CYLINDER RENORMALIZATION OF SIEGEL DISKS.

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Abstract. We study one of the central open questions in one-dimensional renormalization theory – the conjectural universality of golden-mean Siegel disks. We present an approach to the problem based on cylinder renormalization proposed by the second author. Numerical implementation of this approach relies on the Constructive Measurable Riemann Mapping Theorem proved by the first author. Our numerical study yields a convincing evidence to support the Hyperbolicity Conjecture in this setting.

1. Introduction

One of the central examples of universality on one-dimensional dynamics is provided by Siegel disks of quadratic polynomials. Let us consider, for instance, the mapping

$$P_\theta(z) = z^2 + e^{2\pi i\theta}z, \text{ where } \theta = (\sqrt{5}+1)/2$$

is the golden mean. By a classical result of Siegel, the dynamics of P_θ is linearizable near the origin. The Siegel disk of P_θ, which we will further denote Δ_θ is the maximal neighborhood of zero in which a conformal change of coordinates reduces P_θ to the form $w \mapsto e^{2\pi i\theta}w$. By the results of Douady, Ghys, Herman, and Shishikura, the topological disk Δ_θ extends up to the only critical point of P_θ and is bounded by a Jordan curve.

It has been observed numerically (cf. the work of Manton and Nauenberg [MN]), that the boundary of Δ_θ is asymptotically self-similar near the critical point. Moreover, the scaling factor is universal in a large class of analytic mappings with a golden-mean Siegel disks. In 1983 Widom [Wi] defined a renormalization procedure for P_θ which "blows up" a part of the invariant curve $\partial\Delta_\theta$ near the critical point, and conjectured that the renormalizations of P_θ converge to a fixed point. In addition, he conjectured that in a suitable functional space this fixed point is hyperbolic with one-dimensional unstable direction.

In 1986 MacKay and Persival [MP] extended the conjecture to other rotation numbers, postulating the existence of a hyperbolic renormalization horseshoe corresponding to Siegel disks of analytic maps, analogous to the Lanford's horseshoe for critical circle maps [Lan1, Lan2].

In 1994 Stirnemann [Stir] gave a computer-assisted proof of the existence of a renormalization fixed point with a golden-mean Siegel disk. In 1998, McMullen [McM] proved the asymptotic self-similarity of golden Siegel disks in the quadratic family. He constructed a

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version of renormalization based on holomorphic commuting pairs of de Faria [dF1, dF2], and showed that the renormalizations of a quadratic polynomial with a golden Siegel disk near the critical point converge to a fixed point geometrically fast. More generally, he constructed a renormalization horseshoe for bounded type rotation numbers, and used renormalization to show that the Hausdorff dimension of the corresponding quadratic Julia sets is strictly less than two.

Having thus attracted much attention, the hyperbolicity part of the conjecture of Widom for golden-mean Siegel disks is still open.

In [Ya1] the second author has introduced a new renormalization transformation $\mathcal{R}_{cyl}$, which he called the cylinder renormalization, and used it to prove the Lanford's Hyperbolicity Conjecture for critical circle maps. The main advantage of $\mathcal{R}_{cyl}$ over the renormalization scheme based on commuting pairs is that this operator is analytic in a Banach manifold of analytic maps of a subdomain of $\mathbb{C}/\mathbb{Z}$. It is thus a natural setting to study the hyperbolic properties of a fixed point. In the present paper we study the fixed point of the cylinder renormalization numerically, and empirically confirm the hyperbolicity conjecture, as well as study the dynamical properties of the fixed point. The main numerical challenge in working with cylinder renormalization is a change of coordinate involved in its definition. It is defined implicitly, and uniformizes a dynamically defined fundamental domain to the straight cylinder $\mathbb{C}/\mathbb{Z}$. To handle it, we use the Constructive Measurable Riemann Mapping Theorem developed for numerically solving the Beltrami partial differential equation by the first author in [Gai, GK].

2. Definition and main properties of the cylinder renormalization of Siegel disks.

Some functional spaces. Denote π the natural projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$, and $p : \mathbb{C}/\mathbb{Z} \to \bar{\mathbb{C}}$ the conformal isomorphism given by $p(z) = e^{2\pi i z}$. For a topological disk $W \subset \mathbb{C}$ containing 0 and 1 we will denote $\mathbf{A}_W$ the Banach space of bounded analytic functions in W equipped with the sup norm. Let us denote $\mathbf{C}_W$ the Banach subspace of $\mathbf{A}_W$ consisting of analytic mappings $h : W \to \mathbb{C}$ such that $h(0) = 0$ and $h'(1) = 0$.

In the case when the domain W is the disk $\mathbb{D}_\rho$ of radius $\rho > 1$ centered at the origin, we will denote $\mathbf{A}_{\mathbb{D}_\rho} \equiv \mathbf{A}_\rho$ and $\mathbf{C}_{\mathbb{D}_\rho} \equiv \mathbf{C}_\rho$.

For each $\rho > 1$ we will also consider the collection $\mathbf{B}^1_\rho$ of analytic functions $f(z)$ defined on some neighborhood of the origin with $f(0) = 0$, equipped with the weighted l_1 norm on the coefficients of the Maclaurin's series:

$$\|f\|_\rho = \sum_{n=0}^{\infty} \frac{\left|f^{(n)}(0)\right|}{n!} \rho^n. \tag{2.1}$$

We will further denote $\mathbf{L}^1_\rho$ the subset of $\mathbf{B}^1_\rho$ consisting of maps f with the normalizing condition $f'(1) = 0$.

The proof of the following elementary statement is left to the reader:

Lemma 2.1.

1) *Let* $f \in \mathbf{L}^1_\rho$, *then* $\sup_{\mathbb{D}_\rho} |f(z)| \leq \|f\|_\rho$;
2) *Let* $f \in \mathbf{A}_{\rho'}$ *and* $\rho' > \rho$, *then* $\|f\|_\rho \leq \frac{\rho}{\rho'-\rho} \sup_{\mathbb{D}_{\rho'}} |f(z)|$.

As an immediate consequence, we have:

Corollary 2.2. $\mathbf{L}^1_\rho$ *is a Banach space.*

Cylinder renormalization operator. The cylinder renormalization operator is defined as follows. Let $f \in \mathbf{C}_W$. Suppose that for $n \in \mathbb{N}$ there exists a simple arc l which connects a fixed point a of f^n to 0, and has the property that $f^n(l)$ is again a simple arc whose only intersection with l is at the two endpoints. Let C_f be the topological disk in $\mathbb{C} \setminus \{0\}$ bounded by l and $f^n(l)$. We say that C_f is a *fundamental crescent* if the iterate $f^{-n}|_{C_f}$ mapping $f^n(l)$ to l is defined and univalent, and the quotient of $\overline{C_f \cup f^{-n}(C_f)} \setminus \{0, a\}$ by the iterate f^n is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$. Let us denote R_f the first return map of C_f, and let us assume that this map has a critical point z corresponding to the orbit of 0. Let g be the map R_f becomes under the above isomorphism, mapping z to 0, and $h = p \circ g \circ p^{-1}$. We say that f is *cylinder renormalizable with period* n, if $h \in \mathbf{C}_V$ for some V, and call h a *cylinder renormalization* of f.

