A Weyl-Titchmarsh Function of an Abstract Boundary Value Problem, Operator Colligations, and Linear Systems with Boundary Control

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Abstract

The paper defines the Weyl-Titchmarsh function for an abstract boundary value problem and shows that it coincides with the transfer function of some explicitly described linear boundary control system. On the ground of obtained results we explore interplay among boundary value problems, operator colligations, and the linear systems theory and suggest an approach to the study of boundary value problems based on the open systems theory due to M. S. Livšic. Examples of boundary value problems for partial differential equations and calculations of their Weyl-Titchmarsh functions are offered as illustration. In particular, we give an independent derivation of the Weyl-Titchmarsh function for the three dimensional Schrödinger operator introduced by W. O. Amrein and D. B. Pearson. Relationships to the Schrödinger operator with singular potential supported by the unit sphere are clarified and other possible applications of the developed approach in mathematical physics are noted.

Introduction and notation

One of the mathematical folklore beliefs is the possibility to interpret the Weyl-Titchmarsh function known in the theory of Sturm-Liouville equation [40] as a transfer function of some linear system. Indeed, as follows from the definition, say for the formally selfadjoint one-dimensional Schrödinger differential expression on the half line

$$l[y] := -y'' + q(x)y, \qquad x \in [0, \infty) \tag{0.1}$$

in the case of limit circle at the infinity [40], the Weyl-Titchmarsh function $m(\lambda)$, $\lambda \in \mathbb{C}$ relates values of solutions $y_\lambda \in L^2(0, \infty)$ to the equation $l[y] = \lambda y$ and values of their derivatives y'_λ at the origin $x = 0$ by the formula $y'_\lambda(0) = m(\lambda) y_\lambda(0)$. The resemblance with the setting of system theory becomes clear if one interprets the solution y_λ as the internal state of some system with the state space $L^2(0, \infty)$, and values of y_λ at $x = 0$ as the system's input. Then $y'_\lambda(0)$ are regarded as the system output obtained from the input $y_\lambda(0)$ by the transfer function $m(\lambda)$: $y_\lambda(0) \mapsto y'_\lambda(0)$. These more or less empiric arguments are justified by the theory of boundary control systems [19, 28, 38] and by its counterpart, the so-called BC-method of the inverse problem theory [8, 26]. These theories deal in particular with linear systems governed by partial differential operators defined on some domain Ω in the Euclidian space with the control and observation taking place at the domain's boundary $\partial\Omega$. Applied to (0.1), the positive half-axis $(0, \infty)$ is treated as the domain Ω, whereas the role of the boundary $\partial\Omega$ is played by the point $x = 0$. Then the inverse problem for the Schrödinger expression (0.1) consisting in the recovery of potential q given the Weyl-Titchmarsh function $m(\lambda)$ is interpreted as reconstruction of a linear system from the knowledge of its transfer function. The generalization of this concept to the multidimensional setting is quite obvious, at least for smooth bounded domains [39].

However, a simple argument based on the imbedding theorems shows that in contract with the one-dimensional situation, values of the Weyl-Titchmarsh function in this case ought to be unbounded operators acting in $L^2(\partial\Omega)$. Indeed, the Weyl-Titchmarsh function maps boundary traces of solutions of the second order partial differential equation in the domain Ω to traces of their normal derivatives. According to imbedding theorems and properties of the trace map, values of the Weyl-Titchmarsh function decrease smoothness, therefore are unbounded operators in $L^2(\partial\Omega)$.

The concept of Weyl-Titchmarsh function is known for the general Sturm-Liouville problem, difference operators, orthogonal polynomials, Hamiltonian systems [6, 10, 25], various classes of extensions of symmetric operators studied within the theory of boundary and quasi-boundary triplets [7, 20], and some elliptic partial differential operators [16, 21, 39], where it is conventionally called the Dirichlet-to-Neumann map. The recent remarkable work [4] of W. O. Amrein and D. B. Pearson develops the notion of Weyl function for the three dimensional Schrödinger operator defined in the whole space. Their approach differs from mentioned above in that there is no boundary given *a priori*, and the authors have to introduce it artificially. By doing so they arrive at the Weyl-Titchmarsh function defined as bounded operator-function acting in $L^2(S_1)$, where S_1 is the unit sphere in $\mathbb{R}^3$. From the systems theory point of view their constriction is equivalent to the introduction of some means of control and observation into the otherwise closed system described by the "unperturbed" Schrödinger operator. With these control and observation in place, the Weyl-Titchmarsh function again can be likened to the transfer function of so created linear system. We elaborate more on this example in the last section of paper.

All instances mentioned above share one feature that again hints at the close relationship between Weyl-Titchmarsh functions and linear systems. They all are analytic and possess the positive imaginary part in the upper half plane. Functions with this property are called Herglotz or R-functions and an extensive theory has been developed not only for scalar R-functions, but also for their operator-valued analogues. In the cases cited above all respective Weyl-Titchmarsh functions are Herglotz functions, either scalar, or matrix-, or operator-valued, depending on the nature of the problem. As it turns out, R-functions are also well known in the system theory. Namely, any operator-valued R-function is the transfer function of a certain linear system of special type, the passive conservative resistance systems studied in the theory of electrical circuits [5].

These observations suggest that the relationship between the theory of Weyl-Titchmarsh functions and the theory of linear systems is of general nature. The systematic treatment of this topic is hindered by the lack of a convenient representation of Weyl-Titchmarsh functions. Indeed, neither abstract forms of Herglotz integral, nor more detailed representations found in works [4, 16] for some special cases of partial differential operators, are sufficiently explicit to serve as a foundation of the general theory. The present paper links the Weyl-Titchmarsh functions theory with the theory of linear systems in precise manner. Our study is based on the definition of abstract boundary value problems (BVPs) that the notion of Weyl-Titchmarsh function is applicable to. All examples mentioned above belong to this class. We show that the Weyl-Titchmarsh functions under consideration can be identified with characteristic functions of certain operator colligations [14, 29]. This very fact allows us to pass the treatment to the setting of systems theory. More precisely, we show that any BVP from the class studied in the paper corresponds to some linear boundary control systems such that the Weyl-Titchmarsh fucntion of the former coincides with the transfer function of the latter.

In order to dispel possible confusion the reader may have at this point with regard to the paper's content, a clarification is needed. As mentioned above, the relationship between boundary value problems and the systems theory has been known for a long time and was successfully captured in the notion of boundary control systems, see for instance [19, 28, 38]. The main premise of the paper, however, is not based on these results. Instead, the research conducted below makes use of the open systems theory due to M. S. Livšic [29]. Nevertheless, in the course of investigation we discover a natural connection between the theory of open systems [29] and the "standard" linear boundary control system theory [19, 28, 38]. It is not without interest to notice that this connection is the gist of the method of reciprocals recently suggested in [18, 38]. This method allows one to reduce the study of a system with boundary control, which is described by three typically unbounded mappings, the interior, the control and the observation operators, to the study of another linear system with the same properties, but whose operators are bounded. To give an illustrative example, let us turn to the Schrödinger expression (0.1) again. The boundary control system associated with (0.1) consists of two Hilbert spaces, $H = L^2(0,\infty)$ and $E = \mathbb{C}$, and three operators $L : y \mapsto l[y]$, $C : y \mapsto y(0)$, $O : y \mapsto y'(0)$ defined on sufficiently smooth functions $y \in L^2(0,\infty)$, the interior, the control and the observation operators, respectively. It turns out that if the restriction of L to the null set $\mathcal{N}(C)$ is a selfadjoint boundedly invertible operator L_0, and C restricted to $\mathcal{N}(L)$ possesses a bounded left inverse $C^{[-1]}$, then the study of system $\{L, C, O\}$ can be reduced to the study of four operators L_0^{-1}, $C^{[-1]}$, OL_0^{-1}, and $OC^{[-1]}$ that in turn define another boundary control system, called the reciprocal. These considerations are not limited to the one-dimensional setting of (0.1). We show that any BVP of the class introduced in the paper simultaneously defines a linear boundary control system and its reciprocal. The latter is described as an open system of M. S. Livšic. The transfer functions of these systems essentially coincide with the Weyl-Titchmarsh function of the original BVP.

Interconnections among boundary value problems, open systems, and linear boundary control theory clarify in what sense the Weyl-Titchmarsh function can be interpreted as a transfer function. At the same time the study provides alternative perspective on the individual topics involved in the research. Apart from the mentioned above method of reciprocals, our considerations link the boundary control systems theory with the theory of almost solvable extensions of symmetric operators [20, 22], clarify some principles of the inverse spectral theory [9, 26], and demonstrate applicability of the null extensions approach utilized in the paper to the study of BVPs for partial differential operators traditionally regarded as singular [2]. In particular, below we derive the Weyl-Titchmarsh function of the Schrödinger operator introduced by W. O. Amrein and D. B. Pearson in [4] by independent considerations based on the obtained results.

Let us give a brief outline of the paper. The Section 1 introduces an important notion of the so-called null extensions of linear operators that appears to be a convenient abstraction for our purposes. The null extensions based approach provides a coherent methodological framework that unifies diversity of topics studied in the paper. Here we describe a class of boundary value problems under consideration, define strong and weak solutions, discuss the Green's identity and solvability criteria, give a few equivalent definitions of the Weyl-Titchmarsh function accompanied by a brief discussion of its properties, and briefly consider the extension theory of symmetric operators [3, 11, 27, 41] within in the paper's context. Preliminary variant of some results contained in the Section 1 appears in the paper [37].

Sections 2 is devoted to linking the null extension associated with a BVP to the so-called Brodskiĭ-Livšic operator colligation [14, 29] with the characteristic function uniquely determined by the BVP's Weyl-Titchmarsh function. Properties of colligations corresponding to null extensions under consideration are described. Guided by the works of M. S. Livšic on the theory of open systems [29] we show that the studied BVPs can be put into an one-to-one correspondence with a certain class of open systems. It is shown in Section 3 that this type of open systems is comprised of reciprocals [18, 38] of linear boundary control systems whose transfer functions coincide with the Weyl-Titchmarsh functions of the original BPVs. Thus, we establish connections between null extensions (with their corresponding BVPs), the open system theory, and with theory of linear systems with boundary control. With this result in place, the question of interpretation of the Weyl-Titchmarsh function as the transfer functions of some boundary control system becomes settled on the abstract level. As seen from this explanation and will be elucidated more in the main text, the theory of open systems due to M. S. Livšic's is a crucial component of the study. It brings together the theory of boundary value problems and the theory of boundary control systems in a fruitful manner that emphasizes the unifying role the principal object of the study, the Weyl-Titchmarsh function, plays in both fields.

The last section is devoted to the Schrödinger operator $\mathscr{L} = -\Delta + q(x)$ in the three dimensional space. The Weyl-Titchmarsh function of $\mathscr{L}$ under assumption $q \in L^\infty(\mathbb{R}^3)$ was devised by W. O. Amrein and D. B. Pearson in [4] where it is called the M-function. We show that the same result can be obtained within the paper's framework if in addition the function q is smooth.

To that end we introduce the external control over the system described by $\mathscr{L}$. The control is realized as single layer potentials with densities supported on a smooth closed surface. Then we explicitly calculate all objects of the general theory, including the transfer function of the obtained linear system. By direct comparison with the research of W. O. Amrein and D. B. Pearson we show that it coincides with the M-function obtained in their work [4]. This fact allows one to calculate the Weyl-Titchmarsh function of multidimensional Scrödinger without resorting to the limiting procedure analogous to the one-dimensional case [40] and constructed in [4]. More precisely, we show that the M-function from [4] is in fact the operator of single layer potential associated with the Green's function of $\mathscr{L} = -\Delta + q(x)$ acting on the space $L^2(S_1)$, where S_1 is the unit sphere in $\mathbb{R}^3$. The author is grateful to Prof. R. Froese who pointed out that this result can be derived directly from the properties of Dirichlet-to-Neumann maps of boundary value problems for the interior and the exterior of the unit ball. Irrespective of possible consequences for the linear system theory and the theory of multidimensional Schrödinger operator, this may indicate some relevance of the approach employed in the paper to the theory of partial differential equations. A few remarks regarding the relationship between singular perturbations as per [2] and the linear boundary control systems theory conclude the section.

Results of the paper have common points with other disciplines. One of them is the extension theory of symmetric operators, particularly based on the theory of almost solvable extensions and boundary and quasi-boundary triplets [7, 20, 22]. The framework of the paper furnishes an abstract foundation for the study of characteristics of such extensions. It expands the existing theory to cover more generic situations where assumptions of works [7, 20, 22] are not fulfilled. They include non-elliptic, non-semibounded operators with infinite deficiency indices, instances where boundary mappings are non-surjective, problems with the spectral parameter in boundary conditions, etc. Results of the paper make various methods considered the systems theory specific [19, 34, 38] available to specialists in the area of boundary value problems. For example, the analysis of BVPs with the spectral parameter in boundary conditions can be regarded as the problem of linear systems with non-trivial feedback [19, 34, 34]. One more connection of the paper's topics to other disciplines is due to the formula (1.14) below. It allows one to treat the class BVPs studied in the paper by equating them with additive bounded perturbations of bounded operators, thereby greatly simplify their study. In particular, the functional model of nonselfadjoint perturbations of a selfadjoint operator from the paper [33] is directly applicable in the context of the paper. It covers situations that are not handled by the model from [36] limited to the case of almost solvable extensions. The last area where the paper's approach may prove fruitful is the theory of singular perturbations of differential operators [2] including its relationship with the theory of generalized optimal control, see [30] and references therein. A detailed account of these relationships will be given elsewhere.

