RELATIVE OSCILLATION THEORY FOR STURM–LIOUVILLE OPERATORS EXTENDED

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Abstract. We extend relative oscillation theory to the case of Sturm–Liouville operators $Hu = r^{-1}(-(pu')' + qu)$ with different p's. We show that the weighted number of zeros of Wronskians of certain solutions equals the value of Krein's spectral shift function inside essential spectral gaps.

1. Introduction

In [5] we have developed an analog of classical oscillation theory for Sturm–Liouville operators which, rather than measuring the spectrum of one single operator, measures the difference between the spectra of two different operators. Hence the name relative oscillation theory. The main idea behind this extension is to replace zeros of solutions of one operator by weighted zeros of Wronskians of solutions of two different operators.

The purpose of this paper is to extend relative oscillation theory for two different Sturm–Liouville equations

$$\tau_j = \frac{1}{r}\Big(-\frac{d}{dx} p_j \frac{d}{dx} + q_j \Big), \qquad j = 0, 1. \tag{1.1}$$

In [5] we considered the case $p_0 = p_1$, here we want to extend relative oscillation theory to the case $p_0 \neq p_1$. In particular, for H_j, $j = 0, 1$, self-adjoint operators associated with τ_j, we want to show that the weighted number of zeros of Wronskians of certain solutions equals the value of Krein's spectral shift function $\xi(\lambda, H_1, H_0)$ inside essential spectral gaps. To do this, and to make sure that the spectral shift function is well-defined, we will need to find a continuous path connecting the operators H_0 and H_1 in the metric introduced by the trace norm of resolvent differences.

In Section 2 we will recall the necessary background and fix our notation. Moreover, we will present the basic result for the case of regular operators. In Section 3 we have a quick look at Sturm's classical comparison theorem for zeros of solutions and its extension to zeros of Wronskians of solutions. Section 4 is concerned with relative oscillation theory for singular operators and contains our key result, Theorem 4.11, which connects the weighted zeros of Wronskians with Krein's spectral shift function. The remaining sections contain the proofs for our main results and our final appendix collects some facts on the spectral shift functions plus some abstract results which form the functional analytic core of the proof of our main theorem.

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2. Weighted zeros of Wronskians, Prüfer angles, and regular operators

We begin by fixing our notation and reviewing some simple facts from [5]. In particular, we refer to [5] for further details.

We will consider Sturm-Liouville operators on $L^2((a,b), r\,dx)$ with $-\infty \le a < b \le \infty$ of the form

$$\tau = \frac{1}{r}\Big(-\frac{d}{dx} p \frac{d}{dx} + q\Big), \tag{2.1}$$

where the coefficients p, q, r are real-valued satisfying

$$p^{-1}, q, r \in L^1_{loc}(a,b), \quad p, r > 0. \tag{2.2}$$

We will use τ to describe the formal differentiation expression and H for the operator given by τ with separated boundary conditions at a and/or b.

If a (resp. b) is finite and q, p^{-1}, r are in addition integrable near a (resp. b), we will say a (resp. b) is a *regular* endpoint. We will say τ respectively H is *regular* if both a and b are regular.

For every $z \in \mathbb{C}\backslash\sigma_{ess}(H)$ there is a unique (up to a constant) solution $\psi_-(z,x)$ of $\tau u = zu$ which is in L^2 near a and satisfies the boundary condition at a (if any). Similarly there is such a solution $\psi_+(z,x)$ near b.

One of our main objects will be the (modified) Wronskian

$$W_x(u_0, u_1) = u_0(x)\, p_1(x) u_1'(x) - p_0(x) u_0'(x)\, u_1(x) \tag{2.3}$$

of two functions u_0, u_1 and its zeros. Here we think of u_0 and u_1 as two solutions of two different Sturm–Liouville equations

$$\tau_j = \frac{1}{r}\Big(-\frac{d}{dx} p_j \frac{d}{dx} + q_j\Big), \qquad j = 0, 1. \tag{2.4}$$

Under these assumptions $W_x(u_0, u_1)$ is absolutely continuous and satisfies

$$W'(u_0, u_1) = (q_1 - q_0) u_0 u_1 + \left(\frac{1}{p_0} - \frac{1}{p_1}\right) p_0 u_0' p_1 u_1'. \tag{2.5}$$

Next we recall the definition of Prüfer variables ρ_u, θ_u of an absolutely continuous function u:

$$u(x) = \rho_u(x) \sin(\theta_u(x)) \qquad p(x) u'(x) = \rho_u(x) \cos(\theta_u(x)). \tag{2.6}$$

If $(u(x), p(x)u'(x))$ is never $(0,0)$ and u, pu' are absolutely continuous, then ρ_u is positive and θ_u is uniquely determined once a value of $\theta_u(x_0)$ is chosen by requiring continuity of θ_u.

Notice that

$$W_x(u,v) = -\rho_u(x)\rho_v(x) \sin(\Delta_{v,u}(x)), \qquad \Delta_{v,u}(x) = \theta_v(x) - \theta_u(x). \tag{2.7}$$

Hence the Wronskian vanishes if and only if the two Prüfer angles differ by a multiple of π. We will call the total difference

$$\#_{(c,d)}(u_0, u_1) = \lceil \Delta_{1,0}(d)/\pi \rceil - \lfloor \Delta_{1,0}(c)/\pi \rfloor - 1 \tag{2.8}$$

the number of weighted sign flips in (c,d), where we have written $\Delta_{1,0}(x) = \Delta_{u_1,u_0}$ for brevity.

We take two solutions u_j, $j = 1, 2$, of $\tau_j u_j = \lambda_j u_j$ and associated Prüfer variables ρ_j, θ_j. Since we can replace $q \to q - \lambda r$ it is no restriction to assume $\lambda_0 = \lambda_1 = 0$. We remark, that in (2.6) one has to take p_j as p for u_j, $j = 0, 1$.

Lemma 2.1. *Abbreviate* $\Delta_{1,0}(x) = \theta_1(x) - \theta_0(x)$ *and suppose* $\Delta_{1,0}(x_0) \equiv 0 \mod \pi$. *If* $q_0(x) - q_1(x)$ *and* $p_0(x) - p_1(x)$ *are (i) negative, (ii) zero, or (iii) positive for a.e.* $x \in (x_0, x_0 + \varepsilon)$ *respectively for a.e.* $x \in (x_0 - \varepsilon, x_0)$*, then the same holds true for* $(\Delta_{1,0}(x) - \Delta_{1,0}(x_0))/(x - x_0)$.

Proof. By (2.5) we have

$$\begin{aligned} W_x(u_0, u_1) &= -\rho_0(x)\rho_1(x)\sin(\Delta_{1,0}(x)) \\ &= -\int_{x_0}^{x} (q_0(t) - q_1(t))u_0(t)u_1(t) + (p_1^{-1}(t) - p_0^{-1}(t))p_0u_0'(t)p_1u_1'(t)dt. \end{aligned} \tag{2.9}$$

Case (ii) follows. For (i) and (iii), first note that if $u_j(x_0) = 0$, $j = 0, 1$, we have that u_j and $p_j u_j'$, $j = 0, 1$ have the same sign close to x_0, and thus the result follows.

