THE ALGEBRO-GEOMETRIC INITIAL VALUE PROBLEM FOR THE ABLOWITZ–LADIK HIERARCHY

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Dedicated with great pleasure to Percy Deift on the occasion of his 60th birthday

Abstract. We discuss the algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data and prove unique solvability globally in time for a set of initial (Dirichlet divisor) data of full measure. To this effect we develop a new algorithm for constructing stationary complex-valued algebro-geometric solutions of the Ablowitz–Ladik hierarchy, which is of independent interest as it solves the inverse algebro-geometric spectral problem for generally non-unitary Ablowitz–Ladik Lax operators, starting from a suitably chosen set of initial divisors of full measure. Combined with an appropriate first-order system of differential equations with respect to time (a substitute for the well-known Dubrovin-type equations), this yields the construction of global algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy.

The generally non-unitary behavior of the underlying Lax operator associated with general coefficients for the Ablowitz–Ladik hierarchy poses a variety of difficulties that, to the best of our knowledge, are successfully overcome here for the first time. Our approach is not confined to the Ablowitz–Ladik hierarchy but applies generally to $1+1$-dimensional completely integrable (discrete and continuous) soliton equations.

1. Introduction

The principal aim of this paper is an explicit construction of unique global solutions of the algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy for a general class of initial data. However, to put this circle of ideas into a proper perspective, we first very briefly recall the origins of this subject: In the mid-seventies, Ablowitz and Ladik, in a series of papers [3]–[6] (see also [1], [2, Sect. 3.2.2], [7, Ch. 3]), used inverse scattering methods to analyze certain integrable differential-difference systems. One of their integrable variants of such systems included a discretization of the celebrated AKNS-ZS system, the pair of coupled nonlinear differential difference equations,

$$\begin{aligned} &-i\alpha_t - (\alpha^+ - 2\alpha + \alpha^-) + \alpha\beta(\alpha^+ + \alpha^-) = 0, \\ &-i\beta_t + (\beta^+ - 2\beta + \beta^-) - \alpha\beta(\beta^+ + \beta^-) = 0 \end{aligned} \tag{1.1}$$

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with $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Here we used the notation $f^{\pm}(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, for complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}}$. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where $\beta = -\overline{\alpha}$, and in the defocusing case, where $\beta = \overline{\alpha}$, (1.1) yields the discrete analog of the nonlinear Schrödinger equation

$$-i\alpha_t + 2\alpha - (1 \pm |\alpha|^2)(\alpha^+ + \alpha^-) = 0. \tag{1.2}$$

Since then there has been an enormous activity in this area and we refer, for instance, to [7, Ch. 3], [21], [25], [26], [27], [29], [30], [33], [35], [36], [37], [41] and the extensive literature cited therein, for developments leading up to current research in this particular area of completely integrable differential-difference systems. Particularly relevant to this paper are algebro-geometric (and periodic) solutions of the AL system (1.1) and its associated hierarchy of integrable equations. The first systematic and detailed treatment of algebro-geometric solutions of the AL system (1.1) was performed by Miller, Ercolani, Krichever, and Levermore [33] (see also [9], [10], [13], [14], [32], [43]). Algebro-geometric solutions of the AL hierarchy were discussed in great detail in [26] (see also [19], [20], [44]). In addition to these developments within integrable systems and their applications to fields such as nonlinear optics, the study of AL systems recently gained considerable momentum due to its connections with the theory of orthogonal polynomials. Especially, the particular defocusing case $\beta = \overline{\alpha}$ and the associated CMV matrices and orthogonal polynomials on the unit circle attracted great interest. In this context we refer the interested reader to the two-volume treatise by Simon [39] (see also [40]) and the survey by Deift [16] and the detailed reference lists therein.

Returning to the principal scope of this paper, we intend to describe a solution of the following problem: Given $p \in \mathbb{N}_0$, assume $\alpha^{(0)}, \beta^{(0)}$ to be solutions of the pth stationary Ablowitz–Ladik system s-$\mathrm{AL}_p(\alpha,\beta) = 0$ associated with a prescribed hyperelliptic curve $\mathcal{K}_{2p-1}$ of genus $2p-1$ (with nonsingular affine part) and let $r \in \mathbb{N}_0$; we want to construct unique global solutions $\alpha = \alpha(t_r), \beta = \beta(t_r)$ of the rth AL flow $\mathrm{AL}_r(\alpha,\beta) = 0$ with $\alpha(t_{0,r}) = \alpha^{(0)}, \beta(t_{0,r}) = \beta^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. Thus, we seek a unique global solution of the initial value problem

$$\begin{aligned} &\mathrm{AL}_r(\alpha,\beta) = 0, \\ &(\alpha,\beta)\big|_{t_r = t_{0,r}} = \big(\alpha^{(0)}, \beta^{(0)}\big), \end{aligned} \tag{1.3}$$

$$\text{s-AL}_p\big(\alpha^{(0)}, \beta^{(0)}\big) = 0 \tag{1.4}$$

for some $t_{0,r} \in \mathbb{R}$, $p, r \in \mathbb{N}_0$, where $\alpha = \alpha(n,t_r), \beta = \beta(n,t_r)$ satisfy (2.2).

In the special case of a unitary Lax operator L (cf. (2.57) for the definition of L) in the defocusing case, where $\beta = \overline{\alpha}$, the actual solution of this algebro-geometric initial value problem consists of the following two-step procedure familiar from other soliton equations (particularly familiar in the case of real-valued algebro-geometric solutions of the Toda hierarchy, see, e.g., [15], [23, Sect. 1.3], [42, Sect. 8.3] and the extensive literature cited therein):[1]

(i) An algorithm that constructs admissible (cf. Section 4) nonspecial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)} \in \mathrm{Sym}^{2p-1}\mathcal{K}_{2p-1}$ in spectral gaps of L on the unit circle for all $n \in \mathbb{Z}$, starting from an initial Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathrm{Sym}^{2p-1}\mathcal{K}_{2p-1}$ in an appropriate

[1] We freely use the notation of divisors of degree $2p-1$ as introduced in Appendix A (see also the beginning of Section 3).

position in spectral gaps of L on the unit circle. "Trace formulas" of the type (3.30) and (3.31) (the latter requires prior construction of the Neumann divisor $\mathcal{D}_{\underline{\hat{\nu}}}$ from the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$, though, cf. (4.3)) then construct the stationary real-valued solutions $\alpha^{(0)}, \beta^{(0)}$ of s-$\mathrm{AL}_p(\alpha, \beta) = 0$.

(ii) The first-order Dubrovin-type system of differential equations (5.42), augmented by the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} = \mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ (cf. step (i)) together with the analogous "trace formulas" (3.30), (3.31) in the time-dependent context then yield unique global real-valued solutions $\alpha = \alpha(t_r), \beta = \beta(t_r)$ of the rth AL flow $\mathrm{AL}_r(\alpha, \beta) = 0$ satisfying $\alpha(t_{0,r}) = \alpha^{(0)}, \beta(t_{0,r}) = \beta^{(0)}$.

This approach works perfectly in the special unitary case where the Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)} \in \mathrm{Sym}^{2p-1} \mathcal{K}_{2p-1}$, $\underline{\hat{\mu}}(n, t_r) = (\hat{\mu}_1(n, t_r), \dots, \hat{\mu}_{2p-1}(n, t_r))$, $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$, yield Dirichlet eigenvalues $\mu_1, \dots, \mu_{2p-1}$ of the Lax operator L situated in $2p - 1$ (out of a total number of $2p$) different spectral gaps of L on the unit circle. In particular, for fixed $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$, the Dirichlet eigenvalues $\mu_j(n, t_r)$, $j = 1, \dots, 2p-1$, are pairwise distinct and formula (5.42) for μ_{j,t_r}, $j = 1, \dots, 2p-1$, is well-defined. The analogous considerations apply to the Neumann divisors $\mathcal{D}_{\underline{\hat{\nu}}} \in \mathrm{Sym}^{2p-1} \mathcal{K}_{2p-1}$.

This situation changes drastically if general initial data $\alpha^{(0)}, \beta^{(0)}$ or divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$ in general complex position with projections off the unit circle in the general case $\beta \neq \overline{\alpha}$ (where the spectrum of L is no longer confined to a subset of the unit circle) are permitted. In this case the Dirichlet eigenvalues μ_j, $j = 1, \dots, 2p-1$, are no longer confined to well separated spectral gaps of L on the unit circle and, in particular, they are in general no longer pairwise distinct and "collisions" between them can occur at certain values of $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$. Thus, the stationary algorithm in step (i) as well as the Dubrovin-type first-order system of differential equations (5.42) in step (ii) above, breaks down at such values of (n, t_r). A priori, one has no control over such collisions, especially, it is not possible to identify initial conditions $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$ at some $(n_0, t_{0,r}) \in \mathbb{Z} \times \mathbb{R}$ which avoid collisions for all $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$. We solve this problem head on by explicitly permitting collisions in the stationary as well as time-dependent context from the outset. In the stationary context, we properly modify the algorithm described above in step (i) in the unitary case by alluding to a more general interpolation formalism (cf. Appendix B) for polynomials, going beyond the usual Lagrange interpolation formulas. In the time-dependent context we replace the first-order system of Dubrovin-type equations (5.42), augmented with the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$, by a different first-order system of differential equations (6.14), (6.22), and (6.23) with initial conditions (6.24) which focuses on symmetric functions of $\mu_1(n, t_r), \dots, \mu_{2p-1}(n, t_r)$ rather than individual Dirichlet eigenvalues $\mu_j(n, t_r)$, $j = 1, \dots, 2p - 1$. In this manner it will be shown that collisions of Dirichlet eigenvalues no longer pose a problem.

In addition, there is a second nontrivial complication in the non-unitary case: Since the Dirichlet eigenvalues $\mu_j(n, t_r)$, $j = 1, \dots, 2p - 1$, are no longer confined to spectral gaps of L on the unit circle as (n, t_r) vary in $\mathbb{Z} \times \mathbb{R}$, it can no longer be guaranteed that $\mu_j(n, t_r)$ and $\nu_j(n, t_r)$, $j = 1, \dots, 2p - 1$, stay finite and nonzero for all $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$. We solve this particular problem in the stationary as well as time-dependent case by properly restricting the initial Dirichlet and Neumann divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}, \mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,r})} \in \mathrm{Sym}^{2p-1} \mathcal{K}_{2p-1}$ to a dense set of full measure.

Summing up, we offer a new algorithm to solve the inverse algebro-geometric spectral problem for general (unitary and non-unitary) Ablowitz–Ladik Lax operators, starting from a properly chosen dense set of initial divisors of full measure. Combined with an appropriate first-order system of differential equations with respect to time (a substitute for the Dubrovin-type equations), this yields the construction of global algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy.

We emphasize that the approach described in this paper is not limited to the Ablowitz–Ladik hierarchy but applies universally to constructing algebro-geometric solutions of $1+1$-dimensional integrable soliton equations. In particular, it applies to the Toda lattice hierarchy as discussed in [24]. Moreover, the principal idea of replacing Dubrovin-type equations by a first-order system of the type (6.14), (6.22), and (6.23) is also relevant in the context of general non-unitary Lax operators for the continuous models in $1+1$-dimensions. (In particular, the models studied in detail in [22] can be revisited from this point of view.) We also note that while the periodic case with complex-valued α, β is of course included in our analysis, we throughout consider the more general algebro-geometric case (in which α, β need not be quasi-periodic).

Finally we briefly describe the content of each section. Section 2 presents a quick summary of the basics of the Ablowitz–Ladik hierarchy, its recursive construction, Lax pairs, and zero-curvature equations. The stationary algebro-geometric Ablowitz–Ladik hierarchy solutions, the underlying hyperelliptic curve, trace formulas, etc., are the subject of Section 3. A new algorithm solving the algebro-geometric inverse spectral problem for generally non-unitary Ablowitz–Ladik Lax operators is presented in Section 4. In Section 5 we briefly summarize the properties of algebro-geometric time-dependent solutions of the Ablowitz–Ladik hierarchy and formulate the algebro-geometric initial value problem. Uniqueness and existence of global solutions of the algebro-geometric initial value problem as well as their explicit construction are then presented in our final and principal Section 6. Appendix A reviews the basics of hyperelliptic Riemann surfaces of the Ablowitz–Ladik-type and sets the stage of much of the notation used in this paper. Finally, various interpolation formulas of fundamental importance to our stationary inverse spectral algorithm developed in Section 4 are summarized in Appendix B. These appendices support our intention to make this paper reasonably self-contained.

2. The Ablowitz–Ladik Hierarchy in a Nutshell

We briefly review the recursive construction of the Ablowitz–Ladik hierarchy and zero-curvature equations following [25] and [27].

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that α, β satisfy*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0,1\},\ n \in \mathbb{Z}. \tag{2.1}$$

In the time-dependent case we assume that α, β satisfy

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,), \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{2.2}$$

Here $\mathbb{C}^{\mathbb{Z}}$ denotes the set of complex-valued sequences indexed by $\mathbb{Z}$.

We denote by $S^{\pm}$ the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ according to

$$(S^{\pm}f)(n) = f(n\pm 1), \quad n\in\mathbb{Z}. \tag{2.3}$$

Moreover, we will frequently use the notation

$$f^{\pm} = S^{\pm}f, \quad f\in\mathbb{C}^{\mathbb{Z}}. \tag{2.4}$$

Next, we define sequences $\{f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ recursively by

$$g_{0,+} = \tfrac{1}{2}c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^{+}, \quad h_{0,+} = c_{0,+}\beta, \tag{2.5}$$
$$g_{\ell+1,+} - g_{\ell+1,+}^{-} = \alpha h_{\ell,+}^{-} + \beta f_{\ell,+}, \quad \ell\in\mathbb{N}_0, \tag{2.6}$$
$$f_{\ell+1,+}^{-} = f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^{-}), \quad \ell\in\mathbb{N}_0, \tag{2.7}$$
$$h_{\ell+1,+} = h_{\ell,+}^{-} + \beta(g_{\ell+1,+} + g_{\ell+1,+}^{-}), \quad \ell\in\mathbb{N}_0, \tag{2.8}$$

and

$$g_{0,-} = \tfrac{1}{2}c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^{+}, \tag{2.9}$$
$$g_{\ell+1,-} - g_{\ell+1,-}^{-} = \alpha h_{\ell,-} + \beta f_{\ell,-}^{-}, \quad \ell\in\mathbb{N}_0, \tag{2.10}$$
$$f_{\ell+1,-} = f_{\ell,-}^{-} + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^{-}), \quad \ell\in\mathbb{N}_0, \tag{2.11}$$
$$h_{\ell+1,-}^{-} = h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^{-}), \quad \ell\in\mathbb{N}_0. \tag{2.12}$$

Here $c_{0,\pm}\in\mathbb{C}$ are given constants. For later use we also introduce

$$f_{-1,\pm} = h_{-1,\pm} = 0. \tag{2.13}$$

Remark 2.2. *The sequences $\{f_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,+}\}_{\ell\in\mathbb{N}_0}$ can be computed recursively as follows: Assume that $f_{\ell,+}$, $g_{\ell,+}$, and $h_{\ell,+}$ are known. Equation* (2.6) *is a first-order difference equation in $g_{\ell+1,+}$ that can be solved directly and yields a local lattice function that is determined up to a new constant denoted by $c_{\ell+1,+}\in\mathbb{C}$. Relations* (2.7) *and* (2.8) *then determine $f_{\ell+1,+}$ and $h_{\ell+1,+}$, etc. The sequences $\{f_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,-}\}_{\ell\in\mathbb{N}_0}$ are determined similarly.*

Upon setting

$$\gamma = 1-\alpha\beta, \tag{2.14}$$

one explicitly obtains

$$\begin{aligned}
&f_{0,+} = c_{0,+}(-\alpha^{+}), \quad f_{1,+} = c_{0,+}\big(-\gamma^{+}\alpha^{++} + (\alpha^{+})^2\beta\big) + c_{1,+}(-\alpha^{+}),\\
&g_{0,+} = \tfrac{1}{2}c_{0,+}, \quad g_{1,+} = c_{0,+}(-\alpha^{+}\beta) + \tfrac{1}{2}c_{1,+},\\
&h_{0,+} = c_{0,+}\beta, \quad h_{1,+} = c_{0,+}\big(\gamma\beta^{-} - \alpha^{+}\beta^{2}\big) + c_{1,+}\beta,\\
&f_{0,-} = c_{0,-}\alpha, \quad f_{1,-} = c_{0,-}\big(\gamma\alpha^{-} - \alpha^{2}\beta^{+}\big) + c_{1,-}\alpha,\\
&g_{0,-} = \tfrac{1}{2}c_{0,-}, \quad g_{1,-} = c_{0,-}(-\alpha\beta^{+}) + \tfrac{1}{2}c_{1,-},\\
&h_{0,-} = c_{0,-}(-\beta^{+}), \quad h_{1,-} = c_{0,-}\big(-\gamma^{+}\beta^{++} + \alpha(\beta^{+})^2\big) + c_{1,-}(-\beta^{+}), \text{ etc.}
\end{aligned} \tag{2.15}$$

Here $\{c_{\ell,\pm}\}_{\ell\in\mathbb{N}}$ denote summation constants which naturally arise when solving the difference equations for $g_{\ell,\pm}$ in (2.6), (2.10). Subsequently, it will also be useful to work with the corresponding homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, defined by the vanishing of all summation constants $c_{k,\pm}$ for $k=1,\dots,\ell$, and choosing $c_{0,\pm}=1$,

$$\hat{f}_{0,+} = -\alpha^{+}, \quad \hat{f}_{0,-} = \alpha, \quad \hat{f}_{\ell,\pm} = f_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}, \tag{2.16}$$

$$\hat g_{0,\pm} = \tfrac{1}{2}, \quad \hat g_{\ell,\pm} = g_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.17}$$
$$\hat h_{0,+} = \beta, \quad \hat h_{0,-} = -\beta^+, \quad \hat h_{\ell,\pm} = h_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{2.18}$$

By induction one infers that

$$f_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat f_{k,\pm}, \quad g_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat g_{k,\pm}, \quad h_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat h_{k,\pm}. \tag{2.19}$$

Next we define the 2×2 zero-curvature matrices

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix} \tag{2.20}$$

and

$$V_p(z) = i \begin{pmatrix} G_p^-(z) & -F_p^-(z) \\ H_p^-(z) & -K_p^-(z) \end{pmatrix}, \quad p \in \mathbb{N}_0, \tag{2.21}$$

for appropriate Laurent polynomials $F_p(z)$, $G_p(z)$, $H_p(z)$, and $K_p(z)$ in the spectral parameter $z \in \mathbb{C}\setminus\{0\}$ to be determined shortly. By postulating the stationary zero-curvature relation,

$$0 = UV_p - V_p^+ U, \tag{2.22}$$

one concludes that (2.22) is equivalent to the following relations

$$z(G_p^- - G_p) + z\beta F_p + \alpha H_p^- = 0, \tag{2.23}$$
$$z\beta F_p^- + \alpha H_p - K_p + K_p^- = 0, \tag{2.24}$$
$$-F_p + zF_p^- + \alpha(G_p + K_p^-) = 0, \tag{2.25}$$
$$z\beta(G_p^- + K_p) - zH_p + H_p^- = 0. \tag{2.26}$$

In order to make the connection between the zero-curvature formalism and the recursion relations (2.5)–(2.12), we now define Laurent polynomials F_p, G_p, H_p, and K_p by

$$F_p(z) = \sum_{\ell=1}^{p} f_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+} z^{\ell}, \tag{2.27}$$
$$G_p(z) = \sum_{\ell=1}^{p} g_{p-\ell,-} z^{-\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+} z^{\ell}, \tag{2.28}$$
$$H_p(z) = \sum_{\ell=0}^{p-1} h_{p-1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+} z^{\ell}, \tag{2.29}$$
$$K_p(z) = \sum_{\ell=0}^{p} g_{p-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p} g_{p-\ell,+} z^{\ell} = G_p(z) + g_{p,-} - g_{p,+}. \tag{2.30}$$

The stationary zero-curvature relation (2.22), $0 = UV_p - V_p^+ U$, is then equivalent to

$$-\alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- = 0, \tag{2.31}$$
$$\beta(g_{p,+}^- + g_{p,-}) + h_{p-1,+}^- - h_{p-1,-} = 0. \tag{2.32}$$

Thus, varying $p \in \mathbb{N}_0$, equations (2.31) and (2.32) give rise to the stationary Ablowitz–Ladik (AL) hierarchy which we introduce as follows

$$\text{s-AL}_p(\alpha,\beta) = \begin{pmatrix} -\alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ \beta(g_{p,+}^- + g_{p,-}) + h_{p-1,+}^- - h_{p-1,-} \end{pmatrix} = 0, \quad p \in \mathbb{N}_0. \tag{2.33}$$

Explicitly (recalling $\gamma = 1 - \alpha\beta$),

$$\begin{aligned}
\text{s-AL}_0(\alpha,\beta) &= \begin{pmatrix} -c_0\alpha \\ c_0\beta \end{pmatrix} = 0, \\
\text{s-AL}_1(\alpha,\beta) &= \begin{pmatrix} -\gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_1\alpha \\ \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_1\beta \end{pmatrix} = 0, \\
\text{s-AL}_2(\alpha,\beta) &= \begin{pmatrix} -\gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix} \\
&\quad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_2\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_2\beta \end{pmatrix} = 0, \text{ etc.,}
\end{aligned} \tag{2.34}$$

represent the first few equations of the stationary Ablowitz–Ladik hierarchy. Here we introduced

$$c_p = (c_{p,+} + c_{p,-})/2, \quad p \in \mathbb{N}_0. \tag{2.35}$$

By definition, the set of solutions of (2.33), with p ranging in $\mathbb{N}_0$ and $c_{\ell,\pm} \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, represents the class of algebro-geometric Ablowitz–Ladik solutions.

