta(n,t_r) &= \exp\big(\omega_0^{\infty_+}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big)\\
&\quad \times \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n,t_r)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(n,t_r)))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n,t_r)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n,t_r)))}.
\end{aligned} \quad (4.85)$$

Proof. As in Theorem 3.7 one concludes that $\phi(P,n,t_r)$ is of the form (4.76) and that for $t_{0,r} = t_r$, $\psi_1(P,n,n_0,t_r,t_r)$ is of the form

$$\psi_1(P,n,n_0,t_r,t_r) = C(n,n_0,t_r,t_r)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_r)))} \exp\bigg((n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}}\bigg). \quad (4.86)$$

To discuss $\psi_1(P,n,n_0,t_r,t_{0,r})$ we recall (4.27), that is,

$$\psi_1(P,n,n_0,t_r,t_{0,r}) = \psi_1(P,n,n_0,t_r,t_r)\psi_1(P,n_0,n_0,t_r,t_{0,r}), \quad (4.87)$$

and hence it remains to study

$$\psi_1(P,n_0,n_0,t_r,t_{0,r}) = \exp\bigg(i\int_{t_{0,r}}^{t_r} ds\big(\widetilde{G}_r(z,n_0,s) - \widetilde{F}_r(z,n_0,s)\phi(P,n_0,s)\big)\bigg). \quad (4.88)$$

Introducing $\hat{\psi}_1(P)$ on $\mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}\}$ by

$$\begin{aligned}
\hat{\psi}_1(P,n_0,t_r,t_{0,r}) &= C(n_0,n_0,t_r,t_{0,r})\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{0,r})))}\\
&\quad \times \exp\bigg(-(t_r - t_{0,r})\int_{Q_0}^P \widetilde{\Omega}_r^{(2)}\bigg),
\end{aligned} \quad (4.89)$$

we intend to prove that

$$\begin{aligned}
&\psi_1(P,n_0,n_0,t_r,t_{0,r}) = \hat{\psi}_1(P,n_0,t_r,t_{0,r}),\\
&P \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}\},\ n_0 \in \mathbb{Z},\ t_r, t_{0,r} \in \mathbb{R},
\end{aligned} \quad (4.90)$$

for an appropriate choice of the normalization constant $C(n_0,n_0,t_r,t_{0,r})$ in (4.89). We start by noting that a comparison of (4.49), (4.50) and (4.69), (4.70) (4.72), (4.77) shows that ψ_1 and $\hat{\psi}_1$ have the same essential singularities at $P_{\infty_\pm}$ and $P_{0,\pm}$. Thus, we turn to the local behavior of ψ_1 and $\hat{\psi}_1$. By (4.89), $\hat{\psi}_1$ has zeros and poles at $\underline{\hat{\mu}}(n_0,t_r)$ and $\underline{\hat{\mu}}(n_0,t_{0,r})$, respectively. Similarly, by (4.88), ψ_1 can have zeros and poles only at poles of $\phi(P,n_0,s)$, $s\in[t_{0,r},t_r]$ (resp., $s\in[t_r,t_{0,r}]$). In the following we temporarily restrict $t_{0,r}$ and t_r to a sufficiently small nonempty interval $I\subseteq\mathbb{R}$ and pick $n_0\in\mathbb{Z}$ such that for all $s\in I$, $\mu_j(n_0,s)\neq\mu_k(n_0,s)$ for all $j\neq k$, $j,k=1,\dots,2p-1$. One computes

$$
\begin{aligned}
&i\widetilde{G}_r(z,n_0,s)-i\widetilde{F}_r(z,n_0,s)\phi(P,n_0,s)\\
&\quad=i\widetilde{G}_r(z,n_0,s)-i\widetilde{F}_r(z,n_0,s)\frac{(c_{0,+}/2)z^{-p}y+G_p(z,n_0,s)}{F_p(z,n_0,s)}\\
&\quad\underset{P\to\hat{\mu}_j(n_0,s)}{=}\frac{i\widetilde{F}_r(\mu_j(n_0,s),n_0,s)y(\hat{\mu}_j(n_0,s))}{\alpha^+(n_0,s)(z-\mu_j(n_0,s))\prod_{\substack{k=1\\k\neq j}}^{2p-1}(\mu_j(n_0,s)-\mu_k(n_0,s))}+O(1)\\
&\quad\underset{P\to\hat{\mu}_j(n_0,s)}{=}\frac{\partial}{\partial s}\ln\big(\mu_j(n_0,s)-z\big)+O(1).
\end{aligned}\tag{4.91}
$$