Now, look at $P(u_0, u_1) = \frac{u_0}{u_1} W(u_0, u_1)$ (compare (3.1) below) (resp. $P(u_1, u_0)$) and note that u_0/u_1 has constant sign near x_0. The result now follows using the fact that the derivate $P'(u_0, u_1)$ is always negative by the Picone identity (3.2) below. □

Hence $\#_{(c,d)}(u_0, u_1)$ counts the weighted sign flips of the Wronskian $W_x(u_0, u_1)$, where a sign flip is counted as $+1$ if $q_0 - q_1$ and $p_0 - p_1$ are positive in a neighborhood of the sign flip, it is counted as -1 if $q_0 - q_1$ and $p_0 - p_1$ are negative in a neighborhood of the sign flip. In particular, we obtain

Lemma 2.2. *Let* u_0, u_1 *solve* $\tau_j u_j = 0$, $j = 0, 1$, *where* $p_0 - p_1 \ge 0$ *and* $q_0 - q_1 \ge 0$. *Then* $\#_{(a,b)}(u_0, u_1)$ *equals the number sign flips of* $W(u_0, u_1)$ *inside the interval* (a, b).

Finally, we have the following extension of [5, Thm. 2.3] to the case $p_0 \ne p_1$.

Theorem 2.3. *Let* H_0, H_1 *be regular Sturm–Liouville operators associated with (2.4) and the same boundary conditions at* a *and* b. *Then*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#_{(a,b)}(\psi_{0,\pm}(\lambda_0), \psi_{1,\mp}(\lambda_1)). \tag{2.10}$$

The proof will be given in Section 5.

3. Sturm's comparison theorem

One of the core ingredients of oscillation theory is Sturm's comparison theorem for zeros of solutions. We begin by recalling this classical result.

Let u_j solve $\tau_j u_j = \lambda_j u_j$, where without loss of generality we assume $\lambda_0 = \lambda_1 = 0$. For x with $u_1(x) \ne 0$ we introduce

$$P_x(u_0, u_1) = \frac{u_0(x)}{u_1(x)} W_x(u_0, u_1) = -\rho_0^2(x) \frac{\sin(\theta_0(x))\sin(\Delta_{1,0}(x))}{\sin(\theta_1(x))}. \tag{3.1}$$

Obviously $P(u_0, u_1)$ is zero if either u_0 or the Wronskian $W(u_0, u_1)$ vanishes. Moreover, a straightforward computation, verifies the Picone identity (see [13, (2.6.4)])

$$P'(u_0, u_1) = (q_1 - q_0)u_0^2 + (p_1 - p_0)u_0'^2 - p_1 \left(u_0' - \frac{u_0 u_1'}{u_1} \right)^2, \tag{3.2}$$

which shows that $P(u_0, u_1)$ is a nonincreasing function if $q_1 \le q_0$ and $0 < p_1 \le p_0$.

Theorem 3.1 (Sturm's Comparison theorem)**.** *Let $q_0 - q_1 \geq 0$, $p_0 - q_1 \geq 0$, with once strict inequality, and $\tau_j u_j = 0$, $j = 0, 1$. Then between any two zeros of u_0 or $W(u_0, u_1)$, there is a zero of u_1.*

Similarly, between two zeros of u_1, which are not at the same time zeros of u_0, there is at least one zero of u_0 or $W(u_0, u_1)$.

Proof. Assume that u_1 has no zero, $P(u_0, u_1)$ would be well defined on the closed interval between the zeros, and be zero at its end points. This contradicts monotonicity of $P(u_0, u_1)$. The second claim is similar. □

Note that this version is slightly more general then the one usually found in the literature (cf., e.g, [13]) since it includes the case of zeros of Wronskians. For the case $p_0 = p_1$ this was already pointed out in [1]. Moreover, in this case one can also allow zeros of the Wronskian at singular endpoints [1, Cor. 2.3].

Next, the comparison theorem for Wronskians from [5] carries over to the case $p_0 \neq p_1$ without modifying the proof.

Theorem 3.2 (Comparison theorem for Wronskians)**.** *Suppose u_j satisfies $\tau_j u_j = \lambda_j u_j$, $j = 0, 1, 2$, where $\lambda_0 r - q_0 \leq \lambda_1 r - q_1 \leq \lambda_2 r - q_2$, $p_0 \leq p_1 \leq p_2$.*

If $c < d$ are two zeros of $W_x(u_0, u_1)$ such that $W_x(u_0, u_1)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0, u_2)$ in (c, d). Similarly, if $c < d$ are two zeros of $W_x(u_1, u_2)$ such that $W_x(u_1, u_2)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0, u_2)$ in (c, d).

4. Relative Oscillation Theory

After these preparations we are now ready to extend relative oscillation theory to the case $p_0 \neq p_1$. Except for Lemma 4.7 and our key result Theorem 4.11, all results in this section are straightforward modifications of the analog results in [5] and hence we omit the corresponding proofs.

Definition 4.1. *For τ_0, τ_1 possibly singular Sturm–Liouville operators as in (2.4) on (a, b), we define*

$$\underline{\#}(u_0, u_1) = \liminf_{d \uparrow b,\, c \downarrow a} \#_{(c,d)}(u_0, u_1) \quad \text{and} \quad \overline{\#}(u_0, u_1) = \limsup_{d \uparrow b,\, c \downarrow a} \#_{(c,d)}(u_0, u_1), \tag{4.1}$$

where $\tau_j u_j = \lambda_j u_j$, $j = 0, 1$.

We say that $\#(u_0, u_1)$ exists, if $\overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1)$, and write

$$\#(u_0, u_1) = \overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1). \tag{4.2}$$

in this case.

By Lemma 2.1 one infers that $\#(u_0, u_1)$ exists if $p_0 - p_1$ and $q_0 - \lambda_0 r - q_1 + \lambda_1 r$ have the same definite sign near the endpoints a and b.

Theorem 4.2 (Triangle inequality for Wronskians)**.** *Suppose u_j, $j = 0, 1, 2$ are given functions with u_j, $p_j u_j'$ absolutely continuous and $(u_j(x), p_j(x) u_j'(x)) \neq (0, 0)$ for all x. Then*

$$\underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) - 1 \leq \underline{\#}(u_0, u_2) \leq \underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) + 1 \tag{4.3}$$

and similarly for $\underline{\#}$ replaced by $\overline{\#}$.

We recall that in classical oscillation theory τ is called oscillatory if a solution of $\tau u = 0$ has infinitely many zeros.

Definition 4.3. *We call τ_1 relatively nonoscillatory with respect to τ_0, if the quantities $\underline{\#}(u_0,u_1)$ and $\overline{\#}(u_0,u_1)$ are finite for all solutions $\tau_j u_j = 0$, $j = 0,1$. We call τ_1 relatively oscillatory with respect to τ_0, if one of the quantities $\underline{\#}(u_0,u_1)$ or $\overline{\#}(u_0,u_1)$ is infinite for some solutions $\tau_j u_j = 0$, $j = 0,1$.*

Note that this definition is in fact independent of the solutions chosen as a straightforward application of our triangle inequality (cf. Theorem 4.2) shows.

Corollary 4.4. *Let $\tau_j u_j = \tau_j v_j = 0$, $j = 0,1$. Then*

$$|\underline{\#}(u_0,u_1) - \underline{\#}(v_0,v_1)| \le 4, \quad |\overline{\#}(u_0,u_1) - \overline{\#}(v_0,v_1)| \le 4. \tag{4.4}$$

The bounds can be improved using our comparison theorem for Wronskians to be ≤ 2 in the case of perturbations of definite sign.