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A few words regarding notation and conventions accepted in the paper are in order. For two Hilbert spaces H_1 and H_2 the sign $A : H_1 \to H_2$ is used to denote a bounded linear operator A defined everywhere in H_1 with the range in the space H_2. Symbols $\mathbb{R}$, $\mathbb{C}$, $\mathfrak{Im}\,(z)$ stand for the real axis, the complex plane, and the imaginary part of a complex number $z \in \mathbb{C}$, respectively. Furthermore, $\mathbb{C}_\pm := \{z \in \mathbb{C} \mid \pm\,\mathfrak{Im}\,(z) > 0\}$. The domain, the range and the null set of a linear operator A are denoted $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$; the symbol $\rho(A)$ is used for the resolvent set of A. For a Hilbert space the term *subspace* denotes a closed linear set. The orthogonal complement to a linear set in Hilbert space is always closed, i. e. is a subspace. If the operator A acting on the Hilbert space H is closed, the null set $\mathcal{N}(A)$ is a subspace in H. All Hilbert spaces below are separable.

1 A boundary value problem and its Weyl-Titchmarsh functions

There exist a few ways to introduce the class of problems studied in the paper. The most straightforward approach seems to be based on the concept of null extensions of a linear operator. Apart from its simplicity, it clarifies the construction of associated boundary value problem and underlines the close relationship between these problems and the theory of linear open systems. Furthermore, the notion of null extensions naturally leads to the definition of Weyl-Titchmarsh functions. Intrinsic relationships among null extensions, boundary value problems, and Weyl-Titchmarsh functions are in the main focus of this section.

1.1 Null extensions

The formal definition of null extensions is as follows.

Definition 1.1. *Let T be a linear operator on the Hilbert space H with domain $\mathcal{D}(T)$. Linear operator S is called a null extension of T if $\mathcal{D}(T) \subset \mathcal{D}(S)$, $Sx = Tx$ for $x \in \mathcal{D}(T)$, and $Sx = 0$ for $x \in \mathcal{D}(S) \setminus \mathcal{D}(T)$.*

Let H be a Hilbert space. Suppose $\mathscr{H} \subset H$ is an arbitrary linear manifold of elements in H, and A_0 is a non-bounded linear selfadjoint operator in H defined on the domain $\mathcal{D}(A_0)$. Everywhere in the paper we assume that A_0 is boundedly invertible with the inverse $A_0^{-1} : H \to H$ and the pair $\{A_0, \mathscr{H}\}$ satisfies the following assumption.

Assumption 1. *1. Intersection of $\mathscr{H}$ and $\mathcal{D}(A_0) = A_0^{-1}H$ is trivial , i. e. $\mathcal{D}(A_0) \cap \mathscr{H} = \{0\}$.*

2. *There exists a linear operator γ that maps $\mathscr{H}$ to some auxiliary Hilbert space E. The linear set $\gamma\mathscr{H}$ is dense in E and the only solution to the equation $\gamma h = 0$, $h \in \mathscr{H}$ is the null vector $h = 0$.*

3. *The left inverse of γ is bounded. Denote it $\Pi : E \to H$ so that $\Pi\gamma h = h$ for any $h \in \mathscr{H}$.*

Basic objects of the paper are the null extension A of the operator A_0 to the domain $\mathcal{D}(A) := \mathcal{D}(A_0)\dot{+}\mathscr{H}$ and the the null extension Γ_0 of the operator γ originally defined on the set $\mathscr{H}$ to the linear manifold $\mathcal{D}(A)$. In other words, A and Γ_0 are defined on $\mathcal{D}(A) := \mathcal{D}(A_0)\dot{+}\mathscr{H}$ by

$$A : x + h \mapsto A_0 x, \qquad \Gamma_0 : x + h \mapsto \gamma h, \qquad x \in \mathcal{D}(A_0), \quad h \in \mathscr{H}$$

Since A_0 is selfadjoint, its domain $\mathcal{D}(A_0)$ is dense in H, therefor A and Γ_0 are densely defined. At the same time they are not assumed closed on $\mathcal{D}(A)$, or even closable in H. By the construction, the domain $\mathcal{D}(A_0)$ is the null set of Γ_0, $\mathcal{D}(A_0) = \mathcal{N}(\Gamma_0)$ where $\mathcal{N}(\Gamma_0) = \{u \in \mathcal{D}(A) \mid \Gamma_0 u = 0\}$. Vectors $h \in \mathscr{H}$ are distinguished from other elements of $\mathcal{D}(A)$ by the equality $\Pi\Gamma_0 h = h$ or by its equivalent $\Gamma_0\Pi\varphi = \varphi$, where $\varphi = \Gamma_0 h$ with some $h \in \mathscr{H}$. These observations lead to the representation for $\mathcal{D}(A) = \mathcal{D}(A_0)\dot{+}\mathscr{H}$

$$\mathcal{D}(A) = \{A_0^{-1}f + \Pi\varphi \mid f \in H, \varphi \in \Gamma_0\mathscr{H}\}$$

accompanied by following refined definitions of A and Γ_0

$$A : A_0^{-1}f + \Pi\varphi \mapsto f, \qquad \Gamma_0 : A_0^{-1}f + \Pi\varphi \mapsto \varphi, \qquad f \in H, \quad \varphi \in \Gamma_0\mathscr{H}$$

and equalities

$$\mathcal{N}(\Gamma_0) = \{A_0^{-1}f \mid f \in H\}, \qquad A|_{\mathcal{N}(\Gamma_0)} = A_0,$$
$$\mathcal{N}(A) = \{\Pi\varphi \mid \varphi \in \Gamma_0\mathscr{H}\}, \qquad \Gamma_0|_{\mathcal{N}(A)} = \gamma.$$

Since A and Γ_0 need not be closed, $\mathcal{N}(\Gamma_0)$ and $\mathcal{N}(A)$ are not necessarily subspaces in H.

For purposes of the paper we have to introduce one more operator in addition to A and Γ_0. Let us fix an arbitrary symmetric map Λ on the space E with the dense domain $\mathcal{D}(\Lambda) = \Gamma_0\mathscr{H}$ and define the linear operator Γ_1 on $\mathcal{D}(A)$ by

$$\Gamma_1 : A_0^{-1}f + \Pi\varphi \longmapsto \Pi^* f + \Lambda\varphi, \qquad f \in H, \quad \varphi \in \Gamma_0\mathscr{H}. \tag{1.1}$$

where $\Pi^* : H \to E$ is the adjoint to Π. From (1.1) with $f = 0$ follows $\Gamma_1\Pi\varphi = \Lambda\varphi$, where $\varphi \in \Gamma_0\mathscr{H}$. Assuming $\varphi = \Gamma_0 h$ with $h \in \mathscr{H}$ we conclude that $\Gamma_1 h = \Lambda\Gamma_0 h$ for any $h \in \mathscr{H}$. Further, $\Pi^* = \Gamma_1 A_0^{-1}$, as seen from (1.1) with $\varphi = 0$. Note again that Γ_1 and Λ are not assumed closed nor closable.

The rationale behind the definition (1.1) is the special role operators Γ_0 and Γ_1 play as "boundary operators" that map $\mathcal{D}(A)$ into the "boundary space", the space E. More precisely, for the collection $\{A, \Gamma_0, \Gamma_1, H, E\}$ the following version of the Green's formula holds.

Theorem 1.2. *For any* $u, v \in \mathcal{D}(A)$

$$(Au, v)_H - (u, Av)_H = (\Gamma_1 u, \Gamma_0 v)_E - (\Gamma_0 u, \Gamma_1 v)_E, \qquad u, v \in \mathcal{D}(A), \tag{1.2}$$

Proof. Let $u \in \mathcal{D}(A)$ be a vector $u = A_0^{-1}f + h$, where $f \in H$, $h \in \mathscr{H}$. Then

$$\begin{aligned}(Au, u) - (u, Au) &= (A(A_0^{-1}f + h), A_0^{-1}f + h) - (A_0^{-1}f + h, A(A_0^{-1}f + h)) \\ &= (f, A_0^{-1}f + h) - (A_0^{-1}f + h, f) = (f, h) - (h, f).\end{aligned}$$

From the other side, since $\Pi\Gamma_0 h = h$, $\Gamma_1 h = \Lambda\Gamma_0 h$, and Λ is symmetric,

$$\begin{aligned}&(\Gamma_1 u, \Gamma_0 v)_E - (\Gamma_0 u, \Gamma_1 v)_E \\ &\quad = (\Gamma_1(A_0^{-1}f + h), \Gamma_0(A_0^{-1}f + h)) - (\Gamma_0(A_0^{-1}f + h), \Gamma_1(A_0^{-1}f + h)) \\ &\quad = (\Pi^* f + \Gamma_1 h, \Gamma_0 h) - (\Gamma_0 h, \Pi^* f + \Gamma_1 h) \\ &\quad = (f, \Pi\Gamma_0 h) - (\Pi\Gamma_0 h, f) + (\Lambda\Gamma_0 h, \Gamma_0 h) - (\Gamma_0 h, \Lambda\Gamma_0 h) = (f, h) - (h, f).\end{aligned}$$

The proof is complete. ∎

1.2 Boundary value problem

Considerations above, the Green's identity (1.2) in particular, reveal similarity of the introduced objects to the classical setting of boundary value problems. This analogy suggests the following definition of abstract spectral value problem for operator A,

$$\begin{cases} (A - zI)u = f \\ \Gamma_0 u = \varphi \end{cases} \tag{1.3}$$

Here $f \in H$ and $\varphi \in E$ are given elements of responding spaces, the vector $u \in \mathcal{D}(A)$ is unknown, and the complex number $z \in \mathbb{C}$ is the spectral parameter. Before we proceed to the main result regarding solvability of (1.3) note that the condition $\Gamma_0 u = \varphi$ imposed on $u \in \mathcal{D}(A)$ implies the inclusion $\varphi \in \Gamma_0\mathscr{H}$. However, a weak variant of (1.3) that allows one to extend the concept of solutions to (1.3) to the case of all $\varphi \in E$ can be offered.

Definition 1.3. *Given* $f \in H$, $\varphi \in E$ *the vector* $u \in H$ *is called the* ***weak solution*** *to the problem (1.3) if*

$$(u, (A_0 - \bar{z}I)v) = (f, v) + (\varphi, \Gamma_1 v), \qquad \textit{for any } v \in \mathcal{D}(A_0). \tag{1.4}$$

The definition is justified by the next observation. If the vector $u \in H$ solves the equation (1.3) with some $f \in H$, $\varphi \in E$, then for any $v \in \mathcal{D}(A_0)$ by virtue of the Green's formula (1.2),

$$\begin{aligned}(u, (A_0 - \bar{z}I)v) &= (u, A_0 v) - (zu, v) = (u, A_0 v) + (f - Au, v) \\ &= (f, v) + (u, Av) - (Au, v) = (f, v) + (\Gamma_0 u, \Gamma_1 v) - (\Gamma_1 u, \Gamma_0 v) \\ &= (f, v) + (\varphi, \Gamma_1 v).\end{aligned}$$

Thus, any solution to (1.3) at the same time solves the problem (1.4). According to the established terminology, sometimes in the sequel the problem (1.4) is referred to as the variational form of (1.3).

Next result concerns the solvability of (1.3) for $f = 0$. In this case all solutions to (1.3) are obtained from vectors $h \in \mathscr{H}$ by the bounded map $h \mapsto (I - zA_0^{-1})^{-1}h$.

Lemma 1.4. *Suppose $z \in \rho(A_0)$. The map*

$$h \longmapsto (I - zA_0^{-1})^{-1}h, \quad h \in \mathscr{H},\ z \in \rho(A_0)$$

establishes an one-to-one correspondence between $\mathscr{H} = \mathcal{N}(A)$ and $\mathcal{N}(A - zI)$. For vectors $h \in \mathscr{H}$ and $h_z := (I - zA_0^{-1})^{-1}h \in \mathcal{N}(A - zI)$ the equality $\Gamma_0 h = \Gamma_0 h_z$ holds. If two arbitrary vectors $u_1, u_2 \in \mathcal{N}(A - zI)$ satisfy the condition $\Gamma_0 u_1 = \Gamma_0 u_2$, then $u_1 = u_2$.

Proof. Since A is an extension of A_0, we have $(A - zI)(A_0 - zI)^{-1} = I$ where $z \in \rho(A_0)$. Therefore, for any $h \in \mathscr{H}$

$$(A - zI)(I - zA_0^{-1})^{-1}h = (A - zI)[I + z(A_0 - zI)^{-1}]h = (A - zI)h + zh = 0.$$

Conversely, if $h_z \in \mathcal{N}(A - zI)$ and $h := (I - zA_0^{-1})h_z$, then

$$Ah = Ah_z - zAA_0^{-1}h_z = (A - zI)h_z = 0.$$

The equality $\Gamma_0 h = \Gamma_0 h_z$ follows from the relations $h = (I - zA_0^{-1})h_z$ and $\Gamma_0\mathcal{D}(A_0) = 0$. Finally, if $u_1, u_2 \in \mathcal{N}(A - zI)$ then $u_j = (I - zA_0^{-1})^{-1}h_j$, $j = 1, 2$ with some $h_1, h_2 \in \mathscr{H}$. Assumption $\Gamma_0 u_1 = \Gamma_0 u_2$ leads to the equality $\Gamma_0 h_1 - \Gamma_0 h_2 = 0$. Applying operator Π to both sides of this identity and recalling that $\Pi\Gamma_0 h = h$ for any $h \in \mathscr{H}$, we conclude that $h_1 = h_2$, hence $u_1 = u_2$.

The proof is complete. ∎

Now we can formulate the solvability criteria for the problem (1.3) with $f \neq 0$. As one may expect by analogy with the classical boundary value problems theory, the solution to (1.3) is a sum of solutions to two semi-homogeneous problems obtained from (1.3) by assuming $f = 0$, $\varphi \neq 0$ and $f \neq 0$, $\varphi = 0$.

Theorem 1.5. *Suppose $z \in \rho(A_0)$, $\varphi \in \Gamma_0\mathscr{H}$, $f \in H$. Then the solution $u = u_z^{f,\varphi}$ to the problem (1.3) exists and is unique. It is represented in the form*

$$u_z^{f,\varphi} = (A_0 - zI)^{-1}f + (I - zA_0^{-1})^{-1}\Pi\varphi. \tag{1.5}$$

If $\varphi \in E$ is arbitrary, the vector $u_z^{f,\varphi}$ defined by (1.5) is a weak solution to (1.3).