Restricting P to a sufficiently small neighborhood $\mathcal{U}_j(n_0)$ of $\{\hat{\mu}_j(n_0,s)\in\mathcal{K}_{2p-1}\mid s\in[t_{0,r},t_r]\subseteq I\}$ such that $\hat{\mu}_k(n_0,s)\notin\mathcal{U}_j(n_0)$ for all $s\in[t_{0,r},t_r]\subseteq I$ and all $k\in\{1,\dots,2p-1\}\setminus\{j\}$, (4.89) and (4.91) imply

$$
\begin{aligned}
&\psi_1(P,n_0,n_0,t_r,t_{0,r})\\
&\quad=\begin{cases}(\mu_j(n_0,t_r)-z)O(1) & \text{as } P\to\hat{\mu}_j(n_0,t_r)\neq\hat{\mu}_j(n_0,t_{0,r}),\\ O(1) & \text{as } P\to\hat{\mu}_j(n_0,t_r)=\hat{\mu}_j(n_0,t_{0,r}),\\ (\mu_j(n_0,t_{0,r})-z)^{-1}O(1) & \text{as } P\to\hat{\mu}_j(n_0,t_{0,r})\neq\hat{\mu}_j(n_0,t_r),\end{cases}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P=(z,y)\in\mathcal{K}_{2p-1},
\end{aligned}\tag{4.92}
$$

with $O(1)\neq 0$. Thus, ψ_1 and $\hat{\psi}_1$ have the same local behavior and identical essential singularities at $P_{\infty_\pm}$. By Lemma A.4, ψ_1 and $\hat{\psi}_1$ coincide up to a multiple constant (which may depend on $n_0,t_r,t_{0,r}$). This proves (4.90) for $t_{0,r},t_r\in I$ and for n_0 as restricted above. By continuity with respect to divisors this extends to all $n_0\in\mathbb{Z}$ since by hypothesis $\mathcal{D}_{\underline{\hat{\mu}}(n,s)}$ remain nonspecial for all $(n,s)\in\mathbb{Z}\times\mathbb{R}$. Moreover, since by (4.88), for fixed P and n_0, $\psi_1(P,n_0,n_0,.,t_{0,r})$ is entire in t_r (and this argument is symmetric in t_r and $t_{0,r}$), (4.90) holds for all $t_r,t_{0,r}\in\mathbb{R}$ (for an appropriate choice of $C(n_0,n_0,t_r,t_{0,r})$). Together with (4.87), this proves (4.77) for all $(n,t_r),(n_0,t_{0,r})\in\mathbb{Z}\times\mathbb{R}$. The expression (4.78) for ψ_2 then immediately follows from (4.76) and (4.77).

To determine the constant $C(n,n_0,t_r,t_{0,r})$ one compares the asymptotic expansions of $\psi_1(P,n,n_0,t_r,t_{0,r})$ for $P\to P_{\infty_+}$ in (4.49) and (4.77)

$$
\begin{aligned}
C(n,n_0,t_r,t_{0,r})&=\exp\big((t_r-t_{0,r})\widetilde{\Omega}_r^{\infty_+}-(n-n_0)\omega_0^{\infty_+}(P_{0,-},P_{\infty_+})\big)\\
&\quad\times\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n,t_r)))}.
\end{aligned}\tag{4.93}
$$

