THE GLOBAL RENORMALIZATION GROUP TRAJECTORY IN A CRITICAL SUPERSYMMETRIC FIELD THEORY ON THE LATTICE $\mathbb{Z}^3$

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Abstract: We consider an Euclidean supersymmetric field theory in $\mathbb{Z}^3$ given by a supersymmetric Φ^4 perturbation of an underlying massless Gaussian measure on scalar bosonic and Grassmann fields with covariance the Green's function of a (stable) Lévy random walk in $\mathbb{Z}^3$. The Green's function depends on the Lévy-Khintchine parameter $\alpha = \frac{3+\varepsilon}{2}$ with $0 < \alpha < 2$. For $\alpha = \frac{3}{2}$ the Φ^4 interaction is marginal. We prove for $\alpha - \frac{3}{2} = \frac{\varepsilon}{2} > 0$ sufficiently small and initial parameters held in an appropriate domain the existence of a global renormalization group trajectory uniformly bounded on all renormalization group scales and therefore on lattices which become arbitrarily fine. At the same time we establish the existence of the critical (stable) manifold. The interactions are uniformly bounded away from zero on all scales and therefore we are constructing a non-Gaussian supersymmetric field theory on all scales. The interest of this theory comes from the easily established fact that the Green's function of a (weakly) self-avoiding Lévy walk in $\mathbb{Z}^3$ is a second moment (two point correlation function) of the supersymmetric measure governing this model. The control of the renormalization group trajectory is a preparation for the study of the asymptotics of this Green's function.

0. Introduction

It was observed long ago, [PS, Mc], that the Green's function of weakly self avoiding simple random walks (SAW) on a lattice $\mathbb{Z}^d$ can be expressed as a correlation function in a supersymmetric field theory. This can be shown rigorously by the same derivation as in [BEZ, BI1, BI2] for SAWs on hierarchical lattices. Consider instead of simple random walks the more general case of continuous time (stable) Lévy walks whose scaling limits are stable Lévy distributions, [KG, F]. Such walks can be realized as jump processes with probability distributions permitting long range jumps, [F]. Their characteristic functions are given by the Lévy-Khintchine formula with characteristic exponent α, $0 < \alpha \leq 2$, [F], $\alpha = 2$ corresponding to simple random walks. The Green's function of continuous time weakly self avoiding Lévy walks (SALW) can also be realized as a two point correlation function in a supersymmetric field theory by the same derivation as in [BEI, BI1, BI2]. We are interested in SALWs in $\mathbb{Z}^3$ with the Lévy parameter α in the range $\frac{3}{2} < \alpha < 2$ and in particular when α is very close to $\frac{3}{2}$ from above. Heuristic reasoning indicates that $\alpha = \frac{3}{2}$ corresponds to mean field behaviour. This paper is concerned with proving the existence of a uniformly bounded renormalization group (RG) trajectory for the interactions in the underlying supersymmetric field theory. Uniformity is with respect to the lattice scale which changes with each step of the renormalization group map. This gives the foundation for the study of the Green's function which is postponed to the sequel.

The supersymmetric field theory in question is a lattice supersymmetric generalization of the model considered in [BMS]. We describe it informally here and leave the details for the next section. Let Δ be the standard Laplacian in $\mathbb{Z}^3$. Then for $x,\ y \in \mathbb{Z}^3$, and $0 < \alpha < 2$, $C(x-y) = (-\Delta)^{-\alpha/2}(x-y)$ is the Green's function of a stable Lévy walk. Let $\varphi_1,\ \varphi_2$ be independent identically distributed Gaussian random fields in $\mathbb{Z}^3$ with covariance $\frac{1}{2}C$. Let $\varphi = \varphi_1 + i\varphi_2$ and $\bar{\varphi}$ its complex conjugate. Introduce a pair of Grassmann fields $\psi, \bar{\psi}$ of degree 1 and -1 respectively. Let $\Phi = (\varphi, \psi)$ and $\bar{\Phi} = (\bar{\varphi}, \bar{\psi})$. The inner product is given by $(\Phi, \Phi) = \Phi\bar{\Phi} = \varphi\bar{\varphi} + \psi\bar{\psi}$. Let $\Lambda \subset \mathbb{Z}^3$ be a finite subset. Define

$$V_0(\Lambda, \Phi) = g_0 \int_\Lambda dx (\Phi\bar{\Phi})^2(x) + \tilde{\mu}_0 \int_\Lambda dx \Phi\bar{\Phi}(x) \tag{0.0}$$

where the coupling constant $g_0 > 0$ and dx is the counting measure in $\mathbb{Z}^3$. Then our model in finite volume Λ is defined by the the measure

$$d\mu_\Lambda(\Phi) = d\mu_{C_\Lambda}(\Phi) e^{-V_0(\Lambda,\Phi)} \tag{0.1}$$

where C_Λ is the restriction of C to the points of Λ and $d\mu_{C_\Lambda}(\Phi)$ is the Gaussian measure

$$d\mu_{C_\Lambda}(\Phi) = \prod_{x\in\Lambda} d\varphi_1(x) d\varphi_2(x) d\psi(x) d\bar{\psi}(x)\, e^{-(\Phi, C_\Lambda^{-1}\bar{\Phi})_{L^2(\Lambda)}} \tag{0.2}$$

Integration over the Grassmann fields is Berezin integration and $d\mu_\Lambda(\Phi)$ is interpreted as a linear functional on the Grassman algebra (generated by the $\psi, \bar{\psi}$ with coefficients which are functionals of the $\varphi, \bar{\varphi}$). An important fact is that the potential $V_0(\Lambda, \Phi)$ is supersymmetric (supersymmetry in this context and some of its consequences are given in the Section 1.1). As a consequence we have that the measure $d\mu_\Lambda(\Phi)$ is normalized :

$$\int d\mu_\Lambda(\Phi)\, 1 = 1 \tag{0.3}$$

We give an informal description of the results of this paper. We will choose $\alpha = \frac{3+\varepsilon}{2}$ with $0 < \varepsilon < 1$, in particular we hold $\varepsilon > 0$ very small. We will take Λ to be a very large cube. By successive RG transformations we will get a sequence of measures (the RG trajectory of measures) living in smaller and smaller cubes in finer and finer lattices till we arrive at a fixed small cube in a very fine lattice. This will take $\log\Lambda$ steps. At every step the measure is a new gaussian measure times a new supersymmetric density. The Gaussian measure is characterized by a covariance and the sequence of covariances converge to a smooth continuum covariance. The supersymmetric density incorporates the interactions. The principal information is in the local interactions incorporated in local potentials of the above type albeit with new parameters (coupling constants) and on a finer lattice. The other interactions are contracting (irrelevant)in an appropriate sense and are expressed in the form of polymer activities. The coupling constants and polymer activities give coordinates of the RG trajectory. The goal of this paper is to study the RG trajectory of these coordinates in the infinite volume limit which makes sense for these coordinates. The true infinite volume limit and the scaling limit will be taken at the level of correlation functions.

In section 1 we define the model, introduce supersymmetry and develop some of its consequences. The RG analysis of this paper is based on the finite range multiscale expansion of covariances of [BGM]. We summarize the basic results of [BGM] pertinent to this paper in Theoerem 1.1. This is an alternative to the Kadanoff- Wilson block spin RG developed extensively by Gawedzki and Kupiainen [GK 1,2], and Balaban [Bal 1,2,3]. A crucial simplification arises due to the finite range of the fluctuation covariances: Cluster expansions are no longer needed in the control of the fluctuation integration which is an essential step of RG transformations. As a result all estimates are local in character. In this section we also define lattice polymers and polymer activities.

In section 2 we introduce norms which will measure the size of polymer activities. These norms are suggested by those in the continuum analysis of [BMS] (1) but now include the Grassmann variables. The definition that we have hit upon owes much to a suggestion of David Brydges.
In section 3 we define the RG map as we will use it and in section 4 apply it to our model. In particular we develop second order perturbation theory. The task is to control the contributions from the remainder and this is taken up in the next section.
Section 5 gives the basic estimates that we will need for the control of the RG trajectory. These estimates resemble those in [BMS] but now take account of the presence of Grassmann variables as well as the lattice. The upshot is Theorem 5.1.
In section 6 we prove the existence of a uniformly bounded RG trajectory at all scales (Theorem 6.2). This establishes the existence of a critical mass but not its uniqueness. This is remedied by first iterating the RG map a finite number n_0 of times and then restarting the trajectory. There exists a uniformly bounded trajectory at all scales $n \geq n_0$. For n_0 sufficiently large the critical mass is a continuous function of the contracting variables. The stable (critical) manifold, appropriately interpreted for a sequence of non-autonomous maps, is constructed (Theorem 6.4). Finally we observe that the coupling constant is uniformly bounded away from 0 at all scales. As a result the global RG trajectory gives rise to a non-Gaussian field theory. We remark that in a continuum version of this model with a cutoff modelled on that of [BMS] one can prove more: the continuum RG trajectory ends at a non-trivial fixed point. But the notion of a fixed point is devoid of meaning for lattice field theories because the RG map even in infinite volume does not give an autonomous action on a fixed Banach space.

1.1 Definitions, model, supersymmetry

Let e_1, e_2, e_3 be the standard basis of unit vectors specifying the orientation of $\mathbb{Z}^3$. We let ∂_μ denote the forward lattice derivative in direction e_μ and ∂_μ^* its $L^2(\mathbb{Z}^3)$ adjoint. The latter is the backward derivative. Then the lattice Laplacian Δ in $\mathbb{Z}^3$ is defined by

$$-\Delta = \partial_\mu^* \partial_\mu \tag{1.1}$$

Let $\hat{\Delta}(p)$ be the Fourier transform of the integral kernel of Δ in $\mathbb{Z}^3$, namely

$$\hat{\Delta}(p) = 2\sum_{\mu=1}^{3}(cos(p_\mu) - 1) \tag{1.2}$$

.

Let $\alpha = \frac{3+\varepsilon}{2}$ be a real number with $0 < \alpha < 2$. Then the Green's function of a (stable) Lévy walk in $\mathbb{Z}^3$ is given by

$$C(x-y) = (-\Delta)^{-\alpha/2}(x-y) = \int_{[-\pi,\pi]^3} \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}\, (-\hat{\Delta}(p))^{-\alpha/2} \tag{1.3}$$

C is positive-definite and therefore qualifies as the covariance of a Gaussian random field in $\mathbb{Z}^3$. We introduce a pair of independent identically distributed Gaussian random fields $\varphi_1,\ \varphi_2$ with mean 0 and covariance

$$E(\varphi_j(x)\varphi_j(y)) = \frac{1}{2}C(x-y) \tag{1.4}$$

(1) A. Abdesselam in [A] corrected an error which occurs in [BDH-eps, BMS] where it is wrongly asserted that certain normed spaces are complete. This problem was resolved in [A] by some changes in definitions which fortunately are such that, as noted in [A], the estimates, theorems and proofs of [BDH-eps, BMS] remain true without change. On the lattice however the function space subtleties encountered in [A] disappear.

for $j = 1, 2$.
Let $\varphi(x) = \varphi_1(x) + i\varphi_2(x)$ be a complex scalar field and $\bar\varphi(x)$ its compex conjugate. On the space of functionals of $\varphi, \bar\varphi$ we have the Gaussian probability measure

$$d\mu_C(\phi) = d\mu_{\frac{1}{2}C}(\phi_1) d\mu_{\frac{1}{2}C}(\phi_2) \tag{1.5}$$

Then each of $\varphi, \bar\varphi$ has zero covariance and

$$E(\bar\varphi(x)\varphi(y)) = C(x-y) \tag{1.6}$$

The φ_j are Osterwalder-Schrader (O-S) positive generalised free fields. To see this notice that for $0 < \alpha < 2$ we can write

$$C(x-y) = c_\alpha^{-1} \int_0^\infty da\, a^{-\alpha/2}\, G^a(x-y) \tag{1.7}$$

where $c_\alpha = \int_0^\infty da\, a^{-\alpha/2}\, (a+1)^{-1}$ and

$$G^a(x-y) = \int_{(-\pi,\pi)^3} \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}\, (a - \hat\Delta(p))^{-1} \tag{1.8}$$

is the resolvent of the simple random walk on $\mathbb{Z}^3$. From this convergent representation for C the O-S positivity follows since it is well known that G^a is O-S positive.

Grassmann algebra and integration
Let $\Lambda \subset \mathbb{Z}^3$ be a bounded subset. $\mathcal{F}_\Lambda$ represents the algebra of $\mathbb{C}$ valued functionals of the fields $\{\phi, \bar\phi : \Lambda \to \mathbb{C}\}$. Let $\psi(x), \bar\psi(x)$ for all $x \in \Lambda$ be the (anti-commuting) Grassman elements of degree $1, -1$ respectively. Following standard usage we will refer to them as (scalar) *fermions*. We denote by Ω_Λ the Grassman algebra generated by the $\psi(x), \bar\psi(y)$ by multiplication and linear sums for all $x, y \in \Lambda$ with coefficients in $\mathcal{F}_\Lambda$. The Grassmann algebra is naturally graded $\Omega_\Lambda = \oplus_p \Omega^p_\Lambda$ where the integer p is the degree and each Ω^p_Λ is a $\mathcal{F}_\Lambda$ module. Ω^0_Λ is an algebra. Because of the anticommuting property of the generators, and because Λ is a finite lattice an element of Ω^p_Λ is a finite sum of degree p elements with coefficients in $\mathcal{F}_\Lambda$. An element F_Λ of Ω_Λ can be represented as

$$F_\Lambda(\varphi, \psi) = \sum_{k\geq 0} \sum_{\mathbf{a}} \int_{\Lambda^n} dx_1 ... dx_n F^{k,\mathbf{a}}_\Lambda(\varphi; x_1, ..., x_k) \prod_{j=1}^k \psi_{a_j}(x_j) \tag{1.9}$$

where $a_j \in \{-1, 1\}$, $\psi_1 = \psi$, $\psi_{-1} = \bar\psi$, and $\mathbf{a} = (a_1, .., a_k)$ is a multi-index, we take the ordering specified by $a_j = 1$ if j is odd and $a_j = -1$ and dx is the counting measure in $\mathbb{Z}^3$ When Λ is finite subset the above multiple sum is obviously finite. If $F_\Lambda \in \Omega^p_\Lambda$ we constrain the $\mathbf{a}$ sum to satisfy $a_1 + ... a_k = p$. In the following we will often refer to the coefficients $F^{k,\mathbf{a}}_\Lambda$ above as *bosonic coefficients*. Here and in the following we suppress indicating the dependence of the bosonic coefficients on $\bar\varphi$.
These considerations are ofcourse valid for a lattice $(\delta\mathbb{Z})^3$ for any lattice spacing δ with the corresponding notations $\Lambda_\delta, \mathcal{F}_{\Lambda_\delta}, \Omega_{\Lambda_\delta}$, dx being δ^3 times the counting measure in $(\delta\mathbb{Z})^3$.

Now we define fermionic expectation (integration) using Berezin integration which we review briefly and set up our conventions.
Berezin integration is a linear map $\Omega_\Lambda \to \Omega_\Lambda$ which satisfies

$$\int d\psi(x) F_\Lambda(\psi, \bar\psi, \phi, \bar\phi) = \pi^{-1/2} \frac{\partial}{\partial\psi(x)} F_\Lambda \tag{1.10}$$

where $F_\Lambda \in \Omega_\Lambda$ and the fermionic derivative $\frac{\partial}{\partial\psi(x)}$ is an antiderivation. Integration with respect to $d\bar\psi(x)$ is given by the same formula with $\frac{\partial}{\partial\bar\psi(x)}$ on the right hand side. Multiple integration is repeated integration using the above rule, keeping in mind that fermionic derivatives anticommute.
Define $C_\Lambda(x-y) = C(x-y)\ :\ x,y \in \Lambda$. We consider this as a $|\Lambda|\times|\Lambda|$ dimensional positive definite symmetric matrix with x,y labelling the entries. Then we define the fermionic expectation $E_{f,\Lambda}$ as a linear map $\Omega_\Lambda \to \mathcal{F}_\Lambda$ as follows: Let $F_\Lambda \in \Omega_\Lambda$. We adopt the convention $F_\Lambda(\psi,\phi) \equiv F_\Lambda(\psi,\bar\psi,\phi,\bar\phi)$. Then

$$E_{f,\Lambda}(F_\Lambda) = \int d\mu_{C_\Lambda}(\psi) F_\Lambda(\psi,\phi) \tag{1.11}$$

where

$$\int d\mu_{C_\Lambda}(\psi)F_\Lambda(\psi,\phi)) = (det\,\pi C_\Lambda)^{|\Lambda|}\int \prod_{x\in\Lambda}(d\psi(x)d\bar\psi(x))\, e^{-(\psi,C_\Lambda^{-1}\bar\psi)_{L^2(\Lambda)}}F_\Lambda(\psi,\phi) \tag{1.12}$$

We call $d\mu_{C_\Lambda}(\psi)$ a fermionic Gaussian measure and use the terminology measure and expectation interchangeably. It is not difficult to show that we have a fermionic counterpart of the bosonic gaussian formula, namely

$$\int d\mu_{C_\Lambda}(\psi)F_\Lambda(\psi,\phi) = e^{\int_{\Lambda\times\Lambda} dxdy C_\Lambda(x-y)\frac{\partial}{\partial\psi(x)}\frac{\partial}{\partial\bar\psi(y)}}F_\Lambda(\psi,\phi)|_{\psi=\bar\psi=0} \tag{1.13}$$

where dx is the counting measure. The fermionic expectation above annihilates the component of $F_\Lambda \notin \Omega^0_\Lambda$.

Note that the expectation of a product of two ψ or of two $\bar\psi$ vanishes whereas if $x,y\in\Lambda$

$$E_{f,\Lambda}(\bar\psi(x)\psi(y)) = C(x-y) \tag{1.14}$$

More generally, if $x_j, y_j \in \Lambda,\ j = 1,2,..n$

$$E_{f,\Lambda}(\prod_{j=1}^n \bar\psi(x_j)\psi(y_j)) = \det\,(C(x_j-y_k))^n_{j,k=1} \tag{1.15}$$

We define the field $\Phi(x)$ (called superfield in anticipation) as the pair

$$\Phi(x) = (\varphi(x),\psi(x)) \tag{1.16}$$

with the scalar product

$$(\Phi(x),\Phi(y)) = \Phi(x)\bar\Phi(y) = \varphi(x)\bar\varphi(y) + \psi(x)\bar\psi(y) \tag{1.17}$$

More generally if $A(x,y)$ is a matrix for $x,y\in\Lambda$ we define

$$(\Phi, A\Phi)_{L^2(\Lambda)} = \int_{\Lambda\times\Lambda} dxdy\ \Phi(x)A(x,y)\bar\Phi(y) = \int_{\Lambda\times\Lambda} dxdy\ (\varphi(x)A(x,y)\bar\varphi(y) + \psi(x)A(x,y)\bar\psi(y)) \tag{1.18}$$

Let $F_\Lambda(\Phi)$ belong to Ω_Λ. F_Λ also depends on $\bar\Phi$ but here and in the following this is not explicitly indicated. Since $F_\Lambda(\Phi)\in\Omega_\Lambda$ it has the representation (1.9). We define the expectation E_Λ as a linear map $\Omega_\Lambda \to \mathbf{C}$ obtained by combining the bosonic and fermionic expectations: If $F_\Lambda \in \Omega_\Lambda$ with μ_C integrable bosonic coefficients then

$$E_\Lambda(F_\Lambda(\Phi)) = \int d\mu_{C_\Lambda}(\Phi)F_\Lambda(\Phi) = \int d\mu_{C_\Lambda}(\phi)d\mu_{C_\Lambda}(\psi)F_\Lambda(\Phi) \tag{1.19}$$

Thus

$$E_\Lambda(F_\Lambda(\Phi)) = \int \prod_{x\in\Lambda} d\varphi(x)d\bar\varphi(x) \prod_{x\in\Lambda} d\psi(x) \prod_{x\in\Lambda} d\bar\psi(x)\, e^{-(\Phi, C_\Lambda^{-1}\bar\Phi)_{L^2(\Lambda)}} F_\Lambda(\Phi) \tag{1.20}$$

Notice that the determinant in the fermionic integration formula (1.12) has cancelled out with the inverse of the same determinant which appears in the bosonic integration measure.
The expectation defined above is normalized. In other words if $1_\Lambda(\Phi)$ is the indicator function of Ω_Λ then

$$E_\Lambda(1_\Lambda(\Phi)) = 1 \tag{1.21}$$

We have the natural order relation $\Omega_\Lambda \subset \Omega_{\Lambda'}$ if $\Lambda \subset \Lambda'$. Moreover if $\Lambda \subset \Lambda'$ and $F_\Lambda \in \Omega_\Lambda$ then $E_{\Lambda'}(F_\Lambda) = E_\Lambda(F_\Lambda)$ as is not difficult to show. We define Ω as the inductive limit of the Ω_Λ as $\Lambda \subset \mathbb{Z}^3$ varies over increasing subsets tending to $\mathbb{Z}^3$ respecting the order relation above. The $\{E_\Lambda, \Omega_\Lambda\}$ constitute a projective family. We denote by E the projective limit: Let $F \in \Omega$ with μ_C integrable bosonic coefficients. We have

$$E(F) = \int d\mu_C(\Phi)\, F(\Phi) = \lim_{\Lambda'\uparrow\mathbb{Z}^3} E_{\Lambda'}(F) \tag{1.22}$$

and this limit exists since $F \in \Omega_\Lambda$ for some finite set Λ and therefore $E(F) = E_\Lambda(F)$ which exists.
Remark : The above construction is motivated by analogous considerations in [BEI].

Lattice integration : In the following and throughout this paper we will represent lattice sums as integrals where for the $(\delta\mathbb{Z})^3$ lattice the integration measure is the counting measure in $(\delta\mathbb{Z})^3$ times a factor δ^3. Thus if f is a function on $(\delta\mathbb{Z})^3$ we define

$$\int_{(\delta\mathbb{Z})^3} dx\, f(x) = \delta^3 \sum_{x\in(\delta\mathbb{Z})^3} f(x) \tag{1.23}$$

We now define a Laplacian acting on functionals in Ω

$$\begin{aligned}\mathbf{\Delta}_C &= \int_{\mathbb{Z}^3\times\mathbb{Z}^3} dxdy\, C(x-y)\frac{\partial}{\partial\Phi(x)}\cdot\frac{\partial}{\partial\bar\Phi(y)} = \\ &= \int_{\mathbb{Z}^3\times\mathbb{Z}^3} dxdy\, C(x-y)[\frac{\partial}{\partial\phi(x)}\frac{\partial}{\partial\bar\phi(y)} + \frac{\partial}{\partial\psi(x)}\frac{\partial}{\partial\bar\psi(y)}]\end{aligned} \tag{1.24}$$

This integration automatically restricts to $\Lambda\times\Lambda$ when applied to functionals of Φ which live in a bounded subset Λ and the fermionic part of the exponential becomes a finite sum by the Grassmann property. It follows from (1.13) and its bosonic counterpart that if $F_\Lambda(\Phi) \in \Omega_\Lambda$ with integrable bosonic coefficients then

$$E(F_\Lambda(\Phi)) = e^{\mathbf{\Delta}_C} F_\Lambda(\Phi)|_{\varphi=\bar\varphi=\psi=\bar\psi=0} \tag{1.25}$$

Note that the action of $e^{\mathbf{\Delta}_C}$ is well defined. In fact since $F_\Lambda(\Phi)$ is in Ω_Λ and Λ is a finite lattice, it can be expressed as a finite sum of Grassmann elements with coefficients in $\mathcal{F}_\Lambda$. $e^{\mathbf{\Delta}_C}$ factorises into bosonic and Grassmann exponentials. The expansion of the Grassman exponential acting on $F_\Lambda(\Phi)$ evaluated at $\psi = \bar\psi = 0$ thus terminates and we are left with a purely bosonic expectation which is well defined.
We have in particular

$$E(\Phi(x)\bar\Phi(y)) = 0 \tag{1.26}$$

and more generally

$$E(\prod_{j=1}^{n}\Phi(x_j)\bar\Phi(y_j))=0 \tag{1.27}$$

This can be proved by computation or more simply using supersymmetry (introduced later). The integrand is supersymmetric and Lemma 1.1 below gives the result.
Wick polynomials $P(\Phi)$ are defined by the formula

$$:P(\Phi):_C = e^{-\boldsymbol{\Delta}_C}P(\Phi) \tag{1.28}$$

This implies in particular that

$$:\Phi(x)\bar\Phi(y):_C = \Phi(x)\bar\Phi(y) \tag{1.29}$$

and

$$:(\Phi\bar\Phi)^2:_C(x) = (\Phi\bar\Phi)^2(x) - 2C(0)(\Phi\cdot\bar\Phi)(x) \tag{1.30}$$

For future reference we note that for $\alpha=1,2$

$$:(\Phi\bar\Phi)\Phi_\alpha:_C(x) = (\Phi\bar\Phi)\Phi_\alpha)(x) - C(0)\Phi_\alpha(x) \tag{1.31}$$

where $\Phi_1=\varphi$, $\Phi_2=\psi$.

Remark : The considerations from (1.10) to (1.30) remain valid on a lattice $(\delta\mathbb{Z})^3$ if we replace in the above Λ by a bounded subset $\Lambda_\delta\subset(\delta\mathbb{Z})^3$ and the positive definite matrix C by an arbitrary positive definite matrix $C_\delta(x,y)$ with $x,y\in(\delta\mathbb{Z})^3$. The functional Laplacian Δ_C in (1.24) is replaced by Δ_{C_δ} with the integration over $(\delta\mathbb{Z})^3\times(\delta\mathbb{Z})^3$.

