ON SPECTRAL PROPERTIES OF TRANSLATIONALLY INVARIANT MAGNETIC SCHRÖDINGER OPERATORS

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Abstract. We consider a class of translationally invariant magnetic fields such that the corresponding potential has a constant direction. Our goal is to study basic spectral properties of the Schrödinger operator $\mathbf{H}$ with such a potential. In particular, we show that the spectrum of $\mathbf{H}$ is absolutely continuous and we find its location. Then we study the long-time behaviour of solutions $\exp(-i\mathbf{H}t)f$ of the time dependent Schrödinger equation. It turnes out that a quantum particle remains localized in the plane orthogonal to the direction of the potential. Its propagation in this direction is determined by group velocities. It is to a some extent similar to a evolution of a one-dimensional free particle but "exits" to $+\infty$ and $-\infty$ might be essentially different.

1. Introduction

1.1. Traslationally invariant magnetic fields $B(x) = (b_1(x), b_2(x), b_3(x))$, $x = (x_1, x_2, x_3)$, $\operatorname{div} B(x) = 0$, give important examples where a non-trivial information can be obtained about spectral properties of the corresponding Schrödinger operators $\mathbf{H}$. We suppose for definiteness that $B(x)$ does not depend on the x_3-variable so that $\mathbf{H}$ commute with translations along the x_3-axis. There are two essentially different (and in some sense extreme) classes of traslationally invariant magnetic fields.

The first class consists of fields $B(x) = (0, 0, b_3(x_1, x_2))$ of constant direction. For such fields, the momentum p of a classical particle in the x_3-direction is conserved, and in the Schrödinger equation the variable x_3 can be separated. Thus, we arrive to a two-dimensional problem in the (x_1, x_2)-plane. Furthermore, if b_3 is a function of $r = (x_1^2 + x_2^2)^{1/2}$ only, then we get a set of problems on the half-line $r > 0$ labelled by the magnetic quantum number m. The most important example of this type is a constant magnetic field $b_3(r) = const$ (see [10]). Some class of functions $b_3(r)$ decaying as $r \to \infty$ was discussed in [12] (see also [2]) where new interesting effects were found. Another famous case is $b_3(x_1, x_2) = \delta(x_1, x_2)$ ($\delta(\cdot)$ is the Dirac delta-function) studied in [1]. Scattering by an arbitrary short-range (decaying faster than $|x|^{-2-\varepsilon}$, $\varepsilon > 0$, as $|x| \to \infty$) magnetic field $b_3(x_1, x_2)$ turns out to be rather similar to this particular case (see [16]).

The second class consists of fields $B(x) = (b_1(x_1, x_2), b_2(x_1, x_2), 0)$ orthogonal to the x_3-axis. In this case the corresponding magnetic potential $A(x)$, defined (up to gauge

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transformations) by the equation $\operatorname{curl} A(x) = B(x)$, can be chosen as

$$A(x) = (0, 0, -a(x_1, x_2)) \tag{1.1}$$

so that it has the constant direction. In contrast to fields of the first class, now the variable x_3 cannot be separated in the Schrödinger equation. Nevertheless due to the invariance with respect to translations along the x_3-axis the operator (we always suppose that the charge of a particle is equal to 1)

$$\mathbf{H} = (i\nabla + A(x))^2 \tag{1.2}$$

can be realized, after the Fourier transform in the variable x_3, in the space $L^2(\mathbb{R}; L^2(\mathbb{R}^2))$ as the operator of multiplication by the operator-valued function

$$H(p) = -\Delta + (a(x_1, x_2) + p)^2 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad p \in \mathbb{R}. \tag{1.3}$$

Moreover, if $a(x_1, x_2) = a(r)$, then the subspaces with fixed magnetic quantum number $m \in \mathbb{Z}$ are invariant subspaces of $H(p)$ so that the operator $H(p)$ reduces to the orthogonal sum over $m \in \mathbb{Z}$ of the operators

$$H_m(p) = -\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{m^2}{r^2} + (a(r) + p)^2 \tag{1.4}$$

acting in the space $\mathcal{H} = L^2(\mathbb{R}_+; rdr)$. In this case the field is given by the equation

$$B(x) = b(r)(-\sin\theta, \cos\theta, 0) \tag{1.5}$$

where $b(r) = a'(r)$ and θ is the polar angle. Thus, vectors $B(x)$ are tangent to circles centered at the origin. An important example of such type is a field created by a current along an infinite straight wire (coinciding with the x_3-axis). In this case $b(r) = b_0 r^{-1}$ so that $a(r) = b_0 \ln r$. The Schrödinger operator with such magnetic potential was studied in [15].

1.2. In this article we consider magnetic fields (1.5) with a sufficiently arbitrary function $b(r)$. Our goal is to study basic spectral properties of the corresponding Schrödinger operator $\mathbf{H}$ such as the absolute continuity, location and multiplicity of the spectrum, as well as the long-time behaviour of the unitary group $\exp(-i\mathbf{H}t)$. We emphasize that for magnetic fields considered here, the problem is genuinely three-dimensional, and actually the motion of a particle in the x_3-direction is of a particular interest.

Using the cylindrical invariance of field (1.5), we can start either from translational or from rotational (around the x_3-axis) symmetries. The rotational invariance implies that the operator $\mathbf{H}$ is the orthogonal sum of its restrictions $\mathbf{H}_m$ on the subspaces of functions with magnetic quantum number $m \in \mathbb{Z} = \{0, \pm1, \pm2, \dots\}$. It can be identified with the operator (we keep the same notation for this operator)

$$\mathbf{H}_m = -\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{m^2}{r^2} + (i\frac{d}{dx_3} + a(r))^2 \tag{1.6}$$

acting in the space $\mathfrak{H} = L^2(\mathbb{R}_+ \times \mathbb{R}; rdrdx_3)$. In view of the translation invariance, every operator $\mathbf{H}_m$ can be realized (again after the Fourier transform in the variable x_3) in the space $L^2(\mathbb{R}; L^2(\mathbb{R}_+; rdr))$ as the operator of multiplication by the operator-valued function $H_m(p)$ defined by (1.4).

Suppose now that $b(r)$ does not tend to zero too fast so that $a(r) \to \infty$ as $r \to \infty$. Then the spectrum of each operator $H_m(p)$ is discrete. Let $\lambda_{n,m}(p)$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, be the increasing sequence of its eigenvalues (they are simple and positive), and let $\psi_{n,m}(r, p)$ be the corresponding sequence of its eigenfunctions. The functions $\lambda_{n,m}(p)$ are known as dispersion curves of the problem. They determine the spectral properties of the operator $\mathbf{H}_m$.

Note that if $a(r)$ is replaced by $-a(r)$, then $\lambda_{n,m}(p)$ is replaced by $-\lambda_{n,m}(p)$, so that the case $a(r) \to -\infty$ as $r \to \infty$ is automatically included in our considerations. Recall that, for a magnetic field $B(x)$, the magnetic potential $A(x)$ such that $\operatorname{curl} A(x) = B(x)$ is defined up to a gauge term $\operatorname{grad} \varphi(x)$. In particular for magnetic fields (1.5) in the class of potentials $A(x) = (0, 0, -a(r))$ one can always add to $a(r)$ an arbitrary constant c. This leads to the transformations $\lambda_{n,m}(p) \mapsto \lambda_{n,m}(p-c)$ and $\psi_{n,m}(r, p) \mapsto \psi_{n,m}(r, p-c)$.

1.3. The precise definitions of the operators $\mathbf{H}_m$ and $\mathbf{H}$ and their decompositions into the direct integrals over the operators $H_m(p)$ and $H(p)$ are given in Section 2. To put it differently, we construct a complete set of eigenfunctions of the operator $\mathbf{H}$. They are parametrized by the magnetic quantum number m, the momentum p in the direction of the x_3-axis and the number n of an eigenvalue $\lambda_{m,n}(p)$ of the operator $H_m(p)$. Thus, if we set

$$\boldsymbol{\psi}_{n,m,p}(r, \theta, x_3) = e^{ipx_3} e^{im\theta} \psi_{n,m}(r, p), \tag{1.7}$$

then

$$\mathbf{H}\boldsymbol{\psi}_{n,m,p} = \lambda_{n,m}(p)\boldsymbol{\psi}_{n,m,p}. \tag{1.8}$$

In Section 3, we show that for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$:

- Under very general assumptions $\lambda_{n,m}(p) \to \infty$ as $p \to \infty$ (Proposition 3.3).
- If $b(r) \to 0$ as $r \to \infty$, then $\lambda_{n,m}(p) \to 0$ as $p \to -\infty$ (Proposition 3.5).
- If $b(r)$ admits a finite positive limit b_0 as $r \to \infty$, then $\lambda_{n,m}(p) \to (2n-1)b_0$ for all m as $p \to -\infty$ (Proposition 3.6).
- If $b(r) \to \infty$ as $r \to \infty$, then $\lambda_{n,m}(p) \to \infty$ as $p \to -\infty$ (Proposition 3.6).

Related results concerning the dispersion curves for Schrödinger operator with constant magnetic fields defined on unbounded domains $\Omega \subset \mathbb{R}^2$ have been obtained in [5] (the case where Ω is a strip) and in [4], [7, Section 4.3] (the case where Ω is a half-plane).

In Theorem 3.8 we formulate the main spectral results which follow from the asymptotic properties of the dispersion curves $\lambda_{n,m}(p)$, $p \in \mathbb{R}$. First, the analyticity and the asymptotics as $p \to \infty$ of $\lambda_{n,m}(p)$ imply immediately that the spectra $\sigma(\mathbf{H}_m)$ and $\sigma(\mathbf{H})$ of the operators $\mathbf{H}_m$, $m \in \mathbb{Z}$, and $\mathbf{H}$ are purely absolutely continuous. Moreover,

$$\sigma(\mathbf{H}_m) = [\mathcal{E}_m, \infty), \quad m \in \mathbb{Z}, \quad \sigma(\mathbf{H}) = [\mathcal{E}_0, \infty), \tag{1.9}$$

where

$$\mathcal{E}_m = \inf_{p \in \mathbb{R}} \lambda_{1,m}(p) \geq 0. \tag{1.10}$$

Next, in the case where the magnetic field tends to 0 as $r \to \infty$, the spectra of $\mathbf{H}_m$, $m \in \mathbb{Z}$, coincide with $[0, \infty)$ and have infinite multiplicity. On the other hand, in the case where the magnetic field tends as $r \to \infty$ to a positive finite limit, or to infinity, we have that $\mathcal{E}_m > 0$ for all $m \in \mathbb{Z}$ and each of the spectra $\sigma(\mathbf{H}_m)$ contains infinitely many thresholds.

