THE GEOMETRY OF MESOSCOPIC PHASE TRANSITION INTERFACES

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Abstract. We consider a mesoscopic model of phase transitions and we investigate the geometric properties of the interfaces of the associated minimal solutions. We provide density estimates for level sets and, in the periodic setting, we construct minimal interfaces at a universal distance from any given hyperplane.

Contents

Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $u \in W^{1,2}(\Omega)$, we consider the energy functional

$$E_\Omega(u) := \int_\Omega \Big(|\nabla u(x)|^2 + F(x,u) + H(x)\, u(x) \Big)\, dx\,. \tag{1}$$

The function F here above is supposed to be a so-called "double-well potential". More precisely, we assume that:

- F is non-negative, locally bounded and $F(x,1) = F(x,-1) = 0$;
- for any $\theta \in [0,1)$, $\inf_{|u|\le\theta} F(x,u) > 0$;
- there exist $\ell \in (0,1/2)$ so that:
  - $F(x,t) \ge \text{const}\,(1-|t|)^2$, if $|t| \in (\ell,1)$;
  - F is C^1 and, if $|s| < \ell$, then
  $$F_u(x,-1+s) \ge \text{const}\, s\,, \qquad F_u(x,1-s) \le -\text{const}\, s\,;$$
  - $F_u(x,u)$ is increasing for $u \in [-1-\ell,-1+\ell] \cup [1-\ell,1+\ell]$.

The function $H \in L^\infty(\mathbb{R}^n)$ in (1) will be thought as a small perturbation of the standard Ginzburg-Landau-Allen-Cahn functional. To this extent, we suppose that

$$\sup_{\mathbb{R}^n} |H| \le \eta\,,$$

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where η will be taken suitably small (in dependence of n and of the structural constants of the problem). We also assume that H is $\mathbb{Z}^n$-periodic and with zero-average, that is

$$\begin{aligned} H(x+k) &= H(x) \qquad \forall k \in \mathbb{Z}^n \\ \text{and } \int_{[0,1]^n} H(x)\, dx &= 0\,. \end{aligned} \tag{2}$$

The functional in (1) has been introduced in [DLN05] (see also [CCO05] and references therein) as a mesoscopic model for phase transitions, and its limiting behavior in the Γ-convergence sense in relation with suitable anisotropic surface energy has been established there (under more severe hypotheses then the ones in this paper). Heuristically, one may think that the functional in (1) is given by three terms (the first two being the ingredients of the standard Ginzburg-Landau model):

- a "kinetic interaction term" (namely, $|\nabla u|^2$), which penalizes the phase changes of the system,
- a double-well potential term (i.e., F), which penalizes sensitive deviations from the "pure phases" ± 1,
- a "mesoscopic" term (namely, Hu) which, at each point, prefers one of the two phases, but is "neutral" in the average.

We say that u is a local minimizer in Ω if

$$E_\Omega(u+\phi) \ \ge\ E_\Omega(u) \tag{3}$$

for any $\phi \in W_0^{1,2}(\Omega)$.

We denote by $\mathcal{L}$ the n-dimensional Lebesgue measure on $\mathbb{R}^n$. We prove the following density estimates for local minimizers:

Theorem 1. *Fix $\delta > 0$. Let u be a local minimizer in a domain Ω, with $|u| \le 3/2$. Then, there exist positive constants c and r_0, depending only on δ and on the structural constants, in such a way that*

$$E_{B_r(\xi)}(u) \ \le\ c\, r^{n-1}\,,$$

for any $r \ge r_0$, provided that $B_{r+\delta}(\xi) \subseteq \Omega$.

Theorem 2. *Fix $\delta > 0$. Let u be a local minimizer in a domain Ω, with $|u| \le 1 + \eta'$, for some $\eta' \ge 0$. Then, for any $\theta_0 \in (0,1)$, for any $\theta \in [-\theta_0, \theta_0]$ and for any $\mu_0 > 0$, if*

$$\mathcal{L}\Big(B_K(\xi) \cap \{u \ge \theta\}\Big) \ \ge\ \mu_0\,, \tag{4}$$

then there exist positive constants $\widehat{c}$, $c^\star$ and r_0, depending on K, δ, μ_0, θ_0 and on the structural constants, such that

$$\mathcal{L}\Big(B_r(\xi) \cap \{u \ge \theta\}\Big) \ge c^\star r^n\,,$$

for any $r \in [r_0, \widehat{c}/\eta]$, provided that η and η' are suitably small (depending on n, μ_0, θ_0, δ and the structural constants of F) and that $B_{r+\delta}(x) \subseteq \Omega$.

Analogously, if

$$\mathcal{L}\Big(B_K(\xi) \cap \{u \le \theta\}\Big) \ \ge\ \mu_0\,, \tag{5}$$

then

$$\mathcal{L}\Big(B_r(\xi) \cap \{u \le \theta\}\Big) \ge c^\star r^n\,,$$

for any $r \in [r_0, \widehat{c}/\eta]$, provided that η and η' are suitably small (depending on n, μ_0, θ_0, δ and the structural constants of F) and that $B_{r+\delta}(\xi) \subseteq \Omega$.

The original idea of such density estimates goes back to [CC95]. An analogue of Theorem 1 when $H = 0$ plays also an important rôle in [AAC01]. Related techniques have been exploited in [Val04], [PV05a] and [PV05b]. Analogous density estimates for Caccioppoli sets are also crucial in the study of minimal surface functionals penalized by a volume term (see [CdlL01]).
As a consequence of Theorems 1 and 2, we show that, once the minimizer is controlled at a given point, the levels sets suitably far from ± 1 occupy a "small portion" of the space, at a suitably large scale. This will also allow to replace the measure theoretic assumptions (4) and (5) by pointwise assumptions, that are often easier to deal with in applications.

Theorem 3. *Fix $\delta > 0$ and $\theta_0 \in (0,1)$. Let u be a local minimizer in a domain Ω, with $|u| \le 1 + \eta'$, for some $\eta' \ge 0$. Suppose that $|u(x)| \le \theta_0$ for some $x \in \Omega$. Then, there exist positive constants c, $\widehat{c}$, and r_0, possibly depending on θ_0, δ and on the structural constants, such that*

$$\min \Big\{ \mathcal{L}\Big(B_r(x) \cap \{u > \theta_0\}\Big) \, , \, \mathcal{L}\Big(B_r(x) \cap \{u < -\theta_0\}\Big) \Big\} \ge c r^n \tag{6}$$

and

$$\mathcal{L}\Big(B_r(x) \cap \{|u| < \theta_0\}\Big) \ge c r^{n-1} \, , \tag{7}$$

for any $r \in [r_0, \widehat{c}/\eta]$, provided that η and η' are suitably small (depending on n, μ_0, θ_0, δ and the structural constants of F) and that $B_{r+\delta}(x) \subseteq \Omega$.

We now consider the problem of finding minimizers of our functional in a periodic setting, whose level sets lie in a strip of universal width and assigned slope. These kind of problems are related with a PDE version of Mather theory, as recently developed (among others) in [Mos86], [Ban89], [CdlL01], [Val04] and [RS04]. In this framework, we prove the following result:

Theorem 4. *Let F satisfy the assumptions on page 1 and suppose also that*

$$F(x+k, u) \; = \; F(x, u) \tag{8}$$

for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $k \in \mathbb{Z}^n$, that

$$F(x, -1+s) \; = \; F(x, 1+s) \tag{9}$$

for any $s \in [-\delta_0, \delta_0]$ and that

$$F_u(x, -1-s) \le -c \quad \text{and} \quad F_u(x, 1+s) \ge c \tag{10}$$

for any $s \ge \delta_0$, for suitable $c > 0$ and $\delta_0 \in (0, 1/10)$.
Then, there exists a positive constant M_0, depending only on n and on the structural constants of the functional, so that the following holds.
Fixed any $\omega \in \mathbb{R}^n \setminus \{0\}$, there exists a function

$$u_\omega : \mathbb{R}^n \longrightarrow [-1-\delta_0, 1+\delta_0]$$

which is a local minimizer in any bounded domain of $\mathbb{R}^n$ and so that

$$\{|u_\omega| \le 1 - \delta_0\} \subseteq \left\{ \xi \in \mathbb{R}^n \text{ such that } \left| \xi \cdot \frac{\omega}{|\omega|} \right| \le M_0 \right\} , \tag{11}$$

provided that η is suitably small (possibly in dependence of δ_0).
Moreover, u_ω enjoys the following quasi-periodicity and monotonicity properties:

- *if* $\omega \in \mathbb{Q}^n$*, then*

$$u_\omega(x+k) = u_\omega(x)\,, \tag{12}$$

for any $x \in \mathbb{R}^n$ *and any* $k \in \mathbb{Z}^n$ *such that* $\omega \cdot k = 0$*, and*

$$u_\omega(x+k) \leq u_\omega(x) \tag{13}$$

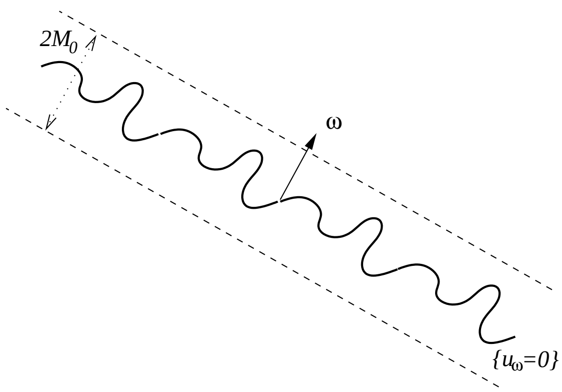
for any $x \in \mathbb{R}^n$ *and any* $k \in \mathbb{Z}^n$ *such that* $\omega \cdot k \geq 0$*;*

- *if* $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$*, then given any sequence of vectors* $\omega_j \in \mathbb{Q}^n$ *so that*

$$\lim_{j\to+\infty} \omega_j = \omega\,,$$

there exists a sequence of functions $u_{\omega_j} : \mathbb{R}^n \longrightarrow [-1-\delta_0, 1+\delta_0]$*, which are local minimizers in any bounded domain of* $\mathbb{R}^n$*, which satisfy the level set constraint and the periodicity and monotonicity properties in* (11)*,* (12) *and* (13) *(with* ω_j *replacing* ω *there), and which converge to* u_ω *uniformly on compact subsets of* $\mathbb{R}^n$*, up to subsequences.*

Theorem 4 may be seen as an extension of Theorem 8.1 of [Val04] (and it reduces to it when $H = 0$). Roughly speaking, it says that, given any hyperplane π in $\mathbb{R}^n$, it is possible to construct a minimal interface of the mesoscopic model lying at a bounded universal distance from π (namely, in the statement of Theorem 4, ω is just a normal vector to π, the interface is given by the level sets $\{|u_\omega| \leq 1-\delta\}$ and the universal distance from π is given by M_0).



