On the dense point and absolutely continuous spectrum for Hamiltonians with concentric δ shells

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Abstract. We consider Schrödinger operator in dimension $d \geq 2$ with a singular interaction supported by an infinite family of concentric spheres, analogous to a system studied by Hempel and coauthors for regular potentials. The essential spectrum covers a halfline determined by the appropriate one-dimensional comparison operator; it is dense pure point in the gaps of the latter. If the interaction is radially periodic, there are absolutely continuous bands; in contrast to the regular case the measure of the p.p. segments does not vanish in the high-energy limit.

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1 Introduction

Operators the spectrum of which consists of interlaced components of different spectral types are always of interest. One of the situations where they can occur concerns radially symmetric and periodic potentials.

The idea can be traced back to the paper [1] by Hempel, Hinz, and Kalf who asked whether the gaps in the spectrum of the one-dimensional Schrödinger operator

$$-\frac{\mathsf{d}^2}{\mathsf{d}r^2} + q(r), \tag{1.1}$$

with an even potential, $q(-r) = q(r)$, are preserved or filled up as one passes to the spherically symmetric operator

$$-\triangle + q(|\cdot|) \quad \text{in} \quad L^2(\mathbb{R}^\nu), \quad \nu \geq 2. \tag{1.2}$$

They proved that for a potential which not oscillate too rapidly and belongs to $L^1_{\text{loc}}(\mathbb{R})$, the negative part having this property uniformly, the gaps are filled, i.e. the essential spectrum covers the half-line $[\lambda_0, \infty)$, where λ_0 is the essential-spectrum threshold of the associated one-dimensional operator (1.1). In the subsequent paper [2] Hempel, Herbst, Hinz, and Kalf proved that if q is periodic on the half-line the absolutely continues spectra is preserved and the gaps are filled with a dense point spectrum.

The spectrum of such systems has been studied further from the viewpoint of the eigenvalue distribution in the gaps [3] and it was also show that the system has a family of *isolated* eigenvalues accumulating at the essential-spectrum threshold [4]. An extension to magnetic Schrödinger operators [5] and Dirac operators [6] were also considered.

A characteristic property of such an interlaced spectrum is that the intervals of the dense pure point spectrum shrink as the energy increases. The aim of this letter is to present an example where the width of the dense-point "bands" *remains nonzero* in the high-energy limit. Since the asymptotic behavior is determined by that of the underlying one-dimensional problem, and thus by the regularity of the potential q, it is clear that we have choose a singular one; we will investigate a family of Schrödinger operators given formally by

$$\mathsf{H} = -\triangle + \alpha \sum_n \delta(|x| - R_n) \quad \text{in} \quad L^2(\mathbb{R}^\nu), \quad \nu \geq 2,$$

with a δ interaction supported by a family of concentric spheres. We will describe the model properly in the next section, then we determine its essential spectrum, and in Section 4 we will show the indicated spectral property.

2 Description of the model

Let us first briefly recall properties of the one-dimensional systems with δ interactions [7]. The operator $\mathsf{h} = -\triangle + \alpha \sum_{n\in\mathbb{Z}} \delta(x - x_n)$ can be given meaning if we require that the points supporting the interaction do not accumulate, $\inf |x_n - x_m| > 0$. Then one can check that the symmetric form t_α

defined by

$$\mathsf{t}_\alpha[f,\, g] = (f',\, g') + \alpha \sum_{n\in\mathbb{Z}} f(x_n)\bar{g}(x_n), \quad D(\mathsf{t}_\alpha) = \mathcal{H}^{1,\,2}(\mathbb{R}), \tag{2.1}$$

is closed and bounded from below [7, 8], and we identify the corresponding self-adjoint operator h_α, in the sense of first representation theorem [9] with the formal operator mentioned above. One can describe it explicitly in terms of boundary conditions: it acts as $\mathsf{h}_\alpha f = -f''$ on the domain

$$D(\mathsf{h}_\alpha) = \left\{ f \in \mathcal{H}^{2,\,2}\Big(\mathbb{R}\backslash \bigcup_{n\in\mathbb{Z}} \{x_n\}\Big) :\ f'(x_n+) - f'(x_n-) = \alpha f(x_n) \right\}.$$

The Kronig-Penney model corresponds to a periodic arrangement of the δ-interactions, for instance, $x_n = \left(n - \frac{1}{2}\right) a$ for some $a > 0$. It has a purely absolutely continuous spectrum with the known band structure [7] and these properties do not change when we pass to such a system on a half-line with any boundary condition at the origin, the only change is that the spectral multiplicity will be one instead of two.