Further, in Section 4, we obtain a convenient formula for the derivatives $\lambda'_{n,m}(p)$ which play the role of asymptotic group velocities. Our formula for $\lambda'_{n,m}(p)$ yields sufficient conditions (see Theorem 4.3) for positivity of these functions. The leading example when these conditions are met, is

$$b(r) = b_0 r^{-\delta}, \quad b_0 > 0, \quad \delta \in [0, 1], \tag{1.11}$$

and $m \neq 0$. If $\delta = 1$, this result remains true for all m (cf. [15]). On the contrary, if $\delta = 0$ and $m = 0$, then $\lambda'_{n,0}(p) < 0$ for all n on some interval of p (lying on the negative half-axis). Similar results concerning for the dispersion curves for the Schrödinger operator with constant magnetic field, defined on the half-plane with Dirichlet (resp., Neumann) boundary conditions, can be found in [4] (resp., [3] and [7, Section 4.3]).

Finally, in Section 5 we discuss the long-time behaviour of a quantum particle. The time evolution of a quantum system is determined by the unitary groups $\exp(-i\mathbf{H}_m t)$, $m \in \mathbb{Z}$, so that an analysis of its asymptotics as $t \to \pm\infty$ relies on spectral properties of the operators $\mathbf{H}_m$. Since these operators have discrete spectra, a quantum particle remains localized in the (x_1, x_2)-plane. Its propagation in the x_3-direction is governed by the group velocities $\lambda'_{n,m}(p)$. In particular, the condition $\lambda'_{n,m}(p) > 0$ for all $n \in \mathbb{N}$ and $p \in \mathbb{R}$ implies that a quantum particle with the magnetic quantum number m propagates as $t \to +\infty$ in the positive direction of the x_3-axis.

Let us compare these results with the long-time behaviour of a classical particle in magnetic field (1.5). As shown in [15], the function $x_3'(t)$ is periodic with period T determined by initial conditions. Its drift $x_3(T) - x_3(0)$ over the period is nonnegative if $b(r) \geq 0$ and $b'(r) \geq 0$. Morever, it is strictly positive if $b'(r) > 0$ for all r. In the case $b(r) = \text{const}$ it is still strictly positive if the angular momentum m of a particle is not zero. Thus, our results for functions (1.11) correspond completely to the classical picture if $\delta = 1$ or $\delta \in [0, 1)$ and $m \neq 0$. In the case $\delta = 0$ and $m = 0$ the behaviour of quantum and classical particles turn out to be qualitatively different.

2. Hamiltonians and their diagonalizations

Here we give precise definitions of the Hamiltonians and discuss their reductions due to the cylindrical symmetry.

2.1. For an arbitrary magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)^3$, the self-adjoint Schrödinger operator (1.2) can be defined via its quadratic form

$$\mathbf{h}[\mathbf{u}] = \int_{\mathbb{R}^3} |i(\nabla \mathbf{u})(x) + A(x)\mathbf{u}(x)|^2 \, dx, \quad x = (x_1, x_2, x_3). \tag{2.1}$$

It is easy to see that this form is closed on the set of functions $u \in L^2(\mathbb{R}^3)$ such that $\nabla \mathbf{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$ and $i\nabla\mathbf{u} + A\mathbf{u} \in L^2(\mathbb{R}^3)^3$. Similarly, if $a \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, then the self-adjoint Schrödinger operator (1.3) can be defined via its quadratic form

$$h[u; p] = \int_{\mathbb{R}^2} \left(|(\nabla u)(x)|^2 + (a(x) + p)^2 |u(x)|^2 \right) dx, \quad x = (x_1, x_2), \quad p \in \mathbb{R}. \tag{2.2}$$

This form is closed on the set of functions $u \in L^2(\mathbb{R}^2)$ such that integral (2.2) is finite. Clearly, this set does not depend on the parameter $p \in \mathbb{R}$.

Let $\mathcal{F} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}; L^2(\mathbb{R}^2))$ be the Fourier transform with respect to x_3, i.e.

$$(\mathcal{F}\mathbf{u})(x_1, x_2, p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix_3 p} \mathbf{u}(x_1, x_2, x_3) dx_3.$$

If $A(x)$ is given by formula (1.1), then

$$\mathbf{h}[\mathbf{u}] = \int_{-\infty}^{\infty} h[(\mathcal{F}\mathbf{u})(p); p] dp$$

which implies the equation

$$(\mathcal{F}\mathbf{H}\mathbf{u})(x_1, x_2; p) = (H(p)\mathcal{F}\mathbf{u})(x_1, x_2; p). \tag{2.3}$$

This equation can be regarded as a "working" definition of the operator $\mathbf{H}$.

2.2. Assume now that the function a in (1.1) depends only on r, and

$$a \in L^2_{\text{loc}}([0, \infty); rdr). \tag{2.4}$$

If we separate variables in the cylindrical coordinates (r, θ, x_3) and denote by $\mathfrak{H}_m \subset L_2(\mathbb{R}^3)$ the subspace of functions $\mathbf{f}(r, x_3)e^{im\theta}$ where $\mathbf{f} \in L_2(\mathbb{R}_+ \times \mathbb{R}; rdrdx_3)$ and $m \in \mathbb{Z}$ is the magnetic quantum number, then

$$L_2(\mathbb{R}^3) = \bigoplus_{m \in \mathbb{Z}} \mathfrak{H}_m.$$

The subspaces $\mathfrak{H}_m$ are invariant with respect to $\mathbf{H}$ so that restrictions $\mathbf{H}_m$ of $\mathbf{H}$ on $\mathfrak{H}_m$ are related with $\mathbf{H}$ by formula

$$\mathbf{H} = \bigoplus_{m \in \mathbb{Z}} \mathbf{H}_m. \tag{2.5}$$

Every $\mathfrak{H}_m$ can obviously be identified with the space $L^2(\mathbb{R}_+ \times \mathbb{R}; rdrdx_3) =: \mathfrak{H} =$; then $\mathbf{H}_m$ is identified with operator (1.6).

Quite similarly, if $\mathcal{H}_m \subset L_2(\mathbb{R}^2)$ is the subspace of functions $f(r)e^{im\theta}$ where $f \in L_2(\mathbb{R}_+; rdr)$, then

$$L_2(\mathbb{R}^2) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m.$$

The subspaces $\mathcal{H}_m$ are invariant with respect to $H(p)$ so that restrictions $H_m(p)$ of $H(p)$ on $\mathcal{H}_m$ are related with $H(p)$ by formula

$$H(p) = \bigoplus_{m \in \mathbb{Z}} H_m(p). \tag{2.6}$$

Every $\mathcal{H}_m$ can obviously be identified with the space $L^2(\mathbb{R}_+; rdr) =: \mathcal{H}$; then $H_m(p)$ is identified with operator (1.4).

Let $\mathcal{F}_m : \mathfrak{H}_m \to L^2(\mathbb{R}; L^2(\mathbb{R}_+; rdr))$ be the restriction of $\mathcal{F}$ on the subspace $\mathfrak{H}_m$. Then we have (cf. (2.3))

$$(\mathcal{F}_m \mathbf{H}_m \mathbf{f})(r; p) = (H_m(p)\mathcal{F}_m \mathbf{f})(r; p). \tag{2.7}$$

Sometimes it is more convenient to consider instead of $H_m(p)$ the operator

$$L_m(p) = r^{1/2} H_m(p) r^{-1/2} = -\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2} + (a(r) + p)^2 \tag{2.8}$$

acting in the space $L^2(\mathbb{R}_+)$ and unitarily equivalent to the operator $H_m(p)$. It is easy to see that the operator $L_m(p)$ corresponds to the quadratic form

$$l_m[g;p] = \int_0^\infty \big(|g'(r)|^2 + (m^2 - 1/4) r^{-2} |g(r)|^2 + (a(r) + p)^2 |g(r)|^2\big)\, dr, \tag{2.9}$$

defined originally on $C_0^\infty(\mathbb{R}_+)$, and then closed in $L^2(\mathbb{R}_+)$.

2.3. If

$$a(r) \to \infty \quad \text{as} \quad r \to \infty, \tag{2.10}$$

then the spectrum of the operator $H_m(p)$, $p \in \mathbb{R}$, $m \in \mathbb{Z}$, is discrete. Thus, it consists of the increasing sequence $\lambda_{n,m}(p)$ of simple eigenvalues. Since $H_m(p)$, $p \in \mathbb{R}$, is a Kato analytic family of type (B) (see [9, Chapter VII, Section 4]), all the eigenvalues $\lambda_{n,m}(p)$ are real analytic functions of $p \in \mathbb{R}$. Moreover, $\lambda_{n,m}(p) > 0$ because form (2.2) is strictly positive.

In view of formula (2.7) spectral analysis of the operators $\mathbf{H}_m$ reduces to a study of a family of functions $\lambda_{n,m}(p)$, $n \in \mathbb{N}$. Indeed, let $\Lambda_{n,m}$ be the operator of multiplication by the function $\lambda_{n,m}(p)$ in the space $L^2(\mathbb{R})$. We denote by $\psi_{n,m}(r;p)$ real normalized eigenfunctions (defined up to signs) of the operators $H_m(p)$ and introduce an isometric mapping

$$\Psi_{n,m} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+ \times \mathbb{R}; r dr dp)$$

by the formula

$$(\Psi_{n,m} w)(p) = \psi_{n,m}(r,p) w(p). \tag{2.11}$$

Then

$$L^2(\mathbb{R}_+ \times \mathbb{R}; r dr dp) = \bigoplus_{n \in \mathbb{N}} \operatorname{Ran} \Psi_{n,m}$$

and

$$\mathbf{H}_m = \bigoplus_{n \in \mathbb{N}} \mathcal{F}_m^* \Psi_{n,m} \Lambda_{n,m} \Psi_{n,m}^* \mathcal{F}_m. \tag{2.12}$$

Together with (2.5), formulas (2.11) and (2.12) justify equations (1.8) for functions (1.7).

3. Dispersion curves and spectral analysis

3.1. In this subsection we consider the operators $H(p)$ acting in the space $L^2(\mathbb{R}^2)$ by formula (1.3). Under the assumption $a \in L^2_{\rm loc}(\mathbb{R}^2)$ they are correctly defined by their quadratic forms (2.2). If

$$a(x) \to \infty \quad \text{as} \quad |x| \to \infty, \quad x = (x_1, x_2), \tag{3.1}$$

then the spectrum of $H(p)$ consists of eigenvalues $\lambda_n(p)$, $n \in \mathbb{N}$. We enumerate them in the increasing order with multiplicity taken into account. Our goal is to investigate the asymptotic behaviour of the eigenvalues $\lambda_n(p)$ as $p \to \infty$. Below we denote by C and c different positive constants whose precise values are of no importance.