The zero level set of u_ω, as in Theorem 4.

Theorem 4 is also related to Mather theory, in the sense that it constructs minimal solutions of any given "frequency" ω, as well as minimal measures of any given rotation vector are constructed in Lagrangian dynamical systems (see [Mat91]).

The proof of Theorem 4 relies on the construction given in [CdlL01] and [Val04]. It will make use of the density estimates of Theorem 3 and of a suitable energy renormalization. We point out that assumption (9) implies that the two (periodic) global minimizers $u_\pm$ given by Lemma 7 below have the same energy on bounded periodic domains, and this fact is in turn necessary for the existence of the minimizer u_ω in Theorem 4. Indeed, assumption (10) is necessary to let the minimization method work (see, e.g., Lemma 7 below), while assumption (9)

makes it possible to appropriately define a rescaled energy functional (see formula (49) below). In case $H = 0$, assumptions (9) and (10) are not needed (see [Val04]).
To end this introduction, we note that, while the density estimates in the usual Ginzburg-Landau setting are valid for any $r \geq r_0$ (see [CC95] and [Val04]), the results in Theorems 2 and 3 here hold on the intermediate (i.e., "mesoscopic") range of radii between r_0 and $O(1/\eta)$. A similar mesoscopic effect happens in the density estimates for volume penalized minimal surface functionals: see [CdlL01]. On the other hand, the minimization property in Theorem 4 does hold at any scale.

Proof of Theorem 1

First, we show that the integral of H in large balls grows way less than the size of the balls:
Lemma 5. *There exists suitably large positive constants r_0 and C in such a way*

$$\left| \int_{B_r(x)} H(y)\, dy \right| \leq C\, \eta\, r^{n-1} ,$$

for any $x \in \mathbb{R}^n$ and $r \geq r_0$.

Proof. For any $k \in \mathbb{Z}^n$, we set $Q_k := k + [0,1]^n$. As a consequence of (2), we have that

$$\int_{Q_k} H = 0 , \qquad \forall k \in \mathbb{Z}^n .$$

We denote by Y the collection of the cubes Q_k which lie inside the ball $B_r(x)$. In this way,

$$\bigcup_{Q_k \in Y} Q_k \subseteq B_r(x)$$

and the above union is non-overlapping.
Moreover, if we set

$$D_r := B_r(x) \setminus \bigcup_{Q_k \in Y} Q_k ,$$

we get that

$$D_r \subseteq \{ y \in \mathbb{R}^n \text{ s.t. } r - \sqrt{n} \leq |x - y| \leq r \} .$$

and so $\mathcal{L}(D_r) \leq \text{const}\, r^{n-1}$ for large r. Then,

$$\left| \int_{B_r(x)} H \right| = \left| \int_{D_r} H \right| \leq \text{const}\, \eta\, r^{n-1} ,$$

as desired. □

We now observe that

$$\Delta u = F_u(x, u) + H(x) \tag{14}$$

in Ω, due to (3). Therefore, given any domains $V \subset U$ contained in B_r, we have that

$$\|u\|_{W^{1,2}(V)} \leq \text{const}\, \sqrt{\mathcal{L}(U)} , \tag{15}$$

thanks to interior elliptic estimates (see, e.g., Theorem 1 on page 309 of [Eva98]; the constant in (15) may depend on the boundary distance of V and U). Let now $h \in C^\infty(\Omega)$ be so that $h = -1$ in B_{r-1} and $h = 2$ in $\Omega \setminus B_r$. Let also $\tau \in C^\infty(\Omega)$ be so that $\tau = -1$ in B_{r-1} and $\tau = -2$ in $\Omega \setminus B_r$. Of course, we can take $\|h\|_{C^1(\Omega)}$ and $\|\tau\|_{C^1(\Omega)}$ to be less than a suitably large constant.

We also define

$$\tilde{u}(x) := \max\{u(x),\, \tau(x)\} \ \text{ and } \ \sigma(x) \,:=\, \min\{\tilde{u}(x),\, h(x)\}\,.$$

Let $\phi := \sigma - u$. We have that ϕ is in $W^{1,2}(\Omega)$ since u, h, τ, $\tilde{u}$ and σ do. Also, $\sigma = \tilde{u} = u$ in $\Omega \setminus B_r$, since we assumed that $|u| \le 3/2$. Therefore, $\phi \in W_0^{1,2}(B_r)$ and so, by (3),

$$E_\Omega(u) \le E_\Omega(u + \phi) = E_\Omega(\sigma)\,.$$

Then, since $\sigma = -1$ in B_{r-1},

$$E_\Omega(u) \le \int_{B_r \setminus B_{r-1}} \Big(|\nabla \sigma|^2 + F(x, \sigma) + H(x)\sigma(x) \Big)\, dx - \int_{B_{r-1}} H(x)\, dx\,. \tag{16}$$

Also, by applying (15) with $V := B_r \setminus B_{r-1}$ and $U := B_{r+\delta/2} \setminus B_{r-1-\delta/2}$, we get that

$$\begin{aligned} \int_{B_r \setminus B_{r-1}} |\nabla \sigma|^2 &\le \int_{B_r \setminus B_{r-1}} \Big(|\nabla u|^2 + |\nabla h|^2 + |\nabla \tau|^2 \Big) \\ &\le \mathrm{const}\, \mathcal{L}(B_{r+\delta/2} \setminus B_{r-1-\delta/2}) \\ &\le \mathrm{const}\, r^{n-1}\,, \end{aligned} \tag{17}$$

as long as r is conveniently large. Also, $|\sigma| \le 2$ by construction, and so

$$\int_{B_r \setminus B_{r-1}} F(x, \sigma) + H(x)\sigma(x)\, dx \ \le\ \mathrm{const}\, r^{n-1} \tag{18}$$

for large r.
By collecting the estimates in (16), (17) and (18), and by exploiting Lemma 5, the claim in Theorem 1 plainly follows.

Proof of Theorem 2

We begin with a technical observation:

Lemma 6. *Fix $\nu \in \mathbb{N}$. Let $a_k \ge 0$ be a sequence such that $a_1 \ge c_0$, $a_k \le C_0\, L^\nu k^{n-1}$, and*

$$\Big(\sum_{1 \le j \le k} a_j \Big)^{(n-1)/n} \ \le\ C_0 \Big(a_{k+1} + \sum_{1 \le j \le k} e^{-L(k+1-j)} a_j + \hat{c} L^\nu k^{n-1} \Big) \tag{19}$$

for any $k \in \mathbb{N}$ and some positive constants $\hat{c}$, L, c_0, and C_0. Then, if L is suitably large (in dependence of ν, n, c_0 and C_0) and $\hat{c}$ is suitably small (in dependence of ν, n, c_0 C_0 and L), there exists $\bar{c} > 0$, depending on n, c_0 and C_0, such that

$$a_k \ge \bar{c}\, k^{\nu-1}$$

for any $k \in \mathbb{N}$.