If τ_0 is nonoscillatory our definition reduces to the classical one.

Lemma 4.5. *Suppose τ_0 is a nonoscillatory operator, then τ_1 is relatively nonoscillatory (resp. oscillatory) with respect to τ_0, if and only if τ_1 is nonoscillatory (resp. oscillatory).*

To demonstrate the usefulness of Definition 4.3, we now establish its connection with the spectra of self-adjoint operators associated with τ_j, $j = 0,1$.

Theorem 4.6. *Let H_j be self-adjoint operators associated with τ_j, $j = 0,1$. Then*

(i) *$\tau_0 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_1$ if and only if* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$.

(ii) *Suppose* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$ *and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for one $\lambda \in [\lambda_0,\lambda_1]$. Then it is relatively nonoscillatory for all $\lambda \in [\lambda_0,\lambda_1]$ if and only if* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) < \infty$.

For a practical application of this theorem one needs of course criteria when $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for λ inside an essential spectral gap. Without loss of generality we only consider the case where one endpoint is regular.

Lemma 4.7. *Let H_0 be bounded from below. Suppose a is regular (b singular) and*

(i) $\lim_{x\to b} r(x)^{-1}(q_0(x) - q_1(x)) = 0$, *$\frac{q_0}{r}$ is bounded near b, and*

(ii) $\lim_{x\to b} p_1(x)p_0(x)^{-1} = 1$.

Then $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for every $\lambda \in \mathbb{R}\backslash\sigma_{ess}(H_0)$.

The analogous result holds for a singular and b regular.

The proof will be given in Section 5.

Our next task is to reveal the precise relation between the number of weighted sign flips and the spectra of H_1 and H_0. The special case $H_0 = H_1$ is covered by [1]:

Theorem 4.8 ([1])**.** *Let H_0 be a self-adjoint operator associated with τ_0 and suppose $[\lambda_0,\lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) = \#(\psi_{0,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_1)). \tag{4.5}$$

Combining this result with our triangle inequality already gives some rough estimates.

Lemma 4.9. *Let H_0, H_1 be self-adjoint operators associated with τ_0, τ_1, respectively, and separated boundary conditions. Suppose that $(\lambda_0,\lambda_1)\subseteq\mathbb{R}\backslash(\sigma_{ess}(H_0)\cup\sigma_{ess}(H_1))$, then*

$$\begin{aligned}&\dim\operatorname{Ran}P_{(\lambda_0,\lambda_1)}(H_1)-\dim\operatorname{Ran}P_{(\lambda_0,\lambda_1)}(H_0)\\ &\qquad\le\underline{\#}(\psi_{1,\mp}(\lambda_1),\psi_{0,\pm}(\lambda_1))-\overline{\#}(\psi_{1,\mp}(\lambda_0),\psi_{0,\pm}(\lambda_0))+2,\end{aligned}\tag{4.6}$$

respectively,

$$\begin{aligned}&\dim\operatorname{Ran}P_{(\lambda_0,\lambda_1)}(H_1)-\dim\operatorname{Ran}P_{(\lambda_0,\lambda_1)}(H_0)\\ &\qquad\ge\overline{\#}(\psi_{1,\mp}(\lambda_1),\psi_{0,\pm}(\lambda_1))-\underline{\#}(\psi_{1,\mp}(\lambda_0),\psi_{0,\pm}(\lambda_0))-2.\end{aligned}\tag{4.7}$$

To extend Theorem 2.3 to the singular case, we need to require the following hypothesis similar to [5, H.3.15].

Hypothesis H.4.10. *Suppose H_0 and H_1 are self-adjoint operators associated with τ_0 and τ_1 and separated boundary conditions (if any). Introduce*

$$\begin{aligned}A_0&=\frac{1}{r}(rp_0)^{1/2}\frac{d}{dx},\\ \mathfrak{D}(A_0)&=\{f\in L^2((a,b),r\,dx)|f\in AC_{loc}(a,b),\ \sqrt{p_0}f'\in L^2(a,b)\}\end{aligned}\tag{4.8}$$

(i) *$r^{-1}q_0$ is infinitesimally form bounded with respect to $A_0^*A_0$.*
(ii) *$r^{-1}(q_1-q_0)$ is infinitesimally form bounded with respect to H_0.*
(iii) *There is a $C_1>1$ such that $C_1^{-1}\le p_0(x)^{-1}p_1(x)\le C_1$ for all x.*
(iv) *$r^{-1}|r(p_0-p_1)|^{1/2}\frac{d}{dx}R_{H_0}(z)$ and $|(r^{-1}(q_1-q_0)|^{1/2}R_{H_0}(z)$ are Hilbert–Schmidt for one (and hence for all) $z\in\rho(H_0)$.*

We note that the conditions of the last hypothesis are for example satisfied for periodic operators if the coefficients are continuous and $p_0^{-1}-p_1^{-1}$ and q_0-q_1 are integrable.

It will be shown in Section 7 that these conditions ensure that we can interpolate between H_0 and H_1 using operators H_ε, $\varepsilon\in[0,1]$, such that the resolvent difference of H_0 and H_ε is continuous in ε with respect to the trace norm. Hence we can fix the spectral shift function $\xi(\lambda,H_1,H_0)$ by requiring $\varepsilon\mapsto\xi(\lambda,H_\varepsilon,H_0)$ to be continuous in $L^1(\mathbb{R},(\lambda^2+1)^{-1}d\lambda)$, where we of course set $\xi(\lambda,H_0,H_0)=0$ (see Lemma 7.7). While ξ is only defined a.e., it is constant on the intersection of the resolvent sets $\mathbb{R}\cap\rho(H_0)\cap\rho(H_1)$, and we will require it to be continuous there. In particular, note that by Weyl's theorem the essential spectra of H_0 and H_1 are equal, $\sigma_{ess}(H_0)=\sigma_{ess}(H_1)$.

Theorem 4.11. *Let H_0, H_1 satisfy Hypothesis 4.10. Then for every $\lambda\in\rho(H_0)\cap\rho(H_1)\cap\mathbb{R}$, we have*

$$\xi(\lambda,H_1,H_0)=\#(\psi_{0,\pm}(\lambda),\psi_{1,\mp}(\lambda)).\tag{4.9}$$

5. Proofs of Lemma 4.7 and the regular case

To prove Lemma 4.7, we need the following modification of [5, Lem. 3.9]:

Lemma 5.1. *Let $(\lambda_0,\lambda_1)\subseteq\mathbb{R}\backslash\sigma_{ess}(H_0)$, $\lambda\in(\lambda_0,\lambda_1)$. If $p_0=p_1$ and $\lambda_0<r^{-1}(q_1-q_0)-\lambda<\lambda_1$ (at least near singular endpoints), then $\tau_1-\lambda$ is relatively nonoscillatory with respect to $\tau_0-\lambda$.*

Proof. Using our comparison theorem, we have that from $\#(u_0(\lambda_0), u_0(\lambda_1)) < \infty$, we obtain

$$\overline{\#}(u_0(\lambda_1), u_1(\lambda)) < \infty, \quad \underline{\#}(u_0(\lambda_0), u_1(\lambda)) > -\infty$$

now the result follows as in [5, Lem. 3.9] by

$$\overline{\#}(u_0(\lambda), u_1(\lambda)) \le \#(u_0(\lambda), u_0(\lambda_1)) + \overline{\#}(u_0(\lambda_1), u_1(\lambda)) + 1$$

as follows from the triangle inequality for Wronskians ([5, Thm. 3.4]) and [5, Thm. 3.8 (i)]. □

Our next proof will require the following resolvent relation for form perturbations. It is a special case from [3, Sect. VI.3] (see in particular equation (VI.3.10); compare also Sect. II.3. in [7]).