The model :

Let L be a dyadic integer, $L=2^p$ with integer $p\ge 2$. Let $\Lambda_N=[-\frac{L^N}{2},\frac{L^N}{2})^3\subset\mathbb{R}^3$, with N large be a large half open cube in $\mathbb{R}^3$. Distances in $\mathbb{R}^3$ and lattices $(\delta\mathbb{Z})^3$ will be measured in the norm

$$|x-y| = \max_{1\le j\le 3}|x_j-y_j| \tag{1.32}$$

Define $\Lambda_{N,0}=\Lambda_N\cap\mathbb{Z}^3$. This is a (large) cube in $\mathbb{Z}^3$ of edge length L^N. The second index 0 in $\Lambda_{N,0}$ emphasizes that this is a cube in $\mathbb{Z}^3$. The local potential (0.0) will be written in a C-Wick ordered form by using (1.30) and (1.29). This gives

$$V_0(\Lambda_{N,0},\Phi) = \int_{\Lambda_{N,0}} dx\, g_0 :(\Phi\bar\Phi)^2:_C(x) + \mu_0\int_{\Lambda_{N,0}} dx\; :\Phi\bar\Phi:_C(x) \tag{1.33}$$

where $\mu_0=\tilde\mu_0+2C(0)g_0$.
Define

$$\mathcal{Z}_0(\Lambda_{N,0},\Phi) = e^{-V(\Lambda_{N,0},\Phi)} \tag{1.34}$$

We define the measure

$$d\mu_{N,0}(\Phi) = d\mu_C(\Phi)\mathcal{Z}_0(\Lambda_{N,0},\Phi) \tag{1.35}$$

Note that the measure is normalized

$$\int d\mu_{N,0}(\Phi) = 1 \tag{1.36}$$

This follows from Lemma 1.1 below which exploits supersymmetry introduced later. However heuristically this is evident if we formally expand the exponential, integrate term by term and use (1.27). This measure defines our model.

Supersymmetry

The density of the measure $d\mu_{N,0}(\Phi)$ as well as its RG evolution have the important property of being *supersymmetric*. This will restrict considerably the form of the evolved density.
A *supersymmetry* transformation $\mathcal{Q} : \Omega_\Lambda \to \Omega_\Lambda$ is a derivation on the bosonic fields and an antiderivation on the grassman fields which acts on the fields as follows :

$$\begin{aligned} \mathcal{Q}\varphi &= \psi \\ \mathcal{Q}\bar\varphi &= -\bar\psi \\ \mathcal{Q}\psi &= \varphi \\ \mathcal{Q}\bar\psi &= \bar\varphi \end{aligned} \tag{1.37}$$

Let $F_\Lambda(\Phi) = F_\Lambda(\varphi, \bar\varphi, \psi, \bar\psi)$ belong to Ω_Λ with bosonic coefficients differentiable in the bosonic fields $\varphi(x)$, $x \in \Lambda$. Then the action of $\mathcal{Q}$ on F_Λ is given by a super vector field denoted by the same symbol $\mathcal{Q}$

$$\mathcal{Q}F_\Lambda = \int_\Lambda dx \,\Big(\psi(x)\frac{\partial}{\partial\varphi(x)} - \bar\psi(x)\frac{\partial}{\partial\bar\varphi(x)} + \varphi(x)\frac{\partial}{\partial\psi(x)} + \bar\varphi(x)\frac{\partial}{\partial\bar\psi(x)}\Big)F_\Lambda \tag{1.38}$$

We say that a functional F_Λ is supersymmetric if $\mathcal{Q}F = 0$.
Remark: A super vector field is not a vector field because fermionic derivatives are antiderivations.

An (infinitesimal) *gauge transformation* $\mathcal{G} : \Omega_\Lambda \to \Omega_\Lambda$ is a derivation whose action is given by

$$\begin{aligned} \mathcal{G}\varphi &= i\varphi \\ \mathcal{G}\bar\varphi &= -i\bar\varphi \\ \mathcal{G}\psi &= i\psi \\ \mathcal{G}\bar\psi &= -i\bar\psi \end{aligned} \tag{1.39}$$

This induces on an Ω_Λ function F_Λ the action of a vector field denoted by the same symbol $\mathcal{G}$

$$\mathcal{G}F_\Lambda = i\int_\Lambda dx \,\Big(\varphi(x)\frac{\partial}{\partial\varphi(x)} - \bar\varphi(x)\frac{\partial}{\partial\bar\varphi(x)} + \psi(x)\frac{\partial}{\partial\psi(x)} + \bar\psi(x)\frac{\partial}{\partial\bar\psi(x)}\Big)F_\Lambda \tag{1.40}$$

We say that a functional F_Λ is gauge invariant if $\mathcal{G}F_\Lambda = 0$.

From (1.37) we see that $\mathcal{Q}^2$ engenders an infinitesimal gauge transformation (1.39). Thus acting on gauge invariant functionals

$$\mathcal{Q}^2 = 0 \tag{1.41}$$

An important property of the super vector field $\mathcal{Q}$ which we will exploit later is that it commutes with the super Laplacian $\mathbf{\Delta}_C$ defined in (1.24):

$$[\,\mathcal{Q},\ \mathbf{\Delta}_C\,] = 0$$

as is easy to verify.
It is easy to verify that any polynomial in $\Phi\bar\Phi$ and their (lattice) derivatives is supersymmetric. This follows from

$$\mathcal{Q}(\Phi(x)\bar\Phi(y)) = \mathcal{Q}\prod_{j=1}^{n}(\Phi(x_j)\bar\Phi(y_j)) = 0 \tag{1.42}$$

and lattice derivatives commute with $\mathcal{Q}$.
As a consequence we have $\mathcal{Q}V(\Lambda,\Phi)=0$ where V is given in (1.33) and thus the starting interaction potential is supersymmetric.

Let $\Gamma(x,y)$ be any positive definite symmetric matrix. Let $\mathbf{\Delta}_\Gamma$ be a super Laplacian given by (1.24) with C replaced by Γ. Let $F_\Lambda(\Phi)$ be an Ω_Λ functional with μ_C integrable bosonic coefficients. Let $\xi=(\zeta,\eta)$ be another superfield. Define the convolution

$$\mu_\Gamma * F_\Lambda(\Phi)=\int d\mu_\Gamma(\xi)F_\Lambda(\Phi+\xi)=e^{\mathbf{\Delta}_\Gamma}F_\Lambda(\Phi) \tag{1.43}$$

Since $\mathcal{Q}$ commutes with $\mathbf{\Delta}_\Gamma$, $\mathcal{Q}$ *also commutes with convolution with the measure* μ_Γ :

$$\mu_\Gamma * \mathcal{Q}F_\Lambda(\Phi)=\mathcal{Q}\mu_\Gamma * F_\Lambda(\Phi) \tag{1.44}$$

Therefore if F_Λ is supersymmetric so is $\mu_\Gamma * F_\Lambda$. This observation prefigures the supersymmetry invariance of the renormalization group map which we will introduce later.

It follows by evaluating (1.44) at $\Phi=0$ that

$$\int d\mu_\Gamma(\Phi)\ \mathcal{Q}F_\Lambda(\Phi)=0 \tag{1.45}$$

since the left hand side is given by $(\mu_\Gamma * \mathcal{Q}F_\Lambda(\Phi))\Big|_{\Phi=0}$ and this vanishes by virtue of (1.44) since the coefficients of the super vector field $\mathcal{Q}$ vanish when the fields vanish. ■

Lemma 1.1 : Let $F_\Lambda(\Phi)$ be a supersymmtric Ω_Λ functional with differentiable bosonic coefficients which are μ_Γ integrable. Then

$$\int d\mu_\Gamma(\Phi)\ F_\Lambda(\Phi)=F_\Lambda(0) \tag{1.46}$$

Proof : λ be a real parameter. Define

$$f(\lambda)=\int d\mu_\Gamma(\Phi)\ F_\Lambda(\lambda\Phi) \tag{1.47}$$

We will prove

$$\frac{d}{d\lambda}f(\lambda)=0 \tag{1.48}$$

This implies that $f(\lambda)$ is a constant and hence evaluating at $\lambda=0$ gives (1.46).
Taking the λ derivative in (1.47) we get

$$\frac{d}{d\lambda}f(\lambda)=\int d\mu_\Gamma(\Phi)\ (\mathcal{D}F_\Lambda)(\lambda\Phi) \tag{1.49}$$

where

$$\mathcal{D}=\int_\Lambda dx\Big(\phi(x)\frac{\partial}{\partial\phi(x)}+\bar\phi(x)\frac{\partial}{\partial\bar\phi(x)}+\psi(x)\frac{\partial}{\partial\psi(x)}+\bar\psi(x)\frac{\partial}{\partial\bar\psi(x)}\Big) \tag{1.50}$$

Note that the four coefficients of $\mathcal{D}$ can also be written as $(\mathcal{Q}\psi(x),\mathcal{Q}\bar\psi(x),\mathcal{Q}\phi(x),-\mathcal{Q}\bar\phi(x))$ which we have taken in the same order as above. This suggests that we consider the operator

$$\mathcal{L}=\int_\Lambda dx\Big(\psi(x)\frac{\partial}{\partial\phi(x)}+\bar\psi(x)\frac{\partial}{\partial\bar\phi(x)}+\phi(x)\frac{\partial}{\partial\psi(x)}-\bar\phi(x)\frac{\partial}{\partial\bar\psi(x)}\Big) \tag{1.51}$$

and act with $\mathcal{Q}$ on it. We also consider the action of $\mathcal{L}$ on $\mathcal{Q}$. A straight forward computation gives the nice formula

$$\mathcal{D} = \frac{1}{2}(\mathcal{Q}\mathcal{L} + \mathcal{L}\mathcal{Q}) \tag{1.52}$$

We substitute for $\mathcal{D}$ in (1.49) the right hand side of (1.52). The contribution of the first term vanishes by (1.45). The contribution of the second term vanishes because F_Λ is supersymmetric by hypothesis. This proves (1.48) and we are done. ∎

Remark : The special case of Lemma 1.1 for a hierarchical lattice is Lemma 2.1 of [BI]. *This Lemma has the important consequence that no field independant relevant parts (defined later) will arise in the renormalization group analysis to follow.*

1.2 Lattice renormalization group transformations

We say that a function $f(x,y)$ has finite range L if $f(x,y) = 0 \,:\, |x-y| \geq L$. Lattice renormalization group transformations will be based on the finite range multiscale expansion of the covariance C established in [BGM].

Let L be a large dyadic integer, i.e. a large integer power of 2. Define $\delta_n = L^{-n}$. We have a sequence of compatible lattices $(\delta_n\mathbb{Z})^3 \subset \mathbb{R}^3$, $(\delta_n\mathbb{Z})^3 \subset (\delta_{n+1}\mathbb{Z})^3$, with $n = 0, 1, 2, ...$ and eventually passing to $\mathbb{R}^3$. $B_{\delta_n} = [-\frac{\pi}{\delta_n}, \frac{\pi}{\delta_n}]^3$ denotes the first Brillouin zone of the dual of the δ_n lattice. We have the following theorem which gives the multiscale expansion of the covariance C on $\mathbb{Z}^3$ as a sum of finite range *fluctuation* covariances living on increasingly finer lattices, together with their properties which we will need later :

Theorem 1.1 (finite range multiscale expansion) : For $0 < \alpha < 2$, $d_s = \frac{(3-\alpha)}{2}$ and $n = 0, 1, 2, ...$ there exist positive definite functions $\Gamma_n(x)$ defined for $x \in (\delta_n\mathbb{Z})^3$ and a smooth positive definite function $\Gamma_{c,*}$ in $\mathbb{R}^3$ such that for all $k \geq 0$, constants $c_{k,L}$, $c_{L,m}$ independent of n and $q = \frac{1}{48}$

(1) $$C(x-y) = \sum_{n\geq 0} L^{-2nd_s}\Gamma_n\Big(\frac{x-y}{L^n}\Big) \text{ and the series converges in } L^\infty(\mathbb{Z}^3)$$

(2) $$\Gamma_n(x) = 0 \quad \text{for} \quad |x| \geq \frac{L}{2}$$

(3) $$\Big|\hat{\Gamma}_n(p)\Big| \leq c_{k,L}(1+p^2)^{-2k} \quad \text{for} \quad p \in B_{\delta_n}, \quad \forall k \geq 0$$

(4a) $$\hat{\Gamma}_{c,*}(p) = \lim_{n\to\infty}\hat{\Gamma}_n(p) \quad \text{exists pointwise in } p$$

(4b) $$\Big|\hat{\Gamma}_n(p) - \hat{\Gamma}_{c,*}(p)\Big| \leq c_{k,L}(1+p^2)^{-2k}L^{-qn} \quad \forall n \geq 2,\ \forall k \geq 0$$

(5a) $$\|\partial^m_{\delta_n}\Gamma_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \leq c_{L,m}, \forall m \geq 0$$

(5b) $$\partial^m_c\Gamma_{c,*} = \lim_{n\to\infty}\partial^m_{\delta_n}\Gamma_n \quad \text{exists in} \quad L^\infty((\delta_l\mathbb{Z})^3) \quad \text{for any fixed } l \geq 0 \text{ and } \forall\, m \geq 0$$

(5c) $$\|\partial^m_{\delta_n}\Gamma_n - \partial^m_c\Gamma_{c,*}\|_{L^\infty((\delta_l\mathbb{Z})^3)} \leq c_{L,m}L^{-qn},\ \forall n \geq l,\ \forall m \geq 0$$

where ∂_c is a continuum partial derivative and ∂_{δ_n} is a forward lattice partial derivative in $(\delta_n\mathbb{Z})^3$.

Proof : The theorem is for the most part a combination of results obtained in various theorems in [BGM]. For the reader's convenience we record them here. The references pertain to [BGM]. Thus the multiscale expansion in part (1) and the finite range property of part (2) were given in Section 4. The range was stated as being $6L$ but we can easily arrange matters in section 2 so that the range is $L/2$ which turns out to be more convenient. Convergence of (1) in $L^\infty(\mathbb{Z}^3)$ follows on using $d_s > 0$, Corollary 5.6 and the lattice Sobolev embedding inequality with k in the Corollary sufficiently large. Part (3) follows from (5.10) of Theorem 5.5 by integration on a with the measure $da\, a^{-\alpha/2}$ (see (4.3) of section 4). Corollary 5.6 and lattice Sobolev embedding gives (5a). Corollary 6.2 gives parts (4a) and (5b). The convergence rate estimates of parts (4b) and (5c) which were not given in [BGM] also follow from the results therein. The proof is given elsewhere, [M]. ■

Define for all $n \geq 0$ the positive definite functions C_n, $C_{c,*}$ on $(\delta_n \mathbb{Z})^3)$ and $\mathbb{R}^3$ respectively :

$$C_n(x) = \sum_{j=0}^{\infty} L^{-2jd_s}\, \Gamma_{n+j}\Big(\frac{x}{L^j}\Big) \tag{1.53}$$

$$C_{c,*}(x) = \sum_{j=0}^{\infty} L^{-2jd_s}\, \Gamma_{c,*}\Big(\frac{x}{L^j}\Big) \tag{1.54}$$

Corollary 1.2 : The series (1.53) for C_n together with that for its multiple lattice derivatives in $(\delta_n \mathbb{Z})^3$ converge in $L^\infty((\delta_n \mathbb{Z})^3)$. For every integer $m \geq 0$ we have a constant $c_{L,m}$ such that

$$\|\partial^m_{\delta_n} C_n\|_{L^\infty((\delta_n \mathbb{Z})^3)} \leq c_{L,m} \tag{1.55}$$

The series (1.53) defining $C_{c,*}$ and its multiple continuum derivatives of arbitrary order converge in $L^\infty(\mathbb{R}^3)$ so that $C_{c,*}$ is a smooth continuum function. For all $m \geq 0$ and ∂_c the continuum partial derivative

$$\sup_{x \in \mathbb{R}^3} |\partial^m_c C_{c,*}(x)| \leq c_{L,m} \tag{1.56}$$

Moreover for any fixed $l \geq 0$ and $\forall m \geq 0$, there exists a constant $c_{L,m}$ such that for all $n \geq l$

$$\|\partial^m_{\delta_l} C_n - \partial^m_c C_{c,*}\|_{L^\infty((\delta_l \mathbb{Z})^3)} \leq c_{L,m} L^{-qn} \tag{1.57}$$

Proof : The first part together with the bound (1.55) follow from (5a) of Theorem 1.1. To prove the next statement apply Theorem 6.1 of [BGM] to (4.3) of Section 4, followed by Sobolev embedding . Therefore (1.54) and its multiple continuum derivatives of arbitrary order converge in $L^\infty(\mathbb{R}^3)$ and thus $C_{c,*}$ is smooth. Finally the estimate (1.57) follows on using part (5c) of Theorem 1.1. This also establishes that $\partial^m_{\delta_l} C_n \to \partial^m_c C_{c,*}$ in $L^\infty((\delta_l \mathbb{Z})^3)$. ■

From the definition (1.53) we get for $x,\ y \in (\delta_n \mathbb{Z})^3$

$$C_n(x-y) = \Gamma_n(x-y) + L^{-2d_s} C_{n+1}\Big(\frac{x-y}{L}\Big) \tag{1.58}$$

This relation generates by iteration the multiscale expansion of the covariance $C = C_0$ in part (1) of Theorem 1.1.

We consider the finite sequence of compatible lattices $\{(\delta_n \mathbb{Z})^3\}$ for $0 \leq n \leq N$. The considerations in Section 1.1 for fields in $\mathbb{Z}^3$ remain valid for every lattice $(\delta_n \mathbb{Z})^3$ provided for the expectations we replace the covariance C by C_n. Let the fields $\varphi, \psi, \bar{\psi}$ be defined in $(\delta_N \mathbb{Z})^3$. These fields restrict to the coarser lattices $(\delta_n \mathbb{Z})^3$ for every n with $0 \leq n \leq N$.

We introduce a parameter ε with $0 < \varepsilon \leq 1$ and define

$$\alpha = \frac{3+\varepsilon}{2} \tag{1.59}$$

Let $x \in (\delta_n \mathbb{Z})^3$. For every $n \leq N-1$ we define the scale transformation S_L by

$$S_L \Phi(x) = \Phi_{L^{-1}}(x) = L^{-d_s} \Phi\Big(\frac{x}{L}\Big) \tag{1.60}$$

where

$$d_s = \frac{(3-\alpha)}{2} = \frac{3-\varepsilon}{4} \tag{1.61}$$

is the *dimension* of the field Φ. The fields $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ are thus assigned the same dimension d_s and the same transformation law (1.60). Note that the scale transformed fields now live in $(\delta_{n+1}\mathbb{Z})^3$.
The C_n and Γ_n are positive definite and therefore qualify as covariances of Gaussian measures. For $x, y \in (\delta_n\mathbb{Z})^3$ we define the scale transformation of the covariance C_{n+1} by

$$S_L C_{n+1}(x-y) = L^{-2d_s} C_{n+1}(\frac{x-y}{L}) \tag{1.62}$$

.

which permits us to write (1.58) as

$$C_n(x-y) = \Gamma_n(x-y) + S_L C_{n+1}(x-y) \tag{1.63}$$

Let $\Lambda \subset (\delta_n\mathbb{Z})^3$ be a bounded subset. Then (1.63) implies upon using (1.25) (with C replaced by C_n) that

$$\int d\mu_{C_n}(\Phi) F_\Lambda(\Phi) = \int d\mu_{S_L C_{n+1}}(\Phi) \int d\mu_{\Gamma_n}(\xi) F_\Lambda(\xi + \Phi) \tag{1.64}$$

Let $L = 2^p$ with integer $p \geq 2$ and let $\Lambda_m = [-\frac{L^m}{2}, \frac{L^m}{2})^3 \subset \mathbb{R}^3$ be a half open cube in $\mathbb{R}^3$ centered at the origin. We denote by

$$\Lambda_{m,n} = \Lambda_m \cap (\delta_n\mathbb{Z})^3 \tag{1.65}$$

the induced cube of side length L^m in $(\delta_n\mathbb{Z})^3$ centered at the origin. Let $F_0(\Lambda_{N,0}\,, \Phi)$ be functional of Φ and $(\bar{\Phi})$ belonging to $\Omega^0(\Lambda_{N,0})$. By virtue of (1.64) we have for $n = 0$

$$\int d\mu_{C_0}(\Phi) F_0(\Lambda_{N,0}\,, \Phi) = \int d\mu_{C_1}(\Phi) F_1(\Lambda_{N-1,1}\,, \Phi) \tag{1.66}$$

where

$$F_1(\Lambda_{N-1,1}\,, \Phi) = S_L \mu_{\Gamma_0} * F_0(\Lambda_{N,0}\,, \Phi) = \int d\mu_{\Gamma_0}(\xi) F_0(\Lambda_{N,0}\,, \xi + S_L\Phi) \tag{1.67}$$

The final scale transformation takes us to a finer lattice as well as scaling down the size of the volume.
The iteration of (1.67) using (1.66) gives after n steps

$$\int d\mu_{C_0} F_0(\Lambda_{N,0}\,, \Phi) = \int d\mu_{C_n} F_n(\Lambda_{N-n,n}\,, \Phi) \tag{1.68}$$

where

$$F_n(\Lambda_{N-n,n}\,, \Phi) = S_L \mu_{\Gamma_{n-1}} * F_{n-1}(\Lambda_{N-n+1,n-1}\,, \Phi) \tag{1.69}$$

(1.69) defines for $N > 0$ fixed and $1 \leq n \leq N-1$ a sequence of maps

$$T_{N-n,n} \,:\, \Omega^0(\Lambda_{N-n+1,n-1}) \to \Omega^0(\Lambda_{N-n,n}) \tag{1.70}$$

any member of which we call a *renormalization group (RG) transformation*. The map is clearly not autonomous. The first index refers to the volume which has gotten reduced because of the rescaling. The second index refers to the lattice spacing which has gotten finer because of the rescaling. In the following we will apply the RG transformation iteratively to the (interaction) density $\mathcal{Z}_0(\Lambda_{N,0}\,, \Phi)$ of the measure $d\mu_{N,0}(\Phi)$ defined in (1.35) generating thereby the sequence $\mathcal{Z}_n(\Lambda_{N-n,n}\,, \Phi)$ for $0 \leq n \leq N-1$. After $N-1$

steps we arrive at $\mathcal{Z}_{N-1}(\Lambda_{1,N-1}\ ,\Phi)$ where $\Lambda_{1,N-1}$ is the cube of edge length L in $(\delta_{N-1}\mathbb{Z})^3$ centered at the origin. The fundamental goal in this paper is to control this sequence of transformations when N is indefinitely large in the infinite volume limit (as explained at the end of section 3).

1.3. **Polymer gas representation**.

In order to analyze the RG evolution we will write the densities $\mathcal{Z}_n$ in a *polymer gas* representation whose form is preserved under RG transformations.

Polymers :
To this end pave $\mathbb{R}^3$ with *half-open cubes* Δ *of edge length 1 and with vertices in* $\mathbb{Z}^3$ *called unit cubes or* 1-*cubes.* $\mathbb{R}^3$ is then the union of disjoint half-open unit cubes. A unit cube $\Delta\subset\mathbb{R}^3$ is of the form $\Delta=[m_1,m_1+1)\times[m_2,m_2+1)\times[m_3,m_3+1)$ where $m_j\in\mathbb{Z}$. *We say two unit cubes from the paving are connected if their closures share at least a vertex in common.* If they are not connected (i.e. their closures are disjoint) we say that they are *strictly disjoint.* A continuum (connected) 1- polymer X is a (connected) union of a finite subset of unit cubes chosen from the paving and is thus half-open. Henceforth unless otherwise mentioned a polymer is connected by default.
We will measure distances in $\mathbb{R}^3$ and in all embedded lattices in the norm

$$|x-y|=\max_{1\le j\le 3}|x_j-y_j| \tag{1.71}$$

If Δ_1 and Δ_2 are two unit cubes from the paving then the distance between them is

$$d(\Delta_1,\Delta_2)=\inf_{x\in\Delta_1,\,y\in\Delta_2}|x-y| \tag{1.72}$$

If Δ_1 and Δ_2 are strictly disjoint than $d(\Delta_1,\Delta_2)\ge 1$.
Let δ be any member of the sequence δ_n. Recall that $\delta_n=L^{-n}$ where L is a dyadic integer. Define the *unit block or* 1-*block* in $(\delta\mathbb{Z})^3$ by

$$\Delta_\delta=\Delta\cap(\delta\mathbb{Z})^3 \tag{1.73}$$

and the *lattice* 1-*polymer* X_δ by

$$X_\delta=X\cap(\delta\mathbb{Z})^3 \tag{1.74}$$

where X is a continuum 1-polymer. Note that as point sets $X_{\delta_n}\subset X_{\delta_{n+1}}$. We say a lattice polymer X_{δ_n} is *convex* if the continuum polymer X of which it is a restriction is a a convex set.
We denote by $|X_\delta|$ the volume of X_δ measured in accordance with (1.23). The 1-blocks are lattice restrictions of half-open continuum unit cubes defined above. Therefore $|\Delta_\delta|=1$ and

$$|X_\delta|=\sharp\{\Delta_\delta\ :\ \Delta_\delta\subset X_\delta\} \tag{1.75}$$

the total number of 1-blocks in X_δ. This is equal to $|X|$ the total number of 1-cubes in X by our construction. As a consequence we have $|X_{\delta_n}|=|X_{\delta_{n+1}}|$.