Proof. The argument we present here is a modification of the one given on page 10 of [CC95]. We define

$$\bar{c} \,:=\, \min\Big\{ c_0\,,\ \frac{1}{2^{n^2} C_0^n n^{(n-1)}} \Big\}\,. \tag{20}$$

We also suppose that L is so large that

$$L^\nu e^{-L} \ \le\ \frac{\bar{c}^{(n-1)/n}}{8 C_0^2 n^{(n-1)/n}} \tag{21}$$

and

$$e^L \geq 2\,. \tag{22}$$

Further, we assume that $\hat{c}$ is so small that

$$\hat{c} \leq \frac{\bar{c}^{(n-1)/n}}{4C_0\, L^\nu\, n^{(n-1)/n}}\,. \tag{23}$$

The proof is by induction. If $k=1$, the claim is true, thanks to (20). Thus, we now take $k \geq 1$, we suppose the claim to hold for any $j = 1, \ldots, k$ and we prove it for a_{k+1}. To this effect, we observe that, from the inductive hypothesis,

$$\begin{aligned}
\Big(\sum_{1\leq j\leq k} a_j\Big)^{(n-1)/n} &\geq \bar{c}^{(n-1)/n}\Big(\sum_{1\leq j\leq k} j^{n-1}\Big)^{(n-1)/n}\\
&\geq \bar{c}^{(n-1)/n}\Big(\int_0^k t^{n-1}\,dt\Big)^{(n-1)/n}\\
&= \frac{\bar{c}^{(n-1)/n}}{n^{(n-1)/n}}\cdot k^{n-1}\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{1\leq j\leq k} e^{-L(k+1-j)}a_j &\leq C_0L^\nu \sum_{1\leq j\leq k} e^{-L(k+1-j)}j^{n-1}\\
&\leq C_0L^\nu k^{n-1}e^{-L}\sum_{i\geq 0} e^{-Li}\\
&\leq 2C_0L^\nu k^{n-1}e^{-L}\,,
\end{aligned}$$

due to (22).
By collecting the above estimates, we thus deduce from (19) that

$$\begin{aligned}
a_{k+1} &\geq \Big(\frac{\bar{c}^{(n-1)/n}}{C_0\, n^{(n-1)/n}} - 2C_0L^\nu e^{-L} - \hat{c}L^\nu\Big)\, k^{n-1}\\
&\geq \frac{\bar{c}^{(n-1)/n}}{2\,C_0\, n^{(n-1)/n}}\cdot k^{n-1}
\end{aligned}$$

due to (21) and (23).
We also notice that (20) and the fact that $k \geq 1$ imply that

$$\bar{c}^{1/n} \leq \frac{1}{2\,C_0\, n^{(n-1)/n}}\cdot\Big(\frac{k}{k+1}\Big)^{n-1}\,.$$

Then, the above inequalities give that $a_{k+1} \geq \bar{c}(k+1)^{n-1}$, as desired. □

We now deal with the proof of the first claim in Theorem 2, the second claim being analogous. For this, we borrow several ideas from [CC95] and [Val04]. First, we observe that, with no loss of generality, we may assume θ to be as close to -1 as we wish. Indeed: assume the result to be true for $\theta^\star$ (say, close to -1), and let $\theta \in [-\theta_0, \theta_0]$, with $\theta^\star \leq -\theta_0$. Then,

$$\mu_0 \leq \mathcal{L}\Big(\{u \geq \theta\} \cap B_K\Big) \leq \mathcal{L}\Big(\{u \geq \theta^\star\} \cap B_K\Big)\,,$$

therefore, using the result for $\theta^\star$ and Theorem 1, we conclude that

$$\begin{aligned}
\mathrm{const}\, r^n &\leq \mathcal{L}\Big(\{u \geq \theta^\star\} \cap B_r\Big) \\
&\leq \mathcal{L}\Big(\{u \geq \theta\} \cap B_r\Big) + \mathcal{L}\Big(\{\theta^\star \leq u < \theta\} \cap B_r\Big) \\
&\leq \mathcal{L}\Big(\{u \geq \theta\} \cap B_r\Big) + \frac{1}{\inf\limits_{u\in[\theta^\star,\theta_0]} F} \int_{B_r} F(\xi,u)\, d\xi \\
&\leq \mathcal{L}\Big(\{u \geq \theta\} \cap B_r\Big) + \mathrm{const}\, E_{B_r}(u) + \mathrm{const}\, \eta\, r^n \\
&\leq \mathcal{L}\Big(\{u \geq \theta\} \cap B_r\Big) + \mathrm{const}\, (r^{n-1} + \eta\, r^n)\,,
\end{aligned}$$

which gives that

$$\mathcal{L}\Big(\{u \geq \theta\} \cap B_r\Big) \geq \mathrm{const}\, r^n$$

for large r and small η. Thus, in the rest of the proof, we may and do assume that θ is as close to -1 as we wish.

In what follows, A is a suitably large positive parameter; we will also make use of two further parameters Θ and T: we will fix Θ small enough and then choose T so that ΘT is suitably large (possibly depending on θ_0). We also set

$$\bar\theta = \theta - C_* e^{-\Theta T}\,, \tag{24}$$

where C_* denotes a suitably large constant.
Let $k \in \mathbb{N}$. On page 183 of [Val04], a function $\tilde h \in C^{1,1}([0,(k+1)T])$ was constructed so that $-1 \leq \tilde h \leq 1$, $\tilde h((k+1)T) = 1$, $\tilde h'(0) = 0$,

$$\tilde h(\tau) + 1 \leq \mathrm{const}\, e^{-\Theta T(k+1-j)} \tag{25}$$

if $\tau \in [(j-1)T, jT]$, for $j = 1, \dots, k+1$,

$$|\tilde h'(\tau)| \leq \mathrm{const}\, \Theta\tau(\tilde h(\tau) + 1)$$

if $\tau \in [0,1]$,

$$|\tilde h'(\tau)| \leq \mathrm{const}\, \Theta(\tilde h(\tau) + 1)$$

if $\tau \in [1,(k+1)T]$, and

$$|\tilde h''(\tau)| \leq \mathrm{const}\, \Theta(\tilde h(\tau) + 1) \tag{26}$$

if $\tau \in [0,(k+1)T]$. We then define

$$\begin{aligned}
&h(x) := (1+\eta')(\tilde h(|x|) + 1) - 1\,, \qquad \sigma(x) := \min\{u(x),\, h(x)\} \\
&\text{and} \qquad \beta(x) := \min\{u(x) - \sigma(x),\, 1 + \bar\theta\}\,.
\end{aligned}$$

Since $h \geq 1 + \eta' \geq u$ on $\partial B_{(k+1)T}$, it follows that $\sigma = u$ on $\partial B_{(k+1)T}$ and so

$$E_{B_{(k+1)T}}(u) \leq E_{B_{(k+1)T}}(\sigma) \tag{27}$$

as long as $B_{(k+1)T} \subset \Omega$, due to (3). We use the Cauchy and Sobolev Inequalities and (27), to gather that

$$
\begin{aligned}
\Big(\int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} \Big)^{\frac{n-1}{n}} &\leq \text{const} \int_{B_{(k+1)T}\cap\{u-\sigma\leq 1+\bar\theta\}} |\beta|\,|\nabla\beta| \\
&\leq \text{const}\, A\Big(\int_{B_{(k+1)T}\cap\{u>\sigma\}} (|\nabla u|^2 - |\nabla\sigma|^2 - 2\nabla(u-\sigma)\cdot\nabla\sigma)\Big) \\
&\quad + \frac{\text{const}}{A} \int_{B_{(k+1)T}\cap\{u-\sigma\leq 1+\bar\theta\}} (u-\sigma)^2 \\
&= \text{const}\, A\Big(\int_{B_{(k+1)T}} (|\nabla u|^2 - |\nabla\sigma|^2) \\
&\quad +2\int_{B_{(k+1)T}\cap\{u>\sigma\}} (u-\sigma)\,\Delta\sigma\Big) \\
&\quad + \frac{\text{const}}{A} \int_{B_{(k+1)T}\cap\{u-\sigma\leq 1+\bar\theta\}} (u-\sigma)^2 \\
&\leq \text{const}\, A\Big[\int_{B_{(k+1)T}\cap\{u>\sigma\}} \Big(F(x,\sigma)-F(x,u)+H(x)(\sigma-u)\Big) \\
&\quad +2\int_{B_{(k+1)T}} (u-\sigma)\,\Delta\sigma\Big] + \frac{\text{const}}{A} \int_{B_{(k+1)T}\cap\{u-\sigma\leq 1+\bar\theta\}} (u-\sigma)^2 .
\end{aligned} \tag{28}
$$

We now estimate the left hand side of (28). If ΘT is large enough and η' is small enough, we see from (25) that $\theta - h \geq (1-\theta_0)/2$ in B_{kT}. Consequently,

$$
\beta \geq \frac{1-\theta_0}{2} \text{ in } B_{kT}\cap\{u>\theta\}. \tag{29}
$$

Thus, given $\rho \geq 0$, if we set

$$
V(\rho) := \mathcal{L}\Big(B_\rho \cap \{u>\theta\}\Big),
$$

we deduce from (29) that the left hand side of (28) is bigger than

$$
\text{const}\, V(kT)^{\frac{n-1}{n}} .
$$

Let us now estimate the right hand side of (28). To this extent, we denote the right hand side of (28) by

$$
I_1 + I_2 ,
$$

with

$$
\begin{aligned}
I_1 &:= \text{const}\, A \int_{B_{(k+1)T}\cap\{u>\sigma\}} \Big(H(x)(\sigma-u)\Big) \qquad \text{and} \\
I_2 &:= \text{const}\, A\Big[\int_{B_{(k+1)T}\cap\{u>\sigma\}} \Big(F(x,\sigma)-F(x,u)\Big) \\
&\quad +2\int_{B_{(k+1)T}} (u-\sigma)\,\Delta\sigma\Big] + \frac{\text{const}}{A} \int_{B_{(k+1)T}\cap\{u-\sigma\leq 1+\bar\theta\}} (u-\sigma)^2 .
\end{aligned}
$$

First of all, we estimate I_1. To this effect, we recall that $r := (k+1)T \in [r_0, \widehat{c}/\eta]$ and so

$$\begin{aligned} I_1 &\le \operatorname{const} \eta\, \mathcal{L}(B_{(k+1)T}) \le \operatorname{const} \eta\, (k+1)^n T^n \\ &\le \operatorname{const} \widehat{c}\, (k+1)^{n-1} T^{n-1} \le \operatorname{const} \widehat{c}\, k^{n-1} T^{n-1} . \end{aligned}$$