It remains to compute the expressions for α and β. Comparing the asymptotic expansions of $\psi_1(P,n,n_0,t_r,t_{0,r})$ for $P\to P_{0,+}$ in (4.50) and (4.77) shows

$$
\frac{\alpha(n,t_r)}{\alpha(n_0,t_{0,r})}=C(n,n_0,t_r,t_{0,r})\exp\big((n-n_0)(\omega_0^{0,+}(P_{0,-},P_{\infty_+})-(t_r-t_{0,r})\widetilde{\Omega}_r^{0,+}\big)
$$

$$\times \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n_0,t_{0,r})))} \tag{4.94}$$

and inserting $C(n,n_0,t_r,t_{0,r})$ proves (4.83). Equation (4.79) for $C(n,t_r)$ follows as in the stationary case since t_r can be viewed as an additional but fixed parameter. By the first line of (3.71),

$$\alpha(n,t_r) = \frac{1}{C(n,t_r)} \exp\big(-\omega_0^{0,+}(P_{0,-},P_{\infty_-})\big) \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n,t_r)))}. \tag{4.95}$$

Inserting the result (4.83) for $\alpha(n,t_r)$ into (4.95) then yields (using Lemma 3.6)

$$\begin{aligned} C(n,t_r) &= \frac{1}{\alpha(n_0,t_{0,r})} \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0,t_{0,r})))} \exp\big((t_r-t_{0,r})(\widetilde{\Omega}_r^{0,+}-\widetilde{\Omega}_r^{\infty_+})\big) \\ &\quad \times \exp\big((n-n_0)(\omega_0^{\infty_+}(P_{0,-},P_{\infty_+})-\omega_0^{0,+}(P_{0,-},P_{\infty_+}))-\omega_0^{0,+}(P_{0,-},P_{\infty_-})\big). \end{aligned} \tag{4.96}$$

Also, since the first line of (3.72) holds,

$$\beta(n,t_r) = C(n,t_r) \exp\big(\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})\big) \frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n,t_r)))}, \tag{4.97}$$

an insertion of (4.96) into (4.97), observing Lemma 3.6, yields equation (4.84) for $\beta(n,t_r)$. Finally, multiplying (4.95) and (4.97) proves (4.85).

Single-valuedness of $\psi_1(\,\cdot\,,n_0,n_0,t_r,t_{0,r})$ on $\mathcal{K}_{2p-1}$ implies

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}) + i(t_r-t_{0,r})\underline{\widetilde{U}}_r^{(2)}. \tag{4.98}$$

Inserting (4.98) into (3.61),

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_r)}) + \underline{A}_{P_{0,-}}(P_{\infty_+})(n-n_0), \tag{4.99}$$

one obtains the result (4.81). ☐

Appendix A. Hyperelliptic Curves and Their Theta Functions

We provide a very brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [27], [28], and [44], as well as monographs dedicated to integrable systems such as [15, Ch. 2], [36, App. A, B].

Fix $g \in \mathbb{N}$. The hyperelliptic curve $\mathcal{K}_g$ of genus g used in Sections 2 and 3 is defined by

$$\mathcal{K}_g \colon \mathcal{F}_g(z,y) = y^2 - R_{2g+2}(z) = 0, \quad R_{2g+2}(z) = \prod_{m=0}^{2g+1}(z-E_m), \tag{A.1}$$

$$\{E_m\}_{m=0,\dots,2g+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m',\ m,m' = 0,\dots,2g+1. \tag{A.2}$$

The curve (A.1) is compactified by adding the points P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $g+1$ nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$ and denotes

$$\mathcal{C} = \bigcup_{j\in\{1,\dots,g+1\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Defining the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.4}$$

and introducing the holomorphic function

$$R_{2g+2}(\,\cdot\,)^{1/2}\colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2g+1}(z-E_m)\right)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5), one considers

$$\mathcal{M}_g = \{(z, \sigma R_{2g+2}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{\pm 1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.6}$$

by extending $R_{2g+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_g$ is then the set $\mathcal{M}_g$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_g$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_g)$ of $\mathcal{K}_g$ is given by

