SPECTRAL AND DYNAMICAL PROPERTIES OF CERTAIN RANDOM JACOBI MATRICES WITH GROWING PARAMETERS

JONATHAN BREUER

Abstract. In this paper, a family of random Jacobi matrices, with off-diagonal terms that exhibit power-law growth, is studied. Since the growth of the randomness is slower than that of these terms, it is possible to use methods applied in the study of Schrödinger operators with random decaying potentials. A particular result of the analysis is the existence of operators with arbitrarily fast transport whose spectral measure is zero dimensional. The results are applied to the infinite Gaussian β Ensembles and their spectral properties are analyzed.

1. Introduction

For a sequence of positive numbers, $\{a(n)\}_{n=1}^{\infty}$, and a sequence, $\{b(n)\}_{n=1}^{\infty}$, of real numbers, let $J\left(\{a(n)\}_{n=1}^{\infty}, \{b(n)\}_{n=1}^{\infty}\right)$ denote the Jacobi matrix with off-diagonal elements given by $\{a(n)\}_{n=1}^{\infty}$ and diagonal elements given by $\{b(n)\}_{n=1}^{\infty}$ on the diagonal. That is:

$$J\left(\{a(n)\}_{n=1}^{\infty}, \{b(n)\}_{n=1}^{\infty}\right) = \begin{pmatrix} b(1) & a(1) & 0 & 0 & \dots \\ a(1) & b(2) & a(2) & 0 & \dots \\ 0 & a(2) & b(3) & a(3) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.1}$$

For $\eta_1 \in (0,1)$ and $\lambda_1 > 0$, let J_{λ_1,η_1} be the Jacobi matrix whose parameters are $a_{\lambda_1,\eta_1}(n) = \lambda_1 n^{\eta_1}$ and $b_{\lambda_1,\eta_1}(n) \equiv 0$. Since $\eta_1 < 1$, J_{η_1} is a self-adjoint operator on $\ell^2(\mathbb{N})$ [1]. For any such operator we may define the spectral measure, μ, as the unique measure satisfying

$$\left(\delta_1, (J-z)^{-1}\delta_1\right) = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z} \quad z \in \mathbb{C} \setminus \mathbb{R}$$

where $(\cdot,\cdot)$ denotes the inner product in ℓ^2. It follows from the work of Janas and Naboko [8] that J_{λ_1,η_1} has absolutely continuous spectrum covering the whole real line.

This paper deals with random perturbations of J_{λ_1,η_1} that are weak, in the sense that the variance of the perturbing random parameters grows like $\sim n^{\eta_2}$ with $\eta_2 < \eta_1$.

The case $a(n) \equiv 1$, $\eta_2 < 0$ ("$\eta_1 = 0$") and no perturbation off the diagonal, is the extensively studied family of one-dimensional discrete Schrödinger operators with a random decaying potential [3, 4, 5, 13, 14, 19]. For these Schrödinger operators it has been established that when $\eta_2 < -\frac{1}{2}$, the absolutely continuous spectrum of the Laplacian is a.s. preserved. When $\eta_2 > -\frac{1}{2}$, however, the disorder wins over and the spectrum is pure point with eigenfunctions that decay at a super-polynomial rate. At the critical point ($\eta_2 = -\frac{1}{2}$) the spectral behavior exhibits a sensitive dependence on the coupling constant: The generalized eigenfunctions decay polynomially and the spectral measure is pure point or singular continuous according to whether these generalized eigenfunctions are in ℓ^2 or not (for a comprehensive treatment of discrete Schrödinger operators with random decaying potentials, see Section 8 of [13]).

From this perspective, the extension presented in this paper is in allowing growth of the off-diagonal terms. Intuitively, these terms are responsible for transport, and thus, their growth should have an effect on the spectrum similar to that of the decay of the potential (diagonal terms). Indeed, a particular case of our analysis is that of $\eta_2 = 0$, namely, the diagonal terms are i.i.d. random variables. We show that the critical point here is $\eta_1 = \frac{1}{2}$. Below this value, the spectrum is pure point, whereas above it the spectral measure is one-dimensional.

More generally, let $\{X_\omega(n)\}_{n=1}^\infty$ be a sequence of i.i.d. random variables. Let $\{Y_\omega(n)\}_{n=1}^\infty$ be another such sequence (the distributions of the X's and the Y's need not be the same). Assume the following is satisfied:

(i) For all n

$$\langle X_\omega(n)\rangle = \langle Y_\omega(n)\rangle = 0 \tag{1.2}$$

(where $\langle f(\omega)\rangle \equiv \int_\Omega f(\omega)dp$ and $(\Omega, \mathcal{F}, dp)$ is the underlying probability space.)

(ii) For any $k \in \mathbb{N}$

$$\left\langle |Y_\omega(n)|^k \right\rangle < \infty, \quad \left\langle |X_\omega(n)|^k \right\rangle < \infty. \tag{1.3}$$

(iii)

$$\left\langle (X_\omega(n))^2 \right\rangle = 1, \quad \left\langle (Y_\omega(n))^2 \right\rangle = \frac{1}{4} \tag{1.4}$$

(iv) The common distribution of $X_\omega(n)$ is absolutely continuous with respect to Lebesgue measure.

Given the quadruple $\Upsilon = (\eta_1, \eta_2, \lambda_1, \lambda_2)$ with $0 < \eta_1 < 1$, $\eta_2 < \eta_1$ and $\lambda_1, \lambda_2 > 0$, define the two random sequences

$$b_{\Upsilon,\omega}(n) \equiv b_{\lambda_2,\eta_2;\omega}(n) \equiv \lambda_2 n^{\eta_2} X_\omega(n), \quad \alpha_{\lambda_2,\eta_2;\omega}(n) \equiv \lambda_2 n^{\eta_2} Y_\omega(n).$$

Let

$$a_{\Upsilon,\omega}(n) \equiv a_{\lambda_1,\eta_1}(n) + \alpha_{\lambda_2,\eta_2;\omega}(n)$$

and define

$$J_{\Upsilon,\omega} = J\left(\{a_{\Upsilon,\omega}(n)\}_{n=1}^{\infty}, \{b_{\Upsilon,\omega}(n)\}_{n=1}^{\infty}\right). \tag{1.5}$$

We shall prove

Theorem 1.1. *For the model above, let $\gamma = \eta_1 - \eta_2$ and let $\Lambda = \frac{1}{2}(\lambda_2/\lambda_1)^2$. The following holds with probability one:*

(1) *If $\gamma > \frac{1}{2}$ the spectrum of $J_{\Upsilon,\omega}$ is $\mathbb{R}$ and $\mu_{\Upsilon,\omega}$, the spectral measure of $J_{\Upsilon,\omega}$, is one-dimensional, meaning that it does not give weight to sets of Hausdorff dimension less than 1.*

(2) *In the case $\gamma = \frac{1}{2}$ the spectrum of $J_{\Upsilon,\omega}$ is $\mathbb{R}$ and we have the following two possibilities:*

   (a) *If $\Lambda > 1 - \eta_1$, then $\mu_{\Upsilon,\omega}$ is pure point with eigenfunctions decaying like*
   $$|\psi_\omega^E(n)|^2 \sim n^{-(\Lambda+\eta_1)}.$$

   (b) *If $\Lambda \le 1 - \eta_1$, then $\mu_{\Upsilon,\omega}$ is purely singular continuous with exact Hausdorff dimensions equal to $1 - \frac{\Lambda}{(1-\eta_1)}$.*

(3) *If $\gamma < \frac{1}{2}$ then the spectrum is pure point with eigenfunctions decaying like*
$$|\psi_\omega^E(n)|^2 \sim e^{-\Lambda n^{1-2\gamma}}.$$
In this case, if $\eta_1 > 2\eta_2$ then the spectrum fills $\mathbb{R}$.

Remark. We say that a measure, μ, has exact Hausdorff dimension, ϱ, if it is supported on a set of Hausdorff dimension ϱ and does not give weight to sets of Hausdorff dimension less than ϱ. For more information concerning the decomposition of general measures with respect to their Hausdorff-dimensional properties, consult [15] and references therein.

Remark. In analogy to the Schrödinger case, one would expect to have absolutely continuous spectrum for $\gamma > 1/2$. Unfortunately, though we believe this is true, one-dimensional spectral measure is all we could get.

In resemblance of the Schrödinger case, the proof of this theorem follows by analysing the asymptotics of solutions to the formal eigenfunction equation "$J\psi = E\psi$". Namely, we shall analyze the solutions to the difference equation:

$$a(n)\psi(n+1)+b(n)\psi(n)+a(n-1)\psi(n-1)=E\psi(n) \quad n>1. \tag{1.6}$$

By a theorem of Kiselev and Last [12, Theorem 1.2], the results obtained have implications for the quantum dynamics associated with J; namely the behavior of a given vector ψ under the operation of the one parameter unitary group e^{-itJ}. More precisely, let $\hat{X}$ be the position operator defined by

$$\left(\hat{X}\psi\right)(n)=n\psi(n).$$

Theorem 1.2 of [12], which can be seen to hold in our setting, says that Theorem 1.1 and Proposition 2.7 below lead to

Theorem 1.2. *Assume $\gamma=\frac{1}{2}$ and $\Lambda\le 1-\eta_1$ (namely, case 2(b) of Theorem 1.1). Then for any $\varepsilon>0$, $m\in\mathbb{N}$, $T>0$ and $\psi\in\ell^2$,*

$$\frac{1}{T}\int_0^T \left|\left(\hat{X}^m e^{-itJ_{\Upsilon,\omega}}\psi, e^{-itJ_{\Upsilon,\omega}}\psi\right)\right| dt \ge C(\omega,\psi,m,\varepsilon)T^{(m/(1-\eta_1))-\varepsilon} \tag{1.7}$$

with probability one.