We now estimate I_2. For this scope, we first consider the contribution of I_2 in $\{u \le \theta\}$. Since $h \ge -1$, we have that $-1 \le h = \sigma \le u$ at any point of $\{u > \sigma\}$, and so

$$\begin{aligned} & (u+1)^2 - (\sigma+1)^2 - \frac{1}{2}(u-\sigma)^2 \\ = \;& (u-\sigma)\left(\frac{1}{2}u + \frac{3}{2}\sigma + 2\right) \ge 0 \end{aligned}$$

in $\{u > \sigma\}$. Accordingly, in $\{\sigma < u \le \theta\}$,

$$\begin{aligned} F(x,u) - F(x,\sigma) &= \int_\sigma^u F_u(x,\zeta)\, d\zeta \\ &\ge \operatorname{const} \int_\sigma^u (\zeta+1)\, d\zeta \\ &= \operatorname{const} \Big[(u+1)^2 - (\sigma+1)^2\Big] \\ &\ge \operatorname{const} (u-\sigma)^2 . \end{aligned}$$

The latter estimate and (26) imply that the contribution of I_2 in $\{u \le \theta\}$ is controlled by

$$\int_{B_{(k+1)T} \cap \{\sigma < u \le \theta\}} \Big(F(x,\sigma) - F(x,u) + \operatorname{const} \sqrt{\Theta} F_u(x,\sigma)(u-\sigma) \Big) \tag{30}$$

as long as A is sufficiently large.
We now show that this quantity is indeed negative. Since we assumed θ to be close to -1, we have that F and F_u are monotone in $\{\sigma < u \le \theta\}$, that $F(x,\sigma) - F(x,u)$ is negative and that

$$|F_u(x,\sigma)(u-\sigma)| \le |F(x,\sigma) - F(x,u)| .$$

Since we assumed Θ to be small, we conclude that the quantity in (30) is negative, and then so is the contribution of I_2 in $\{u \le \theta\}$.
Let us now bound the contribution of I_2 in $\{u > \theta\}$. The contribution in $B_{(k+1)T} \setminus B_{kT}$ of such term is bounded by

$$\int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{u > \theta\}} \Big(|F(x,\sigma) - F(x,u)| + (\sigma+1)(u-\sigma) + (u-\sigma)^2 \Big) ,$$

thanks to (26). The above quantity is then bounded by

$$\begin{aligned} & \mathcal{L}\Big(\{u > \theta\} \cap (B_{(k+1)T} \setminus B_{kT}) \Big) \\ = \;& V((k+1)T) - V(kT) . \end{aligned}$$

Let us now look at the contribution of I_2 in $\{u > \theta\} \cap B_{kT}$. We observe that

$$B_{kT} \cap \{\sigma < u \le \sigma + 1 + \bar{\theta}\} \subseteq B_{kT} \cap \{\sigma < u \le \theta\} ,$$

due to (25), provided that C_* in (24) is large enough.
Consequently,

$$\int_{B_{kT} \cap \{u - \sigma \le 1 + \bar{\theta}\} \cap \{u > \theta\}} (u-\sigma)^2 = 0$$

and so the contribution of I_2 in $\{u>\theta\}\cap B_{kT}$ is controlled by

$$\int_{B_{kT}\cap\{u>\theta\}}\Big(F(x,\sigma)-F(x,u)+|\Delta h|\Big)\leq$$

$$\leq \sum_{j=1}^{k}\int_{B_{jT}\setminus B_{(j-1)T}\cap\{u>\theta\}}\Big(F(x,h)+|\Delta h|\Big)\,. \tag{31}$$

By our assumption on F, we have that

$$F(x,-1+s)\leq \text{const}\, s\,,$$

provided that $s>0$ is small enough. Thus, we bound the above term in (31) by

$$\sum_{j=1}^{k}e^{-\Theta T(k+1-j)}\Big[V(jT)-V((j-1)T)\Big]\,,$$

thanks to (25). Thus, the quantity above provides a bound for the contribution of I_2 in $\{u>\theta\}\cap B_{kT}$.

By collecting all theses estimates, we get that

$$\begin{aligned}&\text{const}\,(V(kT))^{\frac{n-1}{n}}\\ \leq\;& V((k+1)T)-V(kT)+\sum_{j=1}^{k}e^{-\Theta T(k+1-j)}\Big[V(jT)-V((j-1)T)\Big]\\ +\;&\widehat{c}\,k^{n-1}T^{n-1}\,.\end{aligned}$$

Then, the desired result follows from Lemma 6, applied here with $a_j:=V(jT)-V((j-1)T)$.

Proof of Theorem 3

This is a modification of some arguments on pages 167–169 of [Val04].

We first prove (6). To this effect, we define $\hat\theta:=(1+\theta_0)/2$. Exploiting (14) and interior elliptic regularity theory (see, e.g. Theorem 3.13 in [HL97]), we have that u is uniformly Lipschitz continuous in $B_1(x)$, with Lipschitz constant, say, $\Lambda\geq 1$. Thus,

$$|u(y)|\leq|u(x)|+\Lambda|x-y|<\hat\theta\,,$$

as long as $|x-y|<(1-\theta_0)/(2\Lambda)=:K$. Then,

$$\min\Big\{\mathcal{L}\Big(B_K(x)\cap\{u\geq-\hat\theta\}\Big)\,,\;\mathcal{L}\Big(B_K(x)\cap\{u\leq\hat\theta\}\Big)\Big\}\;=\;\mathcal{L}(B_K(x))\,,$$

which gives the analogous of assumptions (4) and (5). Accordingly, by Theorem 2,

$$\min\Big\{\mathcal{L}\Big(B_r(x)\cap\{u\geq-\hat\theta\}\Big)\,,\;\mathcal{L}\Big(B_r(x)\cap\{u\leq\hat\theta\}\Big)\Big\}\;\geq\;\text{const}\,r^n\,,$$

for $r\in[r_0,\widehat{c}/\eta]$.

Consequently, exploiting Theorems 1 and 2,

$$\begin{aligned}
& \mathcal{L}\Big(B_r(x)\cap\{u>\theta_0\}\Big) \\
\geq\; & \mathcal{L}\Big(B_r(x)\cap\{u\geq -\hat\theta\}\Big) - \mathcal{L}\Big(B_r(x)\cap\{\theta_0\geq u\geq -\hat\theta\}\Big) \\
\geq\; & \mathrm{const}\, r^n - \frac{1}{\inf_{u\in[-\hat\theta,\theta_0]} F}\int_{B_r(x)\cap\{\theta_0\geq u\geq-\hat\theta\}} F(x,u)\,dx \\
\geq\; & \mathrm{const}\, r^n - \mathrm{const}\, E_{B_r(x)}(u) - \mathrm{const}\,\eta\, r^n \\
\geq\; & \mathrm{const}\, r^n - \mathrm{const}\, r^{n-1} \\
\geq\; & \mathrm{const}\, r^n\,,
\end{aligned}$$

for large r and small η.
Analogously,

$$\mathcal{L}\Big(B_r(x)\cap\{u<-\theta_0\}\Big) \;\geq\; \mathrm{const}\, r^n\,,$$

as desired. The latter two estimates complete the proof of (6).
We now prove (7). For this scope, we denote by $\mathrm{Per}_U(E)$ the perimeter of the (Caccioppoli) set E in the (open) set U (see, e.g., [Giu84]). We also define

$$\bar u(x) \;:=\; \begin{cases} u(x) & \text{if } |u(x)|\leq\theta_0, \\ \theta_0 & \text{if } u(x)>\theta_0, \\ -\theta_0 & \text{if } u(x)<-\theta_0 \end{cases}$$

and

$$\mu(t,r) := \min\Big\{\mathcal{L}\Big(B_r(x)\cap\{\bar u\geq t\}\Big)\,,\ \mathcal{L}\Big(B_r(x)\cap\{\bar u<t\}\Big)\Big\}\,.$$

Exploiting (6), we have that, if $t\in(-\theta_0,\theta_0)$,

$$\begin{aligned}
\mu(t,r) \;\geq\;& \min\Big\{\mathcal{L}\Big(B_r(x)\cap\{\bar u\geq\theta_0\}\Big)\,,\ \mathcal{L}\Big(B_r(x)\cap\{\bar u\leq-\theta_0\}\Big)\Big\} \\
\geq\;& \min\Big\{\mathcal{L}\Big(B_r(x)\cap\{u>\theta_0\}\Big)\,,\ \mathcal{L}\Big(B_r(x)\cap\{u<-\theta_0\}\Big)\Big\} \\
\geq\;& \mathrm{const}\, r^n\,.
\end{aligned}$$

We now use the above estimate and the Coarea and Isoperimetric Formulas (see, e.g., [Giu84]) to deduce that

$$\begin{aligned}
\int_{B_r(x)\cap\{|u|<\theta_0\}} |\nabla u| \;=\;& \int_{B_r(x)} |\nabla\bar u| \\
\geq\;& \int_{-\theta_0}^{\theta_0} \mathrm{Per}_{B_r(x)}\Big(\{\bar u<t\}\Big)\,dt \\
\geq\;& \mathrm{const}\int_{-\theta_0}^{\theta_0}\Big(\mu(t,r)\Big)^{(n-1)/n}\,dt \\
\geq\;& \mathrm{const}\, r^{n-1}\,.
\end{aligned}$$

Consequently, taking a suitably large additional parameter A, by the Cauchy Inequality and Theorem 1, we have that

$$\begin{aligned}
\mathrm{const}\, r^{n-1} &\leq \frac{1}{A}\int_{B_r(x)} |\nabla u|^2 + A\,\mathcal{L}\Big(B_r(x)\cap\{|u|<\theta_0\}\Big) \\
&\leq \frac{1}{A} E_{B_r(x)}(u) + \mathrm{const}\,\eta\, r^n + A\,\mathcal{L}\Big(B_r(x)\cap\{|u|<\theta_0\}\Big) \\
&\leq \frac{\mathrm{const}\, r^{n-1}}{A} + \mathrm{const}\,\eta\, r^n + A\,\mathcal{L}\Big(B_r(x)\cap\{|u|<\theta_0\}\Big)\,.
\end{aligned}$$

Using that ηr is assumed to be small and choosing A appropriately large, (7) follows. This ends the proof of Theorem 3.

