Lieb-Thirring inequalities with improved constants

Jean Dolbeault, Ari Laptev, and Michael Loss

Abstract

Following Eden and Foias we obtain a matrix version of a generalised Sobolev inequality in one-dimension. This allow us to improve on the known estimates of best constants in Lieb-Thirring inequalities for the sum of the negative eigenvalues for multi-dimensional Schrödinger operators.

Key-words: Sobolev inequalities; Schrödinger operator; Lieb-Thirring inequalities.
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1 Introduction

Let H be a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - V \tag{1}$$

For a real-valued potential V we consider Lieb-Thirring inequalities for the negative eigenvalues $\{\lambda_n\}$ of the operator H

$$\sum |\lambda_n|^\gamma \le L_{d,\gamma} \int_{\mathbb{R}^d} V_+^{d/2+\gamma}(x)\, dx\,, \tag{2}$$

where $V_+ = (|V| + V)/2$ is the positive part of V.

Eden and Foias have obtained in [3] a version of a one-dimensional generalised Sobolev inequality which gives best known estimates for the constants in the inequality (2) for $1 \le \gamma < 3/2$. The aim of this short article is to extend the method from [3] to a class of matrix-valued potentials. By using ideas from [6] this automatically improves on the known estimates of best constants in (2) for multidimensional Schrödinger operators.

Lieb-Thirring inequalities for matrix-valued potentials for the value $\gamma = 3/2$ were obtained in [6] and also in [2]. Here we state a result corresponding to $\gamma = 1$.

Theorem 1. *Let $V \ge 0$ be a Hermitian $m \times m$ matrix-function defined on $\mathbb{R}$ and let λ_n be all negative eigenvalues of the operator* (1). *Then*

$$\sum |\lambda_n| \le \frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \mathrm{Tr}\left[V^{3/2}(x)\right] dx\,. \tag{3}$$

Remark 1. *The constant* $\frac{2}{3\sqrt{3}}$ *should be compared with the Lieb-Thirring constant found in [7] for a class of single eigenvalue potentials and with the constant obtained in [5] which is twice as large as the semi-classical one*

$$\frac{4}{3\sqrt{3}\,\pi} < \frac{2}{3\sqrt{3}} < 2 \times \frac{2}{3\pi} = 2 \times \frac{1}{2\pi} \int_{\mathbb{R}} (1 - \xi^2)_+ \, d\xi \,.$$

This is about $0,2450\dots < 0,3849\dots < 0,4244\dots$.

Remark 2. *Note that the values of the best constants for the range* $1/2 < \gamma < 3/2$ *remain unknown.*

Let $\mathcal{A}(x) = (a_1(x), \dots, a_d(x))$ be a magnetic vector potential with real valued entries $a_k \in L^2_{\text{loc}}(\mathbb{R}^d)$ and let

$$H(\mathcal{A}) = (i\,\nabla + \mathcal{A})^2 - V \,,$$

where $V \geq 0$ is a real-valued function.

Denote the ratio of $2/3\sqrt{3}$ and the semi-classical constant by

$$R := \frac{2}{3\sqrt{3}} \times \left(\frac{2}{3\pi}\right)^{-1} = 1.8138\dots.$$

By using the Aizenmann-Lieb argument [1], a "lifting" with respect to dimension [6], [5], and Theorem 1 we obtain the following result:

Theorem 2. *The negative eigenvalues of the operator* $H(\mathcal{A})$ *satisfy inequalities*

$$\sum |\lambda_n|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V^{d/2+\gamma}(x)\, dx \,,$$

where

$$L_{d,\gamma} \leq R \times L^{cl}_{d,\gamma} = R \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|)^\gamma_+ \, d\xi \,.$$

Remark 3. *Theorem 2 allows us to improve on the estimates of best constants in Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials recently obtained in [4].*