Proof. Uniqueness of the solution (1.5) is easily verified. Assume that for $z \in \rho(A_0)$, $\varphi \in \Gamma_0\mathscr{H}$, and $f \in H$ there exist two solutions $u_1, u_2 \in \mathcal{D}(A)$ to the problem (1.3). Then their difference $u_0 := u_1 - u_2$ satisfies both equations (1.3) with $f = 0$, $\varphi = 0$. Because $\Gamma_0 u_0 = 0$, the vector u_0 belongs to the domain of operator A_0. Then it follows from (1.3) than $(A - zI)u_0 = (A_0 - zI)u_0 = 0$. Therefore, $u_0 = 0$ since $z \in \rho(A_0)$.

Now consider (1.5) with $\varphi \in \Gamma_0\mathscr{H}$. According to Lemma 1.4, the second summand in (1.5) belongs to $\mathcal{N}(A - zI)$. The equalities $(A - zI)u_z^{f,\varphi} = (A - zI)(A_0 - zI)^{-1}f = f$ follow from the definition of A. Further, from Lemma 1.4 with $h = \Pi\varphi \in \mathscr{H}$ we have

$$\Gamma_0 u_z^{f,\varphi} = \Gamma_0(I - zA_0^{-1})^{-1}\Pi\varphi = \Gamma_0[I + z(A_0 - zI)^{-1}]\Pi\varphi = \Gamma_0\Pi\varphi = \varphi.$$

Let us now verify the last statement of Theorem. Suppose $\varphi \in E$ and define the element $u_z^{f,\varphi} \in H$ by the formula (1.5). Then for $z \in \rho(A_0)$ and $v \in \mathcal{D}(A_0)$,

$$\begin{aligned}
&(u_z^{f,\varphi}, (A_0 - \bar{z}I)v) \\
&\quad = ((A_0 - zI)^{-1}f, (A_0 - \bar{z}I)v) + ((I - zA_0^{-1})^{-1}\Pi\varphi, (A_0 - \bar{z}I)v) \\
&\quad = (f, v) + (\varphi, \Pi^*(I - \bar{z}A_0^{-1})^{-1}(A_0 - \bar{z}I)v) = (f, v) + (\varphi, \Gamma_1 v),
\end{aligned}$$

since $\Pi^*(I - \bar{z}A_0^{-1})^{-1}(A_0 - \bar{z}I)v = \Gamma_1 A_0^{-1}(I - \bar{z}A_0^{-1})^{-1}(A_0 - \bar{z}I)v = \Gamma_1 v$. According to Definition 1.3, the element $u_z^{f,\varphi}$ is the weak solution to (1.3).

The proof is complete. ∎

In the light of Theorem1.5 the decomposition $\mathcal{D}(A) = \mathcal{D}(A_0)\dot{+}\mathscr{H}$ is a reminiscence of the classical approach of boundary value problems theory (see [12, 42] for instance). We look for a solution to (1.3) in the form of sum of solutions to two semi-homogeneous problems obtained from (1.3) by putting $f = 0$, $\varphi \neq 0$ and $f \neq 0$, $\varphi = 0$.

The last result of this section is the direct description of boundary value problems (1.3) that correspond to null extensions satisfying Assumption 1. It allows one to quickly verify whether results of the paper are applicable to a given BVP.

Theorem 1.6. *Let H, E be two Hilbert spaces and A, Γ_0 are two linear operators with the domain $\mathcal{D}(A)$ dense in H and with the ranges $\mathcal{R}(A) \subset H$, $\mathcal{R}(\Gamma_0) \subset E$. Operators A, Γ_0 define the spectral boundary value problem*

$$\begin{cases} (A - zI)u = 0 \\ \Gamma_0 u = \varphi \end{cases} \tag{1.6}$$

where $\varphi \in E$ is given, and $u \in \mathcal{D}(A)$ is unknown. Assume the next conditions are fulfilled:

1. *Restriction of A to the domain $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_0)$ is a (necessarily unbounded) selfadjoint operator A_0 with the bounded inverse A_0^{-1} defined everywhere on H.*
2. *Linear manifold $\Gamma_0\mathcal{D}(A)$ is dense in E.*
3. *The Green's formula (1.2) is valid for all $u, v \in \mathcal{D}(A)$*

$$(Au, v)_H - (u, Av)_H = (\Gamma_1 u, \Gamma_0 v)_E - (\Gamma_0 u, \Gamma_1 v)_E$$

with some linear operator Γ_1 defined on $\mathcal{D}(A)$ with the range $\mathcal{R}(\Gamma_1) \subset E$.

Then the domain $\mathcal{D}(A)$ is represented as direct sum $\mathcal{D}(A) = \mathcal{D}(A_0)\dot{+}\mathscr{H}$ where $\mathscr{H}$ is defined as the null set of A. The operator A is the null extension of A_0 to the domain $\mathcal{D}(A)$ satisfying Assumption 1 with $\gamma = \Gamma_0|_{\mathcal{N}(A)}$. Moreover, the mapping $\Gamma_1 A_0^{-1}$ defined on H is bounded. Its adjoint $\Pi := (\Gamma_1 A_0^{-1})^*$ is the left inverse to γ and $\Lambda := \Gamma_1\Pi$ is symmetric on $\mathcal{D}(\Lambda) = \Gamma_0\mathscr{H}$.

Proof. Since $\mathcal{N}(A_0) = \{0\}$ and $A_0 \subset A$, it follows form the invertibility of A_0 that $\mathcal{D}(A_0) \cap \mathscr{H} = \{0\}$ and $(A - zI)(A_0 - zI)^{-1} f = f$ for any $f \in H$ and $z \in \rho(A_0)$. In particular, $AA_0^{-1} f = f$. Let $u \in \mathcal{D}(A)$ be an arbitrary vector. Represent u in the form of sum $u = f_u + h_u$ where $f_u := A_0^{-1} Au \in \mathcal{D}(A_0)$ and $h_u := u - f_u = (I - A_0^{-1} A)u$. Obviously, $h_u \in \mathcal{D}(A)$ and moreover $Ah_u = (I - AA_0^{-1})Au = 0$, so that $h_u \in \mathcal{N}(A) = \mathscr{H}$. Therefore, u is represented as a sum of elements from $\mathcal{D}(A_0)$ and $\mathscr{H}$. This representation is unique because the intersection $\mathcal{D}(A_0) \cap \mathscr{H}$ is trivial.

Define the operator γ required by Assumption 1 to be the restriction of Γ_0 to the set $\mathscr{H}$. If $\gamma h = 0$ for some $h \in \mathscr{H}$, then $h \in \mathcal{D}(A_0) \cap \mathscr{H}$, therefore $h = 0$. Density of $\gamma\mathscr{H}$ in E is ensured by the assumption (ii) of the Theorem, since $\gamma\mathscr{H} = \Gamma_0\mathscr{H} = \Gamma_0\mathcal{D}(A)$.

In order to verify existence and boundedness of the left inverse of the operator γ consider the Green's formula with $u = A_0^{-1} f$, $f \in H$ and $v = h \in \mathscr{H}$. Since $\Gamma_0 u = 0$ and $Av = 0$, we obtain the equality $(f, h) = (\Gamma_1 A_0^{-1} f, \Gamma_0 h)$. According to the definition of adjoint operator, this means that $\Gamma_0 h = \gamma h$ belongs to the domain of $(\Gamma_1 A_0^{-1})^*$ and $(\Gamma_1 A_0^{-1})^* : \gamma h \mapsto h$. Therefore, $\Pi := (\Gamma_1 A_0^{-1})^*$ is the left inverse of γ. Furthermore, $\Gamma_1 A_0^{-1}$ is defined on the whole space H since $\mathcal{D}(\Gamma_1) \supset \mathcal{D}(A_0)$ and $A_0^{-1} H = \mathcal{D}(A_0)$. At the same time, its adjoint $(\Gamma_1 A_0^{-1})^*$ is an operator with dense domain $\gamma\mathscr{H} = \Gamma_0\mathscr{H} = \Gamma_0\mathcal{D}(A)$ due to condition (ii). Density of the domain of the adjoint implies closability; thus, $\Gamma_1 A_0^{-1}$ is closable. From the other hand, $\Gamma_1 A_0^{-1}$ is already closed, since its domain is the whole space H. By virtue of Closed Graph Theorem, the mapping $\Gamma_1 A_0^{-1}$ is bounded, so is its adjoint, the operator Π.

The last statement is easily proven by the calculations conducted for $h \in \mathscr{H}$,

$$(\Lambda\Gamma_0 h, \Gamma_0 h) - (\Gamma_0 h, \Lambda\Gamma_0 h) = (\Gamma_1\Pi\Gamma_0 h, \Gamma_0 h) - (\Gamma_0 h, \Gamma_1\Pi\Gamma_0 h) = (Ah, h) - (h, Ah) = 0.$$

Two last equalities are valid due to the Green's formula since $\Pi\Gamma_0 h = h$.

The proof is complete. □

1.3 Weyl-Titchmarsh function

We continue to denote u_z^φ the solution (1.5) with $f = 0$. Obviously, the map

$$R(z) : \varphi \mapsto u_z^\varphi = (I - zA_0^{-1})^{-1}\Pi\varphi, \qquad \varphi \in E, \quad z \in \rho(A_0) \tag{1.7}$$

is bounded and $R(z)\Gamma_0\mathscr{H} = \mathcal{N}(A - zI)$ according to Lemma 1.4. If $\varphi \in \Gamma_0\mathscr{H}$, then the vector $R(z)\varphi = u_z^\varphi$ belongs to the domain $\mathcal{D}(A)$, therefore $\Gamma_1 u_z^\varphi = \Gamma_1 R(z)\varphi$ is well defined. Let us calculate this vector. For a given pair of $\varphi \in \Gamma_0\mathscr{H}$ and $z \in \rho(A_0)$ we have

$$\begin{aligned}\Gamma_1 u_z^\varphi &= \Gamma_1(I - zA_0^{-1})^{-1}\Pi\varphi = \Gamma_1[I + z(A_0 - zI)^{-1}]\Pi\varphi \\ &= \Gamma_1\Pi\varphi + z\Gamma_1(A_0 - zI)^{-1}\Pi\varphi = \Lambda\varphi + z\Gamma_1 A_0^{-1}(I - zA_0^{-1})^{-1}\Pi\varphi \\ &= \Lambda\varphi + z\Pi^*(I - zA_0^{-1})^{-1}\Pi\varphi = [\Lambda + z\Pi^*(I - zA_0^{-1})^{-1}\Pi]\varphi\end{aligned}$$

Introduce the operator-function $M(z)$, $z \in \rho(A_0)$ with values in the set of operators defined on the dense domain $\Gamma_0\mathscr{H}$ in E by

$$M(z) \ : \ \varphi \longmapsto [\Lambda + z\Pi^*(I - zA_0^{-1})^{-1}\Pi]\varphi, \qquad \varphi \in \Gamma_0\mathscr{H}, \quad z \in \rho(A_0) \tag{1.8}$$

Since u_z^φ is the solution to the problem (1.3), the identity $\varphi = \Gamma_0 u_z^\varphi$ holds and the calculations conducted above show that

$$\Gamma_1 u_z^\varphi = M(z)\Gamma_0 u_z^\varphi, \qquad z \in \rho(A_0), \quad \varphi \in \Gamma_0\mathscr{H}. \tag{1.9}$$

The definition $M(z)\varphi = \Gamma_1 R(z)\varphi$ yields another representation

$$M(z)\varphi = \Gamma_1(I - zA_0^{-1})^{-1}\Pi\varphi, \qquad \varphi \in \Gamma_0\mathscr{H}, \quad z \in \rho(A_0), \tag{1.10}$$

and from Lemma 1.4 and Theorem 1.5 follows one more

$$M(z)\Gamma_0 h_z = \Gamma_1 h_z, \qquad h_z \in \mathcal{N}(A - zI), \quad z \in \rho(A_0). \tag{1.11}$$

Definition 1.7. *Function $M(\cdot)$ is called the **Weyl-Titchmarsh function** of the problem (1.3) or of the null extension A satisfying Assumption 1.*

A few remarks concerning this definition are in order.

1. Analytic operator function $m(z) := M(z) - M(0) = M(z) - \Lambda$ defined for $z \in \rho(A_0)$ is bounded and has a non-negative imaginary part in the upper half-plane $z \in \mathbb{C}_+$. In other words, $m(z)$ is an operator-valued R-function. This statement follows from the equality $m(z) = z\Pi^*(I - zA_0^{-1})^{-1}\Pi$ and the formula for $\varphi, \psi \in E$ and $z, \zeta \in \rho(A_0)$

$$(m(z)\varphi, \psi)_E - (\varphi, m(\zeta)\psi)_E = (z - \overline{\zeta})\left((I - zA_0^{-1})^{-1}\Pi\varphi, (I - \zeta A_0^{-1})^{-1}\Pi\psi\right)_H,$$

obtained by direct calculations. In the special case of $\zeta = z$, $z \notin \mathbb{R}$ and $\varphi = \psi$ we have

$$\mathfrak{Im}\,(m(z)\varphi, \varphi) = (\mathfrak{Im}\,z) \cdot \|(I - zA_0^{-1})\Pi\varphi\|^2 = (\mathfrak{Im}\,z) \cdot \|R(z)\varphi\|^2, \quad \varphi \in E, z \notin \mathbb{R}.$$

Therefore, imaginary parts of $m(z)$ are non-negative operators for $z \in \mathbb{C}_+$. Suppose $\mathcal{N}(\Pi)$ is trivial, which is equivalent to the density of $\mathcal{R}(\Pi^*) = \Gamma_1 A_0^{-1}H = \Gamma_1\mathcal{D}(A_0)$ in E. Then the imaginary part of $m(z)$ is strictly positive for $z \in \mathbb{C}_+$.