$$\mathcal{B}(\mathcal{K}_g) = \{(E_m, 0)\}_{m=0,\dots,2g+1} \tag{A.7}$$

and finite points P on $\mathcal{K}_g$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_g$ satisfying $\mathcal{F}_g(z, y) = y^2 - R_{2g+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_g \setminus (\mathcal{B}(\mathcal{K}_g) \cup \{P_{\infty_+}, P_{\infty_-}\})$ are given by $\zeta_{P_0} = z - z_0$, near $P_{\infty_\pm}$ by $\zeta_{P_{\infty_\pm}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_g)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The Riemann surface $\mathcal{K}_g$ defined in this manner has topological genus g. Moreover, we introduce the holomorphic sheet exchange map (involution)

$$*\colon \mathcal{K}_g \to \mathcal{K}_g, \quad P = (z, y) \mapsto P^* = (z, -y),\ P_{\infty_\pm} \mapsto P^*_{\infty_\pm} = P_{\infty_\mp}. \tag{A.8}$$

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_g$ with zeros of order $g-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, g, \tag{A.9}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_g$. Introducing the invertible matrix C in $\mathbb{C}^g$,

$$\begin{gathered} C = (C_{j,k})_{j,k=1,\dots,g}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ \underline{c}(k) = (c_1(k), \dots, c_n(k)), \quad c_j(k) = C^{-1}_{j,k},\ j,k = 1, \dots, g, \end{gathered} \tag{A.10}$$

the corresponding basis of normalized holomorphic differentials ω_j, $j = 1, \dots, g$ on $\mathcal{K}_g$ is given by

$$\omega_j = \sum_{\ell=1}^{g} c_j(\ell)\eta_\ell, \qquad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1, \dots, g. \tag{A.11}$$

Here $\{a_j, b_j\}_{j=1,\dots,g}$ is a homology basis for $\mathcal{K}_g$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k},\ a_j \circ a_k = 0,\ b_j \circ b_k = 0, \quad j,k = 1, \dots, g. \tag{A.12}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,g}$ we also recall the canonical dissection of $\mathcal{K}_g$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_g$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_g$ given by

$$\partial\widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g^{-1} b_g^{-1}. \tag{A.13}$$

Let $\mathcal{M}(\mathcal{K}_g)$ and $\mathcal{M}^1(\mathcal{K}_g)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_g$. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind,

$\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_g)$, are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is, $\int_{a_j} \omega^{(2)} = 0$, $j = 1, \dots, g$. Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_g$ not of the first or second kind is said to be of the third kind. A differential of the third kind $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_g)$ is usually normalized by the vanishing of its a-periods, that is, $\int_{a_j} \omega^{(3)} = 0$, $j = 1, \dots, g$. A normal differential of the third kind $\omega^{(3)}_{P_1,P_2}$ associated with two points $P_1,\ P_2 \in \widehat{\mathcal{K}}_g$, $P_1 \neq P_2$ by definition has simple poles at P_j with residues $(-1)^{j+1}$, $j = 1, 2$ and vanishing a-periods.

Next, define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,g}$ by

$$\tau_{j,\ell} = \int_{b_\ell} \omega_j, \quad j, \ell = 1, \dots, g. \tag{A.14}$$

Then

$$\operatorname{Im}(\tau) > 0 \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j, \ell = 1, \dots, g. \tag{A.15}$$

Associated with τ one introduces the period lattice

$$L_g = \{\underline{z} \in \mathbb{C}^g \mid \underline{z} = \underline{m} + \underline{n}\tau,\ \underline{m}, \underline{n} \in \mathbb{Z}^g\} \tag{A.16}$$

and the Riemann theta function associated with $\mathcal{K}_g$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,g}$,

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^g} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \underline{n}\tau)\big), \quad \underline{z} \in \mathbb{C}^g, \tag{A.17}$$

where $(\underline{u}, \underline{v}) = \overline{\underline{u}}\, \underline{v}^\top = \sum_{j=1}^g \overline{u_j}\, v_j$ denotes the scalar product in $\mathbb{C}^g$. It has the fundamental properties

