$\mathcal{PT}$ symmetric non-selfadjoint operators, diagonalizable and non-diagonalizable, with real discrete spectrum

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Abstract

Consider in $L^2(\mathbb{R}^d)$, $d \geq 1$, the operator family $H(g) := H_0 + igW$. $H_0 = a_1^* a_1 + \ldots + a_d^* a_d + d/2$ is the quantum harmonic oscillator with rational frequencies , W a $\mathcal{P}$ symmetric bounded potential, and g a real coupling constant. We show that if $|g| < \rho$, ρ being an explicitly determined constant, the spectrum of $H(g)$ is real and discrete. Moreover we show that the operator $H(g) = a_1^* a_1 + a_2^* a_2 + ig a_2^* a_1$ has real discrete spectrum but is not diagonalizable.

1 Introduction

A basic fact underlying $\mathcal{PT}$-symmetric quantum mechanics (see e.g. [1-10]; $\mathcal{P}$ is the parity operation, and $\mathcal{T}$ the complex conjugation) is the existence of non self-adjoint, and not even normal, but $\mathcal{PT}$-symmetric Schrödinger operators (a particular case of complex symmetric operators, as remarked in [11]) which have fully real spectrum.

Two natural mathematical questions arising in this context are (i) the determination of conditions under which $\mathcal{PT}$-symmetry actually yields real spectrum (for results in this direction see e.g. [12],[13], [14], [15], [16]) and (ii) the examination of whether or not this phenomenon can still be understood in terms of self-adjoint spectral theory; for example, it has been remarked that if a $\mathcal{PT}$-symmetric Schrödinger operator with real spectrum is diagonalizable, then it is conjugate to a self-adjoint operator through a similarity map (see e.g. [17], [18], [19]). Hence the question arises whether $\mathcal{PT}$-symmetric Schrödinger-type operators with real spectrum are always diagonalizable.

In this paper a contribution is given to both questions. First, we solve in the negative the second one. Namely, we give a very simple, explicit example of a $\mathcal{PT}$ symmetric operator, with purely real and discrete spectrum, which cannot be diagonalized because of occurrence of Jordan blocks. The example is the following Schrödinger operator, acting

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in a domain $D(P(g)) \subset L^2(\mathbb{R}^2)$ to be specified later:

$$H(g) := a_1^* a_1 + a_2^* a_2 + iga_2^* a_1 + 1, \qquad g \in \mathbb{R} \tag{1.1}$$

Here a_i, a_i^*, $i = 1, 2$ are the standard destruction and creation operators of two independent harmonic oscillators:

$$a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{d}{dx_i}\right), \quad a_i^* = \frac{1}{\sqrt{2}}\left(x_i - \frac{d}{dx_i}\right), \tag{1.2}$$

so that (1.1) can be rewritten under the form

$$H(g) = \frac{1}{2}\left[-\frac{d^2}{dx_1^2} + x_1^2\right] + \frac{1}{2}\left[-\frac{d^2}{dx_2^2} + x_2^2\right] + ig\frac{1}{2}\left(x_2 - \frac{d}{dx_2}\right)\left(x_1 + \frac{d}{dx_1}\right) \tag{1.3}$$

which is manifestly invariant under the $\mathcal{PT}$-operation $x_2 \to -x_2$, $ig \to -ig$.

Second, we identify a new class of non self-adoint, $\mathcal{PT}$-symmetric operators with purely real spectrum in $L^2(\mathbb{R}^d)$, $d > 1$. To our knowledge, this is the first example of such operators in dimension higher than one (a preliminary version of this result, without proofs, already appeared in [20]). An example of an operator belonging to this class is represented by a perturbation of the harmonic oscillators in dimension higher than one, namely by the following Schrödinger operators:

$$H(g) = \frac{1}{2}\sum_{k=1}^{d}\left[-\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2\right] + igW(x_1, \ldots, x_d) \tag{1.4}$$

Here $W \in L^\infty(\mathbb{R}^d)$, $W(-x_1, \ldots, -x_d) = -W(x_1, \ldots, x_d)$, $|g| < \rho$, where $\rho > 0$ is an explicitly estimated positive constant, and the frequencies $\omega_k > 0$ are rational multiples of a fixed frequency $\omega > 0$: $\omega_k = \dfrac{p_k}{q_k}\omega$. Here $p_k \in \mathbb{N}, q_k \in \mathbb{N} : k = 1, \ldots, d$ is a pair of relatively prime numbers, with both p_k and q_k odd, $k = 1, \ldots, n$. When $d = 2$, $\dfrac{\omega_1}{\omega_2} = \dfrac{p}{q}$ this result can be strenghtened: if $\omega_1/\omega_2 = p/q$, the spectrum is real if and only if p and q are both odd.

The paper is organized as follows: in the next section we work out the example (1.1) making use of the Bargmann representation, in Section 3 we establish the class of $\mathcal{PT}$-symmetric operators with real spectrum by exploiting the real nature of Rayleigh-Schrödinger perturbation theory (for related work on spectrum of $\mathcal{PT}$-symmetric operators through perturbation theory, see [22], [23]), and in Section 4 we work out the example represented by the perturbation of the resonant harmonic oscillators proving the above statements.

2 A non diagonalizable $\mathcal{PT}$ symmetric operator with real discrete spectrum

Consider the operator $H(g)$ whose action on its domain is specified by (1.1) or, equivalently, (1.3). Denote H_0 the operator corresponding to the two-dimensional harmonic oscillator, namely:

$$H_0 := \frac{1}{2}\left[-\frac{d^2}{dx_1^2}+x_1^2\right]+\frac{1}{2}\left[-\frac{d^2}{dx_2^2}+x_2^2\right], \; D(H_0)=D(-\Delta)\cap D(x_1^2+x_2^2) \tag{2.1}$$

It is immediately verified that $Vu := a_2^*a_1u \in L^2$ if $u \in D(H_0)$. Therefore we can give the following

Definition 2.1 *The operator family $H(g) : g \in \mathbb{R}$ in $L^2(\mathbb{R}^2)$ is the operator $H(g)$ whose action is $H_0 + igV$ on the domain $D(H_0)$.*

Then we have:

Theorem 2.2 *Consider the operator family $H(g)$ defined above. Then, $\forall\, g \in \mathbb{R}$, $|g| < 2$:*

1. *$H(g)$ has discrete spectrum.*
2. *All eigenvalues of $H(g)$ are $\lambda_m = m+1, m = 0,1,2,\dots$. Each eigenvalue λ_m has geometric multiplicity 1 but algebraic multiplicity $m+1$.*

More precisely: for each m there is an m-dimensional subspace $\mathcal{H}_m$ invariant under $H(g)$ such that we have the orthogonal decomposition $L^2 = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$; if we denote $\tilde{H}_m := H|_{\mathcal{H}_m}$ the restriction of $H(g)$ to $\mathcal{H}_m$, then $H(g) = \bigoplus_{m=0}^{\infty} \tilde{H}_m$ and $\tilde{H}_m$ is represented by the $(m+1)\times(m+1)$ matrix:

$$\tilde{H}_m = (m+1)I_{(m+1)\times(m+1)} + igD_m \tag{2.2}$$

Here D_m is a nilpotent of order $m+1$. Explicitly:

$$D_m := \begin{pmatrix} 0 & \sqrt{m} & \cdot & \cdot & 0 \\ 0 & 0 & \sqrt{2(m-1)} & \cdot & 0 \\ \cdot & \cdot & \cdot & \sqrt{3(m-2)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{m} \\ 0 & 0 & \cdot & \cdot & 0 \end{pmatrix} \Longrightarrow D_m^{m+1} = 0 \tag{2.3}$$

Remarks

1. Spec$(H(g))$ is thus real and independent of g.

2. Formula (2.2) is the Jordan canonical form of $\tilde{H}_m$. The algebraic muliplicity is $m+1$. Since $D_m \neq 0$, $\tilde{H}_m$ is not diagonalizable by definition and, a fortiori, neither is $H(g)$.