We summarize below the basic properties of cylinder renormalization proven in [Ya1]:

Proposition 2.3. *Suppose* $f \in \mathbf{C}_W$ *is cylinder renormalizable, and its renormalization* h_f *is contained in* $\mathbf{C}_V$. *Denote* C_f *the fundamental crescent corresponding to the renormalization. Then the following holds.*

- *Every other fundamental crescent* C'_f *with the same endpoints as* C_f, *and such that* $C'_f \cup C_f$ *is a topological disk, produces the same renormalized map* h_f.
- *There exists an open neighborhood* $U(f) \subset \mathbf{C}_W$ *such that every map* $g \in U(f)$ *is cylinder renormalizable, with a fundamental crescent* C_g *which can be chosen to move continuously with* g.
- *Moreover, the dependence* $g \mapsto h_g$ *of the cylinder renormalization on the map* g *is an analytic mapping* $\mathbf{C}_W \to \mathbf{C}_V$.

We now want to discuss the dynamical properties of the cylinder renormalization of maps with Siegel disks derived in [Ya3]. To simplify the exposition let us specialize to the case when the rotation number of the Siegel disk is the golden mean $\theta = (\sqrt{5}+1)/2$. The golden mean is represented by an infinite continued fraction

$$\theta = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}} \equiv 1 + [1, 1, 1, \ldots].$$

As is customary, we will denote p_n/q_n the n-th convergent

$$p_n/q_n = \underbrace{[1,1,1,\ldots,1]}_{n}.$$

Theorem 2.4 ([Ya3]). *There exists a space $\mathbf{C}_U$ and an analytic mapping $\hat{f} \in \mathbf{C}_U$ which has a Siegel disk Δ_θ with rotation number θ whose boundary is a quasicircle passing through the critical point 1, such that the following holds:*

(I) *There exists a branch of cylinder renormalization with period q_k, $k \in \mathbb{N}$, which we denote $\mathcal{R}_{cyl}$ such that*
$$\mathcal{R}_{cyl}\hat{f} = \hat{f};$$
(II) *the quadratic polynomial $P_\theta(z) = e^{2\pi i\theta}z + z^2$ is infinitely cylinder renormalizable, and*
$$\mathcal{R}^k_{cyl}P_\theta \to \hat{f},$$
at a uniform geometric rate;

(III) *the cylinder renormalization $\mathcal{R}_{cyl}$ is an analytic and compact operator mapping a neighborhood of the fixed point $\hat{f}$ in $\mathbf{C}_U$ to $\mathbf{C}_U$. Its linearization $\mathcal{L}$ at $\hat{f}$ is a compact operator, with at least one eigenvalue with the absolute value greater than one.*

A central open questions in the study of $\mathcal{R}_{cyl}$ is the following:

Conjecture 2.5. *Except for the one unstable eigenvalue, the rest of the spectrum of $\mathcal{L}$ is compactly contained in the unit disk.*

Our numerical study of $\mathcal{R}_{cyl}$ will begin with empirically establishing the convergence to $\hat{f}$. We will then make explicit the choice of the neighborhood U in the above Theorem. Experimental evidence suggests that it can be taken as a round disk $\mathbb{D}_\rho$ for some particular value of ρ. Having numerically established this, we will then proceed to experimentally verify the Conjecture.

3. Construction of the conformal isomorphism to the cylinder

The principal difficulty in numerical, as well as analytical, study of cylinder renormalization is the non-explicit nature of the conformal isomorphism

$$\Phi : \overline{C_f \cup f^{-n}(C_f)} \setminus \{0, a\} \underset{\approx}{\longrightarrow} \mathbb{C}^*$$

of a fundamental crescent, which is a part of the definition of $\mathcal{R}_{cyl}$. An analytic approach to this construction based on the Measurable Riemann Mapping Theorem was presented by the first author in [Gai]. It has its roots in the complex-dynamical folklore; similar arguments are found, for instance, in the work of Lyubich [Lyu] and Shishikura [Shi].

In [Gai], the first author demonstrates how this approach can be implemented constructively, with rigorous error bounds. We will give a brief outline here.

Uniformization of the cylinder using the Measurable Riemann Mapping Theorem. We will start with a description of our choice of a fundamental crescent C^n_f with period q_n for a map $f \in \mathbf{C}_U$ sufficiently close to $\hat{f}$.

To construct the boundary curve l_n of C^n_f consider first the union $\tilde{l}_n$ of two parabolas: one passing through points 0 and $f^{q_{n+2}+q_n}(1)$, the other — through $f^{q_{n+2}+q_n}(1)$ and a repelling fixed point a_{q_n}. These two parabolas are defined uniquely after one specifies their common tangent line at $f^{q_{n+2}+q_n}(1)$. While somewhat arbitrary, this choice has the virtue of possessing a simple analytic form. It can be shown rigorously (see [Gai]) that by modifying $\tilde{l}_n$ in sufficiently small neighborhoods of the endpoints (small enough not to influence our numerical experiments) we obtain a curve l_n which together with $f^{-q_n}(l_n)$ bounds a fundamental crescent C^n_f for $\mathcal{R}_{cyl}$.

Now consider the following conformal change of coordinates for $z \in C^n_f$:

$$z = \tau(\xi) = \frac{a_{q_n}}{1 - e^{i\alpha\xi+\beta}}, \tag{3.1}$$

We will choose the normalizing constants α and β so that

$$|\tau^{-1}(f^{q_{n+2}+q_n}(1))| = 1 \text{ and } \tau^{-1}(f^{q_{n+2}}(1)) = 0.$$

The choice of of this coordinate is motivated by the fact that τ^{-1} maps the interior of the fundamental crescent C^n_f conformally onto the interior of an infinite vertical closed strip $\mathcal{S}$, whose width is comparable to one (cf. Figure 1). Next, similarly to [Shi], define a function

$$\tilde{g}_n : \mathcal{U} \equiv \{u + iv \in \mathbb{C} : 0 \le \operatorname{Re} w \le 1\} \longrightarrow \mathcal{S}$$

by setting

$$\tilde{g}_n(u + iv) = (1 - u)\tau^{-1}(f^{-q_n}(\gamma_n(v))) + u\tau^{-1}(\gamma_n(v)),$$

where γ_n is a parametrization

$$\gamma_n : \mathbb{R} \to l_n$$

which we will specify below.

