THE ASYMPTOTIC LIMITS OF ZERO MODES OF MASSLESS DIRAC OPERATORS

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Abstract. Asymptotic behaviors of zero modes of the massless Dirac operator $H = \alpha \cdot D + Q(x)$ are discussed, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices, $D = \frac{1}{i}\nabla_x$, and $Q(x) = \big(q_{jk}(x)\big)$ is a 4×4 Hermitian matrix-valued function with $|q_{jk}(x)| \le C\langle x\rangle^{-\rho}$, $\rho > 1$. We shall show that for every zero mode f, the asymptotic limit of $|x|^2 f(x)$ as $|x| \to +\infty$ exists. The limit is expressed in terms of an integral of $Q(x)f(x)$.

Key words: Dirac operators, Weyl-Dirac operators, zero modes, asymptotic limits

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1. Introduction

In this paper we study asymptotic behaviors of zero modes (i.e., eigenfunctions with the zero eigenvalue; see Definition 1.1) of the massless Dirac operator

$$H = \alpha \cdot D + Q(x), \quad D = \frac{1}{i}\nabla_x, \;\; x \in \mathbb{R}^3, \tag{1.1}$$

where $\alpha = (\alpha_1, \, \alpha_2, \, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \qquad (j = 1, \, 2, \, 3)$$

with the 2×2 zero matrix $\mathbf{0}$ and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $Q(x)$ is a 4×4 Hermitian matrix-valued function decaying at infinity.

Particular emphasis must be placed on the fact that one can view the operator (1.1) as a generalization of the operator

$$\alpha \cdot \big(D - A(x)\big) + q(x)I_4, \tag{1.2}$$

where (q, A) is an electromagnetic potential and I_4 is the 4×4 identity matrix, by taking $Q(x)$ to be $-\alpha \cdot A(x) + q(x)I_4$. In the case where $q(x) \equiv 0$, the operator (1.2) becomes of the form

$$\alpha \cdot \big(D - A(x)\big) = \begin{pmatrix} \mathbf{0} & \sigma \cdot (D - A(x)) \\ \sigma \cdot (D - A(x)) & \mathbf{0} \end{pmatrix}. \tag{1.3}$$

The component $\sigma \cdot (D - A(x))$ is known as the Weyl-Dirac operator. See Balinsky and Evans [5].

The paper by Fröhlich, Lieb and Loss [12] revealed that the existence of zero modes of a Weyl-Dirac operator plays a crucial role in the study of stability of Coulomb systems with magnetic fields. In connection with [12], Loss and Yau [13] constructed, for the first time ever, examples of vector potentials $A(x)$ for which the corresponding Weyl-Dirac operators have zero modes. After the work by Loss and Yau [13] was published, there have been many contributions on the study of zero modes of Weyl-Dirac operators. See Adam, Muratori and Nash [1], [2], [3], Balinsky and Evans [4], [5], [6], Bugliaro, Fefferman and Graf [7], Elton [8] and, Erdös and Solovej [9], [10], [11].

We would like to mention Loss and Yau's example of the zero mode ψ_L and the vector potential A_L:

$$\psi_L(x) = \langle x \rangle^{-3}\big(I_2 + i\sigma \cdot x\big)\phi_0, \tag{1.4}$$

$$A_L(x) = 3\langle x \rangle^{-4}\big\{(1 - |x|^2)w_0 + 2(w_0 \cdot x)x + 2w_0 \times x\big\}, \tag{1.5}$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$, $\phi_0 = {}^t(1, \, 0)$, and

$$w_0 = \big(\phi_0 \cdot (\sigma_1\phi_0), \, \phi_0 \cdot (\sigma_2\phi_0), \, \phi_0 \cdot (\sigma_3\phi_0)\big).$$

It follows from (1.4) that

$$\lim_{r\to+\infty} r^2\psi_L(r\omega) = (i\sigma\cdot\omega)\phi_0, \tag{1.6}$$

where $r = |x|$ and $\omega = x/|x|$.

