Representations of the Quantum Plane and the Quantum Algebra $U_q(\mathrm{sl}_2)$ on $L^2(\mathbb{R}^d)$

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Abstract

A class of representations on the Hilbert space $L^2(\mathbb{R}^d)$ $(d \geq 2)$ of the quantum plane $\mathbb{C}_q^2$ and the quantum algebra $U_q(\mathrm{sl}_2)$ is presented. The boundedness and the unboundedness of the representations are discussed. A physically interesting example of the representations is shown to appear in a two-dimensional quantum system with a magnetic field concentrated on an infinite lattice.

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1 Introduction

In the previous paper [2] (cf. also [1]), the author considered a quantum system in the plane $\mathbb{R}^2$ which is under the influence of a perpendicular magnetic field concentrated on an infinite discrete set $\mathbf{D}$ of $\mathbb{R}^2$ and showed that, in this quantum system, there appear, in a natural way, representations of the canonical commutation relations (CCR) with two degrees of freedom. Interestingly enough, these representations are not necessarily unitarily equivalent to the Schrödinger representation of the CCR with two degrees of freedom and each inequivalent representation physically corresponds to the occurrence of the Aharonov-Bohm effect in the context of the quantum system under consideration. Moreover, in connection with the inequivalent representations with $\mathbf{D}$ being an infinite lattice, bounded operator representations of the quantum plane $\mathbb{C}_q^2$ and the quantum algebra $U_q(\mathrm{sl}_2)$ with $|q| = 1$ $(q \in \mathbb{C})$ were constructed as well as their reductions to lattice quantum systems.

From a mathematical point of view, it would be natural to ask if there is any general structure behind the representations of $\mathbb{C}_q^2$ $(|q| = 1)$ constructed in [2]. This is one of the

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motivations for the present work. In this paper, we show that the answer is affirmative, presenting a general theory of representations of $\mathbb{C}_q^2$ on the Hilbert space $L^2(\mathbb{R}^d)$ $(d \geq 2)$ with $|q|$ not necessarily equal to one. This general theory, of course, includes, as examples, the previously obtained representations of $\mathbb{C}_q^2$ mentioned above. A new feature of the representations of $\mathbb{C}_q^2$ given in this paper is in that they may be unbounded.

The present paper is organized as follows. Section 2 is a preliminary, where we define a class of closed linear operators on $L^2(\mathbb{R}^d)$ constructed from the Schrödinger representation of the CCR with d degrees of freedom. In Section 3 we present a general scheme for the construction of a class of representations of $\mathbb{C}_q^2$ on $L^2(\mathbb{R}^d)$. Here we see that, to obtain an unbounded operator representation, the constituent operators of the representation are required to have a common invariant domain. A possible form of such an invariant domain is given in Section 4 in the case $d = 2$. In the paper [1], we showed a scheme to obtain representations of $U_q(\mathrm{sl}_2)$ from a representation of $\mathbb{C}_q^2$. Using this scheme and the results in Section 3, we present in Section 5 a class of representations of $U_q(\mathrm{sl}_2)$ on $L^2(\mathbb{R}^d)$. In the last section, we consider a two dimensional quantum system with a singular gauge potential determined by a meromorphic function of which poles are on an infinite lattice and show that representations of $\mathbb{C}_q^2$ (resp. $U_q(\mathrm{sl}_2)$) of the type discussed in Section 3 (resp. Section 5) are realized in this system.

2 Preliminary

For a linear operator A on a Hilbert space, we denote its domain by $D(A)$. Let $d \geq 2$ be a natural number, $\mathbb{R}^d = \{\mathbf{x} = (x_1, \cdots, x_d) | x_j \in \mathbb{R}, j = 1, \cdots, d\}$ and D_j be the generalized partial differential operator in x_j on $L^2(\mathbb{R}^d)$. Then, as is well known, the operator

$$p_j := -iD_j \tag{2.1}$$

(i is the imaginary unit) is self-adjoint. We set

$$\mathbf{p} := (p_1, \cdots, p_d). \tag{2.2}$$

It follows that, for each vector $\mathbf{a} = (a_1, \cdots, a_d) \in \mathbb{R}^d$, the operator

$$\mathbf{a} \cdot \mathbf{p} := \sum_{j=1}^{d} a_j p_j \tag{2.3}$$

with $D(\mathbf{a} \cdot \mathbf{p}) = \cap_{j=1}^d D(p_j)$ is essentially self-adjoint and

$$V(\mathbf{a}) := e^{i\overline{\mathbf{a}\cdot\mathbf{p}}} = e^{ia_1p_1} \cdots e^{ia_dp_d} = e^{ia_dp_d} \cdots e^{ia_1p_1}, \tag{2.4}$$

where $\overline{\mathbf{a} \cdot \mathbf{p}}$ denotes the closure of $\mathbf{a} \cdot \mathbf{p}$.

For a Borel measurable function F on $\mathbb{R}^d$ almost everywhere (a.e.) finite with respect to the Lebesgue measure on $\mathbb{R}^d$, we denote the multiplication operator by the function F on $L^2(\mathbb{R}^d)$ by the same symbol F. For a vector $\mathbf{a} \in \mathbb{R}^d$, we define a function $F_{\mathbf{a}}$ on $\mathbb{R}^d$ by

$$F_{\mathbf{a}}(\mathbf{x}) := F(\mathbf{x} + \mathbf{a}), \quad \mathbf{x} \in \mathbb{R}^d. \tag{2.5}$$

The following lemma is well-known (or easily proved).