Proof of Theorem 4

Let $Q := [0,1]^n$. We define the Q-periodic functions in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$\begin{aligned}
W^{1,2}_{\mathrm{per}}(Q) := \Big\{ & u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \text{ such that} \\
& u(x+e_j) = u(x) \text{ for any } x\in\mathbb{R}^n \Big\}\,,
\end{aligned} \tag{32}$$

where $\{e_1,\dots,e_n\}$ is the standard Euclidean base of $\mathbb{R}^n$. With this setting, we have:

Lemma 7. *The functional E_Q attains its minimum in $W^{1,2}_{\mathrm{per}}(Q)$. Also, if u is any of such minimizers, then it is continuous, its modulus of continuity is uniformly bounded, and*

$$\Big| |u(x)| - 1 \Big| \leq \delta_0\,, \tag{33}$$

for any $x \in Q$, as long as η is small enough.

Proof. Let u_k be a minimizing sequence. We may suppose that

$$E_Q(u_k) \leq E_Q(1) = 0\,, \tag{34}$$

due to (2).
Also, it follows from (10) that

$$\begin{aligned}
\min\Big\{F(x,1+s)-F(x,1+\delta_0),\, F(x,-1-s)-F(x,-1-\delta_0)\Big\} &\geq c\,(s-\delta_0) \\
&\geq \Big|H(x)\,(\delta_0 - s)\Big|\,,
\end{aligned} \tag{35}$$

for any $s \geq \delta_0$, and

$$F(x,r) + H(x) r \geq 0\,,$$

as long as $|r| \geq C_0$, with C_0 appropriately large, if η is small enough. Consequently, by (34),

$$\int_Q |\nabla u_k|^2 \leq \int_{Q\cap\{|u_k|\leq C_0\}} |H\,u_k| \leq C_0\,\mathcal{L}(Q)\,\eta\,. \tag{36}$$

Furthermore, if we define

$$u_k^\star(x) := \begin{cases} u_k(x) & \text{if } |u_k(x)| < 1+\delta_0, \\ 1+\delta_0 & \text{if } u_k(x) \geq 1+\delta_0, \\ -1-\delta_0 & \text{if } u_k(x) \leq -1-\delta_0, \end{cases}$$

then $E_Q(u_k^\star) \le E_Q(u_k)$, thanks to (35). Accordingly, by possibly replacing u_k with $u_k^\star$, we may assume that

$$|u_k| \le 1+\delta_0\,. \tag{37}$$

As a consequence of (36) and (37), the compact embedding of $W^{1,2}(Q)$ into $L^2(Q)$ yields that u_k converges to some u in $L^2(Q)$, weakly in $W^{1,2}(Q)$ and almost everywhere, up to subsequences. Accordingly, $u \in W^{1,2}_{\rm per}(Q)$ and

$$\liminf_{k\to+\infty} \int_Q |\nabla u_k|^2 \,\ge\, \int_Q |\nabla u|^2\,.$$

Then, by Fatou Lemma,

$$\inf_{W^{1,2}_{\rm per}(Q)} E_Q \,=\, \liminf_{k\to+\infty} E_Q(u_k) \,\ge\, E_Q(u)\,,$$

thus u is the desired minimizer.
The fact that the minimizers are continuous follows from standard elliptic regularity theory (see, e.g. Theorem 3.13 in [HL97]).
We now prove (33). For this, we assume that $u \in W^{1,2}_{\rm per}(Q)$ is a minimizer for E_Q and we define

$$u^\star(x) \,:=\, \begin{cases} u(x) & \text{if } |u(x)|<1+\delta_0,\\ 1+\delta_0 & \text{if } u(x)\ge 1+\delta_0,\\ -1-\delta_0 & \text{if } u(x)\le -1-\delta_0. \end{cases}$$

Then, by (35) and the minimality of u, we have

$$0 \le E_Q(u^\star)-E_Q(u) \le -\frac{c}{2}\Big[\int_{\{u>1+\delta_0\}} (u-1-\delta_0) + \int_{\{u<-1-\delta_0\}} (-u-1-\delta_0)\Big] \le 0\,,$$

which says that $|u|\le 1+\delta_0$. Moreover, if, by contradiction,

$$-1+\delta_0 \le u(x_0) \le 1-\delta_0$$

for some $x_0\in Q$, then the uniform continuity of u yields that

$$-1+\frac{\delta_0}{2} \le u(x) \le 1-\frac{\delta_0}{2}$$

for any $x\in B_\rho(x_0)$, for a suitable, universal $\rho>0$. Accordingly, $F(x,u(x)) \ge$ const for $x \in B_\rho(x_0)$, which implies that

$$E_Q(u) \ge \text{ const}\,\mathcal{L}(B_\rho(x_0)) - \eta\mathcal{L}(Q) > 0 = E_Q(1) \ge E_Q(u)$$

and this contradiction ends the proof of (33). □

In the light of (9) and Lemma 7, we deduce that the functional E_Q admits two minimizers in $W^{1,2}_{\rm per}(Q)$, say $u_\pm$, so that $u_+ = u_- + 2$, satisfying

$$\Big| u_\pm \mp 1 \Big| \,\le\, \delta_0\,. \tag{38}$$

By elliptic regularity theory (see, e.g., [GT83] or [HL97]), we also have that $u_\pm \in C^{1,\alpha}(Q)$, for all $\alpha<1$. Let us notice that, if $F(x,\cdot)$ is strictly convex in $[1-\delta_0, 1+\delta_0]$ and in $[-1-\delta_0,-1+\delta_0]$, such minimizers are the only global minimizers of E_Q in $W^{1,2}_{\rm per}(Q)$. We will use these minimizers to construct a reduced energy functional (see (49) below).
We now continue with the proof of Theorem 4. For this scope, we take $\omega \in \mathbb{Q}^n\setminus\{0\}$, the irrational case being then easily obtained by a limit argument. We consider the following equivalence relation $\sim$ induced by ω: we say that $x\sim y$ if and only if $x-y\in\mathbb{Z}^n$ and

$\omega \cdot (x-y) = 0$. We will denote by $\mathbb{R}^n/\sim$ the quotient space, which, of course, is topologically equivalent to the product of the $(n-1)$-dimensional torus and the real line.
The equivalence relation $\sim$ may be made explicit by taking an integer base of $\mathbb{R}^n$ given by suitable mutually orthogonal vectors $K^{(1)}, \dots, K^{(n)} \in \mathbb{Z}^n$ in such a way that ω is parallel to $K^{(n)}$ and $K^{(1)}, \dots, K^{(n-1)}$ span the set of the integer vectors orthogonal to ω.
In this setting, given $\nu \in \mathbb{N}$, we consider the rectangle

$$\mathcal{R}_\nu^\omega := \Big\{ \sum_{j=1}^n t_j K^{(j)}, \ 0 \le t_1 < 1, \dots, 0 \le t_{n-1} < 1, \ -\nu \le t_n < \nu \Big\}.$$

We will now show that the minimizers constructed in Lemma 7 are also minimizers under the periodicity induced by $\mathcal{R}_\nu^\omega$. That is, in analogy with (32), we define

$$\begin{aligned} W^{1,2}_{\rm per}(\mathcal{R}_\nu^\omega) \ := \ & \Big\{ u \in W^{1,2}(\mathcal{R}_\nu^\omega) \ \text{such that} \\ & \quad u(x) = u(x+K^{(1)}) = \dots = u(x+K^{(n-1)}) = u(x+2\nu K^{(n)}) \Big\} \end{aligned}$$

and we prove the following result:

Lemma 8. *Any minimizer for E_Q constructed in Lemma 7 is also a minimizer for $E_{\mathcal{R}_\nu^\omega}$ in $W^{1,2}_{\rm per}(\mathcal{R}_\nu^\omega)$.*

Proof. Let u be a minimizer for E_Q in $W^{1,2}_{\rm per}(Q)$. Let also v be a minimizer for $E_{\mathcal{R}_\nu^\omega}$ in $W^{1,2}_{\rm per}(\mathcal{R}_\nu^\omega)$ (whose existence is warranted in analogy with Lemma 7). Our scope is to show that

$$E_{\mathcal{R}_\nu^\omega}(v) \ = \ E_{\mathcal{R}_\nu^\omega}(u)\,. \tag{39}$$

It is elementary to see that, given any $k \in \mathbb{Z}^n$ the function $v_k(x) := v(x+k)$ is also in $W^{1,2}_{\rm per}(\mathcal{R}_\nu^\omega)$ and thus so are the functions $\min\{v, v_k\}$ and $\max\{v, v_k\}$. Consequently,

$$E_{\mathcal{R}_\nu^\omega}(v) \le E_{\mathcal{R}_\nu^\omega}(\min\{v,v_k\}) \ \text{ and } \ E_{\mathcal{R}_\nu^\omega}(v) \le E_{\mathcal{R}_\nu^\omega}(\max\{v,v_k\}).$$