Let σ_0 be the standard conformal structure on $\mathbb{C}$, and let $\sigma = \tilde{g}_n^*\sigma_0$ be its pull-back on $\mathcal{U}$. Extend this conformal structure to $\mathbb{C}$ through

$$\sigma \equiv (T^k)^*\sigma \text{ on } T^{-k}(\mathcal{U}), \text{ where } T(w) = w + 1, \text{ for all } k \in \mathbb{N}.$$

Assuming the mapping $\tilde{g}_n$ is quasiconformal, the dilatation of σ is bounded in the plane. By Measurable Riemann Mapping Theorem (see e.g. [AB]) there exists a unique quasiconformal mapping $\tilde{g} : \mathbb{C} \mapsto \mathbb{C}$ such that $\tilde{g}^*\sigma_0 = \sigma$, normalized so that $\tilde{g}(0) = 0$ and $\tilde{g}(1) = 1$. Notice that $\tilde{g} \circ T \circ \tilde{g}^{-1}$ preserves the standard conformal structure:

$$\left(\tilde{g} \circ T \circ \tilde{g}^{-1}\right)^* \sigma_0 = (\tilde{g}^{-1})^* \circ T^* \circ \tilde{g}^*\sigma_0 = (\tilde{g}^{-1})^* \circ T^*\sigma = (\tilde{g}^{-1})^*\sigma = (\tilde{g}^*)^{-1}\sigma = \sigma_0,$$

and therefore it is a conformal automorphism of $\mathbb{C}$. Liouville's Theorem implies that this mapping is affine. By construction, it does not have any fixed points in $\mathbb{C}$, and hence, is a translation. Finally, $\tilde{g} \circ T \circ \tilde{g}^{-1}(0) = 1$, and thus

$$\tilde{g} \circ T \circ \tilde{g}^{-1} \equiv T.$$

By the definition of $\tilde{g}_n$,

$$\tilde{g}_n^{-1} \circ \tau^{-1} \circ f^{q_n} \circ \tau = T \circ \tilde{g}_n^{-1}$$

on the image of l_n by τ^{-1}. Set $\phi = \tilde{g} \circ \tilde{g}_n^{-1}$, and $\tilde{\Phi} \equiv \phi \circ \tau^{-1}$. Clearly, $\tilde{\Phi} \bmod \mathbb{Z}$ is a desired conformal isomorphism

$$\overline{C_f^n \cup f^{-q_n}(C_f^n)} \setminus \{0, a_{q_n}\} \underset{\approx}{\longrightarrow} \mathbb{C}/\mathbb{Z}.$$

Set $e(z) = e^{-2\pi i z}$, and $g = e \circ \tilde{g} \circ e^{-1}$. Since

$$\frac{g_{\bar{z}}(e(w))}{g_z(e(w))} = \frac{e(w)}{\overline{e(w)}} \frac{\tilde{g}_{\bar{w}}(w)}{\tilde{g}_w(w)},$$

the 1-periodic function $\tilde{g}$ is a solution of the Beltrami equation

$$\tilde{g}_{\bar{w}} = \tilde{\mu} \tilde{g}_w, \quad \tilde{\mu} = (\tilde{g}_n)_{\bar{w}} / (\tilde{g}_n)_w$$

whenever g is a solution of

$$g_{\bar{z}} = \mu g_z, \quad \mu(z) = (z/\bar{z}) \tilde{\mu}(e^{-1}(z)). \tag{3.2}$$

Thus, we have reduced the problem of finding

$$\Phi \equiv e \circ \tilde{\Phi} = g \circ e \circ \tilde{g}_n^{-1} \circ \tau^{-1}$$

to that of finding the properly normalized solution of the Beltrami equation

$$g_{\bar{z}} = \mu g_z, \quad \mu(z) = \frac{z}{\bar{z}} \frac{(\tilde{g}_n)_{\bar{w}}(e^{-1}(z))}{(\tilde{g}_n)_w(e^{-1}(z))} \tag{3.3}$$

on the punctured plane $\mathbb{C}^*$.

It remains to describe the choice of the parametrization of l_n in the definition of $\tilde{g}_n$. It is convenient for us to parametrize l_n using the radial coordinate in $\mathbb{C}$. For $n = 1$ and $f \in \mathbf{C}_U$, sufficiently close the empirical fixed point of the cylinder renormalization with period 1 we use the following parametrization:

$$\lambda_1(r) = \begin{cases} (x(r), Ax(r)^2 + Bx(r)), & r \leq \tilde{r}, \\ (Cy(r)^2 + Dy(r) + E, y(r)), & r > \tilde{r}, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} x(r) &= \frac{\operatorname{Re} f^4(1)}{|f^4(1)|} T(r), \\ y(r) &= \operatorname{Im} f^4(1) \frac{|a_1 - f^4(1)| + |f^4(1)| - T(r)}{|a_1 - f^4(1)|} + \operatorname{Im} a_1 \frac{T(r) - |f^4(1)|)}{|a_1 - f^4(1)|}, \\ T(r) &= \frac{|a_1 - f^4(1)| + |f^4(1)|}{\sqrt{r} + 1} \sqrt{r}, \end{aligned}$$

and $\tilde{r}$ is defined through $T(\tilde{r}) = |f^4(1)|$. Constants A, B, C, D and E are fixed by the conditions $0, f^4(1), a_1 \in l_1$, together with the requirement that the slope of the common tangent line to both parabolas at the point $f^4(1)$ is equal to 1.1.

This particular choice of the parametrization is motivated by the speed of convergence of the iterative scheme in the Measurable Riemann Mapping Theorem.

We define the following function on $\mathbb{C}^*$:

$$g_1(r,\phi) = \left(\eta(-\phi) + \frac{\phi}{2\pi}\right)\tau^{-1}(\lambda_1(r)) + \left(1 - \eta(-\phi) - \frac{\phi}{2\pi}\right)\tau^{-1}(f(\lambda_1(r))), \tag{3.5}$$

where $-\pi < \phi \le \pi$, and η is the Heaviside step function (we have adopted the convention $\eta(0) = 1$). Then, according to (3.3), the Beltrami differential μ is given by the following expression:

$$\mu(re^{i\phi}) = e^{2i\phi}\frac{r\partial_r g_1(r,\phi) + i\partial_\phi g_1(r,\phi)}{r\partial_r g_1(r,\phi) - i\partial_\phi g_1(r,\phi)} \tag{3.6}$$

on $\mathbb{C}\backslash(-\infty, 0]$. This is the expression that we have used to compute the Beltrami differential in our numerical studies.

A constructive Measurable Riemann Mapping Theorem (MRMT). To solve the Beltrami equation numerically, we use the constructive MRMT proved by the first author in [Gai]. Before formulating it, we need to recall two integral operators used in the classical approach to the proof of MRMT (see [AB]).

The first of them is Hilbert Transform:

$$T[h](z) = \frac{i}{2\pi}\lim_{\epsilon\to 0}\int\int_{\mathbb{C}\backslash B(z,\epsilon)} \frac{h(\xi)}{(\xi - z)^2}\, d\bar{\xi}\wedge d\xi, \tag{3.7}$$

the second is Cauchy Transform

$$P[h](z) = \frac{i}{2\pi}\int\int_{\mathbb{C}} \frac{h(\xi)}{(\xi - z)}\, d\bar{\xi}\wedge d\xi.$$

Hilbert Transform is a well-defined bounded operator on $L_p(\mathbb{C})$ for all $2 < p < \infty$. For every such p there exists a constant C_p such that the following holds (cf [CZ]):

$$\|T[h]\|_p \le C_p \|h\|_p \text{ for any } h \in L_p(\mathbb{C}), \text{ and } C_p \to 1 \text{ as } p \to 2.$$

We are now ready to state the constructive Measurable Riemann Mapping Theorem of [Gai]:

Theorem 3.1. *Let $\mu \in L_\infty(\bar{\mathbb{C}})$ and an integer $p > 2$ be such that $\|\mu\|_\infty \le K < 1$ and $KC_p < 1$, where*

$$C_p = \cot^2(\pi/2p).$$

Assume that $\mu = \nu + \eta + \gamma$, where ν and η are compactly supported in $\mathbb{D}_R$, and $\gamma(z)$ is supported in $\bar{\mathbb{C}} \setminus \mathbb{D}_R$. Furthermore, let η be in $L_p(\mathbb{D}_R)$ and $\|\eta\|_p < \delta$ for some sufficiently small δ. Also, let $h^* \in L_p(\mathbb{C})$ and ϵ be such that $B_p(h^*, \epsilon)$, the ball of radius ϵ around h^* in $L_p(\mathbb{C})$, contains $B_p(T[\nu(h^* + 1)], C_p\epsilon')$, with

$$\epsilon' = \delta\ \mathrm{essup}_{\mathbb{D}_R}|h^* + 1| + K\epsilon.$$

Then the solution g^μ of the Beltrami equation $g^\mu_{\bar z} = \mu g^\mu_z$ admits the following bound:

$$|g^\mu(z) - g^\nu_*(z)| \le F(\epsilon', R; z, g^\nu_*(z), p, K, C_p) \tag{3.8}$$

where $g^\nu_*(z) = P[\nu(h^*+1)](z) + z$, and $F(\epsilon', R) = O(\epsilon', R^{-4/p})$ is an explicit function of its arguments.

Given the theorem, the algorithm for producing an approximate solution of the Beltrami equation is as follows. Given a μ as in the condition of the theorem, we first iterate

$$h \to T[\nu(h+1)] \tag{3.9}$$

to find a numerical approximation h^*_a to the solution of the equation $T[\mu(h^*+1)] = h^*$. After that, we compute an approximate solution as

$$g^\nu_a(z) = P[\nu(h^*_a+1)](z) + z. \tag{3.10}$$

One can obtain rigorous computer-assisted bounds on such solution using Theorem 3.1. Such bounds have been indeed implemented in [Gai] for a particular case of the golden mean quadratic polynomial. However, in the present numerical work we will not require such estimates.

In the Appendix, we will discuss several numerical algorithms for the two integral transforms appearing in this scheme.

4. Empirical convergence to a fixed point

An appropriate choice of the domains of analyticity for the renormalized functions is central to a successful numerical implementation of cylinder renormalization. Our numerical approximation to the renormalization fixed point is a finite-degree truncation of a function analytic in $\mathbb{D}_3$ (see Section 5 for a detailed explanation of this choice of the domain). However, for the purposes of obtaining bounds on higher-order terms that are ignored in our numerical experiments, we will consider a smaller analyticity domain, a disk of radius $\rho = 2.266$. Thus the cylinder renormalization will be *a priori* an analytic operator in a neighborhood of the fixed point in the Banach space $\mathbf{L}^1_\rho$ with $\rho = 2.266$. [1]

Given an $f \in \mathbf{L}^1_\rho$, a numerical approximation to its cylinder renormalization of order n is built as follows. As the first step, we construct a fundamental crescent C^n_f as described above, and find the normalized solution g of the Beltrami equation

$$g_{\bar z}(z) = \mu(z) g_z(z) \text{ with } \mu \text{ as in (3.6)}$$

[1] We have implemented the procedure for the cylinder renormalization described in Section 3 and a particular method of solving the Beltrami equation (see Appendix) as a set of routines in in the programming language Ada 95 (cf [ADA1] for the language standard). We have parallelized our programs and compiled them with the public version 3.15p of the GNAT compiler [ADA2]. The programs ([Prog]) have been run on the computational cluster of 92 2.2 GHz AMD Opteron processors located at the University of Texas at Austin.

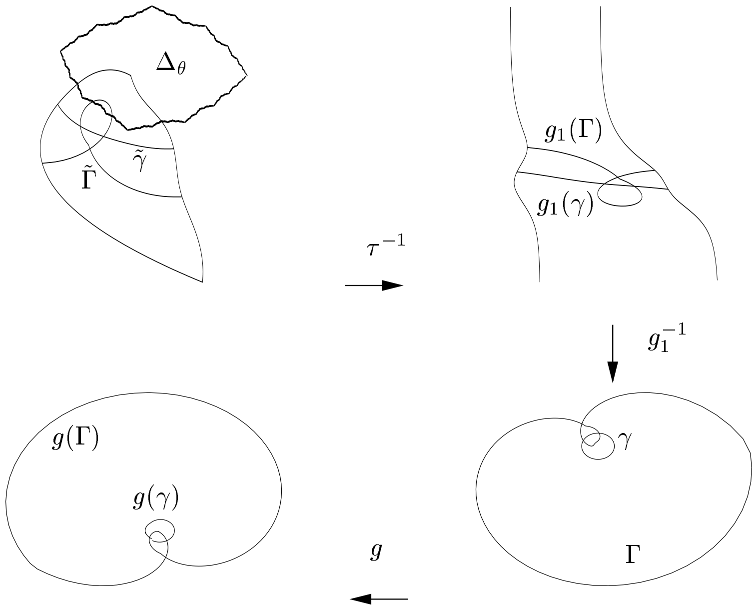

FIGURE 1. Schematics of renormalization. The contours $g(\gamma) \mapsto g(\Gamma)$ are used to find a polynomial approximation of $\mathcal{R}_{cyl}f$ through the Cauchy integral formula.

as described in Section 3 and the Appendix. Next, we choose a contour γ in the domain of g, and map this contour into the fundamental crescent by $\tau \circ g_1$:

$$\tilde{\gamma} \equiv \tau \circ g_1(\gamma).$$

Applying the first return map to the points of this contour, we obtain $\tilde{\Gamma} = R_f(\tilde{\gamma})$, and find the images, $g(\gamma)$ and $g(\Gamma)$ (where $\Gamma = g_1^{-1}(\tau^{-1}(\tilde{\Gamma}))$). The coefficients in a finite order polynomial approximation to

$$\mathcal{R}_{cyl}f = g \circ g_1^{-1} \circ \tau^{-1} \circ R_f \circ \tau \circ g_1 \circ g^{-1}$$

are then found via the Cauchy integral formula using these two contours $g(\gamma)$ and $g(\Gamma)$ (see Figure 1).

As seen in Theorem 2.4, the sequence of the cylinder renormalizations of the quadratic polynomial P_θ converges to a fixed point $\hat{f} = \mathcal{R}_{cyl}\hat{f}$. We have used this fact to compute an approximate renormalization fixed point $\hat{f}_a$ as the cylinder renormalization $\mathcal{R}^k_{cyl}P_\theta$ of order $k = 11$.

Further, we improved this approximation by iterating

$$\hat{f}_a \mapsto \mathbb{P}_s \circ \mathcal{R}_{cyl}\hat{f}_a \tag{4.1}$$

where $\mathbb{P}_s$ is the projection on the candidate stable manifold of $\hat{f}$:

$$\mathrm{W}_s = \{f \in \mathbf{L}^1_\rho : f'(0) = e^{2\pi\theta i}\}.$$

In this way we have obtained a polynomial $\hat{f}_a$ of degree 17, and have estimated that, not taking into account the errors in the solution of the Beltrami equation and due to round-off,

$$\|\mathcal{R}_{cyl}\hat{f}_a - \hat{f}_a\|_\rho \le 1.88 \times 10^{-3} \approx 0.89 \times 10^{-4} ||\hat{f}_a||_\rho. \tag{4.2}$$

Moreover, the iteration (4.1) does not lead to a significant variation in the computed values for the coefficients of $\hat{f}_a$, which indicates that the original approximation is indeed quite accurate. The largest change is in the highest coefficient, which differs by 0.4% for $\hat{f}_a$ and its renormalization. Of course, this represents a negligible correction to the absolute value of the coefficient itself.