2 One-dimensional generalised Sobolev inequality for matrices

Let $\{\phi_n\}_{n=0}^N$ be an ortho-normal system of vector-functions in $L^2(\mathbb{R}, \mathbb{C}^M)$, $M \in \mathbb{N}$,

$$(\phi_n, \phi_m) = (\phi_n, \phi_m)_{L^2(\mathbb{R},\mathbb{C}^M)} = \sum_{j=1}^M \int_{\mathbb{R}} \phi_n(x,j)\, \overline{\phi_m(x,j)}\, dx = \delta_{nm} \,,$$

where δ_{nm} is the Kronecker symbol. Let us introduce a $M \times M$ matrix U with entries

$$u_{j,k}(x,y) = \sum_{n=0}^{N} \phi_n(x,j)\,\overline{\phi_n(y,k)}\,.$$

Clearly

$$U^*(x,y) = U(y,x)\,. \tag{4}$$

The fact that the functions ϕ_n are orthonormal can be written in a compact form

$$\int_{\mathbb{R}} U(x,y)\,U(y,z)\,dy = U(x,y)\,. \tag{5}$$

The latter two properties (4) and (5) prove that $U(x,y)$ could be considered as the integral kernel of an orthogonal projection P in $L^2(\mathbb{R},\mathbb{C}^M)$ whose image is the subspace of vector-functions spanned by $\{\phi_n\}_{n=1}^N$.

Theorem 3. *Let us assume that the vector-function ϕ_n, $n = 1,2,\dots N$, are from the Sobolev class $H^1(\mathbb{R},\mathbb{C}^M)$. Then*

$$\int_{\mathbb{R}} \mathrm{Tr}\left[U(x,x)^3\right] dx \le \sum_{n=1}^{N}\sum_{j=1}^{M} \int_{\mathbb{R}} |\phi_n'(x,j)|^2\,dx\,.$$

Proof.

$$\begin{aligned} &\frac{d}{dy}\,\mathrm{Tr}\left[U(x,y)\,U(y,x)\,U(x,x)\right] \\ &= \mathrm{Tr}\left[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\right] + \mathrm{Tr}\left[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\right] \end{aligned} \tag{6}$$

By integrating (6) and taking absolute values one obtains

$$\begin{aligned} &\frac{1}{2}\,\mathrm{Tr}\left[U(x,z)\,U(z,x)\,U(x,x)\right] \\ &\qquad \le \frac{1}{2}\int_{-\infty}^{z}\Big|\,\mathrm{Tr}\left[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\right] \\ &\qquad\qquad + \mathrm{Tr}\left[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\right]\Big|\,dy \end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}\left[U(x,z)\,U(z,x)\,U(x,x)\right]\\
&\qquad\qquad \le \frac{1}{2}\int_z^{\infty}\left|\,\mathrm{Tr}\left[\left(\frac{d}{dy}\,U(x,y)\right)U(y,x)\,U(x,x)\right]\right.\\
&\qquad\qquad\qquad\qquad \left. +\,\mathrm{Tr}\left[U(x,y)\left(\frac{d}{dy}\,U(y,x)\right)U(x,x)\right]\right| dy\,.
\end{aligned}
$$

Taking absolute values and adding the two inequalities yields for any $z\in\mathbb{R}$

$$
\begin{aligned}
&\left|\,\mathrm{Tr}\left[U(x,z)\,U(z,x)\,U(x,x)\right]\right|\\
&\qquad\qquad \le \frac{1}{2}\int_{\mathbb{R}}\left|\,\mathrm{Tr}\left[\left(\frac{d}{dy}\,U(x,y)\right)U(y,x)\,U(x,x)\right]\right| dy\\
&\qquad\qquad\qquad +\frac{1}{2}\int_{\mathbb{R}}\left|\,\mathrm{Tr}\left[U(x,y)\left(\frac{d}{dy}\,U(y,x)\right)U(x,x)\right]\right| dy\,. \qquad (7)
\end{aligned}
$$

Note that we have reproved Agmon's inequality

$$
|f(x)|^2 \le \int_{\mathbb{R}} |f(y)\,f'(y)|\,dy
$$

for traces of matrices. By using properties of traces, the Cauchy-Schwarz inequality for matrix-functions and also properties (4) and (5), we find that for all $z\in\mathbb{R}$