Furthermore, by the integer periodicity of the functional (namely, by (2) and (8)), we see that $E_{\mathcal{R}_\nu^\omega}(v_k) = E_{\mathcal{R}_\nu^\omega}(v)$. Accordingly,

$$\begin{aligned} 2E_{\mathcal{R}_\nu^\omega}(v) \ &\le \ E_{\mathcal{R}_\nu^\omega}(\min\{v,v_k\}) + E_{\mathcal{R}_\nu^\omega}(\max\{v,v_k\}) \\ &= \ E_{\mathcal{R}_\nu^\omega}(v) + E_{\mathcal{R}_\nu^\omega}(v_k) \\ &= \ 2E_{\mathcal{R}_\nu^\omega}(v)\,, \end{aligned}$$

which gives that

$$E_{\mathcal{R}_\nu^\omega}(\min\{v,v_k\}) = E_{\mathcal{R}_\nu^\omega}(\max\{v,v_k\}) = E_{\mathcal{R}_\nu^\omega}(v)$$

and so both $\min\{v,v_k\}$ and $\max\{v,v_k\}$ minimize $E_{\mathcal{R}_\nu^\omega}$ in $W^{1,2}_{\rm per}(\mathcal{R}_\nu^\omega)$. By repeating the argument, we see that if $\mathcal{Z}$ is any finite subset of $\mathbb{Z}^n$, we have that the function

$$v_{\mathcal{Z}}(x) \ := \ \min\Big\{ v(x+k)\,, \ k \in \mathcal{Z} \Big\}$$

also minimizes $E_{\mathcal{R}_\nu^\omega}$ in $W^{1,2}_{\rm per}(\mathcal{R}_\nu^\omega)$.
We now choose $\mathcal{Z}$ to be the set of all vectors in $\mathbb{Z}^n \cap \mathcal{R}_\nu^\omega$. Since

$$\mathcal{R}_\nu^\omega + m_1 K^{(1)} + \dots + m_{n-1} K^{n-1} + 2\nu m_n K^{(n)}\,, \qquad \text{for } m_1, \dots, m_n \in \mathbb{Z}$$

is a tiling of $\mathbb{R}^n$, we have that for any $k \in e_1 + \mathcal{Z}$ there exists a unique $\kappa(k) \in \mathcal{Z}$ in such a way

$$k - \kappa(k) = m_1 K^{(1)} + \dots + m_{n-1} K^{n-1} + 2\nu m_n K^{(n)}$$

for suitable $m_1, \dots, m_n \in \mathbb{Z}$ and, viceversa, the set $\{\kappa(k)\,,\ k \in e_1 + \mathcal{Z}\}$ agrees with $\mathcal{Z}$. Consequently,

$$\begin{aligned} v_{\mathcal{Z}}(x+e_1) &= \min\Big\{ v(x+k)\,,\ k \in e_1+\mathcal{Z}\Big\} \\ &= \min\Big\{ v(x+\kappa(k))\,,\ k \in e_1+\mathcal{Z}\Big\} \\ &= \min\Big\{ v(x+h)\,,\ h \in \mathcal{Z}\Big\} \\ &= v_{\mathcal{Z}}(x)\,, \end{aligned}$$

due to the periodicity of v. Analogously,

$$v_{\mathcal{Z}}(x+e_1) = v_{\mathcal{Z}}(x+e_2) \cdots = v_{\mathcal{Z}}(x+e_n) = v_{\mathcal{Z}}(x)\,,$$

thence $v_{\mathcal{Z}} \in W^{1,2}_{\rm per}(Q)$. The minimization property of u thus yields that

$$E_Q(u) \le E_Q(v_{\mathcal{Z}})\,. \tag{40}$$

Our next target is to show that

$$E_{\mathcal{R}^\omega_\nu}(w) = \sum_{k\in\mathcal{Z}} E_{k+Q}(w) \tag{41}$$

for any $w \in W^{1,2}_{\rm per}(\mathcal{R}^\omega_\nu)$. Though formula (40) is very close to common intuition (one may just look at some pavement decorations to get convinced), we provide a rigorous proof of it (the expert reader goes straight to (46)). To check (41), we first demonstrate that for any $\xi \in \mathcal{R}^\omega_\nu$ there exist $k \in \mathcal{Z}$ and $\ell_1, \dots, \ell_n \in \mathbb{Z}$ in such a way that

$$\xi - k + \sum_{j=1}^{n-1} \ell_j K^{(j)} + 2\nu \ell_n K^{(n)} \,\in\, Q\,. \tag{42}$$

To confirm this, let $[\cdot]$ denote the integer part of a real number and

$$[\xi] \,:=\, \Big([\xi_1], \dots, [\xi_n]\Big)\,.$$

Let ℓ_j be the unique integer for which

$$\begin{aligned} &[\xi]\cdot \frac{K^{(j)}}{|K^{(j)}|^2} + \ell_j \in [0,1) \quad \text{for } 1 \le j \le n-1 \text{ and} \\ &[\xi]\cdot \frac{K^{(n)}}{2\nu\,|K^{(n)}|^2} + \ell_n \in \Big[-\frac12, \frac12\Big)\,. \end{aligned}$$

Let

$$k := [\xi] + \sum_{j=1}^{n-1} \ell_j K^{(j)} + 2\nu \ell_n K^{(n)}\,. \tag{43}$$

Then, $k \in \mathbb{Z}^n$ and, moreover,

$$\begin{aligned} &k \cdot \frac{K^{(j)}}{|K^{(j)}|} \in \Big[0,\, |K^{(j)}|\Big) \quad \text{for } 1 \le j \le n-1 \text{ and} \\ &k \cdot \frac{K^{(n)}}{|K^{(n)}|} \in \Big[-\nu\,|K^{(n)}|,\, \nu\,|K^{(n)}|\Big)\,, \end{aligned}$$

hence $k \in \mathcal{R}^\omega_\nu$ and so $k \in \mathcal{Z}$. Moreover, the vector on the left hand side of (42) agrees with $\xi - [\xi]$, due to (43), and so it has coordinates lying in $[0,1)$, thus completing the proof of (42).
We now denote $\sim_\nu$ the equivalence relation stating that $x \sim_\nu y$ if and only if

$$x - y = \sum_{j=1}^{n-1} \ell_j K^{(j)} + 2\nu \ell_n K^{(n)}$$

for some $\ell_1, \dots, \ell_n \in \mathbb{Z}$. Let π_ν be the natural projection induced by $\sim_\nu$. Let

$$\mathcal{R} := \bigcup_{k \in \mathcal{Z}} (k + Q)\,.$$

Then, (42) states that $\pi_\nu(\mathcal{R}) = \mathcal{R}^\omega_\nu / \sim_\nu$ (and we may identify the latter with $\mathcal{R}^\omega_\nu$ itself). We now show that

$$\pi_\nu \text{ is, in fact, injective on } \mathcal{R}. \tag{44}$$

Indeed, assume that $\pi_\nu(x) = \pi_\nu(x')$ with $x,\, x' \in \mathcal{R}$. Then, $x = q + k$, $x' = q' + k'$ with $q,\, q' \in Q$, $k,\, k' \in \mathcal{Z}$ and

$$x - x' = \sum_{j=1}^{n-1} \ell_j K^{(j)} + 2\nu \ell_n K^{(n)}$$

for some $\ell_1, \dots, \ell_n \in \mathbb{Z}$. In particular, $q - q' \in \mathbb{Z}^n$ and $q \cdot e_k,\, q' \cdot e_k \in [0,1)$, for any $1 \le k \le n$. Thus, $(q - q') \cdot e_k \in \mathbb{Z} \cap (-1,1) = \{0\}$, and so $q = q'$. Accordingly,

$$k - k' = \sum_{j=1}^{n-1} \ell_j K^{(j)} + 2\nu \ell_n K^{(n)}\,. \tag{45}$$

Since $k \in \mathcal{Z}$, we have that

$$\begin{aligned} & k \cdot \frac{K^{(j)}}{|K^{(j)}|} \in \Big[0,\, |K^{(j)}|\Big) \quad \text{for } 1 \le j \le n-1 \text{ and} \\ & k \cdot \frac{K^{(n)}}{|K^{(n)}|} \in \Big[-\nu\,|K^{(n)}|,\, \nu\,|K^{(n)}|\Big)\,, \end{aligned}$$

for any $1 \le j \le n$ (and the same holds for k'). This and (45) yield that $\ell_j \in (-1,1)$ for $1 \le j \le n$, so $\ell_j = 0$. Consequently, $x = x'$, proving (44).
As a consequence of (44), we have that, if $w \in W^{1,2}_{\rm per}(\mathcal{R}^\omega_\nu)$, then

$$E_{\mathcal{R}^\omega_\nu}(w) = E_{\mathcal{R}^\omega_\nu/\sim_\nu}(w) = E_{\mathcal{R}/\sim_\nu}(w) = E_{\mathcal{R}}(w) = \sum_{k \in \mathcal{Z}} E_{k+Q}(w)\,,$$

that is, (40).
Then, using (40) and the periodicity relations in $W^{1,2}_{\rm per}(Q)$, we gather that

$$E_{\mathcal{R}^\omega_\nu}(u) = \sum_{k \in \mathcal{Z}} E_{k+Q}(u) = \sum_{k \in \mathcal{Z}} E_Q(u) \le \sum_{k \in \mathcal{Z}} E_Q(v_{\mathcal{Z}}) = \sum_{k \in \mathcal{Z}} E_{k+Q}(v_{\mathcal{Z}}) = E_{\mathcal{R}^\omega_\nu}(v_{\mathcal{Z}})\,. \tag{46}$$

We infer from this and (40) that $E_{\mathcal{R}^\omega_\nu}(u) \le E_{\mathcal{R}^\omega_\nu}(v)$. Since $W^{1,2}_{\rm per}(Q) \subseteq W^{1,2}_{\rm per}(\mathcal{R}^\omega_\nu)$, we obviously have also the reverse inequality. This yields the proof of (39), as desired. $\square$

We now address the problem of comparing the energy of the minimizers in $W^{1,2}_{\rm per}(\mathcal{R}^\omega_\nu)$ with the ones in $W^{1,2}(\mathcal{R}^\omega_\nu/\sim)$, where $\sim$ is the equivalence relation introduced on page 14, that is, we estimate how much the periodicity conditions in the direction of ω affect the minimal energy. For this, we will prove an existence result for $W^{1,2}(\mathcal{R}^\omega_\nu/\sim)$-minimizers in Lemma 9 below and then perform the necessary energy estimates in Lemma 10.