We use the following elementary

Lemma 3.1. *Let $v(x) \geq 0$. For an arbitrary $\varepsilon > 0$, we have the inequality*

$$\int_{\mathbb{R}^2} v(x)|u(x)|^2 dx \leq C \sup_{x\in\mathbb{R}^2} \left(\int_{|x-y|\leq\varepsilon} v^2(y)dy \right)^{1/2} \int_{\mathbb{R}^2} \left(\varepsilon|\nabla u(x)|^2 + \varepsilon^{-1}|u(x)|^2 \right) dx \quad (3.2)$$

provided the supremum in the right-hand side is finite.

Proof. Let $\Pi_\varepsilon \subset \mathbb{R}^2$ be a square of length ε. We proceed from the estimate

$$\left(\int_{\Pi_\varepsilon} |u(x)|^4 dx \right)^{1/2} \leq C \left(\varepsilon \int_{\Pi_\varepsilon} |\nabla u(x)|^2 dx + \varepsilon^{-1} \int_{\Pi_\varepsilon} |u(x)|^2 dx \right)$$

which follows from the Sobolev embedding theorem by a scaling transformation. Using the Schwarz inequality, we deduce from this estimate that

$$\int_{\Pi_\varepsilon} v(x)|u(x)|^2 dx \leq C \left(\int_{\Pi_\varepsilon} v^2(x)dx \right)^{1/2} \left(\varepsilon \int_{\Pi_\varepsilon} |\nabla u(x)|^2 dx + \varepsilon^{-1} \int_{\Pi_\varepsilon} |u(x)|^2 dx \right). \quad (3.3)$$

Let us split the space $\mathbb{R}^2$ in the lattice of squares $\Pi_\varepsilon^{(n)}$ of length ε. Applying (3.3) to every $\Pi_\varepsilon^{(n)}$ and summing over all n, we arrive at (3.2). $\square$

In the following assertion we do not assume (3.1).

Proposition 3.2. *Let $a \in L^2_{\rm loc}(\mathbb{R}^2)$. Set $a_-(x) = \max\{-a(x), 0\}$,*

$$\alpha(\varepsilon) = \sup_{x\in\mathbb{R}^2} \int_{|x-y|\leq\varepsilon} a_-^2(y)dy$$

and suppose that $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then we have

$$\liminf_{p\to\infty} p^{-2} \inf \sigma(H(p)) \geq 1. \quad (3.4)$$

Proof. Applying estimate (3.2) with $\varepsilon = p^{-1}$ to the function $v = a_-$, we find that

$$\int_{\mathbb{R}^2} \left(|\nabla u|^2 + (p+a)^2 \right) |u|^2 dx \geq \int_{\mathbb{R}^2} \left(|\nabla u|^2 + (-2pa_- + p^2)|u|^2 \right) dx$$
$$\geq \int_{\mathbb{R}^2} \left(|\nabla u|^2 + p^2 \right) |u|^2 dx - C\sqrt{\alpha(p^{-1})} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + p^2 \right) |u|^2 dx.$$

Since $\alpha(p^{-1}) \to 0$ as $p \to \infty$, this implies (3.4). $\square$

Proposition 3.3. *Let $a \in L^2_{\rm loc}(\mathbb{R}^2)$ and let condition* (3.1) *be satisfied. Then, for all $n \in \mathbb{N}$, we have*

$$\lambda_n(p) = p^2(1 + o(1)), \quad p \to \infty. \quad (3.5)$$

Proof. Under condition (3.1) the function a_- has compact support so that we can use Proposition 3.2 and estimate (3.4) implies

$$\liminf_{p\to\infty} p^{-2} \lambda_n(p) \geq 1. \quad (3.6)$$

Set $G(\varepsilon) = -\Delta + (1+\varepsilon^{-1})a^2(x)$, $\varepsilon > 0$. The spectrum of $G(\varepsilon)$ is discrete; let ν_n, $n \in \mathbb{N}$, be the increasing sequence of its eigenvalues. By the elementary inequality

$$(a+p)^2 \le (1+\varepsilon^{-1})a^2 + (1+\varepsilon)p^2, \quad \varepsilon > 0,$$

we have $H(p) \le G(\varepsilon) + (1+\varepsilon)p^2$ so that by the minimax principle

$$\lambda_n(p) \le \nu_n(\varepsilon) + (1+\varepsilon)p^2.$$

Therefore, for all $\varepsilon > 0$,

$$\limsup_{p\to\infty} p^{-2}\lambda_n(p) \le 1+\varepsilon,$$

which combined with (3.6) yields (3.5). $\square$

Corollary 3.4. *Suppose that the function a depends on r only. Let conditions* (2.4) *and* (2.10) *be satisfied. Then, for all $n \in \mathbb{N}$, $m \in \mathbb{Z}$, we have*

$$\lambda_{n,m}(p) = p^2(1+o(1)), \quad p \to \infty.$$

3.2. *From now on we always assume that the function a depends on r only and that conditions* (2.4) *and* (2.10) *are satisfied.* In this subsection we investigate the asymptotics as $p \to -\infty$ of the eigenvalues $\lambda_{n,m}(p)$ of the operators $H_m(p)$. Actually, it is more convenient to work with the operators $L_m(p)$ acting in the space $L^2(\mathbb{R}_+)$ by formula (2.8). We suppose that the function a is differentiable at least for suffiently big r and formulate the results in terms of the function $b(r) = a'(r)$ related to the magnetic field by formula (1.5).

Remark first that if $k = -p > 0$ is big enough, then the equation

$$a(r) = k \tag{3.7}$$

has at least one solution. We denote by ρ_k the greatest solution of (3.7). Clearly, $\rho_k \to \infty$ as $k \to \infty$.

Proposition 3.5. *Suppose that*

$$\lim_{r\to\infty} b(r) = 0. \tag{3.8}$$

Then for each $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ we have

$$\lim_{k\to\infty} \lambda_{n,m}(-k) = 0. \tag{3.9}$$

Proof. Set

$$\mathbf{b}(r) = \sup_{x\ge r} |b(x)| \quad \text{and} \quad \gamma_k = \mathbf{b}(\rho_k)^{-1/2}. \tag{3.10}$$

Let us fix $n \in \mathbb{N}$. We pick a function $\phi_1 \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\phi_1 = \left[0, \frac{1}{2n}\right]$ and, for $n > 1$, set

$$\phi_j(x) = \phi_1(x-(j-1)/n), \quad x \in \mathbb{R}, \quad j = 2, \dots, n.$$

For $k > 0$ large enough, we put

$$\varphi_j(r;k) = \gamma_k^{-1/2}\phi_j\left(\frac{r-\rho_k}{\gamma_k}\right), \quad r \ge 0, \quad j = 1, \dots, n. \tag{3.11}$$

We will prove now that for quadratic form (2.9)

$$\lim_{k\to\infty} l_m[\varphi_j(k); -k] = 0. \tag{3.12}$$

It follows from (3.11) that

$$\int_0^\infty |\varphi_j'(r;k)|^2 dr \le C\gamma_k^{-2} \tag{3.13}$$

with C independent of k. Further, since $\operatorname{supp}\varphi_j(k) \subset [\rho_k, \rho_k+\gamma_k]$, we have

$$\int_0^\infty r^{-2}|\varphi_j(r;k)|^2 dr \le C\rho_k^{-2}. \tag{3.14}$$

Similarly,

$$\int_0^\infty (a(r)-k)^2|\varphi_j(r;k)|^2 dr \le C \sup_{r\in(\rho_k,\rho_k+\gamma_k)} (a(r)-k)^2. \tag{3.15}$$

Using the condition $a(\rho_k) = k$, we obtain, for $r \ge \rho_k$, the bound

$$(a(r)-k)^2 = (a(r)-a(\rho_k))^2 = \left(\int_{\rho_k}^r b(s)ds\right)^2 \le (r-\rho_k)^2\mathbf{b}^2(\rho_k)$$

where $\mathbf{b}$ is function (3.10). Thus, the right-hand side in (3.15) is bounded by $C\gamma_k^2\mathbf{b}^2(\rho_k)$. Putting together this result with inequalities (3.13), (3.14) and taking into account (3.10), we get

$$l_m[\varphi_j(k); -k] \le C\left(\mathbf{b}(\rho_k) + \rho_k^{-2}\right).$$

This yields (3.12).

Let us use now that the supports of the functions $\varphi_j(k)$, $j = 1, \ldots, n$, are disjoint and set

$$\mathcal{L}_n(k) = \operatorname{span}\left\{\varphi_1(k), \ldots, \varphi_n(k)\right\}. \tag{3.16}$$

Then $\dim \mathcal{L}_n(k) = n$ and according to (3.12) $l_m[\varphi(k); -k] \to 0$ as $k \to \infty$ for all $\varphi(k) \in \mathcal{L}_n(k)$ with $\|\varphi(k)\| = 1$. By the mini-max principle this implies (3.9). ☐

The proof of Proposition 3.6 relies on a comparison of the operator $L_m(-k)$ with the "model" operator

$$T(k) = -\frac{d^2}{dx^2} + b^2(\varrho_k)(x-\varrho_k)^2, \quad x \in \mathbb{R}, \tag{3.17}$$

acting in the space $L^2(\mathbb{R})$. Let f_j be the normalized in $L^2(\mathbb{R})$ real-valued eigenfunctions (defined up to sign) of the harmonic oscillator, i.e.

$$-f_j''(x) + x^2 f_j(x) = (2j-1)f_j(x), \quad x \in \mathbb{R}, \quad j \in \mathbb{N}. \tag{3.18}$$

Then

$$\psi_j(x;k) = b(\varrho_k)^{1/4} f_j(b(\varrho_k)^{1/2}(x-\varrho_k)) \tag{3.19}$$

are normalized eigenfunctions of the operator $T(k)$, that is

$$T(k)\psi_j(k) = b(\varrho_k)(2j-1)\psi_j(k), \quad j \in \mathbb{N}. \tag{3.20}$$

The proof of the following result follows the general lines of the proof of [2, Theorem 11.1].