Note that η_1 may be chosen arbitrarily close to 1 while the spectral measure may have any dimension in $[0,1)$. Thus, by tuning the parameters we obtain operators with any local spectral dimensions having arbitrarily fast transport.

As an application of our general analysis, we study the Gaussian β ensembles arising naturally in the context of Random Matrix Theory: The eigenvalue distribution functions for the three classical Gaussian ensembles are given by

$$f_{\beta,N}(E_1,\cdots,E_N)=\frac{1}{G_{\beta N}}\exp\left(-\frac{1}{2}\sum_{j=1}^N E_j^2\right)\prod_{1\le j<k\le N}|E_j-E_k|^\beta \tag{1.8}$$

with $\beta=1,2$ and 4 for the Gaussian Orthogonal Ensemble, Gaussian Unitary Ensemble and Gaussian Symplectic Ensemble respectively.

A family of random matrix ensembles, indexed by β, having $f_{\beta,N}$ as their eigenvalue distribution function, for *any* positive value of β, was recently constructed by Dumitriu and Edelman [6]. The matrices in these ensembles are finite random Jacobi matrices with the distribution of the off diagonal terms depending on β:

Definition 1.3. Fix $\beta>0$. The random family of Jacobi matrices $J_{\beta,\omega}\equiv J\left(\{a_{\beta,\omega}(n)\}_{n=1}^\infty,\{b_{\beta,\omega}(n)\}_{n=1}^\infty\right)$ is defined by:

(1) The random variables $\{a_{\beta,\omega}(n)\}_{n=1}^\infty,\{b_{\beta,\omega}(n)\}_{n=1}^\infty$ are all independent.

(2) $b_{\beta,\omega}(n)$ are all standard Gaussian variables (that is, with zero mean and variance= 1), irrespective of β and n.

(3) The probability distribution function of $a_{\beta,\omega}(n)$ is given by

$$P\{\omega \mid a_{\beta,\omega}(n) < C\} = \frac{2}{\Gamma\left(\frac{\beta n}{2}\right)} \int_0^C x^{\beta n-1} e^{-x^2} dx. \tag{1.9}$$

In [6], Dumitriu and Edelman showed that the eigenvalue distribution function of the finite matrix, obtained as the restriction of $J_{\beta,\omega}$ to the $N \times N$ upper left corner, is $f_{\beta,N}$ for any $\beta > 0$.

From property 3 above, it follows that

$$\langle a_{\beta,\omega}(n)\rangle \equiv \int_\Omega a_{\beta,\omega}(n) d\omega = \frac{\Gamma\left(\frac{\beta n+1}{2}\right)}{\Gamma\left(\frac{\beta n}{2}\right)} = \sqrt{\frac{\beta n}{2}}\left(1 - \frac{1}{4\beta n}\right) + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right), \tag{1.10}$$

$$\left\langle \left(a_{\beta,\omega}(n) - \langle a_{\beta,\omega}(n)\rangle\right)^2 \right\rangle = \frac{1}{4} + \mathcal{O}\left(\frac{1}{n}\right). \tag{1.11}$$

Thus we see that the family $J_{\beta,\omega}$ corresponds to the case $\eta_1 = 1/2$, $\eta_2 = 0$ of the general matrices introduced above. Technically, the following theorem is not a corollary of Theorem 1.1, because of the $\mathcal{O}\left(n^{-1/2}\right)$ term in (1.10) and the $\mathcal{O}\left(n^{-1}\right)$ term in (1.11). The proof of Theorem 1.1, however, is robust with respect to such a change, and we have

Theorem 1.4. *For any β, the spectrum of $J_{\beta,\omega}$ is $\mathbb{R}$ with probability one.*

If $\beta < 2$, then, with probability one, the spectral measure, $\mu_{\beta,\omega}$, corresponding to $J_{\beta,\omega}$ and δ_1, is pure point with eigenfunctions decaying as

$$|\psi_\omega(n)|^2 \sim n^{-(\frac{1}{2}+\frac{1}{\beta})}.$$

If $\beta \geq 2$, then with probability one, for any $\varepsilon > 0$, $\mu_{\beta,\omega}$ has exact dimension $1 - \frac{2}{\beta}$ with probability one. Furthermore, for $\beta \geq 2$, we have that, almost surely,

$$\frac{1}{T}\int_0^T \left|\left(\hat{X}^m e^{-itJ_{\beta,\omega}}\psi, e^{-itJ_{\beta,\omega}}\psi\right)\right| dt \geq C(\omega,\psi,m,\varepsilon) T^{2-\varepsilon} \tag{1.12}$$

for any ψ, $\varepsilon > 0$ and m.

This result, without the dynamical part, was announced in [2]. We note that the analogous Circular β Ensembles can be realized as eigenvalues of truncated CMV matrices. This was shown by Killip and Nenciu [10] and later used by Killip and Stoiciu in their analysis of level statistics for ensembles of random CMV matrices [11]. The bulk spectral properties of the appropriate matrices were analyzed by Simon [21, Section 12.7].

The proof of Theorem 1.1 is given in the next section. Since the proof of the spectral part of Theorem 1.4 is precisely the same, it is not given separately. As noted earlier, the dynamical part of our analysis (Theorem 1.2 and the correspodning statement in Theorem 1.4) follows immediately from Theorem 1.1 and Proposition 2.7, by Theorem 1.2 of [12].

The method we use is a variation on the one used by Kiselev-Last-Simon [13, Section 8] in their analysis of the Schrödinger case described above. A notable difference is the fact that, due to the growth of the $a(n)$, the effective energy parameter, $\frac{E}{a(n)}$, vanishes in the limit. This, in addition to requiring a modification in the technique of proof (see Lemma 2.4 and Proposition 2.5 below), leads to the fact that the asymptotics of the generalized eigenfunctions are constant over $\mathbb{R}$. At the critical point ($(\eta_1 - \eta_2) = \frac{1}{2}$), this implies uniformity of the local Hausdorff dimensions of the spectral measure.

A modified Combes-Thomas estimate, for operators with unbounded off-diagonal terms, enters our analysis in the identification of the spectrum of $J_{\Upsilon,\omega}$. Such an estimate may be of independent interest and thus is presented in the Appendix.

2. Proof of Theorem 1.1

We begin with a simple lemma that shows that, in a certain sense, $J_{\Upsilon,\omega}$ is a random relatively decaying perturbation of J_{λ_1,η_1}.

Lemma 2.1. *For any $\varepsilon > 0$ there exists, with probability one, a constant $C = C(\omega, \varepsilon)$ for which*

$$\left| \frac{|n^{\eta_2} X_\omega(n)|}{n^{\eta_1}} \right| \leq \frac{C}{n^{\gamma-\varepsilon}} \tag{2.1}$$

and

$$\left| \frac{|n^{\eta_2} Y_\omega(n)|}{n^{\eta_1}} \right| \leq \frac{C}{n^{\gamma-\varepsilon}} \tag{2.2}$$

where $\gamma = \eta_1 - \eta_2$.

Proof. By (1.3) and Chebyshev's inequality we have for any $k \in \mathbb{N}$

$$P_n \equiv \mathcal{P}\left\{ \omega \mid \left| \frac{|n^{\eta_2} X_\omega(n)|}{n^{\eta_1}} \right| \geq \frac{1}{n^{\gamma-\varepsilon}} \right\} \leq \frac{C(k)}{n^{2k\varepsilon}}.$$

By choosing $2k > \varepsilon^{-1}$ we see that

$$\sum_{n=1}^{\infty} P_n < \infty.$$

(2.1) follows now from Borel-Cantelli. The proof of (2.2) is the same. $\square$

As stated in the Introduction, we follow the strategy of [13]. In particular, we will deduce the spectral properties of $J_{\Upsilon,\omega}$ from the asymptotics of the solutions to the corresponding eigenfunction equation.

In order to fix notation, for a given Jacobi matrix $J\left(\{a(n)\}_{n=1}^{\infty}, \{b(n)\}_{n=1}^{\infty}\right)$ and a fixed $E \in \mathbb{R}$, denote by ψ^E a solution to the equation

$$a(n)\psi^E(n+1) + b(n)\psi^E(n) + a(n-1)\psi^E(n-1) = E\psi^E(n) \quad n > 1. \tag{2.3}$$

It is customary to extend this equation to $n = 1$ by defining $a(0) = 1$. Clearly, the space of sequences $\{\psi^E(n)\}_{n=0}^{\infty}$ solving (2.3) is a two-dimensional vector space and any such sequence is completely determined by its values at 0 and 1. We let $\psi_\phi^E(n)$ stand for the solution of (2.3) satisfying

$$\psi_\phi^E(0) = \sin(\phi) \qquad \psi_\phi^E(1) = \cos(\phi). \tag{2.4}$$

We note that, formally, $\psi_0^E(n)$ satisfies

$$J\psi_0^E = E\psi_0^E.$$

Let

$$S^E(n) = \begin{pmatrix} \frac{E-b(n)}{a(n)} & -\frac{a(n-1)}{a(n)} \\ 1 & 0 \end{pmatrix}$$

and

$$T^E(n) = S^E(n) \cdot S^E(n-1) \cdots S^E(1).$$

Then, for any ϕ,

$$\begin{pmatrix} \psi_\phi^E(n+1) \\ \psi_\phi^E(n) \end{pmatrix} = T^E(n) \begin{pmatrix} \psi_\phi^E(1) \\ \psi_\phi^E(0) \end{pmatrix},$$

and so

$$T^E(n) = \begin{pmatrix} \psi_0^E(n+1) & \psi_{\frac{\pi}{2}}^E(n+1) \\ \psi_0^E(n) & \psi_{\frac{\pi}{2}}^E(n) \end{pmatrix}.$$