Proof of Assertion 1

The classical Hamiltonians corresponding to the operators H_0 and $H(g)$ represent their symbols, denoted $\sigma_0(x,\xi)$ and $\sigma_g(x,\xi)$, respectively:

$$\sigma_0(x,\xi) = \frac{1}{2}(\xi_1^2+\xi_2^2+x_1^2+x_2^2), \tag{2.4}$$

$$\sigma_g(x,\xi) = \sigma_0(x,\xi)+ig\widetilde{\sigma}(x,\xi), \quad \widetilde{\sigma}(x,\xi) := \frac{1}{2}(x_2-i\xi_2)(x_1+i\xi_1) \tag{2.5}$$

We have indeed (formally) $\sigma_0(x,-i\nabla_x)=H_0$, $\sigma_g(x,-i\nabla_x)=H(g)$. Since $\sigma_0 \to +\infty$ as $|\xi|+|x| \to +\infty$, by well known results (see e.g.[24], §XIII.14) it is enough to prove that $\forall |g| < g^* = 2$, and $\forall\,(x,\xi)$ outside some fixed ball centered in the origin of $\mathbb{R}^4$:

$$0 < (1-\frac{1}{2}|g|)\sigma_0(x,\xi) \leq |\sigma_g(x,\xi)| \tag{2.6}$$

To see this, we estimate:

$$|\widetilde{\sigma}| \leq \frac{1}{2}|x_2-i\xi_2||x_1+i\xi_1| \leq \frac{1}{4}(|x_2-i\xi_2|^2+|x_1+i\xi_1|^2) = \frac{1}{2}\sigma_0,$$

and hence

$$|\sigma_g| \geq |\sigma_0| - |g||\widetilde{\sigma}| \geq (1-\frac{|g|}{2})\sigma_0.$$

This proves the inequality and hence the assertion.

To prove the remaining assertions of the theorem we make use of the Bargmann representation [21]. To this end, recall the general definition of the Bargmann transform U_B (even though we shall need it only for $d=2$):

$$(U_B u)(z) := f(z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-z^2+2\sqrt{2}\langle z,q\rangle - q^2} u(q)\,dq, \quad z \in \mathbb{C}^d \tag{2.7}$$

Let us recall the relevant properties of the Bargmann transformation.

1. U_B is a unitary map between $L^2(\mathbb{R}^d)$ and $\mathcal{F}=\mathcal{F}_d$, the space of all entire holomorphic functions $f(z):\mathbb{C}^d \to \mathbb{C}$ such that (here $z=x+iy$):

   $$\|f(z)\|_F^2 := \int_{\mathbb{R}^{2d}} |f(z)|^2 e^{-|z|^2}\,dx\,dy = \langle f,f\rangle_F < +\infty \tag{2.8}$$

   where the scalar product $\langle f,g\rangle_{\mathcal{F}}$ in $\mathcal{F}_d$ is defined by

   $$\langle f,g\rangle_{\mathcal{F}} = \int_{\mathbb{R}^{2d}} f(z)\overline{g(z)}e^{-|z|^2}\,dx\,dy \tag{2.9}$$

   Namely, with $f(z):=(U_B u)(z)$: $\|f(z)\|_{\mathcal{F}} = \|u(q)\|_{L^2(\mathbb{R}^d)}$.

2. Let a_i^*, a_i be the destruction and creation operators in the variable x_i defined as in (1.2). Let $N_i := a_i^* a_i$ be the corresponding number operator, $i = 1, \ldots, d$. Denote $N^{(d)} := \sum_{i=1}^{d} N_i$ the total number operator. Then we have:

$$U_B a_i^* U_B^{-1} = z_i, \quad U_B a_i U_B^{-1} = \frac{\partial}{\partial z_i}, \quad U_B N_d U_B^{-1} = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i} \tag{2.10}$$

so that $H_0 = N^{(2)} + 1$. The above operators are defined in their maximal domain in $\mathcal{F}_d$. Moreover:

$$\begin{aligned} Q(g) &:= U_B(H(g) - 1)U_B^{-1} = U_B(N^{(2)} + iga_2^* a_1)U_B^{-1} = \\ &= z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + igz_2 \frac{\partial}{\partial z_1} := Q_0 + igW \end{aligned} \tag{2.11}$$

defined on the maximal domain. Remark that $\mathrm{Spec}\,(Q_0) = \{0, 1, \ldots, m, \ldots\}$. The eigenvalue $\lambda_m = m$ has multiplicity $m + 1$.

3. Let $\psi_k(x)$ be the normalized eigenvectors of the one-dimensional harmonic oscillator in $L^2(\mathbb{R})$. Then:

$$(U_B \psi_k)(z) := e_k(z) = \frac{1}{\sqrt{\pi^{1/2} k!}} z^k, \quad k = 0, 1, \ldots \tag{2.12}$$

Let now $m = 0, 1, 2, \ldots$. Define:

$$\begin{aligned} f_{m,h}(z_1, z_2) &:= e_{m-h}(z_2) e_h(z_1), \ h = 0, \ldots, m; \\ \mathcal{K}_m &:= \mathrm{Span}\{f_{m,h} : h = 0, \ldots, m\} = \\ &= \mathrm{Span}\{e_{l_1}(z_2) e_{l_2}(z_1) : l_1 + l_2 = m\} \end{aligned}$$

Hence the following properties are immediately checked:

$$\dim \mathcal{K}_m = m + 1; \quad \mathcal{K}_m \perp \mathcal{K}_l, \ m \neq l; \quad \bigoplus_{m=0}^{\infty} \mathcal{K}_m = \mathcal{F}_2 \tag{2.13}$$

We then have

Lemma 2.3

1. *For any* $m = 0, 1, \ldots$*:*

$$Q(g) f_{m,h} = m f_{m,h} + igh f_{m,h-1}, \quad h = 0, \ldots, m. \tag{2.14}$$

2. *Let* Π_m *be the orthogonal projection from* $\mathcal{F}_2$ *onto* $\mathcal{K}_m$*. Then:* $[\Pi_m, Q(g)] = 0$*; equivalently,* $\mathcal{K}_m$ *reduces* $Q(g)$*:* $Q(g)\mathcal{K}_m \subset \mathcal{K}_m$*;*

3. *Let $Q(g)_m := Q(g)|_{\mathcal{K}_m} = \Pi_m Q(g)\Pi_m = \Pi_m Q(g) = Q(g)\Pi_m$ be $\mathcal{K}_m$-component of $Q(g)$. Then $Q(g) = \bigoplus_{m=0}^{\infty} Q(g)_m$;*

Proof

1. Just compute the action of $Q(g)$ on $f_{m,h}$:

$$
\begin{aligned}
Q(g)f_{m,h} &= (z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + igz_2\frac{\partial}{\partial z_1})e_{m-h}(z_2)e_h(z_1) \\
&= (m-h)e_{m-h}(z_2)e_h(z_1) + he_{m-h}(z_2)e_h(z_1) \\
&+ ig\sqrt{h(m-h+1)}e_{m-(h-1)}(z_2)e_{h-1}(z_1) \\
&= mf_{m,h} + ig\sqrt{h(m-h+1)}f_{m,h-1}
\end{aligned}
\tag{2.15}
$$

2. Since the vectors $f_{m,h} : h = 0, \ldots, m$ span $\mathcal{K}_m$, by linearity the above formula entails $Q(g)\mathcal{K}_m \subset \mathcal{K}_m$.