The approximate expression for $\hat{f}_a$ is as follows (all numbers truncated to show six significant digits)::

$$\begin{aligned} \hat{f}_a = \quad & x & & e^{2\pi i\theta} + \\ & x^2 & & (\ \ 8.00882 \times 10^{-1} + i4.07682 \times 10^{-1}) + \\ & x^3 & & (-4.12708 \times 10^{-1} + i2.97670 \times 10^{-2}) + \\ & x^4 & & (\ \ 1.02033 \times 10^{-1} - i9.83702 \times 10^{-2}) + \\ & x^5 & & (\ \ 2.61573 \times 10^{-5} + i4.13871 \times 10^{-2}) + \\ & x^6 & & (-8.42868 \times 10^{-3} - i6.96474 \times 10^{-3}) + \\ & x^7 & & (\ \ 2.60095 \times 10^{-3} - i6.58544 \times 10^{-4}) + \\ & x^8 & & (-2.01382 \times 10^{-4} + i5.95113 \times 10^{-4}) + \\ & x^9 & & (-9.40057 \times 10^{-5} - i1.11237 \times 10^{-4}) + \\ & x^{10} & & (\ \ 3.21762 \times 10^{-5} - i4.40144 \times 10^{-6}) + \cdots \end{aligned}$$

5. Domain of Analyticity of the Renormalization Fixed Point

Compactness of $\mathcal{R}_{cyl}$. We have verified experimentally the compactness property of $\mathcal{R}_{cyl}$ stated in Theorem 2.4. More precisely, we observe the following empirical fact:

Set $\rho = 2.266$ and $\rho' = 3$. Then we can take $U \equiv \mathbb{D}_\rho$ in Theorem 2.4. More specifically, the fixed point $\hat{f}$ is a well-defined analytic mapping in $\mathbf{C}_\rho$, and moreover, if we denote

$$g \equiv \hat{f}|_{\mathbb{D}_\rho},\ \text{then } \mathcal{R}_{cyl} g \in \mathbf{C}_{\rho'}.$$

To verify the claim numerically, we have used the approximation $\hat{f}_a$ obtained in the previous section. To estimate ρ', we have chosen a curve γ in the fundamental crescent, such that $\Phi_{\hat{f}_a}(\gamma)$ is a simple closed loop that encircles $\mathbb{D}_3$. We then verify that the orbit under the return map of the component $C^0_{\hat{f}_a}$ of the set $C_{\hat{f}_a} \setminus \gamma$, such that $0 \in \partial C^0_{\hat{f}_a}$, lies within $\mathbb{D}_{2.266}$ (see Figure 2). For our choice of the curve γ, the return map of the set

$$C^0_{\hat{f}_a} = L^0_{\hat{f}_a} \cup R^0_{\hat{f}_a}$$

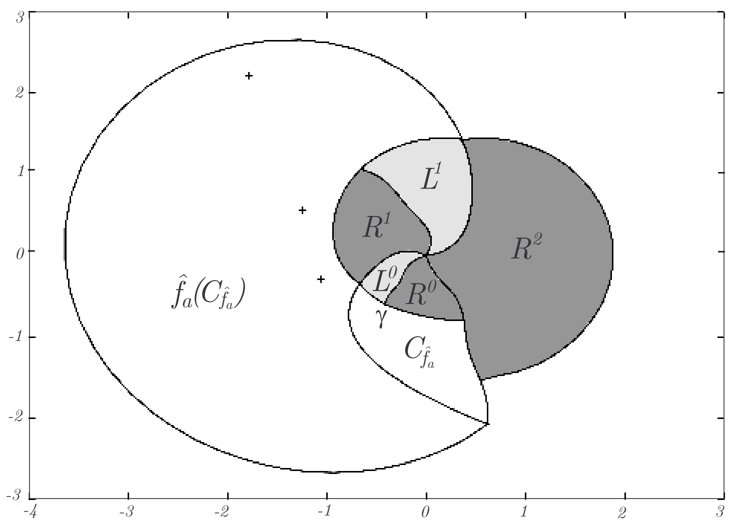

FIGURE 2. The orbit of $C^0_{\hat{f}_a}$: the orbit of $L^0_{\hat{f}_a}$, $L^k \equiv \hat{f}_a^k(L^0_{\hat{f}_a})$, is rendered in light gray, that of $R^0_{\hat{f}_a}$, $R^k \equiv \hat{f}_a^k(R^0_{\hat{f}_a})$, — in dark gray.

is given by the 2-nd and the 3-d iterates of $\hat{f}_a$ on $L^0_{\hat{f}_a}$ and $R^0_{\hat{f}_a}$, respectively.

Maximal analytic extension of $\hat{f}$. The following is an easy consequence of the results of McMullen (see [McM, Theorem 8.1]):

Proposition 5.1. *The fixed point $\hat{f}$ has a maximal analytic extension to a simply-connected domain around the origin.*

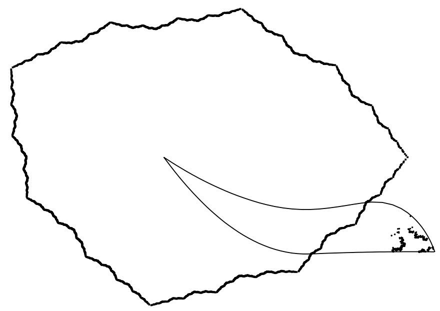

FIGURE 3. Set $S^5_{P_\theta}$ inside a fundamental domain $C^5_{P_\theta}$.

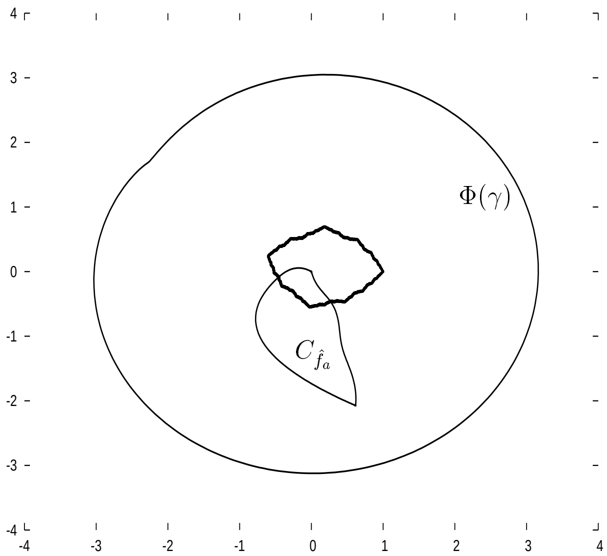

FIGURE 4. The fundamental domain $C_{\hat{f}_a}$ together with $\Phi(\gamma)$.