After this preliminary let us pass to our proper topic and define an operator which can be identified with (1.2); we suppose again that the sequence of radii can accumulate only at infinity, $\inf |R_n - R_m| > 0$. As above we employ the appropriate symmetric form

$$\mathsf{T}_\alpha[f,\, g] = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla \bar{g}(x)\, \mathsf{d}^n x + \alpha \sum_n \int_{S_{R_n}} f(x)\bar{g}(x)\, \mathsf{d}\Omega,$$

with $D(\mathsf{T}_\alpha) = \mathcal{H}^{1,\,2}(\mathbb{R}^n)$, where S_{R_n} is the sphere of radius R_n and $d\Omega$ is the corresponding "area" element. Since the form is spherically symmetric, it is natural to use a partial wave decomposition. Consider the isometry

$$\mathsf{U} \,:\, L^2((0,\, \infty), r^{\nu-1} dr) \,\rightarrow\, L^2(0,\, \infty), \quad \mathsf{U}f(r) = r^{\frac{\nu-1}{2}} f(r),$$

which allows us to write

$$L^2(\mathbb{R}^\nu) = \bigoplus_l \mathsf{U}^{-1} L^2(0,\, \infty) \otimes L^2(S_1)$$

and

$$\mathsf{T}_\alpha = \bigoplus_l \mathsf{U}^{-1} \mathsf{T}_{\alpha,\,l} \mathsf{U} \otimes \mathsf{I}_l,$$

where I_l is the unit operator on $L^2(S_1)$ and

$$\mathsf{T}_{\alpha,l}[f,g] = \int_0^\infty \left(f'(r)\bar{g}'(r) + \frac{1}{r^2}\left[\frac{(n-1)(n-3)}{4} + l(l+n-2)\right] f(r)\bar{g}(r) \right) \mathsf{d}r + \alpha \sum_n f(R_n)\bar{g}(R_n),$$

with $D(\mathsf{T}_{\alpha,l}) = \mathcal{H}^{1,2}(0,\infty)$. The following lemma will help us to find properties of the form $\mathsf{T}_{\alpha,l}$.

Lemma 2.1 *(i) Let $a > 0$. There exists a positive b so that*

$$|\alpha| \sum_n |f(R_n)|^2 \le a \int_0^\infty |f'(x)|^2 \mathsf{d}x + b \int_0^\infty |f(x)|^2 \mathsf{d}x \tag{2.2}$$

holds for all functions f belonging to the Schwartz space $\mathcal{S}(0, \infty)$.

(ii) There exist C such that, for every function f in the domain of $\mathsf{H}_{\alpha,l}$ holds[1]

$$||f'|| \le C(||\mathsf{H}_{\alpha,l}f|| + ||f||) \tag{2.3}$$

Proof: Let $I \subset \mathbb{R}_+$ be an interval and $f \in \mathcal{H}^{1,2}(I)$. By a standard embedding we have $\mathcal{H}^{1,2}(I) \hookrightarrow \mathcal{C}(I)$, more explicitly, there is a $C > 0$ such that

$$|f(x)|^2 \le C\left(\int_I |f(y)|^2\,\mathsf{d}y + \int_I |f'(y)|^2\,\mathsf{d}y\right) \tag{2.4}$$

holds for every $x \in I$. Let $\{y_n\}_{n=0}^\infty$ be an increasing sequence of positive numbers such that $\sup|y_{n+1} - y_n| > 2$ and $y_1 \ge 1$. Then we consider the family of mutually disjoint intervals $I_n = (y_n - 1,\, y_n + 1)$ and summing the inequalities (2.4) for $I = I_n$ over n we get

$$\sum_{n=1}^\infty |f(y_n)|^2 \le C\left(\int_0^\infty |f(y)|^2\,\mathsf{d}y + \int_0^\infty |f'(y)|^2\,\mathsf{d}y\right).$$

[1]The operator $\mathsf{H}_{\alpha,l}$ associated with $\mathsf{T}_{\alpha,l}$ is described explicitly Theorem 2.2 below.