3. The assertion follows from 2. above and the completeness relation (2.13).

Proof of Theorem 2.2

We have to prove Assertion 2.

2. Making $h = 0$ in (2.15) we get:

$$Q(g)f_{m,0} = mf_{m,0}, \quad m = 0, 1, \ldots$$

Hence $\lambda'_m = m$ is an eigenvalue of $Q(g)$ with eigenvector $f_{m,0}$, i.e. with geometric multiplicity one. By the unitary equivalence $H(g) = U_B^{-1}(Q(g)+1)U_B$ we conclude that $\lambda_m = m+1, m = 0, \ldots,$ is an eigenvalue of $H(g)$ of geometric multiplicity one, with eigenvector $U_B^{-1}f_{m,0} = \psi_m(x_1)\psi_0(x_2)$. From (2.14) we read off the matrix representation (2.2), (2.3) and we get the statement about the algebraic multiplicity. On account of the unitary equivalence $\mathcal{K}_m = U_B^{-1}\mathcal{H}_m$ this concludes the proof of the theorem.

3 A class of non self-adjoint $\mathcal{PT}$ symmetric operators with real discrete spectrum

Let H_0 be a selfadjoint operator in $L^2(\mathbb{R}^d), d \geq 1$, bounded below (without loss of generality, positive) with compact resolvent, and let $D(H_0)$ denote its domain. Let $\mathcal{P}$ be the parity operator in $L^2(\mathbb{R}^d)$ defined by

$$(\mathcal{P}\psi)(x) = \psi(-x)\,, \quad \forall\psi \in L^2(\mathbb{R}^d)\,,\ \forall x \in \mathbb{R}^d\,. \tag{3.1}$$

Let us assume that H_0 is $\mathcal{P}$-symmetric, i.e.

$$\mathcal{P}H\psi = H\mathcal{P}\psi\,, \quad \forall\psi \in D(H_0) \tag{3.2}$$

and also $\mathcal{T}$-symmetric, i.e.

$$(\overline{H_0\psi})(x) = (H_0\overline{\psi})(x)\,, \quad \forall \psi \in D(H_0)\,,\, \forall x \in \mathbb{R}^d\,. \tag{3.3}$$

Let $0 < \ell_1 < \ell_2 < \ldots$ be the increasing sequence of the eigenvalues of $\mathcal{H}_0$. Let m_r denote the multiplicity of ℓ_r and $\psi_{r,s}$, $s = 1, ..., m_r$, denote m_r linearly independent eigenfunctions corresponding to ℓ_i, which form a basis of the eigenspace

$$\mathcal{M}_r := \mathrm{Span}\{\psi_{r,s} : s = 1, ..., m_r\} \tag{3.4}$$

corresponding to ℓ_r.

Definition 3.1

1. *An eigenspace $\mathcal{M}_r$ is even (odd) if all basis vectors $\{\psi_{r,s} : s = 1, ..., m_r\}$ are even (odd); i.e., if either $\mathcal{P}\psi_{r,s} = \psi_{r,s}, \forall s = 1, \ldots, m_r$, or $\mathcal{P}\psi_{r,s} = -\psi_{r,s}, \forall s = 1, \ldots, m_r$.*

2. *An eigenvalue ℓ_r is even (odd) if the corresponding eigenspace $\mathcal{M}_r$ is even (odd).*

Now, let $W \in L^\infty(\mathbb{R}^d)$ be an odd real function, i.e. $W(x) = -W(-x)$, $\forall x \in \mathbb{R}^d$. Let $V := iW$; clearly V is $\mathcal{PT}$- even, i.e.

$$\overline{V(-x)} = V(x)\,, \quad \forall x \in \mathbb{R}^d\,. \tag{3.5}$$

Then, $\forall g \in \mathbb{C}$, the operator $H(g) := H_0 + gV$ defined on $D(H(g)) = D(H_0)$ by

$$H(g)\psi = H_0\psi + gV\psi\,, \quad \forall \psi \in D(H_0) \tag{3.6}$$

is closed. More precisely $H(g)$ represents an analytic family of type A of closed operators in the sense of Kato ([25], Ch. VII.2) for $g \in \mathbb{C}$, with compact resolvents. Thus $\mathrm{Spec}(H(g))$ is discrete for all g. For $g \in \mathbb{R}$ the operator $H(g)$ is $\mathcal{PT}$-symmetric, i.e.

$$\overline{\mathcal{P}H(g)\psi}(x) = H(g)\overline{\psi}(-x)\,, \quad \forall \psi \in D(H_0)\,. \tag{3.7}$$

Moreover:

$$H(g)^* = H(-g) \tag{3.8}$$

We want to prove the following result.

Theorem 3.2 *Let H_0 and W enjoy the above listed properties. Assume furthermore:*

(1) $\delta := \dfrac{1}{2}\inf_r(\ell_{r+1} - \ell_r) > 0$;

(2) Each eigenvalue $\ell_r : r = 1, \ldots$ is either even or odd.

Then if $|g| < \dfrac{\delta}{\|W\|_\infty}$ *each eigenvalue* $\lambda(g)$ *of* $H(g)$ *is real, and thus the spectrum of* $H(g)$ *is purely real.*

Example

The d-dimensional harmonic oscillator with equal frequencies

$$H_0 = \frac{1}{2}\sum_{k=1}^{d}\left[-\frac{d^2}{dx_k^2} + \omega^2 x_k^2\right] \tag{3.9}$$

has the properties required by H_0. In this case indeed:

$$\ell_r = \omega(r_1 + \ldots + r_d + d/2) := \omega(r + d/2), \quad r_k = 0, 1, 2, \ldots; \ k = 1, \ldots, d$$

with multiplicity $m_r = (r+1)^d$. Here the corresponding eigenspace is:

$$\mathcal{M}_r := \mathrm{Span}\{\psi_{r,s} : s = 1, ..., m_r\} = \mathrm{Span}\{\psi_{r_1}(x_1)\cdots\psi_{r_d}(x_d) : r_1 + \ldots + r_d = r\}$$

where, as above, $\psi_r(x)$ is an Hermite function. Now if r is odd the sum $r = r_1 + \ldots + r_d$ contains an odd number of odd terms; since $\psi_s(x)$ is an odd function when s is odd, the product $\psi_{r_1}(x_1)\cdots\psi_{r_d}(x_d)$ contains an odd number of odd factors and is therefore odd. ℓ_r is therefore an odd eigenvalue. An analogous argument shows that ℓ_r is an even eigenvalue when r is even. Moreover, $\ell_{r+1} - \ell_r = \omega$ and thus condition (1) above is fulfilled.

Actually, the above example is a particular case of a more general statement, while for $d = 2$ the above application to the perturbation of harmonic oscillators can be considerably strenghtened.