Lemma 2.1 *For all* $\mathbf{a} \in \mathbb{R}^d$ *and each Borel measurable function* F *on* $\mathbb{R}^d$*, the operator equality*

$$V(\mathbf{a})FV(\mathbf{a})^* = F_{\mathbf{a}} \tag{2.6}$$

holds, where $V(\mathbf{a})^*$ *denotes the adjoint of* $V(\mathbf{a})$*.*

Let

$$T(\mathbf{a}) := FV(\mathbf{a}). \tag{2.7}$$

Since the function F is a.e. finite with respect to the Lebesgue measure on $\mathbb{R}^d$, the multiplication operator F is a densely defined closed linear operator on $L^2(\mathbb{R}^d)$. It follows that the same holds for $T(\mathbf{a})$. But $T(\mathbf{a})$ may be unbounded. We first give a characterization for the boundedness of $T(\mathbf{a})$.

Proposition 2.2 *The operator* $T(\mathbf{a})$ *is bounded if and only if the function* F *is essentially bounded on* $\mathbb{R}^d$*. In that case,* $D(T(\mathbf{a})) = D(F) = L^2(\mathbb{R}^d)$*.*

Proof. Suppose that $T(\mathbf{a})$ is bounded. Then it follows from the closedness of $T(\mathbf{a})$ and the denseness of $D(T(\mathbf{a}))$, as mentioned above, that $D(T(\mathbf{a})) = L^2(\mathbb{R}^d)$. Moreover there exists a constant $C > 0$ such that, for all $\psi \in L^2(\mathbb{R}^d)$, $\|FV(\mathbf{a})\psi\| \le C\|\psi\|$, where $\|\cdot\|$ denotes the norm of $L^2(\mathbb{R}^d)$. Since $V(\mathbf{a})$ is unitary, it follows that $D(F) = L^2(\mathbb{R}^d)$ and F is bounded. Hence the function F is essentially bounded on $\mathbb{R}^d$. Proving the converse statement is easy. □

The next proposition is concerned with unitarity of $T(\mathbf{a})$.

Proposition 2.3 *The operator* $T(\mathbf{a})$ *is unitary if and only if* $|F(\mathbf{x})| = 1, \text{a.e.}\mathbf{x} \in \mathbb{R}^d$*.*

Proof. Suppose that $T(\mathbf{a})$ is unitary. Then, $F = T(\mathbf{a})V(\mathbf{a})^{-1}$ is unitary. Hence $|F|$ is a unitary self-adjoint multiplication operator. Therefore the spectrum of $|F|$ is $\{1\}$. This implies that $|F(\mathbf{x})| = 1$, a.e.$\mathbf{x} \in \mathbb{R}^d$.

Conversely, suppose that $|F(\mathbf{x})| = 1$, a.e.$\mathbf{x} \in \mathbb{R}^d$. Then the multiplication operator F is unitary. The operator $V(\mathbf{a})$ is also unitary. Hence $T(\mathbf{a})$ is unitary. □

We also note the following fact:

Proposition 2.4 *Let*

$$N_F := \{\mathbf{x} \in \mathbb{R}^d | F(\mathbf{x}) = 0\}. \tag{2.8}$$

Then $T(\mathbf{a})$ *is injective if and only if the Lebesgue measure* $|N_F|$ *of* N_F *is equal to zero. In that case, the multiplication operator* F *is injective and*

$$T(\mathbf{a})^{-1} = V(-\mathbf{a})F^{-1} \tag{2.9}$$

on $D(F^{-1})$*.*

Proof. Suppose that $T(\mathbf{a})$ is injective. Let $\psi \in \ker F$ so that $F\psi = 0$. Hence, putting $\phi := V(-\mathbf{a})\psi$, we have $\phi \in D(T(\mathbf{a}))$ and $T(\mathbf{a})\phi = 0$. Hence $\phi = 0$. This implies that $\ker F = \{0\}$. Hence F is injective. On the other hand, it is easy to see that

$$\ker F = \{\psi \in L^2(\mathbb{R}^d) | \{\mathbf{x} \in K_\psi^c | \psi(\mathbf{x}) \neq 0\} \subset N_F\}, \tag{2.10}$$

where K_ψ is a null set of $\mathbb{R}^d$ depending on ψ. Hence $|N_F|$ must be zero.

Conversely, suppose that $|N_F| = 0$. Then, by (2.10), $\ker F = \{0\}$. Hence F is injective. Let $\psi \in \ker T(\mathbf{a})$ so that $T(\mathbf{a})\psi = 0$. Then $V(\mathbf{a})\psi \in \ker F$. Hence $V(\mathbf{a})\psi = 0$, implying $\psi = 0$.

If F is injective, then (2.9) easily follows from (2.7). □

For each vector $\mathbf{b} \in \mathbb{R}^d$, we define a unitary operator

$$U(\mathbf{b}) := e^{i\mathbf{b}\cdot\mathbf{x}} \tag{2.11}$$

(the multiplication operator by the function : $\mathbf{x} \mapsto e^{i\mathbf{b}\cdot\mathbf{x}}$) on $L^2(\mathbb{R}^d)$. It is easy to see that

$$V(\mathbf{a})U(\mathbf{b}) = e^{i\mathbf{a}\cdot\mathbf{b}}U(\mathbf{b})V(\mathbf{a}), \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^d. \tag{2.12}$$

Therefore, $V(\mathbf{a})$ and $U(\mathbf{b})$ commute if and only if $\mathbf{a} \cdot \mathbf{b} = 2\pi n$ with some $n \in \mathbb{Z}$ (the set of integers).