In the following we will frequently assume that α, β satisfy the pth stationary Ablowitz–Ladik equations supposing a particular choice of summation constants $c_{\ell,\pm} \in \mathbb{C}$, $\ell = 0, \dots, p$, $p \in \mathbb{N}$, has been made.

In accordance with our notation introduced in (2.16)–(2.18), the corresponding homogeneous stationary Ablowitz–Ladik equations are defined by

$$\text{s-}\widehat{\text{AL}}_p(\alpha,\beta) = \text{s-AL}_p(\alpha,\beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p}, \quad p \in \mathbb{N}_0. \tag{2.36}$$

One can show (cf. [25]) that $g_{p,+} = g_{p,-}$ up to a lattice constant which can be set equal to zero without loss of generality. Thus, we will henceforth assume that

$$g_{p,+} = g_{p,-}, \tag{2.37}$$

which in turn implies that

$$K_p = G_p \tag{2.38}$$

and hence renders V_p in (2.21) traceless in the stationary context. (We note that equations (2.37) and (2.38) cease to be valid in the time-dependent context, though.)

Next, still assuming (2.1) and taking into account (2.38), one infers by taking determinants in the stationary zero-curvature equation (2.22) that the quantity

$$R_{2p} = G_p^2 - F_p H_p \tag{2.39}$$

is a lattice constant, that is, $R_{2p} - R_{2p}^- = 0$. Hence, $R_{2p}(z)$ only depends on z, and assuming in addition to (2.1) that

$$c_{0,\pm} \in \mathbb{C} \setminus \{0\}, \tag{2.40}$$

one may write R_{2p} as[2]

$$R_{2p}(z) = (c_{0,+}^2/4) z^{-2p} \prod_{m=0}^{4p-1} (z - E_m), \quad \{E_m\}_{m=0}^{4p-1} \subset \mathbb{C} \setminus \{0\}. \tag{2.41}$$

In addition, we note that the summation constants $c_{1,\pm}, \dots, c_{p,\pm}$ in (2.33) can be expressed as symmetric functions in the zeros $E_0, \dots, E_{4p-1}$ of the associated Laurent polynomial R_{2p} in (2.41). In fact, one can prove (cf. [25]) that

$$c_{\ell,\pm} = c_{0,\pm} c_\ell\big(\underline{E}^{\pm 1}\big), \quad \ell = 0, \dots, p, \tag{2.42}$$

where

$$\begin{aligned}
& c_0\big(\underline{E}^{\pm 1}\big) = 1, \\
& c_k\big(\underline{E}^{\pm 1}\big) \\
& = - \sum_{\substack{j_0, \dots, j_{4p-1}=0 \\ j_0 + \dots + j_{4p-1} = k}}^{k} \frac{(2j_0)! \cdots (2j_{4p-1})!}{2^{2k} (j_0!)^2 \cdots (j_{4p-1}!)^2 (2j_0 - 1) \cdots (2j_{4p-1} - 1)} E_0^{\pm j_0} \cdots E_{4p-1}^{\pm j_{4p-1}}, \\
& \hspace{25em} k = 1, \dots, p,
\end{aligned} \tag{2.43}$$

are symmetric functions of $\underline{E}^{\pm 1} = (E_0^{\pm 1}, \dots, E_{4p-1}^{\pm 1})$.

Next we turn to the time-dependent Ablowitz–Ladik hierarchy. For that purpose the coefficients α and β are now considered as functions of both the lattice point and time. For each equation in the hierarchy, that is, for each p, we introduce a deformation (time) parameter $t_p \in \mathbb{R}$ in α, β, replacing $\alpha(n), \beta(n)$ by $\alpha(n, t_p), \beta(n, t_p)$. Moreover, the definitions (2.20), (2.21), and (2.27)–(2.30) of U, V_p, and F_p, G_p, H_p, K_p, respectively, still apply. Imposing the zero-curvature relation

$$U_{t_p} + U V_p - V_p^+ U = 0, \quad p \in \mathbb{N}_0, \tag{2.44}$$

then results in the equations

$$\begin{aligned}
\alpha_{t_p} &= i\big(z F_p^- + \alpha(G_p + K_p^-) - F_p\big), && (2.45)\\
\beta_{t_p} &= -i\big(\beta(G_p^- + K_p) - H_p + z^{-1} H_p^-\big), && (2.46)\\
0 &= z(G_p^- - G_p) + z\beta F_p + \alpha H_p^-, && (2.47)\\
0 &= z\beta F_p^- + \alpha H_p + K_p^- - K_p. && (2.48)
\end{aligned}$$

Varying $p \in \mathbb{N}_0$, the collection of evolution equations

$$\begin{aligned}
\mathrm{AL}_p(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_p} - \alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ -i\beta_{t_p} + \beta(g_{p,+}^- + g_{p,-}) - h_{p-1,-} + h_{p-1,+}^- \end{pmatrix} = 0, \\
(n, t_p) \in \mathbb{Z} \times \mathbb{R}, \ p \in \mathbb{N}_0,
\end{aligned} \tag{2.49}$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly,

$$\begin{aligned}
& \mathrm{AL}_0(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_0} - c_0 \alpha \\ -i\beta_{t_0} + c_0 \beta \end{pmatrix} = 0, \\
& \mathrm{AL}_1(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_1} - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_1 \alpha \\ -i\beta_{t_1} + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_1 \beta \end{pmatrix} = 0,
\end{aligned}$$

[2] We use the convention that a product over an empty index set is to be interpreted equal to 1.

$$\mathrm{AL}_2(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_2} - \gamma\big(c_{0,+}\alpha^{++}\gamma^{+} + c_{0,-}\alpha^{--}\gamma^{-} - \alpha(c_{0,+}\alpha^{+}\beta^{-} + c_{0,-}\alpha^{-}\beta^{+}) \\ -\beta(c_{0,-}(\alpha^{-})^2 + c_{0,+}(\alpha^{+})^2)\big) \\ -i\beta_{t_2} + \gamma\big(c_{0,-}\beta^{++}\gamma^{+} + c_{0,+}\beta^{--}\gamma^{-} - \beta(c_{0,+}\alpha^{+}\beta^{-} + c_{0,-}\alpha^{-}\beta^{+}) \\ -\alpha(c_{0,+}(\beta^{-})^2 + c_{0,-}(\beta^{+})^2)\big) \end{pmatrix}$$
$$+ \begin{pmatrix} -\gamma(c_{1,-}\alpha^{-} + c_{1,+}\alpha^{+}) - c_2\alpha \\ \gamma(c_{1,+}\beta^{-} + c_{1,-}\beta^{+}) + c_2\beta \end{pmatrix} = 0, \text{ etc.,} \tag{2.50}$$

represent the first few equations of the time-dependent Ablowitz–Ladik hierarchy.

By (2.49), (2.6), and (2.10), the time derivative of $\gamma = 1 - \alpha\beta$ is given by

$$\gamma_{t_p} = i\gamma\big((g_{p,+} - g_{p,+}^{-}) - (g_{p,-} - g_{p,-}^{-})\big). \tag{2.51}$$

The corresponding homogeneous equations are then defined by

$$\widehat{\mathrm{AL}}_p(\alpha,\beta) = \mathrm{AL}_p(\alpha,\beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p} = 0, \quad p \in \mathbb{N}_0. \tag{2.52}$$

The special case $p = 1$, that is,

$$\widehat{\mathrm{AL}}_1(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_1} - \gamma(\alpha^{-} + \alpha^{+}) \\ -i\beta_{t_1} + \gamma(\beta^{-} + \beta^{+}) \end{pmatrix} = 0, \tag{2.53}$$

represents *the* Ablowitz–Ladik equation.

Remark 2.3. *From* (2.23)–(2.26) *and the explicit computations of the coefficients* $f_{\ell,\pm}$, $g_{\ell,\pm}$, *and* $h_{\ell,\pm}$, *one concludes that the zero-curvature equation* (2.44) *and hence the Ablowitz–Ladik hierarchy is invariant under the scaling transformation*

$$\alpha \to \alpha_c = \{c\,\alpha(n)\}_{n\in\mathbb{Z}}, \quad \beta \to \beta_c = \{\beta(n)/c\}_{n\in\mathbb{Z}}, \quad c \in \mathbb{C}\setminus\{0\}. \tag{2.54}$$

In particular, solutions α, β *of the stationary and time-dependent AL equations are determined only up to a multiplicative constant.*

Finally, we briefly recall the Lax pair (P_p, L) for the Ablowitz–Ladik hierarchy and refer to [27] for detailed discussions of this topic.

In the standard basis $\{\delta_m\}_{m\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ defined by

$$\delta_m = \{\delta_{m,n}\}_{n\in\mathbb{Z}}, \; m \in \mathbb{Z}, \quad \delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases} \tag{2.55}$$

the underlying Lax difference expression L is given by

$$L = \left(\begin{array}{cccccccc} \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & -\alpha(0)\rho(-1) & -\beta(-1)\alpha(0) & -\alpha(1)\rho(0) & \rho(0)\rho(1) & & & \\ & \rho(-1)\rho(0) & \beta(-1)\rho(0) & -\beta(0)\alpha(1) & \beta(0)\rho(1) & 0 & & \\ & & 0 & -\alpha(2)\rho(1) & -\beta(1)\alpha(2) & -\alpha(3)\rho(2) & \rho(2)\rho(3) & \\ & 0 & & \rho(1)\rho(2) & \beta(1)\rho(2) & -\beta(2)\alpha(3) & \beta(2)\rho(3) & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \tag{2.56}$$

$$= \rho^{-}\rho\,\delta_{\mathrm{even}}\,S^{--} + (\beta^{-}\rho\,\delta_{\mathrm{even}} - \alpha^{+}\rho\,\delta_{\mathrm{odd}})S^{-} - \beta\alpha^{+}$$
$$+ (\beta\rho^{+}\,\delta_{\mathrm{even}} - \alpha^{++}\rho^{+}\,\delta_{\mathrm{odd}})S^{+} + \rho^{+}\rho^{++}\,\delta_{\mathrm{odd}}\,S^{++}, \tag{2.57}$$

where δ_{even} and δ_{odd} denote the characteristic functions of the even and odd integers,

$$\delta_{\mathrm{even}} = \chi_{2\mathbb{Z}}, \quad \delta_{\mathrm{odd}} = 1 - \delta_{\mathrm{even}} = \chi_{2\mathbb{Z}+1}. \tag{2.58}$$

In particular, terms of the form $-\beta(n)\alpha(n+1)$ represent the diagonal (n,n)-entries, $n \in \mathbb{Z}$, in the infinite matrix (2.56). In addition, we used the abbreviation

$$\rho = \gamma^{1/2} = (1-\alpha\beta)^{1/2}. \tag{2.59}$$

Next, let T be a bounded operator in $\ell^2(\mathbb{Z})$. Given the standard basis (2.55) in $\ell^2(\mathbb{Z})$, we represent T by

$$T = \big(T(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T(m,n) = (\delta_m, T\,\delta_n)_{\ell^2(\mathbb{Z})}, \quad (m,n) \in \mathbb{Z}^2. \tag{2.60}$$

Actually, for our purpose below, it is sufficient that T is an N-diagonal matrix for some $N \in \mathbb{N}$. Moreover, we introduce the upper and lower triangular parts $T_\pm$ of T by

$$T_\pm = \big(T_\pm(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T_\pm(m,n) = \begin{cases} T(m,n), & \pm(n-m) > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.61}$$

Then, the finite difference expression P_p is given by

$$P_p = \frac{i}{2}\sum_{j=1}^{p}\Big(c_{p-j,+}\big((L^j)_+ - (L^j)_-\big) - c_{p-j,-}\big((L^{-j})_+ - (L^{-j})_-\big)\Big) - \frac{i}{2}c_p\, Q_d, \quad p \in \mathbb{N}_0, \tag{2.62}$$

with Q_d denoting the doubly infinite diagonal matrix

$$Q_d = \big((-1)^k \delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}}. \tag{2.63}$$

The commutator relations $[P_p, L] = 0$ and $L_{t_p} - [P_p, L] = 0$ are then equivalent to the stationary and time-dependent Ablowitz–Ladik equations (2.33) and (2.49), respectively.

3. Properties of Stationary Algebro-Geometric Solutions of the Ablowitz–Ladik Hierarchy

In this section we present a quick review of properties of algebro-geometric solutions of the stationary Ablowitz–Ladik hierarchy. We refer to [25] and [26] for detailed presentations.

We recall the hyperelliptic curve $\mathcal{K}_{2p-1}$ of genus $2p-1$, where

$$\begin{aligned} &\mathcal{K}_{2p-1} \colon \mathcal{F}_{2p-1}(z,y) = y^2 - 4c_{0,+}^{-2} z^{2p} R_{2p}(z) = 0, \\ &R_{2p}(z) = \left(\frac{c_{0,+}}{2z^p}\right)^2 \prod_{m=0}^{4p-1}(z-E_m), \quad \{E_m\}_{m=0}^{4p-1} \subset \mathbb{C}\setminus\{0\}. \end{aligned} \tag{3.1}$$

Throughout this section we make the assumption:

Hypothesis 3.1. *Suppose that*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}} \text{ and } \alpha(n)\beta(n) \notin \{0,1\} \text{ for all } n \in \mathbb{Z}. \tag{3.2}$$

In addition, assume that the affine part of the hyperelliptic curve $\mathcal{K}_{2p-1}$ in (3.1) is nonsingular, that is, suppose that

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 4p-1. \tag{3.3}$$

The curve $\mathcal{K}_{2p-1}$ is compactified by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_{2p-1}$. Points P on $\mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_{2p-1}$ satisfying $\mathcal{F}_{2p-1}(z, y) = 0$. The complex structure on $\mathcal{K}_{2p-1}$ is then defined in the usual way, see Appendix A. Hence, $\mathcal{K}_{2p-1}$ becomes a two-sheeted hyperelliptic Riemann surface of genus $2p-1$ in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_{2p-1}$ (i.e., by fixing $E_0, \dots, E_{4p-1}$), the summation constants $c_{1,\pm}, \dots, c_{p,\pm}$ in $f_{p,\pm}$, $g_{p,\pm}$, and $h_{p,\pm}$ (and hence in the corresponding stationary s-AL$_p$ equations) are uniquely determined as is clear from (2.43) which establishes the summation constants $c_{\ell,\pm}$ as symmetric functions of $E_0^{\pm 1}, \dots, E_{4p-1}^{\pm 1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$.

We denote by $\{\mu_j(n)\}_{j=1,\dots,2p-1}$ and $\{\nu_j(n)\}_{j=1,\dots,2p-1}$ the zeros of $(\,\cdot\,)^p F_p(\,\cdot\,, n)$ and $(\,\cdot\,)^{p-1} H_p(\,\cdot\,, n)$, respectively. Thus, we may write

$$F_p(z) = -c_{0,+}\alpha^+ z^{-p} \prod_{j=1}^{2p-1} (z - \mu_j), \tag{3.4}$$

$$H_p(z) = c_{0,+}\beta z^{-p+1} \prod_{j=1}^{2p-1} (z - \nu_j), \tag{3.5}$$

and we recall that (cf. (2.39))

$$R_{2p} - G_p^2 = -F_p H_p. \tag{3.6}$$

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_j from the complex plane $\mathbb{C}$ to the curve $\mathcal{K}_{2p-1}$. From (3.6) one infers that

$$R_{2p}(z) - G_p(z)^2 = 0, \quad z \in \{\mu_j, \nu_k\}_{j,k=1,\dots,2p-1}. \tag{3.7}$$

We now introduce $\{\hat{\mu}_j\}_{j=1,\dots,2p-1} \subset \mathcal{K}_{2p-1}$ and $\{\hat{\nu}_j\}_{j=1,\dots,2p-1} \subset \mathcal{K}_{2p-1}$ by

$$\hat{\mu}_j(n) = (\mu_j(n), (2/c_{0,+})\mu_j(n)^p G_p(\mu_j(n), n)), \quad j = 1, \dots, 2p-1, \; n \in \mathbb{Z}, \tag{3.8}$$

and

$$\hat{\nu}_j(n) = (\nu_j(n), -(2/c_{0,+})\nu_j(n)^p G_p(\nu_j(n), n)), \quad j = 1, \dots, 2p-1, \; n \in \mathbb{Z}. \tag{3.9}$$

We also introduce the points $P_{0,\pm}$ by

$$P_{0,\pm} = (0, \pm(c_{0,-}/c_{0,+})) \in \mathcal{K}_{2p-1}, \quad \frac{c_{0,-}^2}{c_{0,+}^2} = \prod_{m=0}^{4p-1} E_m. \tag{3.10}$$

We emphasize that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_{2p-1}$. Moreover,

$$y(P) \underset{\zeta \to 0}{=} \begin{cases} \mp\zeta^{-2p}(1 + O(\zeta)), & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm(c_{0,-}/c_{0,+}) + O(\zeta), & P \to P_{0,\pm}, \quad \zeta = z. \end{cases} \tag{3.11}$$

Next we introduce the fundamental meromorphic function on $\mathcal{K}_{2p-1}$ by

$$\phi(P, n) = \frac{(c_{0,+}/2)z^{-p}y + G_p(z, n)}{F_p(z, n)} \tag{3.12}$$

$$= \frac{-H_p(z, n)}{(c_{0,+}/2)z^{-p}y - G_p(z, n)}, \tag{3.13}$$

$$P = (z, y) \in \mathcal{K}_{2p-1}, \; n \in \mathbb{Z},$$

with divisor $(\phi(\,\cdot\,,n))$ of $\phi(\,\cdot\,,n)$ given by

$$(\phi(\,\cdot\,,n)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}, \tag{3.14}$$

using (3.4) and (3.5). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_{2p-1}\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_{2p-1}\} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}). \tag{3.15}$$

For brevity, and in close analogy to the Toda hierarchy, we will frequently refer to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$ as the Dirichlet and Neumann divisors, respectively.

Given $\phi(\,\cdot\,,n)$, the meromorphic stationary Baker–Akhiezer vector $\Psi(\,\cdot\,,n,n_0)$ on $\mathcal{K}_{2p-1}$ is then defined by

$$\begin{aligned}
\Psi(P,n,n_0) &= \begin{pmatrix} \psi_1(P,n,n_0) \\ \psi_2(P,n,n_0) \end{pmatrix}, \\
\psi_1(P,n,n_0) &= \begin{cases} \prod_{n'=n_0+1}^{n} \big(z+\alpha(n')\phi^-(P,n')\big), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z+\alpha(n')\phi^-(P,n')\big)^{-1}, & n \leq n_0-1, \end{cases}
\end{aligned} \tag{3.16}$$

$$\psi_2(P,n,n_0) = \phi(P,n_0) \begin{cases} \prod_{n'=n_0+1}^{n} \big(z\beta(n')\phi^-(z,n')^{-1}+1\big), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z\beta(n')\phi^-(z,n')^{-1}+1\big)^{-1}, & n \leq n_0-1. \end{cases} \tag{3.17}$$

Basic properties of ϕ and Ψ are summarized in the following result.