Theorem 3.3 *Let*

$$H_0 = \frac{1}{2}\sum_{k=1}^{d}\left[-\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2\right] \tag{3.10}$$

Assume the frequencies to be rational multiples of a fixed frequency $\omega > 0$*, namely:*

$$\omega_k = \frac{p_k}{q_k}\omega, \quad k = 1, \ldots, d \tag{3.11}$$

where (p_k, q_k) *are relatively prime natural numbers. Then:*

(i) If p_k *and* q_k *are both odd,* $k = 1, \ldots, d$*, the assumptions of Theorem 3.2 are fulfilled;*

(ii) If $d = 2$*, the condition (3.11)* p_k *and* q_k *both odd is also necessary for the validity of assumption (2) of Theoren 3.2, while assumption (1) holds independently of the parity of* p_k*,* q_k*.*

We will now prove Theorem 3.2 in two steps (Propositions 3.5 and and 3.10), while the proof of Theorem 3.3 is postponed to the next Section. In the first step we show that the degenerate Rayleigh-Schrödinger perturbation theory near each eigenvalue ℓ_r is real and convergent, with a convergence radius independent of r. Thus there exists $\rho > 0$ such that all the m_r eigenvalues near ℓ_r (counted according to their multiplicity) existing for $|g| < \rho$ are real for all r. The second step is the proof that $H(g)$ admits no other eigenvalue for $|g| < \rho$. To formulate the first step, we recall some relevant notions and results of perturbation theory.

Let $g_0 \in \mathbb{C}$ be fixed and let μ be an eigenvalue of $H(g_0)$. Let $c > 0$ be sufficiently small so that

$$\Gamma_c = \{z :| z - \mu |= c\}$$

encloses no other eigenvalue of $H(g_0)$. Then for $|g - g_0|$ small Γ_c is contained in the resolvent set of $H(g)$, $\rho(H(g)) := \mathbb{C} \setminus \mathrm{Spec}(H(g))$. Moreover $\Gamma_c \subset \mathcal{D}$, where

$$\mathcal{D} := \{z \in \mathbb{C} : \exists b(z) > 0 \; s.t. \; (z - H(g))^{-1} := R_g(z) \\ \text{exists and is uniformly bounded for } |g - g_0| < b(z)\}.$$

Then for $|g - g_0|$ sufficiently small

$$P(g) = (2\pi i)^{-1} \oint_{\Gamma_c} R_g(z)\, dz \tag{3.12}$$

is the projection corresponding to the part of the spectrum of $H(g)$ enclosed in Γ_c and $\forall z \in \mathcal{D}$

$$\|R_g(z) - R_{g_0}(z)\| \to 0\,, \quad \text{as } g \to g_0 \tag{3.13}$$

whence

$$\|P(g) - P(g_0)\| \to 0\,, \quad \text{as } g \to g_0 \tag{3.14}$$

(see e.g.[25], §VII.1). In particular, if m denotes the multiplicity of μ, for g close to g_0, $H(g)$ has exactly m eigenvalues (counting multiplicity) inside Γ_c, denoted $\mu_s(g), s = 1, ..., m$, which converge to μ as $g \to g_0$. If we denote by $\mathcal{M}(g)$ the range of the projection operator $P(g)$, then $\dim \mathcal{M}(g) = m$ as $g \to g_0$, and $H(g)\mathcal{M}(g) \subset \mathcal{M}(g)$. Hence the component $P(g)H(g)P(g) = P(g)H(g) = H(g)P(g)$ of $H(g)$ in $\mathcal{M}(g)$ has rank m and its eigenvalues are precisely $\mu_s(g), s = 1, ..., m$.

Assume from now on $g_0 = 0$ so that the unperturbed operator is the self-adjoint operator $H_0 := H(0)$. Let $\ell = \ell_r, r = 1, 2, \ldots,$ be a fixed eigenvalue of H_0, $m = m_r$ its multiplicity and $\psi_s := \psi_{r,s} : s = 1, \ldots, m$ be an orthonormal basis in $\mathcal{M}_r := \mathcal{M}_r(0)$. Then there is

$\bar{g}(r) > 0$ such that the vectors $P_r(g)\psi_{r,s} : s = 1, \ldots, m$ are a basis in the invariant subspace $\mathcal{M}_r(g)$ for $|g| < \bar{g}(r)$. We denote $\phi_{r,s}(g) : s = 1, \ldots, m$ the orthonormal basis in $\mathcal{M}_r(g)$ obtained from $P_r(g)\psi_{r,s} : s = 1, \ldots, m_r$ through the Gram-Schmidt orthogonalization procedure. Then the eigenvalues $\mu_s(g) = \ell_{r,s}(g), s = 1, ..., m_r$, are the eigenvalues of the $m_r \times m_r$ matrix $T_r(g)$ given by:

$$(T_r(g))_{hk} := \langle \phi_{r,h}(g), H(g)P(g)\phi_{r,k}(g)\rangle =$$
$$\langle \phi_{r,h}(g), P_r(g)H(g)P_r(g)\phi_{r,k}(g)\rangle, \quad h, k = 1, \ldots, m_r\,.$$

Let $\phi_{r,s}(g) = \sum_{j=1}^{m} \alpha^r_{sj}(g)P_r(g)\psi_{r,j}\,,\ \alpha^r_{sj}(g) \in \mathbb{C}\,, s, j = 1, \ldots, m_r$. Then

$$(T_r(g))_{hk} = \sum_{j,l=1}^{m} \alpha^r_{hj}(g)\overline{\alpha^r_{kl}(g)}\langle \psi_{r,j}, P_r(-g)H(g)P_r(g)\psi_{r,l}\rangle\,, \quad h, k = 1, \ldots, m\,. \tag{3.15}$$

Consider now the $m_r \times m_r$ matrix $B_r(g) = (B^r_{jl}(g))_{j,l=1,\ldots,m}$, where

$$B^r_{jl}(g) = \langle \psi_{r,j}, P_r(-g)H(g)P_r(g)\psi_{r,l}\rangle\,, \quad j, l = 1, \ldots, m_r\,. \tag{3.16}$$

Its self-adjointness entails the self-adjointness of $T_r(g)$. We have indeed:

Lemma 3.4 *Let $B^r_{jl}(g) = \overline{B^r_{lj}(g)}, \forall j, l = 1, \ldots, m_r$. Then:*

$$(T_r(g))_{hk} = \overline{(T_r(g))_{kh}},\ h, k = 1, \ldots, m_r.$$

Proof

Since $B^r_{jl}(g) = \overline{B^r_{lj}(g)}, \forall j, l$ we can write:

$$\begin{aligned}
\overline{(T_{r,}(g))_{kh}} &= \overline{\sum_{p,s=1}^{m_r} \alpha^r_{kp}(g)\overline{\alpha^r_{hs}(g)}B^r_{ps}(g)} = \sum_{p,s=1}^{m_r} \overline{\alpha^r_{kp}(g)}\alpha^r_{hs}(g)B^r_{sp}(g) \\
&= \sum_{j,l=1}^{m_r} \alpha^r_{hj}(g)\overline{\alpha^r_{kl}(g)}B^r_{jl}(g) = (T_r(g))_{hk}\,.
\end{aligned} \tag{3.17}$$

and this proves the assertion.

In other words the selfadjointness of $T_r(g))$, and thus the reality of the eigenvalues $\ell_{r,s}(g)$ for $|g| < \bar{g}(r)$, follows from the selfadjointness of $B_r(g)$ which will be proved by the construction of the Rayleigh-Schödinger perturbation expansion (RSPE) for the operator $P_r(-g)H(g)P_r(g)$, which we now briefly recall, following ([25], §II.2.7; here $T^{(1)} = V = iW$, $T^{(\nu)} = 0,$, $\nu \geq 2$, $D = 0$).