Adam, Muratori and Nash [1], [2], [3] developed the idea of Loss and Yau [13] and constructed many examples of the pairs of zero modes and vector potentials in a systematic way. Among other things, it is important in the context of the present paper that they constructed the zero modes of the form

$$\psi_A(x) = \langle x\rangle^{-2}U(x)\phi_0, \tag{1.7}$$

where $U(x)$ is a 2×2 matrix-valued function with the limit

$$U_\infty(\omega) := \lim_{r\to+\infty} U(r\omega). \tag{1.8}$$

Thus, it follows from (1.7) and (1.8) that $r^2\psi_A(r\omega)$ has a limit as $r\to+\infty$:

$$\lim_{r\to+\infty} r^2\psi_A(r\omega) = U_\infty(\omega)\phi_0. \tag{1.9}$$

It is apparent from (1.6) and (1.9) that both $\psi_L(x)$ and $\psi_A(x)$ behave in the same manner as $r\to+\infty$. We would like to emphasize that this is not a sheer coincidence. Actually, Theorem 1.1 below asserts that every zero mode $\psi(x)$ of the Weyl-Dirac operator behaves like

$$\psi(r\omega) \sim r^{-2}\, i(\sigma\cdot\omega)\psi_0 \qquad (\psi_0\in\mathbb{C}^2 \text{ a constant vector}) \tag{1.10}$$

for $r\to+\infty$ if the vector potential A satisfies $|A(x)| \le const.\langle x\rangle^{-\rho}\quad(\rho>1)$.

The purpose of the present paper is to show that every zero mode $f(x)$ of the massless Dirac operator (1.1) behaves like

$$f(r\omega) \sim r^{-2}\, i(\alpha\cdot\omega)f_0 \qquad (f_0\in\mathbb{C}^4 \text{ a constant vector}) \tag{1.11}$$

for $r\to+\infty$ if each component of $Q(x)$ satisfies the inequality (1.13) in Assumption (A) below; see Theorem 1.1. We should like to note that Theorem 1.1 can be regarded as a refinement of our previous result [14, Theorem 2.1], where we proved that every zero mode $f(x)$ of the operator (1.1) satisfies the inequality

$$|f(x)| \le const.\langle x\rangle^{-2} \tag{1.12}$$

under the same assumption as in the present paper.

Notation.
By $L^2 = L^2(\mathbb{R}^3)$, we mean the Hilbert space of square-integrable functions on $\mathbb{R}^3$, and we introduce a Hilbert space $\mathcal{L}^2$ by $\mathcal{L}^2 = [L^2(\mathbb{R}^3)]^4$, where the inner product is given by

$$(f,g)_{\mathcal{L}^2} = \sum_{j=1}^{4}(f_j,g_j)_{L^2}$$

for $f = {}^t(f_1, f_2, f_3, f_4)$ and $g = {}^t(g_1, g_2, g_3, g_4)$. By $H^1(\mathbb{R}^3)$ we denote the Sobolev space of order 1, and by $\mathcal{H}^1$ we mean the Hilbert space $[H^1(\mathbb{R}^3)]^4$. When we mention the Weyl-Dirac operator, we must handle two-vectors (two components spinors) which will be denoted by ψ.

Assumption (A).
Each element $q_{jk}(x)$ $(j,\, k = 1, \cdots, 4)$ of $Q(x)$ is a measurable function satisfying

$$|q_{jk}(x)| \le C_q \langle x\rangle^{-\rho} \quad (\rho > 1), \tag{1.13}$$

where C_q is a positive constant. Moreover, $Q(x)$ is a Hermitian matrix for each $x \in \mathbb{R}^3$.

Note that, under Assumption (A), the Dirac operator (1.1) is a self-adjoint operator in $\mathcal{L}^2$ with $\mathrm{Dom}(H) = \mathcal{H}^1$. The self-adjoint realization will be denoted by H again.