Lemma 3.2 ([25])**.** *Suppose that α, β satisfy* (3.2) *and the pth stationary Ablowitz–Ladik system* (2.33)*. Moreover, assume* (3.1) *and* (3.3) *and let $P=(z,y) \in \mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n,n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{3.18}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_p(z)}{F_p(z)}, \tag{3.19}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_p(z)}{F_p(z)}, \tag{3.20}$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p}\frac{y(P)}{F_p(z)}. \tag{3.21}$$

The vector Ψ satisfies

$$U(z,n)\Psi^-(P,n,n_0) = \Psi(P,n,n_0), \tag{3.22}$$

$$V_p(z,n)\Psi^-(P,n,n_0) = -(i/2)c_{0,+}z^{-p}y\Psi^-(P,n,n_0), \tag{3.23}$$

$$\psi_2(P,n,n_0) = \phi(P,n)\psi_1(P,n,n_0), \tag{3.24}$$

$$\psi_1(P,n,n_0)\psi_1(P^*,n,n_0) = z^{n-n_0}\frac{F_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0), \tag{3.25}$$

$$\psi_2(P,n,n_0)\psi_2(P^*,n,n_0) = z^{n-n_0}\frac{H_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0), \tag{3.26}$$

$$\psi_1(P,n,n_0)\psi_2(P^*,n,n_0) + \psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \tag{3.27}$$

$$\begin{aligned}
&= 2z^{n-n_0}\frac{G_p(z,n)}{F_p(z,n_0)}\Gamma(n,n_0),\\
&\psi_1(P,n,n_0)\psi_2(P^*,n,n_0) - \psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \qquad (3.28)\\
&= -c_{0,+}z^{n-n_0-p}\frac{y}{F_p(z,n_0)}\Gamma(n,n_0),
\end{aligned}$$

where we used the abbreviation

$$\Gamma(n,n_0) = \begin{cases} \prod_{n'=n_0+1}^{n}\gamma(n') & n \geq n_0+1,\\ 1 & n = n_0,\\ \prod_{n'=n+1}^{n_0}\gamma(n')^{-1} & n \leq n_0-1. \end{cases} \qquad (3.29)$$

Combining the Laurent polynomial recursion approach of Section 2 with (3.4) and (3.5) readily yields trace formulas for $f_{\ell,\pm}$ and $h_{\ell,\pm}$ in terms of symmetric functions of the zeros μ_j and ν_k of F_p and H_p, respectively. For simplicity we just record the simplest cases.

Lemma 3.3 ([25]). *Suppose that α,β satisfy* (3.2) *and the pth stationary Ablowitz–Ladik system* (2.33)*. Then,*

$$\frac{\alpha}{\alpha^+} = \prod_{j=1}^{2p-1}\mu_j\bigg(\prod_{m=0}^{4p-1}E_m\bigg)^{-1/2}, \qquad (3.30)$$

$$\frac{\beta^+}{\beta} = \prod_{j=1}^{2p-1}\nu_j\bigg(\prod_{m=0}^{4p-1}E_m\bigg)^{-1/2}, \qquad (3.31)$$

$$\sum_{j=1}^{2p-1}\mu_j = \alpha^+\beta - \gamma^+\frac{\alpha^{++}}{\alpha^+} - \frac{c_{1,+}}{c_{0,+}}, \qquad (3.32)$$

$$\sum_{j=1}^{2p-1}\nu_j = \alpha^+\beta - \gamma\frac{\beta^-}{\beta} - \frac{c_{1,+}}{c_{0,+}}. \qquad (3.33)$$

Since nonspecial divisors and the linearization property of the Abel map when applied to $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ will play a fundamental role later on, we also recall the following facts.

Lemma 3.4 ([25], [26]). *Suppose that α,β satisfy* (3.2) *and the pth stationary Ablowitz–Ladik system* (2.33)*. Moreover, assume* (3.1) *and* (3.3) *and let $n \in \mathbb{Z}$. Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_{2p-1}\}$, and $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\underline{\hat{\nu}}_1,\dots,\underline{\hat{\nu}}_{2p-1}\}$, be the pole and zero divisors of degree $2p-1$, respectively, associated with α, β, and ϕ defined according to* (3.8) *and* (3.9)*, that is,*

$$\begin{aligned}
\hat{\mu}_j(n) &= (\mu_j(n), (2/c_{0,+})\mu_j(n)^p G_p(\mu_j(n),n)), \quad j = 1,\dots,2p-1,\\
\hat{\nu}_j(n) &= (\nu_j(n), -(2/c_{0,+})\nu_j(n)^p G_p(\nu_j(n),n)), \quad j = 1,\dots,2p-1.
\end{aligned} \qquad (3.34)$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial for all $n \in \mathbb{Z}$. Moreover, the Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \qquad (3.35)$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \qquad (3.36)$$

where $Q_0 \in \mathcal{K}_{2p-1}$ is a given base point. In addition,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) + \underline{A}_{P_{0,-}}(P_{\infty_-}). \qquad (3.37)$$

For theta function representations of α and β we refer to [26] and the references cited therein. These theta function representations also show that $\gamma(n) \notin \{0,1\}$ for all $n \in \mathbb{Z}$, that is, the second condition in (3.2) is satisfied for the stationary algebro-geometric AL solutions discussed in this section provided the associated Dirichlet and Neumann divisors are admissible.

4. The Stationary Algorithm

The aim of this section is to derive an algorithm that enables one to construct algebro-geometric solutions for the stationary Ablowitz–Ladik hierarchy for general initial data. Equivalently, we offer a solution of the inverse algebro-geometric spectral problem for general (non-unitary) Lax operators L in (2.57), starting with initial divisors in general position.

Up to the end of Section 3 the material was based on the assumption that $\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}$ satisfy the pth stationary AL system (2.33). Now we embark on the corresponding inverse problem consisting of constructing a solution of (2.33) given certain initial data. More precisely, we seek to construct solutions $\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}$ satisfying the pth stationary Ablowitz–Ladik system (2.33) starting from a properly restricted set $\mathcal{M}_0$ of admissible nonspecial Dirichlet divisor initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ at some fixed $n_0 \in \mathbb{Z}$,

$$\begin{aligned} &\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0), \dots, \hat{\mu}_{2p-1}(n_0)\} \in \mathcal{M}_0, \quad \mathcal{M}_0 \subset \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}), \\ &\hat{\mu}_j(n_0) = (\mu_j(n_0), (2/c_{0,+})\mu_j(n_0)^p G_p(\mu_j(n_0), n_0)), \quad j = 1, \dots, 2p-1. \end{aligned} \tag{4.1}$$

For notational convenience we will frequently use the phrase that α, β *blow up* in this manuscript whenever one of the divisors $\mathcal{D}_{\underline{\hat{\mu}}}$ or $\mathcal{D}_{\underline{\hat{\nu}}}$ hits one of the points $P_{\infty_\pm}$, $P_{0,\pm}$.

Of course we would like to ensure that the sequences α, β obtained via our algorithm do not blow up. To investigate when this happens, we study the image of our divisors under the Abel map. The key ingredient in our analysis will be (3.35) which yields a linear discrete dynamical system on the Jacobi variety $J(\mathcal{K}_{2p-1})$. In particular, we will be led to investigate solutions $\mathcal{D}_{\underline{\hat{\mu}}}$, $\mathcal{D}_{\underline{\hat{\nu}}}$ of the discrete initial value problem

$$\begin{aligned} &\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \\ &\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0), \dots, \hat{\mu}_{2p-1}(n_0)\} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}), \end{aligned} \tag{4.2}$$

respectively

$$\begin{aligned} &\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + \underline{A}_{P_{0,-}}(P_{\infty_-}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \\ &\underline{\hat{\nu}}(n_0) = \{\hat{\nu}_1(n_0), \dots, \hat{\nu}_{2p-1}(n_0)\} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}), \end{aligned} \tag{4.3}$$

where $Q_0 \in \mathcal{K}_{2p-1}$ is a given base point. Eventually, we will be interested in solutions $\mathcal{D}_{\underline{\hat{\mu}}}$, $\mathcal{D}_{\underline{\hat{\nu}}}$ of (4.2), (4.3) with initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ satisfying (4.1) and $\mathcal{M}_0$ to be specified as in (the proof of) Lemma 4.1.

Before proceeding to develop the stationary Ablowitz–Ladik algorithm, we briefly analyze the dynamics of (4.2).

Lemma 4.1. *Let $n \in \mathbb{Z}$ and suppose that $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is defined via (4.2) for some divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$.*
(i) If $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial and does not contain any of the points $P_{0,\pm}$, $P_{\infty_\pm}$, and $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ contains one of the points $P_{0,\pm}$, $P_{\infty_\pm}$, then $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ contains $P_{0,-}$ or P_{∞_-}

but not P_{∞_+} *or* $P_{0,+}$.
(*ii*) *If* $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *is nonspecial and* $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ *is special, then* $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *contains at least one of the points* P_{∞_+}, P_{∞_-} *and one of the points* $P_{0,+}$, $P_{0,-}$.
(*iii*) *Item* (*i*) *holds if* $n+1$ *is replaced by* $n-1$, P_{∞_+} *by* P_{∞_-}, *and* $P_{0,+}$ *by* $P_{0,-}$.
(*iv*) *Items* (*i*)*–*(*iii*) *also hold for* $\mathcal{D}_{\underline{\hat{\nu}}(n)}$.

Proof. (*i*) Suppose one point in $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ equals P_{∞_+} and denote the remaining ones by $\mathcal{D}_{\underline{\tilde{\mu}}(n+1)}$. Then (4.2) implies that $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\tilde{\mu}}(n+1)}) + \underline{A}_{Q_0}(P_{\infty_+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) + \underline{A}_{P_{0,-}}(P_{\infty_+})$. Since $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is assumed to be nonspecial one concludes $\mathcal{D}_{\underline{\hat{\mu}}(n)} = \mathcal{D}_{\underline{\tilde{\mu}}(n+1)} + \mathcal{D}_{P_{0,-}}$, contradicting our assumption on $\mathcal{D}_{\underline{\hat{\mu}}(n)}$. The statement for $P_{0,+}$ follows similarly; here we choose Q_0 to be a branch point of $\mathcal{K}_{2p-1}$ such that $\underline{A}_{Q_0}(P^*) = -\underline{A}_{Q_0}(P)$.
(*ii*) We choose Q_0 to be a branch point of $\mathcal{K}_{2p-1}$. If $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ is special, then it contains a pair of points (Q,Q^*) whose contribution will cancel under the Abel map, that is, $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n+1)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\eta}}(n+1)})$ for some $\mathcal{D}_{\underline{\hat{\eta}}(n+1)} \in \operatorname{Sym}^{2p-3}(\mathcal{K}_{2p-1})$. Invoking (4.2) then shows that $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\eta}}(n+1)}) + \underline{A}_{Q_0}(P_{\infty_-}) + \underline{A}_{Q_0}(P_{0,-})$. As $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ was assumed to be nonspecial, this shows that $\mathcal{D}_{\underline{\hat{\mu}}(n)} = \mathcal{D}_{\underline{\hat{\eta}}(n+1)} + \mathcal{D}_{P_{\infty_-}} + \mathcal{D}_{P_{0,-}}$ as claimed.
(*iii*) This is proved as in item (*i*).
(*iv*) Since $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ satisfies the same equation as $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ in (4.2) (cf. (3.36)), items (*i*)–(*iii*) also hold for $\mathcal{D}_{\underline{\hat{\nu}}(n)}$. ☐

We also note the following result:

Lemma 4.2. *Let* $n \in \mathbb{Z}$ *and assume that* $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ *are nonspecial. Then* $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *contains* $P_{0,-}$ *if and only if* $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ *contains* P_{∞_-}. *Moreover,* $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *contains* P_{∞_+} *if and only if* $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ *contains* $P_{0,+}$.

Proof. Suppose a point in $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ equals $P_{0,-}$ and denote the remaining ones by $\mathcal{D}_{\underline{\tilde{\mu}}(n)}$. By (3.37),

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\tilde{\mu}}(n)}) + \underline{A}_{Q_0}(P_{0,-}) + \underline{A}_{P_{0,-}}(P_{\infty_-}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\tilde{\mu}}(n)}) + \underline{A}_{Q_0}(P_{\infty_-}). \tag{4.4}$$

Since $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ is nonspecial, $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ contains P_{∞_-}, and vice versa. The second statement follows similarly. ☐

Let us call the points P_{∞_+}, P_{∞_-}, $P_{0,+}$, and $P_{0,-}$ *exceptional points*. Then Lemma 4.1 yields the following behavior of $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ assuming one starts with some nonspecial initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ without exceptional points: As n increases, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ stays nonspecial as long as it does not include exceptional points. If an exceptional point appears, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is still nonspecial and contains $P_{0,-}$ or P_{∞_-} at least once (but not $P_{0,+}$ and P_{∞_+}). Further increasing n, all instances of $P_{0,-}$ and P_{∞_-} will be rendered into $P_{0,+}$ and P_{∞_+}, until we have again a nonspecial divisor that has the same number of $P_{0,+}$ and P_{∞_+} as the first one had of $P_{0,-}$ and P_{∞_-}. Generically, one expects the subsequent divisor to be nonspecial without exceptional points again.

Next we show that most initial divisors are well-behaved in the sense that their iterates stay away from P_{∞_+}, P_{∞_-}, $P_{0,+}$, and $P_{0,-}$. Since we want to show that this set is of full measure, it will be convenient to identify $\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$ with the Jacobi variety $J(\mathcal{K}_{2p-1})$ via the Abel map and take the Haar measure on $J(\mathcal{K}_{2p-1})$.

Of course, the Abel map is only injective when restricted to the set of nonspecial divisors, but these are the only ones we are interested in.

Lemma 4.3. *The set* $\mathcal{M}_0 \subset \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$ *of initial divisors* $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ *for which* $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n)}$, *defined via* (4.2) *and* (4.3), *are admissible* (*i.e., do not contain* $P_{\infty_\pm}, P_{0,\pm}$) *and hence are nonspecial for all* $n \in \mathbb{Z}$, *forms a dense set of full measure in the set* $\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$ *of positive divisors of degree* $2p-1$.

Proof. Let $\mathcal{M}_{\infty,0}$ be the set of divisors in $\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$ for which (at least) one point is equal to $P_{\infty_\pm}$ or $P_{0,\pm}$. The image $\underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0})$ of $\mathcal{M}_{\infty,0}$ is then contained in the following set,

$$\underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0}) \subseteq \bigcup_{P\in\{P_{0,\pm},P_{\infty_\pm}\}} \big(\underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\operatorname{Sym}^{2p-2}(\mathcal{K}_{2p-1}))\big) \subset J(\mathcal{K}_{2p-1}). \tag{4.5}$$

Since the (complex) dimension of $\operatorname{Sym}^{2p-2}(\mathcal{K}_{2p-1})$ is $2p-2$, its image must be of measure zero by Sard's theorem (see, e.g., [8, Sect. 3.6]). Similarly, let $\mathcal{M}_{\rm sp}$ be the set of special divisors, then its image is given by

$$\underline{\alpha}_{Q_0}(\mathcal{M}_{\rm sp}) = \underline{\alpha}_{Q_0}(\operatorname{Sym}^{2p-3}(\mathcal{K}_{2p-1})), \tag{4.6}$$

assuming Q_0 to be a branch point. In particular, we conclude that $\underline{\alpha}_{Q_0}(\mathcal{M}_{\rm sp}) \subset \underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0})$ and thus $\underline{\alpha}_{Q_0}(\mathcal{M}_{\rm sing}) = \underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0})$ has measure zero, where

$$\mathcal{M}_{\rm sing} = \mathcal{M}_{\infty,0} \cup \mathcal{M}_{\rm sp}. \tag{4.7}$$

Hence,

$$\mathcal{S}_\mu = \bigcup_{n\in\mathbb{Z}} \big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\rm sing}) + n\underline{A}_{P_{0,-}}(P_{\infty_+})\big) \quad \text{and} \quad \mathcal{S}_\nu = \mathcal{S}_\mu + \underline{A}_{P_{0,-}}(P_{\infty_-}) \tag{4.8}$$

are of measure zero as well. But the set $\mathcal{S}_\mu \cup \mathcal{S}_\nu$ contains all initial divisors for which $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ or $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ will hit $P_{\infty_\pm}$ or $P_{0,\pm}$, or become special at some $n \in \mathbb{Z}$. We denote by $\mathcal{M}_0$ the inverse image of the complement of the set $\mathcal{S}_\mu \cup \mathcal{S}_\nu$ under the Abel map,

$$\mathcal{M}_0 = \underline{\alpha}_{Q_0}^{-1}\big(\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}) \backslash (\mathcal{S}_\mu \cup \mathcal{S}_\nu)\big). \tag{4.9}$$

Since $\mathcal{M}_0$ is of full measure, it is automatically dense in $\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$. □

Next, we describe the stationary Ablowitz–Ladik algorithm. Since this is a somewhat lengthy affair, we will break it up into several steps.

The Stationary Ablowitz–Ladik Algorithm:

We prescribe the following data

(*i*) The coefficient $\alpha(n_0) \in \mathbb{C}\backslash\{0\}$ and the constant $c_{0,+} \in \mathbb{C}\backslash\{0\}$.

(*ii*) The set

$$\{E_m\}_{m=0}^{4p-1} \subset \mathbb{C}\backslash\{0\}, \quad E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, \dots, 4p-1, \tag{4.10}$$

for some fixed $p \in \mathbb{N}$. Given $\{E_m\}_{m=0}^{4p-1}$, we introduce the function

$$R_{2p}(z) = \bigg(\frac{c_{0,+}}{2z^p}\bigg)^2 \prod_{m=0}^{4p-1} (z - E_m) \tag{4.11}$$

and the hyperelliptic curve $\mathcal{K}_{2p-1}$ with nonsingular affine part as in (3.1).
(iii) The nonspecial divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1}), \tag{4.12}$$

where $\underline{\hat{\mu}}(n_0)$ is of the form

$$\underline{\hat{\mu}}(n_0) = \{\underbrace{\hat{\mu}_1(n_0), \dots, \hat{\mu}_1(n_0)}_{p_1(n_0) \text{ times}}, \dots, \underbrace{\hat{\mu}_{q(n_0)}, \dots, \hat{\mu}_{q(n_0)}}_{p_{q(n_0)}(n_0) \text{ times}}\} \tag{4.13}$$

with

$$\hat{\mu}_k(n_0) = (\mu_k(n_0), y(\hat{\mu}_k(n_0))), \quad \mu_k(n_0) \neq \mu_{k'}(n_0) \text{ for } k \neq k',\ k, k' = 1, \dots, q(n_0) \tag{4.14}$$

and

$$p_k(n_0) \in \mathbb{N},\ k = 1, \dots, q(n_0), \quad \sum_{k=1}^{q(n_0)} p_k(n_0) = 2p - 1. \tag{4.15}$$

With $\{E_m\}_{m=0}^{4p-1}$, $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$, $\alpha(n_0)$, and $c_{0,+}$ prescribed, we next introduce the following quantities (for $z \in \mathbb{C}$):

$$\alpha^+(n_0) = \alpha(n_0)\bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n_0)} \mu_k(n_0)^{-p_k(n_0)}, \tag{4.16}$$

$$c_{0,-}^2 = c_{0,+}^2 \prod_{m=0}^{4p-1} E_m, \tag{4.17}$$

$$F_p(z, n_0) = -c_{0,+}\alpha^+(n_0) z^{-p} \prod_{k=1}^{q(n_0)} (z - \mu_k(n_0))^{p_k(n_0)}, \tag{4.18}$$

$$\begin{aligned} G_p(z, n_0) = {} & \frac{1}{2}\bigg(\frac{1}{\alpha(n_0)} - \frac{z}{\alpha^+(n_0)}\bigg) F_p(z, n_0) \\ & - \frac{z}{2\alpha^+(n_0)} F_p(z, n_0) \sum_{k=1}^{q(n_0)} \sum_{\ell=0}^{p_k(n_0)-1} \frac{\big(d^\ell\big(\zeta^{-1} y(P)\big)/d\zeta^\ell\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0)}}{\ell!(p_k(n_0) - \ell - 1)!} \\ & \times \Bigg(\frac{d^{p_k(n_0)-\ell-1}}{d\zeta^{p_k(n_0)-\ell-1}} \Bigg((z - \zeta)^{-1} \prod_{k'=1,\, k' \neq k}^{q(n_0)} (\zeta - \mu_{k'}(n_0))^{-p_{k'}(n_0)}\Bigg)\Bigg)\Bigg|_{\zeta = \mu_k(n_0)}. \end{aligned} \tag{4.19}$$

Here the sign of the square root is chosen according to (4.14).