(1) The geometric expansion in powers of g of the resolvent

$$R_g(z) = (z - H(g))^{-1} = (z - H_0 - gV)^{-1} = R_0(z)\sum_{n=0}^{\infty}(-g)^n[VR_0(z)]^n$$

is norm convergent for $|g|$ suitably small. Insertion in (3.12) yields the expansion for $P(g)$:

$$P_r(g) = \sum_{n=0}^{\infty} g^n P_r^{(n)}, \quad P_r^{(0)} = P_r(0) := P_r \tag{3.18}$$

$$P_r^{(n)} = \frac{(-1)^{n+1}}{2\pi i} \oint_{\Gamma_r} R_0(z)[V R_0(z)]^n \, dz, \ n \geq 1 \tag{3.19}$$

whence

$$P_r(-g)H(g)P_r(g) = \sum_{n=0}^{\infty} g^n \hat{T}_r^{(n)}, \quad \hat{T}_r^{(0)} = H_0 P_r \tag{3.20}$$

where

$$\hat{T}_r^{(n)} = \sum_{p=0}^{n} (-1)^p [P_r^{(p)} H_0 P_r^{(n-p)} + P_r^{(p-1)} V P_r^{(n-p)}], \quad n \geq 1, \quad P_r^{(-1)} = 0\,. \tag{3.21}$$

and

$$P_r^{(n)} = (-1)^{n+1} \sum_{\substack{k_1+\ldots+k_{n+1}=n \\ k_j \geq 0}} S_r^{(k_1)} V S_r^{(k_2)} V \ldots V S_r^{(k_n)} V S_r^{(k_{n+1})}\,. \tag{3.22}$$

Here

$$S_r^{(0)} = -P_r\,; \quad S_r = -\sum_{j \neq r} P_j/(\ell_j - \ell_r)\,; \quad S_r^{(k)} = (S_r)^k\,, \quad \forall k = 1, 2, ..., \tag{3.23}$$

where P_j is the projection corresponding to the eigenvalue ℓ_j of H_0.

(2) The series (3.18,3.20) are norm convergent for $|g| < \dfrac{d_r}{2\|W\|_\infty}$, where d_r is the distance of $\ell = \ell_r$ from the rest of the spectrum of H_0. Hence under the present assumptions the convergence takes place a fortiori for

$$|g| < \rho \qquad \rho := \frac{\delta}{\|W\|_\infty}. \tag{3.24}$$

(3) The projection operator $P_r(g)$ is holomorphic for $|g| < \rho$. This entails that its dimension is constant throughout the disk. Therefore $H(g)$ admits exactly m_r eigenvalues $\ell_{r,s}$ (counting multiplicities) inside Γ_r for $|g| < \rho$.

(4) Hence, for $|g| < \rho$ we can write:

$$B_r(g) = \sum_{n=0}^{\infty} g^n \mathcal{G}_r^{(n)}, \quad (\mathcal{G}_r^{(n)})_{jl} := \langle \psi_{r,j}, \hat{T}_r^{(n)} \psi_{r,l} \rangle, \quad j, l = 1, \ldots, m_r\,. \tag{3.25}$$

We can now formulate the first step:

Proposition 3.5 *Let $\ell_r, r = 1, 2, \ldots$ be an eigenvalue of H_0. Then the m_r eigenvalues (counting multiplicity) $\ell_{r,s}$ of $H(g)$ existing for $|g| < \rho$, and converging to ℓ_r as $g \to 0$, are real for $|g| < \bar{g}(r)$, $g \in \mathbb{R}$.*

Proof

We drop the index r because the argument is $r-$independent, i.e. we consider the expansion near the unperturbed eigenvalue $\ell := \ell_r$. Accordingly, we denote by $\psi_s := \psi_{r,s}$ the corresponding eigenvectors. Let us first consider the case of ℓ even. It is enough to prove that $\mathcal{G}^n = 0$ if n is odd and that $\mathcal{G}^n$ is selfadjoint (in fact, real symmetric) when n is even. These assertions will be proved in Lemma 3.7 and 3.9, respectively, which in turn require an auxiliary statement.

Definition 3.6 *The product*

$$\Pi(k_1, \ldots, k_{n+1}) := S^{(k_1)} V S^{(k_2)} V \ldots V S^{(k_n)} V S^{(k_{n+1})} \tag{3.26}$$

containing precisely n factors V and $n+1$ factors $S^{(j)}$, $j \geq 0$, is called string of length n.

Then from (3.21,3.22) we get:

$$(\mathcal{G}^{(n)})_{qs} = (-1)^n \sum_{p=0}^{n} (-1)^p [(\mathcal{G}^{(n)}_{1,p})_{qs} - (\mathcal{G}^{(n)}_{2,p})_{qs}] \tag{3.27}$$

where

$$(\mathcal{G}^{(n)}_{1,p})_{qs} = \langle \psi_q, \sum_{\substack{k_1+\ldots+k_{p+1}=p \\ k_l \geq 0}} \Pi(k_1, \ldots, k_{p+1}) H_0 \sum_{\substack{h_1+\ldots+h_{n-p+1} \\ =n-p;\, h_l \geq 0}} \Pi(h_1, \ldots, h_{n-p+1}) \psi_s \rangle \tag{3.28}$$

$$(\mathcal{G}^{(n)}_{2,p})_{qs} = \langle \psi_q, \sum_{\substack{k_1+\ldots+k_p \\ =p-1;\, k_l \geq 0}} \Pi(k_1, \ldots, k_p) V \sum_{\substack{h_1+\ldots+h_{n-p+1} \\ =n-p;\, h_l \geq 0}} \Pi(h_1, \ldots, h_{n-p+1}) \psi_s \rangle \tag{3.29}$$

Now $S^{(k)}$ is selfadjoint for all k, and $V = iW$ with $W(x) \in \mathbb{R}$. Therefore:

$$(\mathcal{G}^{(n)}_{1,p})_{qs} = (-1)^p \langle \sum_{\substack{k_1+\ldots+k_{p+1}=p;\, k_l \geq 0 \\ h_1+\ldots+h_{n-p+1}=n-p;\, h_l \geq 0}} \Pi(k_{p+1}, \ldots, k_1) \psi_q, H_0 \Pi(h_1, \ldots, h_{n-p+1}) \psi_s \rangle$$

$$(\mathcal{G}^{(n)}_{2,p})_{qs} = (-1)^{p-1} \langle \sum_{\substack{k_1+\ldots+k_p=p-1;\, k_l \geq 0 \\ h_1+\ldots+h_{n-p+1}=n-p;\, h_l \geq 0}} \Pi(k_p, \ldots, k_1) \psi_q, V \Pi(h_1, \ldots, h_{n-p+1}) \psi_s \rangle\,.$$

Since $S^{(k)} \perp P$, $k \geq 1$, in both scalar products (3.28) and (3.29) all terms with $k_1 \neq 0$ or $h_{n-p+1} \neq 0$ vanish. Hence:

$$(\mathcal{G}^{(n)}_{1,p})_{qs} = (-1)^p \langle \sum_{\substack{k_1+\ldots+k_p=p \\ h_1+\ldots+h_{n-p}=n-p \\ k_l \geq 0, h_l \geq 0}} \Pi(k_p, \ldots, k_1) V \psi_q, H_0 \Pi(h_1, \ldots, h_{n-p}) V \psi_s \rangle \tag{3.30}$$

$$(\mathcal{G}^{(n)}_{1,p})_{qs} = (-1)^{p-1} \langle \sum_{\substack{k_1+\ldots+k_{p-1}=p-1 \\ h_1+\ldots+h_{n-p}=n-p \\ k_l \geq 0, h_l \geq 0}} \Pi(k_{p-1}, \ldots, k_1) V \psi_q, V \Pi(h_1, \ldots, h_{n-p}) V \psi_s \rangle \tag{3.31}$$

We now have:

Lemma 3.7 *Let n be odd, and $0 \le p \le n$. Then, $\forall\, k_1, \ldots, k_p \ge 0$, $\forall\, h_1, \ldots, h_{n-p} \ge 0$, $\forall\, q, s = 1, \ldots, m$:*

$$\langle \Pi(k_p, \ldots, k_1) V \psi_q, H_0 \Pi(h_1, \ldots, h_{n-p}) V \psi_s \rangle = 0 \tag{3.32}$$

$$\langle \Pi(k_{p-1}, \ldots, k_1) V \psi_q, V \Pi(h_1, \ldots, h_{n-p}) V \psi_s \rangle = 0 \tag{3.33}$$

Proof

Let us write explicitly (3.32, 3.33):

$$\langle S^{(k_p)} V S^{(k_{p-1})} V \ldots V S^{(k_1)} V \psi_q, H_0 S^{(h_1)} V S^{(h_2)} V \ldots V S^{(h_{n-p})} V \psi_s \rangle = 0 \tag{3.34}$$

$$\langle S^{(k_{p-1})} V S^{(k_{p-2})} V \ldots V S^{(k_1)} V \psi_q, V S^{(h_1)} V S^{(h_2)} V \ldots V S^{(h_{n-p})} V \psi_s \rangle = 0 \tag{3.35}$$

Let us now further simplify the notation as follows. We set:

$$S_+ := - \sum_{j \ne r;\, \ell_j \text{even}} \frac{P_j}{\ell_j - \ell}; \quad S_- = - \sum_{j \ne r;\, \ell_j \text{odd}} \frac{P_j}{\ell_j - \ell}. \tag{3.36}$$

Both series are convergent because $|(\ell_j - \ell)| > \delta$ and $\sum_{j \ne r;} P_j$ is convergent. Hence $S = S_+ \oplus S_-$ and for $k \ne 0$ we have:

$$S^k = S_+^k \oplus S_-^k = (-1)^k \sum_{j \ne r;\, \ell_j \text{even}} \frac{P_j}{(\ell_j - \ell)^k} + (-1)^k \sum_{j \ne r;\, \ell_j \text{odd}} \frac{P_j}{(\ell_j - \ell)^k}. \tag{3.37}$$

Finally we set $S_+^{(0)} := S_+^0 := -P$. Now, the multiplication by V changes the parity of a function, and ψ_j, ψ_l are even. This entails that in both scalar products above $S^{(k_1)}$ can be replaced by $S_-^{(k_1)}$, $S^{(k_2)}$ by $S_+^{(k_2)}$ and so on. The general rule is: $S^{(k_j)}$ can be replaced by $S_-^{k_j}$ (by $S_+^{k_j}$) if and only if j is odd (j is even, respectively). Similarly for the $S^{(h_j)}$. Consider first the scalar product in (3.34). According to the general rule $S_\pm^{(k_p)}$ coincides with $S_+^{(k_p)}$ if p is even and with $S_-^{(k_p)}$ if p is odd. Similarly for $S_\pm^{(h_{n-p})}$. If n is odd p and $n-p$ have opposite parity and since H_0 does not change the parity of a function the scalar product is zero. A similar argument shows that also the scalar product (3.35) is zero if n is odd. Indeed the function in the left hand side has the same parity of the number $p-1$, whereas the function of the right hand side has the same parity of $n-p+1$, and if n is odd $p-1$ and $n-p+1$ have opposite parity. This proves the assertion.

Lemma 3.8 *Let n be odd. Then $\mathcal{G}^{(n)} = 0$.*

Proof

It is an immediate consequence of Lemma 3.7 on account of (3.27,3.30,3.31).

Lemma 3.9 *Let n be even. Then $(\mathcal{G}^{(n)})_{qs} = \overline{(\mathcal{G}^{(n)})_{qs}}$ for all $q, s = 1, \ldots, m$.*

Proof

Once more by (3.27,3.30,3.31) we can write for all n (replacing of course V by iW in the definition (3.26), and denoting Π' the resulting string)

$$
\begin{aligned}
&(\mathcal{G}^{(n)})_{qs} = \\
&(i)^n \sum_{p=0}^{n} [\sum_{\substack{k_1+\ldots+k_p=p; k_j \geq 0 \\ h_1+\ldots+h_{n-p}=n-p;\, h_j \geq 0}} (-1)^p \langle \Pi'(k_1, \ldots, k_p) W\psi_q, H_0 \Pi'(h_1, \ldots, h_{n-p}) W\psi_s \rangle \\
&- \sum_{\substack{k_1+\ldots+k_{p-1}=p-1; k_j \geq 0 \\ h_1+\ldots+h_{n-p}=n-p;\, h_j \geq 0}} (-1)^{p-1} \langle \Pi'(k_1, \ldots, k_{p-1}) W\psi_q, W\Pi'(h_1, \ldots, h_{n-p}) W\psi_s \rangle] = \\
&(i)^n \sum_{p=0}^{n} [\sum_{\substack{k_1+\ldots+k_{n-p}=n-p; \\ h_1+\ldots+h_p=p; h_j \geq 0, k_j \geq 0}} (-1)^{n-p} \langle H_0 \Pi'(k_1, \ldots, k_{n-p}) W\psi_q, \Pi'(h_1, \ldots, h_p) W\psi_s \rangle \\
&- \sum_{\substack{k_1+\ldots+k_{n-p}=n-p; k_j \geq 0 \\ h_1+\ldots+h_{p-1}=p-1; h_j \geq 0}} (-1)^{n-p+1} \langle W\Pi'(k_1, \ldots, k_{n-p}) W\psi_q, \Pi'(h_1), \ldots, h_{p-1}) W\psi_s \rangle] \\
&= \overline{(\mathcal{G}^{(n)})_{qs}}\,.
\end{aligned} \tag{3.38}
$$

To obtain the second equality in (3.38) we have used the selfadjointness of H_0 and W and we have renamed the indices, exchanging p and $n-p$ in the first scalar product, and $p-1$ and $n-p$ in the second scalar product. Finally, to obtain the last equality in (3.38) notice that $(-1)^p = (-1)^{n-p}$ since n is even.

Remarks

1. It is worth noticing that if the $\psi_s, s = 1, \ldots, m$, are chosen to be real valued then $(\mathcal{G}^{(n)})_{qs} \in \mathbb{R}, \forall j, l$, because W is also real valued and the operators $S^{(k)}$ map real valued functions into real valued functions.

2. The argument yielding the real nature of the perturbation expansion is independent of its convergence, namely it holds for all odd potentials V for which the perturbation expansion exists to all orders. In particular, it holds when V is any odd polynomial, i.e. for any odd anharmonic oscillators in any dimension d.

We now proceed to prove that the eigenvalues $\ell_{r,s}(g)$ are real $\forall\, g \in \mathbb{R}$, $|g| < \rho$.