Lemma 5.2. *Let H_0 be a self-adjoint operator which is bounded from below and let λ be below its spectrum. Let V be relatively form bounded with respect to H_0 and with bound less than one. Then, we have that $H = H_0 + V$ is self-adjoint and for its resolvent we have*

$$R_H(z) = R_{H_0}^{1/2}(\lambda)(1 - (z-\lambda)R_{H_0}(\lambda) + C)^{-1} R_{H_0}^{1/2}(\lambda). \tag{5.1}$$

Here C is the bounded operator associated with the quadratic form

$$\psi \mapsto \langle R_{H_0}^{1/2}(\lambda)\psi, V R_{H_0}^{1/2}(\lambda)\psi\rangle. \tag{5.2}$$

We remark, that here and in what follows sums of operators have to be understood as forms sums. Now we come to the

Proof of Lemma 4.7. We first show that $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$. First of all, note that imposing an additional Dirichlet boundary condition at some point $b_n \in (a,b)$ implies that the resolvents of the original and the perturbed operator differ by a rank one perturbation (cf., e.g, [11]). Furthermore, the perturbed operator decomposes into a direct sum of two operators, one regular part on (a, b_n) and one singular part on (b_n, b). Since the resolvent of a regular Strum–Liouville operator is Hilbert–Schmidt, the only interesting part for the essential spectrum is the singular operator on (b_n, b). Denote the corresponding operators by H_j^n, $j = 1, 2$. (i.e., H_j^n is H_j restricted to (b_n, b) with a Dirichlet boundary condition at b_n). Then it suffices to show that the resolvent difference of H_1^n and H_0^n can be made arbitrarily small by choosing b_n close to b.

Recall the definition of A_0 from (4.8) and note that since $\frac{q_0}{r}$ is bounded (for b_n sufficiently large), $A_0 R_{H_0}^{1/2}(-\lambda)$ is bounded for $-\lambda < \sigma(H_0)$. By virtue of Lemma 5.2 we conclude

$$R_{H_1^n}(\lambda) = R_{H_0^n}(\lambda)^{1/2}(1 + C^n)^{-1} R_{H_0^n}(\lambda)^{1/2}$$

for λ below the spectrum of H_0, where

$$C^n = (A_0^n R_{H_0^n}(\lambda)^{1/2})^* \frac{p_1 - p_0}{p_0} (A_0^n R_{H_0^n}(\lambda)^{1/2}) + R_{H_0^n}(\lambda)^{1/2} \frac{q_1 - q_0}{r} R_{H_0^n}(\lambda)^{1/2}$$

and A_0^n denotes the restriction of A_0 to (b_n, b) with a Dirichlet boundary condition at b_n.

By assumption, $\frac{p_1 - p_0}{p_0}$ respectively $\frac{q_1 - q_0}{r}$ and thus $\|C^n\|$ can be made arbitrarily small. Hence $(1 + C^n)^{-1} \to 1$ and the first claim follows.

Now, we come to the proof of the relatively nonoscillation part. Our condition on p_1/p_0 imply that

$$p_0(x)(1-\varepsilon_-(y)) \le p_1(x) \le p_0(x)(1+\varepsilon_+(y)), \quad x \ge y,$$

where

$$\varepsilon_\pm(y) = \pm \sup_{x\ge y}(\pm(p_1(x)/p_0(x)-1)) \to 0, \quad y \to b.$$

Now it follows, from our comparison theorem, that solutions $u_\pm$ of $\tau_\pm u_\pm = 0$ on (y,b), where

$$\tau_\pm = \frac{1}{r}\left(-\frac{d}{dx}(1+\varepsilon_\pm(y))p_0\frac{d}{dx} + q_1 - \lambda r\right),$$

satisfy $\#(u_-,u_0) \ge \#(u_1,u_0) \ge \#(u_+,u_0)$. Since $u_\pm$ also solve $\tilde\tau_\pm u_\pm = 0$ on (y,b), where

$$\tilde\tau_\pm = \frac{1}{r}\left(-\frac{d}{dx}p_0\frac{d}{dx} + \frac{q_1-\lambda r}{1+\varepsilon_\pm(y)}\right),$$

the result follows from our previous lemma since

$$\frac{r^{-1}q_1-\lambda}{1+\varepsilon_\pm(y)} - (r^{-1}q_0-\lambda) = \frac{r^{-1}(q_1-q_0)-\varepsilon_\pm(y)(r^{-1}q_0-\lambda)}{1+\varepsilon_\pm(y)} \to 0$$

as $y \to b$. □

Our next aim is to prove Theorem 2.3. The main ingredient will be Prüfer variables and the formula (2.5) for the derivative of the Wronskian. Let us suppose that $\tau_{0,1}$ are both regular at a and b with boundary conditions

$$\begin{matrix} \cos(\alpha)f(a)-\sin(\alpha)p_j(a)f'(a)=0 \\ \cos(\beta)f(b)-\sin(\beta)p_j(b)f'(b)=0 \end{matrix}, \quad j=0,1. \tag{5.3}$$

Abbreviate $p_\varepsilon = p_0+\varepsilon(p_1-p_0)$. Note that p_ε^{-1} is locally integrable, since $p_\varepsilon^{-1} \le \max(p_0^{-1},p_1^{-1})$. Hence we can choose $\psi_{\pm,\varepsilon}(\lambda,x)$ such that $\psi_{-,\varepsilon}(\lambda,a)=\sin(\alpha)$, $p_\varepsilon(a)\psi'_{-,\varepsilon}(\lambda,a)=\cos(\alpha)$ respectively $\psi_{+,\varepsilon}(\lambda,b)=\sin(\beta)$, $p_\varepsilon(b)\psi'_{+,\varepsilon}(\lambda,b)=\cos(\beta)$. In particular, we may choose

$$\theta_-(\lambda,a)=\alpha\in[0,\pi), \quad -\theta_+(\lambda,b)=\pi-\beta\in[0,\pi). \tag{5.4}$$

Next we introduce

$$\tau_\varepsilon = \tau_0+\varepsilon(\tau_1-\tau_0) = \frac{1}{r}\left(-\frac{d}{dx}p_\varepsilon\frac{d}{dx}+q_\varepsilon\right), \quad \begin{matrix} q_\varepsilon = q_0+\varepsilon(q_1-q_0) \\ p_\varepsilon = p_0+\varepsilon(p_1-p_0) \end{matrix} \tag{5.5}$$

and investigate the dependence with respect to $\varepsilon\in[0,1]$.

If u_ε solves $\tau_\varepsilon u_\varepsilon = 0$, then the corresponding Prüfer angles satisfy

$$\dot\theta_\varepsilon(x) = -\frac{W_x(u_\varepsilon,\dot u_\varepsilon)}{\rho_\varepsilon^2(x)}, \tag{5.6}$$

where the dot denotes a derivative with respect to ε.

As in [5, Lem. 5.1], we obtain by integrating (2.5) and using this to evaluate the corresponding difference quotient the following lemma.