To conclude the argument we employ a scaling. The last inequality applied to $f_\varepsilon :\ f_\varepsilon(x) = f(\varepsilon x)$ gives

$$\sum_{n=1}^{\infty} |f(\varepsilon y_n)|^2 \le C \left(\varepsilon^{-1} \int_0^\infty |f(y)|^2\, \mathsf{d}y + \varepsilon \int_0^\infty |f'(y)|^2\, \mathsf{d}y \right);$$

the claim (i) then follows by substitution $y_n = R_n\, \varepsilon^{-1}$ with ε such that $C\varepsilon < a|\alpha|^{-1}$ and $\sup|R_{n+1} - R_n| > 2\varepsilon$, since without loss of generality we may suppose that $\alpha \neq 0$. The claim (ii) in turn follows from (i) with a fixed $a < 1$ together with the inequality

$$\begin{aligned} ||f'||^2 = (\mathsf{H}_{\alpha,l} f, f) - \int_0^\infty \frac{1}{r^2} \left(\frac{(n-1)(n-3)}{4} + l(l+n-2) \right) |f(r)|^2\, \mathsf{d}r \\ - \alpha \sum_n |f(R_n)|^2 \le \frac{1}{2} ||\mathsf{H}_{\alpha,l} f||^2 + \frac{1}{2} ||f||^2 + a||f'||^2 + b||f||^2, \end{aligned}$$

where we used Cauchy-Schwarz inequality, $(\mathsf{H}_{\alpha,l} f, f) \le \frac{1}{2}(||\mathsf{H}_{\alpha,l} f||^2 + ||f||^2)$, and the nonnegativity of the second term. ■

This allows us to describe the model Hamiltonian explicitly in terms of boundary conditions at the singular points.

Theorem 2.2 *(i) The quadratic form $\mathsf{T}_{\alpha,l}$ is bounded from below and closed on $L^2(0,\infty)$ and the space $C_0^\infty(0,\infty)$ of infinitely differentiable functions of compact support is a core of $\mathsf{T}_{\alpha,l}$.*

(ii) The self-adjoint operator corresponding to $\mathsf{T}_{\alpha,l}$ by the first representation theorem is

$$\mathsf{H}_{\alpha,l} = -\frac{\mathsf{d}^2}{\mathsf{d}^2 r} + \frac{1}{r^2} \left(\frac{(n-1)(n-3)}{4} + l(l+n-2) \right),$$

with the domain $D(\mathsf{H}_{\alpha,l})$ given by

$$\left\{ f \in \mathcal{H}^{2,2} \left(\mathbb{R}^+ \setminus \bigcup_n \{R_n\} \right) :\ f'(R_n+) - f'(R_n-) = \alpha f(R_n) \right\},$$

and the self-adjoint operator associated with the T_α is thus

$$\mathsf{H}_\alpha = \bigoplus_l \mathsf{U}^{-1} \mathsf{H}_{\alpha,l} \mathsf{U} \otimes \mathsf{I}_l. \tag{2.5}$$

Proof: The first claim follows from Ref. [8] in combination with the previous lemma, the second one can be verified directly. ■

3 The essential spectrum

Let us first introduce some notation which we will use throughout this section. We need a one-dimensional comparison operator. For simplicity we take an operator on the whole axis extending the family $\{R_n\}_{n=1}^{\infty}$ of the radii to $\{R_n\}_{n\in\mathbb{Z}}$ by putting $R_{-n} = -R_{n+1}$ for $n = 0, 1, \dots$. By h_α we denote the self-adjoint operator defined in the opening of the previous section in which we now put $x_n := R_n$; the corresponding quadratic form will be again denoted as t_α. By $\mathsf{h}_{\alpha,\,R}$ we denote the self-adjoint operator obtained from h_α by adding the Dirichlet boundary conditions at the points $\pm R$. Since h_α and $\mathsf{h}_{\alpha,\,R}$ have a common symmetric restriction with finite deficiency indices we have

$$\sigma_{ess}(\mathsf{h}_\alpha) = \sigma_{ess}(\mathsf{h}_{\alpha,\,R}). \tag{3.1}$$

Furthermore, by $\mathsf{h}_{\alpha,\,(a,\,b)}$ and $\mathsf{h}_{\alpha,\,R,\,(a,\,b)}$ we denote the self-adjoint operator which is a restriction of h_α, $\mathsf{h}_{\alpha,\,R}$ to $L^2(a,\,b)$, respectively, with Dirichlet boundary conditions at the interval endpoints. We note that

$$\mathsf{h}_{\alpha,\,R,\,(0,\,\infty)} = \mathsf{h}_{\alpha,\,(0,\,R)} \oplus \mathsf{h}_{\alpha,\,(R,\,\infty)}. \tag{3.2}$$

We use a similar notation, namely $\mathsf{H}_{\alpha,\,l,\,R}$ and $\mathsf{H}_{\alpha,\,l,\,(a,\,b)}$, for operators in every partial wave. Furthermore $\mathsf{H}_{\alpha,\,(\rho,\,R)}$ denotes the restriction of H_α to the spherical shell $B_R \setminus B_\rho$. Our main result in this section reads as follows.