We say two 1-*blocks in* X_δ *are connected if the continuum* 1-*cubes of which they are the lattice restrictions are connected (see above).* If the 1-blocks are not connected we say that they are *strictly disjoint.* The distance between two strictly disjoint 1-blocks is ≥ 1. The lattice (connected) polymer X_δ is a (connected) union of a finite subset of disjoint 1-blocks Δ_δ. Let X_δ and Y_δ be each a connected polymer. *We say that* X_δ,Y_δ *are strictly disjoint if they are mutually disconnected i.e. if every* 1-*block from* X_δ *is strictly disjoint from every* 1-*block from* Y_δ . Then distance$(X_\delta,Y_\delta)\ge 1$.
Given a polymer X_δ we denote by ∂X_δ the set of its boundary points. Given an integer $n\ge 1$ we define the *n-collar* of X_δ, denoted $\partial_n X_\delta$ by

$$\partial_n X_\delta = \{y \notin X_\delta\ : |x-y| \le n\delta,\ \text{some}\ x \in X_\delta\} \tag{1.76}$$

where $|\cdot|$ is the distance function inherited from $\mathbb{R}^3$. We define

$$\tilde{X}_\delta^{(n)} = X_\delta \cup \partial_n X_\delta \tag{1.77}$$

Let $f : (\delta\mathbb{Z})^3 \to \mathbb{C}$. We define the forward lattice partial derivative $\partial_{\delta,\mu}$ and the backward lattice derivative $\partial_{\delta,-\mu}$ by

$$\partial_{\delta,\mu} f(x) = \delta^{-1}(f(x+\delta e_\mu) - f(x)) \tag{1.78}$$

$$\partial_{\delta,-\mu} f(x) = \partial^*_{\delta,\mu} f(x) = \delta^{-1}(f(x-\delta e_\mu) - f(x)) \tag{1.79}$$

where e_1, e_2, e_3 is the standard basis of unit vectors which provides the orientation of $\mathbb{R}^3$ and thus of all the embedded lattices we will encounter. $\partial^*_{\delta,\mu}$ is the $L^2((\delta\mathbb{Z})^3)$ adjoint of $\partial_{\delta,\mu}$.

Polymer activity :
A *polymer activity* $K(X_\delta, \Phi) = \tilde{K}(X_\delta, \varphi, \psi)$, where it is henceforth understood that it also depends on $\bar{\varphi}, \bar{\psi}$, is a map $X_\delta, \Phi \to \Omega^0_{\tilde{X}_\delta^{(1)}}$ where the fields Φ depend only on the points of $\tilde{X}_\delta^{(1)}$.

Remark : The reason for attaching a 1-collar is that lattice derivatives on fermionic fields at the boundary points of X_δ which occur in the representation for $\tilde{K}$ (given below) take us into $\partial_1 X_\delta$.

The polymer activities of this paper are of degree 0*, gauge invariant and supersymmetric, and invariant under translations and rotations which leave the lattice invariant. In addition they satisfy the condition* $K(X_\delta, \Phi) = K(X_\delta, -\Phi)$ *together with the support condition :* $K(X_\delta, \Phi) = 0$ *if* X *is not connected. Furthermore* $K(X_\delta, 0) = 0$.

We write the generic density $\mathcal{Z}(\Lambda_\delta)(\Phi)$ in the form

$$\mathcal{Z}(\Lambda_\delta) = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-V(X_\delta^{(c)})} \sum_{X_{\delta,1},..,X_{\delta,N}} \prod_{j=1}^{N} K(X_{\delta,j}) \tag{1.80}$$

where the connected polymers $X_{\delta,j} \subset \Lambda_\delta$ are strictly disjoint, $X_\delta = \cup_1^N X_{\delta,j}$, $X_\delta^{(c)} = \Lambda_\delta \backslash X_\delta$ and $V(Y_\delta) = V(Y_\delta, \Phi, C, g, \mu)$ is given by(1.33) with parameters g, μ and integration over Y_δ with measure dx defined as the counting measure in $(\delta\mathbb{Z})^3$ times δ^3. The Wick ordering covariance $C = C_n$ (see(1.53)) if $\delta = \delta_n$. We have suppressed the field dependence in (1.80). Initially the activities K vanish but they do arise under RG transformations. The representation (1.80) remains stable under RG transformations as we will see in Section 3.

Polymer activities $K(X_\delta, \Phi) = \tilde{K}(X_\delta, \varphi, \psi)$ will be expanded out as a (finite) series in the fermionic fields $\psi, \bar{\psi}$ and their (lattice) derivatives with coefficients which are functionals of the bosonic fields φ. This leads to the representation

$$K(X_\delta, \Phi) = \tilde{K}(X_\delta, \varphi, \psi) = \sum_{p \ge 0} \sum_{\mathbf{l},\mathbf{a}} \int_{X_\delta^{2p}} d\mathbf{x}\ \tilde{K}_{2p}^{\mathbf{l},\mathbf{a}}(X_\delta, \varphi, \mathbf{x})\ \partial_\delta^{\mathbf{l}} \prod_{j=1}^{2p} \psi_{a_j}(x_j) \tag{1.81}$$

with the condition

$$K(X_\delta, 0) = 0 \tag{1.82}$$

where:

$2p$ is the number of Grassmann fields.
A field ψ_a, $a \in \{-1,1\}$, is a two component field defined by $\psi_1 = \psi$, $\psi_{-1} = \bar{\psi}$. The sum on $\mathbf{a} = (a_1, ...a_{2p})$ with $a_j \in \{-1,1\}$ is constrained by $\sum_j a_j = 0$.
$\mathbf{l} = (l_1, ..., l_{2p})$, $l_i \in \{0,1,2,3\}$, $\partial_\delta^{\mathbf{l}} = \partial_{\delta,l_1}.......\partial_{\delta,l_{2p}}$ with the convention $\partial_{\delta,l_j} = 1$ if $l_j = 0$ and otherwise if $l_j = \mu \neq 0$ then ∂_{δ,l_j} is the forward lattice derivative in $(\delta\mathbb{Z})^3$ defined in (1.78).
$\mathbf{x} = (x_1, ..., x_{2p})$, and $d\mathbf{x} = \prod_{i=1}^{2p} dx_i$ where dx_i is the counting measure multiplied by δ^3 on $(\delta\mathbb{Z})^3$.
The kernel $\tilde{K}_{2p}^{\mathbf{l},\mathbf{a}}$ is gauge invariant as is the Grassmann monomial of degree 0.

Remarks :

1.It is ofcourse possible to write the polymer representation such that the fields in the Grassman monomial is always in a specified order (thus specifying the a_j). The more general form that we have introduced turns out to be convenient for later purposes.
2.We will see that the form (1.81) is preserved by renormalization group transformations. *The RG transformations are gauge invariant, preserve supersymmetry by virtue of (1.44), as well as the vanishing condition (1.82) by virtue of Lemma 1.1.*
3.To simplify notation we will often write K instead of $\tilde{K}$ if there is no risk of confusion.

2. REGULATORS, DERIVATIVES AND NORMS

In this section we will introduce Banach spaces of polymer activities. These are lattice analogues of the continuum constructions in [BDH-est, BMS, A] albeit with changes because of the presence of Grassman variable. The Banach space norms that we will presently introduce measure differentiability properties of the activities with respect to fields φ, ψ, as well as the behaviour with respect to large fields $\partial\varphi$ and large sets. The behaviour for large φ itself will be controlled with the help of lattice Sobolev inequalities and the local potential.

Regulators

Let $\partial_{\delta,\mu}$ and $\partial_{\delta,-\mu}$ be respectively the forward and backward lattice derivatives in $(\delta\mathbb{Z})^3$ along the unit vector e_μ defined in (1.78) and (1.79). Here as before δ is any member of the sequence $\{\delta_n\}$ where $\delta_n = L^{-n}$ and $L = 2^p$ with integer $p \geq 2$. Define

$$\partial^j_{\mu_1,\mu_2,..,\mu_j} = \partial_{\delta,\mu_1}\partial_{\delta,\mu_2}...\partial_{\delta,\mu_j}$$

Let X be a connected polymer in $\mathbb{R}^3$ and $X_\delta = X \cap (\delta\mathbb{Z})^3$. Let $\tilde{X}_\delta^{(n)} = X_\delta \cup \partial_n X_\delta$ as defined earlier ((1.76) and (1.77)). Let $\varphi : \tilde{X}_\delta^{(5)} \to \mathbb{C}$. We define a norm $\|\cdot\|_{X_\delta,1,5}$:

$$\|\varphi\|^2_{X_\delta,1,5} = \sum_{j=1}^{5} \frac{1}{2^j} \sum_{\mu_j \in S, \forall j} \int_{X_\delta} dx\, |\partial^j_{\mu_1,\mu_2,..,\mu_j}\varphi(x)|^2 \tag{2.1}$$

where $S = \{1,-1,2,-2,3,-3\}$. This is a lattice Sobolev norm of the type introduced in Section 5, page 421 of [BGM] but now without the L^2 piece.
We define now the *large field regulator*

$$G_\kappa \; : \; X_\delta \times \mathcal{F}_{\tilde{X}_\delta^{(5)}} \to \mathbb{R} \tag{2.2}$$

where $\mathcal{F}_{\tilde{X}_\delta^{(5)}}$ is the algebra of $\mathbf{C}$ valued functions on $\tilde{X}_\delta^{(5)}$ by

$$G_\kappa(X_\delta, \varphi) = e^{\kappa\|\varphi\|^2_{X_\delta,1,5}} \tag{2.3}$$

G_κ satisfies the multiplicative property: If X_δ, Y_δ are disjoint sets then

$$G_\kappa(X_\delta \cup Y_\delta, \varphi) = G_\kappa(X_\delta, \varphi) G_\kappa(Y_\delta, \varphi) \tag{2.4}$$

G_κ^{-1} will be a weight function in polymer activity norms. The norm $\|\cdot\|_{X_\delta,1,5}$ can be used in lattice Sobolev inequalities, in conjunction with the stability provided by the local potential, to control φ and its first two lattice derivatives pointwise. The parameter $\kappa = \kappa(L) > 0$ is chosen so that for all $L \geq 2$ the large field regulator satisfies the stability property given in the following Lemma :

Lemma 2.1 (stability property) : There exists a constant $\kappa_0 = \kappa_0(L) > 0$ independent of n such that for all κ with $0 < \kappa \leq \kappa_0$

$$\int d\mu_{\Gamma_n}(\zeta)\, G_\kappa(X_{\delta_n}, \zeta + \varphi) \leq 2^{|X_\delta|} G_{2\kappa}(X_{\delta_n}, \varphi) \tag{2.5}$$

where $|X_{\delta_n}|$ is the number of unit blocks in X_{δ_n}.
Proof : (2.5) is proved in exactly the same way as in the proof of the stability property of the continuum large field regulator in Lemma 3 of [BDH-est]. The proof uses a flow equation for the measure convolution with interpolated covariance which remains true for the lattice. Another ingredient is Young's convolution inequality for functions which is also true on the lattice. In the cited proof we replace the covariance C by Γ_n and continuum derivatives by lattice derivatives. From the proof of Lemma 3 of [BDH-est] we see that two conditions have to be satisfied by κ_0, namely : 1) $\kappa_0 \max_{2\leq m\leq 10} \|\partial_{\delta_n}^m \Gamma_n\|_{L^\infty((\delta_n \mathbf{Z})^3)}$ is sufficiently small and 2) $\kappa_0 \|\Gamma_n\|_{L^1((\delta_n \mathbf{Z})^3)}$ is sufficiently small. Parts (5a) of Theorem 1.1 shows that that 1) and 2) above can be assured by a κ_0 independent of n. From (5a) we have

$$\kappa_0 \max_{2\leq m\leq 10} \|\partial_{\delta_n}^m \Gamma_n\|_{L^\infty((\delta_n \mathbf{Z})^3)} \leq \kappa_0 c_L$$

and from (5a) and the finite range property

$$\kappa_0 \|\Gamma_n\|_{L^1((\delta_n \mathbf{Z})^3)} \leq \kappa_0 L^3 c'_L$$

It is sufficient to choose κ_0 so that the right hand side of both inequalities are sufficiently small. This is achieved independent of n. ∎

Now hold L sufficiently large and recall that $\alpha < 2$. Then we get after rescaling

$$\int d\mu_\Gamma(\zeta)\, G_\kappa(X_{\delta_n}, \zeta + S_L\varphi) \leq 2^{|X_{\delta_n}|} G_\kappa(L^{-1} X_{\delta_{n+1}}, \varphi) \tag{2.6}$$

because from the scaling property of the fields φ , see (1.60), (1.61)we have

$$\|S_L\varphi\|^2_{X_{\delta_n},1,5} \leq L^{-(2-\alpha)} \|\varphi\|^2_{L^{-1}X_{\delta_{n+1}},1,5} \tag{2.7}$$

Next we introduce a *large set regulator*. Let X_δ be a connected 1-polymer in $(\delta\mathbb{Z})^3$. This is a connected union of 1-blocks defined earlier. We define

$$\mathcal{A}_p(X_\delta) = 2^{p|X_\delta|} L^{(D+2)|X_\delta|} \tag{2.8}$$

where for us the dimension of space $D = 3$, and p is an integer.

Small sets : We call a connected polymer X_δ *small* if $|X_\delta| \leq 2^D$. A connected polymer which is not small is called *large*.

L-polymers and L-closure : Pave $\mathbb{R}^3$ with half-open cubes of edge length L with vertices in $\mathbb{Z}^3$, called L-cubes. Each L-cube is a union of 1-cubes. Take the restriction of these L-cubes to $(\delta\mathbb{Z})^3$ and call the latter cubes

L-blocks. Each L-block is a union of 1-blocks. The paving of $\mathbb{R}^3$ by L-cubes induces a paving of $(\delta\mathbb{Z})^3$ by L-blocks. An L-polymer is a union of L- blocks. *We define the L-closure of the 1-polymer X_δ, denoted $\bar{X_\delta}^{(L)}$, as the L-polymer given by the smallest union of L-blocks containing X_δ.* The notions of connectedness and strict disjointness carry over from the case of 1 blocks and 1-polymers. Thus we say two L-blocks from the L-paving are connected if the closures of the corresponding continuum L-cubes are connected (i.e. share at least a vertex in common). If they are not connected we say that they are strictly disjoint. Strictly disjoint L-blocks are separated by a distance $\geq L$. A connected L-polymer is a connected union of L-blocks. If two connected L-polymers are not connected to each other we say they are strictly disjoint. Strictly disjoint L-polymers are separated by a distance $\geq L$.

Lemma 2.2 : Fix any integer $p \geq 0$ and let L be sufficiently large depending on p. Then for any connected 1-polymer X_δ

$$\mathcal{A}(L^{-1}\bar{X_\delta}^{(L)}) \leq c_p \mathcal{A}_{-p}(X_\delta) \tag{2.9}$$

For X_δ a *large* connected 1-polymer,

$$\mathcal{A}(L^{-1}\bar{X_\delta}^{(L)}) \leq c_p L^{-D-1} \mathcal{A}_{-p}(X_\delta) \tag{2.10}$$

Here $c_p = O(1)$ is a constant independent of L and δ.
Remark : This is the lattice version of Lemma 1 of [BDH-est]. It is purely geometrical and proved in the same way.

Field derivatives and norms: The polymer activities in question are degree 0 gauge invariant supersyymetric functionals of the complex bosonic fields $\varphi, \bar{\varphi}$ and the fermionic fields $\psi, \bar{\psi}$. Lattice field derivatives are partial derivatives with respect to the fields at different points of the lattice. The fermionic derivative is an antiderivation. We see from the definition of the polymer activities in (1.81) that the fermionic field derivatives are taken at the origin of the fermionic field space. However in order to measure the size of the lattice field derivatives it turns out to be useful to generalize the notion of field derivatives as directional derivatives (directional in field space). For the bosonic coefficient this is the lattice transcription of that given in [BDH]. For the fermionic part there is no clear sense of direction and the definition we give below suggested to us by David Brydges turns out to be particularly useful.

Let $X_\delta \subset (\delta\mathbb{Z})^3$ be a connected polymer. Let f_j for $j = 1, ..., m$ be $\mathbf{C}$ valued functions on $\tilde{X}_\delta^{(2)} = X_\delta \cup \partial_2 X_\delta$. Let g_{2p} be a $\mathbf{C}$ valued function on $(\tilde{X}_\delta^{(2)})^{2p}$. A polymer activity $K(X_\delta, \Phi)$ has the representation (1.81). We consider it as a function of $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ denoted as $K(X_\delta, \varphi, \psi)$ where we have suppressed the dependence on $\bar{\varphi}, \bar{\psi}$. We define:

$$D^{2p,m}K(X_\delta, \varphi, 0; f^{\times m}, g_{2p}) = \sum_{\mathbf{l},\mathbf{a}} \int_{X_\delta^{2p}} d\mathbf{x} D^m K_{2p}^{\mathbf{l},\mathbf{a}}(X_\delta, \varphi, \mathbf{x}; f^{\times m}) \partial_\delta^{\mathbf{l}} g_{2p}(x_1, ...x_{2p}) \tag{2.11}$$

where $f^{\times m} = (f_1, .., f_m)$ and

$$D^m K_{2p}^{\mathbf{l},\mathbf{a}}(X_\delta, \varphi, \mathbf{x}; f^{\times m}) = \partial_{s_1} ....\partial_{s_m} K_{2p}^{\mathbf{l},\mathbf{a}}(X_\delta, \varphi, \mathbf{x}; \varphi + s_1 f_1, ..., \varphi + s_m f_m)|_{s_1 = .. = s_m = 0} \tag{2.12}$$

and the s_j are real parameters.
We endow the linear space of $\mathbf{C}$ valued functions f as above with the norm

$$\|f\|_{C^2(X_\delta)} = \sup_{1 \leq \mu, \nu \leq 3} (\|\partial_{\delta,\mu} f\|_{L^\infty(X_\delta)}, \|\partial_{\delta,\mu}\partial_{\delta,\nu} f\|_{L^\infty(X_\delta)}) \tag{2.13}$$

and call the resulting normed space $C^2(X_\delta)$. $\partial_{\delta,\mu}$ is the lattice derivative defined in (1.78).
Let ∂_{δ,μ_j} acting on $g_{2p}(x_1, ..., x_{2p})$ denote the lattice derivative with respect to x_j in the direction e_{μ_j}. We endow the linear space of $\mathbf{C}$ valued functions g_{2n} with the norm

$$\|g_{2p}\|_{C^2(X_\delta^{2p})} = \sup_{\substack{1\le\mu_j,\mu_k,\le 3\\ 1\le j,k\le 2p}} (\|\partial_{\delta,\mu_j} g_{2p}\|_{L^\infty(X_\delta^{2p})}, \|\partial_{\delta,\mu_j}\partial_{\delta,\mu_k} g_{2p}\|_{L^\infty(X_\delta^{2p})}) \tag{2.14}$$

and call the resulting normed space $C^2(X_\delta^{2p})$. The above norms always exist for lattice functions since X_δ is a finite set.
(2.11) then defines a $\mathbf{C}$ valued multilinear functional on $C^2(X_\delta)^m \times C^2(X_\delta^{2p})$ whose norm is defined to be

$$\|D^{2p,m}K(X_\delta,\varphi,0)\| = \sup_{\substack{\|f_j\|_{C^2(X_\delta)}\le 1\\ \|g_{2p}\|_{C^2(X_\delta^{2p})}\le 1\\ \forall 1\le j\le m}} \left|D^{2p,m}K(X_\delta,\varphi,0;f^{\times m},g_{2p})\right| \tag{2.15}$$

The space of $\mathbf{C}$ valued multilinear functionals defined in (2.11) which are bounded in the norm (2.15) is complete and thus a Banach space. The completeness follows on using the completeness of the number field $\mathbf{C}$ by a standard argument.

Let $\mathbf{h} = (h_F, h_B)$ where $h_F, h_B > 0$ are strictly positive real numbers. We define the following set of norms. The $\mathbf{h}$ norm is defined by

$$\|K(X_\delta,\varphi,0)\|_{\mathbf{h}} = \sum_{p=0}^{\infty}\sum_{m=0}^{m_0} h_F^{2p}\frac{h_B^m}{m!}\|D^{2p,m}K(X_\delta,\varphi,0)\| \tag{2.16}$$

In addition we define a *kernel* norm with $\mathbf{h}_* = (h_F, h_{B*})$

$$|K(X_\delta)|_{\mathbf{h}_*} = \sum_{p=0}^{\infty}\sum_{m=0}^{m_0} h_F^{2p}\frac{h_{B*}^m}{m!}\|D^{2p,m}K(X_\delta,0,0)\| \tag{2.17}$$

$\mathbf{h}, \mathbf{h}_*$ will be chosen later in Section 5. We now define the $\mathbf{h}, G_\kappa$ norm by

$$\|K(X_\delta)\|_{\mathbf{h},G_\kappa} = \sup_{\varphi\in\mathcal{F}_{\tilde{X}_\delta^{(5)}}} \|K(X_\delta,\varphi,0)\|_{\mathbf{h}} G_\kappa^{-1}(X_\delta,\varphi) \tag{2.18}$$

Let $\mathcal{A}(X_\delta)$ be the large set regulator defined earlier. We then have our final set of norms

$$\|K\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta} = \sup_{\Delta_\delta}\sum_{X_\delta\supset\Delta_\delta} \|(K(X_\delta)\|_{\mathbf{h},G_\kappa}\mathcal{A}(X_\delta) \tag{2.19}$$

where $\Delta_\delta = \Delta\cap(\delta\mathbb{Z})^3$ and Δ is a unit cube in $\mathbb{R}^3$ as defined earlier, and

$$|K|_{\mathbf{h}_*,\mathcal{A},\delta} = \sup_{\Delta_\delta}\sum_{X_\delta\supset\Delta_\delta} |K(X_\delta)|_{\mathbf{h}_*}\mathcal{A}(X_\delta) \tag{2.20}$$

The index δ in our final norms (2.19) and (2.20) indicate that the large set norm is being taken over polymers in $(\delta\mathbb{Z})^3$. Under each of above norms we have Banach spaces. Moreover it is easy to verify that the *multiplicative (Banach algebra) property* holds for the polymer activities $\tilde{K}(X_\delta)$ under the $\mathbf{h}$-norm (2.16), the kernel norm (2.17), and, for activities supported on disjoint polymers , under the $\mathbf{h}, \mathbf{G}_\kappa$ norm. The multiplicative property plays a very important role in the estimates in the rest of the paper. We therefore state it as *Proposition 2.3* below and supply a proof.