Lemma 9. *The functional $E_{\mathcal{R}^\omega_\nu}$ attains the minimum in $W^{1,2}(\mathcal{R}^\omega_\nu/\sim)$ at a suitable u_ν satisfying*

$$\|u_\nu\|_{C^1(\mathcal{R}^\omega_{\nu-1})} \le C\,, \tag{47}$$

for a suitable universal $C>0$.

Proof. By performing a standard minimization argument as in formulas (34)–(37), we get the existence of a minimizer $u_\nu \in W^{1,2}(\mathcal{R}^\omega_\nu/\sim)$ which is pointwise uniformly bounded. Then, (47) is a consequence of the interior elliptic regularity theory (see, e.g. Theorem 3.13 in [HL97]). □

Lemma 10. *Let $\nu \ge 4$. Let u_ν a minimizer for $E_{\mathcal{R}^\omega_\nu}$ in $W^{1,2}(\mathcal{R}^\omega_\nu/\sim)$, as constructed in Lemma 9. Then,*

$$E_{\mathcal{R}^\omega_\nu}(u_+) \le E_{\mathcal{R}^\omega_\nu}(u_\nu) + C_\omega\,,$$

for a suitable $C_\omega>0$ possibly depending on ω, n and on the structural constants of the problem (but independent of ν).

Proof. Let τ be a smooth cut-off functions, so that $0\le\tau\le1$, $|\nabla\tau|\le10$, $\tau(x)=1$ for any $x\in\mathcal{R}^\omega_{\nu-2}$ and $\tau(x)=0$ for any $x\in\mathcal{R}^\omega_\nu\setminus\mathcal{R}^\omega_{\nu-1}$. Let $v_\nu:=\tau u_\nu$. By construction, v_ν may be extended periodically in the ω-direction outside $\mathcal{R}^\omega_\nu$, that is, there exists $\tilde v_\nu \in W^{1,2}_{\rm per}(\mathcal{R}^\omega_\nu)$ so that $\tilde v_\nu = v_\nu$ in $\mathcal{R}^\omega_\nu$. As a consequence,

$$E_{\mathcal{R}^\omega_\nu}(v_\nu) = E_{\mathcal{R}^\omega_\nu}(\tilde v_\nu) \ge E_{\mathcal{R}^\omega_\nu}(u_+)\,, \tag{48}$$

thanks to Lemma 8.
On the other hand, recalling (47),

$$E_{\mathcal{R}^\omega_\nu\setminus\mathcal{R}^\omega_{\nu-3}}(v_\nu) \le C_{0\omega}$$

for a suitable $C_{0\omega}>0$ independent of ν. Thence, from (48),

$$\begin{aligned} E_{\mathcal{R}^\omega_\nu}(u_+) &\le E_{\mathcal{R}^\omega_{\nu-3}}(v_\nu) + C_{0\omega} \\ &\le E_{\mathcal{R}^\omega_{\nu-3}}(u_{\nu-3}) + C_{0\omega}\,, \end{aligned}$$

being $u_{\nu-3}$ be the minimizer in $W^{1,2}(\mathcal{R}^\omega_{\nu-3}/\sim)$. Since

$$E_{\mathcal{R}^\omega_\nu}(u_+) \ge E_{\mathcal{R}^\omega_{\nu-3}}(u_+) - C_{1\omega}\,,$$

for a suitable $C_{1\omega}$ not depending on ν, we thus conclude that

$$E_{\mathcal{R}^\omega_{\nu-3}}(u_+) \le E_{\mathcal{R}^\omega_{\nu-3}}(u_{\nu-3}) + C_{0\omega} + C_{1\omega}\,,$$

which yields the desired result up to replacing $\nu-3$ by ν. □

Given $u\in W^{1,2}_{\rm loc}(\mathbb{R}^n/\sim)$, we define

$$\begin{aligned} \mathcal{G}_{\mathbb{R}^n/\sim}(u) := \liminf_{\nu\to+\infty} \int_{\mathcal{R}^\omega_\nu} \Big(&|\nabla u(x)|^2 + F(x,u(x)) + H(x)\,u(x) \\ &- |\nabla u_+(x)|^2 - F(x,u_+(x)) - H(x)\,u_+(x)\Big)\,dx\,. \end{aligned} \tag{49}$$

We consider the space of periodic (with respect to the identification $\sim$) functions for which the above functional is well-defined, that is, we define

$$\mathcal{D}_\omega := \Big\{ u \in W^{1,2}_{\rm loc}(\mathbb{R}^n/\sim) \ \text{ for which the lim inf in (49) is finite} \Big\}.$$

Given $\Omega \subseteq \mathbb{R}^n/\sim$, we also define the functional $\mathcal{G}_\Omega$ by replacing the domain of integration in (49) by Ω. Of course, if $u \in \mathcal{D}_\omega$, $\mathcal{G}_\Omega$ is well-defined for any open Ω.
We observe that, given any $u \in \mathcal{D}_\omega$, from Lemma 10 we have

$$\begin{aligned}\mathcal{G}_{\mathbb{R}^n/\sim}(u) &= \liminf_{\nu\to+\infty} \mathcal{G}_{\mathcal{R}^\omega_\nu}(u) \\ &= \liminf_{\nu\to+\infty} \big(E_{\mathcal{R}^\omega_\nu}(u) - E_{\mathcal{R}^\omega_\nu}(u_+) \big) \geq -C_\omega\,.\end{aligned} \tag{50}$$

We fix now $M \geq 8|\omega|$, and we let $\bar{u}$ to be a smooth function such that $\bar{u}(x) = u_+$ if $\omega \cdot x \leq 0$ and $\bar{u}(x) = u_-$ if $\omega \cdot x \geq M$. We define

$$\begin{aligned}\mathcal{Y}^\omega_M \ := \ \Big\{ & u \in \mathcal{D}_\omega \ \text{ such that } |u(x)| \leq 1+\delta_0 \text{ for any } x \in \mathbb{R}^n\,, \\ & u(x) \geq 1-\delta_0 \ \text{ if } \omega\cdot x \leq 0\,, \\ & u(x) \leq -1+\delta_0 \ \text{ if } \omega \cdot x \geq M \text{ and} \\ & u - \bar{u} \in W^{1,1}(\mathbb{R}^n/\sim) \cap W^{1,\infty}(\mathbb{R}^n/\sim) \Big\}.\end{aligned}$$

Note that $\bar{u} \in \mathcal{Y}^\omega_M$ due to (38), and

$$\mathcal{G}_{\mathbb{R}^n/\sim}(\bar{u}) < +\infty. \tag{51}$$

Also, the lim inf in (49) is in fact a full limit for all $u \in \mathcal{Y}^\omega_M$. Consequently, if $u, v \in \mathcal{Y}^\omega_M$,

$$\begin{aligned} & \mathcal{G}_{\mathbb{R}^n/\sim}(u) + \mathcal{G}_{\mathbb{R}^n/\sim}(v) \\ = \ & \lim_{\nu\to+\infty} \mathcal{G}_{\mathcal{R}^\omega_\nu}(u) + \mathcal{G}_{\mathcal{R}^\omega_\nu}(v) \\ = \ & \lim_{\nu\to+\infty} \mathcal{G}_{\mathcal{R}^\omega_\nu}(\min\{u,v\}) + \mathcal{G}_{\mathcal{R}^\omega_\nu}(\max\{u,v\}) \\ \geq \ & \liminf_{\nu\to+\infty} \mathcal{G}_{\mathcal{R}^\omega_\nu}(\min\{u,v\}) + \liminf_{\nu\to+\infty} \mathcal{G}_{\mathcal{R}^\omega_\nu}(\max\{u,v\}) \\ = \ & \mathcal{G}_{\mathbb{R}^n/\sim}(\min\{u,v\}) + \mathcal{G}_{\mathbb{R}^n/\sim}(\max\{u,v\})\,,\end{aligned} \tag{52}$$

that is, we recovered Lemma 4.1 of [Val04].
We would like now to investigate minimizers of $\mathcal{G}_{\mathbb{R}^n/\sim}$ in $\mathcal{Y}^\omega_M$ (see Lemma 12 below). Since the latter is not one of the standard functional spaces, some PDE trickeries will be needed. Namely, we will join the direct minimization methods with a decay estimate for critical points, which may be expressed as follows:

Lemma 11. *Suppose that*[1]

$$F(x,u) \text{ is } C^2 \text{ in } u \text{ and strictly convex for } u \in [-1-\ell, -1+\ell] \cup [1-\ell, 1+\ell]. \tag{53}$$