Proposition 3.6. *Suppose that $a(r)$ is locally semibounded from above. For $r > 0$ large enough, we assume that the function $b(r)$ is differentiable and that conditions*

$$b(r) > 0, \tag{3.21}$$

$$\lim_{r\to\infty} r^2 b(r) = \infty, \tag{3.22}$$

as well as

$$\lim_{r\to\infty} b(r)^{-3}\mathbf{b}_1^2(r) = 0, \quad \text{where} \quad \mathbf{b}_1(r) = \sup_{r/2\leq x\leq 3r/2} |b'(x)|, \tag{3.23}$$

are satisfied. Let also

$$\lim_{k\to\infty} k^{-2} b(\rho_k) = 0. \tag{3.24}$$

Then, for all $n \in \mathbb{N}$, $m \in \mathbb{Z}$, we have

$$\lambda_{n,m}(-k) = b(\varrho_k)(2n - 1 + o(1)), \quad k \to \infty. \tag{3.25}$$

Proof. Due to the minimax principle, it suffices to show that:
(i) For each $n \in \mathbb{N}$ and sufficiently large k there exists a subspace $\mathcal{L}_n(k)$ of $L^2(\mathbb{R}_+)$ such that $\dim \mathcal{L}_n(k) = n$, $\mathcal{L}_n(k) \subset D(L_m(-k))$, and for each $\varphi(k) \in \mathcal{L}_n(k)$ we have

$$\langle L_m(-k)\varphi(k), \varphi(k)\rangle \leq b(\varrho_k)(2n - 1 + o(1))\|\varphi(k)\|^2, \quad k \to \infty. \tag{3.26}$$

(ii) For each $n \in \mathbb{N}$ there exists a bounded operator $R_n(k)$ such that $\operatorname{rank} R_n(k) \leq n-1$ (hence, $R_1(k) = 0$), and

$$L_m(-k) \geq b(\varrho_k)(2n - 1 + o(1))I + R_n(k), \quad k \to \infty. \tag{3.27}$$

We pick $\gamma_k > 0$ such that

$$\gamma_k \to 0, \tag{3.28}$$

$$\gamma_k\, \varrho_k b(\varrho_k)^{1/2} \to \infty, \tag{3.29}$$

$$\gamma_k^{-3}\, b(\varrho_k)^{-3/2}\mathbf{b}_1(\varrho_k) \to 0 \tag{3.30}$$

as $k \to \infty$. Note that (3.29) is compatible with (3.28) due to (3.22), and (3.30) is compatible with (3.28) due to (3.23).

Proof of (i). Let $\zeta \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \zeta(x) \leq 1$, $\zeta(x) = 1$ for $|x| \leq 1/2$ and $\operatorname{supp}\zeta = [-1, 1]$. For k large enough, set

$$\zeta(r; k) = \zeta(\gamma_k b(\varrho_k)^{1/2}(r - \varrho_k)), \quad r \in \mathbb{R}_+, \tag{3.31}$$

and

$$\varphi_j(r; k) = \psi_j(r; k)\zeta(r; k), \quad r \in \mathbb{R}_+, \quad j \in \mathbb{N}, \tag{3.32}$$

the functions $\psi_j(r; k)$ being defined in (3.19). It follows from (3.29) that

$$\operatorname{supp}\varphi_j(k) = [\varrho_k - \gamma_k^{-1}b(\varrho_k)^{-1/2}, \varrho_k + \gamma_k^{-1}b(\varrho_k)^{-1/2}] \subset [\varrho_k/2, 3\varrho_k/2]$$

and, in particular, $\varphi_j(k) \in D(L_m(-k))$. Note that

$$\langle\varphi_j(k), \varphi_l(k)\rangle_{L^2(\mathbb{R}_+)} = \delta_{jl} - \int_{\mathbb{R}} \psi_j(x; k)\psi_l(x; k)(1 - \zeta^2(x; k))dx = \delta_{jl} + o(1) \tag{3.33}$$

as $k \to \infty$. Indeed, the integral here can be estimated by

$$\int_{\mathbb{R}} |f_j(x) f_l(x)|(1 - \zeta^2(\gamma_k x))dx \le \int_{|x| \ge (2\gamma_k)^{-1}} |f_j(x) f_l(x)| dx$$

which tends to zero according to (3.28). In particular, (3.33) implies that for all $n \in \mathbb{N}$ the functions $\varphi_1(k), \ldots, \varphi_n(k)$ are linearly independent if k is large enough. Thus, the space $\mathcal{L}_n(k)$ defined by (3.16) has dimension n.

Let us set

$$\psi(x;k) = \sum_{j=1}^{n} c_j \psi_j(x;k), \quad c_j \in \mathbb{C}, \tag{3.34}$$

$\varphi(r;k) = \psi(r;k)\zeta(r;k)$ and consider $\langle L_m(-k)\varphi(k), \varphi(k)\rangle$. Integrating by parts, we find that

$$-2\mathrm{Re}\,\langle \psi'(k)\zeta'(k), \psi(k)\zeta(k)\rangle - \langle \psi(k)\zeta''(k), \psi(k)\zeta(k)\rangle = \|\psi(k)\zeta'(k)\|^2$$

so that

$$\begin{aligned} \langle L_m(-k)\varphi(k), \varphi(k)\rangle &= \mathrm{Re}\,\langle -\psi''(k) + (a(r) - k)^2\psi(k), \psi(k)\zeta^2(k)\rangle \\ &+ \|\psi(k)\zeta'(k)\|^2 + (m^2 - 1/4)\|r^{-1}\varphi(k)\|^2. \end{aligned} \tag{3.35}$$

We assume that $\|\varphi(k)\| = 1$ and hence according to (3.33) $\|\psi(k)\| = 1 + o(1)$. The second and third terms in the right-hand side of (3.35) are negligible. Indeed, differentiating (3.31) and using condition (3.28), we find that

$$\|\psi(k)\zeta'(k)\|^2 = O(b(\varrho_k)\gamma_k^2) = o(b(\varrho_k)). \tag{3.36}$$

Since $r^{-1} \le 2\varrho_k^{-1}$ on the support of $\varphi(k)$, relation (3.22) implies

$$\|r^{-1}\varphi(k)\|^2 = O(\varrho_k^{-2}) = o(b(\varrho_k)), \quad k \to \infty. \tag{3.37}$$

Further we consider the first term in the right-hand side of (3.35). It follows from equation (3.20) that

$$-\psi_j''(k) + (a(r) - k)^2\psi_j(k) = b(\varrho_k)(2j - 1)\psi_j(k) + \alpha(k)\psi_j(k) \tag{3.38}$$

where the function

$$\alpha(r;k) = (a(r) - k)^2 - b^2(\varrho_k)(r - \varrho_k)^2. \tag{3.39}$$

Let us estimate the right-hand side. In view of the equation $a(\rho_k) = k$, a second-order Taylor expansion of a at ϱ_k yields

$$a(r) = k + b(\varrho_k)(r - \varrho_k) + \int_{\varrho_k}^{r} b'(s)(r - s)ds.$$

Therefore,

$$\alpha(r;k) = 2b(\varrho_k)(r - \varrho_k)\int_{\varrho_k}^{r} b'(s)(r - s)ds + \left(\int_{\varrho_k}^{r} b'(s)(r - s)ds\right)^2,$$

and hence

$$\begin{aligned} |\alpha(r;k)| &\le b(\varrho_k)\mathbf{b}_1(\varrho_k)|r - \varrho_k|^3 + 4^{-1}\mathbf{b}_1^2(\varrho_k)(r - \varrho_k)^4 \\ &\le \gamma_k^{-3} b(\varrho_k)^{-1/2}\mathbf{b}_1(\varrho_k) + 4^{-1}\gamma_k^{-4} b(\varrho_k)^{-2}\mathbf{b}_1^2(\varrho_k), \end{aligned}$$

provided that $|r-\varrho_k|\leq \gamma_k^{-1}b(\varrho_k)^{-1/2}$. In view of conditions (3.28) and (3.30), this gives us the estimate

$$\sup_{|r-\varrho_k|\leq \gamma_k^{-1}b(\varrho_k)^{-1/2}} |\alpha(r;k)| = o(b(\rho_k)) \tag{3.40}$$

so that

$$\big(-\psi_j''(k)+(a(r)-k)^2\psi_j(k)\big)\,\zeta(k) = b(\varrho_k)(2j-1)\varphi_j(k)+o(b(\rho_k)).$$

Thus, using also (3.33) we obtain that

$$\begin{aligned}\operatorname{Re}\langle -\psi''(k)+(a(r)-k)^2\psi(k),\psi(k)\zeta^2(k)\rangle &= b(\varrho_k)\sum_{j,l=1}^{n}(2j-1)c_j\bar{c}_l\langle\varphi_j(k),\varphi_l(k)\rangle + o(b(\rho_k))\\ &\leq b(\varrho_k)(2n-1)+o(b(\rho_k)). \end{aligned} \tag{3.41}$$

Together with (3.36) and (3.37), this implies estimate (3.26) for each $\varphi(k)\in\mathcal{L}_n(k)$.

Proof of (ii). Let functions $\zeta\in C_0^\infty(\mathbb{R})$ and $\eta\in C^\infty(\mathbb{R})$ satisfy $\zeta^2(x)+\eta^2(x)=1$, $x\in\mathbb{R}$; moreover, as before, we require that $0\leq\zeta(x)\leq 1$, $\zeta(x)=1$ for $|x|\leq 1/2$ and $\operatorname{supp}\zeta=[-1,1]$. By analogy with (3.31) set

$$\eta(r;k)=\eta(\gamma_k b(\varrho_k)^{1/2}(r-\varrho_k)),\quad r\in\mathbb{R}_+. \tag{3.42}$$

Then we have

$$\zeta^2(r;k)+\eta^2(r;k)=1,\quad r\in\mathbb{R}_+.$$

We proceed from the localization formula (known as the IMS formula – see e.g. [2, Section 3.1])

$$L_m(-k)=\zeta(k)L_m(-k)\zeta(k)+\eta(k)L_m(-k)\eta(k)-\zeta'(k)^2-\eta'(k)^2,$$

where $\zeta(k)$, $\eta(k)$, $\zeta'(k)$ and $\eta'(k)$ are understood as operators of multiplication by the functions $\zeta(r,k)$, $\eta(r,k)$, $\zeta'(r,k)$ and $\eta'(r,k)$, respectively. According to (3.28) it follows from definitions (3.31) and (3.42) that

$$\max_{r\in\mathbb{R}_+}(\zeta'(r,k)^2+\eta'(r,k)^2)=O\left(\gamma_k^2 b(\varrho_k)\right)=o\left(b(\varrho_k)\right),\quad k\to\infty. \tag{3.43}$$

Next, we check that

$$\eta(k)L_m(-k)\eta(k)\geq \nu_k b(\varrho_k)\eta^2(k) \tag{3.44}$$

with $\nu_k\to\infty$ as $k\to\infty$. By virtue of the Hardy inequality

$$\eta(k)\left(-\frac{d^2}{dr^2}+\frac{m^2-1/4}{r^2}\right)\eta(k)\geq 0,$$

it suffices to check that

$$(a(r)-k)^2\geq \nu_k b(\varrho_k) \tag{3.45}$$

for

$$r\geq \varrho_k+\left(2\gamma_k b(\varrho_k)^{1/2}\right)^{-1}=:\varrho_k^{(+)}\quad\text{and}\quad r\leq \varrho_k-\left(2\gamma_k b(\varrho_k)^{1/2}\right)^{-1}=:\varrho_k^{(-)}. \tag{3.46}$$

According to (3.21) there exists r_0 such that the function $a(r)$ is increasing for $r\geq r_0$.