We call the matrices $S^E(n)$ defined above *one-step transfer matrices*, and for the matrices $T^E(n)$, we use the name *n-step transfer matrices*. Our main technical result is

Theorem 2.2. *Let $J_{\Upsilon,\omega}$ be the family of random Jacobi matrices described in the Introduction. Then, for any $E \in \mathbb{R}$, the following holds with probability one:*

(1) *If $\gamma > \frac{1}{2}$*
$$\lim_{n\to\infty} \frac{\log \parallel T_\omega^E(n) \parallel^2}{\log(n)} = -\eta_1 \tag{2.5}$$
(2) *If $\gamma = \frac{1}{2}$*
$$\lim_{n\to\infty} \frac{\log \parallel T_\omega^E(n) \parallel^2}{\log(n)} = \Lambda - \eta_1. \tag{2.6}$$
(3) *If $\gamma < \frac{1}{2}$*
$$\lim_{n\to\infty} \frac{\log \parallel T_\omega^E(n) \parallel^2}{n^{1-2\gamma}} = \Lambda. \tag{2.7}$$

The EFGP transform (see [13]) is a useful tool for studying the asymptotic behavior of $\parallel T^E(n) \parallel$ in the Schrödinger case ($a(n) \equiv 1$). For $a(n) \to \infty$, certain modifications are needed. We proceed to present a version that is suitable for our purposes.

Let $J\left(\{a_\omega(n)\}_{n=1}^\infty \{b_\omega(n)\}_{n=1}^\infty\right)$ be a Jacobi matrix whose entries are all independent random variables. Let $\tilde{a}(n) = \langle a_\omega(n) \rangle$ and $\alpha_\omega = a_\omega(n) - \tilde{a}(n)$, and assume that
$$\lim_{n\to\infty} \tilde{a}(n) = \infty \tag{2.8}$$
and that
$$\lim_{n\to\infty} \frac{\alpha_\omega(n)}{\tilde{a}(n)} = 0 \tag{2.9}$$
with probability one. These properties clearly hold for $J_{\Upsilon,\omega}$ (see Lemma 2.1). In the analysis that follows we keep $E \in \mathbb{R}$ fixed so we omit it from the notation. Define
$$K_\omega(n) = \begin{pmatrix} 1 & 0 \\ 0 & a_\omega(n) \end{pmatrix}.$$
Then,
$$\tilde{S}_\omega(n) \equiv K_\omega(n) S(n) K_\omega(n-1)^{-1} = \begin{pmatrix} \frac{E-b_\omega(n)}{a_\omega(n)} & -\frac{1}{a_\omega(n)} \\ a_\omega(n) & 0 \end{pmatrix}$$
and
$$\tilde{T}_\omega(n) \equiv \tilde{S}_\omega(n) \cdot \tilde{S}_\omega(n-1) \cdots \tilde{S}_\omega(1) = K_\omega(n) T_\omega(n).$$
For any ϕ, define the sequences $\{u_{\omega,\phi}(n)\}_{n=1}^\infty$ and $\{v_{\omega,\phi}(n)\}_{n=1}^\infty$ by
$$\begin{pmatrix} u_{\omega,\phi}(n) \\ v_{\omega,\phi}(n) \end{pmatrix} = \tilde{T}_\omega(n) \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix},$$

so that, from the definition of $\psi_{\omega,\phi}(= \psi_\phi$ for the random Jacobi parameters), we see that

$$u_{\omega,\phi}(n) = \psi_{\omega,\phi}(n+1) \qquad v_{\omega,\phi}(n) = a_\omega(n)\psi_{\omega,\phi}(n). \tag{2.10}$$

By (2.8) we see that for any $E \in \mathbb{R}$ and sufficiently large n, we may define $k_n \in (0,\pi)$ by

$$2\cos(k_n) = \frac{E}{\tilde{a}(n)}. \tag{2.11}$$

Clearly, $k_n \to \frac{\pi}{2}$ as $n \to \infty$.

Now, define $R_{\omega,\phi}(n)$ and $\theta_{\omega,\phi}(n)$ through

$$R_{\omega,\phi}(n)\sin(\theta_{\omega,\phi}(n)) = v_{\omega,\phi}(n)\sin(k_n) \tag{2.12}$$

and

$$R_{\omega,\phi}(n)\cos(\theta_{\omega,\phi}(n)) = \tilde{a}(n)u_{\omega,\phi}(n) - v_{\omega,\phi}(n)\cos(k_n) \tag{2.13}$$

so that (using (2.10))

$$\begin{aligned} R_{\omega,\phi}(n)^2 &= v_{\omega,\phi}(n)^2 + \tilde{a}(n)^2 u_{\omega,\phi}(n)^2 - 2\tilde{a}(n)u_{\omega,\phi}(n)v_{\omega,\phi}(n)\cos(k_n) \\ &= a_\omega(n)^2\psi_{\omega,\phi}(n)^2 + \tilde{a}(n)^2\psi_{\omega,\phi}(n+1)^2 \\ &\quad - a_\omega(n)E\psi_{\omega,\phi}(n+1)\psi_{\omega,\phi}(n), \end{aligned}$$

which leads to

$$\begin{aligned} \frac{R_{\omega,\phi}(n)^2}{\tilde{a}(n)^2\left(\psi_{\omega,\phi}(n)^2 + \psi_{\omega,\phi}(n+1)^2\right)} &= 1 + \frac{2\alpha_\omega(n)\tilde{a}(n)\psi_{\omega,\phi}(n)^2}{\tilde{a}(n)^2\left(\psi_{\omega,\phi}(n)^2 + \psi_{\omega,\phi}(n+1)^2\right)} \\ &\quad + \frac{2\alpha_\omega(n)^2\psi_{\omega,\phi}(n)^2}{\tilde{a}(n)^2\left(\psi_{\omega,\phi}(n)^2 + \psi_{\omega,\phi}(n+1)^2\right)} \\ &\quad - \frac{Ea_\omega(n)\psi_{\omega,\phi}(n)\psi_{\omega,\phi}(n+1)}{\tilde{a}(n)^2\left(\psi_{\omega,\phi}(n)^2 + \psi_{\omega,\phi}(n+1)^2\right)}. \end{aligned} \tag{2.14}$$

By (2.9), it follows that the right hand side converges to one with probability 1, uniformly in ϕ, so that almost surely, for sufficiently large n, there are constants $C_1, C_2 > 0$ such that

$$C_1 R_{\omega,\phi}(n)^2 \le \tilde{a}(n)^2\left(\psi_{\omega,\phi}(n)^2 + \psi_{\omega,\phi}(n+1)^2\right) \le C_2 R_{\omega,\phi}(n)^2$$

Now, by a straightforward adaptation of Lemma 2.2 of [13] it follows that for any two angles $\phi_1 \neq \phi_2$, there are constants $C_3, C_4 > 0$ such that

$$\begin{aligned} C_3\max(R_{\omega,\phi_1}(n)^2, R_{\omega,\phi_2}(n)^2) &\le \tilde{a}(n)^2 \parallel T_\omega(n) \parallel^2 \\ &\le C_4\max(R_{\omega,\phi_1}(n)^2, R_{\omega,\phi_2}(n)^2). \end{aligned} \tag{2.15}$$

Thus, we are led to examine the asymptotic properties of $\log R_{\omega,\phi}(n)$.

Let us formulate a recursion relation for $R_{\omega,\phi}(n)^2$: (2.12) and (2.13) mean

$$\begin{aligned}
&\begin{pmatrix} R_{\omega,\phi}(n)\sin(\theta_{\omega,\phi}(n)) \\ R_{\omega,\phi}(n)\cos(\theta_{\omega,\phi}(n)) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin(k_n) \\ \tilde{a}(n) & -\cos(k_n) \end{pmatrix} \begin{pmatrix} u_{\omega,\phi}(n) \\ v_{\omega,\phi}(n) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin(k_n) \\ \tilde{a}(n) & -\cos(k_n) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a_\omega(n) \end{pmatrix} \begin{pmatrix} \psi_{\omega,\phi}(n+1) \\ \psi_{\omega,\phi}(n) \end{pmatrix}.
\end{aligned}$$

We also know

$$\begin{pmatrix} \psi_{\omega,\phi}(n+2) \\ \psi_{\omega,\phi}(n+1) \end{pmatrix} = S_\omega(n+1) \begin{pmatrix} \psi_{\omega,\phi}(n+1) \\ \psi_{\omega,\phi}(n) \end{pmatrix},$$

so

$$\begin{aligned}
&\begin{pmatrix} R_{\omega,\phi}(n+1)\sin(\theta_{\omega,\phi}(n+1)) \\ R_{\omega,\phi}(n+1)\cos(\theta_{\omega,\phi}(n+1)) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin(k_{n+1}) \\ \tilde{a}(n+1) & -\cos(k_{n+1}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a_\omega(n+1) \end{pmatrix} S_\omega(n+1) \\
&\quad \cdot \begin{pmatrix} 1 & 0 \\ 0 & a_\omega(n) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \sin(k_n) \\ \tilde{a}(n) & -\cos(k_n) \end{pmatrix}^{-1} \begin{pmatrix} R_{\omega,\phi}(n)\sin(\theta_{\omega,\phi}(n)) \\ R_{\omega,\phi}(n)\cos(\theta_{\omega,\phi}(n)) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin(k_{n+1}) \\ \tilde{a}(n+1) & -\cos(k_{n+1}) \end{pmatrix} \tilde{S}_\omega(n+1) \\
&\quad \cdot \begin{pmatrix} 0 & \sin(k_n) \\ \tilde{a}(n) & -\cos(k_n) \end{pmatrix}^{-1} \begin{pmatrix} R_{\omega,\phi}(n)\sin(\theta_{\omega,\phi}(n)) \\ R_{\omega,\phi}(n)\cos(\theta_{\omega,\phi}(n)) \end{pmatrix}.
\end{aligned} \tag{2.16}$$