2. As follows form the definition, the function $M(\cdot)$ depends on the particular choice of Λ in (1.1). It is clear however, that all functions corresponding to different values of this parameter differ from one another by additive constant operators with the domain $\Gamma_0\mathscr{H}$.

3. Since $[R(z)]^* = [(I - zA_0^{-1})^{-1}\Pi]^* = \Gamma_1 A_0^{-1}(I - \bar{z}A_0^{-1})^{-1} = \Gamma_1(A_0 - \bar{z}I)^{-1}$, the function $M(\cdot)$ can be rewritten in a compact, but somewhat more obscure form

$$M(z) = \Gamma_1[\Gamma_1(A_0 - \bar{z}I)^{-1}]^*, \qquad z \in \rho(A_0). \tag{1.12}$$

Such representations when the operator A_0 is the Dirichlet or Neumann Laplacian in a region of $\mathbb{R}^n$, $n = 2, 3$ can be found in the literature (cf. [4, 21]). In comparison with (1.12), the formula (1.7) separates out the singular part of $M(\cdot)$, that is, the potentially unbounded term $M(0) = \Lambda$. This decomposition of $M(\cdot)$ allows one to study properties of $M(\cdot) - M(0)$ by more elementary means of the bounded operators theory. In addition, the summand $M(0) = \Lambda$ ultimately is not a characteristic of the spectral problem (1.3) or the extension A. It is merely an arbitrary parameter in the definition of boundary operator Γ_1. Therefore, by studying $M(\cdot) - \Lambda$ rather than $M(\cdot)$ one eliminates this arbitrariness from the analysis.

4. An additive representation similar to (1.8) in a special case of operator A_0 was obtained in the work [16], formula (2.6). This paper uses another (unspecified) form of the bounded mapping from E to H whose role in our considerations is played by the operator Π.

5. Consider asymptotic behavior of $M(\cdot)$ along the imaginary axis in the upper half plane. Since $m(z) = M(z) - M(0)$ is an R-function and $M(0) = \Lambda$ is symmetric, we expect $M(iy)$ to possess some kind of limit as $y \to \infty$. Analogy with the theory of bounded R-functions and bounded operators suggests that this limit is likely to be the null operator. For $\varphi \in \Gamma_0\mathscr{H}$ and $h = \Pi\varphi$ represent $M(z)\varphi$ in the form

$$M(z)\varphi = \Gamma_1(I - zA_0^{-1})^{-1}\Pi\varphi = \Gamma_1[I + z(A_0 - zI)^{-1}]h$$

By the Spectral Theorem, $z(A_0 - zI)^{-1} \to -I$ for $z = iy$ when $y \to \infty$ in the strong operator topology. Denote $F(h, z) := [I + z(A_0 - zI)^{-1}]h$. Then we have $F(h, iy) \to 0$ in H as $y \to \infty$ for any $h \in \mathscr{H}$ (in fact, for any $h \in H$). The vector function $F(h, iy)$ can be seen as an approximation error of $h \in \mathscr{H}$ by vectors $-iy(A_0 - iyI)^{-1}h$ from the dense set $\mathcal{D}(A_0)$. If the operator Γ_1 is closable on its domain $\mathcal{D}(A)$, then $F(h, iy) \to 0$ implies $M(iy)\varphi = \Gamma_1 F(h, iy) \to 0$. However, if Γ_1 is not closable, this implication may be not valid and there may exist vectors $\varphi \in \Gamma_0\mathscr{H}$ such that $M(iy)\varphi$ does not converge when $y \to \infty$. From the other side, the requirement of closability of Γ_1 is too generous for the existence of $\lim_{y\to\infty} M(iy)\varphi$ with $\varphi \in \Gamma_0\mathscr{H}$. It is sufficient to request the convergence of $\Gamma_1 F(h, iy)$ for any $h \in \mathscr{H}$ in order to conclude the existence of $\lim_{y\to\infty} M(iy)\varphi$ for any $\varphi \in \Gamma_0\mathscr{H}$. If, in addition, $\Gamma_1 F(h, iy) \to 0$ for each $h \in \mathscr{H}$, then the Weyl-Titchmarsh function $M(z)$ has the expected behavior along the imaginary. This condition is not as restrictive as the closability of Γ_1, since the implication $f_n \to 0 \Longrightarrow \Gamma_1 f_n \to 0$ for $n \to \infty$ and any sequence $\{f_n\} \in \mathcal{D}(\Gamma_1)$, which is equivalent to the closabilty of Γ_1, is not assumed to be fulfilled for any vectors from the domain $\mathcal{D}(\Gamma_1)$; only vectors of the special form $F(h, iy) = h + iy(A_0 - iyI)^{-1}h$, $h \in \mathscr{H}$ are considered. The obtained condition

$$\Gamma_1[h + iy(A_0 - iyI)^{-1}h] \to 0, \quad \text{when } y \to \infty \quad \text{for any } h \in \mathscr{H} \tag{1.13}$$

guarantees that $M(iy)\varphi \to 0$ for any $\varphi \in \Gamma_0\mathscr{H}$ when $y \to \infty$.

6. The last remark is important in applications where Assumption 1 is not fulfiled. It allows one to define the Weyl-Titchmarsh function in cases when A_0 is not boundedly invertible, but $\rho(A_0) \cap \mathbb{R} \neq \{\emptyset\}$.

Remark 1.8. *For a number $t \in \mathbb{R}$ denote $A_t := A + tI$ the "shifted" operator A. Then $\mathcal{N}(A - zI) = \mathcal{N}(A_t - \zeta I)$ where $\zeta = z + t$. For $h_z \in \mathcal{N}(A - zI)$ the Weyl-Titchmarsh function definition $\Gamma_1 h_z = M(z)\Gamma_0 h_z$ may be rewritten in the form $\Gamma_1 v_\zeta = M(\zeta - t)\Gamma_0 v_\zeta$ where $v_\zeta = h_{\zeta - t} \in \mathcal{N}(A_t - \zeta I)$. Therefore, the operator function $M_t(\zeta) := M(\zeta - t)$ for $\zeta \in \rho(A_0 + tI)$ is naturally interpreted as the Weyl-Titchmarsh function of $A_t = A + tI$.*

1.4 Minimal symmetric operator and its Krein extension

As is well known, study of a boundary value problem in many cases can be reduced to analysis of ceratin extensions of a certain symmetric operator conventionally called minimal. In this short section we give a brief account of this reduction carried out in the paper's setting.

Introduce the minimal operator A_{00} as a restriction of A to the domain $\mathcal{D}(A_{00})$ where $\mathcal{D}(A_{00}) := \{u \in \mathcal{D}(A) \mid \Gamma_0 u = \Gamma_1 u = 0\}$. As follows from the Green's formula (1.2), the operator A_{00} is symmetric, but not necessarily densely defined. The operator A_0 can be seen as a selfadjoint extension of A_{00} to the domain $\mathcal{D}(A_0)$. Another important extension of A_{00} is the operator A_K defined as a restriction of A to the set $\mathcal{D}(A_K)$, where $\mathcal{D}(A_K) := \{u \in \mathcal{D}(A) \mid (\Gamma_1 - \Lambda\Gamma_0)u = 0\}$. It is remarkable that neither of operators A_{00} or A_K depends on the particular choice of Λ and can be expressed solely in terms of the pair $\{A_0, \mathscr{H}\}$. More precisely, the following Theorem is valid.

Theorem 1.9. *Domains of A_{00} and A_K are represented by formulae*

$$\begin{aligned}
\mathcal{D}(A_{00}) &= \{u \in \mathcal{D}(A) \mid \Gamma_1 u = \Gamma_0 u = 0\} &&= A_0^{-1}\mathscr{H}^{\perp},\\
\mathcal{D}(A_K) &= \{u \in \mathcal{D}(A) \mid (\Gamma_1 - \Lambda\Gamma_0)u = 0\} &&= A_0^{-1}\mathscr{H}^{\perp}\dot{+}\mathscr{H}.
\end{aligned}$$

Proof. Let us begin by noting that $\mathcal{N}(\Pi^*) = \mathscr{H}^{\perp}$. Indeed, from identity $\Pi\Gamma_0 h = h$, we obtain $(f, h) = (f, \Pi\Gamma_0 h) = (\Pi^* f, \Gamma_0 h)$ for any $h \in \mathscr{H}$ and $f \in H$. Since $\Gamma_0\mathscr{H}$ is dense in E, the inclusion $f \in \mathcal{N}(\Pi^*)$ is equivalent to the orthogonality $f \perp \mathscr{H}$.

Let $u = A_0^{-1}f + h$ be an arbitrary element of $\mathcal{D}(A)$ with some $f \in H$, $h \in \mathscr{H}$. Conditions $\Gamma_0 u = 0$ and $\Gamma_1 u = 0$ result in the equality $\Gamma_1 A_0^{-1} f = 0$, which is equivalent to $f \in \mathcal{N}(\Pi^*) = \mathscr{H}^{\perp}$ since $\Pi^* = \Gamma_1 A_0^{-1}$. Therefore, $\mathcal{D}(A_{00}) \subset A_0^{-1}\mathscr{H}^{\perp}$. The inverse inclusion holds true according to the equalities $\Gamma_0 A_0^{-1} = 0$ and $\Gamma_1 A_0^{-1}\mathscr{H}^{\perp} = \Pi^*\mathscr{H}^{\perp} = 0$. Further, for the domain of $\mathcal{D}(A_K)$ the identity $(\Gamma_1 - \Lambda\Gamma_0)h = 0$ follows directly from the relations $\Lambda = \Gamma_1\Pi$ and $\Pi\Gamma_0 h = h$, $h \in \mathscr{H}$. Therefore, $A_0^{-1}\mathscr{H}^{\perp}\dot{+}\mathscr{H} \subset \mathcal{D}(A_K)$. From the other side, if the element $u = A_0^{-1}f + h \in \mathcal{D}(A)$ belongs to $\mathcal{D}(A_K)$, then it is necessary that $f \in \mathscr{H}^{\perp}$, hence $\mathcal{D}(A_K) \subset A_0^{-1}\mathscr{H}^{\perp}\dot{+}\mathscr{H}$. □

Operator A_K is an analogue of the Krein extension of A_{00} (see [3, 27]), which explains the notation. In this respect, the selfadjoint operator A_0 can be interpreted as the Friedrichs extension of A_{00}. Operator Λ was studied by M. Vishik in the context of elliptic boundary value problems in [41]. The same paper introduces the boundary operator $\Gamma_1 - \Lambda\Gamma_0$ associated with the Krein extension A_K as an alternative to the more customary map Γ_1 equal to the trace of the normal derivative on the domain's boundary. Later $\Gamma_1 - \Lambda\Gamma_0$ was rewritten as $\Gamma_1(I - \Pi\Gamma_0)$ by G. Grubb [23] and independently by A. Alonso and B. Simon in [3]. See the note [24] for further references regarding boundary conditions for the Krein extension.

In conclusion of the section we note that it is possible to develop an analogue of extensions theory of symmetric operators for the pair of operators A_{00} and A. In particular, consider an extension A_B of A_{00} defined as a restriction of A to the domain $\mathcal{D}(A_B) := \{u \in \mathcal{D}(A) \mid (\Gamma_1 - \Lambda\Gamma_0)u = B\Gamma_0 u\}$ with some bounded and boundedly invertible operator B. Under assumption $B^{-1}E \subset \Gamma_0\mathscr{H}$, it can be shown that A_B is also boundedly invertible and

$$A_B^{-1} = A_0^{-1} + \Pi B^{-1}\Pi^*. \tag{1.14}$$

Formally, $B = \infty$ in (1.14) describes the operator A_0, and the case $B = 0$ corresponds to the Krein extension A_K, cf. [3]. Results regarding other types of extensions will be published elsewhere.

2 Associated colligation and corresponding open system

In the previous sections we studied the null extension A of the selfadjoint operator A_0 to the set $\mathcal{D}(A_0)\dot{+}\mathscr{H}$ subject to Assumption 1. The mapping Γ_1 in (1.1) plays the role of the boundary map complementary to Γ_0. Its definition involves one parameter, a symmetric operator Λ with the dense domain $\Gamma_0\mathscr{H}$. It was shown that the extension A defines the spectral boundary value problem (1.3). In this section we connect the problem (1.3) with the so called operator d-node, or an operator colligation. In general, an operator colligation is a collection of two Hilbert spaces and three bounded mappings. Subsequently, there are many ways to incorporate objects related to the boundary value problem into a colligation. The guidance is provided by the book [29] where an interpretation of an operator colligation as a mathematical abstraction for an open system is offered. Roughly speaking, open systems are systems coupled to the external world by means of some kind of channels attached to it. Notions of input, output, and internal state are fully applicable to open systems. In fact, internal states are represented as vectors from the interior space, one of the Hilbert spaces comprising the colligation, whereas inputs and outputs are modeled as elements of the second Hilbert space, the coupling space. One of three mappings of a colligation encapsulates the interior of the corresponding system considered in isolation from the external word, that is, with the coupling channels cut off. The second operator depicts interactions of the interior of the system with the channels, and the third operator describes the metric nature of the channels. Usually it is an involution, i. e. a selfadjoint unitary operator acting in the coupling space. The key element of open systems theory is the system's transfer function. As one may expect, it is an analytic function that maps inputs into outputs.

Our goal in this section therefore can be stated as follows. Given boundary value problem (1.3) we are looking for a suitable operator colligation that would correspond to an open system effectively encapsulating principal characteristics of this BVP. One of these characteristics is undoubtedly the Weyl-Titchmarsh function $M(\cdot)$, and the open system constructed in this section possesses the transfer function that coincides with the function $m(z) := M(z) - M(0)$, $z \in \rho(A_0)$. Once this colligation (or open system) is obtained, other objects such as spaces E, H and operators Π, A_0, and $R(\cdot)$ become endowed with the clear physical meaning expressed in relative terms of this system. Connections among the null extension A, the BVP (1.3), and the open system establish the relationship of the Weyl-Titchmarsh functions theory to the theory of open system we are looking for. The link from open systems theory to the linear systems with boundary control is the subject of the next section where we show that the open system associated with a given BVP essentially is the reciprocal of the boundary control system defined by this BVP in accordance with the mainstream theory [19, 28, 38]. With this result in place, we accomplish the paper's promise by connecting boundary value problems and their Weyl-Titchmarsh functions to the open systems theory due to M. S. Livšic, and then by going a bit further, to the linear boundary control systems.