Definition 1.1. By a zero mode, we mean a function $f \in \mathrm{Dom}(H)$ which satisfies $Hf = 0$.

We are now in a position to state the main result of the present paper.

Theorem 1.1. *Suppose Assumption* (A) *is satisfied. Let f be a zero mode of the operator* (1.1). *Then for any $\omega \in \mathbb{S}^2$*

$$\lim_{r\to+\infty} r^2 f(r\omega) = -\frac{i}{4\pi}\,(\alpha\cdot\omega)\int_{\mathbb{R}^3} Q(y)f(y)\,dy, \tag{1.14}$$

where the convergence being uniform with respect to $\omega \in \mathbb{S}^2$.

In connection with the expression $f(r\omega)$ in (1.14), it is worthy to note that every zero mode is a continuous function (see Theorem 2.1 in the beginning of section 2).

Since $\alpha\cdot\omega$ is a unitary matrix, we have an immediate corollary to Theorem 1.1.

Corollary 1.1. *For any $\omega \in \mathbb{S}^2$*

$$\lim_{r\to+\infty} r^2 |f(r\omega)| = \frac{1}{4\pi}\,\Big|\int_{\mathbb{R}^3} Q(y)f(y)\,dy\Big|. \tag{1.15}$$

One should note that Corollary 1.1 assures the ω-independence of the limit of $r^2|f(r\omega)|$ for $r \to \infty$. In particular we see that Corollary 1.1 implies an interesting fact:

$$\lim_{r\to+\infty} r^2 f(r\omega) = 0 \ \text{ for some (any) } \omega \iff \int_{\mathbb{R}^3} Q(y)f(y)\,dy = 0. \tag{1.16}$$

As for a zero mode of the Weyl-Dirac operator, we have the following theorem, which is also a corollary to Theorem 1.1.

Theorem 1.2. *Suppose*

$$|A(x)| \le C\langle x\rangle^{-\rho} \quad (\rho > 1), \tag{1.17}$$

where C is a positive constant. Let ψ be a zero mode of the Weyl-Dirac operator $\sigma \cdot (D - A(x))$. Then for any $\omega \in \mathbb{S}^2$

$$\begin{aligned} &\lim_{r\to+\infty} r^2\psi(r\omega) \\ &= \frac{i}{4\pi}\int_{\mathbb{R}^3} \big\{\big(\omega\cdot A(y)\big)I_2 + i\sigma\cdot\big(\omega\times A(y)\big)\big\}\psi(y)\,dy, \end{aligned} \tag{1.18}$$

where the convergence being uniform with respect to $\omega \in \mathbb{S}^2$.

Erdös and Solovey [9] generalized the examples by Loss and Yau [13] from the geometrical point of view, and proposed an intrinsic way of producing magnetic fields on $\mathbb{S}^3$ and $\mathbb{R}^3$ for which the corresponding Weyl-Dirac operators have zero modes. They did not mention asymptotic properties of their zero modes, which were obviously not their concern though.

It is interesting from our point of view that Elton [8] showed that for any integer $m \ge 0$ and an open subset $\Omega \subset \mathbb{R}^3$ there exists a vector potential $A \in [C_0^\infty(\mathbb{R}^3)]^3$ such that supp $A \subset \Omega$ and the corresponding Weyl-Dirac operator has a degeneracy of zero modes with multiplicity m. This fact, together with Theorem 1.2, indicates that the asymptotic behavior of vector potential A does not affect the asymptotic behavior of zero modes of the corresponding Weyl-Dirac operator as long as A satisfies the hypothesis (1.17).

2. Proofs

The proof of Theorem 1.1 is based on an estimate, which was established in our previous paper [14, Theorem 2.1].