Now, write

$$\begin{aligned}
\tilde{S}_\omega(n+1) &= \begin{pmatrix} \frac{E-b_\omega(n+1)}{a_\omega(n+1)} & -\frac{1}{a_\omega(n+1)} \\ a_\omega(n+1) & 0 \end{pmatrix} \\
&= \frac{\tilde{a}(n+1)}{a_\omega(n+1)} \left(\begin{pmatrix} \frac{E}{\tilde{a}(n+1)} & -\frac{1}{\tilde{a}(n+1)} \\ \tilde{a}(n+1) & 0 \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} -\frac{b_\omega(n+1)}{\tilde{a}(n+1)} & 0 \\ \frac{a_\omega(n+1)^2-\tilde{a}(n+1)^2}{\tilde{a}(n+1)} & 0 \end{pmatrix} \right).
\end{aligned}$$

We define

$$\mathcal{Z}_\omega(n+1) = \begin{pmatrix} 0 & \sin(k_{n+1}) \\ \tilde{a}(n+1) & -\cos(k_{n+1}) \end{pmatrix} \begin{pmatrix} \frac{E}{\tilde{a}(n+1)} & -\frac{1}{\tilde{a}(n+1)} \\ \tilde{a}(n+1) & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \sin(k_n) \\ \tilde{a}(n) & -\cos(k_n) \end{pmatrix}^{-1} \begin{pmatrix} \sin(\theta_{\omega,\phi}(n)) \\ \cos(\theta_{\omega,\phi}(n)) \end{pmatrix},$$

and

$$\mathcal{W}_\omega(n+1) = \begin{pmatrix} 0 & \sin(k_{n+1}) \\ \tilde{a}(n+1) & -\cos(k_{n+1}) \end{pmatrix} \begin{pmatrix} -\frac{b_\omega(n+1)}{\tilde{a}(n+1)} & 0 \\ \frac{a_\omega(n+1)^2-\tilde{a}(n+1)^2}{\tilde{a}(n+1)} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \sin(k_n) \\ \tilde{a}(n) & -\cos(k_n) \end{pmatrix}^{-1} \begin{pmatrix} \sin(\theta_{\omega,\phi}(n)) \\ \cos(\theta_{\omega,\phi}(n)) \end{pmatrix}$$

(we ignore the dependence on ϕ since we keep it fixed). Then, from (2.16) we see that

$$\frac{R_{\omega,\phi}(n+1)^2}{R_{\omega,\phi}(n)^2} = \frac{\tilde{a}(n+1)^2}{a_\omega(n+1)^2} \parallel \mathcal{Z}_\omega(n+1) + \mathcal{W}_\omega(n+1) \parallel^2 . \qquad (2.17)$$

$\theta_{\omega,\phi}(n)$ satisfies a recurrence relation as well: From (2.10) we have that

$$v_{\omega,\phi}(n+1) = a_\omega(n+1)u_{\omega,\phi}(n) \qquad (2.18)$$

and

$$a_\omega(n+1)u_{\omega,\phi}(n+1) + b_\omega(n+1)u_{\omega,\phi}(n) + a_\omega(n)u_{\omega,\phi}(n-1) = Eu_{\omega,\phi}(n). \qquad (2.19)$$

Write, using (2.12)-(2.13)

$$\begin{aligned} \cot(\theta_{\omega,\phi}(n+1)) &= \frac{\tilde{a}(n+1)u_{\omega,\phi}(n+1) - \cos(k_{n+1})v_{\omega,\phi}(n+1)}{\sin(k_{n+1})v_{\omega,\phi}(n+1)} \\ &= \frac{\tilde{a}(n+1)u_{\omega,\phi}(n+1) - a_\omega(n+1)\cos(k_{n+1})u_{\omega,\phi}(n)}{\sin(k_{n+1})a_\omega(n+1)u_{\omega,\phi}(n)}. \end{aligned} \qquad (2.20)$$

Furthermore, observing that

$$R_{\omega,\phi}(n)\sin(\theta_{\omega,\phi}(n) + k_n) = \tilde{a}(n)\sin(k_n)u_{\omega,\phi}(n),$$

and

$$\begin{aligned} R_{\omega,\phi}(n)\cos(\theta_{\omega,\phi}(n) + k_n) &= \tilde{a}(n)\cos(k_n)u_{\omega,\phi}(n) - v_{\omega,\phi}(n) \\ &= \tilde{a}(n)\cos(k_n)u_{\omega,\phi}(n) - a_\omega(n)u_{\omega,\phi}(n-1), \end{aligned}$$

we may write

$$\cot(\theta_{\omega,\phi}(n) + k_n) = \frac{\tilde{a}(n)\cos(k_n)u_{\omega,\phi}(n) - a_\omega(n)u_{\omega,\phi}(n-1)}{\tilde{a}(n)\sin(k_n)u_{\omega,\phi}(n)}. \qquad (2.21)$$

Substituting $u_{\omega,\phi}(n+1)$ from (2.19) into (2.20) and then $u_{\omega,\phi}(n-1)$ from (2.21) into the resulting equation, we get

$$
\begin{aligned}
\cot(\theta_{\omega,\phi}(n+1)) = {} & \frac{\tilde{a}(n+1)\tilde{a}(n)}{a_\omega(n+1)^2}\frac{\sin(k_n)}{\sin(k_{n+1})}\cot(\theta_{\omega,\phi}(n)+k_n) \\
& + \cot(k_{n+1})\left(\frac{\tilde{a}(n+1)^2}{a_\omega(n+1)^2} - 1\right) \\
& - \frac{\tilde{a}(n+1)}{\sin(k_n)a_\omega(n+1)^2} b_\omega(n+1) \\
& \equiv \kappa_\omega(n+1)\cot(\bar{\theta}_{\omega,\phi}(n)) + \zeta_\omega(n+1)
\end{aligned}
\tag{2.22}
$$

where

$$\bar{\theta}_{\omega,\phi}(n) = \theta_{\omega,\phi}(n) + k_n.$$

By (2.15) and picking two angles $\phi_1 \neq \phi_2$, Theorem 2.2 follows from

Proposition 2.3. *Let $J_{\Upsilon,\omega}$ be the family of random Jacobi matrices described in the Introduction. Then, for any $E \in \mathbb{R}$, and for any ϕ, the following holds with probability one:*

(1) *If $\gamma > \frac{1}{2}$*

$$\lim_{n\to\infty} \frac{\log R^E_{\omega,\phi}(n)^2}{\log(n)} = \eta_1 \tag{2.23}$$

(2) *If $\gamma = \frac{1}{2}$*

$$\lim_{n\to\infty} \frac{\log R^E_{\omega,\phi}(n)^2}{\log(n)} = \Lambda + \eta_1. \tag{2.24}$$

(3) *If $\gamma < \frac{1}{2}$*

$$\lim_{n\to\infty} \frac{\log R^E_{\omega,\phi}(n)^2}{n^{1-2\gamma}} = \Lambda. \tag{2.25}$$

Proof. As in [13] we shall prove the statement by using the recursion relation for $R_\omega(n)^2$ (equation (2.17)). Namely, we shall prove that

$$\frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \left(\log\left(\| \mathcal{Z}_\omega(j) + \mathcal{W}_\omega(j) \|^2 \right) - \log\left(\frac{a_\omega(j)^2}{\tilde{a}(j)^2} \right) \right) \tag{2.26}$$

converges to the appropriate limit, where $F_\gamma(n) = \log(n)$ for $\gamma \geq \frac{1}{2}$ and $F_\gamma(n) = n^{1-2\gamma}$ otherwise. From this point on, $a_\omega(n) = a_{\Upsilon,\omega}(n)$ and $b_\omega(n) = b_{\Upsilon,\omega}(n)$.

We shall need some estimate on the behavior of $\theta_\omega(n)$. We start with

Lemma 2.4. *For any $\varepsilon > 0$, there exists, with probability one, a constant $\tilde{C} = \tilde{C}(\omega, \varepsilon)$, such that*

$$|\theta_{\omega,\phi}(n+1) - \bar{\theta}_{\omega,\phi}(n)| \leq \tilde{C}\max(n^{-\gamma+\varepsilon}, n^{-1}). \tag{2.27}$$

Proof of the Lemma. We start by proving a similar statement for $|\kappa_\omega(n+1)-1|+|\zeta_\omega(n+1)|$:

Recall that $k_n \to \frac{\pi}{2}$. Thus, for large enough n, $\sin(k_n) > \frac{1}{2}$. Moreover, $\cos(k_n) \sim \frac{1}{n^{\eta_1}}$. It follows that

$$|\sin(k_n)-\sin(k_{n+1})| = \frac{|\cos(k_n)^2-\cos(k_{n+1})^2|}{|\sin(k_n)+\sin(k_{n+1})|} \leq \frac{C'}{n^{1+2\eta_1}}.$$

Thus,

$$\begin{aligned}
|\kappa_\omega(n+1)-1|+|\zeta_\omega(n+1)| \leq{}& \left|\frac{\sin(k_n)}{\sin(k_{n+1})}\left(\frac{\tilde{a}(n+1)\tilde{a}(n)}{a_\omega(n+1)^2}-1\right)\right| \\
&+\left|\frac{\sin(k_n)-\sin(k_{n+1})}{\sin(k_{n+1})}\right| \\
&+\left|\cot(k_{n+1})\left(\frac{\tilde{a}(n+1)^2}{a_\omega(n+1)^2}-1\right)\right| \\
&+\left|\frac{\tilde{a}(n+1)}{\sin(k_n)a_\omega(n+1)^2}b_\omega(n+1)\right| \\
={}& I_1+I_2+I_3+I_4.
\end{aligned}$$