2.1 Associated operator colligation

The following definition of operator colligation is taken from [14, 29].

Definition 2.1. ***Operator colligation*** *is the collection of five objects traditionally written in the form*

$$\mathfrak{M} = \begin{pmatrix} T & \sqrt{2}\,K & J \\ H & & E \end{pmatrix} \tag{2.1}$$

where H and E are two Hilbert spaces, and T, K, and $J = J^* = J^{-1}$ are bounded linear operators:

$$T : H \to H, \quad K : E \to H, \quad J : E \to E$$

The mapping $T : H \to H$ is called the ***interior*** *operator; the operator $K : E \to H$ and its adjoint $K^* : H \to E$ are called the* ***coupling operators****.*

In comparison [14, 29], we single out the multiplier $\sqrt{2}$, which is convenient for our purposes. Following an alternative word usage, sometimes we shall employ the term **operator node** for the colligation (2.1), and sometimes shall denote the colligation (2.1) as the list of its components:

$$\mathfrak{M} = \{T, K, H, E, J\}$$

According to the plan of section, below we put the null extension A into a correspondence with a certain operator colligation referred to as its associated colligation.

Definition 2.2. *For the pair $\{A_0, \mathscr{H}\}$ satisfying Assumption 1 the* ***associated colligation (node)*** *is defined by*

$$\mathfrak{M} = \left\{A_0^{-1}, \Pi, H, E, I_E\right\} \tag{2.2}$$

Definition 2.2 is a foundation for the subsequent interpretation of (1.3), or the null extension A, as a problem of the open systems theory. In order to make the relationship (2.2) between BPVs and colligations precise, a characterization of colligations associated with boundary value problems studied in Section 1 is needed. It is given in the next theorem.

Theorem 2.3. *Suppose the pair $\{A_0, \mathscr{H}\}$ satisfies Assumption 1, hence defines a boundary value problem (1.3). The mapping*

$$\{A_0, \mathscr{H}\} \mapsto (\mathfrak{M}, \mathscr{E})$$

where $\mathscr{E} := \Gamma_0\mathscr{H} \subset E$ is a linear set and $\mathfrak{M}$ is defined by (2.2) is an one-to-one correspondence between the pairs $\{A_0, \mathscr{H}\}$ subject to Assumption 1 and the pairs $(\mathfrak{M}, \mathscr{E})$ made of a colligation $\mathfrak{M} = \left\{T, K, H, E, J\right\}$ and a linear set $\mathscr{E} \subset E$ with following properties

1. *$\mathcal{N}(T) = \{0\}$, $\mathcal{R}(T) \neq H$, $\overline{\mathcal{R}(T)} = H$, and $T = T^*$, so that there exists the unbounded (selfadjoint) operator T^{-1} with the domain $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$.*
2. *The equality $\mathcal{R}(T) \cap K\mathscr{E} = \{0\}$ holds.*
3. *The implication $K\varphi = 0,\ \varphi \in \mathscr{E} \Longrightarrow \varphi = 0$ is valid.*
4. *The linear set $\mathscr{E}$ is dense in E.*
5. *$J = I_E$.*

Proof. Let $\mathfrak{M} = \{T, K, H, E, J\}$ be the node (2.2) associated with the null extension A corresponding to $\{A_0, \mathscr{H}\}$ and $\Lambda = 0$. Then $T = A_0^{-1}$ is selfadjoint, $\mathcal{D}(A_0) = \mathcal{R}(T)$, $K = \Pi$, and $J = I_E$. Define the set $\mathscr{E} := \gamma\mathscr{H} = \Gamma_0\mathscr{H}$, so that $\mathscr{H} = \Pi\mathscr{E} = K\mathscr{E}$. Therefore, all four statements about the pair $(\mathfrak{M}, \mathscr{E})$ hold true due to Assumption 1.

In order to prove the inverse let $\mathfrak{M} = \left\{T, K, H, E, I_E\right\}$ be some colligation (2.1) with $J = I_E$ and let $\mathscr{E}$ be a dense linear set in E satisfying all conditions of the Theorem. Then the pair $(\mathfrak{M}, \mathscr{E})$ uniquely defines a null extension A of the selfadjoint operator $A_0 := T^{-1}$ to the set $\mathcal{R}(T)\dot{+}\mathscr{H}$ with $\mathscr{H} := K\mathscr{E}$. Let us show that Assumption 1 for this extension is valid. Indeed, $\mathcal{D}(A_0) \cap \mathscr{H} = \mathcal{R}(T) \cap K\mathscr{E} = \{0\}$ and K is the left inverse to the linear mapping $\Gamma_0 : Tf\dot{+}K\varphi \mapsto \varphi$, $f \in H$, $\varphi \in \mathscr{E}$ restricted to the set $K\mathscr{E}$. Thus, $\Pi = K$, $\mathscr{H} = K\mathscr{E}$, and $\Gamma_0\mathscr{H} = \mathscr{E}$.

The proof is complete. ☐

Theorem 2.3 implies that the triplet $\{A_0, \mathscr{H}, \Lambda\}$ is uniquely determined by $(\mathfrak{M}, \Lambda)$, where Λ is the parameter in (1.1) and $\mathfrak{M}$ is the colligation (2.2). Therefore the pair $(\mathfrak{M}, \Lambda)$ where $\mathfrak{M}$ satisfies conditions of Theorem 2.3 and Λ is some symmetric operator on the domain $\mathcal{D}(\Lambda) := \mathscr{E}$ determines the Weyl-Titchmarsh function of BVP constructed by $\{A_0, \mathscr{H}, \Lambda\}$ uniquely. The inverse statement is valid only partially and under some additional conditions imposed on the operator A.

Theorem 2.4. *In the notation introduced above, let $\widetilde{\mathfrak{M}} = \{\widetilde{T}, \widetilde{K}, \widetilde{H}, E, I_E\}$ be an operator colligation and $\widetilde{\mathscr{E}} \subset E$ be a linear set that satisfy conditions of Theorem 2.3. For $z \in \rho(\widetilde{A}_0)$ and $\widetilde{A}_0 := \widetilde{T}^{-1}$ denote $\widetilde{M}(z)$ some Weyl-Titchmarsh function of the corresponding null extension $\widetilde{A}$. Suppose both linear spans $\bigvee_{n\geqslant 0} A_0^{-n}E$, $\bigvee_{n\geqslant 0} \widetilde{A}_0^{-n}E$ are dense in H and $\widetilde{H}$, respectively. Assume that for some neighborhood of the origin $\mathscr{X} \subset \rho(A_0) \cap \rho(\widetilde{A}_0)$ the identity $M(z) - M(0) = \widetilde{M}(z) - \widetilde{M}(0)$, $z \in \mathscr{X}$ holds. Then operators A_0 and $\widetilde{A}_0$ are unitarily equivalent, that is $UA_0 = \widetilde{A}_0U$, where $U : H \to \widetilde{H}$ is an isometry. Moreover, $UK = \widetilde{K}$.*

Proof. According to [14], the analytic operator function

$$S(\lambda) = I + i[M(1/\lambda) - M(0)] = I - iK^*(T - \lambda I)^{-1}K, \quad \lambda \in \rho(T)$$

coincides with the so called characteristic function of the operator colligation $\mathfrak{M}$. The required result is the known fact of the operator colligations theory (see [14], Theorem 3.2).

The proof is complete. □

Theorems 2.3 and 2.4 show the associated node defined by (2.2) has properties that are sufficient to recover the boundary value problem associated with it with the only exception of the second boundary operator Γ_1. Complemented with a symmetric densely defined map Λ, the associated node $\mathfrak{M}$ fully represents the BVP and the operator Γ_1. For convenience, let us summarize the connections of the triplet $\{A_0, \mathscr{H}, \Lambda\}$ with the corresponding pair $(\mathfrak{M}, \Lambda)$ in a few formulae. Below we assume $f \in H$, $\varphi \in \mathscr{E}$.

$$\begin{aligned}
&A_0 = T^{-1}, \qquad \Pi = K, \qquad \mathscr{H} = \mathscr{E}, \qquad \mathcal{D}(A) = \mathcal{R}(T) \dot{+} K\mathscr{E} \\
&A : Tf + K\varphi \mapsto f \\
&\Gamma_0 : Tf + K\varphi \mapsto \varphi \\
&\Gamma_1 : Tf + K\varphi \mapsto K^*f + \Lambda\varphi \\
&M(z) : \varphi \mapsto \Lambda\varphi + zK^*(I - zT)^{-1}K\varphi
\end{aligned} \tag{2.3}$$

It is clear that any particular choice of Λ affects the operator Γ_1 and the Weyl-Titchmarsh function $M(\cdot)$, whereas the associated colligation $\mathfrak{M}$ does not depend on such choices. Note as well that one can introduce a null extension and corresponding spectral boundary value problem (1.3) by presenting the colligation $\mathfrak{M}$ and the dense set $\mathscr{E} \subset E$ that satisfy conditions of Theorem 2.3. Assuming a symmtric operator Λ, $\mathcal{D}(\Lambda) = \mathscr{E}$ is given, two last formulae in (2.3) serve as definitions of the boundary operator Γ_1 and Weyl-Titchmarsh function $M(\cdot)$. It is worth mentioning that the choice $\Lambda = 0$, which is equivalent to $M(0) = 0$, is always possible.

2.2 M. S. Livšic's open system

The open systems theory developed by M. S. Livšic in his seminal book [29] states that any node $\mathfrak{M} = \big\{T, K, H, E, J\big\}$ corresponds to a certain open stationary dynamic system connected with the external world via so-called coupling channels. The system's internal states are represented by vectors from the interior space H often called the state space, whereas the system's input and output are represented by vectors from the coupling (exterior) space E. The stationarity of the system signifies that its properties do not depend on time. In the case of system corresponding to the node $\mathfrak{M} = \big\{T, K, H, E, J\big\}$ the stationarity means that the operators T, K, and J are constants, i. e. are independent on the parameter $z \in \mathbb{C}$. Let us set forth relevant definitions derived from [29].

Definition 2.5. *(M. S. Livšic). Let H, E be two Hilbert spaces. An **open system** $\mathscr{F}$ with **coupling space** E and **interior space** H is comprised of two linear mappings: the **input–interior transformation** $R : \phi^- \mapsto \psi$ and **input-output transformation** $S : \phi^- \mapsto \phi^+$, $\phi^\pm \in E$, $\psi \in H$. The vectors ϕ^-, ϕ^+ and ψ are called **input, output** and **internal state** of the system $\mathscr{F}$, respectively. An open system is denoted by the symbol*

$$\mathscr{F}\begin{pmatrix} S & & \phi^+ \\ & \phi^- \nearrow & \\ & \searrow & \\ R & & \psi \end{pmatrix} \tag{2.4}$$

The book [29] describes various ways to relate a given operator node to an open system. One of them is commonly known; it is used when the interior operator T of the node is nonselfadjoint and operators K and J from the definition (2.1) satisfy the equation $2iKJK^* = T - T^*$. Another method is suitable if T is selfadjoint. Then the mappings K and J can be chosen arbitrary and we are going to use this fact to define an open system corresponding to the colligation (2.2) associated with a given BVP. Let us cite the relevant definition from [29].

Definition 2.6. *(M. S. Livšic). A node $\mathfrak{M} = \{T, K, H, E, J\}$ as in (2.1) is called the* **d-node of system $\mathscr{F}$ with respect to the number** $z_0 \in \mathbb{C}$ *if the input–output and input–interior transformations $S : \phi^- \mapsto \phi^+$, $R : \phi^- \mapsto \psi$ are connected with $\mathfrak{M}$ by relations:*

$$\left.\begin{aligned} [I-(z-z_0)T]\,\psi &= K\phi^-, \\ \phi^+ &= -i(z-z_0)JK^*\psi \\ &= -i(z-z_0)JK^*[I-(z-z_0)T]^{-1}K\phi^- \end{aligned}\right\} \tag{2.5}$$

If so, it is said that the d-node $\mathfrak{M}$ ***belongs*** *to the system $\mathscr{F} = \mathscr{F}[\mathfrak{M}]$. The transformation $S : \phi^- \mapsto \phi^+$ is an operator function defined on all input vectors $\phi^- \in E$ and analytic for all z such that $\frac{1}{z-z_0} \in \rho(T)$. It is called a* ***transfer function*** *of the system $\mathscr{F}[\mathfrak{M}]$ or its d-node $\mathfrak{M}$. Vectors $\{K\phi^-\}$ are termed* ***channel vectors****.*

Now we return to the triplet $\{A_0, \mathscr{H}, \Lambda\}$ and the corresponding pair $(\mathfrak{M}, \Lambda)$. Assuming $z \in \rho(A_0)$ and putting $z_0 = 0$ in (2.5) we see that the colligation $\mathfrak{M}$ defined in (2.2) and (2.3) is a d-node of the system $\mathscr{F}[\mathfrak{M}]$ described by relations

$$\left.\begin{aligned} \psi &= (I - zA_0^{-1})^{-1}\Pi\phi^-, \\ \phi^+ &= -iz\Pi^*\psi = -iz\Pi^*(I - zA_0^{-1})^{-1}\Pi\phi^- \end{aligned}\right\} \tag{2.6}$$

Formulae (2.6) express the input–interior transformation R and the transfer function S of this system. They are

$$R(z) = (I - zA_0^{-1})^{-1}\Pi, \qquad S(z) = -iz\Pi^*(I - zA_0^{-1})^{-1}\Pi, \quad z \in \rho(A_0) \tag{2.7}$$

The input and output of system $\mathscr{F}[\mathfrak{M}]$ are vectors $\phi^\pm$ of the space E. Now we can make the fundamental observation, namely that the function $R(z) : E \to H$, $z \in \rho(A_0)$ in fact was introduced earlier by (1.7). It coincides with the mapping $\varphi \mapsto u_z^\varphi$, where u_z^φ is a solution to (1.3) with $f = 0$. Moreover, the transfer function S from (2.7) is the z-dependent part of the Weyl-Titchmarsh function multiplied by $-i$, in other words, $S(z) = -i[M(z) - M(0)]$, $z \in \rho(A_0)$. Let us summarize these and some other observations directly obtained by comparison of results of Section 1 with the established relationship between the BVP (1.3) (or the null extension A) and the open system $\mathscr{F}[\mathfrak{M}]$.