Theorem 2.1 (Saitō and Umeda)**.** *Suppose Assumption* (A) *is satisfied. Let f be a zero mode of the operator* (1.1)*. Then*

(i) *the inequality*

$$|f(x)| \le C\langle x\rangle^{-2} \tag{2.1}$$

holds for all $x \in \mathbb{R}^3$, where the constant $C(= C_f)$ depends only on the zero mode f;

(ii) *the zero mode f is a continuous function on $\mathbb{R}^3$.*

Also, the proof of Theorem 1.1 is based on a fact that every zero mode f of the operator (1.1) satisfies the integral equation

$$f(x) = -\frac{i}{4\pi}\int_{\mathbb{R}^3} \frac{\alpha\cdot(x-y)}{|x-y|^3}\, Q(y)f(y)\,dy. \tag{2.2}$$

This fact was established in our previous paper [14] too; see (5.3) in Section 5 of [14].

Remark 2.1. *If we formally take the limit of* (2.3) *below as* $r \to +\infty$*, then we can readily obtain* (1.14)*. Unfortunately, this argument is not rigorous.*

Proof of Theorem 1.1. We begin with the integral equation (2.2) with $x = r\omega\ \ (\omega \in \mathbb{S}^2)$, and multiply the both sides of (2.2) by r^2:

$$r^2 f(r\omega) = -\frac{i}{4\pi}\int_{\mathbb{R}^3} \frac{\alpha\cdot(\omega - r^{-1}y)}{|\omega - r^{-1}y|^3}\, Q(y)f(y)\,dy. \tag{2.3}$$

We then see from (2.3) that

$$\begin{aligned} &r^2 f(r\omega) + \frac{i}{4\pi}\int_{\mathbb{R}^3} (\alpha\cdot\omega)\, Q(y)f(y)\,dy \\ &= \frac{i}{4\pi}\int_{\mathbb{R}^3} \alpha\cdot\Big\{\omega - \frac{\omega - r^{-1}y}{|\omega - r^{-1}y|^3}\Big\}\, Q(y)f(y)\,dy. \end{aligned} \tag{2.4}$$

Now let $\varepsilon > 0$ be given so that $0 < \varepsilon < 1/2$, and choose R_0 so that

$$R_0^{-\rho+1} < \varepsilon. \tag{2.5}$$

Note that $\rho > 1$; see Assumption (A). For $r \geq 2R_0$, we define

$$E_1 := \{\, y \in \mathbb{R}^3 \mid |y| \leq R_0 \,\}, \tag{2.6}$$

$$E_2 := \{\, y \in \mathbb{R}^3 \mid |y| > R_0,\ |r\omega - y| \leq \frac{r}{2} \,\}, \tag{2.7}$$

$$E_3 := \{\, y \in \mathbb{R}^3 \mid |y| > R_0,\ |r\omega - y| > \frac{r}{2} \,\}, \tag{2.8}$$

and accordingly we decompose the integral on the right hand side of (2.4) into three parts:

$$I_r(\omega) := \frac{i}{4\pi}\int_{E_1} \alpha\cdot\Big\{\omega - \frac{\omega - r^{-1}y}{|\omega - r^{-1}y|^3}\Big\}\, Q(y)f(y)\,dy, \tag{2.9}$$

$$II_r(\omega) := \frac{i}{4\pi}\int_{E_2} \alpha\cdot\Big\{\omega - \frac{\omega - r^{-1}y}{|\omega - r^{-1}y|^3}\Big\}\, Q(y)f(y)\,dy, \tag{2.10}$$

$$III_r(\omega) := \frac{i}{4\pi}\int_{E_3} \alpha\cdot\Big\{\omega - \frac{\omega - r^{-1}y}{|\omega - r^{-1}y|^3}\Big\}\, Q(y)f(y)\,dy. \tag{2.11}$$

We thus have

$$\begin{aligned} &r^2 f(r\omega) + \frac{i}{4\pi}\int_{\mathbb{R}^3} (\alpha\cdot\omega)\, Q(y)f(y)\,dy \\ &= I_r(\omega) + II_r(\omega) + III_r(\omega). \end{aligned} \tag{2.12}$$