Now, choosing $\varepsilon < \eta_1-\eta_2$, by Lemma 2.1

$$\begin{aligned}
I_1 \leq{}& 2\left|\frac{\tilde{a}(n+1)\left(\tilde{a}(n)-\tilde{a}(n+1)\right)}{a_\omega(n+1)^2}\right| + 4\left|\frac{\alpha_\omega(n+1)\tilde{a}(n+1)}{a_\omega(n+1)^2}\right| \\
&+2\left|\frac{\alpha_\omega(n+1)^2}{a_\omega(n+1)^2}\right| \leq C_1(\omega,\varepsilon)\max(n^{-\gamma+\varepsilon},n^{-1})
\end{aligned}$$

$$I_3 \leq 2|\cos(k_{n+1})|\left|\frac{2\alpha_\omega(n+1)\tilde{a}(n+1)+\alpha_\omega(n+1)^2}{a_\omega(n+1)^2}\right| \leq \frac{C_3(\omega,\varepsilon)}{n^{\gamma-\varepsilon}},$$

and

$$I_4 \leq 2\left|\frac{\tilde{a}(n+1)b_\omega(n+1)}{a_\omega(n+1)^2}\right| \leq \frac{C_4(\omega,\varepsilon)}{n^{\gamma-\varepsilon}}$$

almost surely. We have shown above that

$$I_2 \leq \frac{C_2}{n^{1+\eta_1}}$$

so we see that there exists, with probability one, a constant $C_5(\omega,\varepsilon)$ for which

$$|\kappa_\omega(n+1)-1|+|\zeta_\omega(n+1)| \leq \frac{C_5}{\max}(n^{-\gamma+\varepsilon},n^{-1}). \tag{2.28}$$

Now, use

$$e^{2ix} = 1-\frac{2}{1+i\cot x}$$

and (2.22) to see that, if $|\kappa_\omega(n+1)-1|+2|\zeta_\omega(n+1)|<\frac{1}{2}$, (which indeed happens almost surely, for large enough n), then

$$\left|e^{2i\theta_{\omega,\phi}(n+1)}-e^{2i\bar{\theta}_{\omega,\phi}(n)}\right|\leq 4\left(|\kappa_\omega(n+1)-1|+|\zeta_\omega(n+1)|\right),$$

which implies (by $|e^{ix}-1|\geq\frac{2|x|}{\pi}$) that

$$|\theta_{\omega,\phi}(n+1)-\bar{\theta}_{\omega,\phi}(n)|\leq\pi\left(|\kappa_\omega(n+1)-1|+|\zeta_\omega(n+1)|\right).$$

This, together with (2.28), implies (2.27) and concludes the proof of the lemma. □

The direct consequence of this is

Proposition 2.5. *Assume $f(n)$ is a function that satisfies*

$$f(n)=o\left(n^r\right)$$

and

$$f(n+1)-f(n)=o\left(n^{r-1}\right),$$

with

$$r=\begin{cases}\gamma-1 & \text{if}\gamma\geq 1/2 \text{ and } \eta_2>0\\ -\gamma & \text{if}\gamma<1/2 \text{ and } \eta_2>0\\ \eta_1-1 & \text{if}\gamma\geq 1/2 \text{ and } \eta_2\leq 0\\ \eta_1-2\gamma & \text{if}\gamma<1/2 \text{ and } \eta_2\leq 0.\end{cases} \tag{2.29}$$

Then, for any ϕ,

$$\lim_{n\to\infty}\frac{1}{F_\gamma(n)}\sum_{j=1}^{n}f(j)\cos(2\theta_{\omega,\phi}(j))=0 \tag{2.30}$$

almost surely. The same statement holds with θ replaced by $\bar{\theta}$.

Proof of the Proposition. We shall prove the statement for θ. By summation by parts,

$$\left|\sum_{j=1}^{n}f(j)\cos(2\theta_{\omega,\phi}(j))\right|$$
$$=\left|f(n)\sum_{j=1}^{n}\cos(2\theta_{\omega,\phi}(j))-\sum_{j=1}^{n-1}\sum_{l=1}^{j}\cos(2\theta_{\omega,\phi}(l))\left(f(j+1)-f(j)\right)\right|. \tag{2.31}$$

Thus we are led to examine $\sum_{l=1}^{j}\cos(2\theta_{\omega,\phi}(l))$. Assume that $j=2m$ is even. Then

$$\begin{aligned}\left|\sum_{l=1}^{j}\cos(2\theta_{\omega,\phi}(l))\right| &= \left|\sum_{l=1}^{m}(\cos(2\theta_{\omega,\phi}(2l))-\cos(2\theta_{\omega,\phi}(2l-1)+\pi))\right|\\ &\le \sum_{l=1}^{m}|2\theta_{\omega,\phi}(2l)-2\theta_{\omega,\phi}(2l-1)-\pi|\\ &\le 2\sum_{l=1}^{m}\left|\theta_{\omega,\phi}(2l)-\bar{\theta}_{\omega,\phi}(2l-1)\right|+\left|k_{2l-1}-\frac{\pi}{2}\right|\\ &\le C_\omega\max(j^{1-\gamma+\varepsilon},j^{1-\eta_1})\end{aligned}\tag{2.32}$$

almost surely, by Lemma 2.4 and by

$$\left|k_n-\frac{\pi}{2}\right|\le 2|\cos(k_n)|,$$

which holds for sufficiently large n. Thus, for any j, we get

$$\begin{aligned}\left|\sum_{l=1}^{j}\cos(2\theta_{\omega,\phi}(l))\right| &\le C_\omega\max(j^{1-\gamma+\varepsilon},j^{1-\eta_1})+1\\ &\le (C_\omega+1)\max(j^{1-\gamma+\varepsilon},j^{1-\eta_1}).\end{aligned}\tag{2.33}$$

A simple calculation finishes the proof for θ. The proof for $\bar{\theta}$ follows the same argument, with an additional $|k_{2l}-k_{2l-1}|$ term in (2.32). $\square$

We abbreviate

$$\begin{aligned}A_\omega(j) &= \frac{a_\omega(j)^2-\tilde{a}(j)^2}{\tilde{a}(j)^2}=2\frac{\alpha_\omega(j)}{\tilde{a}(j)}+\frac{\alpha_\omega(j)^2}{\tilde{a}(j)^2}\\ &= 2\frac{\lambda_2}{\lambda_1}\frac{Y_\omega(j)}{j^\gamma}+\left(\frac{\lambda_2}{\lambda_1}\right)^2\frac{Y_\omega(j)^2}{j^{2\gamma}},\end{aligned}\tag{2.34}$$

and

$$B_\omega(j)=\frac{b_\omega(j)}{\tilde{a}(j)}=\frac{\lambda_2}{\lambda_1}\frac{X_\omega(j)}{j^\gamma}\tag{2.35}$$

By a straightforward calculation we get

$$\begin{aligned}\|\,\mathcal{W}_\omega(j)\,\|^2 &= \frac{\sin^2(\bar{\theta}_\omega(j-1))\tilde{a}(j)^2}{\sin^2(k_{j-1})\tilde{a}(j-1)^2}\\ &\quad\times\left(A_\omega(j)^2+B_\omega(j)^2+2\cos(k_j)B_\omega(j)A_\omega(j)\right),\end{aligned}\tag{2.36}$$

$$\begin{aligned}\| \mathcal{Z}_\omega(j) \|^2 = 1 + \sin^2(\bar{\theta}_\omega(j-1))&\frac{\tilde{a}(j)^2 - \tilde{a}(j-1)^2}{\tilde{a}(j-1)^2} \\ &+ \frac{\tilde{a}(j)^2 \sin^2(\bar{\theta}_\omega(j-1))}{\tilde{a}(j-1)^2}\frac{\sin^2(k_j) - \sin^2(k_{j-1})}{\sin^2(k_{j-1})}\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j)) = &\frac{\tilde{a}(j)}{\tilde{a}(j-1)} \\ &\times \Bigg(\frac{A_\omega(j)\sin(\bar{\theta}_\omega(j-1))}{\sin^2(k_{j-1})}\Big(\frac{\tilde{a}(j)}{\tilde{a}(j-1)}\sin(\bar{\theta}_\omega(j-1)) \\ &- \cos(k_j)\sin(\theta_\omega(j-1))\Big) - \frac{B_\omega(j)\sin(2\bar{\theta}_\omega(j-1))}{2\sin(k_{j-1})}\Bigg)\end{aligned} \tag{2.38}$$

(Recall $\bar{\theta}_\omega(j) \equiv \bar{\theta}_{\omega,\phi}(j) = \theta_{\omega,\phi}(j) + k_j \equiv \theta_\omega(j) + k_j$.)