Proposition 2.5 *For all* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, $U(\mathbf{b})D(T(\mathbf{a})) = D(T(\mathbf{a}))$ *and the operator equality*

$$U(\mathbf{b})T(\mathbf{a}) = e^{-i\mathbf{a}\cdot\mathbf{b}}T(\mathbf{a})U(\mathbf{b}) \tag{2.13}$$

holds. In particular, if $\mathbf{a} \cdot \mathbf{b} \in 2\pi\mathbb{Z} := \{2\pi n | n \in \mathbb{Z}\}$. *then*

$$U(\mathbf{b})T(\mathbf{a}) = T(\mathbf{a})U(\mathbf{b}).$$

Proof. By definition, $\psi \in L^2(\mathbb{R}^d)$ is in $D(T(\mathbf{a}))$ if and only if

$$\int_{\mathbb{R}^d} |F(\mathbf{x})|^2 |(V(\mathbf{a})\psi)(\mathbf{x})|^2 d\mathbf{x} < \infty.$$

Let $\psi \in D(T(\mathbf{a}))$ and $\phi = U(\mathbf{b})\psi$. Then, by (2.12), we have

$$V(\mathbf{a})\phi = e^{i\mathbf{a}\cdot\mathbf{b}}U(\mathbf{b})V(\mathbf{a})\psi.$$

Hence

$$\int_{\mathbb{R}^d} |F(\mathbf{x})|^2 |(V(\mathbf{a})\phi)(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |F(\mathbf{x})|^2 |(V(\mathbf{a})\psi)(\mathbf{x})|^2 d\mathbf{x} < \infty.$$

Therefore $\phi \in D(T(\mathbf{a}))$. Thus $U(\mathbf{b})D(T(\mathbf{a})) \subset D(T(\mathbf{a}))$. This implies that

$$D(T(\mathbf{a})) \subset U(-\mathbf{b})D(T(\mathbf{a})) \subset D(T(\mathbf{a})).$$

Hence it follows that $U(\mathbf{b})D(T(\mathbf{a})) = D(T(\mathbf{a}))$. By this result, we have $D(U(\mathbf{b})T(\mathbf{a})) = D(e^{-\mathbf{a}\cdot\mathbf{b}}T(\mathbf{a})U(\mathbf{b}))$. It is straightforward to see that

$$U(\mathbf{b})T(\mathbf{a})\psi = e^{-I\mathbf{a}\cdot\mathbf{b}}T(\mathbf{a})U(\mathbf{b})\psi, \quad \psi \in D(T(\mathbf{a})).$$

□

Proposition 2.6 *For all* $\mathbf{a} \in \mathbb{R}^d$,

$$T(\mathbf{a})^* = (F^*)_{-\mathbf{a}} V(-\mathbf{a}).$$

In particular, $D(F) \subset D(T(\mathbf{a})^*)$.

Proof. By Lemma 2.1, we have $T(\mathbf{a}) = V(\mathbf{a}) F_{-\mathbf{a}}$. Since $V(\mathbf{a})$ is bounded, it follows that

$$T(\mathbf{a})^* = F^*_{-\mathbf{a}} V(\mathbf{a})^* = (F^*)_{-\mathbf{a}} V(-\mathbf{a}).$$

Let $\psi \in D(F)$. Then

$$\int_{\mathbb{R}^d} |(F^*)_{-\mathbf{a}}(\mathbf{x})|^2 |(V(-\mathbf{a})\psi)(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |F(\mathbf{x})|^2 |\psi(\mathbf{x})|^2 d\mathbf{x} < \infty.$$

Hence $\psi \in D(T(\mathbf{a})^*)$. □

3 A Class of Representations of the Quantum Plane on $L^2(\mathbb{R}^d)$

Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be linearly independent vectors in $\mathbb{R}^d$. Suppose that there exist non-zero Borel measurable functions $F_j : \mathbb{R}^d \to \mathbb{C}$ a.e. finite with respect to the Lebesgue measure on $\mathbb{R}^d$ and a constant $q \in \mathbb{C} \setminus \{0\}$ satisfying

$$F_1(\mathbf{x}) F_2(\mathbf{x} + \mathbf{w}_1) = q F_1(\mathbf{x} + \mathbf{w}_2) F_2(\mathbf{x}), \quad \text{a.e.} \mathbf{x} \in \mathbb{R}^d. \tag{3.14}$$

A simple example for such F_1 and F_2 is given as follows:

Example 3.1 Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \alpha, \beta \in \mathbb{C}$ and

$$q := e^{\beta \mathbf{w}_1 \cdot \mathbf{v} - \alpha \mathbf{w}_2 \cdot \mathbf{u}}.$$

Let

$$F_1(\mathbf{x}) := e^{\alpha \mathbf{x} \cdot \mathbf{u}}, \quad F_2(\mathbf{x}) := e^{\beta \mathbf{x} \cdot \mathbf{v}}.$$

Then it is easily checked that these F_1 and F_2 satisfy (3.14).

We define

$$T_j := F_j V(\mathbf{w}_j), \quad j = 1, 2. \tag{3.15}$$

A simple application of Proposition 2.2 gives the following fact:

Proposition 3.2 *For each* $j = 1, 2$, T_j *is bounded if and only if the function* F_j *is essentially bounded on* $\mathbb{R}^d$. *In that case,* $D(T_j) = D(F_j) = L^2(\mathbb{R}^d)$.

Proposition 2.4 yields the following result:

Proposition 3.3 *For each* $j = 1, 2$, T_j *is injective if and only if* $|N_{F_j}| = 0$. *In that case,* F_j *is injective and* $T_j^{-1} = V(-\mathbf{w}_j) F_j^{-1}$ *on* $D(F_j^{-1})$.