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_g) = \theta(\underline{z}), \tag{A.18}$$

$$\theta(\underline{z} + \underline{m} + \underline{n}\tau) = \exp\big(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \underline{n}\tau)\big)\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^g. \tag{A.19}$$

Next, fix a base point $Q_0 \in \mathcal{K}_g \setminus \{P_{0,\pm}, P_{\infty_\pm}\}$, denote by $J(\mathcal{K}_g) = \mathbb{C}^g/L_g$ the Jacobi variety of $\mathcal{K}_g$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_g \to J(\mathcal{K}_g), \quad \underline{A}_{Q_0}(P) = \bigg(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_g\bigg) \pmod{L_g}, \quad P \in \mathcal{K}_g. \tag{A.20}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_g) \to J(\mathcal{K}_g), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P)\underline{A}_{Q_0}(P), \tag{A.21}$$

where $\operatorname{Div}(\mathcal{K}_g)$ denotes the set of divisors on $\mathcal{K}_g$. Here $\mathcal{D} \colon \mathcal{K}_g \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_g$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_g$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_g)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on $\mathcal{K}_g$ we shall employ the following (additive) notation,

$$\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \tag{A.22}$$

$$\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_g, \quad Q_0 \in \mathcal{K}_g,\ m \in \mathbb{N},$$

where for any $Q \in \mathcal{K}_g$,

$$\mathcal{D}_Q \colon \mathcal{K}_g \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_g \setminus \{Q\}, \end{cases} \tag{A.23}$$

and $\operatorname{Sym}^n \mathcal{K}_g$ denotes the nth symmetric product of $\mathcal{K}_g$. In particular, $\operatorname{Sym}^m \mathcal{K}_g$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ of degree m.

For $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_g)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_g) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(g-1),\ f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}, \omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}, \tag{A.24}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_g) \mid f = 0 \text{ or } (f) \geq \mathcal{D}\}, \quad r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \tag{A.25}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_g) \mid \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \quad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}) \tag{A.26}$$

with $i(\mathcal{D})$ the index of speciality of $\mathcal{D}$, one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$, $\omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}$. Then,*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad g \in \mathbb{N}_0. \tag{A.27}$$

The Riemann–Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - g + 1, \quad g \in \mathbb{N}_0. \tag{A.28}$$

By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$, $g \in \mathbb{N}$ is principal if and only if

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.29}$$

Finally, assume $g \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_g) \to J(\mathcal{K}_g)$ is surjective (Jacobi's inversion theorem).

Theorem A.2. *Let $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^g \mathcal{K}_g$, $\underline{Q} = \{Q_1, \dots, Q_g\}$. Then,*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \tag{A.30}$$

if and only if $\{Q_1, \dots, Q_g\}$ contains s pairings of the type $\{P, P^*\}$. (This includes, of course, branch points for which $P = P^*$.) One has $s \leq g/2$.

Denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,g}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{g} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, g. \tag{A.31}$$

Theorem A.3. *Let $\underline{Q} = \{Q_1, \dots, Q_g\} \in \operatorname{Sym}^g \mathcal{K}_g$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_g\}. \tag{A.32}$$

Lemma A.4. [18, Lemmas 5.4 and 6.1] *Let $(n,t_r),(n_0,t_{0,r})\in\Omega$ for some $\Omega\subseteq\mathbb{Z}\times\mathbb{R}$. Assume $\psi(\,\cdot\,,n,t_r)$ to be meromorphic on $\mathcal{K}_g\setminus\{P_{\infty_+},P_{\infty_-}\}$ with essential singularities at P_{∞_+}, P_{∞_-} such that $\tilde\psi(\,\cdot\,,n,t_r)$ defined by*

$$\tilde\psi(P,n,t_r)=\psi(P,n,t_r)\exp\bigg((t_r-t_{0,r})\int_{Q_0}^{P}\widetilde\Omega_{r-1}^{(2)}\bigg) \tag{A.33}$$