To understand how this domain is formed, consider the quadratic polynomial P_θ. This map is cylinder renormalizable with period q_n for each $n \in \mathbb{N}$ (see [Ya3]). Consider the first return map R_n of the corresponding fundamental crescent C_n. At each of the points of C_n which R_n maps to the origin, the cylinder renormalization has a singularity. Denote the set of all such points $S^n_{P_\theta}$. When $n \to \infty$, the images $\Phi_{P_\theta^{q_n}}(S^n_{P_\theta})$ converge to a compact subset K of $\mathbb{C}^*$ The domain of analyticity of $\hat{f}$ is the connected component of $\mathbb{C} \setminus K$ which contains the origin.

6. Hyperbolic properties of cylinder renormalization

The expanding direction of $\mathcal{R}_{cyl}$. It is not difficult to see that the operator $\mathcal{R}_{cyl}$ possesses an expanding direction at $\hat{f}$ (cf. [Ya3]):

Proof of Theorem 2.4, Part (III). Let $v(z)$ be a vector field in $\mathbf{C}_U$,

$$v(z) = v'(0)z + o(z).$$

Denote γ_v the quantity

$$\gamma_v = \frac{v'(0)}{\hat{f}'(0)} \underset{1}{=} e^{-2\pi i\theta} v'(0).$$

For a smooth family

$$\hat{f}_t(z) = \hat{f}(z) + tv(z) + o(t),$$

we have

$$\hat{f}_t(z) = \alpha^v_t(z)(\hat{f}'(0)z + o(z)), \text{ where } \alpha_t(0) = 1 + t\gamma_v + o(t).$$

The q_{m+1}-st iterate

$$\hat{f}_t^{q_{m+1}}(z) = (\alpha_t^v(z))^{q_{m+1}}((\hat{f}'(0))^{q_{m+1}}z + o(z)).$$

In the neighborhood of 0 the renormalized vector field $\mathcal{L}v$ is obtained by applying a uniformizing coordinate

$$\Phi(z) = (z + o(z))^\beta, \text{ where } \beta = \frac{1}{\theta q_m \bmod 1}.$$

Hence,

$$\alpha_t^{\mathcal{L}v}(0) = [(\alpha_t^v(0))^{q_{m+1}}]^\beta,$$

so

$$\gamma_{\mathcal{L}v} = \Lambda\gamma_v, \text{ where } \Lambda = \beta q_{m+1} > 1.$$

Hence the spectral radius

$$R_{\mathrm{Sp}}(\mathcal{L}v) > 1,$$

and since every non-zero element of the spectrum of a compact operator is an eigenvalue, the claim follows. □

Numerical verification of hyperbolicity of $\mathcal{R}_{cyl}$. It is natural to conjecture:

Conjecture 6.1. *There exists an open neighborhood $\mathcal{U} \subset \mathbf{C}_U$ containing $\hat{f}$ such that $\mathcal{R}_{cyl}$ is a strong contraction in*

$$W = \{f \in \mathcal{U} \mid f'(0) = e^{2\pi i\theta}\}.$$

Thus, $W = W^s_{loc}(\hat{f})$.

To verify this conjecture numerically, we have to justify using a finite-dimensional approximation to $\mathcal{L}$ to test for contraction. For this we rely on a numerical observation discussed in the previous section:

$$\mathcal{L} : \mathbf{L}^1_\rho \to \mathbf{L}^1_{\rho'}, \text{ with } \rho = 2.266, \text{ and } \rho' = 3.$$

This implies, that the finite-dimensional approximations of $\mathcal{L}$ obtained by truncating all the powers higher that z^N will converge geometrically fast in N.

Set h_j to be the coordinate vectors $h_j(z) = z^j/\rho^j$, so that

$$||\mathcal{L}||_\rho = \sup ||\mathcal{L}h_j||_\rho.$$

Since a perturbation $\hat{f} + \epsilon h_j$ does not lie in $\mathbf{L}^1_\rho$, we introduce a different basis in $\mathbf{L}^1_\rho$:

$$e_j = \frac{g_j}{||g_j||_\rho}, \quad g_j(z) = z^j - \frac{j}{j+1}z^{j+1}, \quad 1 \le j \le N, \tag{6.1}$$

$$e_j = h_j, \quad j > N. \tag{6.2}$$

Numerically, to estimate the spectral radius

$$R_{\mathrm{Sp}}\left(\mathcal{L}|_{T_{\hat{f}}W}\right)$$

we can fix a large enough N and a small ϵ, compute for each e_j, $2 \le j \le N$, the finite difference

$$\frac{1}{\epsilon}(\mathcal{R}_{cyl}(\hat{f}_a + \epsilon e_j) - \mathcal{R}_{cyl}\hat{f}_a),$$

truncate past the N-th power – and expand over the basis vectors e_j to obtain an $(N-1) \times (N-1)$ matrix A_N. Below we present the approximate expression for A_6 (the numbers have been truncated to the fifth decimal).

$-0.45879+$ $0.97624i$	$-0.68789+$ $0.46254i$	$-0.11338+$ $0.09738i$	$-0.13041-$ $0.07490i$	$-0.15824-$ $0.11616i$
$0.13666-$ $1.72834i$	$0.64474-$ $0.54306i$	$-0.33937+$ $0.50837i$	$-0.14710+$ $0.13849i$	0.21006- $0.02552i$
$0.90634+$ $1.37322i$	$-0.27155+$ $0.38270i$	$0.05765-$ $1.09081i$	$0.22948-$ $0.14700i$	$-0.18338+$ $0.39078i$
$-1.23970-$ $0.20634i$	$0.08549-$ $0.23219i$	$0.68227+$ $0.81153i$	$-0.12817+$ $0.14685i$	$-0.10981-$ $0.47861i$
$0.63443-$ $0.58168i$	$0.02489+$ $0.18893i$	$-0.80205-$ $0.00357i$	$0.04014-$ $0.12864i$	$0.34685+$ $0.19936i$

Table 1. Matrix A_6.

This matrix has the spectral radius

$$R_{\mathrm{Sp}}(A_6) \approx 0.53.$$

Estimating the spectral radius. We now proceed to produce a justification for the above numerical experiment. We will equip $\mathbf{L}^1_\rho$, viewed as a vector space, with a new l_1-norm

$$|f|_\rho = \sum_{k=1}^{\infty} |f_k|, \tag{6.3}$$

where f_k are the coefficients in the expansion of f in the basis $\{e_j\}$: $f = \sum_{k=1}^{\infty} f_k e_k$; and denote the new Banach space by $\tilde{\mathbf{L}}^1_\rho$. The projection $\mathbb{P}_{\le N}$ on $\mathrm{span}_{1 \le j \le N}\{e_j\}$ will be defined by setting

$$\mathbb{P}_{\le N} f = \sum_{j=1}^{N} f_j e_j. \tag{6.4}$$

We will also abbreviate $\mathbb{I} - \mathbb{P}_{\le N}$ as $\mathbb{P}_{>N}$.