Proposition 2.3 : Let $X_{\delta,1}$, $X_{\delta,2}$ denote two connected polymers, not necessarily disjoint, and $K_1(X_{\delta,1},\varphi,\psi)$, $K_2(X_{\delta,2},\varphi,\psi)$ the corresponding polymer activities. Define a new polymer activity

$$\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,\psi) = K_1(X_{\delta,1},\varphi,\psi)K_2(X_{\delta,2},\varphi,\psi)$$

Then

$$\|\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0)\|_{\mathbf{h}}\leq\|K_1(X_{\delta,1},\varphi,0)\|_{\mathbf{h}}\|K_2(X_{\delta,2},\varphi,0)\|_{\mathbf{h}}$$

If $X_{\delta,1}$ and $X_{\delta,2}$ are disjoint

$$\|\mathbf{K}(X_{\delta,1}\cup X_{\delta,2})\|_{\mathbf{h},\mathbf{G}_\kappa}\leq\|K_1(X_{\delta,1})\|_{\mathbf{h},\mathbf{G}_\kappa}\|K_2(X_{\delta,2})\|_{\mathbf{h},\mathbf{G}_\kappa}$$

Proof

Let $\tilde{X}^{(2)}_{\delta,j}=X_{\delta,j}\cup\partial_2X_{\delta,j}$. Let $f_j,\ j=1,...,m$ be functions on $\tilde{X}^{(2)}_{\delta,1}\cup\tilde{X}^{(2)}_{\delta,2}$ and g_p a function on $(\tilde{X}^{(2)}_{\delta,1}\cup\tilde{X}^{(2)}_{\delta,2})^p$. By definition

$$\|D^{p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0)\|=\sup_{\substack{\|f_j\|_{C^2(X_{\delta,1}\cup X_{\delta,2})}\leq1\\ \|g_p\|_{C^2((X_{\delta,1}\cup X_{\delta,2})^p)}\leq1}}|D^{p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;f^{\times m},g_p)|$$

We denote now $\mathbf{x}_{j,p_j}=(x_{j,1},...,x_{j,p_j}),\ f^{\times m}=(f_1,...,f_m),\ f^{\times|I|}=\{f_i\}_{i\in I}$ where $I\subset\{1,2,.....,p\}$. Using the representation (1.81) for the polymer activities and the the definition (2.11) for the derivatives we get

$$D^{p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;f^{\times m},g_p)=\sum_{p_1+p_2=p}\sum_{m_1+m_2=m}\sum_{\substack{I\cup J=\{1,\dots,m\}\\ I\cap J=\emptyset\\ |I|=m_1,|J|=m_2}}\sum_{\mathbf{l_1},\mathbf{l_2},\mathbf{a_1},\mathbf{a_2}}$$

$$\int_{X^{p_1}_{\delta,1}\times X^{p_2}_{\delta,2}}d\mathbf{x}_{1,p_1}d\mathbf{x}_{2,p_2}D^{m_1}K^{\mathbf{l}_1,a_1}_{1,p_1}(X_{\delta,1},\varphi,\mathbf{x}_{1,p_1};f^{\times|I|})D^{m_2}K^{\mathbf{l_2},\mathbf{a_2}}_{2,p_2}(X_{\delta,2},\varphi,\mathbf{x}_{2,p_2};f^{\times|J|})\partial^{\mathbf{l}_1}_\delta\partial^{\mathbf{l}_2}_\delta g_p(\mathbf{x}_{1,p_1},\mathbf{x}_{2,p_2})$$

$$=\sum_{p_1+p_2=p}\sum_{m_1+m_2=m}\sum_{\substack{I\cup J=\{1,\dots,m\}\\ I\cap J=\emptyset\\ |I|=m_1,|J|=m_2}}\sum_{\mathbf{l}_1,\mathbf{a}_1}$$

$$\int_{X^{p_1}_{\delta,1}}d\mathbf{x}_{1,p_1}D^{m_1}K^{\mathbf{l}_1,\mathbf{a}_1}_{1,p_1}(X_{\delta,1},\varphi,\mathbf{x}_{1,p_1};f^{\times|I|})\partial^{\mathbf{l}_1}_\delta D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0;g_p(\mathbf{x}_{1,p_1},\cdot),f^{\times|J|})\tag{2.21}$$

Define

$$F_{p_1}(\mathbf{x}_{1,p_1})=D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0;g_p(\mathbf{x}_{1,p_1},\cdot),f^{\times|J|})\tag{2.22}$$

as a function on $(\tilde{X}^{(2)}_{\delta,1})^{p_1}$, (recall the definition of g_p). We have suppressed the dependence of F_{p_1} on the other variables. Then for $0\leq k\leq2$

$$\partial^k_\delta F_{p_1}(\mathbf{x}_{1,p_1})=D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0;\partial^k_\delta g_p(\mathbf{x}_{1,p_1},\cdot),f^{\times|J|})$$

where ∂^k_δ is the lattice partial derivative of degree k with respect to $(x_{1,1},...,x_{1,p_1})$ in multi-index notation, whence

$$|\partial^k_\delta F_{p_1}(\mathbf{x}_{1,p_1})|=\|D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0)\|\prod_{j\in J}\|f_j\|_{C^2(X_{\delta,2})}\|\partial^k_\delta g_p(\mathbf{x}_{1,p_1},\cdot)\|_{C^2(X^{p_2}_{\delta,2})}$$

Therefore

$$\|F_{p_1}\|_{C^2(X^{p_1}_{\delta,1})}\leq\|D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0)\|\prod_{j\in J}\|f_j\|_{C^2(X_{\delta,2})}\|g_p\|_{C^2(X^{p_2}_{\delta,2}\times X^{p_1}_{\delta,1})}$$

Since $X^{p_2}_{\delta,2}\times X^{p_1}_{\delta,1}\subset(X_{\delta,1}\cup X_{\delta,2})^p$ we have

$$\|g_p\|_{C^2(X^{p_2}_{\delta,2}\times X^{p_1}_{\delta,1})}\leq\|g_p\|_{C^2((X_{\delta,1}\cup X_{\delta,2})^p)}$$

and

$$\|f_j\|_{C^2(X_{\delta,2})}\leq\|f_j\|_{C^2(X_{\delta,1}\cup X_{\delta,2})}$$

Hence

$$\|F_{p_1}\|_{C^2(X_{\delta,1}^{p_1})} \le \|D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0)\| \prod_{j\in J} \|f_j\|_{C^2(X_{\delta,1}\cup X_{\delta,2})}\|g_p\|_{C^2((X_{\delta,1}\cup X_{\delta,2})^p)} \tag{2.23}$$

Now from (2.21), and the definition (2.22) of F_{p_1}

$$D^{p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;f^{\times m},g_p) = \sum_{\substack{p_1+p_2=p\\ m_1+m_2=m}} \sum_{\substack{I\cup J=\{1,\dots,m\}\\ I\cap J=\emptyset\\ |I|=m_1,|J|=m_2}} D^{p_1,m_1}K_1(X_{\delta,1},\varphi,0;f^{\times|I|},F_{p_1})$$

Therefore

$$|D^{p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;f^{\times m},g_p)| \le \sum_{\substack{p_1+p_2=p\\ m_1+m_2=m}} \sum_{\substack{I\cup J=\{1,\dots,m\}\\ I\cap J=\emptyset\\ |I|=m_1,|J|=m_2}} |D^{p_1,m_1}K_1(X_{\delta,1},\varphi,0;f^{\times|I|},F_{p_1})| \le$$

$$\le \sum_{\substack{p_1+p_2=p\\ m_1+m_2=m}} \sum_{\substack{I\cup J=\{1,\dots,m\}\\ I\cap J=\emptyset\\ |I|=m_1,|J|=m_2}} \|D^{p_1,m_1}K_1(X_{\delta,1},\varphi,0)\| \prod_{i\in I}\|f_i\|_{C^2(X_{\delta,1})}\|F_{p_1}\|_{C^2(X_{\delta,1}^{p_1})} \tag{2.24}$$

From (2.23), (2.24) and observing $\|f_i\|_{C^2(X_{\delta,1})} \le \|f_i\|_{C^2(X_{\delta,1}\cup X_{\delta,2})}$ we get

$$|D^{p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;f^{\times m},g_p)| \le \sum_{\substack{p_1+p_2=p\\ m_1+m_2=m}} \sum_{\substack{I\cup J=\{1,\dots,m\}\\ I\cap J=\emptyset\\ |I|=m_1,|J|=m_2}} \|D^{p_1,m_1}K_1(X_{\delta,1},\varphi,0)\|\|D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0)\|\cdot$$

$$\cdot \prod_{i=1}^{m}\|f_i\|_{C^2(X_{\delta,1}\cup X_{\delta,2})}\|g_p\|_{C^2((X_{\delta,1}\cup X_{\delta,2})^p)}$$

We perform the sum over I and J using

$$\sum_{\substack{I\cup J=\{1,\dots,m\}\\ I\cap J=\emptyset\\ |I|=m_1,|J|=m_2}} 1 = \frac{m!}{m_1!m_2!}$$

Therefore

$$\|D^{p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0)\| \le \sum_{\substack{p_1+p_2=p\\ m_1+m_2=m}} \frac{m!}{m_1!m_2!}\|D^{p_1,m_1}K_1(X_{\delta,1},\varphi,0)\|\|D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0)\|$$

Multiply both sides of the previous inequality by $h_B^m/m!$ and h_F^p. Sum over integers m, $0\le m\le m_0$, and over all integers $p\ge 0$ to obtain

$$\|\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0)\|_{\mathbf{h}} \le \|K_1(X_{\delta,1},\varphi,0)\|_{\mathbf{h}}\|K_2(X_{\delta,2},\varphi,0)\|_{\mathbf{h}}$$

This proves the first inequality of Proposition 2.1. The second inequality follows from the first because for union of disjoint sets

$$G_\kappa(X_\delta\cup Y_\delta,\varphi) = G_\kappa(X_\delta,\varphi)G_\kappa(Y_\delta,\varphi)$$

■

3. THE RG MAP

In this section we describe the RG map applied to the generic density in the polymer representation given in (1.80). This is a lattice transcription of the continuum RG map described in [BMS], (see also [M]). This goes in several steps. First we must perform the fluctuation integration and rescaling (see (1.69))

$$\mathcal{Z}'(L^{-1}\Lambda_{L^{-1}\delta},\varphi) = S_L\mu_\Gamma * \mathcal{Z}(\Lambda_\delta,\Phi) \tag{3.1}$$

where $\Lambda_\delta \subset (\delta\mathbb{Z})^3$ is the volume arrived at after a certain number of previous RG steps and Γ is the fluctuation covariance for the next step. Γ is one of the covariances Γ_n of Theorem 1.1 and has the finite range property stated in that theorem. Thus after n RG steps (see (1.65)-(1.70)) $\delta = \delta_n$, $\Gamma = \Gamma_n$, $\Lambda_\delta = \Lambda_{N-n,n}$ and $L^{-1}\Lambda_{L^{-1}\delta} = \Lambda_{N-n-1,n+1}$.

The polymer representation (1.80) for $\mathcal{Z}(\Lambda_\delta)$ is parametrized by the *coordinates* (V,K) on the scale δ where V is a local functional (potential):

$$V(X_\delta) = \sum_{\Delta_\delta\subset X_\delta} V(\Delta_\delta) \tag{3.2}$$

Let $\tilde V(X_\delta,\Phi)$ be an arbitrary local supersymmetric functional with $\tilde V(X_\delta,0) = 0$. We will see that the polymer representation is preserved under the RG transformation (3.1) with new coordinates $\tilde V_L, \mathcal{F}(K)$ on the next scale $L^{-1}\delta$. $\mathcal{F}$ depends on $\tilde V$. The finite range property of Γ leads to a simple description of this map :

$$V \to \tilde V_L, \quad \tilde V_L(\Delta_{L^{-1}\delta}) = S_L\tilde V(L\Delta_\delta)$$

$$K \to \mathcal{F}(K):\ \mathcal{F}(K)(X_{L^{-1}\delta},\Phi) = \int d\mu_\Gamma(\xi)\mathcal{B}K(LX_\delta,\xi,S_L\Phi) \tag{3.3}$$

where $\mathcal{B}K$ is a $\tilde V$ dependent nonlinear functional of K to be presently described. We call this map the *fluctuation map*.

We can take advantage of the arbitrariness of the local potential $\tilde V$ in the above map so as to remove the expanding (relevant) parts F in the polymer activity $\mathcal{F}(K)$ and compensate by a change $\tilde V_L(F)$ in the local potential $\tilde V_L$ in such a way that the evolved density $\mathcal{Z}'(L^{-1}\Lambda_{L^{-1}\delta})$ on the left hand side of (3.1) remains unchanged. This operation gives rise to the *extraction* map, [BDH-est]

$$\tilde V_L \to V'(F) = \tilde V_L - \tilde V_L(F), \quad \mathcal{F}(K) \to K' = \mathcal{E}(\mathcal{F}(K),F) \tag{3.4}$$

where the image is on the same scale $L^{-1}\delta$. $V'(F)$ and the nonlinear map $\mathcal{E}$ have simple expressions which are lattice transcriptions of those given in [BDH-est]. The composition of the fluctation map (3.3) and the extraction map (3.4) gives the *RG map*

$$f: \quad f(V,K) = (f_V(V,K), f_K(V,K))$$

where

$$f_V : V \to \tilde V_L \to V'(F)$$

$$f_K : K \to \mathcal{F}(K) \to K' = \mathcal{E}(\mathcal{F}(K),F) \tag{3.5}$$

The operation of extraction leads in particular to a discrete flow of the coupling constants in V on scale $L^{-1}\delta$ provided we choose $F, \tilde V_L(F)$ appropriately. The expanding functionals will be gathered in the local potential $V'(F)$ whereas the polymer activity $\mathcal{E}(\mathcal{F}(K),F)$ will be a contracting (irrelevant) error term.

3.1 The Fluctuation Map

We now construct the map (3.3) starting from (3.1) with the density in the polymer representation (1.80). In performing the fluctuation integration

$$\mu_\Gamma * \mathcal{Z}(\Lambda_\delta, \Phi) = \int d\mu_\Gamma(\xi) \sum_N \frac{1}{N!} e^{-V(X_\delta^{(c)}, \Phi+\xi)} \sum_{X_{\delta,1},..,X_{\delta,N}} \prod_{j=1}^N K(X_{\delta,j}, \Phi+\xi) \tag{3.6}$$

we will exploit the independence of $\xi(x)$ and $\xi(y)$ when $|x-y| \geq L$. To this end we construct an L-paving of Λ_δ and the L-closure of $\bar{X_\delta}^{(L)}$ of a connected 1-polymer X_δ as in the paragraph preceding Lemma 2.1. The 1-polymers will be combined into larger connected $L-$polymers which by definition are connected unions of L-blocks, (for the relevant definitions intervening here and in the following see the paragraph on *L-polymers and L-closures* before Lemma 2.1). The combination is performed in such a way that the new polymers are associated to independent functionals of ξ. This is the lattice adaptation of Section 3.1 of [BMS].

Define the polymer activity P, supported on unit blocks, by:

$$P(\Delta_\delta, \xi, \Phi) = e^{-V(\Delta_\delta, \xi+\Phi)} - e^{-\tilde{V}(\Delta_\delta, \Phi)} \tag{3.7}$$

with $\tilde{V}$, to be chosen. $\tilde{V}(\Delta_\delta, \Phi)$ is required to satisfy $\tilde{V}(\Delta_\delta, 0) = 0$. *In the following* V, K *has field argument* $\xi + \Phi$ *whereas* $\tilde{V}$ *depends only on* Φ . The dependence of P on ξ, Φ is as defined above.
$X_\delta^c = \Lambda_\delta \setminus \cup_{j=1}^N X_{\delta,j}$ is a union of disjoint 1-blocks Δ_δ. Therefore

$$e^{-V(X_\delta^c)} = \prod\nolimits_{\Delta_\delta \subset X_\delta^c} [e^{-\tilde{V}(\Delta_\delta)} + P(\Delta_\delta)]$$

Expand the product and insert the expansion into the integrand of in (3.6) which gives

$$\text{integrand} = \sum_N \frac{1}{N!} \sum\nolimits_{(X_{\delta,j}),(\Delta_{\delta,i})} e^{-\tilde{V}(X_{\delta,0})} \prod\nolimits_{j=1}^N K(X_{\delta,j}) \prod\nolimits_{i=1}^M P(\Delta_{\delta,i}) \tag{3.8}$$

where $X_{\delta,0} = \Lambda_\delta \setminus (\cup X_{\delta,j}) \cup (\cup \Delta_{\delta,i})$. Let Y_δ be the $L-$closure of $(\cup X_{\delta,j}) \cup (\cup \Delta_{\delta,i})$ and let $Y_{\delta,1}, \ldots, Y_{\delta,P}$ be the connected components of Y_δ. These are $L-$ polymers. Let f be the function that maps $\pi := (X_{\delta,j}), (\Delta_{\delta,i})$ into $\{Y_{\delta,1}, \ldots, Y_{\delta,P}\}$. Now we perform the sum over $(X_{\delta,j}), (\Delta_{\delta,i})$ in (3.8) by summing over $\pi \in f^{-1}(\{Y_{\delta,1}, \ldots, Y_{\delta,P}\})$ and then $\{Y_{\delta,1}, \ldots, Y_{\delta,P}\}$. The result is:

$$\text{integrand} = \sum_N \frac{1}{N!} \sum\nolimits_{(Y_{\delta,j})} e^{-\tilde{V}(Y_\delta^c)} \prod\nolimits_{j=1}^N \mathcal{B}K(Y_{\delta,j}) \tag{3.9}$$

where the sum is over strictly disjoint connected L polymers and

$$(\mathcal{B}K)(Y_\delta) = \sum_{N+M\geq 1} \frac{1}{N!M!} \sum\nolimits_{(X_{\delta,j}),(\Delta_{\delta,i}) \to \{Y\}} e^{-\tilde{V}(X_{\delta,0})} \prod\nolimits_{j=1}^N K(X_{\delta,j}) \prod\nolimits_{i=1}^M P(\Delta_{\delta,i}) \tag{3.10}$$

where $X_{\delta,0} = Y \setminus (\cup X_{\delta,j}) \cup (\cup \Delta_{\delta,i})$ and the $\to$ is the map f. In other words the sum in (3.10) is over distinct $\Delta_{\delta,i}$ and disjoint 1-polymers $X_{\delta,j}$ such that their L-closure is the connected L-polymer Y_δ.
We now perform the fluctuation integration of (3.9) over ξ followed by rescaling. Since $\tilde{V}(Y_\delta^c)$ is independent of ξ the ξ integration factors through and acts on the product of polymer activities $\prod_j (\mathcal{B}K)(Y_{\delta,j})$. A polymer activity $(\mathcal{B}K)(Y_{\delta,j})$ belongs to $\Omega^0(\tilde{Y}_{\delta,j}^{(1)})$ where $\tilde{Y}_{\delta,j}^{(1)}$ is $Y_{\delta,j}$ to which a 1-collar has been attached (see (1.76), (1.77)). The $Y_{\delta,j}$ are strictly disjoint connected L-polymers and thus necessarily separated from each other by a distance $\geq L$. The collar attached L-polymers $\tilde{Y}_{\delta,j}^{(1)}$ are therefore separated from each other by a distance $\geq L-2$. The fluctation covariance Γ has finite range $L/2$ and for L sufficiently large $L-2 \geq L/2$. Therefore the fluctuation integration over the product of polymer activities factorizes. We now follow this up by applying the rescaling operator to both sides. This has the effect of bringing us back to 1-polymers but on the scale $L^{-1}\delta$. Therefore we obtain

$$S_L\mu_\Gamma * \mathcal{Z}(\Lambda_\delta,\Phi)=\sum_N \frac{1}{N!}\sum\nolimits_{(X_{L^{-1}\delta,j})} e^{-\tilde{V}_L(X^c_{L^{-1}\delta},\Phi)}\prod\nolimits_{j=1}^N \int d\mu_\Gamma(\xi)\mathcal{B}K(LX_{\delta,j},S_L\Phi,\xi) \tag{3.11}$$

where $X_{L^{-1}\delta,j}=X_j\cap(L^{-1}\delta\mathbb{Z})^3$ (as well as $X_{\delta,j}=X_j\cap(\delta\mathbb{Z})^3$) are disjoint 1-polymers, $X^c_{L^{-1}\delta}=L^{-1}\Lambda_{L^{-1}\delta}\setminus\cup_j X_{L^{-1}\delta,j}$. and $\tilde{V}_L(\Delta_{L^{-1}\delta})=S_L\tilde{V}(L\Delta_\delta)$. This gives the fluctuation map (3.3) : $V\to\tilde{V}_L$, $K\to\mathcal{F}(K)$ with $\mathcal{B}K$ defined as above. At the same time we have shown that the polymer representaion is stable with respect to the RG transformation.
Consider

$$\mathcal{F}(K)(X_{L^{-1}\delta},\Phi)=\int d\mu_\Gamma(\xi)\mathcal{B}K(LX_\delta,\xi,S_L\Phi)$$

By construction $\mathcal{B}K$ is supersymmetric. Therefore since the supersymmetry operator commutes with the measure $\mathcal{F}(K)$ is also supersymmetric. Now since $P(\Delta_\delta,\xi,0)$ and $K(X_\delta,\xi)$ vanish for $\xi=0$ (the latter by hypothesis, see (1.82)) it follows that $\mathcal{B}K(LX_\delta,\xi,0)$ also vanishes for $\xi=0$. Therefore by Lemma 1.1

$$\mathcal{F}(K)(X_{L^{-1}\delta},0)=\int d\mu_\Gamma(\xi)\mathcal{B}K(LX_\delta,\xi,0)=\mathcal{B}K(LX_\delta,0,0)=0 \tag{3.12}$$

Thus the condition (1.82) is satisfied by the new polymer activities. This implies in particular that no field independent relevant parts are generated by the fluctuation integration as a consequence of supersymmetry.

3.2 Extraction

Let $\delta'=L^{-1}\delta$ and let $\Lambda'=L^{-1}\Lambda_{\delta'}$. The fluctuation map gave us $\tilde{V}_L,\mathcal{F}(K)$ as the coordinates of the evolved density $\mathcal{Z}(\Lambda')$. We want to change the local potential $\tilde{V}_L$ and the polymer activity $\mathcal{F}(K)$ simultaneously such that $\mathcal{Z}(\Lambda')$ remains invariant. To this end let $P(\Phi(x))$ be a *local* polynomial, which means that it is a polynomial in $\Phi(x)$ for $x\in\Lambda'$. Furthemore we require that $P(0)=0$, i.e. P has no field independent part. Given $\Delta_{\delta'}$ a unit block in Λ' we consider a change in $\tilde{V}_L(\Delta_{\delta'})$ of the form

$$\tilde{V}_L(F)(\Delta_{\delta'})=\sum_P\int_{\Delta_{\delta'}} dx\ \alpha_P(x)P(\Phi(x)) \tag{3.13}$$

where the sum ranges over finitely many local polynomials and, for each such P, $\alpha_P(x)$ has the form

$$\alpha_P(x)=\sum_{X_{\delta'}\supset x}\alpha_P(X_{\delta'},x) \tag{3.14}$$

such that $\alpha_P(X_{\delta'},x)=0$ if $x\notin X_{\delta'}$, $\alpha_P(X_{\delta'},x)=0$ if $X_{\delta'}\not\subset\Lambda'$ and $\alpha_P(X_{\delta'},x)=0$ if $X_{\delta'}$ is not a small set (see definition after (2.8)). The corresponding change in $\mathcal{F}(K)$ is given in terms of the *relevant parts*

$$F(X_{\delta'},\Phi)=\sum_P\int_{X_{\delta'}} dx\ \alpha_P(X_{\delta'},x)P(\Phi(x)),\quad F(X_{\delta'},\Delta_{\delta'})=\sum_P\int_{\Delta_{\delta'}} dx\ \alpha_P(X_{\delta'},x)P(\Phi(x)) \tag{3.15}$$

Note that $F(X_{\delta'},0)=0$.

Extraction Map:

Theorem 3.1 (after Brydges,Dimock and Hurd, [BDH-est]) : Given F, $\tilde{V}_L(F)$ as above there exists a polymer activity which is a non-linear functional $\mathcal{E}(\mathcal{F}(K),F)$ of $\mathcal{F}(K)$, F such that

$$V_L\to V'(F)=\tilde{V}_L-\tilde{V}_L(F),\qquad \mathcal{F}(K)\to K'=\mathcal{E}(\mathcal{F}(K),F) \tag{3.16}$$

preserves the polymer representation for the density $\mathcal{Z}(\Lambda')$ with new coordinates coordinates $(V',\ K')$ satisfing $V'(F)(\Delta_{\delta'},0)=K'(X_{\delta'},0)=0$. Let $\mathcal{E}_1$ denote the linearization of $\mathcal{E}$. Then the linearization of the extraction map is given by

$$\mathcal{E}_1(\mathcal{F}(K),F)=\mathcal{F}(K)-Fe^{-\tilde V_L},\qquad V'(F)=\tilde V_L-\tilde V_L(F) \tag{3.17}$$

Assume $\tilde V_L$ to be stable with respect to perturbation F: There are positive numbers $f(X)$ such that

$$\|e^{-\tilde V_L(\Delta_{\delta'})-\sum_{X_{\delta'}\supset\Delta_{\delta'}} z(X)F(X_{\delta'},\Delta_{\delta'})}\|_{\mathbf{h},G_\kappa}\le 2 \tag{3.18}$$

for all complex numbers $z(X_{\delta'})$ with $|z(X_{\delta'})|f(X_{\delta'})\le 2$. Then $\mathcal{E}(\mathcal{F}(K),F)$ is norm analytic and satisfies the bounds

$$\|\mathcal{E}(\mathcal{F}(K),F)\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta'}\le O(1)(\|\mathcal{F}(K)\|_{\mathbf{h},G_\kappa,\mathcal{A}_1,\delta'}+\|f\|_{\mathcal{A}_3,\delta'} \tag{3.19}$$

$$|\mathcal{E}(\mathcal{F}(K),F)|_{\mathbf{h},\mathcal{A},\delta'}\le O(1)(\|\mathcal{F}(K)\|_{\mathbf{h},\mathcal{A}_1,\delta'}+\|f\|_{\mathcal{A}_3,\delta'} \tag{3.20}$$

Proof : This is a restatement of Theorem 5 in Sec. 4.2 of [BDH-est] with the substitution $(\tilde V_L,\mathcal{F}(K))$ for (V,K), adapted to the lattice. The proof of Theorem 5 exploited Lemmas 10, 11, 12, 13 the last of them providing the extraction formula in equation (121), page 781 of [BDH-est]. In [BDH-est] the continuum unit blocks are open. Our unit lattice blocks are lattice restrictions of continuum half open cubes. The complement of a union of (strictly) disjoint connected lattice polymers in Λ' is given by a disjoint union of lattice unit blocks. Overlap connectedness is replaced by connectedness. With this in mind the proofs of Lemmas 10, 11, 12, 13 go through intact on the lattice providing the extraction map above. The estimates in Theorem 5 on the norms of $\mathcal{E}(K,F)$ together with norm analyticity remain valid on the lattice. ∎.

Remark : The stability property (3.18) is proved in Section 5 once we have chosen $\tilde V$ appropriately. The estimate (3.19) on the extraction operator $\mathcal{E}$ plays an essential role and is exploited in Section 5.

Formal infinite volume limit: We reestablish the notations leading to (1.70). Choose $\delta=\delta_n$, $\Gamma=\Gamma_n$, $\delta'=L^{-1}\delta=\delta_{n+1}$. $\Lambda_\delta=\Lambda_{N-n,n}$, $\Lambda_{\delta'}=\Lambda_{N-n-1,n+1}$ and $\mathcal{F}=\mathcal{F}_{n+1}$ in (3.3). The RG transformation $T_{N-n-1,n+1}$ of (1.70) induces the RG map $f_{N-n-1,n+1}(V,K)$ of (3.5) for the coordinates of the density $\mathcal{Z}_{n-1}(\Lambda_{N-n,n})$ in the polymer representation. $\alpha_P(X_{\delta_{n+1}},x)$ in (3.14) is chosen later in Section 4. This choice will be local, in the sense that it is determined by $\tilde V_L(\Delta_{\delta_{n+1}})$, $\Delta_{\delta_{n+1}}\subset X_{\delta_{n+1}}$ and by $\mathcal{F}_{n+1}(K)(X_{\delta_{n+1}})$. Lemma 13 and equation (112) of [BDH-est] imply that $\mathcal{E}(\mathcal{F}_{n+1}(K),F))(X_{\delta_{n+1}})$ also is local: it is determined by $\mathcal{F}_{n+1}(K)(Y_{\delta_{n+1}})$, $Y_{\delta_{n+1}}\subset X_{\delta_{n+1}}$ and $\tilde V_L(\Delta_{\delta_{n+1}})$, $\Delta_{\delta_{n+1}}\subset X_{\tilde{\delta}_{n+1}}$, where $X_{\tilde{\delta}_{n+1}}$ is a neighbourhood of $X_{\delta_{n+1}}$, namely the union of $X_{\delta_{n+1}}$ with all small sets that intersect X_{δ_n}. Therefore the K component of the map $f_{N-n-1,n+1}$ representing the action of the $n+1$th step of RG, namely $f_{N-n-1,n+1,K}(K,V)(X_{\delta_{n+1}},\Phi)$ is independent of N for all N large enough so that $\Lambda_{N-n-1,n+1}$ contains $X_{\tilde{\delta}_{n+1}}$. Thus $\lim_{N\to\infty}f_{N-n-1,n+1,K}(K,V)(X_{\delta_{n+1}},\Phi)$ exists pointwise in $X_{\delta_{n+1}}$. In this paper we are studying the action of this pointwise infinite volume limit called the *formal infinite volume limit*.