Lemma 5.3. *We have*

$$W_x(\psi_{\varepsilon,\pm},\dot\psi_{\varepsilon,\pm}) = \begin{cases} \int_x^b (q_0(t)-q_1(t))\psi_{\varepsilon,+}(t)^2 dt \\ \qquad + \int_x^b (p_1^{-1}(t)-p_0^{-1}(t))p_\varepsilon \psi'_{\varepsilon,+}(t)^2 dt \\ \\ -\int_a^x (q_0(t)-q_1(t))\psi_{\varepsilon,-}(t)^2 dt \\ \qquad + \int_a^x (p_1^{-1}(t)-p_0^{-1}(t))p_\varepsilon \psi'_{\varepsilon,-}(t)^2 dt, \end{cases} \tag{5.7}$$

where the dot denotes a derivative with respect to ε, $\psi_{\varepsilon,\pm}(x) = \psi_{\varepsilon,\pm}(0,x)$, and $p_\varepsilon = p_0 + \varepsilon(p_1 - p_0)$.

Since we assumed a and b to be regular, all integrals exist.

Denote the Prüfer angles of $\psi_{\varepsilon,\pm}(x) = \psi_{\varepsilon,\pm}(0,x)$ by $\theta_{\varepsilon,\pm}(x)$. The last lemma implies for $q_0 - q_1 \ge 0$, $p_0 - p_1 \ge 0$, that

$$\dot\theta_{\varepsilon,+}(x) \le 0, \qquad \dot\theta_{\varepsilon,-}(x) \ge 0. \tag{5.8}$$

Now we are ready to investigate the associated operators H_0 and H_1. In addition, we will choose the same boundary conditions for H_ε as for H_0 and H_1. The next lemma follows as in [5, Lem. 5.2].

Lemma 5.4. *Suppose $q_0 - q_1 \ge 0$, $p_0 - p_1 \ge 0$ (resp. both ≤ 0). Then the eigenvalues of H_ε are analytic functions with respect to ε and they are decreasing (resp. increasing).*

In particular, this implies that $\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_\varepsilon)$ is continuous from below (resp. above) in ε for every λ. Now we are ready for the

Proof of Theorem 2.3. Without restriction it suffices to assume $\lambda_0 = \lambda_1 = 0$ and to prove the result only for $\#(\psi_{0,+},\psi_{\varepsilon,-})$.

We can split $q_0 - q_1$, $p_0 - p_1$ in the form

$$\begin{aligned} q_0 - q_1 &= q_+ - q_-, &\quad q_+, q_- \ge 0,\\ p_0 - p_1 &= p_+ - p_-, &\quad p_+, p_- \ge 0, \end{aligned}$$

and introduce the operator

$$\tau_- = \frac{1}{r}\left(-\frac{d}{dx}(p_0 - p_-)\frac{d}{dx} + (q_0 - q_-)\right).$$

Now τ_- is a negative perturbation of τ_0 and τ_1 is a positive perturbation of τ_-.

Furthermore, define τ_ε by

$$\tau_\varepsilon = \begin{cases} \tau_0 + 2\varepsilon(\tau_- - \tau_0), & \varepsilon \in [0,1/2] \\ \tau_- + 2(\varepsilon - 1/2)(\tau_1 - \tau_-), & \varepsilon \in [1/2,1]. \end{cases}$$

Let us look at

$$N(\varepsilon) = \#(\psi_{0,+},\psi_{\varepsilon,-}) = \lceil \Delta_\varepsilon(b)/\pi \rceil - \lfloor \Delta_\varepsilon(a)/\pi \rfloor - 1, \quad \Delta_\varepsilon(x) = \Delta_{\psi_{0,+},\psi_{\varepsilon,-}}(x)$$

and consider $\varepsilon \in [0,1/2]$. At the left boundary $\Delta_\varepsilon(a)$ remains constant whereas at the right boundary $\Delta_\varepsilon(b)$ is increasing by Lemma 5.3. Moreover, it hits a multiple of π whenever $0 \in \sigma(H_\varepsilon)$. So $N(\varepsilon)$ is a piecewise constant function which is continuous from below and jumps by one whenever $0 \in \sigma(H_\varepsilon)$. By Lemma 5.4 the same is true for

$$P(\varepsilon) = \dim \operatorname{Ran}\, P_{(-\infty,0)}(H_\varepsilon) - \dim \operatorname{Ran}\, P_{(-\infty,0]}(H_0)$$

and since we have $N(0) = P(0)$, we conclude $N(\varepsilon) = P(\varepsilon)$ for all $\varepsilon \in [0, 1/2]$. To see the remaining case $\varepsilon = [1/2, 1]$, simply replace increasing by decreasing and continuous from below by continuous from above. □

6. Approximation in trace norm

Now we begin with the result for singular operators by proving the case where $q_1 - q_0$ and $p_1 - p_0$ have compact support.

Lemma 6.1. *Let H_j, $j = 0, 1$, be Sturm–Liouville operators on (a, b) associated with τ_j, and suppose that $r^{-1}(q_1 - q_0)$ and $p_1 - p_0$ have support in a compact interval $[c, d] \subseteq (a, b)$, where $a < c$ if a is singular and $d < b$ if b is singular. Moreover, suppose H_0 and H_1 have the same boundary conditions (if any).*

Suppose $\lambda_0 < \inf \sigma_{ess}(H_0)$. Then

$$\dim \operatorname{Ran} P_{(-\infty,\lambda_0)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \tag{6.1}$$

Suppose $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$. Then

$$\begin{aligned}\dim \operatorname{Ran} P_{[\lambda_0,\lambda_1)}(H_1) &- \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1]}(H_0) \\ &= \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)).\end{aligned} \tag{6.2}$$

Proof. Define $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon)H_0$ as usual and observe that $\psi_{\varepsilon,-}(z, x) = \psi_{0,-}(z, x)$ for $x \le c$ respectively $\psi_{\varepsilon,+}(z, x) = \psi_{0,+}(z, x)$ for $x \ge d$. Furthermore, $\psi_{\varepsilon,\pm}(z, x)$ is analytic with respect to ε and $\lambda \in \sigma_p(H_\varepsilon)$ if and only if $W_d(\psi_{0,+}(\lambda), \psi_{\varepsilon,-}(\lambda)) = 0$. Now the proof can be done as in the regular case. □

Lemma 6.2. *Suppose H_0, H_1 satisfy the same assumptions as in the previous lemma and that there is a constant $C_1 > 1$ such that $C_1^{-1} \le p_1(x)p_0(x)^{-1} \le C_1$ for all $x \in (a, b)$. Furthermore, set $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon)H_0$. Then*

$$\|\sqrt{r^{-1}|q_0 - q_1|} R_{H_\varepsilon}(z)\|_{\mathcal{J}_2} \le C(z), \qquad \varepsilon \in [0, 1], \tag{6.3}$$

and

$$\|\sqrt{|p_1 - p_0|} \frac{d}{dx} R_{H_\varepsilon}(z)\|_{\mathcal{J}_2} \le C(z), \qquad \varepsilon \in [0, 1]. \tag{6.4}$$

In particular, H_0 and H_1 are resolvent comparable and

$$\xi(\lambda, H_1, H_0) = \#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda)) \tag{6.5}$$

for every $\lambda \in \mathbb{R}\backslash(\sigma(H_0) \cup \sigma(H_1))$. Here $\xi(H_1, H_0)$ is assumed to be constructed such that $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ is a continuous mapping $[0, 1] \to L^1((\lambda^2 + 1)^{-1}d\lambda)$.