Let first $r\geq r_0$. Then

$$|a(r)-k|=|a(r)-a(\varrho_k)|\geq \pm(a(\varrho_k^{(\pm)})-a(\varrho_k)) \tag{3.47}$$

if $\pm(r-\varrho_k^{(\pm)})\geq 0$ and $r\geq r_0$. It follows from definition (3.46) of the numbers $\varrho_k^{(\pm)}$ that

$$a(\varrho_k^{(\pm)})-a(\varrho_k)=\int_{\varrho_k}^{\varrho_k^{(\pm)}} b(s)ds=\pm(2\gamma_k)^{-1}b(\varrho_k)^{1/2}+\int_{\varrho_k}^{\varrho_k^{(\pm)}}(b(s)-b(\varrho_k))ds. \quad (3.48)$$

The absolute value of the integral in the right-hand side can be estimated by

$$\left|\int_{\varrho_k}^{\varrho_k^{(\pm)}} ds\int_{\varrho_k}^{s}|b'(\sigma)|d\sigma\right|\leq 2^{-1}\mathbf{b}_1(\rho_k)(\varrho_k^{(\pm)}-\varrho_k)^2=8^{-1}\mathbf{b}_1(\rho_k)\gamma_k^{-2}b(\varrho_k)^{-1}$$

where the function $\mathbf{b}_1$ is defined in (3.23). By virtue of conditions (3.28) and (3.30) this expression is $o(\gamma_k^{-1}b(\varrho_k)^{1/2})$ as $k\to\infty$. Therefore the absolute value of expression (3.48) is bounded from below by $(3\gamma_k)^{-1}b(\varrho_k)^{1/2}$. Thus, for $r\geq r_0$, estimate (3.45) with $\nu_k=(3\gamma_k)^{-2}\to\infty$ is a consequence of (3.47).

If $r\leq r_0$, we take into account that $a(r)$ is semibounded from above so that $(a(r)-k)^2\geq 2^{-1}k^2$. Hence estimate (3.47) with $\nu_k=2^{-1}k^2b(\varrho_k)^{-1}\to\infty$ is satisfied according to condition (3.24).

Putting together definitions (2.8) and (3.17) of the operators $L_m(-k)$ and $T(k)$, we see that

$$\zeta(k)L_m(-k)\zeta(k)=\zeta(k)T(k)\zeta(k)+\alpha(k)\zeta^2(k), \quad (3.49)$$

where $\alpha(k)$ is the operator of multiplication by function (3.39). The first term in the right-hand side is bounded from below by $b(\varrho_k)\zeta^2(k)$ because $b(\varrho_k)$ is the first eigenvalue of the operator $T(k)$. By virtue of (3.40) the second term satisfies the estimate

$$\|\alpha(k)\zeta^2(k)\|=o(b(\varrho_k)). \quad (3.50)$$

It follows that operator (3.49) is bounded from below by $b(\varrho_k)\zeta^2(k)-o(b(\varrho_k))I$. Combining this result with (3.43) and (3.44), we get estimate (3.27) in the case $n=1$.

If $n\geq 2$, we denote by $P_n(k)$ the orthogonal projection onto the span of the first $n-1$ eigenfunctions of the operator $T(k)$. Then $T(k)(I-P_n(k))\geq(2n-1)(I-P_n(k))$ and hence

$$\begin{gathered}\zeta(k)T(k)\zeta(k)=\zeta(k)T(k)(I-P_n(k))\zeta(k)+\zeta(k)T(k)P_n(k)\zeta(k)\\ \geq b(\varrho_k)(2n-1)\zeta(k)(I-P_n(k))\zeta(k)+\zeta(k)T(k)P_n(k)\zeta(k)=b(\varrho_k)(2n-1)\zeta^2(k)+R_n(k)\end{gathered} \quad (3.51)$$

where

$$R_n(k)=\zeta(k)(T(k)-b(\varrho_k)(2n-1)I)P_n(k)\zeta(k).$$

Clearly, $\operatorname{rank}R_n(k)\leq n-1$. Putting together (3.43), (3.44) and (3.49) – (3.51), we obtain (3.27) in the case $n\geq 2$. □

Example 3.7. Let $b(r)=b_0r^{-\delta}$, $b_0>0$, $\delta\leq 1$, for sufficiently large r. Then $\mathbf{b}_1(r)=b_0\delta r^{-\delta-1}$ and conditions (3.21) – (3.23) are satisfied. Moreover, $\rho_k=c_1k^\nu$ and $k^2b(\rho_k)=c_2k^{-1-\nu}$ where $\nu=(1-\delta)^{-1}$ and $c_1,c_2>0$ if $\delta<1$. If $\delta=1$, then $\rho_k=\exp(b_0^{-1}k)$ and $k^2b(\rho_k)=k^2b_0\exp(-b_0^{-1}k)$. In both cases condition (3.24) is also satisfied. Thus, Proposition 3.6 implies the following results. If $\delta>0$, then $\lambda_{n,m}(p)\to 0$ as $p\to-\infty$ (this result follows also from Proposition 3.5). If $\delta=0$, then the functions $\lambda_{n,m}(p)$ have finite limits $b_0(2n-1)$ as $p \to -\infty$. If $\delta < 0$, then these functions tend to $+\infty$ as $p \to -\infty$.

3.3. Let us return to the Hamiltonians $\mathbf{H}_m$ and $\mathbf{H}$ defined in Section 2.

Theorem 3.8. *Assume* (2.4) *and* (2.10).

(*i*) *Then all operators* $\mathbf{H}_m$, $m \in \mathbb{Z}$, *and hence* $\mathbf{H}$ *are absolutely continuous and their spectra coincide with the half-axes defined by equations* (1.9) *and* (1.10).

(*ii*) *If the hypotheses of Proposition* 3.5 *hold true, then* $\mathcal{E}_m = 0$ *for all* $m \in \mathbb{Z}$. *Moreover, the multiplicities of all spectra* $\sigma(\mathbf{H}_m)$ *and hence of* $\sigma(\mathbf{H})$ *are infinite.*

(*iii*) *Let the hypotheses of Proposition* 3.6 *hold true. If* $b(r) \to \infty$, *then the infimum in* (1.10) *is attained* (*at a finite point*) *so that for all* $m \in \mathbb{Z}$

$$\mathcal{E}_m = \min_{p \in \mathbb{R}} \lambda_{1,m}(p) > 0.$$

(*iv*) *Let the hypotheses of Proposition* 3.6 *hold true. If* $b(r)$ *admits a finite positive limit* b_0 *as* $r \to \infty$, *then* $\mathcal{E}_m \in (0, b_0]$ *for all* $m \in \mathbb{Z}$.

Proof. It suffices to prove only the assertions concerning the operators $\mathbf{H}_m$. In view of decomposition (2.12) they reduce to corresponding statements about the operators $\Lambda_{n,m}$. These operators are absolutely continuous because the eigenvalues $\lambda_{n,m}(p)$ are real analytic functions of $p \in \mathbb{R}$ which are non constants since according to Corollary 3.4 $\lambda_{n,m}(p) \to \infty$ as $p \to \infty$. Moreover, we have that

$$\sigma(\Lambda_{n,m}) = [\mathcal{E}_{n,m}, \infty) \quad \text{where} \quad \mathcal{E}_{n,m} = \inf_{p \in \mathbb{R}} \lambda_{n,m}(p) \geq 0 \tag{3.52}$$

because $\lambda_{n,m}(p) > 0$ for all $p \in \mathbb{R}$. This implies relations (1.9) with $\mathcal{E}_m$ defined by (1.10).

In case (ii) it suffices to use that according to (3.9) $\mathcal{E}_{n,m} = 0$ and hence $\sigma(\Lambda_{n,m}) = [0, \infty)$ for all m and n.

In case (iii) Proposition 3.6 implies that $\lambda_{n,m}(p) \to \infty$ as $p \to -\infty$ for all n and m so that

$$\mathcal{E}_{n,m} = \min_{p \in \mathbb{R}} \lambda_{n,m}(p) > 0 \tag{3.53}$$

and hence infimum in (1.10) can be replaced by minimum.

In case (iv) we use that according to (3.25) $\mathcal{E}_{n,m} \leq (2n-1)b_0$. Moreover, $\mathcal{E}_{n,m} > 0$ because $\lambda_{n,m}(p) > 0$ for all $p \in \mathbb{R}$. For $n = 1$, this gives the desired result. $\square$

Remark 3.9. According to (2.12) and (3.52) the spectrum of the operator $\mathbf{H}_m$ consists of the "branches" $[\mathcal{E}_{n,m}, \infty)$ where the points $\mathcal{E}_{n,m}$ are called thresholds. In cases (iii) and (iv)

$$\mathcal{E}_{n,m} < \mathcal{E}_{n+1,m} \tag{3.54}$$

for all $n \in \mathbb{N}$. Indeed, in case iii) (3.54) is a consequence of the estimate $\lambda_{n,m}(p) < \lambda_{n+1,m}(p)$ valid for all $p \in \mathbb{R}$ and of formula (3.53). In case (iv) one has to take additionally into account that the limit of $\lambda_{n,m}(p)$ as $p \to -\infty$ is strictly smaller than that of $\lambda_{n+1,m}(p)$. Inequality (3.54) means that there are infinitely many distinct thresholds in each of the spectra $\sigma(\mathbf{H}_m)$, $m \in \mathbb{Z}$, and hence in $\sigma(\mathbf{H})$.

Remark 3.10. In case(iii) the multiplicity of the spectrum of all operators $\Lambda_{n,m}$ equals at least to 2 whereas in cases (ii) and (iv) it might be equal to 1.

4. Group velocities

4.1. In this subsection we obtain a formula for the derivative $\lambda'_{n,m}(p)$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$, which yields sufficient conditions for the monotonicity of $\lambda_{n,m}(p)$ as a function of p. Recall that the operators $H_m(p)$, $m \in \mathbb{N}$, $p \in \mathbb{R}$, were defined in the space $\mathcal{H}$ by formula (1.4).

The proof of Theorem 4.3 relies on integration by parts. To prove that non-integral terms disappear at $r = 0$, we use standard bounds on $\psi_{n,m}(r;p)$. Unfortunately, we were unable to find necessary results in the literature and therefore give their brief proofs.

Let us consider the differential equation of Bessel type

$$-r^{-1}(ry')' + m^2 r^{-2} y + q(r) y = 0, \quad m = 0, 1, 2, \ldots, \tag{4.1}$$

in a neighborhood $(0, r_0)$ of the point $r = 0$. If $q(r) = 0$, then it has the regular $y_0^{(reg)}(r) = r^m$ and singular $y_0^{(sing)}(r) = r^{-m}$ solutions for $m \neq 0$ and $y_0^{(reg)}(r) = 1$ and $y_0^{(sing)}(r) = \ln r$ for $m = 0$.