3.3 Appendix:

We introduce some notations which will be used later. We define for $x,y\in(L^{-1}\delta\mathbb{Z})^3$ and any covariance u on $(\delta\mathbb{Z})^3$

$$u_L(x-y)=S_{L^{-1}}u(x-y)=L^{2d_s}u(L(x-y)) \tag{3.21}$$

Since the fluctuation covariance Γ defined on $(\delta\mathbb{Z})^3$ has finite range $L/2$ we have that Γ_L defined on $(L^{-1}\delta\mathbb{Z})^3$ has finite range $1/2$. We recall from section 1.3 that a polymer X_δ is defined by $X_\delta=X\cap(\delta\mathbb{Z})^3$ where X is a continuum polymer. We define the rescaling of polymer activities by

$$S_LK(X_{L^{-1}\delta},\Phi)=K_L(X_{L^{-1}\delta},\Phi)=K(LX_\delta,S_L\Phi) \tag{3.22}$$

We write the fluctuation integration of the polymer activity $K(X_\delta,\Phi,\xi)$ with respect to μ_Γ as

$$K^{\sharp}(X_{\delta},\Phi)=\int d\mu_{\Gamma}(\xi)K(X_{\delta},\Phi,\xi) \tag{3.23}$$

.

We write the fluctuation integration of the polymer activity $K(X_{L^{-1}\delta},\Phi,\xi)$ with respect to μ_{Γ_L} as

$$K^{\natural}(X_{L^{-1}\delta},\Phi)=\int d\mu_{\Gamma_L}(\xi)K(X_{L^{-1}\delta},\Phi,\xi) \tag{3.24}$$

.

We define

$$\mathcal{S}_L=S_L\mathcal{B} \tag{3.25}$$

.

With these notations it is easy to see that the fluctuation map can be written as

$$\mathcal{F}(K)(X_{L^{-1}\delta})=S_L(\mathcal{B}K)^{\sharp}(LX_{\delta})=(\mathcal{S}_LK)^{\natural}K(X_{L^{-1}\delta}) \tag{3.26}$$

4. THE RENORMALIZATION GROUP MAP APPLIED

In this section we specify the RG map of Section 3 by making choices for the local potential $\tilde{V}$, and relevant parts F. $\tilde{V}$ is chosen via first order perturbation theory. F is chosen so as to remove the expanding part of the fluctuation map. This is the extraction step. This will be done in second order perturbation theory as well as in the error term. We will follow closely the strategy in Section 4 of [BMS]. We will use the notations established in the Appendix to section 3.3, (3.21)-(3.26). We take $\delta=\delta_n$, $\Gamma=\Gamma_n$. Recall that, see (1.59), $\alpha=\frac{3+\varepsilon}{2}$ where we take $0<\varepsilon<1$. The field scaling dimension is $d_s=\frac{3-\varepsilon}{4}$, see (1.60),(1.61).

We assume that starting from the unit lattice where only the local potential (1.33) is present n steps of the renormalization group map has been carried out. This produces a new local potential

$$V_n(\Delta_{\delta_n},\Phi)=V(\Delta_{\delta_n},\Phi,C_n,g_n,\mu_n)=g_n\int_{\Delta_{\delta_n}}dx:(\Phi\bar{\Phi})^2(x):_{C_n}+\mu_n\int_{\Delta_{\delta_n}}dx(\Phi\bar{\Phi})(x) \tag{4.1}$$

together with a polymer activity K_n supported on connected polymers in $(\delta_n\mathbb{Z})^3$. Note that $(\Phi\bar{\Phi})(x)=:\Phi\bar{\Phi}:_{C_n}(x):$ by virtue of (1.29). We write K_n in the form

$$K_n=Q_ne^{-V_n}+R_n \tag{4.2}$$

where Q_n is a polymer activity which is given by second order perturbation theory in g assuming that μ is $O(g^2)$. Q_n is specified below. R_n is the remainder which is formally of $O(g^3)$. Q_n,R_n vanish when $\Phi=0$ by hypothesis. The RG map will preserve this property.

In order to carry through the next step of the RG map as described in Section 3 we must also specify $\tilde{V}(\Delta_{\delta_n},\Phi)$. We define

$$\tilde{V}_n(\Delta_{\delta_n},\Phi)=V(\Delta_{\delta_n},\Phi,C_{n+1,L^{-1}},g_n,\mu_n)=g_n\int_{\Delta_{\delta_n}}dx:(\Phi\bar{\Phi})^2(x):_{C_{n+1,L^{-1}}}+\mu_n\int_{\Delta_{\delta_n}}d^3x(\Phi\bar{\Phi})(x) \tag{4.3}$$

where we have used the notation $C_{n+1,L^{-1}}=S_LC_{n+1}$. Here and in what follows we adopt the notations introduced in the Appendix of Section 3.3. Thus $\sharp$ denotes fluctuation integration with respect to the measure $d\mu_{\Gamma_n}(\xi)$ and $\natural$ denotes fluctuation integration with respect to the measure $d\mu_{\Gamma_{n,L}}(\xi)$, with $\Gamma_{n,L}=S_{L^{-1}}\Gamma_n$. We recall (see section 3.1) that when we perform the fluctuation integration the fluctuation field ξ enters V through $V(\Delta_{\delta_n},\Phi+\xi)$ but $\tilde{V}$ will remain independent of ξ.

We now define Q_n: Q_n is supported on connected polymers X_{δ_n} such that $|X_{\delta_n}| \le 2$. We assume it can be written in the form

$$Q_n(X_{\delta_n}, \Phi) = Q(X_{\delta_n}, \Phi; C_n, \mathbf{w}_n, g_n) = g_n^2 \sum_{j=1}^{3} Q^{(j,j)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-j)}) \tag{4.4}$$

where $\mathbf{w}_n = (w_n^{(1)}, w_n^{(2)}, w_n^{(3)})$ is a triple of integral kernels to be obtained inductively and

$$\hat{X}_{\delta_n} = \begin{cases} \Delta_{\delta_n} \times \Delta_{\delta_n} & \text{if } X_{\delta_n} = \Delta_{\delta_n} \\ (\Delta_{\delta_n,1} \times \Delta_{\delta_n,2}) \cup (\Delta_{\delta_n,2} \times \Delta_{\delta_n,1}) & \text{if } X_{\delta_n} = \Delta_{\delta_n,1} \cup \Delta_{\delta_n,2} \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

$$\begin{aligned} Q^{(1,1)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(3)}) &= -2 \int_{\hat{X}_{\delta_n}} dxdy (\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y)) w_n^{(3)}(x-y) \\ Q^{(2,2)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(2)}) &= - \int_{\hat{X}_{\delta_n}} dxdy \big[:(\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y))(\Phi(x) + \Phi(y))(\bar{\Phi}(x) + \bar{\Phi}(y)) :_{C_n} + \\ &\qquad +3 : [(\Phi\bar{\Phi})(x) - (\Phi\bar{\Phi})(y)]^2 :_{C_n} \big] w_n^{(2)}(x-y) \\ Q^{(3,3)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(1)}) &= 4 \int_{\hat{X}_{\delta_n}} dxdy \ : \Phi(x)\bar{\Phi}(x)\Phi(x)\bar{\Phi}(y)\Phi(y)\bar{\Phi}(y) :_{C_n} w_n^{(1)}(x-y) \end{aligned} \tag{4.6}$$

Note that in the expression for $Q^{(1,1)}$ is equal to its C_n Wick ordered form because of (1.29).
Next we define the second order approximation to the RG map. Let p_n be the activity supported on unit blocks defined by

$$p_n(\Delta_{\delta_n}, \xi, \Phi) = V_n(\Delta_{\delta_n}, \xi + \Phi) - \tilde{V}_n(\Delta_{\delta_n}, \Phi) = p_{n,g} + p_{n,\mu} \tag{4.7}$$

where

$$\begin{aligned} p_{n,g} = g \int_{\Delta_{\delta_n}} dx \Big(&: (\xi\bar{\xi})^2 :_{\Gamma_n} (x) + 2 \sum_{\alpha} \big[\Phi_\alpha(x) : \bar{\xi}_\alpha (\xi\bar{\xi}) :_{\Gamma_n} (x) + : (\xi\bar{\xi})\xi_\alpha :_{\Gamma_n} (x) \Phi_a(x) \big] + \\ &+2(\Phi\bar{\Phi})(x)(\xi\bar{\xi})(x) + (\Phi\bar{\xi})^2(x) + (\xi\bar{\Phi})^2(x) + 2 \sum_{\alpha,\beta} : (\xi_\alpha\bar{\xi}_\beta) :_{\Gamma_n} (x) : (\bar{\Phi}_\alpha \Phi_\beta)(x) :_{C_{n+1,L^{-1}}} + \\ &+2 \sum_{\alpha} \big[\xi_\alpha(x) : \bar{\Phi}_\alpha (\Phi\bar{\Phi}) :_{C_{n+1,L^{-1}}} (x) + : (\Phi\bar{\Phi})\Phi_\alpha :_{C_{n+1,L^{-1}}} (x) \bar{\xi}_\alpha(x) \big] \Big) \end{aligned}$$

$$p_{n,\mu} = \mu \int_{\Delta_{\delta_n}} dx \left((\xi\bar{\Phi})(x) + (\Phi\bar{\xi})(x) + (\xi\bar{\xi})(x) \right) \tag{4.8}$$

In (4.8) we have used a component notation. Thus $\Phi_1 = \varphi$, $\Phi_2 = \psi$. Similarly for the fluctuation field ξ, $\xi_1 = \zeta$, $\xi_2 = \eta$. ζ is bosonic (degree 0)and η fermionic (degree 1). In deriving (4.8) from (4.7) have used $C_n = \Gamma_n + C_{n+1,L^{-1}}$ (see (1.58)), the independence of Φ, ξ in the sense that their components are independent and distributed with covariances $C_{n+1,L^{-1}}$, Γ_n respectively. The unordered objects in (4.8) are equal to their Wick ordered form.

We will effectuate the RG map of Section 3 following closely the strategy in [BMS]. Namely, we insert a complex parameter λ into our previous definitions in such a way that (i) at $\lambda = 1$ our λ dependent objects correspond with the previous definitions. (ii) The expansion through order λ^2 is second order perturbation theory in g_n counting $\mu_n = O(g_n^2)$. (iii) Powers of λ are determined so as to correspond with leading powers of g_n buried inside polymer activities. (iv) All functions will turn out to be norm analytic in λ and this will enable us in section 5 to profit from Cauchy estimates.
We define

$$P_n(\lambda) = e^{-\tilde{V}_n}\big(- \lambda p_{n,g} - \lambda^2 p_{n,\mu} + \frac{1}{2}\lambda^2 p_{n,g}^2\big) + \lambda^3 r_{n,1} \tag{4.9}$$

where $r_{n,1}$ is defined by the condition $P_n(\lambda = 1) = P_n = e^{-V_n} - e^{-\tilde{V}_n}$. Similarly, we define

$$K_n(\lambda) = \lambda^2 e^{-\tilde{V}_n} Q_n + \lambda^3\big([e^{-V_n} - e^{-\tilde{V}_n}]Q_n + R_n\big) \tag{4.10}$$

which, for $\lambda = 1$ coincides with $K_n = e^{-V_n} Q_n + R_n$, where Q_n is an explicit polymer activity called the second order avtivity (second order in the coupling constant) motivated by perturbation theory. R_n is the remainder. Corresponding to (3.10) we define

$$\mathcal{B}(\lambda, K_n)(Y_{\delta_n}) = \sum_{N+M\geq 1} \frac{1}{N!M!} \sum\nolimits_{(X_{\delta_n,j}),(\Delta_{\delta_n,i})\to\{Y_{\delta_n}\}} e^{-\tilde{V}_n(X_{\delta_n,0})} \prod\nolimits_{j=1}^{N} K_n(\lambda, X_{\delta_n,j}) \prod\nolimits_{i=1}^{M} P_n(\lambda, \Delta_{\delta_n,i}) \tag{4.11}$$

where $X_{\delta_n,0} = Y_{\delta_n} \setminus (\cup X_{\delta_n,j}) \cup (\cup \Delta_{\delta_n,i})$. Let $\mathcal{S}(\lambda, K_n) = S_L \mathcal{B}(\lambda, K_n)$, where S_L is the rescaling defined in the last section.

The RG map (see section 3) for K_n with parameter λ is $K_n \mapsto f_{n+1,K}(\lambda, K_n) = \mathcal{E}(\mathcal{S}(\lambda, K_n)^\natural, F_n(\lambda))$, where the superscript $\natural$ denotes integration over the fluctuation field $\xi = (\zeta, \eta)$ with the measure $d\mu_{\Gamma_{n,L}}$ and $\Gamma_{n,L}$ is the rescaled covariance $S_{L^{-1}}\Gamma_n$ as in the Appendix to Section 3. The relevant part $F_n(\lambda))$ is defined on polymers in $(\delta_{n+1}\mathbb{Z})^3$ and will be written as

$$F_n(\lambda) = \lambda^2 F_{Q_n} + \lambda^3 F_{R_n} \tag{4.12}$$

and $F_n(\lambda) = F_n$, when $\lambda = 1$.

Perturbative contribution to f_{n+1}.

Given a function $f(\lambda)$ let

$$T_\lambda f = f(0) + f'(0) + \frac{1}{2} f''(0) \tag{4.13}$$

be the Taylor expansion to second order evaluated at $\lambda = 1$. Then the second order approximation to the RG map is $f_{n+1}^{(\leq 2)} = (f_{n+1,K}^{(\leq 2)}, f_{n+1,V}^{(\leq 2)})$ with

$$f_{n+1,K}^{(\leq 2)}(K_n, V_n) = T_\lambda \mathcal{E}(\mathcal{S}(\lambda, K_n)^\natural, F_n(\lambda)) = \mathcal{E}_1(T_\lambda \mathcal{S}(\lambda, K_n)^\natural, F_{Q_n}), \qquad f_{n+1,V}^{(\leq 2)}(K_n, V_n) = V_{n+1}^{(\leq 2)} \tag{4.14}$$

where

$$V_{n+1}^{(\leq 2)} = \tilde{V}_{n,L} - \tilde{V}_{n,L}(F_{Q_n})$$

Note also that only the linearized $\mathcal{E}_1$ intervenes, because it will turn out that the nonlinear part of extraction generates terms only at order λ^3 or higher.

Proposition 4.1: There is a choice of F_Q such that the form of Q remains invariant under the RG evolution at second order. In more detail, $f_{n+1}^{(\leq 2)}(V_n, Q_n e^{-V_n}) = (V_{n+1}^{(\leq 2)}, Q_{n+1}^{(\leq 2)} e^{-\tilde{V}_{n,L}})$ where the parameters in

$$V_{n+1}^{(\leq 2)}(\Delta_{\delta_{n+1}}) = V(\Delta_{\delta_{n+1}}, C_{n+1}, g'_{n+1,(\leq 2)}, \mu'_{n+1,(\leq 2)})$$

evolved according to

$$g_{n+1,(\leq 2)} = L^\varepsilon g_n(1 - L^\varepsilon a_n g_n) \qquad \mu'_{n+1,(\leq 2)} = L^{\frac{3+\varepsilon}{2}} \mu_n - L^{2\varepsilon} b_n g_n^2 \tag{4.15}$$

The parameters in $Q^{(\leq 2)}_{n+1} = Q(C_{n+1}, \mathbf{w_{n+1}}, g_{n,L})$, *where* $g_{n,L} = L^{\varepsilon} g_n$, *evolved according to*

$$\mathbf{w_{n+1}} = \mathbf{v_{n+1}} + \mathbf{w}_{n,L} \qquad v^{(1)}_{n+1} = \Gamma_{n,L} \qquad v^{(p)}_{n+1} = (C_{n,L})^p - (C_{n+1})^p \qquad p = 2, 3 \tag{4.16}$$

The constants a_n, b_n *are given by*

$$a_n = 4 \int_{(\delta_{n+1}\mathbf{Z})^{\mathbf{3}}} dy\, v^{(2)}_{n+1}(y), \qquad b_n = 2 \int_{(\delta_{n+1}\mathbf{Z})^{\mathbf{3}}} dy\, v^{(3)}_{n+1}(y) \tag{4.17}$$

Proof: We define a polymer activity $\hat{Q}_{n,L}$ supported on connected polymers $X_{\delta_{n+1}}$ with $|X_{\delta_{n+1}}| \leq 2$ as follows: if $|X_{\delta_{n+1}}| = 1$, say $X_{\delta_{n+1}} = \Delta_{\delta_{n+1}}$, then

$$\hat{Q}_{n,L}(\Delta_{\delta_{n+1}}, \xi, \Phi) = \frac{1}{2}(p_{n,L}(\Delta_{\delta_{n+1}}, \xi, \Phi))^2$$

If $|X_{\delta_{n+1}}| = 2$ then

$$\hat{Q}_{n,L}(X_{\delta_{n+1}}, \xi, \Phi) = \frac{1}{2} \sum_{\substack{\Delta_{n+1,1}, \Delta_{n+1,2} \\ \Delta_{n+1,1} \cup \Delta_{n+1,2} = X_{\delta_{n+1}}}} p_{n,L,g}(\Delta_{n+1,1}, \xi, \Phi) p_{n,L,g}(\Delta_{n+1,2}, \xi, \Phi) \tag{4.18}$$

where $p_{n,L,g}$ is defined by replacing in (4.7) and (4.8) $(g_n, \mu_n, \Gamma_n, C_{n+1,L^{-1}})$ by $(g_{n,L}, \mu_{n,L}, \Gamma_{n,L}, C_{n+1})$ with $g_{n,L} = L^{\varepsilon} g_n$ and $\mu_{n,L} = L^{\frac{3+\varepsilon}{2}} \mu_n$.
It is easy to check that

$$T_\lambda \mathcal{S}(K_n, \lambda) = -p_{n,L} e^{-\tilde{V}_{n,L}} + (e^{-\tilde{V}_{n,L}} \hat{Q}_{n,L} + e^{-\tilde{V}_{n,L}} Q_{n,L}) \tag{4.19}$$

where

$$Q_{n,L}(X_{\delta_{n+1}}, \xi + \Phi) = Q(X_{\delta_{n+1}}, \xi + \Phi, C_{n,L}, \mathbf{w_{n,L}}, g_{n,L})$$

Using $C_{n,L} = \Gamma_{n,L} + C_{n+1}$ and remembering that $\tilde{V}$ depends only on Φ we get

$$(e^{-\tilde{V}_{n,L}} Q_{n,L})^\natural = e^{-\tilde{V}_{n,L}} Q(X_{\delta_{n+1}}, \Phi, C_{n+1}, \mathbf{w_{n,L}}, g_{n,L})$$

Therefore

$$T_\lambda \mathcal{S}(K_n, \lambda)^\natural (X_{\delta_{n+1}}, \Phi) = e^{-\tilde{V}_{n,L}} \Big(Q(X_{\delta_{n+1}}, \Phi, C_{n+1}, \mathbf{w_{n,L}}, g_{n,L}) + \tilde{Q}_n(X_{\delta_{n+1}}, \Phi, \mathbf{v}_{n+1}, C_{n+1}, g_{n,L}) \Big) \tag{4.20}$$

where $\tilde{Q}_n = \hat{Q}^\natural_{n,L}$ is given after a straightforward but lengthy computation by

$$\tilde{Q}_n(X_{\delta_{n+1}}, \Phi, \mathbf{v}_{n+1}, C_{n+1}, g_{n,L}) = g^2_{n,L} \sum_{j=1}^{3} \tilde{Q}^{(j,j)}(\hat{X}_{\delta_{n+1}}, \Phi; v^{(4-j)}_{n+1}, C_{n+1}) \tag{4.21}$$

where

$$\begin{aligned}
\tilde{Q}^{(1,1)}(\hat{X}_{\delta_{n+1}}, \Phi; u) =& 2 \int_{\hat{X}_{\delta_{n+1}}} dxdy [\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x)] u(x-y) \\
\tilde{Q}^{(2,2)}(\hat{X}_{\delta_{n+1}}, \Phi; u) =& \int_{\hat{X}_{\delta_{n+1}}} dxdy\, \Big\{ : [\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x)]^2 :_{C_{n+1}} \\
& + 4 : (\Phi(x)\bar{\Phi}(x))(\Phi(y)\bar{\Phi}(y)) :_{C_{n+1}} \Big\} u(x-y) \\
\tilde{Q}^{(3,3)}(\hat{X}_{\delta_{n+1}}, \Phi; u) =& 4 \int_{\hat{X}_{\delta_{n+1}}} dxdy : \Phi(x)\bar{\Phi}(x)\Phi(x)\bar{\Phi}(y)\Phi(y)\bar{\Phi}(y) :_{C_{n+1}} u(x-y)
\end{aligned} \tag{4.22}$$

Define

$$F_{Q_n} = \tilde{Q}(C_{n+1}, \mathbf{v}_{n+1}, g_{n,L}) - Q(C_{n+1}, \mathbf{v}_{n+1}, g_L) \tag{4.23}$$

evaluated on $X_{\delta_{n+1}}, \Phi$.
Then we have from (4.20) and (4.23)

$$\mathcal{E}_1\Big(T_\lambda \mathcal{S}(\lambda, K_n)^\natural, F_n\Big) = T_\lambda \mathcal{S}(\lambda, K_n)^\natural - F_{Q_n} e^{-\tilde{V}_{n,L}} = e^{-\tilde{V}_{n,L}} Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n,L}) \tag{4.24}$$

which shows that Q is stable under RG evolution and verifies (4.16). It remains to show that the chosen perturbative relevant part F_{Q_n} given by (4.23) is of the form (3.15) and thus suitable for extraction.
To compute the difference in (4.23) we will make use of the following *localization formulae*

$$\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x) = \Phi(x)\bar{\Phi}(x) + \Phi(y)\bar{\Phi}(y) - (\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y)) \tag{4.25}$$

$$(\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x))^2 + 4(\Phi(x)\bar{\Phi}(x))(\Phi(y)\bar{\Phi}(y)) = 4[(\Phi\bar{\Phi})^2(x) + (\Phi\bar{\Phi})^2(y)] -$$

$$-(\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y))(\Phi(x) + \Phi(y))(\bar{\Phi}(x) + \bar{\Phi}(y)) - 3[(\Phi\bar{\Phi})(x) - (\Phi\bar{\Phi})(y)]^2 \tag{4.26}$$

that are immediate to check. We get

$$F_{Q_n}(X_{\delta_{n+1}}) = 2\int_{\hat{X}_{\delta_{n+1}}} dxdy[(\Phi\bar{\Phi})(x) + (\Phi\bar{\Phi})(y)]v^{(3)}(x-y) +$$

$$+4\int_{\hat{X}_{\delta_{n+1}}} dxdy[:(\Phi\bar{\Phi})^2:_{C_{n+1}}(x) + :(\Phi\bar{\Phi})^2:_{C_{n+1}}(y)]v^{(2)}(x-y) \tag{4.27}$$

Note that due to supersymmetry there is no field independent part in F_Q.
$F_{Q_n}(X_{\delta_{n+1}})$ can be written as:

$$F_{Q_n}(X_{\delta_{n+1}}) = \sum_{\Delta_{\delta_{n+1}} \subset X_{\delta_{n+1}}} F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \tag{4.28}$$

where

$$F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = 4g_L^2 F_{Q_n}^{(2)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) + 2g_L^2 F_{Q_n}^{(1)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \tag{4.29}$$

and

$$F_{Q_n}^{(m)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx :(\Phi\bar{\Phi})^m(x):_{C_{n+1}} f_{Q_n}^{(m)}(x, X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \tag{4.30}$$

with

$$f_{Q_n}^{(m)}(x, X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \begin{cases} \int_{\Delta_{\delta_{n+1}}} dy v^{(m')}(x-y) & X_{\delta_{n+1}} = \Delta_{\delta_{n+1}} \\ \int_{\Delta'_{\delta_{n+1}}} dy v^{(m')}(x-y) & X_{\delta_{n+1}} = \Delta_{\delta_{n+1}} \cup \Delta'_{\delta_{n+1}}, \text{ connected} \end{cases} \tag{4.31}$$

and $m' = 4 - m$.

$$V(F_{Q_n},\Delta_{\delta_{n+1}}) = \sum_{X_{\delta_{n+1}}\supset\Delta_{\delta_{n+1}}} F_{Q_n}(X_{\delta_{n+1}},\Delta_{\delta_{n+1}}) = 4g_L^2 \sum_{X_{\delta_{n+1}}\supset\Delta_{\delta_{n+1}}} F^{(2)}_{Q_n}(X_{\delta_{n+1}},\Delta_{\delta_{n+1}})+$$

$$+2g_L^2 \sum_{X_{\delta_{n+1}}\supset\Delta_{\delta_{n+1}}} F^{(1)}_{Q_n}(X_{\delta_{n+1}},\Delta_{\delta_{n+1}}) \tag{4.32}$$

In the following we will use the fact that the $v^{(j)}_{n+1}(x-y),\ 1\le j\le 3$ vanish for $|x-y|\ge 1/2$. This follows from the fact that $\Gamma_{n,L}(x-y)$ appears as a factor in the expression (4.16) for $v^{(j)}_{\delta_{n+1}}(x-y)$ and $\Gamma_{n,L}$ has range $\frac{1}{2}$. Returning to (4.32) we have

$$\sum_{X_{\delta_{n+1}}\supset\Delta_{\delta_{n+1}}} F^{(m)}_{Q_n}(X_{\delta_{n+1}},\Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx :(\Phi\bar\Phi)^m(x):_{C_{n+1}} \Big[\int_{\Delta_{\delta_{n+1}}} dy v^{(m')}_{n+1}(x-y)+$$

$$+ \sum_{\substack{\Delta'_{\delta_{n+1}}\neq\Delta_{\delta_{n+1}}\\ (\Delta_{\delta_{n+1}},\Delta'_{\delta_{n+1}})\ \text{connected}}} \int_{\Delta'_{\delta_{n+1}}} dy v^{(m')}_{n+1}(x-y)\Big]$$

On the r.h.s. use $v^{(m')}_{n+1}(x-y)=0$ for $|x-y|\ge 1/2$ to extend the sum on $\Delta'_{\delta_{n+1}}$ to *all* the $\Delta'_{\delta_{n+1}}\neq\Delta_{\delta_{n+1}}$. We then get

$$\sum_{X_{\delta_{n+1}}\supset\Delta_{\delta_{n+1}}} F^{(m)}_{Q_n}(X_{\delta_{n+1}},\Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx :(\Phi\bar\Phi)^m(x):_{C_{n+1}} \int dy v^{(m')}(x-y)$$

Hence from (4.32) and above we get

$$V(F_{Q_n},\Delta_{\delta_{n+1}}) = a_n g^2_{n,L}\int_{\Delta_{\delta_{n+1}}} dx :(\Phi\bar\Phi)^2(x):_{C_{n+1}} + b_n g^2_{n,L}\int_{\Delta_{\delta_{n+1}}} dx(\Phi\bar\Phi)(x) \tag{4.33}$$

where

$$a_n = 4\int_{(\delta_{n+1}\mathbb{Z})^3} dy\ v^{(2)}_{n+1}(y) \qquad\qquad b = 2\int_{(\delta_{n+1}\mathbb{Z})^3} dy\ v^{(3)}_{n+1}(y) \tag{4.34}$$

■

Remark : a_n and b_N are well defined since the v^j_{n+1} have compact support. They are positive and their properties are discussed in Lemma 5.12 of Section 5.