Let $\varepsilon < \frac{\gamma}{8}$. By Theorem 2.1, the fact that $\cos(k_j) \sim \frac{1}{\tilde{a}(j)}$ and the identity

$$|\sin^2(x) - \sin^2(y)| = |\cos^2(x) - \cos^2(y)|,$$

it follows that $\| \mathcal{W}_\omega(j) \|^2 = \mathcal{O}_\omega(j^{-2\gamma+2\varepsilon})$, $(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j)) = \mathcal{O}_\omega(j^{-\gamma+\varepsilon})$ and $(\| \mathcal{Z}_\omega(j) \|^2 - 1) = \mathcal{O}(j^{-1})$ with probability 1, where the notation $\mathcal{O}_\omega$ indicates that the implicit constant depends on ω. Thus, we can use

$$\log(1+x) = x - \frac{x^2}{2} + \mathcal{O}(x^3),$$

together with the observation that (almost surely)

$$\begin{aligned}&\Big(\| \mathcal{Z}_\omega(j) \|^2 - 1 + 2(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j)) + \| \mathcal{W}_\omega(j) \|^2 \Big)^2 \\ &\quad = 4(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j))^2 + \mathcal{O}_\omega\left(\frac{1}{j^{3\gamma-\varepsilon}}\right),\end{aligned}$$

to see that, with probability one, for large enough j we have that

$$\begin{aligned}&\log\Big(1 + \| \mathcal{Z}_\omega(j) \|^2 - 1 + 2(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j)) + \| \mathcal{W}_\omega(j) \|^2\Big) \\ &\quad = (\| \mathcal{Z}_\omega(j) \|^2 - 1) + 2(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j)) + \| \mathcal{W}_\omega(j) \|^2 \\ &\qquad - 2(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j))^2 + \mathcal{O}_\omega\left(\frac{1}{j^{3\gamma-\varepsilon}} + \frac{1}{j^{1+\gamma-\varepsilon}}\right).\end{aligned} \tag{2.39}$$

Therefore, since the edition of a finite number of terms is inconsequential, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \log\left(\| \mathcal{Z}_\omega(j) + \mathcal{W}_\omega(j) \|^2\right) \\
&\quad = \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \Big((\| \mathcal{Z}_\omega(j) \|^2 - 1) + 2(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j)) + \| \mathcal{W}_\omega(j) \|^2 \\
&\qquad - 2(\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j))^2 \Big)
\end{aligned} \tag{2.40}$$

with probability one, in the sense that both limits exist together and are equal if they do.

Similarly,

$$\begin{aligned}
\lim_{n\to\infty} \frac{-1}{F_\gamma(n)} \sum_{j=1}^{n} \log\left(\frac{a_\omega(j)^2}{\tilde{a}(j)^2}\right) &= \lim_{n\to\infty} \frac{-1}{F_\gamma(n)} \sum_{j=1}^{n} \log\left(1 + A_\omega(j)\right) \\
&= \lim_{n\to\infty} \frac{-1}{F_\gamma(n)} \sum_{j=1}^{n} \left(A_\omega(j) - \frac{A_\omega(j)^2}{2}\right).
\end{aligned} \tag{2.41}$$

Thus, our problem is reduced to computing the limits:

$$\xi_{\mathcal{Z}} \equiv \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \left(\| \mathcal{Z}_\omega(j) \|^2 - 1\right)$$

$$\xi_{\mathcal{W}} \equiv \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \| \mathcal{W}_\omega(j) \|^2$$

$$\xi_{\mathcal{Z}\mathcal{W}} \equiv \lim_{n\to\infty} \frac{2}{F_\gamma(n)} \sum_{j=1}^{n} (\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j))$$

$$\xi_{\mathcal{Z}\mathcal{W}_2} \equiv \lim_{n\to\infty} \frac{-2}{F_\gamma(n)} \sum_{j=1}^{n} (\mathcal{Z}_\omega(j), \mathcal{W}_\omega(j))^2$$

$$\xi_A \equiv \lim_{n\to\infty} \frac{-1}{F_\gamma(n)} \sum_{j=1}^{n} \left(A_\omega(j) - \frac{A_\omega(j)^2}{2}\right).$$

By (2.37),

$$\| \mathcal{Z}_\omega(j) \|^2 - 1 = \sin^2(\bar{\theta}_\omega(j-1)) \frac{\tilde{a}(j)^2 - \tilde{a}(j-1)^2}{\tilde{a}(j-1)^2} + \frac{\tilde{a}(j)^2 \sin^2(\bar{\theta}_\omega(j-1))}{\tilde{a}(j-1)^2} \frac{\sin^2(k_j) - \sin^2(k_{j-1})}{\sin^2(k_{j-1})}.$$

The last term on the right is absolutely summable ($= \mathcal{O}(n^{-1-2\eta_1})$) so we only need to look at $\sin^2(\bar{\theta}_\omega(j-1)) \frac{\tilde{a}(j)^2 - \tilde{a}(j-1)^2}{\tilde{a}(j-1)^2}$. But, by $\sin^2(\alpha) = \frac{1}{2} - \frac{\cos(2\alpha)}{2}$ and by Proposition 2.5, we see that (recall $\tilde{a}(j) = \lambda_1 j^{-\eta_1}$)

$$\begin{aligned} \xi_{\mathcal{Z}} &= \frac{1}{2} \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \frac{\tilde{a}(j)^2 - \tilde{a}(j-1)^2}{\tilde{a}(j-1)^2} \\ &= \frac{1}{2} \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \frac{2\eta_1}{j} = \begin{cases} \eta_1 & \text{if } \gamma \geq 1/2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.42}$$

For the other four limits, we shall use extensively Lemma 8.4 of [13], in order to replace A_ω and B_ω by their means. For example, write

$$\begin{aligned} \| \mathcal{W}_\omega(j) \|^2 &= \frac{\sin^2(\bar{\theta}_\omega(j-1))\tilde{a}(j)^2}{\sin^2(k_{j-1})\tilde{a}(j-1)^2} \left(A_\omega(j)^2 - \left\langle A_\omega(j)^2 \right\rangle \right) \\ &+ \frac{\sin^2(\bar{\theta}_\omega(j-1))\tilde{a}(j)^2}{\sin^2(k_{j-1})\tilde{a}(j-1)^2} \left(B_\omega(j)^2 - \left\langle B_\omega(j)^2 \right\rangle \right) \\ &+ \frac{\sin^2(\bar{\theta}_\omega(j-1))\tilde{a}(j)^2}{\sin^2(k_{j-1})\tilde{a}(j-1)^2} \left(2\cos(k_j) B_\omega(j) A_\omega(j) \right) \\ &+ \frac{\sin^2(\bar{\theta}_\omega(j-1))\tilde{a}(j)^2}{\sin^2(k_{j-1})\tilde{a}(j-1)^2} \left(\left\langle A_\omega(j)^2 \right\rangle + \left\langle B_\omega(j)^2 \right\rangle \right). \end{aligned} \tag{2.43}$$

Then the first three terms have mean zero and so, by Lemma 8.4 of [13] we get that

$$\xi_{\mathcal{W}} \equiv \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{\infty} \frac{\sin^2(\bar{\theta}_\omega(j-1))\tilde{a}(j)^2}{\sin^2(k_{j-1})\tilde{a}(j-1)^2} \left(\left\langle A_\omega(j)^2 \right\rangle + \left\langle B_\omega(j)^2 \right\rangle \right).$$

Expanding $\langle A_\omega(j)^2\rangle$, throwing out terms that are $o(j^{-2\gamma})$, and applying Proposition 2.5 (writing, again, $\sin^2(\alpha) = \frac{1}{2} - \frac{\cos(2\alpha)}{2}$), we get

$$\begin{aligned} \gamma_{\mathcal{W}} &= \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \frac{j^{2\eta_1}}{(j-1)^{2\eta_1}\sin^2(k_{j-1})} \Lambda j^{-2\gamma} \\ &\quad \times \left(\left\langle 4Y_\omega(j)^2\right\rangle + \left\langle X_\omega(j)^2\right\rangle\right) \qquad (2.44) \\ &= \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} 2\Lambda j^{-2\gamma} = \begin{cases} 0 & \text{if } \gamma > 1/2 \\ 2\Lambda & \text{otherwise,} \end{cases} \end{aligned}$$

(recall $\langle Y_\omega(j)^2\rangle = \frac{1}{4}$). Applying the same procedure to $\xi_{\mathcal{ZW}}$ and ξ_A we get

$$\begin{aligned} \xi_{\mathcal{ZW}} &= \lim_{n\to\infty} \frac{2}{F_\gamma(n)} \sum_{j=1}^{n} \frac{j^{2\eta_1}}{(j-1)^{2\eta_1}} \frac{\sin^2(\bar\theta_\omega(j-1))}{\sin^2(k_{j-1})} \left\langle 2\Lambda \frac{Y_\omega(j)^2}{j^{2\gamma}} \right\rangle \\ &= \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \frac{\Lambda}{2}\frac{1}{j^{2\gamma}} = \begin{cases} 0 & \text{if } \gamma > 1/2 \\ \frac{\Lambda}{2} & \text{otherwise} \end{cases} \end{aligned} \qquad (2.45)$$

and

$$\begin{aligned} \xi_A &= \lim_{n\to\infty} \frac{-1}{F_\gamma(n)} \sum_{j=1}^{n} \left\langle 2\Lambda \frac{Y_\omega(j)^2}{j^{2\gamma}} \right\rangle \\ &= \lim_{n\to\infty} \frac{1}{F_\gamma(n)} \sum_{j=1}^{n} \left(\frac{\Lambda}{2}\frac{1}{j^{2\gamma}}\right) = \begin{cases} 0 & \text{if } \gamma > 1/2 \\ \frac{\Lambda}{2} & \text{otherwise.} \end{cases} \end{aligned} \qquad (2.46)$$