is meromorphic on $\mathcal{K}_g$ and its divisor satisfies

$$(\tilde\psi(\,\cdot\,,n,t_r))\geq-\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}+(n-n_0)\big(\mathcal{D}_{P_{0,+}}-\mathcal{D}_{P_{\infty_-}}\big) \tag{A.34}$$

for some positive divisor $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}$ of degree g. Here $\widetilde\Omega_{r-1}^{(2)}$ is defined in (4.72) *and the path of integration is chosen identical to that in the Abel maps* (A.20) *and* (A.21)[2]. *Define a divisor $\mathcal{D}_0(n,t_r)$ by*

$$(\tilde\psi(\,\cdot\,,n,t_r))=\mathcal{D}_0(n,t_r)-\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}+(n-n_0)\big(\mathcal{D}_{P_{0,+}}-\mathcal{D}_{P_{\infty_-}}\big). \tag{A.35}$$

Then

$$\mathcal{D}_0(n,t_r)\in\operatorname{Sym}^g\mathcal{K}_g,\quad \mathcal{D}_0(n,t_r)>0,\quad \deg(\mathcal{D}_0(n,t_r))=g. \tag{A.36}$$

Moreover, if $\mathcal{D}_0(n,t_r)$ is nonspecial for all $(n,t_r)\in\Omega$, that is, if

$$i(\mathcal{D}_0(n,t_r))=0,\quad (n,t_r)\in\Omega, \tag{A.37}$$

then $\psi(\,\cdot\,,n,t_r)$ is unique up to a constant multiple (*which may depend on the parameters $(n,t_r),(n_0,t_{0,r})\in\Omega$*).

Appendix B. Asymptotic Spectral Parameter Expansions and Nonlinear Recursion Relations

In this appendix we consider asymptotic spectral parameter expansions of F_p/y, G_p/y, and H_p/y, the resulting recursion relations for the homogeneous coefficients $\hat f_\ell$, $\hat g_\ell$, and $\hat h_\ell$, their connection with the nonhomogeneous coefficients f_ℓ, g_ℓ, and h_ℓ, and the connection between $c_{\ell,\pm}$ and $c_\ell(\underline{E}^{\pm1})$ (cf. (B.6)). For detailed proofs of the material in this section we refer to [38]. We will employ the notation

$$\underline{E}^{\pm1}=\big(E_0^{\pm1},\dots,E_{4p-1}^{\pm1}\big). \tag{B.1}$$

We start with the following elementary results (consequences of the binomial expansion) assuming $\eta\in\mathbb{C}$ such that $|\eta|<\min\{|E_0|^{-1},\dots,|E_{4p-1}|^{-1}\}$:

$$\bigg(\prod_{m=0}^{4p-1}\big(1-E_m\eta\big)\bigg)^{-1/2}=\sum_{k=0}^{\infty}\hat c_k(\underline{E})\eta^k, \tag{B.2}$$

where

$$\begin{aligned}&\hat c_0(\underline{E})=1,\\&\hat c_k(\underline{E})=\sum_{\substack{j_0,\dots,j_{4p-1}=0\\ j_0+\cdots+j_{4p-1}=k}}^{k}\frac{(2j_0)!\cdots(2j_{4p-1})!}{2^{2k}(j_0!)^2\cdots(j_{4p-1}!)^2}E_0^{j_0}\cdots E_{4p-1}^{j_{4p-1}},\quad k\in\mathbb{N}.\end{aligned} \tag{B.3}$$

[2]This is to avoid multi-valued expressions and hence the use of the multiplicative Riemann–Roch theorem in the proof of Lemma A.4.