Lemma 4.1. *Let $m \neq 0$, and let the function $rq(r)$ belong to the class $L^1(0, r_0)$. Then equation (4.1) has a solution $y^{(reg)}(r)$ satisfying the relation*

$$y^{(reg)}(r) = r^m + o(r^m), \quad r \to 0. \tag{4.2}$$

For its derivative, we have the bound

$$dy^{(reg)}(r)/dr = O(r^{m-1}). \tag{4.3}$$

Let $m = 0$. Suppose that the function $r \ln r q(r)$ belongs to the class $L^1(0, r_0)$. Then equation (4.1) has a solution $y^{(reg)}(r)$ satisfying relation (4.2) where $m = 0$. For its derivative, we have the bound

$$dy^{(reg)}(r)/dr = O\left(\int_0^r |q(s)| ds\right). \tag{4.4}$$

Moreover, if the function $r \ln^2 r q(r)$ belongs to the class $L^1(0, r_0)$, then equation (4.1) has a solution $y^{(sing)}(r)$ satisfying the relation

$$y^{(sing)}(r) = \ln r + o(1), \quad r \to 0. \tag{4.5}$$

In this case any bounded solution of equation (4.1) coincides (up to a constant factor) with the regular solution $y^{(reg)}(r)$.

Proof. We construct the function $y^{(reg)}(r)$ as the solution of the Volterra integral equation

$$y^{(reg)}(r) = y_0^{(reg)}(r) + \varkappa_m \int_0^r s\big(y_0^{(reg)}(r) y_0^{(sing)}(s) - y_0^{(reg)}(s) y_0^{(sing)}(r)\big) q(s) y^{(reg)}(s) ds \tag{4.6}$$

where $\varkappa_m = (2m)^{-1}$ for $m \neq 0$ and $\varkappa_0 = -1$. Differentiating it twicely, we see that $y^{(reg)}(r)$ satisfies equation (4.1). Equation (4.6) can be solved by iterations, that is

$$y^{(reg)}(r) = \sum_{n=0}^{\infty} y_n^{(reg)}(r). \tag{4.7}$$

Hereby the n^{th}-iteration obeys the bound

$$|y_n^{(reg)}(r)| \leq \frac{C^n}{n!} r^m \Big(\int_0^r s|q(s)|ds \Big)^n$$

if $m \neq 0$; if $m = 0$, then $s|q(s)|$ should be replaced by $s|\ln s||q(s)|$. This ensures the convergence of series (4.7) as well as relation (4.2). Differentiating equation (4.6) and using (4.2), we get bounds (4.3) and (4.4) on the derivative of $y^{(reg)}(r)$.

If $m = 0$, we can construct the function $y^{(sing)}(r)$ as the solution of equation (4.6) where the first term, $y_0^{(reg)}(r)$, in the right-hand side is replaced by $y_0^{(sing)}(r)$, that is

$$y^{(sing)}(r) = \ln r + \int_0^r s \ln(r/s) q(s) y^{(sing)}(s) ds.$$

This equation can again be solved by iterations which, in particular, implies estimate (4.5). $\square$

This result can be supplemented by the following

Lemma 4.2. *Let $m \neq 0$, and let the function $rq^2(r)$ belong to the class $L^1(0, r_0)$. Assume additionally that $q = \bar{q}$. If ψ is a solution of equation* (4.1) *from the class $L^2((0, r_0); rdr)$, then it coincides* (*up to a constant factor*) *with the regular solution $y^{(reg)}(r)$ and hence satisfies estimates* (4.2) *and* (4.3).

Proof. Let us extend the function $q(r)$ to (r_0, ∞) by zero, and let us consider the differential operator

$$hy = -r^{-1}(ry')' + m^2 r^{-2} y + q(r) y$$

in the space $L^2(\mathbb{R}_+; rdr)$ on domain $C_0^\infty(\mathbb{R}_+)$. If $q = 0$, we denote this operator by h_0. The operator h_0 is essentially self-adjoint. To prove the same for h, it suffices to check that

$$\int_{\mathbb{R}_+} q^2(r)|f(r)|^2 rdr \leq \varepsilon \|h_0 f\|^2 + C\|f\|^2, \quad f \in C_0^\infty(\mathbb{R}_+), \quad \varepsilon < 1. \tag{4.8}$$

Let us use the estimate

$$\begin{aligned} \int_{|x|\leq r_0} q^2(|x|)|u(x)|^2 dx &\leq \int_{|x|\leq r_0} q^2(|x|) dx \max_{x\in\mathbb{R}^2} |u(x)|^2 \\ &\leq \varepsilon \int_{\mathbb{R}^2} |(\Delta u)(x)|^2 dx + C\varepsilon^{-1} \int_{\mathbb{R}^2} |u(x)|^2 dx, \quad \forall \varepsilon > 0. \end{aligned}$$

Restricting it on the subspace of functions $u(x) = f(r)e^{im\theta}$, we obtain estimate (4.8) which implies that h is essentially self-adjoint as well as h_0. Thus, equation (4.1) has at most one solution from $L^2((0, r_0); rdr)$ which is necessarily proportional to $y^{(reg)}(r)$. $\square$

Now we are in a position to obtain a formula for the derivative $\lambda'_{n,m}(p)$. In addition to our usual assumptions that $b(r)$ is not too singular at $r = 0$, an integration-by-parts marchinery requires that $b(r)$ does not vanish too rapidly as $r \to 0$. The precise conditions are formulated rather differently in the cases $m \neq 0$ and $m = 0$. We start with the first case.

Theorem 4.3. *Let $m \neq 0$. Suppose that $b \in C^3(\mathbb{R}_+)$ and $b(r) > 0$, $r \in \mathbb{R}_+$. Assume (2.10) and that $b(r) = O(e^{cr})$ for some $c > 0$ as $r \to \infty$. At $r = 0$ we suppose that $b(r) = O(r^{-\gamma})$ where $\gamma < 3/2$. Moreover, we assume that for some $\beta < 2|m| - 1$*

$$|(b(r)^{-1})^{(k)}| \leq Cr^{-\beta-k}, \quad k = 0, 1, 2, 3, \quad r \to 0. \tag{4.9}$$

Put

$$v(r) = r(r^{-1}(rb(r)^{-1})')'.$$

Then

$$\lambda'_{n,m}(p) = -2\int_0^\infty rb^{-2}(r)b'(r)\psi'_{n,m}(r;p)^2 dr$$

$$-2^{-1}\int_0^\infty v'(r)\psi^2_{n,m}(r;p)dr + 2m^2\int_0^\infty r^{-2}b^{-1}(r)\psi^2_{n,m}(r;p)dr, \tag{4.10}$$

where the eigenfunctions $\psi_{n,m}(r;p)$ of the operator $H_m(p)$ are real and normalized, that is $\|\psi_{n,m}\| = 1$.

Proof. In view of the equation

$$(a(r) + p)^2\psi_{n,m} = r^{-1}(r\psi'_{n,m})' - m^2r^{-2}\psi_{n,m} + \lambda_{n,m}\psi_{n,m} \tag{4.11}$$

we can apply to the function $\psi_{n,m}$ the results of Lemmas 4.1 and 4.2 where $q(r) = (a(r) + p)^2 - \lambda_{n,m}$. Thus, Lemma 4.2 implies that $\psi_{n,m}(r;p) = O(r^{|m|})$ and $\psi'_{n,m}(r;p) = O(r^{|m|-1})$ as $r \to 0$ which ensures that non-integral terms disappear at $r = 0$.

To prove the same for non-integral terms corresponding to $r \to \infty$, we use super-exponential decay of eigenfunctions $\psi_{n,m}(r;p)$ of the operators $H_m(p)$. This result is valid [14] (see also [6]) for all one-dimensional Schrödinger operators with discrete spectra. In view of the condition $a(r) = O(e^{cr})$, it follows from equation (4.11) that the derivatives $\psi'_{n,m}(r;p)$ also decay super-exponentially.

Let us proceed from the formula of the first order perturbation theory (known as the Feynman-Hellman formula)

$$\lambda'_{n,m}(p) = \int_0^\infty \frac{\partial(a(r) + p)^2}{\partial p}\psi^2_{n,m}(r;p)rdr = \int_0^\infty \frac{\partial(a(r) + p)^2}{\partial r}\psi^2_{n,m}(r;p)\tau(r)dr \tag{4.12}$$

where $\tau(r) = rb(r)^{-1}$. Using that $a(r) = O(r^{1-\gamma})$ and $\tau(r) = O(r^{1-\beta})$, we integrate by parts and get

$$\lambda'_{n,m}(p) = -\int_0^\infty (a(r) + p)^2\psi_{n,m}(r;p)(\tau'(r)\psi_{n,m}(r;p) + 2\tau(r)\psi'_{n,m}(r;p))dr.$$

Now it follows from equation (4.11) that

$$\lambda_{n,m}'(p) = -\lambda_{n,m}(p)\int_0^\infty (\tau(r)\psi_{n,m}^2(r;p))'dr + m^2\int_0^\infty r^{-2}(\tau(r)\psi_{n,m}^2(r;p))'dr$$
$$-\int_0^\infty r^{-1}(r\psi_{n,m}'(r;p))'(\tau'(r)\psi_{n,m}(r;p) + 2\tau(r)\psi_{n,m}'(r;p))dr. \quad (4.13)$$

By the condition $\tau(r)\psi_{n,m}^2(r;p) \to 0$ as $r \to 0$, the first term in the right-hand side equals zero. In the second term we integrate by parts which yields

$$\int_0^\infty r^{-2}(\tau(r)\psi_{n,m}^2(r;p))'dr = 2\int_0^\infty r^{-3}\tau(r)\psi_{n,m}^2(r;p)dr$$

because $r^{-2}\tau(r)\psi_{n,m}^2(r;p) \to 0$.