The exact RG map f_{n+1} for $K_n = Q_n e^{-V_n} + R_n$.

$$K_n \mapsto K_{n+1} = f_{n+1,K}(\lambda,K_n,V_n)|_{\lambda=1} = \mathcal{E}(\mathcal{S}(\lambda,K_n)^\natural, F_n(\lambda))|_{\lambda=1} \tag{4.35}$$

induces an evolution of the remainder R_n which is studied by Taylor series around $\lambda=0$ with remainder written using the Cauchy formula:

$$f_{n+1,K}(\lambda=1) = \sum_{j=0}^{3}\frac{f^{(j)}_{n+1,K}(0)}{j!} + \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4(\lambda-1)} f_{n+1,K}(\lambda)$$

The terms $j=0,1,2$ are the second order part $f^{(\le 2)}_{n+1,K}$. In the $j=3$ term there are no terms mixing R_n with Q_n, P_n because of the λ^3 in front of R_n. Therefore it splits $\frac{f^{(3)}_K(0)}{3!} = R_{n+1,1} + R_{n+1,2}$ into the third order derivative at $R_n=0$, which we write using the Cauchy formula as

$$R_{n+1,1} \equiv R_{n+1,\mathrm{main}} = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4}\mathcal{E}\Big(\mathcal{S}(\lambda, Q_n e^{-V_n})^\natural, F_{Q_n}(\lambda)\Big) \tag{4.36}$$

and terms linear in R_n:

$$\begin{aligned} &R_{n+1,2} \equiv R_{n+1,\mathrm{linear}} = \big(\mathcal{S}_1 R_n\big)^\natural - F_{R_n} e^{-\tilde V_{L,n}} \\ &\mathcal{S}_1 R_n(Z_{\delta_{n+1}}) = \sum_{X_{\delta_{n+1}}:L^{-1}\bar X^L_{\delta_{n+1}} = Z_{\delta_{n+1}}} e^{-\tilde V_{n,L}(Z_{\delta_{n+1}}\setminus L^{-1}X_{\delta_{n+1}})} R_{n,L}(L^{-1}X_{\delta_{n+1}}) \end{aligned} \tag{4.37}$$

The remainder term in the Taylor expansion is

$$R_{n+1,3} = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4(\lambda-1)}\mathcal{E}(\mathcal{S}(\lambda, K_n)^\natural, F_n(\lambda)) \tag{4.38}$$

In Proposition 4.1 the coupling constant in $Q^{(\le 2)}_{n+1}$ is not the same as the coupling constant in $V^{(\le 2)}_{n+1}$. Furthermore, the coupling constant in $V^{(\le 2)}_{n+1}$ will further change because of the contribution from F_R. To take this into account we introduce

$$\begin{aligned} &V_{n+1}(\Delta_{\delta_{n+1}}) = V(\Delta_{\delta_{n+1}}, C_{n+1}, g_{n+1}, \mu_{n+1}) \\ &g_{n+1} = L^\varepsilon g_n(1 - L^\varepsilon a_n g_n) + \xi_n(u_n) \\ &\mu_{n+1} = L^{\frac{3+\varepsilon}{2}}\mu_n - L^{2\varepsilon} b_n g_n^2 + \rho_n(u_n) \end{aligned} \tag{4.39}$$

where $u_n = (g_n, \mu_n, R_n)$ and the remainders $\xi_n(u_n), \rho_n(u_n)$ anticipate the effects of a yet-to-be-specified F_{R_n}. Then we set

$$R_{n+1,4} = e^{-V_{n+1}}Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) - e^{-\tilde V_{n,L}}Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n,L}) \tag{4.40}$$

and define

$$\begin{aligned} &Q_{n+1} = Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) \\ &R_{n+1} = R_{n+1,\mathrm{main}} + R_{n+1,\mathrm{linear}} + R_{n+1,3} + R_{n+1,4} \\ &K_{n+1} = Q_{n+1}e^{-V_{n+1}} + R_{n+1} \end{aligned} \tag{4.41}$$

With these definitions we have obtained the RG map

$$f_{n+1,V}(V_n, K_n) = V_{n+1}, \quad f_{n+1,K}(V_n, K_n) = K_{n+1} \tag{4.42}$$

Definition of F_{R_n}

To complete the RG step we must specify the relevant part F_{R_n} from the remainder R_n. The goal is to choose F_{R_n} so that the map $R_n \to R_{n+1,\mathrm{linear}}$ will be contractive in the following sense. R_n is measured in the norm (2.19), and the kernel norm (2.20), with $\delta = \delta_n$, with a choice of $\mathbf{h}$ and $\mathbf{h}'$ (to be made in section 5). R_{n+1} is measured in the same norms but on the lattice scale δ_{n+1}. We will say that the map is contractive if the size of $R_{n+1,\mathrm{linear}}$ is less than the size of R_n.

F_{R_n} will have the form given in (3.15) with P a supersymmetric polynomial vanishing at $\Phi = 0$. The coefficients α_P will be identified via *normalization conditions* on the small set part of $R_{n+1,\mathrm{linear}}$. This means that certain derivatives with respect to $\Phi = (\varphi, \psi)$ vanish when $\Phi = 0$. That the map in question is contractive when $R_{n+1,\mathrm{linear}}$ is suitably normalized is proven in Section 5.

For given coefficients $\tilde\alpha_{n,P}(X)$, we define

$$\tilde{F}_{R_n}(X_{\delta_n},\Phi)=\sum_P\int_{X_{\delta_n}}dx\ \tilde{\alpha}_{n,P}(X_{\delta_n})P(\Phi(x),\partial_{\delta_n}\Phi(x)) \tag{4.43}$$

$$\tilde{F}_{R_n}(X_{\delta_n},\Phi)=0\ :X_{\delta_n}\text{is not a small set} \tag{4.44}$$

P runs over the *relevant* monomials which in this model are $P=\Phi\bar{\Phi},(\Phi\bar{\Phi})^2,\Phi\partial_{\delta_n,\mu}\bar{\Phi},\partial_{\delta_n,\mu}\Phi\bar{\Phi}$ with $\mu=1,2,3$ with the corresponding coefficients $\tilde{\alpha}_P(X_{\delta_n})=\tilde{\alpha}_{n,2,0}(X_{\delta_n}),\tilde{\alpha}_{n,4}(X_{\delta_n}),\tilde{\alpha}_{n,2,\bar{1}}(X_{\delta_n},\mu)$,
$\tilde{\alpha}_{n,2,1}(X_{\delta_n},\mu)$. Note that $P=1$ is not a relevant monomial in this model: R_n vanishes when $\Phi=0$ vanishes by hypothesis. Then $R^\sharp_n(X_{\delta_n},\Phi)$ vanishes when $\Phi=0$ by supersymmetry, (Lemma 1.1) so that no subtraction is necessary at $\Phi=0$.
Choose the coefficients $\tilde{\alpha}_{n,P}$ so that

$$J_n=R^\sharp_n-\tilde{F}_{R_n}e^{-\tilde{V}_n} \tag{4.45}$$

is normalized (details are given below). Note that $J(X_{\delta_n},0)=0$. We define the relevant part, supported on small sets, by

$$F_{R_n}(Z_{\delta_{n+1}},\Phi)=\sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}}\tilde{F}_{R_n,L}(L^{-1}X_{\delta_{n+1}},\Phi)=\sum_{\substack{X_{\delta_n}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_n}=Z_{\delta_n}}}\tilde{F}_{R_n}(X_{\delta_n},S_L\Phi) \tag{4.46}$$

F_{R_n} is supported on small sets by construction. From the definition of $R_{n+1,\text{linear}}$ in (4.37) we get

$$R_{n+1,\text{linear}}(Z_{\delta_{n+1}})=\sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}}e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus L^{-1}X_{\delta_{n+1}})}J_{n,L}(L^{-1}X_{\delta_{n+1}})+$$
$$+\sum_{\substack{X_{\delta_{n+1}}:\text{large sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}}e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus L^{-1}X_{\delta_{n+1}})}J_{n,L}(L^{-1}X_{\delta_{n+1}}) \tag{4.47}$$

Therefore the first sum in R_{linear} is also normalized because normalization as defined below is preserved under multiplication by smooth functionals of Φ and rescaling.
Substitution of (4.43) in (4.46) shows that F_{R_n} is of the form required in (3.15). We have

$$F_{R_n}(Z_{\delta_{n+1}},\Phi)=\sum_P\int_{Z_{\delta_{n+1}}}dx\ \alpha_{n,P}(Z_{\delta_{n+1}},x)P(\Phi(x),\partial_{\delta_{n+1}}\Phi(x)) \tag{4.48}$$

where

$$\alpha_{n,P}(Z_{\delta_{n+1}},x)=\sum_{\substack{X_{\delta_{n+1}}\ \text{small set}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}}\tilde{\alpha}_{n,P}(X_{\delta_n})L^{-[P]+3}1_{L^{-1}X_{\delta_{n+1}}}(x) \tag{4.49}$$

In (4.49) 1_X is the characteristic function of the set X. Note that $X_{\delta_{n+1}}$ fixes X_{δ_n} by restriction by our construction of polymers in section 1.3. $[P]$ is the dimension of the monomial P, ($2kd_s$ for $(\Phi\bar{\Phi})^k$ and $2d_s+1$ for $\Phi\partial_{\delta_{n+1}}\bar{\Phi}$). $\alpha_{n,P}(Z_{\delta_{n+1}},x)$ is supported on small sets $Z_{\delta_{n+1}}$ and vanishes if $x\notin Z_{\delta_{n+1}}$.
We now compute V_{F_R} following (3.13). Define

$$\alpha_{n,P}:=\sum_{Z_{\delta_{n+1}}\supset x}\alpha_{n,P}(Z_{\delta_{n+1}},x) \tag{4.50}$$

This is independent of x by translation invariance. In fact given an x it belongs uniquely to a block $\Delta_{\delta_{n+1}}$, since our blocks which are restrictions of half open continuum cubes are always disjoint (see section 1.3). The sum over all polymers containing a block $\Delta_{\delta_{n+1}}$ is independent of $\Delta_{\delta_{n+1}}$ by translation invariance.
From (4.49) and (4.50) we get

$$\alpha_{n,P} = L^{-[P]+3} \sum_{X_{\delta_{n+1}} \text{ small set}: L^{-1}X_{\delta_{n+1}} \supset x} \tilde{\alpha}_{n,P}(X_{\delta_n}) \tag{4.51}$$

$\alpha_{n,P} = 0$ for $P = \Phi\partial_{\delta_{n+1}}\bar{\Phi}$ or $\partial_{\delta_{n+1}}\Phi\bar{\Phi}$ by reflection invariance of polymer activities.
Therefore

$$\begin{aligned}\tilde{V}_L(F_{R_n}, \Delta_{\delta_{n+1}}) &= \int_{\Delta_{\delta_{n+1}}} dx \Big\{ \alpha_{n,2,0}\Phi\bar{\Phi} + \alpha_{n,4,0}(\Phi\bar{\Phi})^2 \Big\} \\ &= \int_{\Delta_{\delta_{n+1}}} dx \Big\{ \rho_n(u_n) : \Phi\bar{\Phi} :_{C_{n+1}} + \xi_n(u_n) : (\Phi\bar{\Phi})^2 :_{C_{n+1}} \Big\}\end{aligned} \tag{4.52}$$

where $u_n = (g_n, \mu_n, R_n)$ and

$$\rho_n = \alpha_{n,2,0} + 2C_{n+1}(0)\alpha_{n,4,0} \qquad\qquad \xi_n = \alpha_{4,0} \tag{4.53}$$

which are formulas for the error terms in (4.39).

Normalization conditions

By an abuse of notation let 1 denote the constant function in $C^2(X_{\delta_n})$ equal to 1. Similarly let 1^{2p} denote the constant function in $C^2(X^{2p}_{\delta_n})$ equal to 1. We will identify the $C^2(X_{\delta_n})$ function $f(x) = x_\mu$ with x_μ. Similarly we will identify $C^2(X^2_{\delta_n})$ functions $g_2(x_1,x_2) = x_{1,\mu}$, $g_2(x_1,x_2) = x_{2,\mu}$ with $x_{1,\mu}$, $x_{2,\mu}$ respectively. Suppose the polymer activity $J(X_{\delta_n},\Phi) = J(X_{\delta_n},\varphi,\psi)$ is of degree 0, gauge invariant and supersymmetric. We have the following identities:

$$D^{2,0}J(X_{\delta_n},0,0;1^2) = D^{0,2}J(X_{\delta_n},0,0,;1,1) \tag{4.54}$$

$$D^{2,0}J(X_{\delta_n},0,0;x_{1,\mu}) = D^{0,2}J(X_{\delta_n},0,0,;x_\mu,1) \tag{4.55}$$

$$D^{2,0}J(X_{\delta_n},0,0;x_{2,\mu}) = D^{0,2}J(X_{\delta_n},0,0,;1,x_\mu) \tag{4.56}$$

$$D^{2,2}J(X_{\delta_n},0,0;1,1,1^2) = 2D^{0,4}J(X_{\delta_n},0,0;1,1,1,1) \tag{4.57}$$

$$D^{4,0}J(X_{\delta_n},0,0;1^4) = 0 \tag{4.58}$$

where the field derivatives are taken according to (2.11). The identities (4.54)-(4.57) follow by expanding $J(X_{\delta_n},\Phi)$ in the fields, retaining a degree 4 supersymmetric polynomial in Φ and $\partial_{\delta_n}\Phi$ which is all that enters into the computation. Then express it in the Grassmann representation (1.81). (4.58) is trivial. Because J is of degree 0 the only term that survives for the computation of (4.58) is of the form $\int_{X^4_{\delta_n}} dx\, a(x_1,x_2,x_3,x_4)\psi(x_1)\bar{\psi}(x_2)\psi(x_3)\bar{\psi}(x_4)$ where the kernel a is antisymmetric in x_1,x_3 and in x_2,x_4. The integral vanishes if we replace the grassmann piece by 1^4. Derivatives on the grassmann fields annihilate 1^4.

We say that a degree 0, gauge invariant, supersymmetric polymer activity $J(X_{\delta_n},\Phi) = J(X_{\delta_n},\varphi,\psi)$ with $J(X_{\delta_n},0) = 0$ is *normalized* if, for all small sets X_{δ_n},

$$
\begin{aligned}
D^{2,0}J(X_{\delta_n},0,0;1^2) =& D^{0,2}J(X_{\delta_n},0,0,;1,1) = 0 \\
D^{2,0}J(X_{\delta_n},0,0;x_{1,\mu}) =& D^{2,0}J(X_{\delta_n},0,0;x_{2,\mu}) = 0 \\
D^{0,2}J(X_{\delta_n},0,0;1,x_\mu) =& D^{0,2}J(X_{\delta_n},0,0;x_\mu,1) = 0 \\
2D^{0,4}J(X_{\delta_n},0,0;1,1,1,1) =& D^{2,2}J(X_{\delta_n},0,0;1,1,1^2) = 0
\end{aligned}
\tag{4.59}
$$

Determining coefficients from (4.59)

We will apply the normalization conditions to $J = J_n$ defined in (4.45). This will determine the dependence of the error terms ξ_n, ρ_n on R_n. Lemma 5.17 will show that these terms are bounded by the kernel norm of R_n.

In doing the following computations note that $J_n(X_{\delta_n},0,0) = 0$ as shown earlier. Moreover the odd derivatives $D^{0,j}J_n(X_{\delta_n},0;f^{\times j})$, j=odd integer, vanish identically by gauge invariance. It is enough to take derivatives with respect to the bosonic fields φ because of the identities stated above, (4.54) et seq. Taking derivatives of (4.45) and remembering that $\tilde{F}_{R_n}(X_{\delta_n},0) = 0$, $\tilde{V}_n(X_{\delta_n},0) = 0$ we get

$$
\begin{aligned}
D^{0,2}J_n(X_{\delta_n},0,0;f,\bar{f}) &= D^{0,2}R_n^\sharp(X_{\delta_n},0,0;f,\bar{f}) - D^{0,2}\tilde{F}_{R_n}(X_{\delta_n},0,0;f,\bar{f}) \\
D^{0,4}J_n(X_{\delta_n},0,0;f_1,\bar{f}_1,f_2,\bar{f}_2) &= D^{0,4}R_n^\sharp(X_{\delta_n},0,0;f_1,\bar{f}_1,f_2,\bar{f}_2) + D^{0,4}\tilde{F}_{R_n}(X_{\delta_n},0,0;f_1,\bar{f}_1,f_2,\bar{f}_2) + \\
&\quad + 4D^{0,2}\tilde{F}_{R_n}(X_{\delta_n},0,0;f,\bar{f})D^{0,2}\tilde{V}_n(X_{\delta_n},0,0;f,\bar{f})
\end{aligned}
\tag{4.60}
$$

where the f are complex valued functions in $C^2(X_{\delta_n})$. A variation of φ along f implies that we vary $\bar{\varphi}$ along $\bar{f}$. Note that from (4.43)

$$
\begin{aligned}
D^{0,2}F_{R_n}(X_{\delta_n},0,0;1,1) &= |X_{\delta_n}|\tilde{\alpha}_{n,2,0}(X_{\delta_n}) \\
D^{0,2}F_{R_n}(X_{\delta_n},0,0;1,x_\mu) &= |X_{\delta_n}|\tilde{\alpha}_{n,2,\bar{1}}(X_{\delta_n},\mu) + \tilde{\alpha}_{n,2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu \\
D^{0,2}F_{R_n}(X_{\delta_n},0,0;x_\mu,1) &= |X_{\delta_n}|\tilde{\alpha}_{n,2,1}(X_{\delta_n},\mu) + \tilde{\alpha}_{n,2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu \\
D^{0,4}F_{R_n}(X_{\delta_n},0,0;1,1,1,1) &= 4|X_{\delta_n}|\tilde{\alpha}_{n,4}(X_{\delta_n})
\end{aligned}
$$

Now imposing successively the conditions (4.59) we get

$$
\begin{aligned}
\tilde{\alpha}_{n,2,0}(X_{\delta_n}) &= \frac{1}{|X_{\delta_n}|}D^{0,2}R_n^\sharp(X_{\delta_n},0,0;1,1) \\
\tilde{\alpha}_{n,2,\bar{1}}(X_{\delta_n},\mu) &= \frac{1}{|X_{\delta_n}|}D^{0,2}R_n^\sharp(X_{\delta_n},0,0;1,x_\mu) - \frac{1}{|X_{\delta_n}|}\tilde{\alpha}_{2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu \\
\tilde{\alpha}_{n,2,1}(X_{\delta_n},\mu) &= \frac{1}{|X_{\delta_n}|}D^{0,2}R_n^\sharp(X_{\delta_n},0,0;,x_\mu,1) - \frac{1}{|X_{\delta_n}|}\tilde{\alpha}_{n,2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu \\
\tilde{\alpha}_{n,4}(X_{\delta_n}) &= \frac{1}{4}\frac{1}{|X_{\delta_n}|}\Big(D^{0,4}R_n^\sharp(X_{\delta_n},0,0;1,1,1,1) \\
&\quad + D^{0,2}\tilde{V}_n(X_{\delta_n},0,0;1,1)D^{0,2}R_n^\sharp(X_{\delta_n},0,0;1,1)\Big)
\end{aligned}
\tag{4.61}
$$

The leading contributions to the $\tilde{\alpha}_P(X_{\delta_n})$ are obtained by setting $\tilde{V}_n = 0$ in the above formulae.

Let $u_n = (g_n, \mu_n, R_n)$. Then $(\mathbf{w}_n, u_n)$ are the coordinates of the measure density in the polymer representation after n successive applications of the RG map f_j, $1 \le j \le n$, of Section 4. The $\mathbf{w}_n$ evolve according to $\mathbf{w}_{n+1} = f_{n+1,\mathbf{w}}(\mathbf{w}_n) = \mathbf{v}_{n+1} + \mathbf{w}_{n,L}$ as given in (4.16). This evolution is independent of u_n and is solved in Lemma 5.9 below. The sequence $\{\mathbf{w}_n, u_n\}$ with $u_{n+1} = f_{n+1}(u_n)$, where the solution for $\mathbf{w}_n$ is incorporated in the map f_n, is the RG trajectory. The index n in R_n also indicates that R_n is supported on polymers in $(\delta_n \mathbb{Z})^3$. Correspondingly the norms for Banach spaces of polymer activities given in Section 2 are indexed by the lattice spacing δ_n. In this section we first set up a uniformly bounded domain $\mathcal{D}_n$ for u_n. The rest of this section is then devoted to the proof of Theorem 5.1 below. This theorem controls the remainders (ξ_n, ρ_n) in the flow equations (4.39) together with R_{n+1} in (4.41) when u_n belongs $\mathcal{D}_n$. It also gives bounds on g_{n+1} and μ_{n+1}. Theorem 5.1 will provide essential ingredients for the proof (in Section 6,) of existence of an initial choice of the mass parameter such that there is a uniformly bounded RG trajectory at all scales labelled by n.

The aforementioned domain will be a ball defined with Banach space norms with the center of the ball fixed i.e. independent of n. To this end we first obtain an approximate discrete flow of the coupling constant g_n from the first equation in (4.39) by ignoring the remainder $\xi_n(g_n, \mu_n, R_n)$. The approximate flow equation has n-dependent coefficients. However we show below (Lemma 5.12), with no assumption about the domain $\mathcal{D}_n$ given below, that the the positive coefficients a_n converge geometrically as $n \to \infty$ to a constant $a_{c,*} > 0$. This leads us to set up a reference approximate discrete flow of the coupling constant

$$g_{c,n+1} = L^\varepsilon g_{c,n}(1 - L^\varepsilon a_{c,*} g_{c,n}) \tag{5.1}$$

This may be thought of as an approximate flow in an underlying continuum theory. This approximate flow has a nontrivial fixed point, namely

$$\bar{g} = \frac{L^\varepsilon - 1}{L^{2\varepsilon} a_{c,*}} > 0 \tag{5.2}$$

The constant $a_{c,*} = a_{c,*}(L, \varepsilon)$ depends on L, ε in such a way that when $\varepsilon \to 0$ with L fixed $a_{c,*}(L, \varepsilon) \to \bar{a}_{c,*}(L)$ which depends only on L. We will assume L large but fixed for the rest of the paper. We then choose ε sufficiently small depending on L.
We have

$$0 < \bar{g} < C_L \varepsilon \tag{5.3}$$

where C_L is a constant which depends only on L. ε is then a measure of smallness of $\bar{g}$.
In the following $O(1)$ denotes a constant *independent of L, ε and n.* Constants C are *independent of ε and n but may depend on L.* These constants may change from line to line. It will not be necessary to keep track of these changes.

The Domain $\mathcal{D}_n$:
We will say that $u_n = (g_n, \mu_n, R_n)$ belongs to the domain $\mathcal{D}_n$ if

$$|g_n - \bar{g}| < \nu \bar{g}, \quad |\mu_n| < \bar{g}^{2-\delta} \tag{5.4}$$

$$|||R_n|||_n < \bar{g}^{11/4-\eta} \tag{5.5}$$

where the constant ν is held in the range $0 < \nu < 1$, *and*

$$|||R_n|||_n = \max\{|R_n|_{\mathbf{h}_*, \mathcal{A}, \delta_n},\ \bar{g}^2 \|R_n\|_{\mathbf{h}, G_\kappa, \mathcal{A}, \delta_n}\} \tag{5.6}$$

We choose $\kappa = \kappa(L)$ as in Lemma 2.1 and $\rho = \rho(L)$ as in Lemma 5.3 (independent of the domain hypothesis). κ, ρ will be held fixed after L has been chosen sufficiently large. $\delta, \eta = O(1) > 0$ are very small fixed numbers, say $1/64$, and $h_B = c\bar{g}^{-1/4}$ with $c = O(1) > 0$ a very small number. Furthermore we take $h_{B*} = (\rho\kappa)^{-1/2}$ and choose $m_0 = 9$. $h_F = h_F(L)$ is an ε independent constant which depends on L and is taken to be sufficiently large. (The dependence of h_F on L enters in the proofs of Lemma 5.15 and Lemma 5.16 below). We recall that $\mathbf{h} = (h_B, h_F)$, $\mathbf{h}_* = (h_{B*}, h_F)$.