Theorem 3.1 *The essential spectrum of the operator (2.5) is equal to*

$$\sigma_{ess}(\mathsf{H}_\alpha) = [\inf \sigma_{ess}(\mathsf{h}_\alpha), \infty) \tag{3.3}$$

The idea of the proof is the same as in [1]. First we check that $\inf \sigma_{ess}(\mathsf{H}_\alpha)$ cannot be smaller then $\inf \sigma_{ess}(\mathsf{h}_\alpha)$, after that we will show that $\sigma_{ess}(\mathsf{H}_\alpha)$ contains the interval $[\inf \sigma_{ess}(\mathsf{h}_\alpha), \infty)$.

Proposition 3.2 *In the stated assumptions we have*

$$\inf \sigma_{ess}(\mathsf{H}_\alpha) \geq \inf \sigma_{ess}(\mathsf{h}_\alpha) \tag{3.4}$$

Proof: The partial-wave decomposition of Theorem 2.2 in combination with the minimax principle imply that the spectral minimum is reached in the s-state subspace, hence we can consider only spherically symmetric functions. Then the idea is to estimate $\inf \sigma_{ess}(\mathsf{H}_\alpha)$ by means of the lowest eigenvalue $\mu_{\rho, R}$ of the operator $\mathsf{H}_{\alpha, (\rho, R)}$ and ρ, R large enough. The associated – spherically symmetric – eigenfunction $u_{\rho R}$ clearly satisfied the δ boundary conditions, hence one can repeat the argument from [1], Proposition 1. ■

Proposition 3.3

$$\sigma_{ess}(\mathsf{H}_\alpha) \supset [\inf \sigma_{ess}(\mathsf{h}_\alpha), \infty) \tag{3.5}$$

Proof: The idea is to employ Weyl criterion. Following [10], let $\lambda_0 \in \sigma_{ess}(h_\alpha)$ and $\lambda > 0$, then we have to show that for every $\epsilon > 0$ there is a function

$$\varphi \in D(\mathsf{H}_\alpha) \quad \text{satisfying} \quad ||\varphi|| \geq 1 \quad \text{and} \quad ||(\mathsf{H}_\alpha - \lambda_0 - \lambda)\varphi|| \leq \epsilon .$$

The key ingredients in the estimates of the regular-case proof – cf. [10], (i), (ii) on the first page – correspond to the equations (2.3) and (3.1) here. In order to use directly the said argument, we have to deal with the boundary conditions. To do this we use the simple observation that whenever

$$f(r) \in D(\mathsf{h}_\alpha) \quad \text{and} \quad g(x) \in D(\mathsf{H}_0) \quad \text{then} \quad \phi(x) = f(|x|)g(x) \in D(\mathsf{H}_\alpha)\,, \tag{3.6}$$

now we consider such a $\phi(x)$ and follow step by step the proof in [10]. ■

4 Character of the spectrum

In this section we will make two claims. One is general, without a specific requirement on the distribution of the δ barriers other that $\inf |R_n - R_m| > 0$. It stems from the fact that the essential spectrum of the associated one-dimensional operator h_α may have gaps; we want to know how the spectrum of H_α looks like in these gaps. First we observe that in every partial wave

$$\sigma_{ess}(\mathsf{H}_{\alpha, l}) = \sigma_{ess}(\mathsf{h}_\alpha). \tag{4.1}$$

Indeed, in view of (3.1) we have

$$\sigma_{ess}(\mathsf{H}_{\alpha,\,l}) = \sigma_{ess}(\mathsf{H}_{\alpha,\,l,\,R})\,,$$

and since $\mathsf{H}_{\alpha,\,l,\,(0,\,R)}$ has a purely discrete spectrum, we use (3.2) to infer that

$$\sigma_{ess}(\mathsf{H}_{\alpha,\,l}) = \sigma_{ess}(\mathsf{H}_{\alpha,\,l,\,(R,\,\infty)})\,. \tag{4.2}$$