The first few coefficients explicitly read

$$\hat{c}_0(\underline{E}) = 1,\ \hat{c}_1(\underline{E}) = \frac{1}{2}\sum_{m=0}^{4p-1} E_m,\ \hat{c}_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{4p-1} E_{m_1}E_{m_2} + \frac{3}{8}\sum_{m=0}^{4p-1} E_m^2, \quad \text{etc.} \tag{B.4}$$

Similarly,

$$\bigg(\prod_{m=0}^{4p-1}\big(1 - E_m\eta\big)\bigg)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E})\eta^k, \tag{B.5}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1,\\ &c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{4p-1}=0\\ j_0+\cdots+j_{4p-1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{4p-1})!\,E_0^{j_0}\cdots E_{4p-1}^{j_{4p-1}}}{2^{2k}(j_0!)^2\cdots(j_{4p-1}!)^2(2j_0-1)\cdots(2j_{4p-1}-1)}, \quad k\in\mathbb{N}. \end{aligned} \tag{B.6}$$

The first few coefficients explicitly are given by

$$c_0(\underline{E}) = 1,\ c_1(\underline{E}) = -\frac{1}{2}\sum_{m=0}^{4p-1} E_m,\ c_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{4p-1} E_{m_1}E_{m_2} - \frac{1}{8}\sum_{m=0}^{4p-1} E_m^2, \quad \text{etc.} \tag{B.7}$$

Next we turn to asymptotic expansions. We recall the convention $y(P) = \mp\zeta^{-2p} + O(\zeta^{-2p+1})$ near $P_{\infty_\pm}$ (where $\zeta = 1/z$) and $y(P) = \pm(c_{0,-}/c_{0,+}) + O(\zeta)$ near $P_{0,\pm}$ (where $\zeta = z$).

Theorem B.1 ([38])**.** *Assume* (3.1), s-$\mathrm{AL}_p(\alpha,\beta) = 0$*, and suppose* $P = (z,y) \in \mathcal{K}_{2p-1}\setminus\{P_{\infty_+}, P_{\infty_-}\}$*. Then* F_p/y*,* G_{p+1}/y*, and* H_{p+1}/y *have the following convergent expansions*

$$\frac{2z^p}{c_{0,+}}\frac{F_p(z)}{y} = \begin{cases} \mp 2\sum_{\ell=0}^{\infty}\hat{f}_{\ell,+}\zeta^{\ell+1}, & P\to P_{\infty_\pm}, \quad \zeta = 1/z,\\ \pm 2\sum_{\ell=0}^{\infty}\hat{f}_{\ell,-}\zeta^{\ell}, & P\to P_{0,\pm}, \quad \zeta = z,\end{cases} \tag{B.8}$$

$$\frac{2z^p}{c_{0,+}}\frac{G_p(z)}{y} = \begin{cases} \mp 2\sum_{\ell=0}^{\infty}\hat{g}_{\ell,+}\zeta^{\ell}, & P\to P_{\infty_\pm}, \quad \zeta = 1/z,\\ \pm 2\sum_{\ell=0}^{\infty}\hat{g}_{\ell,-}\zeta^{\ell}, & P\to P_{0,\pm}, \quad \zeta = z,\end{cases} \tag{B.9}$$

$$\frac{2z^p}{c_{0,+}}\frac{H_p(z)}{y} = \begin{cases} \mp 2\sum_{\ell=0}^{\infty}\hat{h}_{\ell,+}\zeta^{\ell}, & P\to P_{\infty_\pm}, \quad \zeta = 1/z,\\ \pm 2\sum_{\ell=0}^{\infty}\hat{h}_{\ell,-}\zeta^{\ell+1}, & P\to P_{0,\pm}, \quad \zeta = z,\end{cases} \tag{B.10}$$

where $\hat{f}_{\ell,\pm}$*,* $\hat{g}_{\ell,\pm}$ *and* $\hat{h}_{\ell,\pm}$ *are the homogeneous versions of the coefficients* $f_{\ell,\pm}$*,* $g_{\ell,\pm}$*, and* $h_{\ell,\pm}$ *introduced in* (2.17)–(2.19)*.*

Moreover, the E_m*-dependent summation constants* $c_{\ell,\pm}$*,* $\ell = 0,\dots,p$*, in* F_p*,* G_p*, and* H_p *are given by*

$$c_{\ell,\pm} = c_{0,\pm}c_\ell(\underline{E}^{\pm 1}), \quad \ell = 0,\dots,p. \tag{B.11}$$

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