Next we record a series of facts:

(I) By construction (cf. Lemma B.1),

$$\begin{aligned} & \frac{d^\ell\big(G_p(z, n_0)^2\big)}{dz^\ell}\bigg|_{z=\mu_k(n_0)} = \frac{d^\ell R_{2p}(z)}{dz^\ell}\bigg|_{z=\mu_k(n_0)}, \\ & z \in \mathbb{C}, \quad \ell = 0, \dots, p_k(n_0) - 1,\ k = 1, \dots, q(n_0). \end{aligned} \tag{4.20}$$

(II) Since $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ is nonspecial by hypothesis, one concludes that

$$p_k(n_0) \geq 2 \text{ implies } R_{2p}(\mu_k(n_0)) \neq 0, \quad k = 1, \dots, q(n_0). \tag{4.21}$$

(III) By (4.19) and (4.20) one infers that F_p divides $G_p^2 - R_{2p}$.
(IV) By (4.11) and (4.19) one verifies that

$$G_p(z,n_0)^2 - R_{2p}(z) \underset{z\to\infty}{=} O(z^{2p-1}), \tag{4.22}$$

$$G_p(z,n_0)^2 - R_{2p}(z) \underset{z\to 0}{=} O(z^{-2p+1}). \tag{4.23}$$

By **(III)** and **(IV)** we may write

$$G_p(z,n_0)^2 - R_{2p}(z) = F_p(z,n_0)\check{H}_{q,r}(z,n_0), \quad z\in\mathbb{C}, \tag{4.24}$$

for some $r\in\{0,\dots,p\}$, $q\in\{0,\dots,p-1\}$, where $\check{H}_{q,r}(z,n_0)$ is a Laurent polynomial of the form $c_{-q}z^{-q}+\cdots+c_r z^r$. If, in fact, $\check{H}_{0,0}=0$, then $R_{2p}(z)=G_p(z,n_0)^2$ would yield double zeros of R_{2p}, contradicting our basic hypothesis (4.10). Thus we conclude that in the case $r=q=0$, $\check{H}_{0,0}$ cannot vanish identically and hence we may break up (4.24) in the following manner

$$\check{\phi}(P,n_0) = \frac{G_p(z,n_0)+(c_{0,+}/2)z^{-p}y}{F_p(z,n_0)} = \frac{\check{H}_{q,r}(z,n_0)}{G_p(z,n_0)-(c_{0,+}/2)z^{-p}y}, \quad P=(z,y)\in\mathcal{K}_{2p-1}. \tag{4.25}$$

Next we decompose

$$\check{H}_{q,r}(z,n_0) = Cz^{-q}\prod_{j=1}^{r+q}(z-\nu_j(n_0)), \quad z\in\mathbb{C}, \tag{4.26}$$

where $C\in\mathbb{C}\setminus\{0\}$ and $\{\nu_j(n_0)\}_{j=1}^{r+q}\subset\mathbb{C}$ (if $r=q=0$ we replace the product in (4.26) by 1). By inspection of the local zeros and poles as well as the behavior near $P_{0,\pm}$, $P_{\infty_\pm}$ of the function $\check{\phi}(\,\cdot\,,n_0)$ using (3.11), its divisor, $\big(\check{\phi}(\,\cdot\,,n_0)\big)$, is given by

$$\big(\check{\phi}(\,\cdot\,,n_0)\big) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n_0)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n_0)}, \tag{4.27}$$

where

$$\underline{\hat{\nu}}(n_0) = \{\underbrace{P_{0,-},\dots,P_{0,-}}_{p-1-q \text{ times}},\hat{\nu}_1(n_0),\dots,\hat{\nu}_{r+q}(n_0),\underbrace{P_{\infty_+},\dots,P_{\infty_+}}_{p-r\text{ times}}\}. \tag{4.28}$$

In the following we call a positive divisor of degree $2p-1$ *admissible* if it does not contain any of the points $P_{\infty_\pm}, P_{0,\pm}$.

Hence,

$$\mathcal{D}_{\underline{\hat{\nu}}(n_0)} \text{ is an admissible divisor if and only if } r=p \text{ and } q=p-1. \tag{4.29}$$

We note that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + \underline{A}_{P_{0,-}}(P_{\infty_-}), \tag{4.30}$$

in accordance with (3.37).
(V) Assuming that (4.22), (4.23) are precisely of order $z^{\pm(2p-1)}$, that is, assuming $r=p$ and $q=p-1$ in (4.24), we rewrite (4.24) in the more appropriate manner

$$G_p(z,n_0)^2 - R_{2p}(z) = F_p(z,n_0)H_p(z,n_0), \quad z\in\mathbb{C}. \tag{4.31}$$

(We will later discuss conditions which indeed guarantee that $r=p$ and $q=p-1$, cf. (4.29) and the discussion in step **(X)** below.) By construction, $H_p(\,\cdot\,,n_0)$ is then of the type

$$H_p(z,n_0) = c_{0,+}\beta(n_0)z^{-p+1}\prod_{k=1}^{\ell(n_0)}(z-\nu_k(n_0))^{s_k(n_0)}, \quad \sum_{k=1}^{\ell(n_0)}s_k(n_0)=2p-1,$$

$$\nu_k(n_0) \neq \nu_{k'}(n_0) \text{ for } k \neq k',\ k,k' = 1,\dots,\ell(n_0),\quad z \in \mathbb{C}, \tag{4.32}$$

where we introduced the coefficient $\beta(n_0)$. We define

$$\hat{\nu}_k(n_0) = (\nu_k(n_0), -(2/c_{0,+})\nu_k(n_0)^p G_p(\nu_k(n_0), n_0)),\quad k = 1,\dots,\ell(n_0). \tag{4.33}$$

An explicit computation of $\beta(n_0)$ then yields

$$\begin{aligned}\alpha^+(n_0)\beta(n_0) = -\frac{1}{2}\sum_{k=1}^{q(n_0)} &\frac{\big(d^{p_k(n_0)-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n_0)-1}\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0)}}{(p_k(n_0)-1)!}\\ &\times \prod_{k'=1,\,k'\neq k}^{q(n_0)} (\mu_k(n_0) - \mu_{k'}(n_0))^{-p_k(n_0)}\\ &+\frac{1}{2}\bigg(\frac{\alpha^+(n_0)}{\alpha(n_0)} + \sum_{k=1}^{q(n_0)} p_k(n_0)\mu_k(n_0) - \frac{1}{2}\sum_{m=0}^{4p-1} E_m\bigg).\end{aligned} \tag{4.34}$$

The result (4.34) is obtained by inserting the expressions (4.18), (4.19), and (4.32) for $F_p(\,\cdot\,,n_0)$, $G_p(\,\cdot\,,n_0)$, and $H_p(\,\cdot\,,n_0)$ into (4.31) and collecting all terms of order z^{2p-1}.

(VI) Introduce

$$\beta^+(n_0) = \beta(n_0)\prod_{k=1}^{\ell(n_0)} \nu_k(n_0)^{s_k(n_0)}\bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{-1/2}. \tag{4.35}$$

(VII) Using $G_p(z,n_0)$, $H_p(z,n_0)$, $F_p(z,n_0)$, $\beta(n_0)$, $\alpha^+(n_0)$, and $\beta^+(n_0)$, we next construct the $n_0 \pm 1$ terms from the following equations:

$$F_p^- = \frac{1}{z\gamma}(\alpha^2 H_p - 2\alpha G_p + F_p), \tag{4.36}$$

$$H_p^- = \frac{z}{\gamma}(\beta^2 F_p - 2\beta G_p + H_p), \tag{4.37}$$

$$G_p^- = \frac{1}{\gamma}((1+\alpha\beta)G_p - \alpha H_p - \beta F_p), \tag{4.38}$$

respectively,

$$F_p^+ = \frac{1}{z\gamma^+}((\alpha^+)^2 H_p + 2\alpha^+ z G_p + z^2 F_p), \tag{4.39}$$

$$H_p^+ = \frac{1}{z\gamma^+}((\beta^+ z)^2 F_p + 2\beta^+ z G_p + H_p), \tag{4.40}$$

$$G_p^+ = \frac{1}{z\gamma^+}((1+\alpha^+\beta^+)zG_p + \alpha^+ H_p + \beta^+ z^2 F_p). \tag{4.41}$$

Moreover,

$$(G_p^-)^2 - F_p^- H_p^- = R_{2p},\quad (G_p^+)^2 - F_p^+ H_p^+ = R_{2p}. \tag{4.42}$$

Inserting (4.18), (4.19), and (4.32) in (4.36)–(4.38) one verifies

$$F_p^-(z,n_0) \underset{z\to\infty}{=} -c_{0,+}\alpha(n_0)z^{p-1} + O(z^{p-2}), \tag{4.43}$$

$$H_p^-(z,n_0) \underset{z\to\infty}{=} O(z^p), \tag{4.44}$$

$$F_p^-(z,n_0) \underset{z\to 0}{=} O(z^{-p}), \tag{4.45}$$

$$H_p^-(z,n_0) \underset{z\to 0}{=} -c_{0,-}\beta(n_0)z^{-p+1} + O(z^{-p+2}), \tag{4.46}$$

$$G_p^-(z,n_0) = \tfrac{1}{2}c_{0,-}z^{-p} + \cdots + \tfrac{1}{2}c_{0,+}z^p. \tag{4.47}$$

The last equation implies

$$G_p(z,n_0-1)^2 - R_{2p}(z) \underset{z\to\infty}{=} O(z^{2p-1}), \tag{4.48}$$
$$G_p(z,n_0-1)^2 - R_{2p}(z) \underset{z\to 0}{=} O(z^{-2p+1}), \tag{4.49}$$

so we may write

$$G_p(z,n_0-1)^2 - R_{2p}(z) = \check F_{s,p-1}(z,n_0-1)\check H_{p-1,r}(z,n_0-1), \quad z\in\mathbb{C}, \tag{4.50}$$

for some $s\in\{1,\dots,p\}$, $r\in\{1,\dots,p\}$, where

$$\check F_{s,p-1}(n_0-1) = c_{-s}z^{-s} + \cdots - c_{0,+}\alpha(n_0)z^{p-1},$$
$$\check H_{p-1,r}(n_0-1) = -c_{0,-}\beta(n_0)z^{-p+1} + \cdots + c_r z^r.$$

The right-hand side of (4.50) cannot vanish identically (since otherwise $R_{2p}(z) = G_p(z,n_0-1)^2$ would yield double zeros of $R_{2p}(z)$), and hence,

$$\check\phi(P,n_0-1) = \frac{G_p(z,n_0-1) + (c_{0,+}/2)z^{-p}y}{\check F_{s,p-1}(z,n_0-1)} = \frac{\check H_{p-1,r}(z,n_0-1)}{G_p(z,n_0-1) - (c_{0,+}/2)z^{-p}y},$$
$$P=(z,y)\in\mathcal{K}_{2p-1}. \tag{4.51}$$

Next, we decompose

$$\check F_{s,p-1}(z,n_0-1) = -c_{0,+}\alpha(n_0)z^{-s}\prod_{j=1}^{p-1+s}(z-\mu_j(n_0-1)), \tag{4.52}$$
$$\check H_{p-1,r}(z,n_0-1) = Cz^{-p+1}\prod_{j=1}^{p-1+r}(z-\nu_j(n_0-1)), \tag{4.53}$$

where $C\in\mathbb{C}\backslash\{0\}$ and $\{\mu_j(n_0-1)\}_{j=1}^{p-1+s}\subset\mathbb{C}$, $\{\nu_j(n_0-1)\}_{j=1}^{p-1+r}\subset\mathbb{C}$. The divisor of $\check\phi(\,\cdot\,,n_0-1)$ is then given by

$$\big(\check\phi(\,\cdot\,,n_0-1)\big) = \mathcal{D}_{P_{0,-}\underline{\hat\nu}(n_0-1)} - \mathcal{D}_{P_{\infty_-}\underline{\hat\mu}(n_0-1)}, \tag{4.54}$$

where

$$\underline{\hat\mu}(n_0-1) = \{\underbrace{P_{0,+},\dots,P_{0,+}}_{p-s\text{ times}}, \hat\mu_1(n_0-1),\dots,\hat\mu_{p-1+s}(n_0-1)\}, \tag{4.55}$$
$$\underline{\hat\nu}(n_0-1) = \{\hat\nu_1(n_0-1),\dots,\hat\nu_{p-1+r}(n_0-1), \underbrace{P_{\infty_+},\dots,P_{\infty_+}}_{p-r\text{ times}}\}. \tag{4.56}$$

In particular,

$$\mathcal{D}_{\underline{\hat\mu}(n_0-1)} \text{ is an admissible divisor if and only if } s=p, \tag{4.57}$$
$$\mathcal{D}_{\underline{\hat\nu}(n_0-1)} \text{ is an admissible divisor if and only if } r=p. \tag{4.58}$$

(VIII) Assuming that (4.48), (4.49) are precisely of order $z^{\pm(2p-1)}$, that is, assuming $s=p$ and $r=p$ in (4.51), we rewrite (4.51) as

$$G_p(z,n_0-1)^2 - R_{2p}(z) = F_p(z,n_0-1)H_p(z,n_0-1), \quad z\in\mathbb{C}. \tag{4.59}$$

By construction, $F_p(\,\cdot\,,n_0-1)$ and $H_p(\,\cdot\,,n_0-1)$ are then of the type

$$\begin{aligned}
F_p(z,n_0-1) = -c_{0,+}\alpha(n_0)z^{-p} \prod_{k=1}^{q(n_0-1)} (z-\mu_j(n_0-1))^{p_k(n_0-1)},\\
\sum_{k=1}^{q(n_0-1)} p_k(n_0-1) = 2p-1,\\
\mu_k(n_0-1) \neq \mu_{k'}(n_0-1) \text{ for } k\neq k',\ k,k'=1,\dots,q(n_0-1),\ \ z\in\mathbb{C},
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
H_p(z,n_0-1) = c_{0,+}\beta(n_0-1)z^{-p+1} \prod_{k=1}^{\ell(n_0-1)} (z-\nu_k(n_0-1))^{s_k(n_0-1)},\\
\sum_{k=1}^{\ell(n_0-1)} s_k(n_0-1) = 2p-1,\\
\nu_k(n_0-1) \neq \nu_{k'}(n_0-1) \text{ for } k\neq k',\ k,k'=1,\dots,\ell(n_0-1),\ \ z\in\mathbb{C},
\end{aligned} \tag{4.61}$$

where we introduced the coefficient $\beta(n_0-1)$. We define

$$\begin{aligned}
\hat\mu_k(n_0-1) &= (\mu_k(n_0-1), (2/c_{0,+})\mu_k(n_0-1)^p G_p(\mu_k(n_0-1),n_0-1)),\\
&\qquad k=1,\dots,q(n_0-1),\\
\hat\nu_k(n_0-1) &= (\nu_k(n_0-1), -(2/c_{0,+})\nu_k(n_0-1)^p G_p(\nu_k(n_0-1),n_0-1)),\\
&\qquad k=1,\dots,\ell(n_0-1).
\end{aligned} \tag{4.62}$$

(IX) At this point one can iterate the procedure step by step to construct $F_p(\,\cdot\,,n)$, $G_p(\,\cdot\,,n)$, $H_p(\,\cdot\,,n)$, $\alpha(n)$, $\beta(n)$, $\mu_j(n)$, $\nu_j(n)$, etc., for $n\in(-\infty,n_0]\cap\mathbb{Z}$, subject to the following assumption (cf. (4.57), (4.58)) at each step:

$$\mathcal{D}_{\underline{\hat\mu}(n-1)} \text{ is an admissible divisor (and hence } \alpha(n-1)\neq 0) \text{ for all } n\in(-\infty,n_0]\cap\mathbb{Z}, \tag{4.63}$$

$$\mathcal{D}_{\underline{\hat\nu}(n-1)} \text{ is an admissible divisor (and hence } \beta(n-1)\neq 0) \text{ for all } n\in(-\infty,n_0]\cap\mathbb{Z}. \tag{4.64}$$

The formalism is symmetric with respect to n_0 and can equally well be developed for $n\in(-\infty,n_0]\cap\mathbb{Z}$ subject to the analogous assumption

$$\mathcal{D}_{\underline{\hat\mu}(n+1)} \text{ is an admissible divisor (and hence } \alpha(n+2)\neq 0) \text{ for all } n\in[n_0,\infty)\cap\mathbb{Z}, \tag{4.65}$$

$$\mathcal{D}_{\underline{\hat\nu}(n+1)} \text{ is an admissible divisor (and hence } \beta(n+2)\neq 0) \text{ for all } n\in[n_0,\infty)\cap\mathbb{Z}. \tag{4.66}$$

(X) Choosing the initial data $\mathcal{D}_{\underline{\hat\mu}(n_0)}$ such that

$$\mathcal{D}_{\underline{\hat\mu}(n_0)} \in \mathcal{M}_0, \tag{4.67}$$

where $\mathcal{M}_0\subset \operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$ is the set of admissible initial divisors introduced in Lemma 4.3, then guarantees that assumptions (4.63)–(4.66) are satisfied for all $n\in\mathbb{Z}$.

Equations (4.36)–(4.41) (for arbitrary $n\in\mathbb{Z}$) are equivalent to s-$\mathrm{AL}_p(\alpha,\beta)=0$.

At this stage we have verified the basic hypotheses of Section 3 (i.e., (3.2) and the assumption that α, β satisfy the pth stationary AL system (2.33)) and hence all results of Section 3 apply.

In summary, we proved the following result:

Theorem 4.4. *Let $n \in \mathbb{Z}$, suppose the set $\{E_m\}_{m=0}^{4p-1} \subset \mathbb{C}$ satisfies $E_m \neq E_{m'}$ for $m \neq m'$, $m, m' = 0, \dots, 4p-1$, and introduce the function R_{2p} and the hyperelliptic curve $\mathcal{K}_{2p-1}$ as in* (3.1). *Choose $\alpha(n_0) \in \mathbb{C}\backslash\{0\}$, $c_{0,+} \in \mathbb{C}\backslash\{0\}$, and a nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathcal{M}_0$, where $\mathcal{M}_0 \subset \mathrm{Sym}^{2p-1}(\mathcal{K}_{2p-1})$ is the set of admissible initial divisors introduced in Lemma* 4.3. *Then the stationary* (*complex*) *Ablowitz–Ladik algorithm as outlined in steps* (***I***)–(***X***) *produces solutions α, β of the pth stationary Ablowitz–Ladik system,*

$$\text{s-AL}_p(\alpha,\beta) = \begin{pmatrix} -\alpha(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ \beta(g_{p,+}^- + g_{p,-}) + h_{p-1,+}^- - h_{p-1,-} \end{pmatrix} = 0, \quad p \in \mathbb{N}_0, \tag{4.68}$$

satisfying (3.2) *and*

$$\alpha(n) = \bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{(n-n_0)/2} \mathcal{A}(n,n_0)\,\alpha(n_0), \tag{4.69}$$

$$\begin{aligned}
\beta(n) = \Bigg(&-\frac{1}{2}\sum_{k=1}^{q(n)} \frac{\big(d^{p_k(n)-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n)-1}\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n)}}{(p_k(n)-1)!} \\
&\qquad\times \prod_{k'=1,\, k'\neq k}^{q(n)} (\mu_k(n) - \mu_{k'}(n))^{-p_k(n)} \\
&+ \frac{1}{2}\bigg(\bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n)} \mu_k(n)^{-p_k(n)} + \sum_{k=1}^{q(n)} p_k(n)\mu_k(n) - \frac{1}{2}\sum_{m=0}^{4p-1} E_m\bigg)\Bigg) \\
&\times \bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{-(n+1-n_0)/2} \mathcal{A}(n+1,n_0)^{-1}\,\alpha(n_0)^{-1},
\end{aligned} \tag{4.70}$$

where

$$\mathcal{A}(n,n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} \prod_{k=1}^{q(n')} \mu_k(n')^{-p_k(n')}, & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} \prod_{k=1}^{q(n')} \mu_k(n')^{p_k(n')}, & n \leq n_0 - 1. \end{cases} \tag{4.71}$$

Moreover, Lemmas 3.2–3.4 *apply.*

Finally, we briefly illustrate some aspects of this analysis in the special case $p = 1$ (i.e., the case where (3.1) represents an elliptic Riemann surface) in more detail.