Remark : Note that condition (5.5) is equivalent to having both

$$\|R_n\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_n} < \bar{g}^{3/4-\eta} \tag{5.7}$$

$$|R_n|_{\mathbf{h}_*,\mathcal{A},\delta_n} < \bar{g}^{11/4-\eta} \tag{5.8}$$

Recall the definitions of $\rho_n(g_n, \mu_n, R_n)$ and $\xi_n(g_n, \mu_n, R_n)$ from (4.53). We will prove in this section

Theorem 5.1

Let $u_n = (g_n, \mu_n, R_n) \in \mathcal{D}_n$. Let L be large but fixed followed by ε sufficiently small depending on L. $\bar{g}$ is thus sufficiently small. Let $u_{n+1} = f_{n+1}(u_n)$ where f_{n+1} is the RG map of section 4. Then there exist constants C_L independent of n and ε such that

$$|\xi_n| \le C_L \bar{g}^{11/4-\eta} \tag{5.9}$$

$$|\rho_n| \le C_L \bar{g}^{11/4-\eta} \tag{5.10}$$

$$|||R_{n+1}|||_{n+1} \le L^{-1/4} \bar{g}^{11/4-\eta} \tag{5.11}$$

$$|g_{n+1} - \bar{g}| < 2\nu\bar{g}^{3/2}, \quad |\mu_{n+1}| < O(1) L^{\frac{3+\varepsilon}{2}} \bar{g}^{2-\delta} \tag{5.12}$$

Remark : The lemmas which follow will serve to prove Theorem 5.1. They are organized as in Section 5 of [BMS]. We remark that Lemmas 5.1-5.4, 5.9, and Lemma 5.12 are independent of the domain hypothesis. *All the other lemmas are under the assumption that (g_n, μ_n, R_n) belong to the domain $\mathcal{D}_n$.* Lemmas 5.21, 5.22, 5.23 and 5.27 are the major parts of the program. $R_{n+1,\mathrm{main}}$ is bounded in Lemma 5.21 and this result determines the qualitative form of the bound on the remainder. $R_{n+1,3}$ and $R_{n+1,4}$ are seen, in Lemmas 5.22, 5.23 to be negligible in comparison. $R_{n+1,\mathrm{linear}}$ is the crux of the program and it is bounded in Lemma 5.27. The remaining Lemmas are auxiliary results on the way to these Lemmas. These auxiliary lemmas implement some of the following principles: Wick constants $C_n(0)$ are uniformly bounded by constants $C = C_L$. In bounds by $G_\kappa, \mathbf{h}, \mathcal{A}$ norms, a fluctuation field ζ contributes a constant $C = O(1)(\rho(L)\kappa(L))^{-1/2}$ and a field φ contributes a constant $O(1)\bar{g}^{-1/4}$. The contributions of these fields as well as of the Grassmann fields ψ, η are controlled by the structure of the norms defined in section 2 (with above choice of $\mathbf{h}, \mathbf{h}_*$) and later in this section ((5.20) et seq). Integration over the Grassman fluctuation fields η is controlled by the Gramm inequality. In bounds by the $\mathbf{h}_*, \mathcal{A}$ norms, fluctuation fields ζ contribute a constant $C = O(1)(\rho(L)\kappa(L))^{-1/2}$ and fields φ contribute $O(1)$. h_{B*} has been adjusted to take care of the constant C above in the contribution of the fluctuation field.

Lattice Taylor expansions

In the following we will have occasion to estimate the difference of lattice fields at two different points of a hypercubic lattice $(\delta_n\mathbb{Z})^d$. Let f be a lattice function and x, y be two points in the lattice. We write $y - x = \sum_{j=1}^d \delta_n h_j e_j$ where $h_j \in \mathbf{Z}$ and the e_j are the unit vectors of the lattice. We will express the difference $f(y) - f(x)$ as a sum of forward lattice derivatives of f along segments of a specified lattice path. Given $j \in \{1, 2, .., d\}$, $s \in \mathbf{Z}$ and $v \in (\delta_n\mathbb{Z})^d$ we define $p_j(v, s) \in (\delta_n\mathbb{Z})^d$ by

$$p_j(v,s)=\sum_{i=1}^{j-1}(v,e_i)e_i+\delta_n s e_j \tag{5.13}$$

with the convention that if $j=1$ the sum is empty. Then it is a simple matter to verify that

$$f(y)-f(x)=\delta_n\sum_{j=1}^{d}\sum_{s_j=0}^{h_j-1}\partial_{\delta_n,e_j}f(x+p_j(y-x,s_j)) \tag{5.14}$$

By iterating (5.14) we get the second order lattice Taylor expansion

$$f(y)-f(x)=\sum_{j=1}^{d}((y-x),e_j)\partial_{\delta_n,e_j}f(x)+\delta_n^2\sum_{j,k=1}^{d}\sum_{s_j=0}^{h_j-1}\sum_{s_k=0}^{h_k-1}\partial_{\delta_n,e_j}\partial_{\delta_n,e_k}f(x+p_k(p_j(y-x,s_j),s_k)) \tag{5.15}$$

Lemma 5.1
Let $Z_{\delta_n},X_{\delta_n}$ be connected 1-polymers in $(\delta_n\mathbb{Z})^3$. Let $Y_{\delta_n}=\emptyset$ or $Y_{\delta_n}=L^{-1}X_{\delta_n}\subset Z_{\delta_n}$ such that vol($Z_{\delta_n}\backslash Y_{\delta_n}$) $\geq\frac{1}{2}$. Choose any $\gamma=O(1)>0$ and $\kappa=\kappa(L)>0$ as in Lemma 2.1. Let $\varphi:\tilde Z^{(5)}_{\delta_n}\to\mathbb{C}$ where $\tilde Z^{(5)}_{\delta_n}$ is Z_{δ_n} with 5-collar attached (see (1.76), (1.77)). Then there exists a constant $C=\kappa^{-j/2}O(1)$ such that

$$\|\varphi\|^j_{C^2(Z_{\delta_n})}\leq C\,2^{|Z|}\bar g^{-\frac{j}{4}}e^{\gamma\bar g\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy|\varphi(y)|^4}G_\kappa(Z_{\delta_n},\varphi) \tag{5.16}$$

For $Y_{\delta_n}=\emptyset$ the above bound holds without the factor $2^{|Z|}$.
Proof: This is the lattice analogue of Lemma 5.1 in [BMS]. The proof reposes on the Hölder inequality and the lattice Sobolev inequality (see [BGM], Appendix B for an elementary proof). Let $x\in Z_{\delta_n}$. Write

$$\varphi(x)=\frac{1}{vol(Z_{\delta_n}\backslash Y_{\delta_n})}\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy(\varphi(y)+\varphi(x)-\varphi(y))$$

and bound

$$|\varphi(x)|\leq\frac{1}{vol(Z_{\delta_n}\backslash Y_{\delta_n})}\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy|\varphi(y)|+\frac{1}{vol(Z_{\delta_n}\backslash Y_{\delta_n})}\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy|\varphi(x)-\varphi(y)|$$

We bound the first term on the right hand side by $O(1)\|\varphi\|_{L^4(Z_{\delta_n}\backslash Y_{\delta_n})}$. To bound the second term we write the difference $\varphi(y)-\varphi(x)$ as a sum of lattice derivatives along the segments of the path as in (5.14). Any connected polymer Z_{δ_n} as defined in section 1.3 can be represented as $Z_{\delta_n}=Z\cap(\delta_n\mathbb{Z})^3$ where Z is a connected continuum polymer. Recall from section 1.3 that Z_{δ_n} is said to be convex if Z is a convex set. If Z_{δ_n} is convex then the path $p_j(y-x,s_j)$ in (5.14) lies entirely in Z_{δ_n}. If Z_{δ_n} is not convex then it decomposes as a connected union of connected convex polymers. If x,y are not in the same convex component then it suffices to consider the case when they are in adjacent components. We pick a point z_0 in the intersection of the closures, write $f(x)-f(y)=(f(x)-f(z_0))+(f(z_0)-f(y))$ and use the above first order taylor expansion for each summand. The estimates below remain valid. Therefore it is sufficient to consider the case Z_{δ_n} is convex. From

$$\varphi(y)-\varphi(x)=\delta_n\sum_{j=1}^{3}\sum_{s_j=0}^{h_j-1}\partial_{\delta_n,e_j}\varphi(x+p_j(y-x,s_j)) \tag{5.17}$$

we get the bound

$$|\varphi(y)-\varphi(x)| \le \sum_{j=1}^{3}\delta_n|h_j|\sup_{z\in Z_{\delta_n}}|\partial_{\delta_n,e_j}\varphi(z)| \le 3\delta_n(\max_j|h_j|)\max_j\sup_{z\in Z_{\delta_n}}|\partial_{\delta_n,e_j}\varphi(z)|$$
$$\le O(1)|y-x|\ \|\varphi\|_{Z_{\delta_n},1,5} \tag{5.18}$$

where in the last step we have used the lattice Sobolev embedding theorem. We have $|y-x|\le|Z|$. Putting the above bounds together we get

$$|\varphi(x)| \le 0(1)|Z|(\|\varphi\|_{L^4(Z_{\delta_n}\backslash Y_{\delta_n})}+\|\varphi\|_{Z_{\delta_n},1,5})$$

We also have for $k=1,2$, $|\partial^k_{\delta_n,\mu_1,..,\mu_k}\varphi(x)|\le\|\varphi\|_{Z_{\delta_n},1,5}$ by Sobolev embedding. Therefore combining with the previous inequality we get

$$\|\varphi\|_{C^2(Z_{\delta_n})} \le 0(1)|Z|(\|\varphi\|_{L^4(Z_{\delta_n}\backslash Y_{\delta_n})}+\|\varphi\|_{Z_{\delta_n},1,5})$$

Hence

$$\|\varphi\|^j_{C^2(Z_{\delta_n})} \le 0(1)^j|Z|^j(\|\varphi\|^j_{L^4(Z_{\delta_n}\backslash Y_{\delta_n})}+\|\varphi\|^j_{Z_{\delta_n},1,5}) \le C\,2^{|Z|}\bar g^{-j/4}e^{\gamma\bar g\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy\,\varphi^4(y)}G_\kappa(Z_{\delta_n},\varphi)$$

where C depends on κ,γ,j. This proves the bound (5.16). We now prove the statement following (5.16). Let $Y_{\delta_n}=\emptyset$. For $x\in Z_{\delta_n}$ pick the unit block $\Delta_{\delta_n}\subset Z_{\delta_n}$, $\Delta_{\delta_n}\ni x$. We have

$$|\varphi(x)| \le \int_{\Delta_{\delta_n}}dy\,|f(y)|+\int_{\Delta_{\delta_n}}dy\,|\varphi(x)-\varphi(y|$$

Proceeding as before the first term is bounded by the $L^4(\Delta_{\delta_n})$ norm which is less than the $L^4(Z_{\delta_n})$ norm. The second term is bounded as before except that since $x,y\in\Delta_{\delta_n}$ we have $|x-y|\le O(1)$. The rest is as before. ■

In effecting the fluctuation map in Section 3.1 we created polymer activities which depended separately on Φ and the fluctuation field ξ. The following lemmas will enable us to estimate the contributions of the bosonic fluctuation field ζ at various steps. Define a large field regulator for the bosonic fluctuation field $\zeta:\tilde X^5_{\delta_n}\to\mathbf{C}$

$$\tilde G_{\kappa,\rho}(X_{\delta_n},\zeta)=e^{\rho\|\zeta\|^2_{L^2(X_{\delta_n})}}G_\kappa(X_{\delta_n},\zeta),\quad \rho,\kappa>0 \tag{5.19}$$

κ is chosen as in Lemma 2.1 and is held sufficiently small. The choice of $\rho>0$ is dictated by Lemma 5.3 below.

Lemma 5.2
For any $x\in X_{\delta_n}$

$$|\zeta(x)|^j \le C_{\rho,j,\kappa}\tilde G_{\kappa,\rho}(X_{\delta_n},\zeta) \tag{5.20}$$

where $C_{\rho,j}=(\rho\kappa)^{-(j/2)}O(1)$ and $O(1)$ depends on j. We have isolated out the ρ,κ dependence in the bound.
Proof : The proof follows the lines of the proof of Lemma 5.1 for the case $Y_{\delta_n}=\emptyset$. Take the unit block $\Delta_{\delta_n}\subset X_{\delta_n}$ such that $\Delta_{\delta_n}\ni x$. We replace the L^4 norm by the L^2 norm in the appropriate place and estimate $|\zeta(x)-\zeta(y)|$ with $x,y\in\Delta_{\delta_n}$ as before now using the regulator $\tilde G_{\kappa,\rho}$. ■

The parameter $\rho>0$ is chosen such that the following Lemma 5.3 holds. ρ depends on L.

Lemma 5.3 : Let $\kappa>0$ be chosen as in Lemma 2.1. Then there exists $\rho_0=\rho_0(L)>0$ independent of n such that for all ρ, $0<\rho\le\rho_0$

$$\int d\mu_{\Gamma_n}(\zeta)\tilde{G}_{\kappa,\rho}(X_{\delta_n},\zeta) \leq 2^{|X_{\delta_n}|} \tag{5.21}$$

Lemma 5.3 is proved in the same way as Lemma 2.1.

We introduce a new intermediate large field regulator which combines the ones introduced earlier

$$\hat{G}_{\kappa,\rho}(X_{\delta_n},\zeta,\varphi) = G_\kappa(X_{\delta_n},\zeta+\varphi)G_\kappa(X_{\delta_n},\varphi)\tilde{G}_{\kappa,\rho}(X_{\delta_n},\zeta) \tag{5.22}$$

Lemma 5.4 : Let κ, ρ be chosen as in Lemma 2.1 and Lemma 5.3 respectively. Then we have

$$\int d\mu_{\Gamma_n}(\zeta)\hat{G}_{\kappa,\rho}(X_{\delta_n},\zeta,\varphi) \leq 2^{|X_{\delta_n}|}G_{3\kappa}(X_{\delta_n},\varphi) \tag{5.23}$$

Proof: The proof follows from an application of the Cauchy-Schwarz inequality and Lemmas 2.1, 5.3.

In effecting the fluctuation map we have introduced polymer activities which are functions of the four separate fields φ,ζ,ψ,η and not just of $\varphi+\zeta,\psi+\eta$. The polymer representation will have the form

$$\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\tilde{\eta}) = \sum_{p\geq 0}\sum_{I\subset\{1,\dots,2p\}}\sum_{\mathbf{l},\mathbf{a}}\int_{X_{\delta_n}^{2p}} d\mathbf{x}\tilde{K}_{2p}^{\mathbf{l},\mathbf{a},I}(X_{\delta_n},\varphi,\zeta,\mathbf{x})\ \partial_{\delta_n}^{\mathbf{l}}\prod_{j\in I}\psi_{a_j}(x_j)\prod_{j\in I^c}\eta_{a_j}(x_j) \tag{5.24}$$

where we have kept the same conventions as in (1.81).
Note that if $\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta)$ is actually $\tilde{K}_n(X_{\delta_n},\varphi+\zeta,\psi+\eta)$ then

$$\tilde{K}_{2p}^{\mathbf{l},\mathbf{a},I} = \tilde{K}_{n,2p}^{\mathbf{l},\mathbf{a}}, \quad \forall\, I\subset\{1,...,2p\} \tag{5.25}$$

The field derivatives of polymer activities represented in (5.24) are defined by :

$$D^{2p,m}\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0;f^{\times m},g_{2p}) = \sum_{I\subset\{1,\dots,2p\}}\sum_{\mathbf{l},\mathbf{a}}\int_{X_{\delta_n}^{2p}} d\mathbf{x}D^m\tilde{K}_{2p}^{\mathbf{l},\mathbf{a},I}(X_{\delta_n},\varphi,\zeta,\mathbf{x};f^{\times m})\ \partial_{\delta_n}^{\mathbf{l}}g_{2p}(\mathbf{x}) \tag{5.26}$$

where the derivative D^m of the bosonic coefficient is with respect to the field φ (and not the fluctuation field ζ). The fermionic field derivatives (at the origin) are being computed with respect to both ψ *and* the fermionic fluctuation field η.
The norm of the multilinear functional (5.26) is defined as in (2.15), namely

$$\|D^{2p,m}\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0)\| = \sum_{I\subset\{1,\dots,2p\}}\sup_{\substack{\|f_j\|_{C^2(X_{\delta_n})}\leq 1\\ \|g_{2p}\|_{C^2(X_{\delta_n}^{2p})}\leq 1\\ \forall 1\leq j\leq m}}\left|\sum_{\mathbf{l},\mathbf{a}}\int_{X_{\delta_n}^{2p}} d\mathbf{x}D^m\tilde{K}_{2p}^{\mathbf{l},\mathbf{a},I}(X_{\delta_n},\varphi,\zeta,\mathbf{x};f^{\times m})\ \partial^{\mathbf{l}}g_{2p}(\mathbf{x})\right| \tag{5.27}$$

For $I\subset\{1,...,2p\}$ define a polymer activity

$$\tilde{K}_{2p}^{I}(X_{\delta_n},\varphi,\zeta,\psi,\tilde{\eta}) = \sum_{\mathbf{l},\mathbf{a}}\int_{X_{\delta_n}^{2p}} d\mathbf{x}\tilde{K}_{2p}^{\mathbf{l},\mathbf{a},I}(X_{\delta_n},\varphi,\zeta,\mathbf{x})\ \partial_{\delta_n}^{\mathbf{l}}\prod_{j\in I}\psi_{a_j}(x_j)\prod_{j\in I^c}\eta_{a_j}(x_j) \tag{5.28}$$

Then

$$D^{2p,m}\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0;f^{\times m},g_{2p}) = \sum_{I\subset\{1,\dots,2p\}}D^{2p,m}\tilde{K}_{2p}^{I}(X_{\delta_n},\varphi,\zeta,0,0;f^{\times m},g_{2p}) \tag{5.29}$$

and observe that

$$\|D^{2p,m}\tilde K(X_{\delta_n},\varphi,\zeta,0,0)\| = \sum_{I\subset\{1,\dots,2p\}} \|D^{2p,m}\tilde K^I_{2p}(X_{\delta_n},\varphi,\zeta,0,0)\| \tag{5.30}$$

In the beginning of this section we specified $\mathbf{h}=(h_F,h_B)$ and $\mathbf{h}_*=(h_F,h_{B*})$. We define in addition

$$\hat{\mathbf{h}}=(\hat h_F,\hat h_B)=(\frac{h_F}{2},h_B) \tag{5.31}$$

$$\hat{\mathbf{h}}_*=(\hat h_F,\hat h_{B*})=(\frac{h_F}{2},h_{B*}) \tag{5.32}$$

We define the norms

$$\|\tilde K(X_{\delta_n},\varphi,\zeta,0,0)\|_{\hat{\mathbf{h}}}=\sum_{p=0}^{\infty}\sum_{m=0}^{m_0}\hat h_F^{2p}\frac{h_B^m}{m!}\|D^{2p,m}\tilde K(X_{\delta_n},\varphi,\zeta,0,0)\| \tag{5.33}$$

$$\|\tilde K(X_{\delta_n})\|_{\hat{\mathbf{h}},\hat G_{\kappa,\rho}}=\sup_{\varphi,\zeta\in\mathcal{F}_{\tilde X^{(5)}_\delta}}\|\tilde K(X_{\delta_n},\varphi,\zeta,0,0)\|_{\hat{\mathbf{h}}}\hat G^{-1}_{\kappa,\rho}(X_{\delta_n},\varphi,\zeta) \tag{5.34}$$

$$\|\tilde K(X_{\delta_n},0,\zeta,0,0)\|_{\mathbf{h}_*}=\sum_{p=0}^{\infty}\sum_{m=0}^{m_0}h_F^{2p}\frac{h_{B*}^m}{m!}\|D^{2p,m}\tilde K(X_{\delta_n},0,\zeta,0,0)\| \tag{5.35}$$

$$\|\tilde K(X_{\delta_n})\|_{\mathbf{h}_*,\tilde{\mathbf{G}}_{\kappa,\rho}}=\sup_{\zeta\in\mathcal{F}_{\tilde X^{(5)}_\delta}}\|\tilde K(X_{\delta_n},0,\zeta,0,0)\|_{\mathbf{h}_*}G^{-1}_{\kappa,\rho}(X_{\delta_n},\zeta) \tag{5.36}$$

$$|\tilde K(X_{\delta_n})|_{\hat{\mathbf{h}}_*}=\sum_{p=0}^{\infty}\sum_{m=0}^{m_0}\hat{h_F}^p\frac{{h_B}_*^m}{m!}\|D^{p,m}\tilde K(X_{\delta_n},0,0,0,0)\| \tag{5.37}$$

Note that if we represent a polymer activity $\tilde K(X_{\delta_n},\varphi+\zeta,\psi+\eta)$ in the form (5.24) then

$$\|\tilde K(X_{\delta_n},\varphi+\zeta,0)\|_{\mathbf{h}}=\|\tilde K(X_{\delta_n},\varphi,\zeta,0,0)\|_{\hat{\mathbf{h}}} \tag{5.38}$$

where we have used (5.25) and the fact that

$$\sum_{I\subset\{1,\dots,2p\}}1=2^{2p} \tag{5.39}$$

It is straightforward to prove that the above norms satisfy the multiplicative property of Proposition 2.3 (with G_κ replaced by $\hat G_\kappa$).

Lemma 5.5
Let g_n,μ_n belong to $\mathcal{D}_n$. Let Y_{δ_n} be a 1-polymer. Then for $V_n(Y_{\delta_n},\Phi,\xi)=V(Y_{\delta_n},\Phi+\xi,C_n,g_n,\mu_n)$ or $V(Y_{\delta_n},\Phi,S_LC_{n+1},g_n,\mu_n)$, we have

$$\|e^{-V_n(Y_{\delta_n},\Phi,\xi)}\|_{\mathbf{h}}\le 2^{|Y_{\delta_n}|}e^{-\bar g/4\int_{Y_{\delta_n}}dx|\varphi+\zeta|^4(x)} \tag{5.40}$$

$$\|e^{-V_n(Y_{\delta_n},\Phi,\xi)}\|_{\mathbf{h}_*}\le 2^{|Y_{\delta_n}|} \tag{5.41}$$

for $\varepsilon > 0$ sufficiently small depending on L. In the above norms $\mathbf{h} = (h_B, h_F)$ and $\mathbf{h}_* = (h_{B*}, h_F)$ are chosen as in the hypothesis for the domain $\mathcal{D}_{\delta_\backslash}$. Thus $h_F = h_F(L)$, $h_B = c\bar{g}^{-\frac{1}{4}}$ with $c = O(1)$ sufficiently small, and $h_{B*} = (\rho\kappa)^{-1}$. Note that h_{B*} depends on L.