To estimate $I_r(\omega)$, we first note that

$$\frac{1}{2} \le |\omega - r^{-1}y| \le \frac{3}{2} \quad \text{if } |y| \le R_0,\ r \ge 2R_0. \tag{2.13}$$

This implies that

$$\begin{aligned}\Big|\omega - \frac{\omega - r^{-1}y}{|\omega - r^{-1}y|^3}\Big| &\le \Big(\frac{1}{2}\Big)^{-3} \big|\, |\omega - r^{-1}y|^3\omega - (\omega - r^{-1}y)\,\big| \\ &= 2^3 \,\big|\, (|\omega - r^{-1}y|^3 - 1)\omega + r^{-1}y \,\big|\end{aligned} \tag{2.14}$$

when $|y| \le R_0,\ r \ge 2R_0$. Moreover, we have

$$\begin{aligned}\big|\, |\omega - r^{-1}y|^3 - 1 \,\big| &= \big|\, |\omega - r^{-1}y| - 1 \,\big| \\ &\qquad \times \big(|\omega - r^{-1}y|^2 + |\omega - r^{-1}y| + 1\big) \\ &\le \big|\, |\omega - r^{-1}y| - 1 \,\big| \times \frac{19}{4} \\ &= \frac{19}{4}\frac{1}{|\omega - r^{-1}y| + 1} \times \big|\, |\omega - r^{-1}y|^2 - 1 \,\big| \\ &\le \frac{19}{6}\big| -2\omega\cdot(r^{-1}y) + r^{-2}|y|^2 \,\big|\end{aligned} \tag{2.15}$$

provided that $|y| \le R_0,\ r \ge 2R_0$. Combining (2.14) and (2.15), we obtain

$$\Big|\omega - \frac{\omega - r^{-1}y}{|\omega - r^{-1}y|^3}\Big| \le 2^3\Big\{\frac{19}{6}(2R_0r^{-1} + R_0^2r^{-2}) + R_0r^{-1}\Big\} \tag{2.16}$$

whenever $|y| \le R_0,\ r \ge 2R_0$. Now it follows from (2.16), Theorem 2.1, Assumption (A) and the anti-commutation relation that

$$\begin{aligned}|I_r(\omega)| &\le \frac{1}{4\pi} \times 2^3\Big\{\frac{22}{3}R_0r^{-1} + \frac{19}{6}R_0^2r^{-2}\Big\}\int_{E_1} |Q(y)f(y)|\,dy \\ &\le C_1R_0r^{-1}\int_{\mathbb{R}^3} \langle y\rangle^{-\rho-2}\,dy \\ &\le C_1'R_0r^{-1}\end{aligned} \tag{2.17}$$

for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$, where the constant C_1' is dependent only on the constant C_f in Theorem 2.1 and the constant C_q in Assumption (A). Note that in the first inequality in (2.17) we have used the fact that $|(\alpha\cdot x)f| = |x|\,|f|$ for all $x \in \mathbb{R}^3$ and all $f \in \mathbb{C}^4$, and that in the third inequality we have used the fact that $\langle y\rangle^{-\rho-2}$ is integrable on $\mathbb{R}^3$ since $\rho + 2 > 3$.

As for $I\!I_r(\omega)$, it follows again from Theorem 2.1 and Assumption (A) that

$$|I\!I_r(\omega)| \le C_2\int_{E_2}\Big(1 + \frac{1}{|\omega - r^{-1}y|^2}\Big)\langle y\rangle^{-\rho-2}\,dy \tag{2.18}$$

for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$, where the constant C_2 depends only on the constants C_f and C_q. To estimate the right hand side of (2.18), we need the fact that