One of the main results of this section is the following theorem:

Theorem 3.4 *For all* $\psi \in D(T_1^*) \cap D(T_2^*)$ *and* $\phi \in D(T_1) \cap D(T_2)$,

$$\langle T_1^* \psi, T_2 \phi \rangle = q \langle T_2^* \psi, T_1 \phi \rangle , \tag{3.16}$$

where $\langle \cdot , \cdot \rangle$ *denotes the inner product of* $L^2(\mathbb{R}^d)$.

Proof. By Proposition 2.6, we have

$$\langle T_1^* \psi, T_2 \phi \rangle = \int_{\mathbb{R}^d} F_1(\mathbf{x}-\mathbf{w}_1) F_2(\mathbf{x}) \psi(\mathbf{x}-\mathbf{w}_1)^* \phi(\mathbf{x}+\mathbf{w}_2) d\mathbf{x},$$
$$\langle T_2^* \psi, T_1 \phi \rangle = \int_{\mathbb{R}^d} F_2(\mathbf{x}-\mathbf{w}_2) F_1(\mathbf{x}) \psi(\mathbf{x}-\mathbf{w}_2)^* \phi(\mathbf{x}+\mathbf{w}_1) d\mathbf{x}.$$

By change of variables and condition (3.14), one sees that (3.16) holds. □

Theorem 3.4 and Proposition 2.2 imply the following result:

Corollary 3.5 *Suppose that* F_1 *and* F_2 *are essentially bounded on* $\mathbb{R}^d$. *Then* T_1 *and* T_2 *are bounded with* $D(T_j) = L^2(\mathbb{R}^d)$ $(j = 1, 2)$ *and*

$$T_1 T_2 = q T_2 T_1. \tag{3.17}$$

Thus, in the case where F_1 and F_2 are essentially bounded, the set $\{T_1, T_2\}$ of operators yields a bounded representation of the quantum plane $\mathbb{C}_q^2$ with $q \neq 0, \pm 1$ [4].

In the case where at least one of F_1 and F_2 is not essentially bounded on $\mathbb{R}^d$, however, we need some condition for $\{T_1, T_2\}$ to give a representation of $\mathbb{C}_q^2$:

Corollary 3.6 *Suppose that there exists a dense subspace* D *in* $L^2(\mathbb{R}^d)$ *such that, for* $j = 1, 2$, $V(\mathbf{w}_j)D \subset D$ *and* $F_j D \subset D$. *Then* $T_j D \subset D$ $(j = 1, 2)$ *and*

$$T_1 T_2 = q T_2 T_1 \tag{3.18}$$

on D.

Proof. The property $T_j D \subset D$ follows from the present assumption and the definition of T_j. By Proposition 2.6, $D \subset D(T_1^*) \cap D(T_2^*)$. Hence, by Theorem 3.4, we have that, for all $\psi, \phi \in D$, $\langle \psi, T_1 T_2 \phi \rangle = \langle \psi, q T_2 T_1 \phi \rangle$. Since D is dense, we obtain (3.18) on D. □

Suppose that the assumption of Corollary 3.6 holds. Then we can define the algebra generated by $T_1|D$ (the restriction of T_1 to D) and $T_2|D$. We denote it $\mathcal{O}_{\mathbf{w}_1,\mathbf{w}_2}(D)$. We define

$$\mathcal{O}_{\mathbf{w}_1,\mathbf{w}_2}(D)' := \{B \in \mathfrak{B}(L^2(\mathbb{R}^d)) |\, \langle B^* \phi, T_j \psi \rangle = \left\langle T_j^* \phi, B \psi \right\rangle , \phi, \psi \in D, j = 1, 2\}, \tag{3.19}$$

where $\mathfrak{B}(L^2(\mathbb{R}^d))$ denotes the Banach space of all bounded linear operators B on $L^2(\mathbb{R}^d)$ with $D(B) = L^2(\mathbb{R}^d)$.

We can apply Proposition 2.5 to obtain the following fact:

Proposition 3.7 *Let* $\mathbf{a} \in \mathbb{R}^d$ *be such that* $\mathbf{a} \cdot \mathbf{w}_j \in 2\pi\mathbb{Z}, j = 1, 2$ *and* $U(\mathbf{a})D \subset D$. *Then* $U(\mathbf{a}) \in \mathcal{O}_{\mathbf{w}_1,\mathbf{w}_2}(D)'$.

4 Invariant Domains in the Case $d = 2$

We consider the case $d = 2$ and suppose that $\mathbf{w}_1 \times \mathbf{w}_2 > 0$. In this case, a possible invariant domain for T_1 and T_2 can be constructed as follows. We first introduce a subset $\mathbb{L}_j$ of $\mathbb{R}^d$ $(j = 1, 2)$ by

$$\mathbb{L}_1 := \cup_{n\in\mathbb{Z}}\{t\mathbf{w}_1 + n\mathbf{w}_2 | t \in \mathbb{R}\}, \quad \mathbb{L}_2 := \cup_{m\in\mathbb{Z}}\{m\mathbf{w}_1 + t\mathbf{w}_2 | t \in \mathbb{R}\}. \tag{4.20}$$

Note that

$$\mathbb{L}_1 \cap \mathbb{L}_2 = \mathbb{L}_{\mathbf{w}_1,\mathbf{w}_2} := \{m\mathbf{w}_1 + n\mathbf{w}_2 | m, n \in \mathbb{Z}\}, \tag{4.21}$$

a two-dimensional lattice. The set

$$\Omega := \mathbb{L}_1^c \cap \mathbb{L}_2^c = (\mathbb{L}_1 \cup \mathbb{L}_2)^c \tag{4.22}$$

is an open set in $\mathbb{R}^d$. It is easy to see that

$$\Omega \pm \mathbf{w}_j = \Omega, \quad j = 1, 2. \tag{4.23}$$

Hence Ω has the translation symmetry with vectors $\pm\mathbf{w}_j$ $(j = 1, 2)$.