In the last integral in the right-hand side of (4.13), we also integrate by parts using that $\psi_{n,m}'(r;p)\psi_{n,m}(r;p)\tau'(r) \to 0$ as $r \to 0$. Thus, we have that

$$\begin{aligned} -\int_0^\infty r^{-1}(r\psi_{n,m}'(r;p))'\tau'(r)\psi_{n,m}(r;p)dr &= \int_0^\infty \tau'(r)\psi_{n,m}'(r;p)^2dr \\ &+ \int_0^\infty v(r)\psi_{n,m}'(r;p)\psi_{n,m}(r;p)dr. \quad (4.14) \end{aligned}$$

The last integral in the right-hand side equals

$$-2^{-1}\int_0^\infty v'(r)\psi_{n,m}^2(r;p)dr$$

because $v(r)\psi_{n,m}^2(r;p) \to 0$ as $r \to 0$. Similarly, we get that

$$\begin{aligned} -2\int_0^\infty r^{-1}(r\psi_{n,m}'(r;p))'\tau(r)\psi_{n,m}'(r;p)dr &= -\int_0^\infty r^{-2}\tau(r)d(r\psi_{n,m}'(r;p)^2) \\ &= \int_0^\infty r^2(r^{-2}\tau(r))'\psi_{n,m}'(r;p)^2dr \end{aligned}$$

since $\tau(r)\psi_{n,m}'(r;p)^2 \to 0$ as $r \to 0$. Putting the results obtained together, we arrive at representation (4.10). □

Corollary 4.4. *If $b'(r) \le 0$ and $r^2b(r)v'(r) \le 4m^2$ for all $r \ge 0$, then $\lambda_{n,m}'(p) \ge 0$ for all $p \in \mathbb{R}$ and n. If, moreover, one of these inequalities is strict on some interval, then $\lambda_{n,m}'(p) > 0$.*

Corollary 4.5. *If $b(r) = b_0r^{-\delta}$, $\delta \in [0,1]$, then $\tau(r) = b_0^{-1}r^{1+\delta}$, $v(r) = b_0^{-1}(\delta^2-1)r^{\delta-1}$ and*

$$\lambda_{n,m}'(p) = 2b_0^{-1}\delta\int_0^\infty r^\delta\psi_{n,m}'(r;p)^2dr$$
$$+b_0^{-1}(2m^2 - 2^{-1}(1-\delta)^2(1+\delta))\int_0^\infty r^{-2+\delta}\psi_{n,m}^2(r;p)dr.$$

For $b_0 > 0$, this expression is strictly positive (so that the functions $\lambda_{n,m}(p)$ are strictly increasing for all $p \in \mathbb{R}$) for $m \neq 0$ since $(1-\delta)^2(1+\delta) \leq 1$. Moreover, for $\delta = 1$ this result is true for all $m \in \mathbb{Z}$.

In the case $m = 0$ we consider for simplicity only fields (1.11).

Proposition 4.6. *If $b(r) = b_0 r^{-\delta}$, $\delta \in [0,1]$, then*

$$\lambda'_{n,0}(p) = 2b_0^{-1}\delta \int_0^\infty r^\delta \psi'_{n,0}(r;p)^2 dr$$
$$-b_0^{-1}2^{-1}(1-\delta)^2(1+\delta)\int_0^\infty r^{-2+\delta}(\psi^2_{n,0}(r;p) - \psi^2_{n,0}(0;p))dr.$$

If $b_0 > 0$ and $\delta = 1$, then $\lambda'_{n,0}(p) > 0$ for all $p \in \mathbb{R}$.

Proof. Let us proceed again from formula (4.12). We use now that the function $\psi_{n,0}(|x|;p)$ of $x \in \mathbb{R}^2$ belongs to the Sobolev class $\mathsf{H}^2_{loc}(\mathbb{R}^2)$, and therefore $\psi_{n,0}(r;p)$ has a finite limit as $r \to 0$. Thus, by Lemma 4.1 $\psi'_{n,0}(r;p) = O(r^{1-\varepsilon})$ for any $\varepsilon > 0$ as $r \to 0$. These results allow us to intergrate by parts as in the case $m \neq 0$. The only difference is with the second integral in the right-hand side of (4.14). Now $v(r) = b_0^{-1}(\delta^2 - 1)r^{\delta - 1}$ and this integral equals

$$\begin{aligned}\int_0^\infty v(r)\psi'_{n,0}(r;p)\psi_{n,0}(r;p)dr &= 2^{-1}\int_0^\infty v(r)d(\psi^2_{n,0}(r;p) - \psi^2_{n,0}(0;p)) \\ &= -2^{-1}\int_0^\infty v'(r)(\psi^2_{n,0}(r;p) - \psi^2_{n,0}(0;p))dr\end{aligned}$$

because $v(r)(\psi^2_{n,0}(r;p) - \psi^2_{n,0}(0;p))$ as $r \to 0$. □

4.2. In this subsection we show that for linear potentials, that is for magnetic fields not depending on r, all eigenvalues $\lambda_{n,0}(p)$, $n \in \mathbb{N}$, of the operator $H_0(p)$ are not monotonous functions of $p \in \mathbb{R}$. We follow closely the proof of the first part of Proposition 3.6. However we now use that eigenfunctions of the harmonic oscillator decay faster than any power of r^{-1} at infinity (actually, they decay super-exponentially).

Proposition 4.7. *Assume that for sufficiently large r*

$$b(r) = b_0 > 0. \tag{4.15}$$

Then, for all $n \in \mathbb{N}$, some $\gamma_n > 0$ and sufficiently large $k > 0$, we have

$$\lambda_{n,0}(-k) \leq (2n-1)b_0 - \gamma_n k^{-2}. \tag{4.16}$$

Proof. Let ζ be the same function as in the proof of the first part of Proposition 3.6. We set $\rho_k = b_0^{-1}k$, $\gamma_k = 2b_0^{1/2}k^{-1}$ and define the functions $\zeta(r;k)$ and $\varphi_j(r;k)$ by formulas (3.31) and (3.32), respectively. It suffices to check that

$$\langle L_0(-k)\varphi(k), \varphi(k)\rangle \leq 2n - 1 - \gamma_n k^{-2}. \tag{4.17}$$

for sufficiently large k and all normalized functions from subspace (3.16). Let us proceed from formula (3.35). Since the functions $\psi_j(x;k)$ decay faster than any power of $|x|^{-1}$

as $|x| \to \infty$, the term $o(1)$ in (3.33) is actually $O(k^{-\infty})$. Similarly, estimate (3.36) can be formulated in a more precise form as

$$\|\psi(k)\zeta'(k)\|^2 = O(k^{-\infty}). \tag{4.18}$$

Since $r \leq 2^{-1}3k$ on the support of $\varphi(k)$, we have that

$$\|r^{-1}\psi(k)\|^2 \geq (2/3)^2 k^{-2}. \tag{4.19}$$

Now function (3.34) is zero if r and k are large enough. Therefore equation (3.38) yields the exact equality

$$\operatorname{Re}\langle -\psi''(k) + (b_0 r - k)^2\psi(k), \psi(k)\zeta^2(k)\rangle = b_0 \sum_{j,l=1}^{n} (2j-1)c_j\bar{c}_l \langle \varphi_j(k), \varphi_l(k)\rangle$$

(cf. (3.41)). Up to terms $O(k^{-\infty})$, the right-hand side here is estimated by $b_0(2n-1)$. Together with (4.18) and (4.19), this implies estimate (4.17). $\square$

Combining relations (3.25) and (4.16), we see that the eigenvalues $\lambda_{n,0}(p)$ tend as $p \to -\infty$ to their limits $(2n-1)b_0$ from below. On the other hand, according to (3.5) $\lambda_{n,0}(p) \to \infty$ as $p \to \infty$. Thus, all functions $\lambda_{n,0}(p)$ have necessarily local minima. We can obtain an additional information using the following elementary

Lemma 4.8. *Suppose that* (4.15) *is satisfied for all* $r > 0$ *and that* $a(r) = b_0 r$. *Then*

$$\lambda_{n,m}(0) = 2b_0(2n - 1 + |m|) \tag{4.20}$$

for all $n \in \mathbb{N}$ *and* $m \in \mathbb{Z}$.

Proof. Let us consider the two-dimensional harmonic oscillator $\mathbf{T} = -\Delta + b_0^2(x_1^2 + x_2^2)$. Separating the variables x_1, x_2, we see that its spectrum consists of the eigenvalues $2b_0(l_1 + l_2 - 1)$ where $l_1, l_2 \in \mathbb{N}$. It follows that the operator $\mathbf{T}$ has the eigenvalues $2b_0 j$, $j \in \mathbb{N}$, of multiplicity j. On the other hand, separating the variables in the polar coordinates, we see that the spectrum of $\mathbf{T}$ consists of the eigenvalues $\lambda_{n,m}(0)$ of the operators $H_m(0)$. For the proof of (4.20) we take into account that all eigenvalues $\lambda_{n,m}(0)$ are simple and that $\lambda_{n,m+1}(0) > \lambda_{n,m}(0)$ for all n and $m \geq 0$. Clearly, the operator $H_0(0)$ has an eigenvalue $2b_0 j$ if and only if its multiplicity j is odd. This gives formula (4.20) for $m = 0$. We shall show that for every $j \in \mathbb{N}$

$$\lambda_{1,j-1}(0) = \lambda_{2,j-3}(0) = \ldots = \lambda_{2,-j+3}(0) = \lambda_{1,-j+1}(0) = 2b_0 j \tag{4.21}$$

which is equivalent to formula (4.20) for all m. Let us choose some j_0 and suppose that (4.21) holds for all $j \leq j_0$. Then we check it for $j = j_0 + 1$. First we remark that if an operator $H_m(0)$ for some $m > 0$ has n eigenvalues in the interval $[2b_0, 2b_0(j_0+1)]$, then the operator $H_{m-1}(0)$ has at least n eigenvalues in the interval $[2b_0, 2b_0 j_0]$. Then using (4.21) for $j \leq j_0$, we see that if an operator $H_m(0)$ has the eigenvalue $2b_0(j_0+1)$, then necessarily the operator $H_{m-1}(0)$ has the eigenvalue $2b_0 j_0$. Therefore according to (4.21) for $j = j_0$, only the operators $H_m(0)$ with $m = j_0, j_0 - 2, \ldots, -j_0 + 2, -j_0$ might have the eigenvalue $2b_0(j_0+1)$. There are j_0+1 of such operators and the multiplicity of this eigenvalue equals j_0+1. Thus, all the operators $H_m(0)$ for $m = j_0, j_0-2, \ldots, -j_0+2, -j_0$ and only for such m have the eigenvalue $2b_0(j_0+1)$. This proves (4.21) for $j_0 + 1$. $\square$

Comparing this result with (3.27), we see that, for potentials $a(r) = b_0 r$,

$$\lim_{p\to-\infty} \lambda_{n,m}(p) = b_0(2n-1) < 2b_0(2n-1+|m|) = \lambda_{n,m}(0).$$

Together with (4.16), this implies that the functions $\lambda_{n,0}(p)$ have negative local minima.

Thus, we get the following

Theorem 4.9. *Under the hypotheses of Proposition* 4.7 *the eigenvalues* $\lambda_{n,0}(p)$, $n \in \mathbb{N}$, *of the operator* $H_0(p)$ *are not monotonous functions of* $p \in \mathbb{R}$. *Moreover, if* (4.15) *is satisfied for all* $r > 0$ *and* $a(r) = b_0 r$, *then the functions* $\lambda_{n,0}(p)$ *lose their monotonicity for* $p < 0$.

We do not know how many minima have the functions $\lambda_{n,0}(p)$.

The problem of monotonicity of the eigenvalues $\lambda_{n,0}(p)$ for fields $b(r) = b_0 r^{-\delta}$ where $\delta \in (0,1)$ remains also open.