Example 4.5. The case $p = 1$.
In this case one has

$$\begin{aligned}
F_1(z,n) &= -c_{0,+}\alpha(n+1)z^{-1}(z - \mu_1(n)), \\
G_1(z,n) &= \frac{1}{2}\bigg(\frac{1}{\alpha(n)} - \frac{z}{\alpha(n+1)}\bigg)F_1(z,n) + R_2(\hat{\mu}_1(n))^{1/2}, \\
R_2(z) &= \bigg(\frac{c_{0,+}\alpha^+}{z}\bigg)^2 \prod_{m=0}^{3}(z - E_m),
\end{aligned} \tag{4.72}$$

and hence a straightforward calculation shows that

$$\begin{aligned} G_1(z,n)^2 - R_2(z) &= -c_{0,+}^2 \alpha(n+1)\beta(n) z^{-1}(z-\mu_1(n))(z-\nu_1(n)) \\ &= -\frac{c_{0,+}^2}{2z}(z-\mu_1(n))\bigg(\bigg(-\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\sqrt{\widetilde{E}}}{\mu_1(n)} + \mu_1(n) - \frac{\widehat{E}^+}{2}\bigg)z \\ &\quad - \frac{\widetilde{E}}{\mu_1(n)}\bigg(-\frac{1}{\sqrt{\widetilde{E}}}\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\mu_1(n)}{\sqrt{\widetilde{E}}} + \frac{1}{\mu_1(n)} - \frac{\widehat{E}^-}{2}\bigg)\bigg), \end{aligned} \tag{4.73}$$

where

$$\widehat{E}^{\pm} = \sum_{m=0}^{3} E_m^{\pm 1}, \quad \widetilde{E} = \prod_{m=0}^{3} E_m. \tag{4.74}$$

Solving for $\nu_1(n)$ one then obtains

$$\nu_1(n) = \frac{\widetilde{E}}{\mu_1(n)} \frac{-\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\sqrt{\widetilde{E}}}{\mu_1(n)} + \mu_1(n) - \frac{\widehat{E}^+}{2}}{-\frac{1}{\sqrt{\widetilde{E}}}\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\mu_1(n)}{\sqrt{\widetilde{E}}} + \frac{1}{\mu_1(n)} - \frac{\widehat{E}^-}{2}}. \tag{4.75}$$

Thus, $\nu_1(n_0)$ could be 0 or ∞ even if $\mu_1(n_0) \neq 0, \infty$.

5. Properties of Algebro-Geometric Solutions of the Time-Dependent Ablowitz–Ladik Hierarchy

In this section we present a quick review of properties of algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy. Again we omit all proofs and refer to [23], [25], and [26] for details.

For most of this section we assume the following hypothesis.

Hypothesis 5.1. (*i*) *Suppose that* α, β *satisfy*

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,),\, \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z}\times\mathbb{R}. \end{aligned} \tag{5.1}$$

(*ii*) *Assume that the hyperelliptic curve* $\mathcal{K}_{2p-1}$ *satisfies* (3.1) *and* (3.3).

In order to briefly analyze algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy we proceed as follows. Given $p \in \mathbb{N}_0$, consider a complex-valued solution $\alpha^{(0)}, \beta^{(0)}$ of the pth stationary Ablowitz–Ladik system s-$\mathrm{AL}_p(a,b) = 0$, associated with $\mathcal{K}_{2p-1}$ and a given set of summation constants $\{c_{\ell,\pm}\}_{\ell=1,\dots,p} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to consider solutions $\alpha = \alpha(t_r), \beta = \beta(t_r)$ of the rth AL flow $\mathrm{AL}_r(\alpha,\beta) = 0$ with $\alpha(t_{0,r}) = \alpha^{(0)}, \beta(t_{0,r}) = \beta^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the summation constants in the definitions of the stationary and the time-dependent Ablowitz–Ladik equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde{V}_r$, $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_{s,\pm}$, $\tilde{g}_{s,\pm}$, $\tilde{h}_{s,\pm}$, $\tilde{c}_{s,\pm}$, in order to distinguish them from V_p, F_p, G_p, H_p, $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, $c_{\ell,\pm}$, in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth Ablowitz–Ladik flow by a separate time variable $t_r \in \mathbb{R}$. More precisely, we will review properties of solutions α, β of the time-dependent algebro-geometric initial value problem

$$\begin{aligned}
&\widetilde{\mathrm{AL}}_r(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_r} - \alpha(\tilde g_{r,+} + \tilde g_{r,-}^-) + \tilde f_{r-1,+} - \tilde f_{r-1,-}^- \\ -i\beta_{t_r} + \beta(\tilde g_{r,+}^- + \tilde g_{r,-}) - \tilde h_{r-1,-} + \tilde h_{r-1,+}^- \end{pmatrix} = 0, \\
&(\alpha,\beta)\big|_{t=t_{0,r}} = \big(\alpha^{(0)},\beta^{(0)}\big),
\end{aligned} \tag{5.2}$$

$$\text{s-AL}_p\left(\alpha^{(0)},\beta^{(0)}\right) = \begin{pmatrix} -\alpha^{(0)}(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ \beta^{(0)}(g_{p,+}^- + g_{p,-}) - h_{p-1,-} + h_{p-1,+}^- \end{pmatrix} = 0 \tag{5.3}$$

for some $t_{0,r} \in \mathbb{R}$, where $\alpha = \alpha(n,t_r)$, $\beta = \beta(n,t_r)$ satisfy (5.1) and a fixed curve $\mathcal{K}_{2p-1}$ is associated with the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ in (5.3). In terms of the zero-curvature formulation this amounts to solving

$$U_{t_r}(z,t_r) + U(z,t_r)\widetilde V_r(z,t_r) - \widetilde V_r^+(z,t_r)U(z,t_r) = 0, \tag{5.4}$$

$$U(z,t_{0,r})V_p(z,t_{0,r}) - V_p^+(z,t_{0,r})U(z,t_{0,r}) = 0. \tag{5.5}$$

One can show (cf. Lemma 6.2) that the stationary Ablowitz–Ladik system (5.5) is actually satisfied for all times $t_r \in \mathbb{R}$. Thus, we impose

$$U_{t_r} + U\widetilde V_r - \widetilde V_r^+ U = 0, \tag{5.6}$$

$$UV_p - V_p^+ U = 0, \tag{5.7}$$

instead of (5.4) and (5.5). For further reference, we recall the relevant quantities here (cf. (2.20), (2.21), (2.27)–(2.29), (2.38)):

$$\begin{aligned}
&U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \\
&V_p(z) = i\begin{pmatrix} G_p^-(z) & -F_p^-(z) \\ H_p^-(z) & -G_p^-(z) \end{pmatrix}, \quad \widetilde V_r(z) = i\begin{pmatrix} \widetilde G_r^-(z) & -\widetilde F_r^-(z) \\ \widetilde H_r^-(z) & -\widetilde K_r^-(z) \end{pmatrix},
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
F_p(z) &= \sum_{\ell=1}^{p} f_{p-\ell,-}z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+}z^{\ell} = -c_{0,+}\alpha^+ z^{-p}\prod_{j=1}^{2p-1}(z-\mu_j), \\
G_p(z) &= \sum_{\ell=1}^{p} g_{p-\ell,-}z^{-\ell} + \sum_{\ell=0}^{p} g_{p-\ell,+}z^{\ell}, \\
H_p(z) &= \sum_{\ell=0}^{p-1} h_{p-1-\ell,-}z^{-\ell} + \sum_{\ell=1}^{p} h_{p-\ell,+}z^{\ell} = c_{0,+}\beta z^{-p+1}\prod_{j=1}^{2p-1}(z-\nu_j), \\
\widetilde F_r(z) &= \sum_{s=1}^{r} \tilde f_{r-s,-}z^{-s} + \sum_{s=0}^{r-1} \tilde f_{r-1-s,+}z^{s}, \\
\widetilde G_r(z) &= \sum_{s=1}^{r} \tilde g_{r-s,-}z^{-s} + \sum_{s=0}^{r} \tilde g_{r-s,+}z^{s}, \\
\widetilde H_r(z) &= \sum_{s=0}^{r-1} \tilde h_{r-1-s,-}z^{-s} + \sum_{s=1}^{r} \tilde h_{r-s,+}z^{s}, \\
\widetilde K_r(z) &= \sum_{s=0}^{r} \tilde g_{r-s,-}z^{-s} + \sum_{s=1}^{r} \tilde g_{r-s,+}z^{s} = \widetilde G_r(z) + \tilde g_{r,-} - \tilde g_{r,+}
\end{aligned} \tag{5.9}$$

for fixed $p, r \in \mathbb{N}_0$. Here $f_{\ell,\pm}$, $\tilde{f}_{s,\pm}$, $g_{\ell,\pm}$, $\tilde{g}_{s,\pm}$, $h_{\ell,\pm}$, and $\tilde{h}_{s,\pm}$ are defined as in (2.5)–(2.12) with appropriate sets of summation constants $c_{\ell,\pm}$, $\ell \in \mathbb{N}$, and $\tilde{c}_{k,\pm}$, $k \in \mathbb{N}$. Explicitly, (5.6) and (5.7) are equivalent to (cf. (2.23)–(2.26), (2.45)–(2.48)),

$$\alpha_{t_r} = i\big(z\widetilde{F}_r^- + \alpha(\widetilde{G}_r + \widetilde{K}_r^-) - \widetilde{F}_r\big), \tag{5.10}$$

$$\beta_{t_r} = -i\big(\beta(\widetilde{G}_r^- + \widetilde{K}_r) - \widetilde{H}_r + z^{-1}\widetilde{H}_r^-\big), \tag{5.11}$$

$$0 = z(\widetilde{G}_r^- - \widetilde{G}_r) + z\beta\widetilde{F}_r + \alpha\widetilde{H}_r^-, \tag{5.12}$$

$$0 = z\beta\widetilde{F}_r^- + \alpha\widetilde{H}_r + \widetilde{K}_r^- - \widetilde{K}_r, \tag{5.13}$$

$$0 = z(G_p^- - G_p) + z\beta F_p + \alpha H_p^-, \tag{5.14}$$

$$0 = z\beta F_p^- + \alpha H_p - G_p + G_p^-, \tag{5.15}$$

$$0 = -F_p + zF_p^- + \alpha(G_p + G_p^-), \tag{5.16}$$

$$0 = z\beta(G_p + G_p^-) - zH_p + H_p^-, \tag{5.17}$$

respectively. In particular, (2.39) holds in the present t_r-dependent setting, that is,

$$G_p^2 - F_pH_p = R_{2p}. \tag{5.18}$$

As in the stationary context (3.8), (3.9) we introduce

$$\begin{aligned} \hat{\mu}_j(n,t_r) = (\mu_j(n,t_r), (2/c_{0,+})\mu_j(n,t_r)^p G_p(\mu_j(n,t_r),n,t_r)) \in \mathcal{K}_{2p-1}, \\ j = 1,\dots,2p-1,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R}, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} \hat{\nu}_j(n,t_r) = (\nu_j(n,t_r), -(2/c_{0,+})\nu_j(n,t_r)^p G_p(\nu_j(n,t_r),n,t_r)) \in \mathcal{K}_{2p-1}, \\ j = 1,\dots,2p-1,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R}, \end{aligned} \tag{5.20}$$

and note that the regularity assumptions (5.1) on α, β imply continuity of μ_j and ν_k with respect to $t_r \in \mathbb{R}$ (away from collisions of these zeros, μ_j and ν_k are of course C^∞).

In analogy to (3.12), (3.13), one defines the following meromorphic function $\phi(\,\cdot\,,n,t_r)$ on $\mathcal{K}_{2p-1}$,

$$\phi(P,n,t_r) = \frac{(c_{0,+}/2)z^{-p}y + G_p(z,n,t_r)}{F_p(z,n,t_r)} \tag{5.21}$$

$$= \frac{-H_p(z,n,t_r)}{(c_{0,+}/2)z^{-p}y - G_p(z,n,t_r)}, \tag{5.22}$$

$$P = (z,y) \in \mathcal{K}_{2p-1},\ (n,t_r) \in \mathbb{Z}\times\mathbb{R},$$

with divisor $(\phi(\,\cdot\,,n,t_r))$ of $\phi(\,\cdot\,,n,t_r)$ given by

$$(\phi(\,\cdot\,,n,t_r)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n,t_r)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_r)}. \tag{5.23}$$

The time-dependent Baker–Akhiezer vector is then defined in terms of ϕ by

$$\Psi(P,n,n_0,t_r,t_{0,r}) = \begin{pmatrix} \psi_1(P,n,n_0,t_r,t_{0,r}) \\ \psi_2(P,n,n_0,t_r,t_{0,r}) \end{pmatrix}, \tag{5.24}$$

$$\psi_1(P,n,n_0,t_r,t_{0,r}) = \exp\left(i\int_{t_{0,r}}^{t_r} ds\big(\widetilde{G}_r(z,n_0,s) - \widetilde{F}_r(z,n_0,s)\phi(P,n_0,s)\big)\right) \tag{5.25}$$

$$\times \begin{cases} \prod_{n'=n_0+1}^{n} \big(z+\alpha(n',t_r)\phi^-(P,n',t_r)\big), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z+\alpha(n',t_r)\phi^-(P,n',t_r)\big)^{-1}, & n \leq n_0-1, \end{cases}$$

$$\psi_2(P,n,n_0,t_r,t_{0,r}) = \exp\Big(i\int_{t_{0,r}}^{t_r} ds\big(\widetilde G_r(z,n_0,s) - \widetilde F_r(z,n_0,s)\phi(P,n_0,s)\big)\Big) \tag{5.26}$$

$$\times \phi(P,n_0,t_r) \begin{cases} \prod_{n'=n_0+1}^{n} \big(z\beta(n',t_r)\phi^-(P,n',t_r)^{-1}+1\big), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z\beta(n',t_r)\phi^-(P,n',t_r)^{-1}+1\big)^{-1}, & n \leq n_0-1, \end{cases}$$

$$P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,t_r)\in\mathbb{Z}\times\mathbb{R}.$$

One observes that

$$\begin{aligned} &\psi_1(P,n,n_0,t_r,\tilde t_r) = \psi_1(P,n_0,n_0,t_r,\tilde t_r)\psi_1(P,n,n_0,t_r,t_r), \\ &P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,n_0,t_r,\tilde t_r)\in\mathbb{Z}^2\times\mathbb{R}^2. \end{aligned} \tag{5.27}$$

The following lemma records basic properties of ϕ, ψ, and Ψ in analogy to the stationary case discussed in Lemma 3.2.

Lemma 5.2 ([25])**.** *Assume Hypothesis* 5.1 *and suppose that* (5.6), (5.7) *hold. In addition, let* $P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-}\}$, $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$, *and* $r\in\mathbb{N}_0$. *Then* ϕ *satisfies*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{5.28}$$

$$\phi_{t_r}(P) = i\widetilde F_r\phi^2(P) - i(\widetilde G_r(z)+\widetilde K_r(z))\phi(P) + i\widetilde H_r(z), \tag{5.29}$$

$$\phi(P)\phi(P^*) = \frac{H_p(z)}{F_p(z)}, \tag{5.30}$$

$$\phi(P)+\phi(P^*) = 2\frac{G_p(z)}{F_p(z)}, \tag{5.31}$$

$$\phi(P)-\phi(P^*) = c_{0,+}z^{-p}\frac{y(P)}{F_p(z)}. \tag{5.32}$$

Moreover, assuming $P=(z,y)\in\mathcal{K}_{2p-1}\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$, *then* Ψ *satisfies*

$$\psi_2(P,n,n_0,t_r,t_{0,r}) = \phi(P,n,t_r)\psi_1(P,n,n_0,t_r,t_{0,r}), \tag{5.33}$$

$$U(z,n,t_r)\Psi^-(P,n,n_0,t_r,t_{0,r}) = \Psi(P,n,n_0,t_r,t_{0,r}), \tag{5.34}$$

$$V_p(z,n,t_r)\Psi^-(P,n,n_0,t_r,t_{0,r}) = -(i/2)c_{0,+}z^{-p}y\Psi^-(P,n,n_0,t_r,t_{0,r}), \tag{5.35}$$

$$\Psi_{t_r}(P,n,n_0,t_r,t_{0,r}) = \widetilde V_r^+(n,t_r)\Psi(P,n,n_0,t_r,t_{0,r}), \tag{5.36}$$

$$\psi_1(P,n,n_0,t_r,t_{0,r})\psi_1(P^*,n,n_0,t_r,t_{0,r}) = z^{n-n_0}\frac{F_p(z,n,t_r)}{F_p(z,n_0,t_{0,r})}\Gamma(n,n_0,t_r), \tag{5.37}$$

$$\psi_2(P,n,n_0,t_r,t_{0,r})\psi_2(P^*,n,n_0,t_r,t_{0,r}) = z^{n-n_0}\frac{H_p(z,n,t_r)}{F_p(z,n_0,t_{0,r})}\Gamma(n,n_0,t_r), \tag{5.38}$$

$$\begin{aligned} &\psi_1(P,n,n_0,t_r,t_{0,r})\psi_2(P^*,n,n_0,t_r,t_{0,r}) + \psi_1(P^*,n,n_0,t_r,t_{0,r})\psi_2(P,n,n_0,t_r,t_{0,r}) \\ &\quad = 2z^{n-n_0}\frac{G_p(z,n,t_r)}{F_p(z,n_0,t_{0,r})}\Gamma(n,n_0,t_r), \end{aligned} \tag{5.39}$$

$$\psi_1(P,n,n_0,t_r,t_{0,r})\psi_2(P^*,n,n_0,t_r,t_{0,r}) - \psi_1(P^*,n,n_0,t_r,t_{0,r})\psi_2(P,n,n_0,t_r,t_{0,r})$$

$$= -c_{0,+} z^{n-n_0-p} \frac{y}{F_p(z,n_0,t_{0,r})} \Gamma(n,n_0,t_r), \tag{5.40}$$

where

$$\Gamma(n,n_0,t_r) = \begin{cases} \prod_{n'=n_0+1}^{n} \gamma(n',t_r) & n \geq n_0+1, \\ 1 & n = n_0, \\ \prod_{n'=n+1}^{n_0} \gamma(n',t_r)^{-1} & n \leq n_0 - 1. \end{cases} \tag{5.41}$$

In addition, as long as the zeros $\mu_j(n_0,s)$ of $F_p(\,\cdot\,,n_0,s)$ are all simple and distinct from zero for $s \in \mathcal{I}_\mu$, $\mathcal{I}_\mu \subseteq \mathbb{R}$ an open interval, $\Psi(\,\cdot\,,n,n_0,t_r,t_{0,r})$ is meromorphic on $\mathcal{K}_{2p-1} \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ for $(n,t_r,t_{0,r}) \in \mathbb{Z} \times \mathcal{I}_\mu^2$.

The trace formulas recorded in Lemma 3.3 extend to the present time-dependent context without any change as $t_r \in \mathbb{R}$ can be viewed as a fixed parameter. Further details are thus omitted.

For completeness we next mention the Dubrovin-type equations for the time variation of the zeros μ_j of F_p and ν_j of H_p governed by the $\widetilde{\mathrm{AL}}_r$ flow.

Lemma 5.3 ([25]). *Assume Hypothesis* 5.1 *and suppose that* (5.6), (5.7) *hold on $\mathbb{Z} \times \mathcal{I}_\mu$ with $\mathcal{I}_\mu \subseteq \mathbb{R}$ an open interval. In addition, assume that the zeros μ_j, $j = 1,\dots,2p-1$, of $F_p(\,\cdot\,)$ remain distinct and nonzero on $\mathbb{Z}\times\mathcal{I}_\mu$. Then $\{\hat{\mu}_j\}_{j=1,\dots,2p-1}$, defined in* (5.19), *satisfies the following first-order system of differential equations on $\mathbb{Z} \times \mathcal{I}_\mu$,*

$$\mu_{j,t_r} = -i\widetilde{F}_r(\mu_j) y(\hat{\mu}_j) (\alpha^+)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^{2p-1} (\mu_j - \mu_k)^{-1}, \quad j = 1,\dots,2p-1, \tag{5.42}$$

with

$$\hat{\mu}_j(n,\cdot) \in C^\infty(\mathcal{I}_\mu, \mathcal{K}_{2p-1}), \quad j = 1,\dots,2p-1,\ n \in \mathbb{Z}. \tag{5.43}$$

For the zeros ν_j, $j = 1,\dots,2p-1$, of $H_p(\,\cdot\,)$, identical statements hold with μ_j and $\mathcal{I}_\mu$ replaced by ν_j and $\mathcal{I}_\nu$, etc. (*with $\mathcal{I}_\nu \subseteq \mathbb{R}$ an open interval*). *In particular, $\{\hat{\nu}_j\}_{j=1,\dots,2p-1}$, defined in* (5.20), *satisfies the first-order system on $\mathbb{Z} \times \mathcal{I}_\nu$,*

$$\nu_{j,t_r} = i\widetilde{H}_r(\nu_j) y(\hat{\nu}_j) (\beta\nu_j)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^{2p-1} (\nu_j - \nu_k)^{-1}, \quad j = 1,\dots,2p-1, \tag{5.44}$$

with

$$\hat{\nu}_j(n,\cdot) \in C^\infty(\mathcal{I}_\nu, \mathcal{K}_{2p-1}), \quad j = 1,\dots,2p-1,\ n \in \mathbb{Z}. \tag{5.45}$$

When attempting to solve the Dubrovin-type systems (5.42) and (5.44), they must be augmented with appropriate divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in \operatorname{Sym}^{2p-1} \mathcal{K}_{2p-1}$, $t_{0,r} \in \mathcal{I}_\mu$, and $\mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,r})} \in \operatorname{Sym}^{2p-1} \mathcal{K}_{2p-1}$, $t_{0,r} \in \mathcal{I}_\nu$, as initial conditions.