Proposition 3.10 *The eigenvalues $\ell_{r,s}$, $r = 1, 2, \ldots,$ $s = 1, \ldots, m_r$ are holomorphic for $|g| < \rho$ and real for $g \in \mathbb{R}$, $|g| < \rho$.*

Proof

The vectors $U_r(g)P_r\psi_{r,k} = U_r(g)\psi_{r,k} : k = 1, \ldots, m_r$ represent a basis of $\mathcal{M}_r(g)$ for all

$|g| < \rho$ ([25], §II.4.2). Here the similarity operator $U_r(g)P_r$ is recursively defined in the following way:

$$U_r(g)P_r = P_r + \sum_{k=1}^{\infty} U_r^{(k)} g^k, \quad kU_r^{(k)} = kP_r^{(k)} + (k-1)P_r^{(k-1)}U_r^{(1)} + \dots P_r^{(1)}U_r^{(k-1)} \quad (3.39)$$

We denote $\chi_{r,s}(g) : s = 1, \dots, m_r$ the orthonormal basis in $\mathcal{M}_r(g)$ obtained from $U_r(g)\psi_{r,s}$, $s = 1, \dots, m_r$ through the Gram-Schmidt orthogonalization procedure. Then the eigenvalues $\ell_{r,s}(g), s = 1, ..., m_r$, are the eigenvalues of the $m \times m$ matrix $X_r(g)$ given by:

$$\begin{aligned}(X_r(g))_{hk} &:= \langle \chi_{r,h}(g), H(g)P(g)\chi_{r,k}(g)\rangle \\ &= \langle \chi_{r,h}(g), H(g)\chi_{r,k}(g)\rangle, \quad h, k = 1, \dots, m_r \,. \end{aligned} \quad (3.40)$$

because $P(g)\chi_{r,h}(g) = \chi_{r,h}$, $h = 1, \dots, m_r$. For $|g| < \bar{g}(r)$ the orthonormal vectors $\chi_{r,h}(g)$: $h = 1, \dots, m_r$ are linear combinations of the orthonormal vectors $\phi_{r,h}(g) : h = 1, \dots, m_r$ defined above. Since $X_r(g)$ and $T_r(g)$ represent the same operator on two different orthonormal basis, if either one is self-adjoint the second must enjoy the same property. Hence the matrix $(X_r(g))_{hk}$ is self-adjoint, $|g| < \bar{g}(r)$, $g \in \mathbb{R}$. Expand now $(X_r(g))_{hk}$ in power series:

$$(X_r(g))_{hk} = \sum_{m=0}^{\infty} (\theta_{r,m})_{hk} g^m$$

The series converges for $|g| < \rho$. It follows indeed by the standard Gram-Schmidt procedure (we omit the details) that it can be written as the quotient of two functions of g involving only linear combinations of scalar products of the operators $P_r(g)$ on vectors independent of g; the denominator never vanishes for $|g| < \rho$ by construction, on account of the linear independence of the vectors $U_r(g)\psi_{r,s}$, $s = 1, \dots, m_r$ when $|g| < \rho$. Now it necessarily follows from the self-adjointness of $(X_r(g))_{hk}$, valid for $|g| < \bar{g}(r)$ that $(\theta_{r,m})_{hk} = \overline{(\theta_{r,m})}_{kh}$, $m = 0, 1, \dots$. Hence the matrix $X_r(g)$ is self-adjoint for $|g| < \rho$, $g \in \mathbb{R}$, and thus the eigenvalues $\ell_{r,s}$ are real in the same domain. This proves the assertion.

Proof of Theorem 3.2

We have seen that the RSPE associated with the ℓ_r-group of eigenvalues $\ell_{r,s}(g), s = 1, \dots, m_r$, of $H(g)$ which converge to ℓ_r as $g \to 0$, have radius of convergence no smaller than ρ. Hence, $\forall g \in \mathbb{R}$ such that $|g| < \rho$, $H(g)$ admits a sequence of real eigenvalues $\ell_{r,s}(g), s = 1, \dots, m_r, r \in \mathbb{N}$. We want to prove that for $|g| < \rho, g \in \mathbb{R}$, $H(g)$ has no other eigenvalues. Thus all its eigenvalues are real. To this end, for any $r \in \mathbb{N}$ let $\mathcal{Q}_r$ denote the square centered at ℓ_r with side 2δ. Then if $g \in \mathbb{R}$, $|g| < \rho$, and $\ell(g)$ is an eigenvalue of $H(g)$:

$$\ell(g) \in \bigcup_{r \in \mathbb{N}} \mathcal{Q}_r.$$

In fact, for any $z \notin \cup_{r\in\mathbb{N}} \mathcal{Q}_r$ we have

$$\|gVR_0(z)\| \leq |g| \|W\|_\infty \|R_0(z)\| < \rho \|W\|_\infty [\mathrm{dist}(z, \sigma(H_0))]^{-1} \leq \frac{\rho \|W\|_\infty}{\delta} = 1 \qquad (3.41)$$

where $R_0(z) := (H_0 - z)^{-1}$. Thus, $z \in \rho(H(g))$ and

$$R(g,z) := (H(g) - z)^{-1} = R_0(z)[1 + gVR_0(z)]^{-1} \, .$$

Now let $g_0 \in \mathbb{R}$ be fixed with $|g| < \rho$. Without loss of generality we assume that $g_0 > 0$. Let $\ell(g_0)$ be a given eigenvalue of $H(g_0)$. Then $\ell(g_0)$ must be contained in the interior (and not on the boundary) of $\mathcal{Q}_{n_0}$ for some $n_0 \in \mathbb{N}$. Moreover if m_0 is the multiplicity of $\ell(g_0)$, for g close to g_0 there are m_0 eigenvalues $\ell^{(\alpha)}(g), \alpha = 1. \ldots, m_0$, of $H(g)$ which converge to $\ell(g_0)$ as $g \to g_0$ and each function $\ell^{(\alpha)}(g)$ represents a branch of one or several holomorphic functions which have at most algebraic singularities at $g = g_0$ (see [Kato, Thm. VII.1.8]). Let us now follow one of such branches $\ell^{(\alpha)}(g)$ for $0 < g < g_0$, suppressing the index α from now on. First of all we notice that, by continuity, $\ell(g)$ cannot go out of $\mathcal{Q}_{n_0}$ for g close to g_0. Moreover, if we denote Γ_{2t} the boundary of the square centered at ℓ_{n_0} with side $2t$, for $0 < t \leq 1$, we have, for $z \in \Gamma_{2t}$ and $0 < g \leq g_0$,

$$\|gVR_0(z)\| \leq g[\mathrm{dist}(z, \sigma(H_0))]^{-1} \leq g/t \, . \qquad (3.42)$$

Then $t > g$ implies $z \notin \sigma(H(g))$, i.e. if $z \in \sigma(H(g)) \cap \Gamma_{2t}$ then $t \leq g < g_0 < 1$. Hence we observe that as $g \to g_0^-$, $\ell(g)$ is contained in the square centered at ℓ_{n_0} and side $2g$. Suppose that the holomorphic function $\ell(g)$ is defined on the interval $]g_1, g_0]$ with $g_1 > 0$. We will show that it can be continued up to $g = 0$, and in fact up to $g = -1$. From what has been established so far the function $\ell(g)$ is bounded as $g \to g_1^+$. Thus, by the well known properties on the stability of the eigenvalues of the analytic families of operators, $\ell(g)$ must converge to an eigenvalue $\ell(g_1)$ of $H(g_1)$ as $g \to g_1^+$ and $\ell(g_1)$ is contained in the square centered at ℓ_{n_0} and side $2g_1$. Repeating the argument starting now from $\ell(g_1)$, we can continue $\ell(g)$ to a holomorphic function on an interval $]g_2, g_1]$, which has at most an algebraic singularity at $g = g_2$. We build in this way a sequence $g_1 > g_2 > \ldots > g_n > \ldots$ which can accumulate only at $g = -1$. In particular the function $\ell(g)$ is piecewise holomorphic on $]-1, 1]$. But while passing through $g = 0$, $\ell(g)$ coincides with one of the eigenvalues $\ell_{r,s}(g), s = 1, \ldots, m_r$, generated by an unperturbed eigenvalue ℓ_r of H_0 (namely ℓ_{n_0}), which represent m_r real analytic functions defined for $g \in]-1, 1]$. Thus, $\ell(g_0)$ arises from one of these functions and is therefore real. This concludes the proof of the Theorem.