Furthermore, a multiplication by (a multiple of) r^{-2} is $\mathsf{h}_{\alpha,\,(R,\,\infty)}$ compact, which implies by Weyl's theorem that

$$\sigma_{ess}(\mathsf{H}_{\alpha,\,l,(R,\,\infty)}) = \sigma_{ess}(\mathsf{h}_{\alpha,\,(R,\,\infty)})\,,$$

and using once more the "chopping" argument we arrive at (4.1). Now we are ready to state and prove the claim which is a counterpart of the result derived in [2] for regular potential barriers.

Theorem 4.1 *Let H_α be as described above, then for any gap $(\alpha,\,\beta)$ in the essential spectrum of h_α the following is valid:*

(i) H_α has no continuous spectrum in $(\alpha,\,\beta)$;

(ii) eigenvalues of H_α are dense in $(\alpha,\,\beta)$.

Proof: By (4.1), none of the operators $\mathsf{H}_{\alpha,\,l}$, $l = 0,\,1,\,2,\ldots$, has a continuous spectrum in $(\alpha,\,\beta)$, hence H_α has no continuous spectrum in this interval either. On the other hand, the entire interval $(\alpha,\,\beta)$ is contained in the essential spectrum of H_α, and it follows that the spectrum of H_α in $(\alpha,\,\beta)$ consists of eigenvalues, which are necessarily dense in the interval. ∎

Now we pass to a particular case when the δ-sphere interactions are arranged in a periodic way, $R_n = na - a/2$ with $a > 0$, and prove that in this situation there is a purely continuous spectrum in the *bands* of the associated one-dimensional Kronig-Penney model. The argument is similar to Section 2 of [2] so we will concentrate mostly on the changes required by the singular character of the interaction.

Lemma 4.2 *Let $(a,\,b)$ be the interior of a band of the operator h_α in $L_2(\mathbb{R})$. Let further $K \subset (a,\,b)$ be a compact subinterval, $c \in \mathbb{R}$, and $x_0 > 0$. Then there exist numbers $C_1,\,C_2 > 0$ such that for every $\lambda \in K$ any solution u of*

$$-u''(r) + \frac{c}{r^2}u(r) = \lambda u(r)\,, \quad u \in D(\mathsf{h}_\alpha)\,, \tag{4.3}$$

with the normalization $|u(x_0)|^2 + |u'(x_0)|^2 = 1$ *satisfies*

$$C_1^2 \geq |u(x)|^2 + |u'(x)|^2 \,, \qquad \int\limits_{x_0}^{x} |u(t)|^2 \,\mathsf{d}t \geq C_2(x - x_0) \quad \text{for } \; x \geq x_0 + 1 \,. \tag{4.4}$$

Proof: Let $\lambda \in K$. As it is well known [11] the equation $\mathsf{h}_\alpha w = \lambda w$ has two linearly independent solutions $u_0 = u_0(\cdot,\,\lambda)$, $v_0 = v_0(\cdot,\,\lambda)$ such that $u_0,\, v_0 \in D(\mathsf{h}_\alpha)$, and $|u_0|$, $|u_0'|$, $|v_0|$, $|v_0'|$ are periodic, bounded and continuous w.r.t. λ. Without loss of generality we may assume that the Wronski matrix

$$Y = \begin{bmatrix} u_0 & v_0 \\ u_0' & v_0' \end{bmatrix}$$

has determinant equal to one. Let $C_0 > 0$ be a constant such that

$$|u_0(x,\,\lambda)|^2 + |u_0'(x,\,\lambda)|^2 + |v_0(x,\,\lambda)|^2 + |v_0'(x,\,\lambda)|^2 \leq C_0 \quad (x \in \mathbb{R},\, \lambda \in K).$$