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\left|\,\mathrm{Tr}\left[\left(\frac{d}{dy}\,U(x,y)\right)\,U(y,x)\,U(x,x)\right]\right|\,dy\right)^2\\
&\le \int_{\mathbb{R}}\mathrm{Tr}\left[\frac{d}{dy}\,U(x,y)^*\,\frac{d}{dy}\,U(x,y)\right]dy\,\int_{\mathbb{R}}\mathrm{Tr}\,\left[U(x,y)^*\,U^2(x,x)\,U(x,y)\right]dy\\
&= \int_{\mathbb{R}}\mathrm{Tr}\left[\frac{d}{dy}\,U(y,x)\,\frac{d}{dy}\,U(x,y)\right]dy\,\int_{\mathbb{R}}\mathrm{Tr}\,\left[U^2(x,x)\,U(x,y)\,U(y,x)\right]dy\\
&\qquad\qquad\qquad = \int_{\mathbb{R}}\mathrm{Tr}\left[\frac{d}{dy}\,U(x,y)\,\frac{d}{dy}\,U(y,x)\right]dy\;\;\mathrm{Tr}\,\left[U(x,x)^3\right],
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\left|\,\mathrm{Tr}\left[U(x,y)\,\frac{d}{dy}\,U(y,x)\,U(x,x)\right]\right|\,dy\right)^2\\
&\qquad\qquad\qquad \le \int_{\mathbb{R}}\mathrm{Tr}\left[\frac{d}{dy}\,U(x,y)\,\frac{d}{dy}\,U(y,x)\right]dy\;\;\mathrm{Tr}\,\left[U(x,x)^3\right]\,.
\end{aligned}
$$

Thus, using this, and setting $x = z$ in (7), we arrive at

$$\Big| \operatorname{Tr} \Big[U(x,x)^3 \Big] \Big| \le \int_{\mathbb{R}} \operatorname{Tr} \left[\frac{d}{dy} U(x,y) \frac{d}{dy} U(y,x) \right] dy\,.$$

Integrating with respect to x we finally obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \Big| \operatorname{Tr} \Big[U(x,x)^3 \Big] \Big| \, dx \\
&\qquad \le \sum_{n,k=1}^{N} \sum_{i,j=1}^{M} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_n(x,i)\, \overline{\phi_n'(y,j)}\; \phi_k'(y,j)\, \overline{\phi_k(x,i)}\, dx\, dy \\
&\qquad\qquad = \sum_{n=1}^{N} \sum_{j=1}^{M} \int_{\mathbb{R}} |\phi_n'(x,j)|^2 \, dx\,,
\end{aligned}$$

which completes the proof. $\square$

3 Lieb-Thirring inequalities for Schrödinger operators with matrix-valued potentials

Let us assume that $V \in C_0^\infty(\mathbb{R})$, $V \ge 0$, be a $M \times M$ Hermitian matrix-valued potential with entries $\{v_{ij}\}_{i,j=1}^{M}$. Then the negative spectrum of the Schrödinger operator $H = -\frac{d^2}{dx^2} - V$ in $L^2(\mathbb{R})$ is finite.

Denote by $\{\phi_n\}$ an ortho-normal system eigen-vector functions corresponding to the eigenvalues $\{\lambda_n\}_{n=1}^{N}$

$$-\frac{d^2}{dx^2}\, \phi_n - V \phi_n = -\lambda_n\, \phi_n\,.$$

Clearly,

$$-\sum_n \lambda_n = \sum_{n,j} \int_{\mathbb{R}} |\phi_n'(x,j)|^2 \, dx - \operatorname{Tr} \left[\int_{\mathbb{R}} V(x)\, U(x,x)\, dx \right]$$

and by Hölder's inequality for traces,

$$\int_{\mathbb{R}} \operatorname{Tr} \left[V(x)\, U(x,x) \right] dx \le \left(\int_{\mathbb{R}} \operatorname{Tr} \left[V^{3/2}(x) \right] dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}} \operatorname{Tr} \left[U(x,x)^3 \right] dx \right)^{\frac{1}{3}},$$

so that using Theorem 3

$$-\sum_n \lambda_n \ge X - \left(\int_{\mathbb{R}} \operatorname{Tr} \left[V^{3/2}(x) \right] dx \right)^{\frac{2}{3}} X^{\frac{1}{3}}$$

with $X := \int_{\mathbb{R}} \mathrm{Tr}\, \left[U(x,x)^3\right] dx$. Minimising the right hand side with respect to X we finally complete the proof of Theorem 1

$$-\sum_n \lambda_n \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \mathrm{Tr}\, \left[V^{3/2}(x)\right] \, dx \, .$$

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J. Dolbeault: Ceremade UMR CNRS no. 7534, Université Paris Dauphine, F-75775 Paris Cedex 16, France. *E-mail:* `dolbeaul@ceremade.dauphine.fr`

A. Laptev: Department of Mathematics, Imperial College London, London SW7 2AZ, UK, Royal Institute of Technology, 100 44 Stockholm, Sweden. *E-mail:* `a.laptev@imperial.ac.uk`

M. Loss: School of Mathematics, Georgia Institute of Technology, Atlanta GA 30332-0160, USA. *E-mail:* `loss@math.gatech.edu`

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