The computation of $\xi_{\mathcal{ZW}_2}$ involves $\sin^4(\bar\theta_\omega) = \frac{3}{8} - \frac{\cos(2\bar\theta_\omega)}{2} + \frac{\cos(4\bar\theta_\omega)}{8}$, for which Proposition 2.5 is useless. Luckily, the $\cos(4\bar\theta_\omega(j))$ cancels out. As before

$$\begin{aligned} \xi_{\mathcal{ZW}_2} = \lim_{n\to\infty} \frac{-2}{F_\gamma(n)} \sum_{j=1}^{n} \frac{2\Lambda}{j^{2\gamma}} \\ \times \Bigg(\frac{\langle 4Y_\omega(j)^2\rangle \sin^4(\bar\theta_\omega(j-1))}{\sin^2(k_{j-1})} \frac{j^{2\eta_1}}{(j-1)^{2\eta_1}} \\ + \frac{\langle X_\omega(j)^2\rangle \sin^2(2\bar\theta_\omega(j-1))}{4} \Bigg) \end{aligned}$$

Now write

$$\begin{aligned}
&\left(\frac{\langle 4Y_\omega(j)^2\rangle \sin^4(\bar{\theta}_\omega(j-1))}{\sin^2(k_{j-1})}\frac{j^{2\eta_1}}{(j-1)^{2\eta_1}} + \frac{\langle X_\omega(j)^2\rangle \sin^2(2\bar{\theta}_\omega(j-1))}{4}\right) \\
&\quad = \left(\frac{\sin^4(\bar{\theta}_\omega(j-1))}{\sin^2(k_{j-1})}\frac{j^{2\eta_1}}{(j-1)^{2\eta_1}} + \frac{\sin^2(2\bar{\theta}_\omega(j-1))}{4}\right) \\
&\quad = \left(\frac{3j^{2\eta_1}}{8(j-1)^{2\eta_1}\sin^2(k_{j-1})} + \frac{1}{8}\right) \\
&\qquad + \frac{j^{2\eta_1}}{2(j-1)^{2\eta_1}\sin^2(k_{j-1})}\cos(2\bar{\theta}_\omega(j-1)) \\
&\qquad + \cos(4\bar{\theta}_\omega(j-1))\left(\frac{j^{2\eta_1}}{(j-1)^{2\eta_1}\sin^2(k_{j-1})} - 1\right) \\
&\quad = \left(\frac{3j^{2\eta_1}}{8(j-1)^{2\eta_1}\sin^2(k_{j-1})} + \frac{1}{8}\right) \\
&\qquad + \frac{j^{2\eta_1}}{2(j-1)^{2\eta_1}\sin^2(k_{j-1})}\cos(2\bar{\theta}_\omega(j-1)) \\
&\qquad + \cos(4\bar{\theta}_\omega(j-1))\left(\mathcal{O}(n^{-1}) + \mathcal{O}(n^{-2\eta_1})\right).
\end{aligned}$$

to see that

$$\xi_{\mathcal{Z}\mathcal{W}_2} = \begin{cases} 0 & \text{if } \gamma > 1/2 \\ -2\Lambda & \text{otherwise,} \end{cases} \tag{2.47}$$

where the $\cos(2\bar{\theta}_\omega)$ term vanishes by Proposition 2.5. Summing up the various limits, the proposition is proved. □

Proof of Theorem 2.2. By (2.15), the theorem follows from Proposition 2.3. □

Theorem 1.1 almost follows immediately from Theorem 2.2 and Proposition 2.3. As in the Schrödinger situation, the case $\gamma = \frac{1}{2}$ requires some subtle reasoning. We have established that, in this case, with probability one, equation (2.3) has a solution, ψ, with

$$|\psi(n)|^2 \asymp n^{\Lambda - \eta_1}. \tag{2.48}$$

In order to use subordinacy theory, we need the existence of another solution with faster decay at infinity. The following is Lemma 8.7 of [13], formulated for general regular matrices:

Lemma 2.6. *Let $u_\phi = (\cos\phi, \sin\phi) \in \mathbb{R}^2$. For any matrix, $A \in GL_2(\mathbb{R})$ with $\det(A) = d > 0$, let $\phi(A)$ be the unique $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\frac{1}{\sqrt{d}} \parallel Au_\phi \parallel = \sqrt{d} \parallel A \parallel^{-1}$. Define $\rho(A) = \frac{\|Au_0\|}{\|Au_{\pi/2}\|}$.*

Let A_n be a sequence of matrices in $GL_2(\mathbb{R})$ with $\det(A_n) = d_n > 0$, that satisfy $\frac{1}{\sqrt{d_n}} \parallel A_n \parallel \to \infty$ and $d_n \frac{\|A_{n+1}A_n^{-1}\|}{\|A_n\|\|A_{n+1}\|} \to 0$ as $n \to \infty$. Let $\rho_n = \rho(A_n)$ and $\phi_n = \phi(A_n)$. Then:

1. *ϕ_n has a limit ϕ_∞ if and only if $\lim_{n\to\infty} \rho_n = \rho_\infty$ exists ($\rho_\infty = \infty$ is allowed, but then we only have $|\phi_n| \to \frac{\pi}{2}$).*
2. *Suppose ϕ_n has a limit $\phi_\infty \neq 0, \frac{\pi}{2}$ (equivalently, $\rho_\infty \neq 0, \infty$). Then*
$$\lim_{n\to\infty} \frac{\log \parallel A_n u_\infty \parallel - \log\sqrt{d_n}}{\log \parallel A_n \parallel - \log\sqrt{d_n}} = -1$$
if and only if
$$\limsup_{n\to\infty} \frac{\log|\rho_n - \rho_\infty|}{\log \parallel A_n \parallel - \log\sqrt{d_n}} \leq -2.$$

Proposition 2.7. *Let $J_{\Upsilon,\omega}$ be the family of random Jacobi matrices described in Theorem 2.2, with $\gamma = \frac{1}{2}$. Then, for any $E \in \mathbb{R}$, there exists, with probability one, an initial condition $\Psi_{\phi(\omega)} = \begin{pmatrix} \cos(\phi(\omega)) \\ \sin(\phi(\omega)) \end{pmatrix}$ such that*
$$\lim_{n\to\infty} \frac{\log \parallel T_\omega^E(n)\Psi_{\phi(\omega)} \parallel}{\log n} = -\frac{1}{2}\left(\Lambda + \eta\right). \tag{2.49}$$

Proof. We imitate the proof of [13, Lemma 8.8]. By Proposition 2.2
$$\lim_{n\to\infty} \frac{\log|R_{\omega,0}(n)|}{\log n} = \frac{1}{2}\left(\Lambda + \eta_1\right)$$
and
$$\lim_{n\to\infty} \frac{\log|R_{\omega,\pi/2}(n)|}{\log n} = \frac{1}{2}\left(\Lambda + \eta_1\right)$$
for almost every ω. By (2.12)-(2.13),
$$R_{\omega,0}(n)R_{\omega,\pi/2}(n)\sin(\theta_{\omega,\pi/2}(n) - \theta_{\omega,0}(n)) = \tilde{a}(n)\sin k_n$$
so that, for almost every ω,
$$\lim_{n\to\infty} \frac{\log|\theta_{\omega,0}(n) - \theta_{\omega,\pi/2}(n)|}{\log n} = -\Lambda \tag{2.50}$$
(recall $\tilde{a}(n) = \lambda_1 n^{\eta_1}$).

Let $\rho_\omega(n) = \frac{R_{\omega,0}(n)}{R_{\omega,\pi/2}(n)}$. Then
$$L_\omega(n) \equiv \log\rho_\omega(n+1) - \log\rho_\omega(n) = \log(1+\mathcal{X}_{\omega,0}(n)) - \log(1+\mathcal{X}_{\omega,\pi/2}(n)) \tag{2.51}$$

where

$$\mathcal{X}_{\omega,\phi}(n) = \left(\| \mathcal{Z}_{\omega,\phi}(n+1) + \mathcal{W}_{\omega,\phi}(n+1) \|^2 - 1 \right). \tag{2.52}$$

Since $\mathcal{X}_{\omega,\phi}(j) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right)$ almost surely, we may apply a finite Taylor expansion to the above (using (2.36),(2.37) and (2.38)) to see that, with probability one, for large enough n

$$\begin{aligned} L_\omega(n) = {} & \Delta^1_\omega(n) \left(\sin^2(\bar{\theta}_{\omega,0}(n-1)) - \sin^2(\bar{\theta}_{\omega,\pi/2}(n-1)) \right) \\ & + \Delta^2_\omega(n) \left(\sin(2\bar{\theta}_{\omega,0}(n-1)) - \sin(2\bar{\theta}_{\omega,\pi/2}(n-1)) \right) \\ & + \Delta^3_\omega(n) \Big(\sin(\bar{\theta}_{\omega,0}(n-1)) \sin(\theta_{\omega,0}(n-1)) \\ & - \sin(\bar{\theta}_{\omega,\pi/2}(n-1)) \sin(\theta_{\omega,\pi/2}(n-1)) \Big) \\ & + \mathcal{O}\left(n^{-(1+\Lambda-\varepsilon)} \right) \end{aligned}$$

where

$$\left\langle \Delta^1_\omega(n) \right\rangle = \left\langle \Delta^2_\omega(n) \right\rangle = \left\langle \Delta^3_\omega(n) \right\rangle = 0$$

and

$$\left\langle \left(\Delta^i_\omega(n) \right)^2 \right\rangle \leq \frac{C_j}{n}$$

$i = 1, 2, 3$. (The $\mathcal{O}\left(n^{-(1+\Lambda-\varepsilon)}\right)$ for the remainder follows from Lemma 2.1 and (2.50).)