For the t_r-dependence of F_p, G_p, and H_p one obtains the following result.

Lemma 5.4 ([25]). *Assume Hypothesis* 5.1 *and suppose that* (5.6), (5.7) *hold. In addition, let $(z,n,t_r) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{R}$. Then,*

$$F_{p,t_r} = -2iG_p\widetilde{F}_r + i(\widetilde{G}_r + \widetilde{K}_r)F_p, \tag{5.46}$$

$$G_{p,t_r} = iF_p\widetilde{H}_r - iH_p\widetilde{F}_r, \tag{5.47}$$

$$H_{p,t_r} = 2iG_p\widetilde{H}_r - i(\widetilde{G}_r + \widetilde{K}_r)H_p. \tag{5.48}$$

In particular, (5.46)–(5.48) *are equivalent to*

$$V_{p,t_r} = \big[\widetilde{V}_r, V_p\big]. \tag{5.49}$$

It will be shown in Section 6 that Lemma 5.4 yields a first-order system of differential equations for $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$, that serves as a pertinent substitute for the Dubrovin equations (5.42) even (in fact, especially) when some of the μ_j coincide.

Lemma 3.4 on nonspecial divisors and the linearization property of the Abel map extend to the present time-dependent setting. For this fact we need to introduce a particular differential of the second kind, $\widetilde{\Omega}_r^{(2)}$, defined as follows. Let $\omega^{(2)}_{P_{\infty_\pm},q}$ and $\omega^{(2)}_{P_{0,\pm},q}$ be the normalized differentials of the second kind with a unique pole at $P_{\infty_\pm}$ and $P_{0,\pm}$, respectively, and principal parts

$$\omega^{(2)}_{P_{\infty_\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q}+O(1)\big)d\zeta, \quad P\to P_{\infty_\pm},\ \zeta=1/z,\ q\in\mathbb{N}_0, \tag{5.50}$$

$$\omega^{(2)}_{P_{0,\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q}+O(1)\big)d\zeta, \quad P\to P_{0,\pm},\ \zeta=z,\ q\in\mathbb{N}_0, \tag{5.51}$$

with vanishing a-periods,

$$\int_{a_j}\omega^{(2)}_{P_{\infty_\pm},q} = \int_{a_j}\omega^{(2)}_{P_{0,\pm},q} = 0, \quad j=1,\dots,2p-1. \tag{5.52}$$

Moreover, we define

$$\begin{aligned}\widetilde{\Omega}_r^{(2)} = \frac{i}{2}\bigg(&\sum_{s=1}^{r} s\tilde{c}_{r-s,-}\big(\omega^{(2)}_{P_{0,+},s-1}-\omega^{(2)}_{P_{0,-},s-1}\big)\\ &+\sum_{s=1}^{r} s\tilde{c}_{r-s,+}\big(\omega^{(2)}_{P_{\infty_+},s-1}-\omega^{(2)}_{P_{\infty_-},s-1}\big)\bigg),\end{aligned} \tag{5.53}$$

where $\tilde{c}_{\ell,\pm}$ are the summation constants in $\widetilde{F}_r$. The corresponding vector of b-periods of $\widetilde{\Omega}_r^{(2)}/(2\pi i)$ is then denoted by

$$\underline{\widetilde{U}}_r^{(2)} = \big(\widetilde{U}^{(2)}_{r,1},\dots,\widetilde{U}^{(2)}_{r,p}\big), \quad \widetilde{U}^{(2)}_{r,j} = \frac{1}{2\pi i}\int_{b_j}\widetilde{\Omega}_r^{(2)}, \quad j=1,\dots,2p-1. \tag{5.54}$$

The time-dependent analog of Lemma 3.4 then reads as follows.

Lemma 5.5 ([25], [26])**.** *Assume Hypothesis* 5.1 *and suppose that* (5.6), (5.7) *hold. Moreover, let* $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$*. Denote by* $\mathcal{D}_{\underline{\hat{\mu}}}$*,* $\underline{\hat{\mu}}=\{\hat{\mu}_1,\dots,\hat{\mu}_{2p-1}\}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}}$*,* $\underline{\hat{\nu}}=\{\hat{\nu}_1,\dots,\hat{\nu}_{2p-1}\}$*, the pole and zero divisors of degree* $2p-1$*, respectively, associated with* α*,* β*, and* ϕ *defined according to* (5.19) *and* (5.20)*, that is,*

$$\hat{\mu}_j(n,t_r) = (\mu_j(n,t_r),(2/c_{0,+})\mu_j(n,t_r)^pG_p(\mu_j(n,t_r),n,t_r)), \quad j=1,\dots,2p-1, \tag{5.55}$$

$$\hat{\nu}_j(n,t_r) = (\nu_j(n,t_r),-(2/c_{0,+})\nu_j(n,t_r)^pG_p(\nu_j(n,t_r),n,t_r)), \quad j=1,\dots,2p-1. \tag{5.56}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}$ *are nonspecial for all* $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$*. Moreover, the Abel map linearizes the auxiliary divisors* $\mathcal{D}_{\underline{\hat{\mu}}(n,t)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n,t)}$ *in the sense that*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_r-t_{0,r})\underline{\widetilde{U}}_r^{(2)}, \tag{5.57}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,r})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_r-t_{0,r})\underline{\widetilde{U}}_r^{(2)}, \tag{5.58}$$

where $Q_0 \in \mathcal{K}_{2p-1}$ *is a given base point and* $\underline{\widetilde{U}}_r^{(2)}$ *is the vector of b-periods introduced in* (5.54).

Again we refer to [26] (and the references cited therein) for theta function representations of α and β. These theta function representations also show that $\gamma(n,t_r) \notin \{0,1\}$ for all $(n,t_r) \in \mathbb{Z} \times \mathbb{R}$, that is, the last condition in (5.1) is satisfied for the time-dependent algebro-geometric AL solutions discussed in this section provided the associated Dirichlet and Neumann divisors are admissible.

6. The Algebro-Geometric Ablowitz–Ladik Hierarchy Initial Value Problem

In this section we consider the algebro-geometric Ablowitz–Ladik hierarchy initial value problem (5.2), (5.3) with complex-valued initial data. For a generic set of initial data we will prove unique solvability of the initial value problem globally in time.

While it is natural in the special unitary case to base the solution of the algebro-geometric initial value problem on the Dubrovin-type equations (5.42), (5.44) and the trace formulas for α and β, this strategy meets with serious difficulties in the non-unitary case as Dirichlet eigenvalues $\hat{\mu}_j$ may now collide on $\mathcal{K}_{2p-1}$ and hence the denominator of (5.42) can blow up. Hence, we will develop an alternative strategy based on the use of elementary symmetric functions of the variables $\{\mu_j\}_{j=1,\dots,2p-1}$ in this section, which can accommodate collisions of $\hat{\mu}_j$. In short, our strategy will consist of the following:

(*i*) Replace the first-order autonomous Dubrovin-type system (5.42) of differential equations in t_r for the Dirichlet eigenvalues $\mu_j(n,t_r)$, $j = 1,\dots,2p-1$, augmented by appropriate initial conditions, by the first-order autonomous system (6.56), (6.57) for the coefficients $f_{j,\pm}$, $h_{j,\pm}$, $j = 1,\dots,p-1$, and $g_{j,\pm}$, $j = 1,\dots,p$, with respect to t_r. Solve this first-order autonomous system in some time interval $(t_{0,r} - T_0, t_{0,r} + T_0)$ under appropriate initial conditions at $(n_0, t_{0,r})$ derived from an initial (nonspecial) Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$.

(*ii*) Use the stationary algorithm derived in Section 4 to extend the solution of step (*i*) from $\{n_0\} \times (t_{0,r} - T_0, t_{0,r} + T_0)$ to $\mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$ (cf. Lemma 6.2).

(*iii*) Prove consistency of this approach, that is, show that the discrete algorithm of Section 4 is compatible with the time-dependent Lax and zero-curvature equations in the sense that first solving the autonomous system (6.56), (6.57) and then applying the discrete algorithm, or first applying the discrete algorithm and then solving the autonomous system (6.56), (6.57) yields the same result whenever the same endpoint (n,t_r) is reached (cf. Lemma 6.3 and Theorem 6.4).

(*iv*) Prove that there is a dense set of initial conditions of full measure for which this strategy yields global solutions of the algebro-geometric Ablowitz–Ladik hierarchy initial value problem.

To set up this formalism we need some preparations. From the outset we make the following assumption.

Hypothesis 6.1. *Suppose that*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}} \ \text{ and } \ \alpha(n)\beta(n) \notin \{0,1\} \ \text{ for all } \ n \in \mathbb{Z}, \tag{6.1}$$

and assume that α, β satisfy the pth stationary Ablowitz–Ladik equation (2.33). *In addition, suppose that the affine part of the hyperelliptic curve $\mathcal{K}_{2p-1}$ in* (3.1) *is nonsingular.*

We introduce a deformation (time) parameter $t_r \in \mathbb{R}$ in $\alpha = \alpha(t_r)$ and $\beta = \beta(t_r)$ and hence obtain t_r-dependent quantities $f_\ell = f_\ell(t_r)$, $g_\ell = g_\ell(t_r)$, $F_p(z) = F_p(z, t_r)$, $G_p(z) = G_p(z, t_r)$, etc. At a fixed initial time $t_{0,r} \in \mathbb{R}$ we require that

$$(\alpha, \beta)|_{t_r = t_{0,r}} = \big(\alpha^{(0)}, \beta^{(0)}\big), \tag{6.2}$$

where $\alpha^{(0)} = \alpha(\,\cdot\,, t_{0,r}), \beta^{(0)} = \beta(\,\cdot\,, t_{0,r})$ satisfy the pth stationary Ablowitz–Ladik equation (2.33). As discussed in Section 4, in order to guarantee that the stationary solutions (6.2) can be constructed for all $n \in \mathbb{Z}$ one starts from a particular divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,r})} \in \mathcal{M}_0, \tag{6.3}$$

where $\underline{\hat{\mu}}(n_0, t_{0,r})$ is of the form

$$\begin{aligned} &\underline{\hat{\mu}}(n_0, t_{0,r}) \\ &\quad = \{\underbrace{\hat{\mu}_1(n_0, t_{0,r}), \dots, \hat{\mu}_1(n_0, t_{0,r})}_{p_1(n_0, t_{0,r}) \text{ times}}, \dots, \underbrace{\hat{\mu}_{q(n_0, t_{0,r})}(n_0, t_{0,r}), \dots, \hat{\mu}_{q(n_0, t_{0,r})}(n_0, t_{0,r})}_{p_{q(n_0, t_{0,r})}(n_0, t_{0,r}) \text{ times}}\}. \end{aligned} \tag{6.4}$$

Moreover, as in Section 4 we prescribe the data

$$\alpha(n_0, t_{0,r}) \in \mathbb{C}\backslash\{0\} \text{ and } c_{0,+} \in \mathbb{C}\backslash\{0\}, \tag{6.5}$$

and of course the hyperelliptic curve $\mathcal{K}_{2p-1}$ with nonsingular affine part (cf. (4.10)). In addition, we introduce

$$\alpha^+(n_0, t_{0,r}) = \alpha(n_0, t_{0,r}) \bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n_0, t_{0,r})} \mu_k(n_0, t_{0,r})^{-p_k(n_0, t_{0,r})}, \tag{6.6}$$

$$\begin{aligned} F_p(z, n_0, t_{0,r}) &= \sum_{\ell=1}^{p} f_{p-\ell,-}(n_0, t_{0,r}) z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+}(n_0, t_{0,r}) z^\ell \\ &= -c_{0,+}\alpha^+(n_0, t_{0,r}) z^{-p} \prod_{k=1}^{q(n_0, t_{0,r})} (z - \mu_k(n_0, t_{0,r}))^{p_k(n_0, t_{0,r})}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} G_p(z, n_0, t_{0,r}) &= \frac{1}{2}\bigg(\frac{1}{\alpha(n_0, t_{0,r})} - \frac{z}{\alpha^+(n_0, t_{0,r})}\bigg) F_p(z, n_0, t_{0,r}) \\ &\quad - \frac{z}{2\alpha^+(n_0, t_{0,r})} F_p(z, n_0, t_{0,r}) \\ &\quad \times \sum_{k=1}^{q(n_0, t_{0,r})} \sum_{\ell=0}^{p_k(n_0, t_{0,r})-1} \frac{\big(d^\ell\big(\zeta^{-1} y(P)\big)/d\zeta^\ell\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0, t_{0,r})}}{\ell!(p_k(n_0, t_{0,r}) - \ell - 1)!} \\ &\quad \times \Bigg(\frac{d^{p_k(n_0, t_{0,r})-\ell-1}}{d\zeta^{p_k(n_0, t_{0,r})-\ell-1}}\Bigg((z-\zeta)^{-1} \\ &\qquad\qquad \times \prod_{k'=1,\, k'\neq k}^{q(n_0, t_{0,r})} (\zeta - \mu_{k'}(n_0, t_{0,r}))^{-p_{k'}(n_0, t_{0,r})}\Bigg)\Bigg)\Bigg|_{\zeta=\mu_k(n_0, t_{0,r})}, \end{aligned} \tag{6.8}$$

in analogy to (4.16).

Our aim is to find an autonomous first-order system of ordinary differential equations with respect to t_r for $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ rather than for μ_j. We divide the differential equation

$$F_{p,t_r} = -2iG_p\widetilde{F}_r + i(\widetilde{G}_r + \widetilde{K}_r)F_p \tag{6.9}$$

by y and rewrite it using [25, Theorem A2, (2.40)–(2.43)]

$$\begin{aligned}\sum_{\ell=0}^{\infty}\hat{f}_{\ell,+,t_r}\zeta^{\ell+1} &= -2i\bigg(\sum_{j=1}^{r}\tilde{f}_{r-j,-}\zeta^{j} + \sum_{j=0}^{r-1}\tilde{f}_{r-1-j,+}\zeta^{-j}\bigg)\sum_{\ell=0}^{\infty}\hat{g}_{\ell,+}\zeta^{\ell} \\ &\quad + i\bigg(2\sum_{j=1}^{r}\tilde{g}_{r-j,-}\zeta^{j} + 2\sum_{j=1}^{r}\tilde{g}_{r-j,+}\zeta^{-j} + \tilde{g}_{r,-} + \tilde{g}_{r,+}\bigg)\sum_{\ell=0}^{\infty}\hat{f}_{\ell,+}\zeta^{\ell+1}, \\ &\qquad P \to P_{\infty_-},\ \zeta = 1/z.\end{aligned} \tag{6.10}$$

The coefficients of ζ^{-j}, $j = 0, \dots, r-1$, cancel since

$$\sum_{j=0}^{\ell}\tilde{f}_{\ell-j,+}\hat{g}_{j,+} = \sum_{j=0}^{\ell}\tilde{g}_{\ell-j,+}\hat{f}_{j,+}, \quad \ell \in \mathbb{N}_0. \tag{6.11}$$

In (6.11) we used (2.19),

$$\tilde{f}_{\ell,+} = \sum_{k=0}^{\ell}\tilde{c}_{\ell-k,+}\hat{f}_{k,+}, \quad \tilde{g}_{\ell,+} = \sum_{k=0}^{\ell}\tilde{c}_{\ell-k,+}\hat{g}_{k,+}. \tag{6.12}$$

Comparing coefficients in (6.10) then yields

$$\begin{aligned}\hat{f}_{\ell,+,t_r} &= i\hat{f}_{\ell,+}(\tilde{g}_{r,+} - \tilde{g}_{r,-}) + 2i\sum_{j=0}^{\ell}\big(\hat{f}_{j,+}\tilde{g}_{r-\ell+j,-} - \hat{g}_{j,+}\tilde{f}_{r-1-\ell+j,-}\big) \\ &\quad + 2i\sum_{j=0}^{r-1}\big(\tilde{g}_{j,+}\hat{f}_{r+\ell-j,+} - \tilde{f}_{j,+}\hat{g}_{r+\ell-j,+}\big), \quad \ell = 0, \dots, r-1.\end{aligned} \tag{6.13}$$

By (6.11), the last sum in (6.13) can be rewritten as

$$\begin{aligned}\sum_{j=0}^{r-1}\big(\tilde{g}_{j,+}\hat{f}_{r+\ell-j,+} - \tilde{f}_{j,+}\hat{g}_{r+\ell-j,+}\big) &= -\sum_{j=r}^{r+\ell}\big(\tilde{g}_{j,+}\hat{f}_{r+\ell-j,+} - \tilde{f}_{j,+}\hat{g}_{r+\ell-j,+}\big) \\ &= \sum_{j=0}^{\ell}\big(\hat{g}_{j,+}\tilde{f}_{r+\ell-j,+} - \hat{f}_{j,+}\tilde{g}_{r+\ell-j,+}\big).\end{aligned}$$

One can perform a similar computation for $\hat{f}_{\ell,-,t_r}$ using [25, Theorem A2] at $P \to P_{0,+}$. In summary, since $f_{k,\pm} = \sum_{\ell=0}^{k} c_{k-\ell,\pm}\hat{f}_{\ell,\pm}$, (6.13) yields the following autonomous first-order system (for fixed $n = n_0$)

$$f_{k,\pm,t_r} = \mathcal{F}_{k,\pm}(f_{j,-}, f_{j,+}, g_{j,-}, g_{j,+}), \quad k = 0, \dots, p-1, \tag{6.14}$$

with initial conditions

$$\begin{aligned}&f_{j,\pm}(n_0, t_{0,r}), \quad j = 0, \dots, p-1, \\ &g_{j,\pm}(n_0, t_{0,r}), \quad j = 0, \dots, p,\end{aligned} \tag{6.15}$$

where $\mathcal{F}_{k,\pm}$, $k=0,\dots,p-1$, are polynomials in $4p+2$ variables

$$\begin{aligned}\mathcal{F}_{k,\pm} &= 2i\sum_{j=0}^{k}\big(f_{j,\pm}(\tilde g_{r-k+j,\mp}-\tilde g_{r+k-j,\pm})+g_{j,\pm}(\tilde f_{r+k-j,\pm}-\tilde f_{r-1-k+j,\mp})\big)\\ &\quad + if_{k,\pm}(\tilde g_{r,\pm}-\tilde g_{r,\mp}),\quad k=0,\dots,r-1. \end{aligned}\tag{6.16}$$

Explicitly, one obtains

$$\begin{aligned}\mathcal{F}_{0,\pm} &= if_{0,\pm}(\tilde g_{r,\mp}-\tilde g_{r,\pm})+2ig_{0,\pm}(\tilde f_{r,\pm}-\tilde f_{r-1,\mp}), &(6.17)\\ \mathcal{F}_{1,\pm} &= 2if_{0,\pm}(\tilde g_{r-1,\mp}-\tilde g_{r+1,\pm})+if_{1,\pm}(\tilde g_{r,\mp}-\tilde g_{r,\pm})\\ &\quad + 2ig_{0,\pm}(\tilde f_{r+1,\pm}-\tilde f_{r-2,\mp})+2ig_{1,\pm}(\tilde f_{r,\pm}-\tilde f_{r-1,\mp}),\ \text{ etc.}\end{aligned}$$

By (6.6)–(6.8), the initial conditions (6.15) are uniquely determined by the initial divisor $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}$ in (6.3) and by the data in (6.5).