Proof. Denote by

$$G_\varepsilon(z, x, y) = (H_\varepsilon - z)^{-1}(x, y) = \frac{\psi_{\varepsilon,-}(z, x_<), \psi_{\varepsilon,+}(z, y_>)}{W(\psi_{\varepsilon,-}(z), \psi_{\varepsilon,+}(z))},$$

where $x_< = \min(x, y)$, $y_> = \max(x, y)$, the Green's function of H_ε. As pointed out in the proof of the previous lemma, $\psi_{\varepsilon,\pm}(z, x)$ is analytic with respect to ε and hence a simple estimate shows

$$\int_a^b \int_a^b |G_\varepsilon(z, x, y)|^2 |r(y)^{-1}(q_1(y) - q_0(y))| r(x)dx\, r(y)dy \le C(z)^2$$

for $\varepsilon \in [0, 1]$, which establishes the first claim.

For the second claim, we need to show that

$$\begin{aligned}\int_a^b \int_a^b |p_\varepsilon \partial_x G_\varepsilon(z,x,y)|^2 \left| \frac{p_1(x)-p_0(x)}{p_\varepsilon^2(x)} \right| r(x)dx\, r(y)dy \\ \le C(z) \int_c^d \left| \frac{p_1(x) p_0(x)}{p_\varepsilon^2(x)} \right| |p_0^{-1}(x) - p_1^{-1}(x)| r(x)dx\end{aligned}$$

is uniformly bounded in $\varepsilon \in [0,1]$. However, this follows here from the integrand being integrable, since

$$0 < \frac{p_0}{p_\varepsilon} \le C_1, \quad 0 < \frac{p_1}{p_\varepsilon} \le C_1.$$

Moreover, a straightforward calculation (using (2.5)) and

$$\psi_{+,\varepsilon}(c) = \psi_{+,\varepsilon'}(c) W_c(\psi_{+,\varepsilon}, \psi_{-,\varepsilon'}) - \psi_{-,\varepsilon'}(c) W_c(\psi_{+,\varepsilon}, \psi_{+,\varepsilon'})$$

shows

$$\begin{aligned}G_{\varepsilon'}(z,x,y) =& G_\varepsilon(z,x,y) \\ &+ (\varepsilon - \varepsilon') \int_a^b G_{\varepsilon'}(z,x,t) r^{-1}(t)(q_1(t) - q_0(t)) G_\varepsilon(z,t,y) r(t)dt \\ &+ (\varepsilon - \varepsilon') \int_a^b \frac{\partial G_{\varepsilon'}(z,x,t)}{\partial t} r^{-1}(t)(p_1(t) - p_0(t)) \frac{\partial G_\varepsilon(z,t,y)}{\partial t} r(t)dt.\end{aligned}$$

Hence $R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)$ can be written as the sum of two products of two Hilbert–Schmidt operators, whose norm can be estimated by the first claims:

$$\|R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)\|_{\mathcal{J}_1} \le |\varepsilon' - \varepsilon| C(z)^2. \tag{6.6}$$

Thus $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ is continuous. The rest follows from (7.4). □

Before proving Theorem 4.11, we still need to transform Hypothesis 4.10 in a form such that we can apply our operator theoretic results from the appendix. The next lemma will do the job.

Lemma 6.3. *Assume Hypothesis 4.10, and introduce*

$$\mathfrak{Q}_0 = \{ f \in L^2((a,b), rdx) \,|\, f \in AC_{loc}(a,b),\ \sqrt{p_0} f' \in L^2(a,b) \}, \tag{6.7}$$

and $\mathfrak{Q}$ by imposing boundary conditions at a and b if necessary. Furthermore, introduce the following operators on $\mathfrak{Q}$ with $N = \lceil \sup_x (p_1(x) p_0(x)^{-1} - 1)_+ \rceil + 1$

$$A_j = \frac{1}{N^{1/2}} (p_0 - p_1)_+^{1/2} \frac{d}{dx}, \quad j = 1, \dots N, \tag{6.8}$$

$$A_{N+1} = |q_0 - q_1|^{1/2}, \quad A_{N+2} = (p_0 - p_1)_-^{1/2} \frac{d}{dx} \tag{6.9}$$

$$S_1, \dots, S_N = 1, \quad S_{N+1} = \operatorname{sgn}(q_0 - q_1), \quad S_{N+2} = -1 \tag{6.10}$$

Then Hypothesis 7.4 is satisfied with these operators and H_0, H_1 are self-adjoint extensions of τ_0, τ_1, respectively.

Proof. By Lemma 7.6, it is sufficient to check the form bounds with respect to the form of τ_0 with $q_0 = 0$, since we have by [5, Lem. 4.1], that q_0, q_1 will be infinitesimally form bounded.

To see the claims on the other operators, note that $p_0^{1/2} f' \in L^2$ implies $|p_1 - p_0|^{1/2} f' \in L^2$, since $p_0^{-1/2} |p_1 - p_0|^{1/2}$ is essentially bounded by assumption. We are left with computing the form bounds, but again ($1 \le j \le N$, $u \in \mathfrak{Q}$)

$$\|A_j u\|^2 = \frac{1}{N} \|p_0^{-1/2} (p_0 - p_1)_+^{1/2} p_0^{1/2} u'\|^2 \le \frac{\sup_x (p_0(x)^{-1} p_1(x) - 1)_+}{N} \langle u, A_0^* A_0 u\rangle$$

which shows that the form bound with respect to $A_0^* A_0$ is less then one. By Lemma 7.6 the same is true with respect to H_0.

Boundedness from below follows by noting, that the quadratic forms are bounded from below, by the bounds on q_0 (resp. q_1). □

Now we come to the

Proof of Theorem 4.11. We first assume that we have compact support near one endpoint, say a. Define by K_ε the multiplication operator by $\chi_{(a,b_\varepsilon]}$ with $b_\varepsilon \uparrow b$. Then K_ε satisfies the assumptions of Lemma 7.7. The last lemma guarantees that Hypothesis 4.10 implies Hypothesis 7.4, so we can apply Lemma 7.3 by Lemma 7.7.

Denote by

$$\tau_\varepsilon = \frac{1}{r} \left(-\frac{d}{dx} p_\varepsilon \frac{d}{dx} + q_\varepsilon \right), \qquad \begin{array}{l} p_\varepsilon = p_0 + \chi_{(a,b_\varepsilon]}(p_1 - p_0) \\ q_\varepsilon = q_0 + \chi_{(a,b_\varepsilon]}(q_1 - q_0) \end{array}$$

and by $\psi_{\varepsilon,-}$ the corresponding solutions satisfying the boundary condition at a. By Lemma 7.7 we have that $\xi(H_\varepsilon, H_0)$ is constant and equal to $\xi(H_1, H_0)$ once ε is greater then some ε_0.

Now let us turn to the Wronskians. We first prove the $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ case. By Lemma 6.2 we know

$$\xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)$$

for every $\varepsilon < 1$. Concerning the right-hand side observe that

$$W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = W_x(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$$

for $x \le b_\varepsilon$ and that $W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))$ is constant for $x \ge b_\varepsilon$. This implies that for $\varepsilon \ge \varepsilon_0$ we have

$$\begin{aligned} \xi(\lambda, H_1, H_0) &= \xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) \\ &= \#_{(a,b_\varepsilon)}(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = \#_{(a,b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)). \end{aligned}$$

In particular, the last item $\#_{(a,b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ is eventually constant and thus has a limit which, by definition, is $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$.