We denote by $C_0^\infty(\Omega)$ the set of infinitely differentiable functions on $\mathbb{R}^d$ with compact support in Ω.

Proposition 4.1 *Suppose that each F_j $(j = 1, 2)$ is infinitely differentiable on Ω. Then $T_j C_0^\infty(\Omega) \subset C_0^\infty(\Omega)$ and (3.18) holds on $C_0^\infty(\Omega)$.*

Proof. Let $\psi \in C_0^\infty(\Omega)$. We have

$$(V(\mathbf{w}_j)\psi)(\mathbf{x}) = \psi(\mathbf{x} + \mathbf{w}_j), \quad \mathbf{x} \in \mathbb{R}^d. \tag{4.24}$$

This relation and (4.23) imply that $V(\mathbf{w}_j)\psi \in C_0^\infty(\Omega)$. It is obvious from the present assumption for F_j that $F_j C_0^\infty(\Omega) \subset C_0^\infty(\Omega)$. Thus the desired result follows. $\square$

5 A Class of Representations of $U_q(\mathrm{sl}_2)$ on $L^2(\mathbb{R}^d)$

For a complex number $q \in \mathbb{C} \setminus \{0, -1, 1\}$, the quantum algebra $U_q(\mathrm{sl}_2)$ is defined to be the complex associative algebra with unit 1 generated by four elements E, F, K, K^{-1} subject to the following relations [4]:

$$KK^{-1} = K^{-1}K = 1, \tag{5.25}$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{5.26}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \tag{5.27}$$

where $[X, Y] := XY - YX$. As is shown in Lemma 4.1 and Lemma 5.1 in [1], there is a general scheme to construct a representation of $U_q(\mathrm{sl}_2)$ from a representation of the quantum plane $\mathbb{C}_q^2$.

Let T_1 and T_2 be as in Section 3.

Theorem 5.1 *Assume that $|N_{F_j}| = 0$, $j = 1, 2$. Suppose that there exists a dense subspace D in $L^2(\mathbb{R}^d)$ such that, for $j = 1, 2$, $V(\mathbf{w}_j)D \subset D$ and $F_jD \subset D, F_j^{-1}D \subset D$. Let $a, b, a', b' \in \mathbb{C}$ be constants satisfying*

$$abq = \frac{a'b'}{q} = -\frac{1}{(q - q^{-1})^2}. \tag{5.28}$$

and

$$E := T_2(aT_2 + a'T_2^{-1})T_1^{-1}, \tag{5.29}$$
$$F := T_1(bT_2 + b'T_2^{-1})T_2^{-1}, \tag{5.30}$$
$$K := T_2^2, \quad K^{-1} := (T_2^{-1})^2. \tag{5.31}$$

Then the set $\{E, F, K, K^{-1}\}$ generates a representation of $U_q(\mathrm{sl}_2)$ on the vector space D.

Proof. We need only to apply Lemma 4.1 in [1] to the representation $\{T_1, T_2\}$ of $\mathbb{C}_q^2$. □

We denote by $\Pi_{a,b;a',b'}$ the representation of $U_q(\mathrm{sl}_2)$ given in Theorem 5.1.

Proposition 5.2 *There is no finite dimensional subspace M of D which reduces the representation $\Pi_{a,b;a',b'}$ and in which the highest weight of $\Pi_{a,b;a',b'}|M$ is not equal to $-a'/aq^2$.*

Proof. Suppose that there is a finite dimensional subspace M of D which reduces the representation $\Pi_{a,b;a',b'}$. By a general fact on finite dimensional representations of $U_q(\mathrm{sl}_2)$ ([3, Proposition VI.3.3]), The representation $\Pi_{a,b;a',b'}|M$ has a highest weight vector $\psi \in M \setminus \{0\}$: $E\psi = 0$ and $K\psi = \lambda\psi$ with highest weight $\lambda \in \mathbb{C} \setminus \{0\}$. Then

$$T_2(aT_2 + a'T_2^{-1})T_1^{-1}\psi = 0, \tag{5.32}$$
$$F_2(\mathbf{x})F_2(\mathbf{x} + \mathbf{w}_2)\psi(\mathbf{x} + 2\mathbf{w}_2) = \lambda\psi(\mathbf{x}), \quad \text{a.e.}\mathbf{x} \in \mathbb{R}^d. \tag{5.33}$$

By (5.32) and the easily derived relation

$$T_2T_1^{-1} = qT_1^{-1}T_2,$$

we obtain $(aq^2\lambda + a')T_1^{-1}\psi = 0$. Hence $aq^2\lambda + a' \neq 0$. Therefore $\lambda = -a'/aq^2$. This implies the desired assertion. □

Theorem 5.3 *Suppose that the following (i) or (ii) hold:*

(i) For a constant $c > 0$, $|F_2(\mathbf{x})| = c$, a.e.$\mathbf{x} \in \mathbb{R}^d$.

(ii) $|F_2(\mathbf{x})|^2 \geq |a'|/|a||q|^2$, a.e.$\mathbf{x} \in \mathbb{R}^d$.