A standard application of Kolmogorov's inequality and the Borel Cantelli Lemma shows that with probability one, for large enough k, and for each $i = 1, 2, 3$

$$\sup_{m=2^{k-1}+1,\ldots,2^k} \left| \sum_{j=2^{k-1}+1}^{m} \Delta^i_\omega(j) \right| \leq k \tag{2.53}$$

and also

$$\sup_{m=2^{k-1}+1,\ldots,2^k-1} \left| \sum_{m}^{2^k} \Delta^i_\omega(j) \right| \leq k. \tag{2.54}$$

Combining this with the fact that, with probability one, for large enough n,

$$\left(\sin^2(\bar{\theta}_{\omega,0}(n-1)) - \sin^2(\bar{\theta}_{\omega,\pi/2}(n-1)) \right) = \mathcal{O}\left(n^{-\Lambda+\varepsilon} \right),$$
$$\left(\sin(2\bar{\theta}_{\omega,0}(n-1)) - \sin(2\bar{\theta}_{\omega,\pi/2}(n-1)) \right) = \mathcal{O}\left(n^{-\Lambda+\varepsilon} \right)$$

and

$$\begin{aligned} & \Big(\sin(\bar{\theta}_{\omega,0}(n-1)) \sin(\theta_{\omega,0}(n-1)) \\ & \quad - \sin(\bar{\theta}_{\omega,\pi/2}(n-1)) \sin(\theta_{\omega,\pi/2}(n-1)) \Big) = \mathcal{O}\left(n^{-\Lambda+\varepsilon} \right), \end{aligned}$$

it follows that

$$\sum_{n=1}^{\infty} L_\omega(n)$$

exists and

$$\left|\sum_{n=N}^{\infty} L_\omega(n)\right| \leq C_\omega N^{-\Lambda+\varepsilon}$$

almost surely.

Thus, with proability one, $\lim_{n\to\infty} \frac{R_{\omega,0}(n)}{R_{\omega,\pi/2}(n)} = \lim_{n\to\infty} \rho_\omega(n) = \rho_\omega(\infty)$ exists and

$$\limsup_{n\to\infty} \frac{\log|\rho_\omega(n) - \rho_\omega(\infty)|}{\log n} \leq -\Lambda.$$

Lemma 2.6 completes the proof (note that $d_n \equiv \det T_\omega^E(n) = \frac{1}{a_\omega(n)}$). □

We are now ready to complete the

Proof of Theorem 1.1. (1) By Theorem 2.2, Fubini's Theorem, the fact that the distribution of $X_\omega(n)$ is absolutely continuous with respect to Lebesgue measure, and the theory of rank-one perturbations ([20]), it follows that, with probability one, the spectral measure is supported on the set of energies where, for any $\varepsilon > 0$ and sufficiently large n, $\| T_\omega^E(n) \|^2 \leq n^{-\eta_1+\varepsilon}$. From Corollary 4.4 of [9] it follows now that the spectral measure is continuous with respect to $(1-\varepsilon)$-dimensional Hausdorff measure, for any $\varepsilon > 0$. Thus the spectral measure is one-dimensional. The fact that the spectrum is $\mathbb{R}$ follows from Theorem A.1.

(2) In this case again, the fact that the spectrum is $\mathbb{R}$ follows from the polynomial bound on the transfer matrices in Theorem 2.2 and Theorem A.1 below. As for the properties of the spectral measure, these follow from Theorem 1.2 in [9] using (2.48), Proposition 2.7 and the theory of rank one perturbations.

(3) The existence, with probability one, of an exponentially decaying eigenfunction, for every $E \in \mathbb{R}$, follows from Theorem 2.2 and Theorem 8.3 of [16]. Fubini and the theory of rank one perturbations imply that the spectral measure is supported, with probability one, on the set where these eigenfunctions exist. Theorem A.1 implies that, as long as $\eta_1 > 2\eta_2$, the spectrum fills $\mathbb{R}$.

□

Appendix A. A Combes-Thomas Estimate for Jacobi Matrices with Unbounded Weights

This section presents a Combes-Thomas estimate, suitable for application to Jacobi matrices with unbounded off-diagonal terms. Also see [18] for a related result.

Theorem A.1. *Let $J = J\left(\{a(n)\}_{n=1}^{\infty}, \{b(n)\}_{n=1}^{\infty}\right)$ be a self-adjoint Jacobi matrix such that $0 < a(n) \leq f(n)$ for a nondecreasing function $f(n)$. Let $z \in \mathbb{C}$ be such that $\mathrm{dist}(z, Spec(J)) = \sigma$ (where $Spec(J)$ is the spectrum of J). Then*

$$\left|\left(\delta_1, (J-z)^{-1}\delta_N\right)\right| \leq \frac{2e}{\sigma} e^{-\alpha_N \cdot N} \tag{A.1}$$

where $\alpha_N = \min\left(1, \frac{\sigma}{4ef(N)}\right)$. In particular

$$\left|\left(\delta_1, (J_{\Upsilon,\omega}-z)^{-1}\delta_N\right)\right| \leq \frac{2e}{\sigma} e^{-C(\sigma,\omega)\cdot N^{1-\eta_1}} \tag{A.2}$$

almost surely.

Remark. The monotonicity of f is not essential. One may instead replace $f(N)$ in the formula for α_N, by $\max(f(1), \ldots, f(N))$.

Proof. Let R_N be the diagonal matrix defined by

$$R_N(n,n) = \begin{cases} e^{\alpha_N \cdot n} & n \leq N \\ e^{\alpha_N \cdot N} & n > N. \end{cases} \tag{A.3}$$

Then

$$e^{\alpha_n \cdot (N-1)} \left(\delta_1, (J-z)^{-1}\delta_N\right) = \left(\delta_1, R_N^{-1}(J-z)^{-1} R_N \delta_N\right)$$

so it suffices to bound $\| \ R_N^{-1}(J-z)^{-1}R_N \ \|$. Noting that $R_N^{-1}(J-z)^{-1}R_N = \left(R_N^{-1}(J-z)R_N\right)^{-1} \equiv C_N(z)$, we may apply the Resolvent identity to get

$$C_N(z) = (J-z)^{-1} + C_N(z)\cdot\left(J - z - R_N^{-1}(J-z)R_N\right)(J-z)^{-1}.$$

A simple computation shows that $\| \ J - z - R_N^{-1}(J-z)R_N \ \| \leq 2ef(N)\alpha_N$ if $\alpha_N \leq 1$. Thus, by $\| \ (J-z)^{-1} \ \| \leq \frac{1}{\sigma}$, we see that

$$\| \ C_N(z) \ \| \leq \frac{1}{\sigma} + C_N(z)\frac{2ef(N)}{\sigma}\alpha_N$$

so, by $\alpha_N \leq \frac{1}{2}\cdot\frac{\sigma}{2ef(N)}$, we see that

$$\| \ C_N(z) \ \| \leq \frac{2}{\sigma}$$

which finishes the proof. □

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References

[1] J. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Transl. Math. Monographs, Vol. **17**, Amer. Math. Soc., Providence, RI, 1968.

[2] J. Breuer, P. Forrester and U. Smilansky, *Random discrete Schrödinger operators from random matrix theory*, J. Phys. A: Math. Theor. **40** (2007), F1–F8.

[3] F. Delyon, *Appearance of a purely singular continuous spectrum in a class of random Schrödinger operators*, J. Stat. Phys. **40** (1985), 621–630.

[4] F. Delyon, B. Simon and B. Souillard, *From Power-Localized to Extended States in a Class of One-Dimensional Disordered Systems*, Phys. Rev. Lett. **52** (1984), 2187–2189.

[5] F. Delyon, B. Simon and B. Souillard, *From power pure point to continuous spectrum in disordered systems*, Ann. Inst. H. Poincaré Phys. Théor. **42** (1985), 283–309.

[6] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), 5830–5847.

[7] D. J. Gilbert and D. B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), 30–56.

[8] J. Janas and S. Naboko, *Spectral analysis of selfadjoint Jacobi matrices with periodically modulated entries*, J. Funct. Anal. **191** (2002), 318–324.

[9] S. Jitomirskaya and Y. Last, *Power-law subordinacy and singular spectra, I. Half-line operators*, Acta Math. **183** (1999), 171–189.

[10] R. Killip and I. Nenciu, *Matrix models for circular ensembles*, Int. Math. Res. Not. **50** (2004), 2665–2701.

[11] R. Killip and M. Stoiciu, *Eigenvalue statistics for CMV matrices: From Poisson to clock via $C\beta E$*, preprint math-ph/0608002.

[12] A. Kiselev and Y. Last, *Solutions, spectrum and dynamics for Schrödinger operators on infinite domains*, Duke Math. J. **102** (2000), 125–150.

[13] A. Kiselev, Y. Last, and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, Commun. Math. Phys. **194** (1998), 1–45.

[14] S. Kotani and N. Ushiroya, *One-dimensional Schrödinger operators with random decaying potentials*, Commun. Math. Phys. **115** (1988), 247–266.

[15] Y. Last, *Quantum dynamics and decompositions of singular continuous spectra*, J. Funct. Anal. **142** (1996), 406–445.

[16] Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, Invent. Math. **135** (1999), 329–367.

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics,* I. *Functional Analysis*, Academic Press, New York, 1972.

[18] J. Sahbani, *Spectral properties of Jacobi matrices of certain birth and death processes*, J. Operator Theory **56** (2006), 377–390.

[19] B. Simon, *Some Jacobi matrices with decaying potential and dense point spectrum*, Commun. Math. Phys. **87** (1982), 253–258.

[20] B. Simon "Spectral analysis of rank one perturbations and applications". In *Proc. Mathematical Quantum Theory, II. Schrödinger Operators,* CRM Proceedings and Lecture Notes 8, edited by J. Feldman, R. Froese and L. Rosen, 109–149. Providence, RI: American Mathematical Society, (1995).

[21] B. Simon, *Orthogonal Polynomials on the Unit Circle, vol. 2.* American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, Rhode Island, 2004.

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel. Email: jbreuer@math.huji.ac.il