For the corresponding $\#(\psi_{1,+}(\lambda), \psi_{0,-}(\lambda))$ case one simply exchanges the roles of H_0 and H_1.

Hence the result holds if the perturbation has compact support near one endpoint. Now one repeats the argument to remove the compact support assumption near the other endpoint as well.

That the spectral shift function constructed here corresponds to the one constructed by requiring continuity of the mapping $\varepsilon \mapsto \xi(H_0, H_0 + \varepsilon V)$ follows from the compactness of $[0, 1]$ and [12, Thm. 8.7.4]. □

7. Appendix: The Spectral Shift Function

In this appendix we collect some facts on Krein's spectral shift function which are of relevance to us. Most results are taken from [12] (see also [9] for an easy introduction). The first part closely follows the appendix of [5].

Two operators H_0 and H_1 are called resolvent comparable, if

$$R_{H_1}(z) - R_{H_0}(z) \tag{7.1}$$

is trace class for one $z \in \rho(H_1) \cap \rho(H_0)$. By the first resolvent identity (7.1) then holds for all $z \in \rho(H_1) \cap \rho(H_0)$.

Theorem 7.1 (Krein [4]). *Let H_1 and H_0 be two resolvent comparable self-adjoint operators, then there exists a function*

$$\xi(\lambda, H_1, H_0) \in L^1(\mathbb{R}, (\lambda^2+1)^{-1} d\lambda) \tag{7.2}$$

such that

$$\operatorname{tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda, H_1, H_0) f'(\lambda) d\lambda \tag{7.3}$$

for every smooth function f with compact support.

Note: Equation (7.3) holds in fact for a much larger class of functions f. See [12, Thm. 9.7.1] for this and a proof of the last theorem.

The function $\xi(\lambda) = \xi(\lambda, H_1, H_0)$ is called Krein's spectral shift function and is unique up to a constant. Moreover, $\xi(\lambda)$ is clearly constant on every interval $(\lambda_0, \lambda_1) \subset \rho(H_0) \cap \rho(H_1)$. Hence, if $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_j) < \infty$, $j = 0, 1$, then $\xi(\lambda)$ is a step function and

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) = \lim_{\varepsilon \downarrow 0} \Big(\xi(\lambda_1 - \varepsilon) - \xi(\lambda_0 + \varepsilon) \Big). \tag{7.4}$$

This formula clearly explains the name spectral shift function.

Before investigating further the properties of the SSF, we will recall a few things about trace ideals (see for example [8]). First, for $1 \le p < \infty$ denote by $\mathcal{J}^p$ the Schatten p-class, and by $\|.\|_{\mathcal{J}^p}$ its norm. We will use $\|.\|$ for the usual operator norm. Using $\|A\|_{\mathcal{J}^p} = \infty$ if $A \notin \mathcal{J}^p$, we have the following inequalities for all operators:

$$\|AB\|_{\mathcal{J}^p} \le \|A\| \|B\|_{\mathcal{J}^p}, \quad \|AB\|_{\mathcal{J}^1} \le \|A\|_{\mathcal{J}^2} \|B\|_{\mathcal{J}^2}.$$

Furthermore, we will use the notation of $\mathcal{J}^p$-converges to denote convergence in the respective norm. The following result from [2, Thm. IV.11.3] will be needed.

Lemma 7.2. *Let $p > 0$, $A_n \xrightarrow{\mathcal{J}^p} A$, $T_n \xrightarrow{s} T$, $S_n \xrightarrow{s} S$ sequences of strongly convergent bounded linear operators, then:*

$$\|T_n A_n S_n^* - T A S^*\|_{\mathcal{J}^p} \to 0. \tag{7.5}$$

Here $\|.\|_{\mathcal{J}^p}$ are the norms of the Schatten p-classes $\mathcal{J}^p$.

We will also need the following continuity result for ξ. It will also allow us to fix the unknown constant. The second part is [5, Lem. 7.3], the first from [12].

Lemma 7.3. *Suppose H_ε, $\varepsilon \in [0, 1]$, is a family of self-adjoint operators, which is continuous in the metric*

$$\rho(A, B) = \|R_A(z) - R_B(z)\|_{\mathcal{J}^1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{7.6}$$

and abbreviate $\xi_\varepsilon = \xi(H_\varepsilon, H_0)$. Then there exists a unique choice of ξ_ε such that $\varepsilon \mapsto \xi_\varepsilon$ is continuous $[0,1] \to L^1(\mathbb{R}, (\lambda^2+1)^{-1}d\lambda)$ with $\xi_0 = 0$.

If $H_\varepsilon \geq \lambda_0$ is bounded from below, we can also allow $z = \lambda \in (-\infty, \lambda_0)$.

For $\lambda \in \rho(H_1) \cap \mathbb{R}$, we have that there is an ε_0 such that $\xi_\varepsilon(\lambda) = \xi_1(\lambda)$ for $\varepsilon > \varepsilon_0$.

Proof. We just need to proof the third part. For ε close to 1 a whole neighborhood of λ is in $\rho(H_\varepsilon) \cap \mathbb{R}$, since the resolvent sets converge. Furthermore, we know from this that the ξ_ε is integer valued near 1 in a neighborhood of λ. Now the claim follows from the convergence of $\xi_\varepsilon \to \xi_1$ in $L^1(\mathbb{R}, (\lambda^2+1)^{-1}d\lambda)$. □

Our final aim is to find some conditions which allow us to verify the assumptions of this lemma. To do this, we derive some properties of relatively bounded operators multiplied by strongly continuous families of operators.

Hypothesis H. 7.4. *Suppose H_0 is self-adjoint and bounded from below. Let A_j, $j = 1, \dots, n$, be closed operators and S_j, $j = 1, \dots, n$ be bounded operators with $\|S_j\| \leq 1$. Furthermore, suppose that these satisfy for $j = 1, \dots, n-1$ that*

- (i) *$A_j^* A_j$ is relatively form bounded with respect to H_0 with relative form bound less than one and S_j is positive, or*
- (i') *$A_j^* A_j$ is infinitesimally form bounded with respect to H_0.*

Suppose for $j = n$, that

- (ii) *$A_j^* A_j$ is relatively form bounded with respect to H_0 with relative form bound less than one.*

Note that condition (i) implies that $A_j^* S_j A_j$ is a positive operator.

We recall that A^*A being form bounded with respect to H_0 means that we have $\mathfrak{Q}(A^*A) \supseteq \mathfrak{Q}(H_0)$ and

$$\langle \psi, A^*A\psi \rangle \leq a\langle \psi, H_0\psi \rangle + b\|\psi\|^2, \quad \forall \psi \in \mathfrak{Q}(H_0). \tag{7.7}$$

for some $0 \leq a < 1$, $0 \leq b$. The form bound is the infinimum over all a such that (7.7) holds.

The next lemma is modified from [5, Lem. 7.5], to be able to deal with differential operators and sums of operators.