4.3. In a somewhat similar situation the break down of monotonicity of group velocities was exhibited in [7]. In this paper one considers the Schrödinger operator $\mathbf{H}^{(N)} = -\frac{\partial^2}{\partial x^2} + \left(i\frac{\partial}{\partial y} - bx\right)^2$ with constant magnetic field $b > 0$, defined on the semi-plane $\{(x,y) \in \mathbb{R}^2 : x > 0\}$ with the Neumann boundary condition at $x = 0$. Let $H^{(N)}(p) = -d^2/dx^2 + (bx+p)^2$, $p \in \mathbb{R}$, be the self-adjoint operator in the space $L^2(\mathbb{R}_+)$ corresponding to the boundary condition $u'(0) = 0$. Then the operator $\mathbf{H}^{(N)}$ is unitarily equivalent under the partial Fourier transform with respect to y, to the direct integral $\int_{\mathbb{R}}^{\oplus} H^{(N)}(p)dp$. It is shown in [7, Section 4.3] that the lowest eigenvalue $\mu_1(p)$ of $H^{(N)}(p)$ is not monotonous for $p < 0$. This follows from the inequality $\mu_1'(0) > 0$ proven[1] in [3] and the relations

$$\lim_{p\to-\infty} \mu_1(p) = \mu_1(0) = b. \tag{4.22}$$

Our proof of non-monotonicity of the functions $\lambda_{n,0}(p)$ is essentially different since in contrast with (4.22) we have $\lim_{p\to-\infty} \lambda_{n,0}(p) < \lambda_{n,0}(0)$.

5. Asymptotic time evolution

5.1. Combined with the stationary phase method, the spectral analysis of the operators $\mathbf{H} = \mathbf{H}(a)$ allows us to find the asymptotics for large t of solutions $u(t) = \exp(-i\mathbf{H}t)u_0$ of the time dependent Schrödinger equation. It follows from (1.2) that

$$\overline{\exp(-i\mathbf{H}(a)t)u_0} = \exp(i\mathbf{H}(-a)t)\overline{u_0}.$$

Therefore it suffices to consider the case $a(r) \to +\infty$. Moreover, on every subspace $\mathfrak{H}_m$ with a fixed magnetic quantum number m, the problem reduces to the asymptotics of the function $u(t) = \exp(-i\mathbf{H}_m t)u_0$.

Let us proceed from decomposition (2.12). Suppose that $\mathcal{F}_m u_0 \in \operatorname{Ran} \Psi_{n,m}$. Then (see (2.11))

$$(\mathcal{F}_m u_0)(r,p) = \psi_{n,m}(r,p) f(p) \tag{5.1}$$

[1]Note that in [3] and [7] the parameter p is chosen with the opposite sign.

where $f = \Psi^*_{n,m}\mathcal{F}_m u_0$ and $u(t) = \mathcal{F}^*_m \Psi_{n,m} e^{-i\Lambda_{n,m}t} f$, that is

$$u_{n,m}(r, x_3, t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ipx_3 - i\lambda_{n,m}(p)t} \psi_{n,m}(r, p) f(p) dp. \tag{5.2}$$

The analytic function $\lambda''_{n,m}(p)$ might have only a countable set of zeros $p_{n,m,l}$ with possible accumulations at $\pm\infty$ only. The function $\lambda'_{n,m}(p)$ is monotone on every interval $(p_{n,m,l}, p_{n,m,l+1})$ and takes there all values between $\lambda'_{n,m}(p_{n,m,l}) =: \alpha_{n,m,l}$ and $\lambda'_{n,m}(p_{n,m,l+1}) =: \beta_{n,m,l}$. We consider the asymptotics of integral (5.2) on each of the subspaces $L^2(p_{n,m,l}, p_{n,m,l+1})$ separately. Let us set $\gamma = x_3 t^{-1}$. First we suppose that $f \in C_0^\infty(p_{n,m,l}, p_{n,m,l+1})$. The stationary points of integral (5.2) are determined by the equation

$$\lambda'_{n,m}(p) = \gamma. \tag{5.3}$$

If $\gamma \notin (\alpha_{n,m,l}, \beta_{n,m,l})$, it does not have solutions from the interval $(p_{n,m,l}, p_{n,m,l+1})$. Therefore integrating directly by parts, we find that function (5.2) decays in this region of x_3/t faster than any power of $(|x_3| + |t|)^{-1}$ (and r). If $\gamma \in (\alpha_{n,m,l}, \beta_{n,m,l})$, then on the interval $(p_{n,m,l}, p_{n,m,l+1})$ equation (5.3) has a unique solution which we denote by $\nu_{n,m,l}(\gamma)$. Let us set

$$\Phi_{n,m,l}(\gamma) = \nu_{n,m,l}(\gamma)\gamma - \lambda_{n,m}(\nu_{n,m,l}(\gamma))$$

and denote by $\chi_{n,m,l}$ the characteristic function of the interval $(\alpha_{n,m,l}, \beta_{n,m,l})$. For γ from this interval, we apply the stationary phase method to integral (5.2) which yields

$$\begin{aligned} u(r, x_3, t) &= \tau^{(\pm)}_{n,m,l} e^{i\Phi_{n,m,l}(\gamma)t} \psi_{n,m}(r, \nu_{n,m,l}(\gamma)) |\lambda''_{n,m}(\nu_{n,m,l}(\gamma))|^{-1/2} \\ &\times f(\nu_{n,m,l}(\gamma)) \chi_{n,m,l}(\gamma) |t|^{-1/2} + u_\infty(r, x_3, t), \quad \gamma = x_3 t^{-1}, \quad t \to \pm\infty, \end{aligned} \tag{5.4}$$

where $\tau^{(\pm)}_{n,m,l} = e^{\mp\pi i \,\mathrm{sgn}(\lambda''_{n,m}(p))/4}$ for $p \in (p_{n,m,l}, p_{n,m,l+1})$ and

$$\lim_{t \to \pm\infty} \|u_\infty(\cdot, t)\| = 0. \tag{5.5}$$

Since the norm in the space $\mathfrak{H}$ of the first term in the right-hand side of (5.4) equals the norm of f in the space $L^2(p_{n,m,l}, p_{n,m,l+1})$, asymptotics (5.4) extends to all functions (5.1) with an arbitary $f \in L^2(p_{n,m,l}, p_{n,m,l+1})$. Thus, we have proven

Theorem 5.1. *Assume* (2.4) *and* (2.10). *Let $u(t) = \exp(-i\mathbf{H}_m t)u_0$ where u_0 satisfies* (5.1) *with $f \in L^2(p_{n,m,l}, p_{n,m,l+1})$. Then the asymptotics as $t \to \pm\infty$ of this function is given by relations* (5.4), (5.5).

Of course asymptotics (5.4), (5.5) extends automatically to all $f \in L^2(\mathbb{R})$ with compact support and to linear of functions $\psi_{n,m}(r, p) f_n(p)$ over different n.

By virtue of formulas (5.4), (5.5) a quantum particle in magnetic field (1.5) remains localized in the (x_1, x_2)-plane but propagates in the x_3-direction. If $f \in L^2(p_{n,m,l}, p_{n,m,l+1})$, then a particle "lives" as $|t| \to \infty$ in the region where $x_3 \in (\alpha_{n,m,l}t, \beta_{n,m,l}t)$. In particular, if $\lambda'(p) > 0$ ($\lambda'(p) < 0$) for $p \in (p_{n,m,l}, p_{n,m,l+1})$, then a particle propagates in the positive (negative) direction as $t \to +\infty$. Thus, according to Corollary 4.5 if $b(r) = b_0 r^{-\delta}$, $\delta \in [0, 1]$, $b_0 > 0$, then a particle with the magnetic quantum number $m \neq 0$ propagates always in the positive direction of the x_3-axis. If $\delta = 1$, then this result remains true from all m. On the contrary, if $\delta = 0$ and $m = 0$, then a particle will propagate in a negative direction for some interval of momenta p.

5.2. Theorem 5.1 implies the existence of asymptotic velocity in the x_3-direction. The corresponding operator is defined by the equation (cf. (2.12))

$$\mathbf{H}'_m = \bigoplus_{n\in\mathbb{N}} \mathcal{F}^*_m \Psi_{n,m} \Lambda'_{n,m} \Psi^*_{n,m} \mathcal{F}_m,$$

where $\Lambda'_{n,m}$ are the operators of multiplication by the functions $\lambda'_{n,m}(p)$. To put it differently, the operator $\mathbf{H}'_m$ acts as multiplication by $\lambda'_{n,m}(p)$ in the spectral representation of the operator $\mathbf{H}_m$ where it acts as multiplication by the functions $\lambda_{n,m}(p)$.

Proposition 5.2. *Assume* (2.4) *and* (2.10)*. Then, for an arbitrary bounded function* $\mathcal{Q}$*,*

$$\text{s-lim}_{|t|\to\infty} \exp(i\mathbf{H}_m t)\mathcal{Q}(x_3/t)\exp(-i\mathbf{H}_m t) = \mathcal{Q}(\mathbf{H}'_m) \tag{5.6}$$

(*in particular, the strong limit in the left-hand side exists*).

Proof. We shall check that for all $u_0 \in \mathcal{H}_m$

$$\lim_{|t|\to\infty} \|\mathcal{Q}(x_3/t)\exp(-i\mathbf{H}_m t)u_0 - \exp(-i\mathbf{H}_m t)\mathcal{Q}(\mathbf{H}'_m)u_0\| = 0 \tag{5.7}$$

which is equivalent to relation (5.6). Remark that if u_0 satisfies (5.1), then

$$(\Psi_{n,m}\mathcal{Q}(\mathbf{H}'_m)u_0)(r,p) = \psi_{n,m}(r,p)\mathcal{Q}(\lambda'_{n,m}(p))f(p). \tag{5.8}$$

It suffices to prove (5.7) on a dense set of elements u_0 such that equality (5.1) is true with $f \in L^2(p_{n,m,l}, p_{n,m,l+1})$. Applying the operator $\mathcal{Q}(x_3/t)$ to asymptotic relation (5.4), we see that the asymptotics of $\mathcal{Q}(x_3/t)\exp(-i\mathbf{H}_m t)u_0$ is given again by formula (5.4) where the function $f(\nu_{n,m,l}(\gamma))$ in the right-hand side is replaced by the function $\mathcal{Q}(\gamma)f(\nu_{n,m,l}(\gamma))$. Similarly, it follows from Theorem 5.1 and relation (5.8) that the asymptotics of $\exp(-i\mathbf{H}_m t)\mathcal{Q}(\mathbf{H}'_m)u_0$ is given by formula (5.4) where the function $f(\nu_{n,m,l}(\gamma))$ in the right-hand side is replaced by the function $\mathcal{Q}(\lambda'_{n,m}(\nu_{n,m,l}(\gamma)))f(\nu_{n,m,l}(\gamma))$. So for the proof of (5.7), it remains to take equation (5.3) into account. □

Relation (5.6) shows that $\mathbf{H}'_m$ can naturally be interpreted as the operator of asymptotic velocity in the x_3-direction.

Similar results concerning the Iwatsuka model (see [8] or [2]) have been obtained in [11].

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