Similarly, one transforms

$$G_{p,t_r} = iF_p\widetilde H_r - iH_p\widetilde F_r, \tag{6.18}$$

$$H_{p,t_r} = 2iG_p\widetilde H_r - i(\widetilde G_r+\widetilde K_r)H_p \tag{6.19}$$

into (for fixed $n=n_0$)

$$\begin{aligned}\hat g_{0,\pm,t_r} &= 0,\\ \hat g_{\ell,\pm,t_r} &= i\sum_{j=0}^{\ell-1}\big(\hat f_{j,\pm}\tilde h_{r-\ell+j,\mp}-\hat h_{j,\pm}\tilde f_{r-\ell+j,\mp}\big)\\ &\quad + i\sum_{j=0}^{r-1}\big(\tilde h_{j,\pm}\hat f_{r-1+\ell-j,\pm}-\tilde f_{j,\pm}\hat h_{r-1+\ell-j,\pm}\big)\\ &= i\sum_{j=0}^{\ell-1}\big(\hat f_{j,\pm}(\tilde h_{r-\ell+j,\mp}-\tilde h_{r-1+\ell-j,\pm})+\hat h_{j,\pm}(\tilde f_{r-1+\ell-j,\pm}-\tilde f_{r-\ell+j,\mp})\big),\\ &\qquad \ell=1,\dots,r, \end{aligned}\tag{6.20}$$

$$\begin{aligned}\hat h_{\ell,\pm,t_r} &= i\hat h_{\ell,\pm}\big(\tilde g_{r,\mp}-\tilde g_{r,\pm}\big)+2i\sum_{j=0}^{\ell}\big(\hat g_{j,\pm}\tilde h_{r-1-\ell+j,\mp}-\hat h_{j,\pm}\tilde g_{r-\ell+j,\mp}\big)\\ &\quad + 2i\sum_{j=0}^{r-1}\big(\tilde h_{j,\pm}\hat g_{r+\ell-j,\pm}-\tilde g_{j,\pm}\hat h_{r+\ell-j,\pm}\big)\\ &= 2i\sum_{j=0}^{\ell}\big(\hat g_{j,\pm}(\tilde h_{r-1-\ell+j,\mp}-\tilde h_{r+\ell-j,\pm})+\hat h_{j,\pm}(\tilde g_{r+\ell-j,\pm}-\tilde g_{r-\ell+j,\mp})\big)\\ &\quad + i\hat h_{\ell,\pm}\big(\tilde g_{r,\mp}-\tilde g_{r,\pm}\big),\quad \ell=0,\dots,r-1. \end{aligned}\tag{6.21}$$

Summing over ℓ in (6.20), (6.21) then yields the following first-order system

$$g_{k,\pm,t_r} = \mathcal{G}_{k,\pm}(f_{j,-},f_{j,+},h_{j,-},h_{j,+}),\quad k=0,\dots,p, \tag{6.22}$$

$$h_{k,\pm,t_r} = \mathcal{H}_{k,\pm}(g_{j,-},g_{j,+},h_{j,-},h_{j,+}),\quad k=0,\dots,p-1, \tag{6.23}$$

with initial conditions

$$f_{j,\pm}(n_0,t_{0,r}),\quad j=0,\dots,p-1,$$

$$\begin{aligned} & g_{j,\pm}(n_0,t_{0,r}), \quad j=0,\dots,p, \\ & h_{j,\pm}(n_0,t_{0,r}), \quad j=0,\dots,p-1, \end{aligned} \tag{6.24}$$

where $\mathcal{G}_{k,\pm}$, $\mathcal{H}_{k,\pm}$, are polynomials in $4p$, $4p+2$ variables

$$\mathcal{G}_{k,\pm} = i\sum_{j=0}^{k-1} \big(f_{j,\pm}(\tilde h_{r-k+j,\mp} - \tilde h_{r-1+k-j,\pm}) + h_{j,\pm}(\tilde f_{r-1+k-j,\pm} - \tilde f_{r-k+j,\mp}\big), \\ k=1,\dots,r, \tag{6.25}$$

$$\begin{aligned} \mathcal{H}_{k,\pm} = 2i\sum_{j=0}^{k} &\big(g_{j,\pm}(\tilde h_{r-1-k+j,\mp} - \tilde h_{r+k-j,\pm}) + h_{j,\pm}(\tilde g_{r+k-j,\pm} - \tilde g_{r-k+j,\mp})\big) \\ &+ ih_{k,\pm}\big(\tilde g_{r,\mp} - \tilde g_{r,\pm}\big), \quad k=0,\dots,r-1. \end{aligned} \tag{6.26}$$

Explicitly,

$$\begin{aligned} \mathcal{G}_{0,\pm} &= 0, \\ \mathcal{G}_{1,\pm} &= if_{0,\pm}(\tilde h_{r-1,\mp} - \tilde h_{r,\pm}) + ih_{0,\pm}(\tilde f_{r,\pm} - \tilde f_{r-1,\mp}), \\ \mathcal{G}_{2,\pm} &= if_{0,\pm}(\tilde h_{r-2,\mp} - \tilde h_{r+1,\pm}) + if_{1,\pm}(\tilde h_{r-1,\mp} - \tilde h_{r,\pm}) \\ &\quad + ih_{0,\pm}(\tilde f_{r+1,\pm} - \tilde f_{r-2,\mp}) + ih_{1,\pm}(\tilde f_{r,\pm} - \tilde f_{r-1,\mp}), \text{ etc.,} \end{aligned} \tag{6.27}$$

$$\begin{aligned} \mathcal{H}_{0,\pm} &= 2ig_{0,\pm}(\tilde h_{r-1,\mp} - \tilde h_{r,\pm}) + ih_{0,\pm}(\tilde g_{r,\pm} - \tilde g_{r,\mp}), \\ \mathcal{H}_{1,\pm} &= 2ig_{0,\pm}(\tilde h_{r-2,\mp} - \tilde h_{r+1,\pm}) + 2ig_{1,\pm}(\tilde h_{r-1,\mp} - \tilde h_{r,\pm}) \\ &\quad + 2ih_{0,\pm}(\tilde g_{r+1,\pm} - \tilde g_{r-1,\mp}) + ih_{1,\pm}(\tilde g_{r,\pm} - \tilde g_{r,\mp}), \text{ etc.} \end{aligned} \tag{6.28}$$

Again by (6.6)–(6.8), the initial conditions (6.24) are uniquely determined by the initial divisor $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}$ in (6.3) and by the data in (6.5).

Being autonomous with polynomial right-hand sides, there exists a $T_0>0$, such that the first-order initial value problem (6.14), (6.22), (6.23) with initial conditions (6.24) has a unique solution

$$\begin{aligned} f_{j,\pm} &= f_{j,\pm}(n_0,t_r), \quad j=0,\dots,p-1, \\ g_{j,\pm} &= g_{j,\pm}(n_0,t_r), \quad j=0,\dots,p, \\ h_{j,\pm} &= h_{j,\pm}(n_0,t_r), \quad j=0,\dots,p-1, \\ &\text{for all } \ t_r\in(t_{0,r}-T_0,t_{0,r}+T_0) \end{aligned} \tag{6.29}$$

(cf., e.g., [45, Sect. III.10]). Given the solution (6.29), we proceed as in Section 4 and introduce the following quantities (where $t_r\in(t_{0,r}-T_0,t_{0,r}+T_0)$):

$$\alpha^+(n_0,t_r) = \alpha(n_0,t_r)\bigg(\prod_{m=0}^{4p-1}E_m\bigg)^{1/2}\prod_{k=1}^{q(n_0,t_r)}\mu_k(n_0,t_r)^{-p_k(n_0,t_r)}, \tag{6.30}$$

$$\begin{aligned} F_p(z,n_0,t_r) &= \sum_{\ell=1}^{p} f_{p-\ell,-}(n_0,t_r)z^{-\ell} + \sum_{\ell=0}^{p-1} f_{p-1-\ell,+}(n_0,t_r)z^{\ell} \\ &= -c_{0,+}\alpha^+(n_0,t_r)z^{-p}\prod_{k=1}^{q(n_0,t_r)}(z-\mu_k(n_0,t_r))^{p_k(n_0,t_r)}, \end{aligned} \tag{6.31}$$

$$G_p(z,n_0,t_r) = \frac{1}{2}\bigg(\frac{1}{\alpha(n_0,t_r)} - \frac{z}{\alpha^+(n_0,t_r)}\bigg)F_p(z,n_0,t_r) \tag{6.32}$$

$$
\begin{aligned}
&- \frac{z}{2\alpha^+(n_0,t_r)} F_p(z,n_0,t_r) \sum_{k=1}^{q(n_0,t_r)} \sum_{\ell=0}^{p_k(n_0,t_r)-1} \frac{\big(d^\ell\big(\zeta^{-1}y(P)\big)/d\zeta^\ell\big)\big|_{P=(\zeta,\eta)=\hat\mu_k(n_0,t_r)}}{\ell!(p_k(n_0,t_r)-\ell-1)!} \\
&\times \Bigg(\frac{d^{p_k(n_0,t_r)-\ell-1}}{d\zeta^{p_k(n_0,t_r)-\ell-1}}\Bigg((z-\zeta)^{-1} \prod_{k'=1,\,k'\neq k}^{q(n_0,t_r)} (\zeta-\mu_{k'}(n_0,t_r))^{-p_{k'}(n_0,t_r)}\Bigg)\Bigg)\Bigg|_{\zeta=\mu_k(n_0,t_r)}.
\end{aligned}
$$

In particular, this leads to the divisor

$$\mathcal{D}_{\underline{\hat\mu}(n_0,t_r)} \in \mathrm{Sym}^{2p-1}(\mathcal{K}_{2p-1}) \tag{6.33}$$

and the sign of y in (6.31) is chosen as usual by

$$
\begin{aligned}
\hat\mu_k(n_0,t_r) = (\mu_k(n_0,t_r), (2/c_{0,+})\mu_j(n_0,t_r)^p G_p(\mu_k(n_0,t_r),n_0,t_r)),& \\
k=1,\dots,q(n_0,t_r),&
\end{aligned} \tag{6.34}
$$

and

$$\underline{\hat\mu}(n_0,t_r) = \{\underbrace{\mu_1(n_0,t_r),\dots,\mu_1(n_0,t_r)}_{p_1(n_0,t_r)\text{ times}},\dots,\underbrace{\mu_{q(n_0,t_r)}(n_0,t_r),\dots,\mu_{q(n_0,t_r)}(n_0,t_r)}_{p_{q(n_0,t_r)}(n_0,t_r)\text{ times}}\} \tag{6.35}$$

with

$$\mu_k(n_0,t_r) \neq \mu_{k'}(n_0,t_r) \text{ for } k\neq k',\ k,k'=1,\dots,q(n_0,t_r) \tag{6.36}$$

and

$$p_k(n_0,t_r) \in \mathbb{N},\ k=1,\dots,q(n_0,t_r), \qquad \sum_{k=1}^{q(n_0,t_r)} p_k(n_0,t_r) = 2p-1. \tag{6.37}$$

By construction (cf. (6.34)), the divisor $\mathcal{D}_{\underline{\hat\mu}(n_0,t_r)}$ is nonspecial for all $t_r \in (t_{0,r} - T_0, t_{0,r}+T_0)$.

In exactly the same manner as in (4.19)–(4.21) one then infers that $F_p(\,\cdot\,,n_0,t_r)$ divides $R_{2p} - G_p^2$ (since t_r is just a fixed parameter).

As in Section 4, the assumption that the Laurent polynomial $F_p(\,\cdot\,,n_0-1,t_r)$ is of full order is implied by the hypothesis that

$$\mathcal{D}_{\underline{\hat\mu}(n_0,t_r)} \in \mathcal{M}_0 \text{ for all } t_r \in (t_{0,r}-T_0,t_{0,r}+T_0). \tag{6.38}$$

The explicit formula for $\beta(n_0,t_r)$ then reads (for $t_r \in (t_{0,r}-T_0,t_{0,r}+T_0)$)

$$
\begin{aligned}
&\alpha^+(n_0,t_r)\beta(n_0,t_r) \qquad (6.39)\\
&\quad = -\frac{1}{2}\sum_{k=1}^{q(n_0,t_r)} \frac{\big(d^{p_k(n_0,t_r)-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n_0,t_r)-1}\big)\big|_{P=(\zeta,\eta)=\hat\mu_k(n_0,t_r)}}{(p_k(n_0,t_r)-1)!} \\
&\qquad \times \prod_{k'=1,\,k'\neq k}^{q(n_0,t_r)} (\mu_k(n_0,t_r)-\mu_{k'}(n_0,t_r))^{-p_k(n_0,t_r)} \\
&\qquad + \frac{1}{2}\bigg(\frac{\alpha^+(n_0,t_r)}{\alpha(n_0,t_r)} + \sum_{k=1}^{q(n_0,t_r)} p_k(n_0,t_r)\mu_k(n_0,t_r) - \frac{1}{2}\sum_{m=0}^{4p-1} E_m\bigg).
\end{aligned} \tag{6.40}
$$

With (6.20)–(6.40) in place, we can now apply the stationary formalism as summarized in Theorem 4.4, subject to the additional hypothesis (6.38), for each fixed $t_r \in (t_{0,r}-T_0,t_{0,r}+T_0)$. This yields, in particular, the quantities

$$F_p,\ G_p,\ H_p,\ \alpha,\ \beta, \text{ and } \underline{\hat\mu},\ \underline{\hat\nu} \text{ for } (n,t_r) \in \mathbb{Z}\times(t_{0,r}-T_0,t_{0,r}+T_0), \tag{6.41}$$

which are of the form (6.31)–(6.40), replacing the fixed $n_0 \in \mathbb{Z}$ by an arbitrary $n \in \mathbb{Z}$. In addition, one has the following result.

Lemma 6.2. *Assume Hypothesis* 6.1 *and condition* (6.38). *Then the following relations are valid on* $\mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$,

$$G_p^2 - F_p H_p = R_{2p}, \tag{6.42}$$

$$z(G_p^- - G_p) + z\beta F_p + \alpha H_p^- = 0, \tag{6.43}$$

$$z\beta F_p^- + \alpha H_p - G_p + G_p^- = 0, \tag{6.44}$$

$$-F_p + zF_p^- + \alpha(G_p + G_p^-) = 0, \tag{6.45}$$

$$z\beta(G_p + G_p^-) - zH_p + H_p^- = 0, \tag{6.46}$$

and hence the stationary part, (5.7), *of the algebro-geometric initial value problem holds,*

$$UV_p - V_p^+ U = 0 \ \text{ on } \mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0). \tag{6.47}$$

In particular, Lemmas 3.2–3.4 *apply.*

Lemma 6.2 now raises the following important consistency issue: On one hand, one can solve the initial value problem (6.56), (6.57) at $n = n_0$ in some interval $t_r \in (t_{0,r} - T_0, t_{0,r} + T_0)$, and then extend the quantities F_p, G_p, H_p to all $\mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$ using the stationary algorithm summarized in Theorem 4.4 as just recorded in Lemma 6.2. On the other hand, one can solve the initial value problem (6.56), (6.57) at $n = n_1$, $n_1 \neq n_0$, in some interval $t_r \in (t_{0,r} - T_1, t_{0,r} + T_1)$ with the initial condition obtained by applying the discrete algorithm to the quantities F_p, G_p, H_p starting at $(n_0, t_{0,r})$ and ending at $(n_1, t_{0,r})$. Consistency then requires that the two approaches yield the same result at $n = n_1$ for t_r in some open neighborhood of $t_{0,r}$.

Equivalently, and pictorially speaking, envisage a vertical t_r-axis and a horizontal n-axis. Then, consistency demands that first solving the initial value problem (6.56), (6.57) at $n = n_0$ in some t_r-interval around $t_{0,r}$ and using the stationary algorithm to extend F_p, G_p, H_p horizontally to $n = n_1$ and the same t_r-interval around $t_{0,r}$, or first applying the stationary algorithm starting at $(n_0, t_{0,r})$ to extend F_p, G_p, H_p horizontally to $(n_1, t_{0,r})$ and then solving the initial value problem (6.56), (6.57) at $n = n_1$ in some t_r-interval around $t_{0,r}$ should produce the same result at $n = n_1$ in a sufficiently small open t_r interval around $t_{0,r}$.

To settle this consistency issue, we will prove the following result. To this end we find it convenient to replace the initial value problem (6.56), (6.57) by the original t_r-dependent zero-curvature equation (5.6), $U_{t_r} + U\widetilde{V}_r - \widetilde{V}_r^+ U = 0$ on $\mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$.

Lemma 6.3. *Assume Hypothesis* 6.1 *and condition* (6.38). *Moreover, suppose that* (5.46)–(5.48) *hold on* $\mathbb{C} \times \{n_0\} \times (t_{0,r} - T_0, t_{0,r} + T_0)$. *Then* (5.46)–(5.48) *hold on* $\mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$, *that is,*

$$\begin{aligned} F_{p,t_r}(z,n,t_r) &= -2iG_p(z,n,t_r)\widetilde{F}_r(z,n,t_r) \\ &\quad + i\big(\widetilde{G}_r(z,n,t_r) + \widetilde{K}_r(z,n,t_r)\big)F_p(z,n,t_r), \end{aligned} \tag{6.48}$$

$$G_{p,t_r}(z,n,t_r) = iF_p(z,n,t_r)\widetilde{H}_r(z,n,t_r) - iH_p(z,n,t_r)\widetilde{F}_r(z,n,t_r), \tag{6.49}$$

$$H_{p,t_r}(z,n,t_r) = 2iG_p(z,n,t_r)\widetilde{H}_r(z,n,t_r)$$

$$- i\big(\widetilde{G}_r(z,n,t_r) + \widetilde{K}_r(z,n,t_r)\big)H_p(z,n,t_r), \tag{6.50}$$
$$(z,n,t_r) \in \mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0).$$

Moreover,

$$\begin{aligned}
\phi_{t_r}(P,n,t_r) &= i\widetilde{F}_r(z,n,t_r)\phi^2(P,n,t_r) \\
&\quad - i\big(\widetilde{G}_r(z,n,t_r) + \widetilde{K}_r(z,n,t_r)\big)\phi(P,n,t_r) + i\widetilde{H}_r(z,n,t_r), && (6.51)\\
\alpha_{t_r}(n,t_r) &= iz\widetilde{F}_r^-(z,n,t_r) \\
&\quad + i\alpha(n,t_r)\big(\widetilde{G}_r(z,n,t_r) + \widetilde{K}_r^-(z,n,t_r)\big) - i\widetilde{F}_r(z,n,t_r), && (6.52)\\
\beta_{t_r}(n,t_r) &= -i\beta(n,t_r)\big(\widetilde{G}_r^-(z,n,t_r) + \widetilde{K}_r(z,n,t_r)\big) \\
&\quad + i\widetilde{H}_r(z,n,t_r) - iz^{-1}\widetilde{H}_r^-(z,n,t_r), && (6.53)\\
&(z,n,t_r) \in \mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0).
\end{aligned}$$

Proof. By Lemma 6.2 we have (5.21), (5.22), (5.28), (5.30)–(5.32), and (6.42)–(6.46) for $(n,t_r) \in \mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$ at our disposal.

Differentiating (5.21) at $n = n_0$ with respect to t_r and inserting (6.48) and (6.49) at $n = n_0$ then yields (6.51) at $n = n_0$.

We note that the sequences $\tilde{f}_{\ell,\pm}$, $\tilde{g}_{\ell,\pm}$, $\tilde{h}_{\ell,\pm}$ satisfy the recursion relations (2.6)–(2.12) (since the homogeneous sequences satisfy these relations). Hence, to prove (6.52) and (6.53) at $n = n_0$ it remains to show

$$\begin{aligned}
\alpha_{t_r} &= i\alpha(\tilde{g}_{r,+} + \tilde{g}_{r,-}^-) + i(\tilde{f}_{r-1,-}^- - \tilde{f}_{r-1,+}), \\
\beta_{t_r} &= -i\beta(\tilde{g}_{r,+}^- + \tilde{g}_{r,-}) - i(\tilde{h}_{r-1,+}^- - \tilde{h}_{r-1,-}).
\end{aligned} \tag{6.54}$$

But this follows from (6.48), (6.50) at $n = n_0$ (cf. (6.17), (6.28))

$$\begin{aligned}
\alpha_{t_r} &= i\alpha(\tilde{g}_{r,+} - \tilde{g}_{r,-}) + i(\tilde{f}_{r,-} - \tilde{f}_{r-1,+}), \\
\beta_{t_r} &= i\beta(\tilde{g}_{r,+} - \tilde{g}_{r,-}) + i(\tilde{h}_{r-1,-} - \tilde{h}_{r,+}).
\end{aligned}$$

Inserting now (2.11) and (2.8) at $\ell = r-1$ then yields (6.54).

For the step $n = n_0 \mp 1$ we differentiate (4.36)–(4.41) (which are equivalent to (6.42)–(6.46)) and insert (6.48)–(6.50), (5.10)–(5.17) at $n = n_0$. For the case $n > n_0$ we obtain $\alpha_{t_r}^+$ and $\beta_{t_r}^+$ from (6.48), (6.50) at $n = n_0$ as before using the other two signs in (6.17), (6.28). Iterating these arguments proves (6.48)–(6.53) for $(z,n,t_r) \in \mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$. □

We summarize Lemmas 6.2 and 6.3 next.