Proposition 2.7. *Following statements hold true:*

- *The internal state $\psi = R\phi^-$ of the system $\mathscr{F}[\mathfrak{M}]$ defined in (2.6) is the solution u_z^φ to the problem (1.3) with $f = 0$ as described by Theorem 1.5. It corresponds to the choice of input $\phi^- = \varphi \in \Gamma_0\mathscr{H}$, in which case $R\phi^-$ belongs to $\mathcal{N}(A - zI)$.*

- *The set $\mathscr{H}$ consists of the channel vectors of system $\mathscr{F}[\mathfrak{M}]$ obtained by the mapping $\phi^- \mapsto \Pi\phi^-$ from the inputs $\phi^- \in \Gamma_0\mathscr{H}$.*

- *For an arbitrary input $\phi^- \in E$ the internal state $R\phi^-$ is the weak solution to the problem (1.3) with $f = 0$ in the sense of Definition 1.3 and $\Pi\phi^-$ is the corresponding channel vector.*

- *The transfer function S from (2.7) is an analytic bounded operator function. It does not depend on the particular choice of operator Λ and is related to the Weyl-Titchmarsh function M of $\{A_0, \mathscr{H}, \Lambda\}$ by the formula $S = -i(M - \Lambda)$.*

Proposition 2.7 provides the link announced in the beginning of section between BVPs studied in Section 1 and a class of open systems, thereby offering a system-theoretic interpretation for BVPs that correspond to null extensions satisfying Assumption 1.

2.3 Remarks

The established connection between boundary value problems and open systems is further clarified by means of simple observations regarding their relationships.

1. Comparison of (1.8) and (2.7) leads to the representation for Weyl-Titchmarsh function $M(\cdot)$ in terms of the pair $(\mathfrak{M}, \Lambda)$

$$M(z) = \Lambda + iS(z), \quad z \in \rho(A_0), \tag{2.8}$$

where the domain of $M(\cdot)$ is equal to $\mathcal{D}(\Lambda) = \Gamma_0\mathscr{H}$. Noting that the output of system $\mathscr{F}[\mathfrak{M}]$ is given by the equality $\phi^+ = S(z)\phi^-$, we can interpret the Weyl-Titchmarsh function as a transfer function of an open system formally written as

$$\mathfrak{F}\begin{pmatrix} \Lambda + iS & & \upphi^+ \\ & \upphi^- \nearrow & \\ & \searrow & \\ R & & \uppsi \end{pmatrix}, \qquad \upphi^- \in \mathcal{D}(\Lambda) \tag{2.9}$$

This notation is formal because values of $\Lambda + iS$ need not be bounded operators defined everywhere on the coupling space E. Admissible input vectors $\{\upphi^-\}$ of this system for which the input-output mapping makes sense, belong to the domain $\mathcal{D}(\Lambda) = \Gamma_0\mathscr{H}$. The interior states of systems $\mathfrak{F}$ and $\mathscr{F}[\mathfrak{M}]$ coincide and equal to the set of solutions $\{u_z^\varphi\}$ of the spectral problem (1.3) with $f = 0$, $\varphi \in \Gamma_0\mathscr{H}$, $z \in \rho(A_0)$. In other words, the input–interior transformation of system $\mathfrak{F}$ is the restriction of the input–interior transformation R of system $\mathscr{F}[\mathfrak{M}]$ to the set $\Gamma_0\mathscr{H}$. Finally, the output vectors of (2.9) are vectors $\{\Gamma_1 u_z^\varphi\}$, that is, images of admissible inputs under the mapping of Weyl-Titchmarsh function $M(\cdot)$. For $\upphi^- \in \mathcal{D}(\Lambda) = \Gamma_0\mathscr{H}$ inputs, outputs and internal states of systems $\mathfrak{F}$ and $\mathscr{F}[\mathfrak{M}]$ from (2.9) and (2.4) are related as follows

$$\phi^- = \upphi^-, \quad \phi^+ = S\phi^- = i(\Lambda\upphi^- - \upphi^+), \quad \psi = \uppsi, \qquad \upphi^- \in \mathcal{D}(\Lambda).$$

2. For $z \in \rho(A_0)$ and input vector $\varphi \in \Gamma_0\mathscr{H}$ the output $\upphi^+ = \Gamma_1 u_z^\varphi = M(z)\Gamma_0 u_z^\varphi$ of system $\mathfrak{F}$ is represented as a sum of $\Lambda\varphi$ and an analytic vector function $iS(z)\varphi$ of the variable $z \in \rho(A_0)$. In applications, where the spectral parameter $z \in \mathbb{C}$ has the meaning of oscillation frequency, the first summand $\Lambda\varphi$ is interpreted as a “static reaction” of the system, i. e. the reaction at the zero frequency. Following terminology of the system theory, the map Λ is called a **feedthrough operator**. Vector $\Lambda\varphi$ is not defined for all inputs $\varphi \in E$ unless Λ is bounded. For $z \in \rho(A_0)$ in a small vicinity of the origin, the summand $iS(z)\varphi$ describes low-frequency oscillations of the system $\mathfrak{F}$ around its static reaction $\Lambda\varphi$. Obviously, if the static reaction is taken into account by extraction of $\Lambda\varphi$ from the output, the analysis of a such modified system should be greatly simplified. More accurately, according to (2.8), the equality $\Gamma_1 u_z^\varphi = M(z)\Gamma_0 u_z^\varphi$ for the input–output mapping of the system $\mathfrak{F}$ can be rewritten in the form

$$(\Gamma_1 - \Lambda\Gamma_0)\, u_z^\varphi = iS(z)\Gamma_0 u_z^\varphi, \tag{2.10}$$

hence the function $iS(z)$ maps the input vector $\varphi = \Gamma_0 u_z^\varphi$ into $(\Gamma_1 - \Lambda\Gamma_0)\, u_z^\varphi$. Thus we arrive at the system with the input–output transformation $\phi^- \mapsto \phi^+ - \Lambda\phi^-$, the null feedback operator, and the transfer function $iS(z)$ describing small oscillations.

3. In the terminology accepted in the systems theory, if the set of internal states is dense in the interior space then the system $\mathscr{F}[\mathfrak{M}]$ is called **approximately controllable**. According to (2.7), this condition means density of the linear span

$$\bigvee_{e \in E,\ z \in \rho(A_0)} (I - zA_0^{-1})^{-1}\Pi e$$

in the whole space H. The power series expansion of the resolvent combined with the Theorem 2.4 now result in the following observation.

Remark 2.8. *Let S and $\widetilde{S}$ be the transfer functions of two approximately controllable systems with the same coupling space E. If $S(0) = \widetilde{S}(0) = 0$ and $S(z) = \widetilde{S}(z)$ in some neighborhood of the origin, then the d-nodes that belong to these systems are unitarily equivalent in the sense of Theorem 2.4.*

Another important notion of systems theory is the system's **observability**. In application to the system $\mathscr{F}[\mathfrak{M}]$ with the transfer function $S(z)$, $z \in \rho(A_0)$ it seems natural to call an internal state $u_z \in \mathcal{N}(A - zI)$ **unobservable** if its corresponding output $(\Gamma_1 - \Lambda\Gamma_0)u_z$ is equal to zero, cf. (2.10). Note that according to this definition the set of unobservable vectors depends on the parameter $z \in \rho(A_0)$ and can be empty. Since all internal states of $\mathscr{F}[\mathfrak{M}]$ are elements of H uniquely represented in the form $u_z^\varphi = (I - zA_0^{-1})^{-1}\Pi\varphi$ with $z \in \rho(A_0)$, $\varphi \in \Gamma_0\mathscr{H}$, we see that for a given $z \in \rho(A_0)$ unobservable vectors of $\mathscr{F}[\mathfrak{M}]$ are in fact the solutions to the homogeneous BVP

$$\begin{cases} (A - zI)u = 0 \\ (\Gamma_1 - \Lambda\Gamma_0)u = 0 \end{cases} \tag{2.11}$$

In other words, if for $\varphi \in \Gamma_0\mathscr{H}$ and $z \in \rho(A_0)$ the internal state $u_z^\varphi = (I - zA_0^{-1})^{-1}\Pi\varphi$ satisfies the "boundary condition" $(\Gamma_1 - \Lambda\Gamma_0)u_z^\varphi = 0$, then the state u_z^φ is unobservable. Another way to describe unobservable vectors can be based directly on the definition of the transfer function $S(\cdot)$ of the system $\mathscr{F}[\mathfrak{M}]$. Indeed, the state u_z^φ is unobservable if and only if $S(z)\varphi = 0$. Obviously, corresponding unobservable internal states in this case are nothing but weak solutions to the equation $(A - zI)u = 0$. Note as well that any vector from $\mathscr{H}$ satisfies (2.11) with $z = 0$, therefore $\mathscr{H}$ consists of unobservable states of the system $\mathscr{F}[\mathfrak{M}]$ at the zero frequency $z = 0$ (unobservable static reactions).

4. The condition (1.13) of the "regular" behavior of the function $M(z)$ along the imaginary axis in the upper half plane can be used as a definition for a class of "regular" systems (and corresponding boundary value problems). For these systems the summands in the decomposition $M = \Lambda + iS$ are not independent. Equivalently, the components of the pair $(\mathfrak{M}, \Lambda)$, independent in general, in this case are related to each other. Indeed, under assumption $M(iy)\varphi \to 0$ when $y \to \infty$ for any $\varphi \in \Gamma_0\mathscr{H}$, the operator Λ is reconstructed from the operator function $S(z)$ uniquely determined by the colligation $\mathfrak{M}$, by the limiting procedure $\Lambda\varphi = -\lim_{y\to\infty} iS(iy)\varphi$ where $\varphi \in \Gamma_0\mathscr{H}$.

3 Connection with the linear boundary control systems theory

The preceding sections showed that the null extension A uniquely determines the open system $\mathscr{F}[\mathfrak{M}]$ and the spectral boundary value problem (1.3). Subsequent introduction of one more operator $\Gamma_1 = \Pi^* A + \Lambda\Gamma_0$ allowed us to define the Weyl-Titchmarsh function $M(z)$, $z \in \rho(A_0)$ of the problem (1.3) and express its various properties in terms of the open system $\mathscr{F}[\mathfrak{M}]$ whose transfer function in turn is uniquely determined by $M(z)$. In this section we establish relationship between the theory developed in the paper and the theory of linear systems with boundary control. It turns out that the open system (2.2) associated with the BVP (1.3) is the reciprocal [18, 38] of the system with boundary control as defined by the control theory [19, 28, 34, 38]. It will be shown that the transfer function of this boundary control systems coincides with Weyl-Titchmarsh function of the problem (1.3).

The research is based on the obtained in Theorem 1.6 characteristics of BVPs corresponding to null extensions subject to Assumption 1. The problem (1.6) gives rise to the linear system described by the main operator A and control introduced by the vector φ from the boundary space E. In accordance with common practice established in the control theory, consider Γ_1 as the observation operator that maps internal states of the system into its output. In other words, define the output of system as $y = \Gamma_1 u_z^\varphi$ where u_z^φ is the internal state corresponding to the input $\varphi \in E$.

Suppose all conditions of Theorem 1.6 are satisfied for A, Γ_0, Γ_1, and the input φ belongs to $\Gamma_0\mathscr{H}$. Then we can seek the solution u to the problem (1.6) in the form $u = A_0^{-1}x + h$ with some $x \in H$ and $h \in \mathscr{H}$. Substitution into (1.6) results in the equation $(I - zA_0^{-1})x = zh$ for unknown vector x. From $\Gamma_0 u = \varphi$ we obtain $\Gamma_0 h = \varphi$, therefore $h = \Pi\varphi$. It follows that for the solution $u = A_0^{-1}x + h$ the controllability condition $\Gamma_0 u = \varphi$ is fulfilled automatically if we put $h = \Pi\varphi$. At the same time the equation for $x \in H$ takes the form $\left(z^{-1} - A_0^{-1}\right) x = \Pi\varphi$. The output $y = \Gamma_1 u$ now can be rewritten as $y = \Gamma_1(A_0^{-1}x + h) = \Pi^* x + \Lambda\varphi$. Let us sum up these results in a proposition.

Proposition 3.1. *For any control $\varphi \in \Gamma_0\mathscr{H}$ the system with boundary control described by the problem (1.6) with unknown $u \in \mathcal{D}(A)$ and with the observation mapping defined as $u \mapsto y = \Gamma_1 u$ is equivalent to the system associated with the problem*

$$\begin{aligned} z^{-1}x &= A_0^{-1}x + \Pi\varphi \\ y &= \Pi^* x + \Lambda\varphi \end{aligned} \tag{3.1}$$

where unknown vectors $u \in \mathcal{D}(A)$ and $x \in H$ are related by the formula $u = A_0^{-1}x + \Pi\varphi$.