We would like to emphasize that $\mathbb{P}_{\le N} f \in \tilde{\mathbf{L}}^1_\rho$ whenever $f \in \tilde{\mathbf{L}}^1_\rho$, and therefore the operator

$$\mathcal{A} = \mathbb{P}_{\le N} \mathcal{L} \mathbb{P}_{\le N} \tag{6.5}$$

serves as a finite-dimensional approximation to the action of $\mathcal{L}$ on $\tilde{\mathbf{L}}^1_\rho$. We will now make the latter statement more precise.

To this end, notice, that

$$\mathcal{L} = \mathcal{A} + \mathcal{L}\mathbb{P}_{>N} + \mathbb{P}_{>N}\mathcal{L}\mathbb{P}_{\le N} = \mathcal{A} + \mathcal{H}. \tag{6.6}$$

The following Lemma demonstrates how one can obtain an upper bound on the spectral radius of the differential $\mathcal{L}$ at the fixed point in terms of the norm of a power of the finite-rank operator $\mathcal{A}$ and the magnitude of the norm of $\mathcal{H}$.

Lemma 6.2. *Let $\mathcal{L} = \mathcal{A} + \mathcal{H}$ be a bounded operator on some Banach space, such that $\|\mathcal{A}^k\| < \gamma < 1$ for some $k \ge 1$ and, $\|\mathcal{H}\| < \delta < 1$. Then, the spectral radius $R_{Sp}(\mathcal{L})$ satisfies*

$$R_{Sp}(\mathcal{L}) \le \gamma^{1/k}(1 + C\delta/\gamma)^{1/k} \tag{6.7}$$

for some (explicit) constant C.

Proof. The claim follows from the spectral radius formula. First,

$$R_{\mathrm{Sp}}(\mathcal{L}) = \overline{\lim}_{n\to\infty}\|\mathcal{L}^n\|^{\frac{1}{n}} = \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]+k\left\{\frac{n}{k}\right\}}\right\|^{\frac{1}{n}} \le \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]}\right\|^{\frac{1}{n}}\overline{\lim}_{n\to\infty}\|\mathcal{L}^k\|^{1-\frac{k\left[\frac{n}{k}\right]}{n}}.$$

The norm $\|\mathcal{L}^k\|$ is finite, and therefore

$$\overline{\lim}_{n\to\infty}\|\mathcal{L}^k\|^{\frac{n-k\left[\frac{n}{k}\right]}{n}} = 1.$$

Then

$$R_{\mathrm{Sp}}(\mathcal{L}) \le \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]}\right\|^{\frac{1}{n}} \le \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]}\right\|^{\frac{1}{\left[\frac{n}{k}\right]k}}\overline{\lim}_{n\to\infty}\|\mathcal{L}^k\|^{\frac{\left[\frac{n}{k}\right]k-n}{nk}} \le \overline{\lim}_{m\to\infty}\left\|\mathcal{L}^{km}\right\|^{\frac{1}{mk}}.$$

Let ${}_nC_k$ denotes the binomial coefficients, and let $C = \sum_{i=1}^{k} {}_kC_i\|\mathcal{A}^{k-i}\|\|H\|^{i-1}$ then

$$\begin{aligned} R_{\mathrm{Sp}}(\mathcal{L}) &\le \overline{\lim}_{n\to\infty}\left[\|\mathcal{A}^k\|^n + \sum_{i=1}^{n} {}_nC_i\|\mathcal{A}^k\|^{n-i}(C\delta)^i\right]^{\frac{1}{kn}} \\ &\le \overline{\lim}_{n\to\infty}\left[\gamma^n(1 + (C\delta/\gamma+1)^n - 1)\right]^{\frac{1}{kn}} \\ &= \gamma^{1/k}(1 + C\delta/\gamma)^{1/k}. \end{aligned}$$

□

It is left now to bound the difference of $\mathcal{L}$ from $\mathcal{A}$. First, we state the following Cauchy-type estimate, whose straightforward proof will be left to the reader:

Proposition 6.3. *Assume that an operator $\mathcal{R}_{cyl}$ is analytic in an open ball $B_r(\hat{f}) \subset \tilde{\mathbf{L}}^1_\rho$. Let $\epsilon < 1$ be small, and $h \in \tilde{\mathbf{L}}^1_\rho$ be such that $|h|_\rho < r$. Then*

$$|\mathcal{R}_{cyl}(\hat{f} + \epsilon h) - \mathcal{R}_{cyl}\hat{f} - \epsilon\mathcal{L}h|_\rho \le \frac{\epsilon^2}{1-\epsilon}\sup_{|s|\le 1}|\mathcal{R}_{cyl}(\hat{f} + sh) - \mathcal{R}_{cyl}\hat{f}|_\rho. \tag{6.8}$$

Note that

$$|\mathcal{L}|_{T_{\hat f}W}|_\rho \le \sup_{j\ge 2} |\mathcal{L}e_j|_\rho.$$

This, together with the preceding Proposition and the compactness property of renormalization, immediately implies that $|\mathcal{L}|_{T_{\hat f}W}|_\rho$ can be bound by a finite difference. Specifically,

$$\begin{aligned}
|\mathcal{L}e_j|_\rho &\le \sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\le\delta}} \left[\epsilon^{-1}\delta^{-1}|\mathcal{R}_{cyl}(\hat f+\epsilon h)-\hat f|_\rho + \frac{\epsilon}{\delta(1-\epsilon)}|\mathcal{R}_{cyl}(\hat f+h)-\hat f)|_\rho\right]\\
&\le \sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\le\delta}} \left[\epsilon^{-1}\delta^{-1}|\mathbb{P}_{\le N}[\mathcal{R}_{cyl}(\hat f+\epsilon h)-\hat f]|_\rho + \frac{\epsilon}{\delta(1-\epsilon)}|\mathbb{P}_{\le N}[\mathcal{R}_{cyl}(\hat f+h)-\hat f]|_\rho\right]\\
&+ \left(\frac{\rho}{\rho'}\right)^{N+1} \sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\le\delta}} \frac{1}{\delta}\left[\epsilon^{-1}|\mathbb{P}_{>N}[\mathcal{R}_{cyl}(\hat f+\epsilon h)-\hat f]|_{\rho'} + \frac{\epsilon}{1-\epsilon}|\mathbb{P}_{>N}[\mathcal{R}_{cyl}(\hat f+h)-\hat f]|_{\rho'}\right]\\
&\equiv C_1, \qquad (6.9)
\end{aligned}$$

for all $j>N$. Similarly, for all $2\le j\le N$:

$$\begin{aligned}
|\mathbb{P}_{>N}\mathcal{L}e_j|_\rho &\le \sup_{\substack{h\in T_{\hat f}W\\ |h|_\rho\le\delta}} \frac{1}{\delta}\left[\epsilon^{-1}|\mathbb{P}_{>N}[\mathcal{R}_{cyl}(\hat f+\epsilon h)-\hat f]|_\rho + \frac{\epsilon}{1-\epsilon}|\mathbb{P}_{>N}[\mathcal{R}_{cyl}(\hat f+h)-\hat f]|_\rho\right]\\
&\le \left(\frac{\rho}{\rho'}\right)^{N+1} \sup_{\substack{h\in T_{\hat f}W\\ |h|_\rho\le\delta}} \frac{1}{\delta}\left[\epsilon^{-1}|\mathbb{P}_{>N}\mathcal{R}_{cyl}(\hat f+\epsilon h)-\hat f]|_{\rho'} + \frac{\epsilon}{1-\epsilon}|\mathbb{P}_{>N}[\mathcal{R}_{cyl}(\hat f+h)-\hat f]|_{\rho'}\right]\\
&\equiv C_2. \qquad (6.10)
\end{aligned}$$

We would like to emphasize that these bounds use the fact that $\mathcal{L}$ is a compact operator in an essential way. Bounds (6.9) and (6.10) can be used to estimate $|(\mathcal{L}-\mathcal{A})|_{T_{\hat f}W}|_\rho$. In particular, according to equation (6.6),

$$|(\mathcal{L}-\mathcal{A})|_{T_{\hat f}W}|_\rho \le \max\left\{|\mathcal{L}\mathbb{P}_{>N}|_\rho, |\mathbb{P}_{>N}\mathcal{L}|_{T_{\hat f}W}\mathbb{P}_{\le N}|_\rho\right\} \le \max\{C_1, C_2\}.$$

This expression provides a bound on δ in (6.7).