$$y \in E_2 \;\Rightarrow\; |y| \ge \frac{r}{2}. \tag{2.19}$$

Thus, the integral on the right hand side of (2.18) is estimated by

$$\begin{aligned} &C\Big\{ \int_{|y|\ge r/2} \langle y\rangle^{-\rho-2}\,dy + r^{-\rho-2}\int_{|r\omega-y|\le r/2} \frac{r^2}{|r\omega-y|^2}\,dy\Big\} \\ &\quad\le C' r^{-\rho+1} \end{aligned} \tag{2.20}$$

for for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$, with the constant C' independent of ω and r. Combining (2.20) with (2.18), we get

$$|II_r(\omega)| \le C_2'\, r^{-\rho+1} \tag{2.21}$$

for for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$, where C_2' is a constant independent of ω and r.

In the same way as in (2.18) we have

$$|III_r(\omega)| \le C_3 \int_{E_3} \Big(1 + \frac{1}{|\omega - r^{-1}y|^2}\Big)\langle y\rangle^{-\rho-2}\,dy \tag{2.22}$$

for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$. Here the constant C_3 depends only on C_f C_q. The integral on the right hand side of (2.22) is bounded by

$$\begin{aligned} &C''\Big\{ \int_{|y|\ge R_0} \langle y\rangle^{-\rho-2}\,dy + \int_{E_3} \frac{1}{|\omega - r^{-1}y|^2}\langle y\rangle^{-\rho-2}\,dy\Big\} \\ &\quad\le C''' R_0^{-\rho+1} \end{aligned} \tag{2.23}$$

for for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$, with the constant C''' independent of ω and r. Here we have used the fact that $|\omega - r^{-1}y| \ge 1/2$ for all $y \in E_3$. It follows from (2.22) and (2.23) that

$$|III_r(\omega)| \le C_3' R_0^{-\rho+1} \tag{2.24}$$

for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$, with the constant C_3' independent of ω and r.

We are now ready to combine (2.12) with (2.17), (2.21) (2.24), and we can conclude that

$$\begin{aligned} &\Big|r^2 f(r\omega) + \frac{i}{4\pi}\int_{\mathbb{R}^3} (\alpha\cdot\omega)\,Q(y)f(y)\,dy\Big| \\ &\quad\le C_1' R_0 r^{-1} + C_2'\, r^{-\rho+1} + C_3' R_0^{-\rho+1} \\ &\quad\le C_1' R_0 r^{-1} + (C_2' + C_3') R_0^{-\rho+1} \end{aligned} \tag{2.25}$$

for all $r \ge 2R_0$ and all $\omega \in \mathbb{S}^2$. Putting $R_1 := R_0/\varepsilon (> 2R_0)$, and recalling (2.5), we have shown that

$$\Big|r^2 f(r\omega) + \frac{i}{4\pi}\int_{\mathbb{R}^3} (\alpha\cdot\omega)\,Q(y)f(y)\,dy\Big| \le (C_1' + C_2' + C_3')\varepsilon \tag{2.26}$$

for for all $r \ge R_1$ and all $\omega \in \mathbb{S}^2$. Since $\varepsilon > 0$ was arbitrary, (2.26) implies the conclusion of the theorem. $\square$

Proof of Theorem 1.2. In view of (1.3) and Theorem 1.1, we only have to compute

$$-\frac{i}{4\pi}\int_{\mathbb{R}^3}(\sigma\cdot\omega)\big(-\sigma\cdot A(y)\big)\psi(y)\,dy. \tag{2.27}$$

Using the anti-commutation relation $\sigma_j\sigma_k+\sigma_k\sigma_j=2\delta_{jk}I_2$ and the facts that

$$\sigma_1\sigma_2=i\sigma_3,\ \ \sigma_2\sigma_3=i\sigma_1,\ \ \sigma_3\sigma_1=i\sigma_2, \tag{2.28}$$

we get

$$(\sigma\cdot\omega)\big(\sigma\cdot A(y)\big)=\big(\omega\cdot A(y)\big)I_2+i\sigma\cdot\big(\omega\times A(y)\big).$$

This completes the proof. □

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