Equalities (3.1) describe the linear system with the bounded main operator A_0^{-1}, internal state x, and the summand $\Pi\varphi$ representing control. The output of system (3.1) is given by the map $x \mapsto \Pi^* x + \Lambda\varphi$. In this light, Π^* and Λ are the observation and feedthrough operators, respectively. Thus, the system (1.6) with internal states $u \in \mathcal{D}(A)$ is equivalent to the system described by (3.1). Internal states of these two systems are related by the equality $u = A_0^{-1}x + \Pi\varphi$. Moreover, calculations of Section 1 show that the transfer function of (1.6) defined as $\varphi \mapsto \Gamma_1 u_z^\varphi$, where u_z^φ is the solution to (1.6), is the Weyl-Titchmarsh function $M(z)$, $z \in \rho(A_0)$ whose building blocks A_0^{-1}, Π, and Λ are bounded operators defined by the system (3.1). The price paid for this reduction from unbounded maps A, Γ_0 in (1.6) to bounded A_0^{-1}, Π in (3.1) is the possibly unbounded feedthrough operator Λ of (3.1) compared to the null feedthrough operator of (1.6).

One remarkable detail about system (3.1) is that the control and observation maps are mutually adjoint to each other. Consequently, the matrix of this system written according to the standard notation of linear systems theory (see [5, 38])

$$\begin{pmatrix} A_0^{-1} & \Pi \\ \Pi^* & \Lambda \end{pmatrix}$$

is selfadjoint (assuming the symmetric feedthrough map Λ is in fact selfadjoint). Such systems are studied in the theory of electrical circuits, where they are commonly termed **resistance systems**. The meaning of their transfer functions is the electrical impedance that maps "voltage" into "current strength", see [5].

4 Weyl-Titchmarsh function of the Schrödinger operator

Obtained results are equally applicable to the theory of boundary value problems and to the theory of linear systems theory; consequently, it is possible to illustrate the main points of the paper by examples originating in either of these two disciplines. One such example has been already mentioned in the Introduction. It consists of the Dirichlet boundary value problem for the Laplacian defined in the bounded simply connected domain $\Omega \subset \mathbb{R}^3$ with the smooth boundary Γ. The Dirichlet Laplacian A_0 defined on functions from the usual Sobolev class $H^2(\Omega)$ vanishing on the boundary Γ gives rise to the null extension A of A_0 to the set $\mathcal{D}(A) := \mathcal{D}(A_0)\dot{+}\mathscr{H}$ where $\mathscr{H}$ denotes the subset of harmonic functions in Ω with smooth traces on Γ. Assumption 1 is easily verified on the grounds of well-known properties of harmonic functions in Ω and selfadjointness of A_0 that follows from [15]. The boundary map Γ_0 is the trace operator defined on $\mathcal{D}(A)$. Then, obviously Π is the operator of harmonic continuation from the boundary Γ to the interior of domain Ω defined on smooth functions given on Γ. The Green's identity for the Laplacian suggests the second boundary operator Γ_1 chosen as the trace of normal derivative of functions from $\mathcal{D}(A)$ on the boundary Γ. With this choice the Weyl-Titchmarsh function $M(\cdot)$ maps a smooth function φ on Γ to the trace of normal derivative of the solution to the problem $(A - zI)u = 0$ satisfying boundary condition $u|_\Gamma = \varphi$. A representation for $M(\cdot)$ in the form of a pseudodifferential operator acting in $L^2(\Gamma)$ could be found for example, in [1]. According to the established terminology, $M(\cdot)$ is the Dirichlet-to-Neumann map for the Laplace operator on domain Ω, see [39] for more details. At the same time, $M(\cdot)$ is the transfer function of the linear system generated by the Laplacian in Ω with the Dirichlet boundary control on Γ.

Of course, this example can be extended to more general situations of operators and domains. In particular, the case of strongly elliptic operators and systems defined on domains of lesser regularity (see [31], for instance) is the first candidate for such generalizations. However, in this section we will explore another setting where the operator A_0 is not defined in terms of boundary conditions. In fact, its definition does not involve any notion of "boundary" at all. We continue to consider the null extensions framework as a convenient method to introduce coupling channels into the open system described by operator A_0. However, in this section we emphasize another interpretation of these channels as some sort of perturbations of the operator A_0. This "perturbative" aspect of null extensions and their relation to the general singular perturbations theory [2] will be treated in detail elsewhere.

Our underlying motive in this section is the reconciliation of the Weyl-Titchmarsh function $\mathfrak{M}(\cdot)$ for the three-dimensional Schrödinger operator studied by W. O. Amrein and D. B. Pearson in [4] with the theory developed in the paper. The operator A_0 is defined in $L^2(\mathbb{R}^3)$ by the differential expression $\mathscr{L} = -\Delta + q(x)$ with the real-valued bounded potential function $q(x)$. The domain of A_0 is the usual Sobolev class $H^2(\mathbb{R}^3)$. As will be shown, the function $\mathfrak{M}(\cdot)$ from [4] coincides with the Weyl-Titchmarsh function of some null extension of A_0. Under additional smoothness conditions on the function $q(x)$ we also obtain an expression for $\mathfrak{M}(\cdot)$ in the form of single layer potential operator associated with the Green's function of $\mathscr{L} = -\Delta + q(x)$ in $L^2(\mathbb{R}^3)$.

Let us start with some heuristic observations. Assume $H := L^2(\mathbb{R}^3)$ and A_0 is the self-adjoint Schrödinger operator corresponding to the expression $\mathscr{L} = -\Delta + q(x)$ defined on the domain $\mathcal{D}(A_0) = H^2(\mathbb{R}^3)$. Without any loss of generality we will assume that A_0 is boundedly invertible in H. If not, a suitable constant can be added to A_0 to ensure the equality $\mathcal{N}(A_0) = \{0\}$, see Remark 1.8. Suppose the physical problem under consideration is formulated as a certain "perturbation" of A_0 by a smooth closed compact surface $\Gamma \subset \mathbb{R}^3$ that splits $\mathbb{R}^3$ into the interior domain Ω bounded by Γ and the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$. The surface Γ is their common boundary. For example, the problem in hand could be the scattering process by an obstacle Ω with boundary Γ. The common way to introduce the "perturbed" operator would be to consider a restriction of A_0 to the set of smooth functions vanishing in the neighborhood of Γ and then to study various extensions of thus obtained symmetric operator corresponding to different types of boundary conditions on Γ. The problem naturally breaks into two independent boundary subproblems; the first one is for the interior of the scatterer, and the second one is for the external area that includes the infinity. Thus, this approach introduces two separate Hilbert spaces of functions defined inside and outside of the scatterer whose relation to the unperturbed operator acting in the whole space is rather loose. Indeed, by the very nature of scattering process the "free" operator A_0 does not depend on the scatterer at all, whereas two former operators act in the spaces defined exclusively in terms of the scatterer. Although the situation is manageable, having to deal with three spaces instead of the initial one can be perceived as a certain indication of inadequate modeling.

A plausible alternative can be seen in considering null extensions A of A_0 to the direct sum $\mathcal{D}(A) := \mathcal{D}(A_0) \dot{+} \mathscr{H}$ where $\mathscr{H}$ is some linear set of solutions $h \in L^2(\mathbb{R}^3)$ to the equation $\mathscr{L}h = 0$. The set $\mathscr{H}$ automatically belongs to $L^2(\mathbb{R}^3) \setminus H^2(\mathbb{R}^3)$, since otherwise A_0 would be not boundedly invertible. Below we explore the case when $\mathscr{H}$ is composed of single and double layer potentials associated with $\mathscr{L}$ with densities supported by Γ.

Single layer potentials. First, we need to make some assumptions with regard to the function $q(x)$ that would allow us to employ methods of layer potentials [17, 31, 32]. Although cases of less regularity can be considered, we assume for simplicity that $q \in C^\infty(\mathbb{R}^3)$ and Ω is C^∞-domain. It will be clear from the sequel that in fact the only requirements on q and Ω are the possibility to define layer potentials associated with $\mathscr{L}$ and the boundary Γ that possess usual properties of their acoustic counterparts corresponding to $q = 0$, see [17, 31]. Under our smoothness assumptions the resolvent $(A_0 - zI)^{-1}$, $z \in \rho(A_0)$ is an integral operator with the kernel $G(x,y,z)$, $x,y \in \mathbb{R}^3$. The function $G(x,y,z)$ is infinitely differentiable if $x \neq y$ and has singularities like $|x-y|^{-1}$ when $|x-y| \to 0$. It is symmetric in x and y and real-valued for $z \in \rho(A_0) \cap \mathbb{R}$. Traditionally $G(x,y,z)$ is called the Green's function of A_0.

For smooth functions w on Γ the **single-layer potential** $\mathscr{S}_z w$ is defined by

$$(\mathscr{S}_z w)(x) := \int_\Gamma G(x,y,z) w(y) d\sigma_y, \qquad x \in \mathbb{R}^3$$

where $d\sigma_y$ is the Euclidian surface measure on Γ. For $q(x) = 0$ the operator $\mathscr{S}_z$ is the usual acoustic single layer potential for the Helmholtz equation, cf. [17]. Outside the surface Γ functions $\mathscr{S}_z w$ are infinitely differentiable and satisfy the equation $(\mathscr{L} - zI)\mathscr{S}_z w = 0$. Note that since the Lebesque measure of Γ in $\mathbb{R}^3$ is zero, we can say that the layer potential $\mathscr{S}_z w$ satisfy this equation almost everywhere in $\mathbb{R}^3$, hence in $H = L^2(\mathbb{R}^3)$. This makes functions $\mathscr{S}_z w$ at $z = 0$ good candidates to the role of channel vectors $\mathscr{H} = \mathscr{N}(A)$ within the developed above theory. Denote $\partial_\nu = \frac{\partial}{\partial \nu}$ the normal derivative in the direction of outer normal to the domain Ω defined everywhere on Γ. Proofs of the following properties of $\mathscr{S}_z$ and $\partial_\nu \mathscr{S}_z$ when $x \in \mathbb{R}^3 \setminus \Gamma$ tends to some $x_0 \in \Gamma$ can be found for instance in [31, 32]. Denote $\Omega^- := \Omega$ and $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$. For $z \notin \mathbb{R}$ operators $\mathscr{S}_z$ map the space $E := L^2(\Gamma)$ into $L^2(\Omega^\pm)$. Boundary values $(\mathscr{S}_z w)^\pm$, $(\partial_\nu \mathscr{S}_z w)^\pm$ of $\mathscr{S}_z w$ and $\partial_\nu \mathscr{S}_z w$ on the surface Γ from $\Omega^\pm$ exist as elements of $L^2(\Gamma)$. Almost everywhere on Γ these values satisfy the so-called "jump relations" (cf. [17]):

$$(\mathscr{S}_z w)^\pm = \mathsf{S}_z w, \qquad (\partial_\nu \mathscr{S}_z w)^\pm = \mathsf{T}_z w \mp \frac{1}{2} w \tag{4.1}$$

Here w is assumed continuous on Γ and for almost all $x \in \Gamma$

$$(\mathsf{S}_z w)(x) := \int_\Gamma G(x,y,z) w(y) d\sigma_y, \qquad (\mathsf{T}_z w)(x) := \int_\Gamma [\partial_{\nu(x)} G(x,y,z)] w(y) d\sigma_y$$

Operators S_z and T_z are bounded in $L^2(\Gamma)$. From (4.1) and usual density arguments we obtain almost everywhere on Γ

$$(\mathscr{S}_z w)^- - (\mathscr{S}_z w)^+ = 0, \quad (\partial_\nu \mathscr{S}_z w)^- - (\partial_\nu \mathscr{S}_z w)^+ = w, \quad \text{for } w \in L^2(\Gamma) \tag{4.2}$$

Following the schema explained above, we define $\mathscr{H}$ as the set of single layer potentials $\{\mathscr{S}_0 \varphi\}$ with densities φ from some linear manifold $\mathscr{E}$ dense in $L^2(\Gamma)$. Without loss of generality we assume that $\mathscr{E}$ consists of smooth functions. Then functions from $\mathscr{H}$ are infinitely differentiable in $\mathbb{R}^3 \setminus \Gamma$, and continuous in $\mathbb{R}^3$. On the surface Γ normal derivatives of any function from $\mathscr{H}$ is discontinuous according to (4.2). Introduce A as a null extension of A_0 to the domain $\mathcal{D}(A) := \mathcal{D}(A_0) \dot{+} \mathscr{H} = H^2(\mathbb{R}^3) \dot{+} \mathscr{S}_0 \mathscr{E}$ and define the coupling operator $\Pi : E \to H$ as $\varphi \mapsto \mathscr{S}_0 \varphi$ where $\varphi \in \mathscr{E}$. The jump relations (4.2) suggest the following choice for the boundary map: $\Gamma_0 : u \mapsto (\partial_\nu u)^-|_\Gamma - (\partial_\nu u)^+|_\Gamma$, where $(\partial_\nu u)^\pm|_\Gamma$ are the traces of normal derivatives of $u \in \mathcal{D}(A)$ on the surface Γ from $\Omega^\pm$. Then $\Gamma_0 \Pi \varphi = \Gamma_0 \mathscr{S}_0 \varphi = \varphi$ for $\varphi \in \mathscr{E}$, and $\Pi \Gamma_0 h = h$ for $h \in \mathscr{H}$, as required. Furthermore, jumps of normal derivatives of functions from $\mathcal{D}(A_0)$ on the surface Γ are equal to zero due to Sobolev imbedding theorems, therefore $\Gamma_0 \mathcal{D}(A_0) = 0$. We see now that the pair $\{A, \mathscr{H}\}$ defines a certain boundary value problem satisfying Assumption 1, and simultaneously a system with boundary control and an open system of M. S. Livšic.