Suppose that $w \in W^{1,2}_{\rm loc}(\mathbb{R}^n/\sim)$ is a (weak) solution of

$$\Delta w(x) = F_u(x, w(x)) + H(x)$$

in $\mathbb{R}^n/\sim$. Assume that $|w(x)| \leq 1+\delta_0$ for any $x \in \mathbb{R}^n$, $w(x) \geq 1-\delta_0$ if $\omega\cdot x \leq 0$ and $w(x) \leq -1+\delta_0$ if $\omega \cdot x \geq M$.
Then,

$$|w - u_-| \leq 2\delta_0 e^{-c_1(x\cdot\omega/|\omega| - M)} \tag{54}$$

[1]The "auxiliary assumption" (53) will then be removed on page 21.

for any x so that $x \cdot \omega/|\omega| \geq M$, and

$$|w - u_+| \leq 2\delta_0 e^{-c_1(x\cdot\omega/|\omega|-M)} \tag{55}$$

for any x so that $x \cdot \omega/|\omega| \leq -M$, for a suitable universal $c_1 > 0$.
Moreover,

$$|\nabla(w - u_\pm)| \leq c_2 e^{-c_3|x\cdot\omega/|\omega||}\,, \tag{56}$$

for suitable universal c_2, $c_3 > 0$

Proof. We only prove[2] the claim in (54), since the one in (55) is analogous and then (56) follows from elliptic estimates (see, e.g., Theorem 8.32 of [GT83]). Let $v := \pm w \mp u_-$ and

$$\gamma(x) := \int_0^1 F_{uu}\Big(x, \tau w(x) + (1-\tau)u_-(x)\Big)\, d\tau\,.$$

Note that, by (53), if $u \in [-1-2\delta_0, -1+2\delta_0]$, we have that $F_{uu}(x,u) \in [C, C']$, for some $C' \geq C > 0$, as long as δ_0 is small enough. Since $|v(x)| \leq 2\delta_0$ if $x \cdot \omega/|\omega| \geq M$, due to (38), we gather that $\gamma(x) \in [C, C']$ if $x \cdot \omega/|\omega| \geq M$.
Let $a > 0$ and

$$\beta(x) := \frac{2\delta_0(e^{\sqrt{C}a}-1)}{e^{\sqrt{C}a}-e^{-\sqrt{C}a}} e^{-\sqrt{C}(x\cdot\omega/|\omega|-M)} + \frac{2\delta_0(1-e^{-\sqrt{C}a})}{e^{\sqrt{C}a}-e^{-\sqrt{C}a}} e^{\sqrt{C}(x\cdot\omega/|\omega|-M)}\,.$$

Then, if $x \cdot \omega/|\omega| \in \{M, M+a\}$, $\beta(x) = 2\delta_0 \geq v(x)$, while, if $x \cdot \omega/|\omega| \in [M, M+a]$

$$\Delta\beta - \gamma\beta = (C-\gamma)\beta \leq 0 = \Delta v - \gamma v\,.$$

Hence, by the elliptic comparison principle (see, e.g., § 8.7 of [GT83]), $v(x) \leq \beta(x)$ for any x so that $x \cdot \omega/|\omega| \in [M, M+a]$. In particular, if $x \cdot \omega/|\omega| \in [M, M+(a/2)]$,

$$v(x) \leq \frac{2\delta_0(e^{\sqrt{C}a}-1)}{e^{\sqrt{C}a}-e^{-\sqrt{C}a}} e^{-\sqrt{C}(x\cdot\omega/|\omega|-M)} + \frac{2\delta_0(1-e^{-\sqrt{C}a})}{e^{\sqrt{C}a}-e^{-\sqrt{C}a}} e^{\sqrt{C}a/2}\,.$$

By letting $a \to +\infty$, it follows that

$$v(x) \leq 2\delta_0 e^{-\sqrt{C}(x\cdot\omega/|\omega|-M)}\,,$$

as desired. □

We are now in position to minimize $\mathcal{G}_{\mathbb{R}^n/\sim}$ in $\mathcal{Y}^\omega_M$:

Lemma 12. *Assume* (53)*. The functional $\mathcal{G}_{\mathbb{R}^n/\sim}$ attains its minimum on $\mathcal{Y}^\omega_M$.*

Proof. Given $\nu \in \mathbb{N}$, by arguing as in[3] the proof of Lemma 7, one finds v_ν which minimizes $\mathcal{G}_{\mathcal{R}^\omega_\nu}(u)$ among all the functions u so that $u - \bar{u} \in W^{1,2}_0(\mathcal{R}^\omega_\nu)$. Further, by the argument on page 14, we have that $|v_\nu(x)| \leq 1 + \delta_0$. Then, by interior regularity estimates (see, e.g., Theorem 8.32 in [GT83]), we deduce that, up to subsequences,

$$v_\nu \text{ converges in } C^1_{\rm loc}(\mathbb{R}^n/\sim) \text{ to a suitable } v. \tag{57}$$

By construction, v is a local minimizer of $\mathcal{G}_\Omega$ in any bounded subset Ω of $\mathbb{R}^n/\sim$, therefore $v \in \mathcal{Y}^\omega_M$, thanks to Lemma 11. We now show that, indeed, v minimizes $\mathcal{G}_{\mathbb{R}^n/\sim}$ in $\mathcal{Y}^\omega_M$. For this, take any $u \in \mathcal{Y}^\omega_M$. Then, $u - v$ belongs to $W^{1,1}(\mathbb{R}^n/\sim) \cap W^{1,\infty}(\mathbb{R}^n/\sim)$ since the same holds for $u - \bar{u}$

[2] A different proof may be also obtained using the ring-shaped barrier of Lemma 3.3 in [GG03].

[3] Though the energy is bounded by below due to (50) and an upper bound for the minimizing energy is given by (51), standard direct methods do not suffice to prove Lemma 12, since, in principle, the minimizer could jump out of $\mathcal{Y}^\omega_M$. Lemma 11 prevents this to occur.

and $v-\bar{u}$. Hence, we may consider a mollified sequence , say u_j, so that $u_j - v \in C_0^\infty(\mathcal{R}^\omega_{R_j}/\sim)$ for suitable $R_j > 0$, in such a way

$$\begin{aligned} &u_j \text{ approaches } u \text{ in } W^{1,1}(\mathbb{R}^n/\sim) \\ &\text{with } W^{1,\infty}(\mathbb{R}^n/\sim)\text{-norm bounded independently of } j. \end{aligned} \tag{58}$$

We also set $u_{j,\nu} := u_j - v + v_\nu$. Since $u_{j,\nu}$ agrees with v_ν outside $\mathcal{R}^\omega_{R_j}/\sim$, when $\nu > R_j$ the minimizing property of v_ν yields that

$$\begin{aligned} \mathcal{G}_{\mathcal{R}^\omega_{R_j}}(u_{j,\nu}) &= \mathcal{G}_{\mathcal{R}^\omega_\nu}(u_{j,\nu}) - \mathcal{G}_{\mathcal{R}^\omega_\nu \setminus \mathcal{R}^\omega_{R_j}}(u_{j,\nu}) \\ &\geq \mathcal{G}_{\mathcal{R}^\omega_\nu}(v_\nu) - \mathcal{G}_{\mathcal{R}^\omega_\nu \setminus \mathcal{R}^\omega_{R_j}}(u_{j,\nu}) \\ &= \mathcal{G}_{\mathcal{R}^\omega_{R_j}}(v_\nu). \end{aligned}$$

Since, by (57), $u_{j,\nu}$ converges in $C^1_{\rm loc}(\mathbb{R}^n/\sim)$ to u_j when $\nu \to +\infty$, for fixed j, we thus gather that

$$\mathcal{G}_{\mathcal{R}^\omega_{R_j}}(u_j) \geq \mathcal{G}_{\mathcal{R}^\omega_{R_j}}(v)$$

and so, since u_j and v agree outside $\mathcal{R}^\omega_{R_j}$,

$$\mathcal{G}_{\mathbb{R}^n/\sim}(u_j) \geq \mathcal{G}_{\mathbb{R}^n/\sim}(v).$$

Then, by letting $j \to +\infty$, we deduce from the latter formula and (58) that

$$\mathcal{G}_{\mathbb{R}^n/\sim}(u) \geq \mathcal{G}_{\mathbb{R}^n/\sim}(v)$$

and so v is the desired minimizer. □

The proof of Theorem 4 may now be obtained by repeating verbatim the arguments on pages 169–170 and 162–164 of [Val04], replacing the density estimates in Proposition 10.4 of [Val04] with the ones in Theorem 3 here and using (52) here in the place of Lemma 4.1 there. This completes the proof of Theorem 4.

The careful reader noticed that Theorem 4 has been proved under the "auxiliary assumption" (53), but she will be convinced that this hypothesis may be easily dropped by arguing as follows. First of all, notice that even if the constants in Lemma 11 do depend on (53), the constant M_0 in Theorem 4 does not. This is due to the fact that Lemma 11 is only used to show the existence of a minimizer in Lemma 12, while M_0 is obtained by the independent argument of [Val04]. Consequently, we may take a sequence of potential $F^{(\epsilon)}$ satisfying (53) and approaching F in $C^1(\mathbb{R}^n \times [-2,2])$ as $\epsilon \to 0$. Then, we have shown the existence of a suitable $u^{(\epsilon)}_\omega$ satisfying the theses of Theorem 4. Elliptic regularity estimates (see, e.g., Theorem 8.32 in [GT83]) imply that, up to subsequences, $u^{(\epsilon)}_\omega$ approaches a suitable u_ω in $C^1_{\rm loc}(\mathbb{R}^n)$, which is then the minimizer sought by Theorem 4. Under the additional assumption (53), we also get that the two periodic minimizers $u_\pm$ are unique (in the class of functions of constant sign), and the function u_ω is a "heterocline" connecting these two minimizers, with an exponential decay.

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