We have chosen $N=14$, and experimentally bounded C_1 and C_2 by testing on vectors $h=e_{15}$, and $h=e_2$ which empirically maximize the respective suprema. Choosing $k=80$ in (6.7), we have the following values for the constants that enter estimate (6.7):

$$\gamma < 2.07\times 10^{-18}, \quad \delta < 0.24, \quad C < 8.4\times 10^{-6},$$

therefore, according to (6.7),

$$R_{\mathrm{Sp}}(\mathcal{L}|_{T_{\hat f}W}) < 0.85.$$

A better bound can be obtained if one computes all relevant constants for a larger value of N, which requires more computer time. It is plausible that the spectral radius is close to 0.58, since we have observed that as N increases the largest eigenvalue of the operator

$$\mathbb{P}_{\leq N} \mathcal{L}|_{T_{\hat{f}} W} \mathbb{P}_{\leq N}$$

converges to

$$\lambda = -0.15 - i0.56.$$

This eigenvalue has been truncated to 2 decimal places.

As a final comment, note that the following simple observation implies that perturbations in the directions of the vectors h_j can also be used for estimating the spectral radius of $\mathcal{L}|_{T_{\hat{f}} W}$.

Proposition 6.4. *We have*

$$Spec(\mathcal{L}|_{\mathbf{B}^1_\rho}) \subset Spec(\mathcal{L}|_{\mathbf{L}^1_\rho}) \cup \{0\}.$$

To see this, note that the operator $\mathcal{L}|_{\mathbf{B}^1_\rho}$ has an eigendirection corresponding to linear rescaling which corresponds to eigenvalue 0. We leave the straightforward details to the reader.

7. Appendix

The objective of this Appendix will be to describe how the Cauchy (3.8) and the Hilbert (3.7) transforms can be computed numerically.

The Constructive Measurable Riemann Theorem 3.1 deals with L_p functions which generally do not need to be differentiable. Therefore, one has to chose an appropriate representation of the L_p functions that enter the Theorem 3.1; possibly, as a collection of values on a grid, or as a Fourier series with the radially dependent coefficients. The latter choice has been made, for instance, in [Da1], [Gai], [GK], and will be also adopted in the present paper.

Represent h and $P[h]$ in (3.8) as:

$$h(re^{i\theta}) = \sum_{k=-\infty}^{\infty} h_k(r) e^{ik\theta}, \tag{7.1}$$

$$\tilde{P}[h](re^{i\theta}) = \sum_{k=-\infty}^{\infty} p_k(r) e^{ik\theta}, \tag{7.2}$$

where the coefficients of the P-transform are given by

$$p_k(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} P[h](re^{i\theta}) \, d\theta. \tag{7.3}$$

A classical theorem of analysis (cf [Ahl], [Mar]) states that the Cauchy transform of an L_p-function, $p > 2$, is well-defined and is Hölder continuous with exponent $1 - 2/p$. In

[Da1] and [GK] this fact has been used to compute the Fourier coefficients of the Cauchy transform:

$$p_k(r) = \begin{cases} 2\displaystyle\int_0^r \left(\frac{r}{\rho}\right)^k h_{k+1}(\rho)d\rho, & k<0, \\ -2\displaystyle\int_r^\infty \left(\frac{r}{\rho}\right)^k h_{k+1}(\rho)d\rho, & k \geq 0. \end{cases} \tag{7.4}$$

To obtain similar formulae for the Hilbert transform, assume that h is a Hölder continuous function compactly supported in an open disk around zero of radius R, $B(0,R) \subset \mathbb{C}$. The Hilbert transform of such function is known to exist as a Cauchy principle value (cf [Ahl], [Car]). As with the Cauchy transform, represent this transform as a Fourier series:

$$T[h](re^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k(r)e^{ik\theta}, \quad c_k(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}T[h](re^{i\theta})\, d\theta. \tag{7.5}$$

In [Da2], [Da3] and [GK] the authors arrive at the following expressions for these coefficients:

$$c_0(0) = -2\lim_{\epsilon\to 0}\int_\epsilon^R \frac{h_2(\rho)}{\rho}d\rho, \text{ and } c_k(0)=0, \text{ whenever } k\neq 0, \tag{7.6}$$

$$c_k(r) = A_k\int_0^r \frac{r^k}{\rho^{k+1}}h_{k+2}(\rho)d\rho + B_k\int_r^R \frac{r^k}{\rho^{k+1}}h_{k+2}(\rho)d\rho + h_{k+2}(r), \tag{7.7}$$

where

$$A_k = \begin{cases} 0\ , & k\geq 0, \\ 2(k+1)\ , & k<0, \end{cases} \quad \text{and } B_k = \begin{cases} -2(k+1), & k\geq 0, \\ 0, & k<0. \end{cases} \tag{7.8}$$

We would like to mention that the fact that the Hilbert transform is a singular integral operator makes a rigorous justification of the formulas (7.6)–(7.7) significantly more involved than that of (7.4).

Formulas (7.4) and (7.6)–(7.7) can be used to construct an efficient algorithm for solving a Beltrami equation. Given values of h, for instance, on a circular $N\times M$ grid that contains the compact support of h, one can use a fast Fourier transform (FFT, cf [NR]) to find the values of the coefficients h_k at the radii r_i, $1\leq i\leq M$. Next, one can use these values to construct a piecewise constant, a piecewise linear or a spline approximation of the functions h_k (the choice of approximation, of course, depends on the known or expected smoothness of h_k). This allows one to compute integrals in (7.4) and in (7.6)–(7.7). Armed with these implementations of the Hilbert and Cauchy transforms, one can try to solve the Beltrami equation (3.3), first by running iterations (3.9) for some time, and, finally, applying (3.10). It is convenient to use the point-wise multiplication of grid values of h and $\mu+1$ inside the Hilbert transform in (3.9), rather than the multiplication of their Fourier series: the order of the computational complexity of the point-wise multiplication is $O(NM)$, as opposed to $O(NM^2)$ for the series. The transition from the representation of h as a Fourier series to point values at each iteration step can be performed with the help of the FFT. This way, the computational complexity of one iteration step becomes $O(NM \log_2 M)$.

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