Theorem 6.4. *Assume Hypothesis* 6.1 *and condition* (6.38)*. Moreover, suppose that*

$$\begin{aligned}
&f_{j,\pm} = f_{j,\pm}(n_0,t_r), \quad j = 0,\dots,p-1, \\
&g_{j,\pm} = g_{j,\pm}(n_0,t_r), \quad j = 0,\dots,p, \\
&h_{j,\pm} = h_{j,\pm}(n_0,t_r), \quad j = 0,\dots,p-1 \\
&\textit{for all } t_r \in (t_{0,r}-T_0, t_{0,r}+T_0),
\end{aligned} \tag{6.55}$$

satisfy the autonomous first-order system of ordinary differential equations (*for fixed* $n = n_0$)

$$f_{j,\pm,t_r} = \mathcal{F}_{j,\pm}(f_{k,-}, f_{k,+}, g_{k,-}, g_{k,+}), \quad j = 0,\dots,p-1,$$

$$\begin{aligned} g_{j,\pm,t_r} &= \mathcal{G}_{j,\pm}(f_{k,-}, f_{k,+}, h_{k,-}, h_{k,+}), \quad j = 0, \dots, p, \\ h_{j,\pm,t_r} &= \mathcal{H}_{j,\pm}(g_{k,-}, g_{k,+}, h_{k,-}, h_{k,+}), \quad j = 0, \dots, p-1, \end{aligned} \tag{6.56}$$

with $\mathcal{F}_{j,\pm}$, $\mathcal{G}_{j,\pm}$, $\mathcal{H}_{j,\pm}$ *given by* (6.16), (6.25), (6.26), *and with initial conditions*

$$\begin{aligned} &f_{j,\pm}(n_0, t_{0,r}), \quad j = 0, \dots, p-1, \\ &g_{j,\pm}(n_0, t_{0,r}), \quad j = 0, \dots, p, \\ &h_{j,\pm}(n_0, t_{0,r}), \quad j = 0, \dots, p-1. \end{aligned} \tag{6.57}$$

Then F_p, G_p, *and* H_p *as constructed in* (6.31)–(6.41) *on* $\mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$ *satisfy the zero-curvature equations* (5.6), (5.7), *and* (5.49) *on* $\mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$,

$$U_{t_r} + U\widetilde{V}_r - \widetilde{V}_r^+ U = 0, \tag{6.58}$$

$$UV_p - V_p^+ U = 0, \tag{6.59}$$

$$V_{p,t_r} - [\widetilde{V}_r, V_p] = 0 \tag{6.60}$$

with U, V_p, *and* $\widetilde{V}_r$ *given by* (5.8). *In particular,* α, β *satisfy* (5.1) *and the algebro-geometric initial value problem* (5.2), (5.3) *on* $\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$,

$$\begin{aligned} &\widetilde{\mathrm{AL}}_r(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_r} - \alpha(\tilde g_{r,+} + \tilde g_{r,-}^-) + \tilde f_{r-1,+} - \tilde f_{r-1,-}^- \\ -i\beta_{t_r} + \beta(\tilde g_{r,+}^- + \tilde g_{r,-}) - \tilde h_{r-1,-} + \tilde h_{r-1,+}^- \end{pmatrix} = 0, \\ &(\alpha,\beta)\big|_{t=t_{0,r}} = \big(\alpha^{(0)}, \beta^{(0)}\big), \end{aligned} \tag{6.61}$$

$$\text{s-AL}_p\left(\alpha^{(0)}, \beta^{(0)}\right) = \begin{pmatrix} -\alpha^{(0)}(g_{p,+} + g_{p,-}^-) + f_{p-1,+} - f_{p-1,-}^- \\ \beta^{(0)}(g_{p,+}^- + g_{p,-}) - h_{p-1,-} + h_{p-1,+}^- \end{pmatrix} = 0. \tag{6.62}$$

In addition, α, β *are given by*

$$\alpha^+(n,t_r) = \alpha(n,t_r)\bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n,t_r)} \mu_k(n,t_r)^{-p_k(n,t_r)}, \tag{6.63}$$

$$\begin{aligned} \alpha^+(n,t_r)\beta(n,t_r) = -\frac{1}{2}\sum_{k=1}^{q(n,t_r)} &\frac{\big(d^{p_k(n,t_r)-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n,t_r)-1}\big)\big|_{P=(\zeta,\eta)=\hat\mu_k(n,t_r)}}{(p_k(n,t_r)-1)!} \\ &\times \prod_{k'=1,\, k'\neq k}^{q(n,t_r)} (\mu_k(n,t_r) - \mu_{k'}(n,t_r))^{-p_k(n,t_r)} \\ + \frac{1}{2}\bigg(\bigg(\prod_{m=0}^{4p-1} E_m\bigg)^{1/2} &\prod_{k=1}^{q(n,t_r)} \mu_k(n,t_r)^{-p_k(n,t_r)} + \sum_{k=1}^{q(n,t_r)} p_k(n,t_r)\mu_k(n,t_r) \\ - \frac{1}{2}\sum_{m=0}^{4p-1} E_m\bigg), &\quad (z,n,t_r) \in \mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0). \end{aligned} \tag{6.64}$$

Moreover, Lemmas 3.2–3.4 *and* 5.2–5.4 *apply.*

As in Lemma 4.3 we now show that also in the time-dependent case, most initial divisors are well-behaved in the sense that the corresponding divisor trajectory stays away from $P_{\infty_\pm}, P_{0,\pm}$ for all $(n, t_r) \in \mathbb{Z}\times\mathbb{R}$.

Lemma 6.5. *The set $\mathcal{M}_1$ of initial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$ for which $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}$, defined via* (5.57) *and* (5.58)*, are admissible (i.e., do not contain $P_{\infty_\pm}$, $P_{0,\pm}$) and hence are nonspecial for all $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$, forms a dense set of full measure in the set* $\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$ *of nonnegative divisors of degree $2p-1$.*

Proof. Let $\mathcal{M}_{\text{sing}}$ be as introduced in the proof of Lemma 4.3. Then

$$\begin{aligned}
&\bigcup_{t_r\in\mathbb{R}} \Big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\text{sing}}) + t_r \underline{\widetilde{U}}_r^{(2)}\Big)\\
&\quad \subseteq \bigcup_{P\in\{P_{\infty_\pm},P_{0,\pm}\}} \bigcup_{t_r\in\mathbb{R}} \Big(\underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\operatorname{Sym}^{2p-2}(\mathcal{K}_{2p-1})) + t_r\underline{\widetilde{U}}_r^{(2)}\Big)
\end{aligned} \tag{6.65}$$

is of measure zero as well, since it is contained in the image of $\mathbb{R}\times\operatorname{Sym}^{2p-2}(\mathcal{K}_{2p-1})$ which misses one real dimension in comparison to the $4p-2$ real dimensions of $J(\mathcal{K}_{2p-1})$. But then

$$\begin{aligned}
&\bigcup_{(n,t_r)\in\mathbb{Z}\times\mathbb{R}} \Big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\text{sing}}) + n\underline{A}_{P_{0,-}}(P_{\infty_+}) + t_r\underline{\widetilde{U}}_r^{(2)}\Big)\\
&\quad \cup \bigg(\bigcup_{(n,t_r)\in\mathbb{Z}\times\mathbb{R}} \Big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\text{sing}}) + n\underline{A}_{P_{0,-}}(P_{\infty_+}) + t_r\underline{\widetilde{U}}_r^{(2)}\Big) + \underline{A}_{P_{0,-}}(P_{\infty_-})\bigg)
\end{aligned} \tag{6.66}$$

is also of measure zero. Applying $\underline{\alpha}_{Q_0}^{-1}$ to the complement of the set in (6.66) then yields a set $\mathcal{M}_1$ of full measure in $\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$. In particular, $\mathcal{M}_1$ is necessarily dense in $\operatorname{Sym}^{2p-1}(\mathcal{K}_{2p-1})$. $\square$

Theorem 6.6. *Let $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in \mathcal{M}_1$ be an initial divisor as in Lemma* 6.5*. Then the sequences α,β constructed from $\underline{\hat{\mu}}(n_0,t_{0,r})$ as described in Theorem* 6.4 *satisfy Hypothesis* 5.1*. In particular, the solution α,β of the algebro-geometric initial value problem* (6.63)*,* (6.64) *is global in $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$.*

Proof. Starting with $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in \mathcal{M}_1$, the procedure outlined in this section and summarized in Theorem 6.4 leads to $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}$ for all $(n,t_r)\in\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$ such that (5.57) and (5.58) hold. But if α,β should blow up, then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ or $\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}$ must hit one of $P_{\infty_\pm}$ or $P_{0,\pm}$ which is excluded by our choice of initial condition. $\square$

We note, however, that in general (i.e., unless one is, e.g., in the special periodic or unitary case), $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ will get arbitrarily close to $P_{\infty_\pm}$, $P_{0,\pm}$ since straight motions on the torus are generically dense (see e.g. [11, Sect. 51] or [28, Sects. 1.4, 1.5]) and hence no uniform bound (and no uniform bound away from zero) on the sequences $\alpha(n,t_r),\beta(n,t_r)$ exists as (n,t_r) varies in $\mathbb{Z}\times\mathbb{R}$. In particular, these complex-valued algebro-geometric solutions of the Ablowitz–Ladik hierarchy initial value problem, in general, will not be quasi-periodic with respect to n or t_r (cf. the usual definition of quasi-periodic functions, e.g., in [38, p. 31]).

Appendix A. Hyperelliptic Curves in a Nutshell

We provide a very brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [17], [18], and [34], as well as monographs dedicated to integrable systems such as [12, Ch. 2], [23, App. A, B], [42, App. A].

Fix $g \in \mathbb{N}$. The hyperelliptic curve $\mathcal{K}_g$ of genus g used in Sections 3–6 is defined by

$$\mathcal{K}_g \colon \mathcal{F}_g(z,y) = y^2 - R_{2g+2}(z) = 0, \quad R_{2g+2}(z) = \prod_{m=0}^{2g+1} (z - E_m), \tag{A.1}$$

$$\{E_m\}_{m=0,\dots,2g+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 0, \dots, 2g+1. \tag{A.2}$$

The curve (A.1) is compactified by adding the points P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $g+1$ nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$ and denotes

$$\mathcal{C} = \bigcup_{j \in \{1,\dots,g+1\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Defining the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.4}$$

and introducing the holomorphic function

$$R_{2g+2}(\,\cdot\,)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2g+1} (z - E_m) \right)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5), one considers

$$\mathcal{M}_g = \{(z, \sigma R_{2g+2}(z)^{1/2}) \,|\, z \in \mathbb{C},\ \sigma \in \{\pm 1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.6}$$

by extending $R_{2g+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_g$ is then the set $\mathcal{M}_g$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_g$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_g)$ of $\mathcal{K}_g$ is given by

$$\mathcal{B}(\mathcal{K}_g) = \{(E_m, 0)\}_{m=0,\dots,2g+1} \tag{A.7}$$

and finite points P on $\mathcal{K}_g$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_g$ satisfying $\mathcal{F}_g(z,y) = y^2 - R_{2g+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_g \setminus (\mathcal{B}(\mathcal{K}_g) \cup \{P_{\infty_+}, P_{\infty_-}\})$ are given by $\zeta_{P_0} = z - z_0$, near $P_{\infty_\pm}$ by $\zeta_{P_{\infty_\pm}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_g)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The Riemann surface $\mathcal{K}_g$ defined in this manner has topological genus g. Moreover, we introduce the holomorphic sheet exchange map (involution)

$$* \colon \mathcal{K}_g \to \mathcal{K}_g, \quad P = (z,y) \mapsto P^* = (z, -y),\ P_{\infty_\pm} \mapsto P_{\infty_\pm}^* = P_{\infty_\mp}. \tag{A.8}$$

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_g$ with zeros of order $g-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1} dz}{y}, \quad j = 1, \dots, g, \tag{A.9}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_g$. Introducing the invertible matrix C in $\mathbb{C}^g$,

$$\begin{aligned} &C = (C_{j,k})_{j,k=1,\dots,g}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ &\underline{c}(k) = (c_1(k), \dots, c_n(k)), \quad c_j(k) = C_{j,k}^{-1},\ j,k = 1, \dots, g, \end{aligned} \tag{A.10}$$

the corresponding basis of normalized holomorphic differentials ω_j, $j=1,\dots,g$, on $\mathcal{K}_g$ is given by

$$\omega_j = \sum_{\ell=1}^{g} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k=1,\dots,g. \tag{A.11}$$

Here $\{a_j, b_j\}_{j=1,\dots,g}$ is a homology basis for $\mathcal{K}_g$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k},\ a_j \circ a_k = 0,\ b_j \circ b_k = 0, \quad j,k=1,\dots,g. \tag{A.12}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,g}$ we also recall the canonical dissection of $\mathcal{K}_g$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_g$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_g$ given by

$$\partial\widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g^{-1} b_g^{-1}. \tag{A.13}$$

Let $\mathcal{M}(\mathcal{K}_g)$ and $\mathcal{M}^1(\mathcal{K}_g)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_g$. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind, $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_g)$, are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is, $\int_{a_j} \omega^{(2)} = 0$, $j=1,\dots,g$. Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_g$ not of the first or second kind is said to be of the third kind. A differential of the third kind $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_g)$ is usually normalized by the vanishing of its a-periods, that is, $\int_{a_j} \omega^{(3)} = 0$, $j=1,\dots,g$. A normal differential of the third kind $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2 \in \widehat{\mathcal{K}}_g$, $P_1 \neq P_2$, by definition, has simple poles at P_j with residues $(-1)^{j+1}$, $j=1,2$ and vanishing a-periods.

Next, define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,g}$ by

$$\tau_{j,\ell} = \int_{b_\ell} \omega_j, \quad j,\ell=1,\dots,g. \tag{A.14}$$

Then

$$\operatorname{Im}(\tau) > 0 \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j,\ell=1,\dots,g. \tag{A.15}$$

Associated with τ one introduces the period lattice

$$L_g = \{\underline{z} \in \mathbb{C}^g \mid \underline{z} = \underline{m} + \underline{n}\tau,\ \underline{m}, \underline{n} \in \mathbb{Z}^g\}. \tag{A.16}$$

Next, fix a base point $Q_0 \in \mathcal{K}_g \setminus \{P_{0,\pm}, P_{\infty\pm}\}$, denote by $J(\mathcal{K}_g) = \mathbb{C}^g/L_g$ the Jacobi variety of $\mathcal{K}_g$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_g \to J(\mathcal{K}_g), \quad \underline{A}_{Q_0}(P) = \left(\int_{Q_0}^{P} \omega_1, \dots, \int_{Q_0}^{P} \omega_g \right) \pmod{L_g}, \quad P \in \mathcal{K}_g. \tag{A.17}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_g) \to J(\mathcal{K}_g), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P) \underline{A}_{Q_0}(P), \tag{A.18}$$

where $\operatorname{Div}(\mathcal{K}_g)$ denotes the set of divisors on $\mathcal{K}_g$. Here $\mathcal{D} \colon \mathcal{K}_g \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_g$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_g$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_g)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on $\mathcal{K}_g$ we shall employ the following (additive) notation,

$$\begin{aligned} &\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ &\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_g, \quad Q_0 \in \mathcal{K}_g, \ m \in \mathbb{N}, \end{aligned} \tag{A.19}$$

where for any $Q \in \mathcal{K}_g$,

$$\mathcal{D}_Q \colon \mathcal{K}_g \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_g \setminus \{Q\}, \end{cases} \tag{A.20}$$

and $\operatorname{Sym}^n \mathcal{K}_g$ denotes the nth symmetric product of $\mathcal{K}_g$. In particular, $\operatorname{Sym}^m \mathcal{K}_g$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ of degree m.

For $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_g)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_g) \,|\, \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(g-1),\ f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}, \omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}, \tag{A.21}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_g) \,|\, f = 0 \text{ or } (f) \geq \mathcal{D}\}, \quad r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \tag{A.22}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_g) \,|\, \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \quad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}), \tag{A.23}$$

with $i(\mathcal{D})$ the index of speciality of $\mathcal{D}$, one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental fact.

Theorem A.1. *Let $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^g \mathcal{K}_g$, $\underline{Q} = \{Q_1, \dots, Q_g\}$. Then,*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \tag{A.24}$$

if and only if $\{Q_1, \dots, Q_g\}$ contains s pairings of the type $\{P, P^*\}$. (This includes, of course, branch points for which $P = P^*$.) One has $s \leq g/2$.

Appendix B. Some Interpolation Formulas

In this appendix we recall a useful interpolation formula which goes beyond the standard Lagrange interpolation formula for polynomials in the sense that the zeros of the interpolating polynomial need not be distinct.

Lemma B.1 ([24])**.** *Let $p \in \mathbb{N}$ and S_{p-1} be a polynomial of degree $p-1$. In addition, let F_p be a monic polynomial of degree p of the form*

$$F_p(z) = \prod_{k=1}^{q} (z - \mu_k)^{p_k}, \quad p_j \in \mathbb{N},\ \mu_j \in \mathbb{C},\ j = 1, \dots, q, \quad \sum_{k=1}^{q} p_k = p. \tag{B.1}$$

Then,

$$S_{p-1}(z) = F_p(z) \sum_{k=1}^{q} \sum_{\ell=0}^{p_k - 1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!(p_k - \ell - 1)!} \tag{B.2}$$

$$\times \left(\frac{d^{p_k-\ell-1}}{d\zeta^{p_k-\ell-1}} \left((z-\zeta)^{-1} \prod_{k'=1,\, k'\neq k}^{q} (\zeta-\mu_{k'})^{-p_{k'}} \right) \right) \bigg|_{\zeta=\mu_k}, \quad z \in \mathbb{C}.$$

In particular, S_{p-1} is uniquely determined by prescribing the p values

$$S_{p-1}(\mu_k), S'_{p-1}(\mu_k), \dots, S^{(p_k-1)}_{p-1}(\mu_k), \quad k = 1, \dots, q, \tag{B.3}$$

at the given points $\mu_1 \dots, \mu_q$.
Conversely, prescribing the p complex numbers

$$\alpha_k^{(0)}, \alpha_k^{(1)}, \dots, \alpha_k^{(p_k-1)}, \quad k = 1, \dots, q, \tag{B.4}$$

there exists a unique polynomial T_{p-1} of degree $p-1$,

$$T_{p-1}(z) = F_p(z) \sum_{k=1}^{q} \sum_{\ell=0}^{p_k-1} \frac{\alpha_k^{(\ell)}}{\ell!(p_k-\ell-1)!} \tag{B.5}$$

$$\times \left(\frac{d^{p_k-\ell-1}}{d\zeta^{p_k-\ell-1}} \left((z-\zeta)^{-1} \prod_{k'=1,\, k'\neq k}^{q} (\zeta-\mu_{k'})^{-p_{k'}} \right) \right) \bigg|_{\zeta=\mu_k}, \quad z \in \mathbb{C}, \tag{B.6}$$

such that

$$T_{p-1}(\mu_k) = \alpha_k^{(0)}, T'_{p-1}(\mu_k) = \alpha_k^{(1)}, \dots, T^{(p_k-1)}_{p-1}(\mu_k) = \alpha_k^{(p_k-1)}, \quad k = 1, \dots, q. \tag{B.7}$$

We briefly mention two special cases of (B.2). First, assume the generic case where all zeros of F_p are distinct, that is,

$$q = p, \quad p_k = 1, \quad \mu_k \neq \mu_{k'} \text{ for } k \neq k',\ k, k' = 1, \dots, p. \tag{B.8}$$

In this case (B.2) reduces to the classical Lagrange interpolation formula

$$S_{p-1}(z) = F_p(z) \sum_{k=1}^{p} \frac{S_{p-1}(\mu_k)}{((dF_p(\zeta)/d\zeta)|_{\zeta=\mu_k})(z-\mu_k)}, \quad z \in \mathbb{C}. \tag{B.9}$$

Second, we consider the other extreme case where all zeros of F_p coincide, that is,

$$q = 1, \quad p_1 = p, \quad F_p(z) = (z-\mu_1)^p, \quad z \in \mathbb{C}. \tag{B.10}$$

In this case (B.2) reduces of course to the Taylor expansion of S_{p-1} around $z = \mu_1$,

$$S_{p-1}(z) = \sum_{\ell=0}^{p-1} \frac{S^{(\ell)}_{p-1}(\mu_1)}{\ell!} (z-\mu_1)^\ell, \quad z \in \mathbb{C}. \tag{B.11}$$

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