Then there is no finite dimentional subspace M of D which reduces the representation $\Pi_{a,b;a',b'}$.

Proof. By a general fact on finite dimensional representations of $U_q(\mathrm{sl}_2)$ ([3, Proposition VI.3.3]), we need only to show that the representation $\Pi_{a,b;a',b'}$ has no highest weight vector. Let ψ be a vector in D such that $E\psi = 0$ and $K\psi = \lambda\psi$ with $\lambda \in \mathbb{C} \setminus \{0\}$. Then (5.32) and (5.33) hold. By Proposition 5.2, we need only to consider the case $\lambda = -a'/aq^2 \neq 0$. Then, by (5.33), we have $|F_2(\mathbf{x})||F_2(\mathbf{x}+\mathbf{w}_2)||\psi(\mathbf{x}+2\mathbf{w}_2)| = |a'||\psi(\mathbf{x})/|a||q|^2$.

Let condition (i) be satisfied. Then we have $k|\psi(\mathbf{x} + 2\mathbf{w}_2)| = |\psi(\mathbf{x})|$ with $k = |a||q|^2c^2/|a'|$. Since ψ is in $L^2(\mathbb{R}^d)$, it follows that $k = 1$. Then $|\psi(\mathbf{x} + 2\mathbf{w}_2)| = |\psi(\mathbf{x})|$. But this implies $\int_{\mathbb{R}^d} |\psi(\mathbf{x})|^2dx = \infty$ if $\psi \neq 0$. Thus $\psi = 0$.

We next consider the case where condition (ii) holds. In this case we have $|\psi(\mathbf{x} + 2\mathbf{w}_2)| \leq |\psi(\mathbf{x})$. This inequality and condition $\psi \in L^2(\mathbb{R}^d)$ imply that $\psi = 0$. □

The element C defined by

$$C := \frac{qK - 2 + q^{-1}K^{-1}}{(q - q^{-1})^2} + FE \tag{5.34}$$

is called the Casimir element of $U_q(\mathrm{sl}_2)$ and commutes with E, F, K and K^{-1}.

In the representation $\Pi_{a,b;a',b'}$, we have

$$\Pi_{a,b;a',b'}(C) = a'b + ab' - \frac{2}{(q - q^{-1})^2}, \tag{5.35}$$

which is a scalar.

Theorem 5.4 *Let $\Pi_{a,b;a',b'}$ and $\Pi_{c,d;c',d'}$ be two representations of $U_q(\mathrm{sl}_2)$ of the form given in Theorem 5.1. Suppose that $a'b+ab' \neq c'd+cd'$. Then $\Pi_{a,b;a',b'}$ is not unitarily equivalent to $\Pi_{c,d;c',d'}$.*

Proof. If $\Pi_{a,b;a',b'}$ is unitarily equivalent to $\Pi_{c,d;c',d'}$ with a unitary operator U, then $U\Pi_{a,b;a',b'}(C)U^{-1} = \Pi_{c,d;c',d'}(C)$. Hence, by (5.35), we have $a'b + ab' = c'd + cd'$. But this is a contradiction. □

Proposition 5.5 *Let F_1 and F_2 be essentially bounded and $\mathfrak{A}_q$ be the $*$-algebra generated by E, F, K, K^{-1} in Theorem 5.1. Then $\mathfrak{A}_q$ is not irreducible.*

Proof. Let W be the two-dimensional subspace generated by $\mathbf{w}_1$ and $\mathbf{w}_2$ and $\{\mathbf{f}_1, \mathbf{f}_2\}$ be an orthonormal basis of W. We expand $\mathbf{w}_j$ $(j = 1, 2)$ as $\mathbf{w}_j = w_{j1}\mathbf{f}_1 + w_{j2}\mathbf{f}_2$ $(w_{j1}, w_{j2} \in \mathbb{C})$ and define a matrix

$$T := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \tag{5.36}$$

Then T is regular. For each $\mathbf{n} = (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$, we define $(a_{\mathbf{n}1}, a_{\mathbf{n}2}) \in \mathbb{C}^2$ by

$$\begin{pmatrix} a_{\mathbf{n}1} \\ a_{\mathbf{n}2} \end{pmatrix} := 2\pi T^{-1}\mathbf{n}. \tag{5.37}$$

Then the vector $\mathbf{a}_\mathbf{n} := a_{\mathbf{n}1}\mathbf{f}_1 + a_{\mathbf{n}2}\mathbf{f}_2$ satisfies $\mathbf{a}_\mathbf{n} \cdot \mathbf{w}_j = 2\pi n_j \in 2\pi\mathbb{Z}$. Hence $U(\pm\mathbf{a}_\mathbf{n})$ commute with E, F, K and K^{-1}, and hence all those of $\mathfrak{A}_q$ (note that $U(\mathbf{a}_\mathbf{n})^* = U(-\mathbf{a}_\mathbf{n})$). Therefore the commutant of $\mathfrak{A}_q$ is not the set of scalar operators. Thus $\mathfrak{A}_q$ is not irreducible. □

6 Representations of $\mathbb{C}_q^2$ and $U_q(\mathrm{sl}_2)$ in a quantum system with a singular gauge potential

In this section we consider a two-dimensional quantum system with a perpendicular magnetic field concentrated on the infinite lattice $\mathbb{L}_{\mathbf{w}_1,\mathbf{w}_2}$ defined by (4.21).

For a vector $\mathbf{a} = (a_1, a_2)$, we denote its corresponding complex number by $a = a_1 + ia_2$.