Given any solution u of (4.3), the function

$$y := Y^{-1} \begin{bmatrix} u \\ u' \end{bmatrix}$$

satisfies the equation $y' = Ay$ on every interval $\big((n - \frac{1}{2})a,\, (n + \frac{1}{2})a\big)$, where

$$A = -\frac{c}{x^2} \begin{bmatrix} u_0 v_0 & v_0^2 \\ -u_0^2 & -u_0 v_0 \end{bmatrix}$$

in analogy with [2]. By a straightforward calculation we get

$$y = \begin{bmatrix} v_0' u - v_0 u' \\ -u_0' u + u_0 u' \end{bmatrix} \,, \qquad y' = \frac{c}{x^2} \begin{bmatrix} -v_0 u \\ u_0 u \end{bmatrix} \,,$$

which implies that y, y' are continuous at the singular points. Thus

$$y(x) = \exp\left\{ \int\limits_{x_0}^{x} A(t)\,\mathsf{d}t \right\} y(x_0)$$

is a solution of $y' = Ay$ and as in [2] it holds that

$$\frac{1}{2}(|y|^2)' \leq |(y,\, y')| \leq \|A\| |y|^2$$

and so for $x \geq x_0$ we have

$$|y(x)|^2 \leq |y(x_0)|^2 \exp\left\{2\int\limits_{x_0}^{x} \|A(t)\|\,\mathsf{d}t\right\} \leq |Y^{-1}(x_0)|^2 \exp\left\{2\int\limits_{x_0}^{\infty} \|A(t)\|\,\mathsf{d}t\right\}$$

for any solution of (4.3) with the normalization $|u(x_0)|^2+|u'(x_0)|^2 = 1$. From

$$\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix} = Y(x)Y^{-1}(x_0)\begin{bmatrix} u(x_0) \\ u'(x_0) \end{bmatrix} + \int\limits_{x_0}^{x} Y(x)A(t)y(t)\,\mathsf{d}t\,, \quad x \geq x_0\,,$$

we now infer the existence of a number $C_1 > 0$ such that

$$|u(x)|^2 + |u'(x)|^2 \leq C_1^2\,, \quad x \geq x_0\,, \tag{4.5}$$

holds for all solutions of (4.3) which are normalized in the described way. This proves the first inequality in (4.4).

Let u be a real-valued solution of (4.3), again with the same normalization, and suppose that v is a solution such that

$$v(x_0) = -u'(x_0)\,, \quad v'(x_0) = u(x_0)\,.$$

Then the Wronskian of u and v equals one, and therefore

$$1 = [u(x)v'(x) - u'(x)v(x)]^2 \leq [u^2(x) + u'^2(x)][v^2(x) + v'^2(x)]\,, \quad x \geq x_0\,.$$

Since v satisfies (4.5) we find that

$$\frac{x - x_0}{C_1^2} \leq \int_{x_0}^{x} (u^2 + u'^2)(t)\,\mathsf{d}t\,, \quad x \geq x_0\,,$$

and the second assertion in (4.4) follows from Lemma 2.1(ii) ■

In particular, this lemma proves through (4.4) that the operator $\mathsf{H}_{\alpha,l}$ has no embedded eigenvalues in $(a,\,b)$. Next we will derive a Lipschitz bound for the number of eigenvalues of the operator $\mathsf{h}_k \equiv \mathsf{h}_{\alpha,\,(0,\,R_k+a/2)}$; we denote their number in the interval $(\lambda_1,\,\lambda_2)$ by $N_k(\lambda_1,\,\lambda_2)$.

Lemma 4.3 *Let $(a,\,b)$ be a spectra band of the operator h_α in $L^2(\mathbb{R})$ and $\lambda_2 - \lambda_1 > 0$. Then there exists a number $C > 0$ such that*

$$N_k(\lambda_1,\,\lambda_2) \leq C(\lambda_2 - \lambda_1)R_k \tag{4.6}$$

for every $k \in \mathbb{N}$.

Proof: Let $\mathsf{h}^{(\theta)}$ be the operator h_α acting on $L^2(0,\,a)$ with θ-periodic boundary conditions. Then λ is an eigenvalue of h_k if and only if there is an integer $j \in \{0,\,\dots,\,k-1\}$ such that λ is the eigenvalue of $\mathsf{h}^{(j\pi/k)}$. The eigenvalues of $\mathsf{h}^{(\theta)}$ are the roots of Kronig-Penney equation,

$$\cos(\theta a) = \cos(\lambda a) + \frac{\alpha}{2\lambda}\sin(\lambda a)\,.$$

It follows from Theorem III.2.3.1 in [7] that there is precisely one eigenvalue of $\mathsf{h}^{(\theta)}$ in every interval $((k-1)^2\pi^2 a^{-2},\, k^2\pi^2 a^{-2})$. Hence