According to (1.1), the second boundary operator has the form $\Gamma_1 := \Pi^* A + \Lambda \Gamma_0$, where Λ is the feedthrough map of the system. It determines the action of Γ_1 on the set $\mathscr{H} = \mathscr{S}_0 \mathscr{E}$ and always can be chosen arbitrarily as long as it is symmetric on the domain $\mathcal{D}(\Lambda) := \mathscr{E}$. In order to make a reasonable choice let us calculate Π^* and $\Pi^* A$. Then we can discuss possibilities for Λ and Γ_1 more intelligently on the grounds of obtained results. For $\varphi \in \mathscr{E}$ and $f \in L^2(\mathbb{R}^3)$ we have

$$\begin{aligned}(\Pi\varphi, f)_H = (\mathscr{S}_0 \varphi, f)_H &= \int_{\mathbb{R}^3} \left(\int_\Gamma G(x,y,0) \varphi(y) d\sigma_y \right) \overline{f(x)}\, dx \\ &= \int_\Gamma \varphi(y) \left(\int_{\mathbb{R}^3} G(x,y,0) \overline{f(x)} dx \right) d\sigma_y = \left(\varphi,\, A_0^{-1} f\big|_\Gamma\right)_E\end{aligned}$$

by the virtue of Fubini's theorem. Therefore, $\Pi^* : f \mapsto A_0^{-1}f|_\Gamma$ and $\Pi^* A : f_0 \mapsto f_0|_\Gamma$ for $f_0 \in \mathcal{D}(A_0)$. The restrictions $A_0^{-1}f\big|_\Gamma$ of functions from $\mathcal{D}(A_0) = H^2(\mathbb{R}^3)$ to the surface Γ exist due to imbedding theorems. Let us choose Γ_1 on $\mathcal{D}(A)$ as an operator that maps elements from $\mathcal{D}(A) = H^2(\mathbb{R}^3)\dot{+}\mathscr{S}_0\mathscr{E}$ to their traces on the surface Γ. This definition is unambiguous, since single layer potentials of continuous functions are continuous in the whole $\mathbb{R}^3$. Then for $\Lambda = \Gamma_1\Pi$ we obtain $\Lambda\varphi = \mathscr{S}_0\varphi|_\Gamma = \mathsf{S}_0\varphi$, $\varphi \in \mathscr{E}$. Note that Λ is bounded in $L^2(E)$, hence the Weyl-Titchmarsh is also bounded. The input–interior transformation $R(z)$ has the form $R(z) : \varphi \mapsto \mathscr{S}_z\varphi$ for $\mathscr{S}_z\varphi \in \mathcal{N}(A - zI)$ and $\Gamma_0\mathscr{S}_z\varphi = \varphi$. Finally, for the Weyl-Titchmarsh function we have $M(z)\varphi = \Gamma_1 R(z)\varphi = (\mathscr{S}_z\varphi)|_\Gamma = \mathsf{S}_z\varphi$, so that $M(z) = \mathsf{S}_z$, where $\mathsf{S}_{(\cdot)}$ is the operator of single layer potential on the surface Γ.

Let us summarize obtained results. For definiteness we assume $\mathscr{E} := C^\infty(\Gamma)$

$$\begin{gathered}
H = L^2(\mathbb{R}^3), \qquad E := L^2(\Gamma), \qquad \mathscr{E} := C^\infty(\Gamma) \\
\mathscr{L} := -\Delta + q(x), \quad A_0 f \mapsto \mathscr{L}f, \quad f \in \mathcal{D}(A_0) := H^2(\mathbb{R}^3), \\
A : u \mapsto \mathscr{L}u, \quad u \in \mathcal{D}(A) := \mathcal{D}(A_0)\dot{+}\mathscr{H}, \ \text{ where } \ \mathscr{H} := \{\mathscr{S}_0\varphi \mid \varphi \in \mathscr{E}\}, \\
\Gamma_0 : u \mapsto (\partial_\nu u)^-|_\Gamma - (\partial_\nu u)^+|_\Gamma, \qquad \Gamma_1 : u \mapsto u|_\Gamma, \qquad u \in \mathcal{D}(A), \\
\Pi : \varphi \mapsto \mathscr{S}_0\varphi, \qquad \Lambda : \varphi \mapsto \mathscr{S}_0\varphi|_\Gamma = \mathsf{S}_0\varphi, \qquad \varphi \in E, \\
R(z) : \varphi \mapsto \mathscr{S}_z\varphi, \qquad M(z) : \varphi \mapsto \mathsf{S}_z\varphi, \qquad \varphi \in E
\end{gathered} \tag{4.3}$$

Vectors φ on the last two lines belong to E since operators Λ and $M(z)$ are bounded, therefore can be continuously extended from the dense set $\mathscr{E}$ to the whole space E.

Now we can examine a question as to how the Weyl-Titchmarsh function $\mathfrak{M}(\cdot)$ of the Schrödinger operator introduced by W. O. Amrein and D. B. Pearson in [4] relates to the construction above. We will see that for smooth potentials $q(x)$ the function $\mathfrak{M}(\cdot)$ coincides with the Weyl-Titchmarsh function $M(\cdot)$ from (4.3).

Definition 4.1 ([4]). *Let $\mathscr{L}$ be the Schrödinger differential expression $-\Delta + q(x)$ with the real-valued potential function $q(x) \in L^\infty(\mathbb{R}^3)$ acting in $L^2(\mathbb{R}^3)$ and S_1 be the unit sphere in $\mathbb{R}^3$. For $z \in \mathbb{C}_+$, the operator $\mathfrak{M}(z) : L^2(S_1) \to L^2(S_1)$ is defined by*

$$\mathfrak{M}(z)v = -w$$

where $w = \gamma^\pm f$ and $f \in H^2(\mathbb{R}^3 \setminus S_1)$ is the unique solution of $(\mathscr{L} - zI)f = 0$ subject to conditions

$$\gamma^+ f = \gamma^- f, \qquad \gamma^+ \frac{\partial f}{\partial \nu} - \gamma^- \frac{\partial f}{\partial \nu} = v. \tag{4.4}$$

Here $\gamma^\pm$ are trace operators associated with the exterior and the interior of the unit ball B_1 in $\mathbb{R}^3$. It is assumed that the boundary values in (4.4) exists as functions from $L^2(S_1)$.

The equality $M = \mathfrak{M}$ already can be derived by comparison of Definition 4.1 with the formulae (4.3). However, let us give a more detailed account assuming $q \in C^\infty(\mathbb{R}^3)$. Suppose φ is a smooth function on S_1. For $\Gamma = S_1$ the formulae (4.3) and the Weyl-Titchmarsh function definition show that $M(z)\varphi$ can be calculated as follows. First, one need to solve the problem $(\mathscr{L} - zI)u = 0$ inside and outside of the unit ball subject to the "transmission" conditions $u^- = u^+$, $u^- - u^+ = \varphi$ imposed on the boundary values of the solutions $u^\pm$. Here the signs $\pm$ correspond to the domains $\Omega^\pm$ with $\Omega^- = B_1$ and $\Omega^+ = \mathbb{R}^3 \setminus \overline{B}_1$. The solution to this problem in the whole space $\mathbb{R}^3$ is represented as the sum $u_z = u^+ + u^-$. The function u_z is continuous in $\mathbb{R}^3$ and its derivatives discontinue on the support of φ. In terms of corresponding open system, u_z is the interior state obtained from the input φ by the operator $R(z)$. The Weyl-Titchmarsh function $M(z)$ maps φ to the trace of u_z on the surface S_1 defined unambiguously since u_z is continuous. Now it is clear that the functions f, v, and w from Definition 4.1 in this notation are u_z, $-\varphi$, and $u|_{S_1}$, respectively. Therefore, $M(z)\varphi = \mathfrak{M}(z)\varphi$ for continuous φ, hence $M(z) = \mathfrak{M}(z)$, $z \notin \mathbb{R}$ due to boundedness of M and $\mathfrak{M}$.

Double layer potentials. As a natural extension of the argumentation above it is possible to consider the case of double layer potentials with densities supported by the surface Γ. Then the channel vectors from $\mathscr{H}$ are discontinuous across the surface Γ, but their normal derivatives are continuous everywhere in $\mathbb{R}^3$. Analogously to the above, we assume that q and Ω are such that **double-layer potential** $\mathscr{D}_z w$ defined by

$$(\mathscr{D}_z w)(x) := \int_\Gamma [\partial_{\nu(y)} G(x,y,z)] w(y) d\sigma_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma,$$

possesses all usual properties of its acoustic counterpart. In particular, we assume that the potential $\mathscr{D}_z w$ with smooth w defined on Γ has boundary values $(\mathscr{D}_z w)^{\pm}$ from inside and outside of Ω and the jump relations are valid:

$$(\mathscr{D}_z w)^+ - (\mathscr{D}_z w)^- = w, \qquad (\partial_\nu \mathscr{D}_z w)^+ - (\partial_\nu \mathscr{D}_z w)^- = 0$$

We choose $\mathscr{H}$ to be the set of double layer potentials $\{\mathscr{D}_0 \varphi\}$ with smooth densities φ that belong to some linear set $\mathscr{E}$ dense in $E := L_2(\Gamma)$. Noting that the jump on the surface Γ for any functions from $\mathcal{D}(A_0) = H^2(\mathbb{R}^3)$ is always equal to zero, we define the operator Γ_0 on $\mathcal{D}(A) := \mathcal{D}(A_0) \dot{+} \mathscr{H} = H^2(\mathbb{R}^3) \dot{+} \mathscr{D}_0 \mathscr{E}$ to be $\Gamma_0 : u \mapsto u^+|_\Gamma - u^-|_\Gamma$, where $u^{\pm}|_\Gamma$ are limit values on Γ from $\Omega^{\pm}$ of the function $u \in \mathcal{D}(A)$. The inverse $\Pi = (\Gamma_0|_{\mathscr{H}})^{-1}$ is the mapping $\Pi : \varphi \mapsto \mathscr{D}_0 \varphi$ defined on $\mathscr{E} = \Gamma_0 \mathscr{H}$. Calculations similar to conducted above for single layer potentials show that the adjoint Π^* is given by $\Pi^* : f \mapsto \partial_\nu A_0^{-1} f\big|_\Gamma$, $f \in H$. This formula holds for any $f \in H$ due to boundedness of Π^* ensured by imbedding theorems. Therefore, $\Pi^* A : f_0 \mapsto \partial_\nu f_0|_\Gamma$, $f_0 \in \mathcal{D}(A_0)$. Any function $f_0 \in \mathcal{D}(A_0)$ has continuous derivatives, thus the trace $\partial_\nu f_0|_\Gamma$ is defined unambiguously. Obtained result for $\Pi^* A$ suggests a plausible definition for the operator $\Gamma_1 = \Pi^* A + \Lambda \Gamma_0$ on the domain $\mathcal{D}(A) = \mathcal{D}(A_0) \dot{+} \mathscr{D}_0 \mathscr{E}$ as $\Gamma_1 : u \mapsto \partial_\nu u|_\Gamma$, $u \in \mathcal{D}(A)$. Having made this particular choice of Γ_1, we can calculate action of $\Lambda = \Gamma_1 \Pi$ on the domain $\mathscr{E}$. Obviously, $\Lambda = \Gamma_1 \Pi : \varphi \mapsto \partial_\nu \mathscr{D}_0 \varphi|_\Gamma$, $\varphi \in \mathscr{E}$. Introduce **hypersingular operator** R_z defined on smooth functions w from $L^2(\Gamma)$ by

$$(\mathsf{R}_z w)(x) := \partial_{\nu(x)} \int_\Gamma [\partial_{\nu(y)} G_z(x,y,z)] w(y) d\sigma_y \Big|_\Gamma$$

Values of R_z are unbounded operators in $L^2(\Gamma)$ and it can be shown that for $q \in C^\infty$ and Ω of the C^∞ class, operators $\mathsf{R}(z)$ are pseudodifferential of order 1, see [31]. Therefore, $\Lambda = \mathsf{R}_0$ is unbounded Following the line of reasoning employed for the case of single layer potentials, we conclude that the Weyl-Titchmarsh function $M(z)$ corresponding to the problem under consideration is R_z, $z \notin \mathbb{R}$. This form of the Weyl-Titchmarsh function for the three dimensional Schrödinger operator can be treated equally with the M-function $\mathfrak{M}(\cdot)$ of W. O. Amrein and D. B. Pearson discussed above.

The boundary value problem for A_0 with $q = 0$ perturbed by double layer potentials with boundary conditions $\Gamma_0 u = \lambda \Gamma_1 u$ where $\lambda \in \mathbb{C}$ is a complex parameter is closely related to problems arising in acoustics. For the detailed analysis that involves methods of pseudodifferential operators theory see § 9.4 in [1] and references therein.

Remarks on singular perturbations. We conclude this section with the following observation. Arguments of systems theory indicate quite clearly that by introduction of coupling channels realized as layer potentials into the system governed by the operator A_0 one introduces some kind of perturbations of A_0. Obviously, these perturbations are not additive. The adequate mathematical object that describes this type of perturbations is the operator colligation consisting in our case of two independent operators. However, if one take into consideration the relation (1.14), it becomes clear that by adding a boundary condition, in other words by introducing a linear dependency on the inputs and outputs of the system, the setting is reduced to the case of an extension of the minimal symmetric operator.

To clarify this point let Γ_0 and Γ_1 be the boundary operators defined earlier for the BPV associated with single layer potentials and $\gamma(x)$ be a continuous function on Γ. It was shown in [13] that the BVP for the Laplacian $-\Delta$ in $L^2(\mathbb{R}^3)$ with boundary conditions $\Gamma_0 u = \gamma(x)\Gamma_1 u$ can be used as a mathematical model of the quantum mechanical Schrödinger operator perturbed by the singular potential supported by the surface Γ. Denote this operator T. Formula (1.14) now describes T in perturbative terms. It shows that T is not an additive perturbation of A_0, but its inverse T^{-1} is an additive perturbation of A_0^{-1}. More precisely, $T^{-1} = A_0^{-1} + \Pi(\gamma^{-1} - \mathsf{S}_0)^{-1}\Pi^*$ assuming the operator $\gamma^{-1} - \mathsf{S}_0$ is boundedly invertible. The connection between extensions of symmetric operators and the theory of singular perturbations is well known and thoroughly described in the literature [2, 35]. Its interpretation in the terms of systems theory captured in the paper may prove beneficial for the analysis of linear systems with singular control [30].

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