Let f be a meromorphic function on $\mathbb{C} \setminus \{nw_1 + mw_2 | m, n \in \mathbb{Z}\}$ with possible poles on the points $nw_1 + mw_2$ $(m, n \in \mathbb{Z})$ $(w_j = w_{j1} + iw_{j2})$. Then one can define a gauge potential $\mathbf{A}$ on

$$\mathbb{M} := \mathbb{R}^2 \setminus \mathbb{L}_{\mathbf{w}_1,\mathbf{w}_2} \tag{6.38}$$

by

$$\mathbf{A}(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x})) \tag{6.39}$$

with

$$A_1(\mathbf{x}) := \Im f(x_1 + ix_2), \quad A_2(\mathbf{x}) := \Re f(x_1 + ix_2), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{M}. \tag{6.40}$$

The magnetic field B is defined as a distribution on $\mathbb{R}^2$ by

$$B := D_1 A_2 - D_2 A_1. \tag{6.41}$$

Each component A_j $(j = 1, 2)$ is infinitely differentiable on $\mathbb{M}$. By the Cauchy-Riemann equation for f, we have

$$B(\mathbf{x}) = 0, \quad \partial_1 A_1(\mathbf{x}) + \partial_2 A_2(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{M}. \tag{6.42}$$

Hence the magnetic field B is concentrated, as a distribution, on the set $\mathbb{L}_{\mathbf{w}_1,\mathbf{w}_2}$.

Let

$$F_j^{\mathbf{A}}(\mathbf{x}) := e^{\alpha I_{\mathbf{A}}^{(j)}(\mathbf{x})}, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \mathbb{L}_j \tag{6.43}$$

with $\alpha \in \mathbb{C} \setminus \{0\}$ and

$$I_{\mathbf{A}}^{(j)}(\mathbf{x}) := \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{w}_j} \mathbf{A}(\mathbf{x}') \cdot d\mathbf{x}', \quad \mathbf{x} \in \mathbb{R}^2 \setminus \mathbb{L}_j, \tag{6.44}$$

where the integral $\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{w}_j}$ means the line integral along the straightline : $t \mapsto \mathbf{x} + t\mathbf{w}_j$, $t \in [0, 1]$.

Let

$$\Omega_{m,n} := mw_1 + nw_2. \tag{6.45}$$

and $\mathrm{Res}(\Omega_{m,n}, f)$ be the residue of f at $z = \Omega_{m,n}$.

In what follows, we assume that

$$c_f := \Re(\mathrm{Res}(\Omega_{m,n}, f)), \tag{6.46}$$

the real part of the residue $\mathrm{Res}(\Omega_{m,n}, f)$*, is a non-zero constant independently of* $(m, n) \in \mathbb{Z}^2$.

We introduce a constant:

$$q_\alpha := e^{2\pi\alpha c_f} \tag{6.47}$$

Lemma 6.1 *For all* $\mathbf{x} \in \Omega$*,*

$$F_1^{\mathbf{A}}(\mathbf{x})F_2^{\mathbf{A}}(\mathbf{x}+\mathbf{w}_1) = q_\alpha F_1^{\mathbf{A}}(\mathbf{x}+\mathbf{w}_2)F_2^{\mathbf{A}}(\mathbf{x}). \tag{6.48}$$

Proof. It is sufficient to show that, for each $\mathbf{x} \in \Omega$,

$$I_{\mathbf{A}}^{(1)}(\mathbf{x}) + I_{\mathbf{A}}^{(2)}(\mathbf{x}+\mathbf{w}_1) - I_{\mathbf{A}}^{(1)}(\mathbf{x}+\mathbf{w}_2) - I_{\mathbf{A}}^{(2)}(\mathbf{x})) = 2\pi c_f. \tag{6.49}$$

Let L be the left hand side of this equation. It is easy to see that $L = \Im \int_C f(z)dz$, where C is the closed curve starting and ending at $\mathbf{x}$ which is a composition of four straight lines going as $\mathbf{x} \to \mathbf{x}+\mathbf{w}_1 \to \mathbf{x}+\mathbf{w}_1+\mathbf{w}_2 \to \mathbf{x}+\mathbf{w}_2 \to \mathbf{x}$, forming a parallelogram. There exists a unique $(m,n) \in \mathbb{Z}^2$ such that $\Omega_{m,n}$ is in the interior of C. Hence, by the residue theorem, we have $\int_C f(z)dz = 2\pi i \mathrm{Res}(\Omega_{m,n}, f)$. Thus $L = 2\pi c_f$. □

We define

$$T_j^{\mathbf{A}} := F_j^{\mathbf{A}} V(\mathbf{w}_j), \quad j = 1,2. \tag{6.50}$$

Theorem 6.2 *Each* $T_j^{\mathbf{A}}$ *leaves* $C_0^\infty(\Omega)$ *invariant and*

$$T_1^{\mathbf{A}}T_2^{\mathbf{A}} = q_\alpha T_2^{\mathbf{A}}T_1^{\mathbf{A}} \tag{6.51}$$

holds on $C_0^\infty(\Omega)$*.*

Proof. It is easy to see that the function $F_j^{\mathbf{A}}$ is infinitely differentiable on Ω. Hence we can apply Proposition 4.1 with $F_j = F_j^{\mathbf{A}}$ to obtain the desired result. □

Thus $\{T_1^{\mathbf{A}}, T_2^{\mathbf{A}}\}$ gives a representation of the quantum plane $\mathbb{C}_{q_\alpha}^2$ on $C_0^\infty(\Omega)$. We next consider the boundedness of this representation.