$$N_k(\lambda_1,\,\lambda_2) \le k\left\lceil (\sqrt{\lambda_2}-\sqrt{\lambda_1})\frac{a}{\pi}\right\rceil \le k\left((\sqrt{\lambda_2}-\sqrt{\lambda_1})\frac{a}{\pi}+1\right) \le R_k(\lambda_2-\lambda_1)C\,,$$

where

$$C := 2\,\frac{a(\sqrt{\lambda_2}-\sqrt{\lambda_1})+\pi}{a\pi(\lambda_2-\lambda_1)}\,;$$

we have used here the fact that $R_k - \frac{1}{2}a > \frac{1}{2}ka$. ■

With these preliminaries, we are prepared to prove the absolute continuity of the spectrum inside the Kronig-Penney bands.

Theorem 4.4 *The spectrum of* $\mathsf{H}_{\alpha,l}$ *is absolutely continuous in the interior of each spectral band of* h_α.

Proof: Since the argument is similar to [2], [11, Thm 15.3], we just sketch it. The aim is to show that for any fixed $f \in C_0^\infty(0,\,\infty)$ the function $||E(\lambda)f||^2$, where $E(\lambda)$ denotes the spectral measure of $\mathsf{H}_{\alpha,l}$, is Lipschitz continuous for λ in the spectral band $(a,\,b)$. As there are no eigenvalues of $\mathsf{H}_{\alpha,l}$ in $(a,\,b)$ by Lemma 4.2) one has the strong convergence

$$E^{R_n}(\lambda) \to E(\lambda)\,, \quad R_n \to \infty\,,$$

where $E^{R_n}(\lambda)$ denotes the spectral resolution of $\mathsf{H}_k := \mathsf{h}_k + c\,r^{-2}$, and consequently, it is sufficient to prove that for $[\alpha,\,\beta] \subset (a,\,b)$

$$((E^{R_n}(\beta) - E^{R_n}(\alpha))f,\,f) \le \text{const}\,(\beta-\alpha+\epsilon)\,. \tag{4.7}$$

holds for any ϵ. The spectrum of $\mathsf{H}_{\alpha,l,R_n}$ is purely discrete and simple. Let us denote its j-th eigenvalue by λ_j and suppose that the associated eigenfunction ϕ_j has the normalization

$$|\phi_j(R_0)|^2 + |\phi_j'(R_0)|^2 = 1.$$

Lemma 4.2 establishes the existence of numbers C_1, $C_2 > 0$ such that

$$((E^{R_n}(\beta) - E^{R_n}(\alpha))f,\, f) \le \sum_{\alpha<\lambda_j<\beta} |(f,\, \phi_j)|^2 ||\phi_j||^{-2}$$
$$\le \frac{C_1^2}{C_2(R_n - R_0)} ||f||^2 \sum_{\alpha<\lambda_j<\beta} 1 \le \frac{C_3}{R_n - R_0} \#\{j : \alpha < \lambda_j \le \beta\}\,, \quad (4.8)$$

for all $R_n > R_0$. Now we fix ε so small that $[\alpha - \varepsilon/2,\, \beta + \varepsilon/2] \subset (a,\, b)$ and choose $R_{n(\varepsilon)}$ so that

$$\frac{|c|}{r^2} < \frac{\varepsilon}{2} \quad \text{for} \quad r > R_{n(\varepsilon)} \qquad (4.9)$$

and impose an additional Dirichlet boundary condition at the point $R_{n(\varepsilon)}$. Then the interval $(0,\, R_{n(\varepsilon)})$ contributes by a certain number C_ε of eigenvalues. On the other hand, from Lemma 4.3 we know that the number of eigenvalues of the operator $\mathsf{h}_{(k_\varepsilon,k)}$ in $[\alpha - \varepsilon/2, \beta + \varepsilon/2]$ can be estimated by

$$C(\beta - \alpha + \varepsilon)R_n$$

and by the minimax principle and (4.9) the number of eigenvalues of $\mathsf{H}_{R_{n(\varepsilon)}}$ in $[\alpha,\, \beta]$ is estimated with the same relation. In this way we have proved the bound

$$\#\{j : \alpha < \lambda_j \le \beta\} \le C_\varepsilon + C_0(\beta - \alpha + \varepsilon)R_n.$$

Finally, we substitute this result back to the right-hand side of (4.8), and taking into account that R_n can be chosen arbitrarily large, we obtain the needed inequality (4.7) concluding thus the proof. ■

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