Lemma 7.5. *Let $\varepsilon \ni [0,1] \to K_\varepsilon$ be a strongly continuous family of self-adjoint bounded operators which satisfy $0 = K_0 \leq K_\varepsilon \leq K_1 = 1$.*

Let

- (i) *$\varepsilon \mapsto H_\varepsilon$ satisfy the assumptions of Lemma 7.3,*
- (ii) *S be a bounded operator with $\|S\| \leq 1$, and*
- (iii) *A be a closed operator such that A^*A is relatively bounded with respect to H_ε with uniform in ε bound less then one, and $AR_{H_\varepsilon}(z) \in \mathcal{J}^2$ for one $z \in \mathbb{C}\backslash\mathbb{R}$.*

Then $\tilde{H}_\varepsilon = H_\varepsilon + A^* K_\varepsilon S A$ also satisfies the assumptions of Lemma 7.3. Furthermore, for form bounded B with $BR_{H_\varepsilon}(z) \in \mathcal{J}^2$, we have $BR_{\tilde{H}_\varepsilon}(z) \in \mathcal{J}^2$ for all $\varepsilon \in [0,1]$.

Proof. We will abbreviate $V_\varepsilon = A^* K_\varepsilon S A$, $\tilde{H}_\varepsilon = H_\varepsilon + V_\varepsilon$, $R_\varepsilon(z) = R_{H_\varepsilon}(z)$, and $\tilde{R}_\varepsilon(z) = R_{\tilde{H}_\varepsilon}(z)$. By the KLMN theorem ([6, Thm. X.17]), $\tilde{H}_\varepsilon$ is self-adjoint since

$$|\langle \psi, V_\varepsilon \psi \rangle| \leq |\langle A\psi, K_\varepsilon S A\psi \rangle| \leq \langle \psi, A^*A\psi \rangle, \quad \psi \in \mathfrak{Q}(V_\varepsilon).$$

Moreover, using (7.7) we obtain

$$\|AR_\varepsilon(-\lambda)^{1/2}\|^2 \le a, \qquad \frac{b}{a} < \lambda.$$

For $\lambda > \frac{b}{a}$ we have by Lemma 5.2

$$\begin{aligned}\tilde{R}_\varepsilon(-\lambda) &= R_\varepsilon(-\lambda)^{1/2}(1+C_\varepsilon)^{-1}R_\varepsilon(-\lambda)^{1/2},\\ C_\varepsilon &= (AR_\varepsilon(-\lambda)^{1/2})^*(K_\varepsilon SAR_\varepsilon(-\lambda)^{1/2}).\end{aligned}$$

Hence, a straightforward calculation shows

$$\begin{aligned}\tilde{R}_\varepsilon(-\lambda) &= R_\varepsilon(-\lambda) - (AR_\varepsilon(-\lambda))^*(1+\tilde{C}_\varepsilon)^{-1}(K_\varepsilon SAR_\varepsilon(-\lambda)),\\ \tilde{C}_\varepsilon &= (K_\varepsilon SAR_\varepsilon(-\lambda)^{1/2})(AR_\varepsilon(-\lambda)^{1/2})^*.\end{aligned} \tag{7.8}$$

By $\|\tilde{C}_\varepsilon\| \le a < 1$, we have that $(1+\tilde{C}_\varepsilon)^{-1}$ exists. Furthermore, note that (7.8) implies, that $B\tilde{R}_\varepsilon(-\lambda) \in \mathcal{J}^2$, since:

$$B\tilde{R}_\varepsilon(-\lambda) = BR_\varepsilon(-\lambda) - BR_\varepsilon(-\lambda)^{1/2}(AR_\varepsilon(-\lambda)^{1/2})^*(1+\tilde{C}_\varepsilon)^{-1}(K_\varepsilon SAR_\varepsilon(-\lambda)),$$

and $AR_\varepsilon(-\lambda) \in \mathcal{J}^2$. Now, look at

$$\begin{aligned}D_{\varepsilon,\varepsilon'}\psi &= (-\tilde{C}_\varepsilon(1+\tilde{C}_\varepsilon)^{-1} - \tilde{C}_{\varepsilon'}(1+\tilde{C}_{\varepsilon'})^{-1})\psi\\ &= (C_{\varepsilon'} - C_\varepsilon)(1+C_\varepsilon)^{-1}\psi - C_{\varepsilon'}D_{\varepsilon,\varepsilon'}\psi,\end{aligned}$$

where

$$D_{\varepsilon,\varepsilon'} = (1+\tilde{C}_\varepsilon)^{-1} - (1+\tilde{C}_{\varepsilon'})^{-1}.$$

Taking norms we obtain

$$\|D_{\varepsilon,\varepsilon'}\psi\| = \frac{1}{1-a}\|(C_{\varepsilon'} - C_\varepsilon)(1+C_\varepsilon)^{-1}\psi\|,$$

where the last term converges to 0 as $\varepsilon' \to \varepsilon$. This implies, that $(1+\tilde{C}_\varepsilon)^{-1}$ is strongly continuous. Now, we obtain from (7.8) for the difference of resolvents

$$\tilde{R}_\varepsilon(-\lambda) - \tilde{R}_{\varepsilon'}(-\lambda) = (AR_\varepsilon(-\lambda))^*((1+\tilde{C}_\varepsilon)^{-1}K_\varepsilon - (1+\tilde{C}_{\varepsilon'})^{-1}K_{\varepsilon'}))(SAR_\varepsilon(-\lambda))$$

$\mathcal{J}^1$-converges to 0 as $\varepsilon \to \varepsilon'$ by Lemma 7.2 and by $AR_\varepsilon(-\lambda) \in \mathcal{J}^2$. This way we also obtain that $\tilde{H}_\varepsilon$ and $\tilde{H}_{\varepsilon'}$ are indeed resolvent comparable. □

We also recall the following well-known fact on quadratic forms:

Lemma 7.6. *Let v, s, t be quadratic forms, such that s is positive and symmetric, and v is infinitesimal form bounded with respect to s, and t is form bounded with bound less then 1 with respect to s. Then t is also form bounded with bound less then 1 with respect to $s+v$.*

Proof. Using $|v(\psi)| \le \varepsilon s(\psi) + C\|\psi\|^2$ for arbitrary small $\varepsilon > 0$, a direct calculation shows

$$s(\psi) \le \frac{1}{1-\varepsilon}|s(\psi) + v(\psi)| + \frac{C(\varepsilon)}{1-\varepsilon}\|\psi\|^2.$$

Denoting by a the s bound of t, it follows that t is $s+v$ bounded with bound less then $a/(1-\varepsilon)$, implying that the bound is again less then one. □

Lemma 7.7. *Let* $\varepsilon \ni [0,1] \to K_\varepsilon$ *be a strongly continuous family of self-adjoint bounded operators which satisfy* $0 = K_0 \le K_\varepsilon \le K_1 = 1$.

Assume Hypothesis 7.4. Then

$$H_\varepsilon = H_0 + \sum_{j=1}^{n} A_j^* K_\varepsilon S_j A_j \tag{7.9}$$

are self-adjoint operators such that the assumptions of Lemma 7.3 hold.

Proof. Introduce $H_\varepsilon^m = H_0 + V_\varepsilon^m$, $m = 0, \dots, n$, where $V_\varepsilon^m = \sum_{j=1}^m A_j^* K_\varepsilon S_j A_j$. Since all but the last perturbations are either positive or infinitesimal (in which case on has to use Lemma 7.6), we can assume that $A_m^* K_\varepsilon S_m A_m$ is relatively form bounded with uniform bound less then one with respect to H_ε^l with $l < m$.

Now, the result follows by applying the previous lemma with $H = H^{m-1}$, $\tilde{H} = H^m$, $A = A_m$, $S = S_m$ and $B = B_l$, $l = m+1, \dots, n$ and letting m going up from 1 to n. □

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