Theorem 6.3 *Let* $\Re\alpha = 0$*. Then each* $T_j^{\mathbf{A}}$ *is unitary.*

Proof. In the present case we have $|F_j^{\mathbf{A}}| = 1$. Hence, by Proposition 2.3, $T_j^{\mathbf{A}}$ is unitary. □

Remark 6.4 If $\Re\alpha = 0$, then $q_\alpha = e^{2\pi i(\Im\alpha)c_f}$. Hence $|q_\alpha| = 1$.

In the case where $\Re\alpha \neq 0$, for each $j = 1,2$, we introduce two real-valued functions $\mathbb{R}^2 \setminus \mathbb{L}_j$ as follows:

$$K_j(\mathbf{x}) := (\Re\alpha)\Im \int_x^{x+w_j} f(z)dz, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \mathbb{L}_j. \tag{6.52}$$

Proposition 6.5 *For each* $j = 1,2$*,* $T_j^{\mathbf{A}}$ *is bounded if and only if the function* K_j *is bounded above on* $\mathbb{R}^2 \setminus \mathbb{L}_j$*.*

Proof. By Proposition 2.2, $T_1^{\mathbf{A}}$ and $T_2^{\mathbf{A}}$ are bounded if and only if $F_1^{\mathbf{A}}$ and $F_2^{\mathbf{A}}$ are essentially bounded on $\mathbb{R}^2$. Note that

$$I_{\mathbf{A}}^{(j)}(\mathbf{x}) = \Im \int_x^{x+w_j} f(z)dz, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \mathbb{L}_j.$$

Hence $|F_j^{\mathbf{A}}(\mathbf{x})| = e^{K_j(\mathbf{x})}$, $\mathbf{x} \in \mathbb{R}^2 \setminus \mathbb{L}_j$. Obviously each K_j is continuous on Ω. Therefore $F_j^{\mathbf{A}}$ is essentially bounded on $\mathbb{R}^2$ if and only if K_j is bounded above on $\mathbb{R}^2 \setminus \mathbb{L}_j$. $\square$

We can apply Theorem 5.1 to the representation $\{T_1^{\mathbf{A}}, T_2^{\mathbf{A}}\}$ of the quantum plane $\mathbb{C}^2_{q_\alpha}$ to obtain the following result:

Corollary 6.6 *Let $a, b, a', b' \in \mathbb{C}$ be constants satisfying*

$$abq_\alpha = \frac{a'b'}{q_\alpha} = -\frac{1}{(q_\alpha - q_\alpha^{-1})^2}. \tag{6.53}$$

and

$$E_{\mathbf{A}} := T_2^{\mathbf{A}}(aT_2^{\mathbf{A}} + a'(T_2^{\mathbf{A}})^{-1})(T_1^{\mathbf{A}})^{-1}, \tag{6.54}$$

$$F_{\mathbf{A}} := T_1^{\mathbf{A}}(bT_2^{\mathbf{A}} + b'(T_2^{\mathbf{A}})^{-1})(T_2^{\mathbf{A}})^{-1}, \tag{6.55}$$

$$K_{\mathbf{A}} := (T_2^{\mathbf{A}})^2, \quad K_{\mathbf{A}}^{-1} := (T_2^{\mathbf{A}})^{-2}. \tag{6.56}$$

Then the set $\{E_{\mathbf{A}}, F_{\mathbf{A}}, K_{\mathbf{A}}, K_{\mathbf{A}}^{-1}\}$ generates a representation of $U_{q_\alpha}(\mathrm{sl}_2)$ on the vector space $C_0^\infty(\Omega)$.

Example 6.7 Let ζ be the Weierstrass Zeta function :

$$\zeta(z) := \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right\}$$

and P be a polynomial of z. We define

$$f(z) = \zeta(z) + P'(z), \quad z \in M := \mathbb{C} \setminus \{\Omega_{m,n} | m, n \in \mathbb{Z}\}.$$

It is easy to see that f is holomorphic on M with poles at $z = \Omega_{m,n}$, $(m, n) \in \mathbb{Z}^2$ and

$$c_f = 1.$$

Hence

$$q_\alpha = e^{2\pi\alpha}.$$

Therefore, if $\Re\alpha \neq 0$, then $|q_\alpha| \neq 1$.

It is well known that the function

$$\sigma(z) := z \prod_{(m,n)\neq(0,0)} \left\{ \left(1 - \frac{z}{\Omega_{m,n}}\right) \exp\left(\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2}\right) \right\}$$

has the following properties [5, Chapter XX]:

(i)

$$\frac{d}{dz}\log\sigma(z) = \zeta(z).$$

(ii)

$$\sigma(z+w_j) = -e^{\eta_j(2z+w_j)}\sigma(z), \quad j=1,2,$$

where $\eta_j := \zeta(w_j/2)$.

Hence

$$\int_x^{x+w_j} f(z)dz = \pi i + \eta_j(2x+w_j) + P(x+w_j) - P(x).$$

Hence, in the present case, we have

$$K_j(\mathbf{x}) = (\Re\alpha)(\pi + 2(x_1\Im\eta_j + x_2\Re\eta_j) + \Im(\eta_j w_j) + \Im(P(x+w_j) - P(x)).$$

Therefore, if $\Re\alpha \neq 0$ and the degree of P is one or more than 2, then K_j is neither bounded above nor below on $\mathbb{R}^2 \setminus \mathbb{L}_j$. Thus, in this case, the operators $T_1^{\mathbf{A}}$ and $T_2^{\mathbf{A}}$ are unbounded.

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