4 Perturbation of resonant harmonic oscillators

Consider again the d-dimensional harmonic oscillator

$$H_0 = \frac{1}{2}\sum_{k=1}^{d}\left[-\frac{d^2}{dx_k^2} + \omega_k^2 x_k^2\right] \tag{4.1}$$

where now the frequencies $\omega_k > 0 : k = 1, \ldots, d$ may be different. Theorem 3.3 will be a consequence of the following

Proposition 4.1 *The operator (4.1) fulfills Assumption (2) of Theorem 3.2 if and only if the following condition on the frequencies holds:*

(A) $\forall\, k \in \mathbb{Z}^d \setminus \{0\}$ such that the components $k_i : i = 1, \ldots, d$ have no common divisor, and $\omega_1 k_1 + \ldots + \omega_d k_d = 0$, the number $O(k)$ of k_i odd is even.

Proof

We first prove the sufficiency part. Let therefore (A) be fulfilled. First recall the obvious fact that the rational dependence of the frequency entails the degeneracy of any eigenvalue of (4.1). In order to show that each eigenvalue

$$\ell_{n_1,\ldots,n_d} = \omega_1 n_1 + \ldots \omega_d n_d + \frac{1}{2}(\omega_1 + \ldots \omega_d)$$

of H_0 has a definite parity, consider a corresponding eigenfunction

$$\Psi_{n_1,\ldots,n_d}(x_1, \ldots, x_d) = \prod_{s=1}^{d} \psi_{n_s}(x_s)$$

Now $\psi_{n_s}(x)$ is even or odd according to the parity of n_s, and therefore Ψ will be even if and only if the number of odd n_s is even. Since ℓ is degenerate, there exist $(l_1, \ldots, l_d) \neq (n_1, \ldots, n_d)$ such that

$$\omega_1 n_1 + \ldots + \omega_d n_d = \omega_1 l_1 + \ldots + \omega_d l_d \Longrightarrow \langle \omega, k\rangle = 0,\ k := (n_1 - l_1, \ldots, n_d - l_d)$$

and hence the eigenfunction

$$\Psi_{l_1,\ldots,l_d}(x_1, \ldots, x_d) = \prod_{s=1}^{d} \psi_{l_s}(x_s)$$

corresponds to the same eigenvalue. The eigenfunctions $\Psi_{n_1,\ldots,n_d}$ and $\Psi_{l_1,\ldots,l_d}$ have one and the same parity if and only if the number of the odd differences k_i is even: in fact, an even difference $k_i = n_i - l_i$ does not change the relative parity, while an odd difference does. Let us show that if Assumption (A) holds the number of odd differences is even.

The case in which $k_i : i = 1, \ldots, d$ have no common divisor is the Assumption itself. Let therefore $k_i : i = 1, \ldots, d$ have a common divisor. If a common divisor is 2, k_i is even for any i. Hence there are no odd differences. If 2 is not a common divisor, there will be an odd common divisor, denoted b, such that $k_i = bk_i'$, where the numbers k_i' have no common divisor. Now $\langle k', \omega \rangle = \langle k, \omega \rangle / b = 0$. Hence by the assumptions $O(k')$ is even. Since the multiplication by the odd number b does not change the parity of the k_i', the same conclusion applies also to the numbers k_i. Thus the total number of odd differences does not change after multlplication by b: $O(k) = O(k')$ is even.

Conversely, let us assume that Assumption (A) is violated. Therefore there exists $k \in \mathbb{Z}^d \setminus \{0\}$ such that the numbers k_i have no common divisor, $\langle k, \omega \rangle = 0$ and $O(k)$ is odd. Consider again the eigenfunctions $\Psi_{n_1,\ldots,n_d}(x_1, \ldots, x_d)$ and $\Psi_{l_1,\ldots,l_d}(x_1, \ldots, x_d)$ corresponding to the same eigenvalue ℓ, with $k_i = n_i - l_i$ as above. By construction, the two eigenfunctions have opposite parity, and this concludes the proof of the Proposition.

Proof of Theorem 3.3

Let us first prove that Assumption (1) of Theorem 3.2 is fulfilled. Let $\ell_l = \ell_{l_1,\ldots,l_d}$ and $\ell_n = \ell_{n_1,\ldots,n_d}$ denote different eigenvalues. Then, by assumption:

$$|\ell_n - \ell_l| = \omega|(n_1 - l_1)\frac{p_1}{q_1} + \ldots + (n_d - l_d)\frac{p_d}{q_d}| =$$
$$\frac{\omega}{q_1 \cdots q_d}|(n_1 - l_1)p_1 q_2 \cdots q_d + \ldots + (n_d - l_d)p_d q_1 \cdots q_{d-1}|$$
$$\geq \frac{\omega}{q_1 \cdots q_d} := \delta > 0$$

Since this lower bound does not depend on the multi-indices (n, l) the assertion is proved. Let us now check Assertion (i), namely that if the frequencies have the form $\omega_k = \omega p_k / q_k$ with p_k and p_k odd then Assertion (2) of Theorem (3.2) holds; namely, all eigenvalues of (4.1) have a definite parity. By Proposition 4.2, it is enough to prove that Assumption (A) is satisfied. Let indeed $(k_1, \ldots, k_d) \in \mathbb{Z}^d \setminus \{0\}$ be without common divisor and such that $\langle \omega, k \rangle = 0$. Then:

$$\begin{aligned} \frac{p_1}{q_1}k_1 + \ldots + \frac{p_d}{q_d}k_d &= \frac{1}{q_1 \cdots q_d}(p_1 q_2 \cdots q_d k_1 + \ldots + p_d q_1 \cdots q_{d-1} k_d) = \\ &:= \frac{1}{q_1 \cdots q_d}(D_1 k_1 + \ldots + D_d k_d) = 0 \end{aligned}$$

Now the integers $D_k : k = 1, \ldots, d$ are odd; hence the above sum must have an even number of terms. The odd terms are those, and only those, containing an odd k_i; therefore the number of odd k_i must be even. Then the result follows by the above Proposition.

Consider now Assertion (ii) of Theorem 3.3. The only thing left to prove is that the validity of Assumption (A) entails that $\frac{\omega_1}{\omega_2} = \frac{d_1}{d_2}$ where d_1 and d_2 are odd. Suppose indeed $\frac{\omega_1}{\omega_2} = \frac{k_2}{k_1}$ where k_1 is odd and k_2 even, or viceversa. Then $\omega_1 k_1 - \omega_2 k_2 = 0$- However this contradicts Assumption (A) which states that the number $O(k)$ of odd k_i must be even. This concludes the proof of Theorem 3.3.

Corollary 4.2 *Under the conditions of Theorem 3.3 on H_0, assume furthermore that the matrix $\langle \psi_r, W\psi_s \rangle : r, s = 1, \ldots, m_0$ is not identically zero for at least one eigenvalue ℓ_0 of H_0 of multiplicity $m_0 > 1$. Then for $|g| < \frac{\delta}{\|W\|_\infty}$, $H(g)$ has real eigenvalues if and only if p and q are both odd.*

Proof

The sufficiency part is a particular case of Theorem 3.3. As for the necessity, under the present conditions the eigenfunctions have opposite parity. Thefore we can directly apply the argument of [15] and conclude that if p is even and q odd or viceversa $H(g)$ has a pair of complex conjugate eigenvalues near ℓ_0 for $g \in \mathbb{R}$ suitably small.

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