Proof : It is sufficient to prove this when Y_{δ_n} is a 1-block Δ_{δ_n}. Because otherwise we can write Y_{Δ_n} as a disjoint union of 1-blocks and write the left hand side as a product over 1-block contributions. Then the multiplicative property of the $\mathbf{h}$ norm (Proposition 2.3) gives the lemma.
From the definition of V in (4.1) we get on undoing the Wick ordering

$$V_n(\Delta_{\delta_n}, \Phi) = V_{n,u}(\Delta_{\delta_n}, \varphi) + 2g_n \int_{\Delta_{\delta_n}} dx \varphi\bar{\varphi}(x)\psi\bar{\psi}(x) + \tilde{\mu}_n \int_{\Delta_{\delta_n}} dx \psi\bar{\psi}(x) \tag{5.42}$$

where

$$V_{u,n}(\Delta_{\delta_n}, \varphi) = g_n \int_{\Delta_{\delta_n}} dx[(\varphi\bar{\varphi}(x))^2 + \tilde{\mu}_n \varphi\bar{\varphi}(x)] \tag{5.43}$$

and $\tilde{\mu}_n = \mu_n - 2g_n C_n(0)$.
By the multiplicative property of the $\mathbf{h}$ norm,

$$\|e^{-V(\Delta_{\delta_n},\Phi)}\|_{\mathbf{h}} \leq \|e^{-V_{u,n}(\Delta_{\delta_n},\varphi)}\|_{h_B} \|e^{-2g_n \int_{\Delta_{\delta_n}} dx\varphi\bar{\varphi}(x)\psi\bar{\psi}(x)}\|_{\mathbf{h}} \|e^{-\tilde{\mu}_n \int_{\Delta_{\delta_n}} dx\psi\bar{\psi}(x)}\|_{h_F} \tag{5.44}$$

We estimate each of the factors on the right hand side in turn. We observe that by taking ε sufficiently small we can make $\bar{g}$ as small as necessary since $0 < \bar{g} \leq C\varepsilon$. Since g_n, μ_n belong to $\mathcal{D}_n$ we have $g_n = O(\bar{g})$ and $\mu = O(\bar{g}^{2-\delta})$. Moreover from from (5a) of Theorem 1.1 and (1.53) we have the uniform bound $|C_n(0)| \leq C_L$. For the first factor we have the bound

$$\|e^{-V_{u,n}(\Delta_{\delta_n},\varphi)}\|_{h_B} \leq 2^{\frac{1}{2}|\Delta_{\delta_n}|} e^{-g_n/2 \int_{\Delta_{\delta_n}} dx(\varphi\bar{\varphi})^2(x)} \tag{5.45}$$

This can be proved on the lines of the proof of Lemma 5.5 of [BMS] by substituting there $\bar{g}$ for ε and taking account of the previous observations. Now consider the second factor. From

$$e^{-2g_n \int_{\Delta_{\delta_n}} dx\varphi\bar{\varphi}(x)\psi\bar{\psi}(x)} = \sum_{p\geq 0} \frac{(-2g_n)^p}{p!} \int_{\Delta^p_{\delta_n}} d\mathbf{x} K_{2p}(\varphi, \Delta, \mathbf{x}) \prod_{j=1}^{p} \psi\bar{\psi}(x_j)$$

where

$$K_{2p}(\varphi, \Delta, \mathbf{x}) = \prod_{j=1}^{p} \varphi\bar{\varphi}(x_j)$$

we obtain

$$\|e^{-2g_n \int_{\Delta_{\delta_n}} dx\varphi\bar{\varphi}(x)\psi\bar{\psi}(x)}\|_{\mathbf{h}} \leq \sum_{p\geq 0} \frac{(2g_n)^p}{p!} h_F^{2p} \sum_{m=0}^{m_0} \frac{h_B^m}{m!}$$

$$\sup_{\substack{\|f_i\|_{C^2(\Delta_{\delta_n})} \leq 1,\ \forall i \\ \|g_{2p}\|_{C^2(\Delta^{2p}_{\delta_n})} \leq 1}} \left| \int_{\Delta^p_{\delta_n}} d\mathbf{x} D^m K_{2p}(\varphi, \Delta_{\delta_n}, \mathbf{x}; f^{\times m}) g_{2p}(x_1, x_1, ..., x_p, x_p) \right| \leq$$

$$\leq \sum_{p\geq 0} \frac{(2g_n)^p}{p!} h_F^{2p} \int_{\Delta^p_{\delta_n}} d\mathbf{x} \prod_{j=1}^{p} \|\varphi\bar{\varphi}(x_j)\|_{h_B} \leq e^{2g_n h_F^2 \int_\Delta dx \|\varphi\bar{\varphi}(x)\|_{h_B}}$$

It is straightforward to show that

$$2g_n h_F^2\int_\Delta dx\|\varphi\bar\varphi(x)\|_{h_B}\le C_L\bar g^{\frac12}\big(\frac{\bar g}{4}\int_{\Delta_n}dx(\varphi\bar\varphi)^2(x)+O(1)\big)$$

and therefore for the second factor we have the bound

$$\|e^{2g_n\int_{\Delta_{\delta_n}}dx\varphi\bar\varphi(x)\psi\bar\psi(x)}\|_{\mathbf h}\le 2^{C_L\bar g^{\frac12}|\Delta|}e^{C_L\bar g^{\frac12}\frac{\bar g}{4}\int_\Delta dx(\varphi\bar\varphi)^2(x)} \tag{5.46}$$

Finally for the third factor we have straightforwardly the bound

$$\|e^{\tilde\mu_n\int_{\Delta_{\delta_n}}dx\psi\bar\psi(x)}\|_{h_F}\le 2^{C_L\bar g|\Delta|} \tag{5.47}$$

Putting together the bounds for the three factors, we obtain for ε sufficiently small depending on L

$$\|e^{-V(\Delta_{\delta_n},\Phi)}\|_{\mathbf h}\le 2^{|\Delta|}e^{-\bar g/4\int_\Delta dx(\varphi\bar\varphi)^2(x)}$$

which proves the first part of the lemma follows. The proof of the second part follows the same lines. ∎

Lemma 5.6
Let $p_{n,g}(\Delta_{\delta_n},\xi,\Phi)$ and $p_{n,\mu}(\Delta_{d_n},\xi,\Phi)$ be as given in (4.8). Let g_n,μ_n belong to $\mathcal D_n$. Let $h_B=c\bar g^{-1/4}$ and h_{B*} be as in the definition of $\mathcal D_n$. Recall that $\mathbf h=(h_B,h_F)$, and $\mathbf h_*=(h_{B*},h_F)$, where $h_F=h_F(L)$. Let $\kappa=\kappa(L)>0$ and $\rho=\rho(L)>0$, be as specified in Lemmas 2.1 and 5.3. Then for any $\gamma=O(1)>0$, $0\le s<1$ we have constants C_L independent of n and ε but depending on L such that

$$\|p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf h}\le C_L\bar g^{1/4}(1-s)^{-3/4}\tilde G_{\kappa,\rho}(\Delta_{\delta_n},\zeta)G_\kappa(\Delta_{\delta_n},\varphi)e^{\bar g(1-s)\gamma\int_{\Delta_{\delta_n}}dx\ (\varphi\bar\varphi)^2(x)} \tag{5.48}$$

$$\|p_{n,\mu}(\Delta_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf h}\le C_L\bar g^{7/4-\delta}(1-s)^{-1/2}\tilde G_{\kappa,\rho}(\Delta_{\delta_n},\zeta)G_\kappa(\Delta_{\delta_n},\varphi)e^{\bar g(1-s)\gamma\int_{\Delta_{\delta_n}}dx\ (\varphi\bar\varphi)^2(x)} \tag{5.49}$$

$$\|p_{n,g}(\Delta_{\delta_n},0,\zeta,0,0)\|_{\mathbf h_*}\le C_L\ \bar g\tilde G_{\kappa,\rho}(\Delta_{\delta_n},\zeta) \tag{5.50}$$

$$\|p_{n,\mu}(\Delta_{\delta_n},0,\zeta,0,0)\|_{\mathbf h_*}\le C_L\bar g^{2-\delta}\tilde G_{\kappa,\rho}(\Delta_{\delta_n},\zeta) \tag{5.51}$$

Proof : $p_{n,g}(\Delta_{\delta_n},\xi,\Phi)$ is given in (4.7). We undo the Wick ordering which produces constants $C_n(0)$ uniformly bounded by constant C_L from Corollary 1.2. We can then write it in the form (5.24) by expanding out in the Grassmann fields. Since it is a local polynomial of degree four we get,

$$p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,\psi,\eta)=\sum_{p=0}^{2}\sum_{\mathbf a}\sum_{I\subset\{1,\dots,2p\}}\int_{\Delta_{\delta_n}}dx\tilde p^{\mathbf 0,\mathbf a,I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)\prod_{i\in I}\psi_{a_i}(x)\prod_{i\in I^c}\eta_{a_i}(x) \tag{5.52}$$

where $\mathbf 0$ means that $l_i=0\ \forall i$. We have following the definition of the norm in (5.33) with $\hat{\mathbf h}$ replaced by $\mathbf h$

$$\|p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf h}\le\sum_{p=0}^{2}h_F^{2p}\sup_{||g_{2p}||\le1}\sum_{\mathbf a}\sum_{I\subset\{1,\dots,2p\}}\int_{\Delta_{\delta_n}}dx\|\tilde p^{\mathbf 0,\mathbf a,I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)\|_{h_B}|g_{2p}(\mathbf x)|$$

$$\leq \sum_{p=0}^{2} h_F^{2p} \sum_{\mathbf{a}} \sum_{\substack{I\subset\{1,\dots,2n\}\\ |I|\ \text{even}}} \int_{\Delta_{\delta_n}} dx \|\tilde{p}^{\mathbf{0},\mathbf{a},I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)\|_{h_B} \tag{5.53}$$

where $g_{2p}(\mathbf{x}) = g_{2p}(x,x,...,x)$ and $\|g_{2p}\|$ is the $C^2(\Delta^{2p}_{\delta_n})$ norm of g_{2p}. $\tilde{p}^{\mathbf{0},\mathbf{a},I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)$ is a polynomial in φ,ζ and every term in the h_B norm of $\tilde{p}_{n,g,2p}$ can be estimated as in the proof of Lemma 5.6 of [BMS]. Each term carries a factor $g_n = O(\bar{g})$. The fluctuation fields ζ are estimated via Lemma 5.2 and the fields φ via Lemma 5.1. For each field φ we loose $\bar{g}^{-\frac{1}{4}}$. In the $p=0$ term the maximum power of φ in $\tilde{p}_{n,g,p}$ is 3, for $p=1$ the maximum power is 2, and for $p=2$ it is 0. The bound (5.48) now follows as in Lemma 5.6, [BMS]. The bound (5.49) for $p_{n,\mu}(\Delta_{\delta_n},\xi,\Phi)$ is proved in the same way. We just have to remember that $\mu_n = O(\bar{g}^{2-\delta})$ from the domain hypothesis, and that the maximum power of the field φ in the $\tilde{p}_{n,\mu,p}$ is 1. The remaining parts are proved in the same way. ∎

Define $p_n(s) = p_n(s,\Delta_{\delta_n},\Phi,\xi)$ by $p_n(s) = sp_{n,g} + s^2 p_{n,\mu}$. Then $r_{n,1} = r_{n,1}(\Delta_{\delta_n},\Phi,\xi)$ defined by (4.9) is given by

$$r_{n,1} = \frac{1}{2}\int_0^1 ds (1-s)^2 e^{-p_n(s)-\tilde{V}_n}\big(-p'_n(s)^3 + 6p'_n(s)p_{n,\mu}\big) \tag{5.54}$$

with $p'_n(s) = \frac{d}{ds}p_n(s) = p_{n,g} + 2sp_{n,\mu}$ and $p''_n(s) = 2p_{n,\mu}$.

Lemma 5.7
Under the conditions of the domain $\mathcal{D}_n$ there exists a constant C_L independent of n and ε but dependent on L such that

$$\|r_{n,1}(\Delta_{\delta_n})\|_{h,\hat{G}_{\kappa,\rho}} \leq C_L \bar{g}^{3/4} \tag{5.55}$$

$$\|r_{n,1}(\Delta_{\delta_n})\|_{h_*,\tilde{G}_{\kappa,\rho}} \leq C_L \bar{g}^{3-\delta} \tag{5.56}$$

Proof : Follow the proof of the corresponding Lemma 5.7 of [BMS]. Write $\tilde{V}_n + p_n(s) = V_{n,1}(s) + V_{n,2}(s)$ where

$$V_{n,1}(s) = V(\Delta_{\delta_n},\Phi+\xi,C_n,sg_n,s^2\mu_n), \quad V_{n,2}(s) = V(\Delta_{\delta_n},\Phi,S_L C_n,(1-s)g_n,(1-s^2)\mu_n)$$

We have

$$\|r_{n,1}(\Delta_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf{h}} \leq \frac{1}{2}\int_0^1 ds(1-s)^2 \|e^{-V_{n,1}(s)}\|_{\mathbf{h}} \|e^{-V_{n,2}(s)}\|_{\mathbf{h}} \Big(\|p'(s)\|^3_{\mathbf{h}} + 6\|p'(s)\|_{\mathbf{h}}\|p_\mu\|_{\mathbf{h}}\Big)$$

g_n,μ_n belong to $\mathcal{D}_n$. Lemmas 5.5 and 5.6 continue to hold with g_n,μ_n replaced by $sg_n, s^2\mu_n$ or $(1-s)g_n,(1-s^2)\mu_n$. We bound $\|e^{-V_{n,1}(s)}\|_{\mathbf{h}} \leq 2$ and $\|e^{-V_{n,2}(s)}\|_{\mathbf{h}} \leq 2e^{-(1-s)\frac{\bar{g}}{4}\int_{\Delta_{\delta_n}} dx|\varphi|^4(x)}$. We bound the remaining factor (using Lemma 5.6) by $C_L\bar{g}^{\frac{3}{4}}\hat{G}_{\rho,\kappa}(\Delta_{\delta_n},\varphi,\zeta)e^{(1-s)\bar{g}3\gamma\int_{\Delta_{\delta_n}} dx|\varphi|^4(x)}$. We put the three bounds together and choose $0<\gamma<\frac{1}{12}$. This gives the bound (5.55). The proof of (5.56) is similar. ∎

Lemma 5.8
Under the conditions for the domain $\mathcal{D}_{\delta_n}$ there exists a constant C_L, independent of n and ε but dependent on L such that

$$\|P_n(\lambda)\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A},\delta_n} \leq C_L|\lambda\bar{g}^{1/4}| \qquad \text{for } |\lambda\bar{g}^{1/4}|\leq 1 \tag{5.57}$$

$$\|P_n(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n} \leq C_L|\lambda\bar{g}^{1-\delta/2}| \qquad \text{for } |\lambda\bar{g}^{1-\delta/2}| \leq 1 \tag{5.58}$$

Proof: This follows on applying Lemmas 5.5, 5.6 and 5.7 to $P_n(\lambda)$ defined in (4.9). ∎

Estimates for $Q_n e^{-V_n}$

We now turn to the estimate of $Q_n e^{-V_n}$. From (4.4)

$$Q_n(X_{\delta_n},\Phi) = Q(X_{\delta_n},\Phi;C_n,\mathbf{w}_n,g) = g_n^2\sum_{j=1}^{3} Q^{(j,j)}(\hat{X}_{\delta_n},\Phi;C_n,w_n^{(4-j)}) \tag{5.59}$$

where the $Q^{(m,m)}$ are given in (4.6). Under an iteration, see Proposition 4.1, we have

$$w_n^{(p)} \to w_{n+1}^{(p)} = v_{n+1}^{(p)} + w_{n,L}^p$$

,
where $p = 1,2,3$ and the $v_n^{(p)}$ are given in Proposition 4.1. Starting with $w_0^{(p)} = 0$ we get by iterating

$$w_n^{(p)} = \sum_{j=0}^{n-1} v_{n-j,L^j}^{(p)} \tag{5.60}$$

For every integer $n \geq 0$ we consider the Banach spaces $\mathcal{W}_{p,\delta_n}$ of functions $f : (\delta_n\mathbb{Z})^3 \mapsto \mathbf{R}$ with norms $\|\cdot\|_{p,n}$, $p = 1,2,3$:

$$\|f\|_{p,n} = \sup_{x\in(\delta_n\mathbb{Z})^3}\Big((|x|+\delta_n)^{\frac{6p+1}{4}}|f(x)|\Big) \tag{5.61}$$

We define the Banach space $\mathcal{W}_n = \mathcal{W}_{1,n}\times\mathcal{W}_{2,n}\times\mathcal{W}_{3,n}$ consisting of vectors $\mathbf{f} = (f^{(1)},f^{(2)},f^{(3)})$, $f^{(p)} : (\delta_n\mathbb{Z})^3 \mapsto \mathbf{R}$ with the norm

$$\|\mathbf{f}\|_n = \max_p \|f^{(p)}\|_n \tag{5.62}$$

Let $\mathbf{w}_n = (w_n^{(1)},w_n^{(2)},w_n^{(3)})$ as above.

Lemma 5.9

1. For L sufficiently large and $\varepsilon > 0$ sufficiently small there exists a constant k_L independent of n and ε such that for all $n \geq 1$,

$$\|\mathbf{w}_n\|_n \leq k_L/2 \tag{5.63}$$

If we start the sequence $\{\mathbf{w}_n\}_{n\geq 0}$ with $\mathbf{w}_0 \neq 0$, with $\|\mathbf{w}_n\|_{\delta_0} \leq k_L/2$, then

$$\|\mathbf{w}_n\|_n \leq k_L,\ \forall\, n \geq 0 \tag{5.64}$$

2. There exists a function $\mathbf{w}_*$ defined on $\cup_{n\geq 0}(\delta_n\mathbb{Z})^3 \subset \mathbb{Q}^3$ such that for every integer $l \geq 0$ held fixed , the sequence $\{\mathbf{w}_n\}_{l\leq n}$ converges to $\mathbf{w}_*$ in the norm $\|\cdot\|_l$ as $n\to\infty$. The convergence rate is given by

$$\|\mathbf{w}_n - \mathbf{w}_*\|_l \leq \tilde{c}_L L^{-qn} \tag{5.65}$$

where $q > 0$ is the constant in Theoerem 1.1 and Corollary 1.2.. We have $\mathbf{w}_* = \mathbf{v}_{c*} + \mathbf{w}_{*,L}$ in $\mathcal{W}_l$ for every $l \geq 0$.

Proof:
1. Let $m = n - j$ with $0 \leq j \leq n-1$. By definition $v_m^{(p)} = C_{m,L}^p - C_{m+1}^p$ with pointwise multiplication. Since $C_{m,L} = \Gamma_{m,L} + C_{m+1}$, it follows that $v_m^{(p)}$ has $\Gamma_{m,L}$ as factor. From the finite range property of $\Gamma_{m,L}$ it follows that

$$v_m^{(p)}(x) = 0 : \ |x| \geq 1$$

Theorem 1.1, part (5a), and Corollary 1.2 give uniform bounds on the Γ_m and C_m. Therefore there exists a constant $c_{L,p}$ independent of n such that

$$\|v_m^{(p)}\|_{L^\infty((\delta_m \mathbb{Z})^3)} \leq c_{L,p}$$

2. By definition

$$\begin{aligned}\|v_{n-j,L^j}^{(p)}\|_{p,n} &= \sup_{x \in (\delta_n \mathbb{Z})^3} \Big(L^{2d_s j} (|x| + \delta_n)^{\frac{6p+1}{4}} |v_{n-j}^{(p)}(L^j x)| \Big) \\ &= L^{-j(\frac{6p+1}{4} - 2d_s)} \sup_{y \in (\delta_{n-j} \mathbb{Z})^3} \Big((|y| + \delta_{n-j})^{\frac{6p+1}{4}} |v_{n-j}^{(p)}(y)| \Big)\end{aligned}$$

Because of the finite range property of $v_{n-j}^{(p)}$ of paragraph 1, we can bound $|y| \leq 1$ in the weight factor on the right. Because $n - j \geq 1$ we can bound in the weight factor $\delta_{n-j} \leq \delta_1 = L^{-1}$. Therefore on using the bound on $v_{n-j}^{(p)}$ of paragraph 1 we get

$$\|v_{n-j,L^j}^{(p)}\|_{p,n} \leq L^{-j(\frac{6p+1}{4} - 2d_s)} (1 + L^{-1})^{\frac{6p+1}{4}} c_{L,p}$$

We bound the first geometric factor by taking $p = 1$ and $\varepsilon > 0$ very small in $d_s = (3 - \varepsilon)/4$. This gives $L^{-j/5}$. This gives the bound

$$\|v_{n-j,L^j}^{(p)}\|_{p,n} \leq L^{-j/5} \, c_{L,p}$$

with a new constant $c_{L,p}$ independent of n. Using the above bound we get from (5.60) the bound

$$\|w_n^{(p)}\|_{p,n} \leq c_{L,p} \sum_{j=0}^{\infty} L^{-j/5} \leq 2 c_{L,p}$$

for L sufficiently large. Therefore setting $k_L = 4 \max_p c_{L,p}$ we get

$$\|\mathbf{w}_n\|_n \leq k_L/2$$

which proves (5.63). (5.64) is a trivial consequence of the above. This proves the first part of the lemma.
3. Let $v_{c*}^{(p)} = C_{c*,L}^p - C_{c*}^p$, with pointwise multiplication, where C_{c*} is the smooth continuum covariance in $\mathbb{R}^3$ of Corollary 1.2. By factoring out $\Gamma_{c*,l}$, Theorem 1.1 and Corollary 1.2 we have that $v_{c*}^{(p)}$ exists in $L^\infty(\mathbb{R}^3)$ and has finite range: $v_{c*}^{(p)}(x) = 0 : |x| \geq 1$. Moreover by Theorem 1.1 and Corollary 1.2 we have, see the proof of Lemma 5.12 for the detailed argument,

$$\|v_m^{(p)} - v_{c*}^{(p)}\|_{L^\infty((\delta_m \mathbb{Z})^3)} \leq c_{L,p} L^{-q(m-1)}$$

Define

$$w_*^{(p)} = \sum_{j=0}^{n-1} v_{c*,L^j}^{(p)}$$

Fix any integer $l \geq 0$. Then for $n \geq l$ the (p,l) norm is dominated by the (p,n) norm. Then proceeding as in the first part and using the previous inequality we get

$$\|w_n^p - w_*^p\|_{p,l} \leq \sum_{j=0}^{n-1} \|v_{n-j,L^j}^p - v_{c*,L^j}^p\|_{p,n} \leq c_{L,p} \sum_{j=0}^{n-1} L^{-j(\frac{6p+1}{4}-2d_s)} \|v_{n-j}^p - v_{c*}^p\|_{L^\infty((\delta_{n-j}\mathbb{Z})^3)}$$

$$\leq c_{L,p} \sum_{j=0}^{n-1} L^{-j/5} L^{-q(n-j-1)} \leq c'_{L,p} L^{-qn}$$

Now take the maximum over p. This proves (5.61) and at the same time the convergence statement of part 2 of the lemma. The last statement of part 2 is trivial to prove. This completes the proof of the second part of the lemma. ■

Lemma 5.10

Under the conditions of the domain $\mathcal{D}_n$ there exists constants $C_{p,L}$ independent of n such that

$$\|Q_n e^{-V_n}\|_{\mathbf{h},G_\kappa,\mathcal{A}_p,\delta_n} \leq C_{p,L}\, \bar{g}^{1/2} \tag{5.66}$$

$$|Q_n e^{-V_n}|_{\mathbf{h}_*,\mathcal{A}_p,\delta_n} \leq C_{p,L}\, \bar{g}^2 \tag{5.67}$$

Proof

$$Q_n(X_{\delta_n}) e^{-V_n(X_{\delta_n})} = g_n^2 \sum_{m=1}^{3} Q^{(m,m)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-m)}) e^{-V_n(X_{\delta_n})} \tag{5.68}$$

$$\left\| Q_n(X_{\delta_n},\varphi) e^{-V_n(X_{\delta_n},\varphi)} \right\|_{\mathbf{h}} \leq g_n^2 \sum_{m=1}^{3} \left\| Q^{(m,m)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-m)}) \right\|_{\mathbf{h}} \left\| e^{-V_n(X_{\delta_n})} \right\|_{\mathbf{h}} \tag{5.69}$$

Here X_{δ_n} is a small set because of the support properties of Q_n. The last factor will be estimated by Lemma 5.5. From (4.6) we have

$$Q^{(3,3)}(\hat{X}_{\delta_n}, \Phi; w_n^{(1)}) = 4 \int_{\hat{X}_{\delta_n}} dxdy\ \ : \Phi(x)\bar{\Phi}(x)\Phi(x)\bar{\Phi}(y)\Phi(y)\bar{\Phi}(y) :_{C_n}\ w_n^{(1)}(x-y) \tag{5.70}$$

We exhibit (5.70) as an element of the Grassmann algebra:

$$\begin{aligned} Q^{(3,3)}(\hat{X}_{\delta_n}, \Phi; w_n^{(1)}) &= Q_0^{(3,3)}(\hat{X}_{\delta_n}, \varphi; w_n^{(1)}) + \int_{\hat{X}_{\delta_n}} dxdy\ Q_1^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x, y; w_n^{(1)}) : \psi(x)\bar{\psi}(y) :_{C_n} + \\ &+ \int_{X_{\delta_n}} dx\ Q_2^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x; w_n^{(1)}) : \psi(x)\bar{\psi}(x) :_{C_n} + \\ &+ \int_{\hat{X}_{\delta_n}} dxdy\ Q_3^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x, y; w_n^{(1)}) : \psi(x)\bar{\psi}(x)\psi(y)\bar{\psi}(vy) :_{C_n} \end{aligned} \tag{5.71}$$

where

$$
\begin{aligned}
Q_0^{(3,3)}(\hat{X}_{\delta_n},\varphi;w_n^{(1)}) &= 4\int_{\hat{X}_{\delta_n}} dxdy\ :\varphi(x)\bar{\varphi}(x)\varphi(x)\bar{\varphi}(y)\varphi(y)\bar{\varphi}(y):_{C_n} w_n^{(1)}(x-y)\\
Q_1^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)}) &= 4:\varphi(x)\bar{\varphi}(x)\varphi(y)\bar{\varphi}(y):_{C_n} w_n^{(1)}(x-y)\\
Q_2^{(3,3)}(\hat{X}_{\delta_n},\varphi,x;w_n^{(1)}) &= 4\int_{X'_{\delta_n}} dy:(\varphi(x)\bar{\varphi}(y)+\varphi(y)\bar{\varphi}(x))\varphi(y)\bar{\varphi}(y):_{C_n} w_n^{(1)}(x-y)\\
Q_3^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)}) &= 4:\varphi(x)\bar{\varphi}(y):_{C_n} w_n^{(1)}(x-y)
\end{aligned}
\tag{5.72}
$$

where, denoting with $\Delta(x)$ the block Δ such that $x\in\Delta$, we have $X'_{\delta_n}=\begin{cases}\Delta & \text{if } X_{\delta_n}=\Delta\\ X_{\delta_n}\setminus\Delta(x) & \text{if } X_{\delta_n}=\Delta_1\cup\Delta_2\end{cases}$. Undo the Wick ordering, which produces lower order terms with coefficients which are uniformly bounded independent of n by Corollary 1.2. It is therefore enough to estimate with the Wick ordering taken off. We get

$$
\begin{aligned}
&\|Q^{(3,3)}(\hat{X}_{\delta_n},\Phi;w_n^{(1)})\|_{\mathbf{h}} \le \|Q_0^{(3,3)}(\hat{X}_{\delta_n},\varphi;w_n^{(1)})\|_{h_B}+\\
&+h_F^2\sup_{\|g_2\|_{C^2(\hat{X}^2_{\delta_n})}\le 1}\int_{\hat{X}_{\delta_n}} dxdy\ \|Q_1^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)})\|_{h_B}|g_2(x,y)|+\\
&+h_F^2\sup_{\|g_2\|_{C^2(\hat{X}^2_{\delta_n})}\le 1}\int_{X_{\delta_n}} dx\ \|Q_2^{(3,3)}(\hat{X}_{\delta_n},\varphi,x;w_n^{(1)})\|_{h_B}|g_2(x,x)|+\\
&+h_F^4\sup_{\|g_4\|_{C^2(\hat{X}^4_{\delta_n})}\le 1}\int_{\hat{X}_{\delta_n}} dxdy\ \|Q_3^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)})\|_{h_B}|g_4(x,x,y,y)|
\end{aligned}
$$

To estimate the h_B norm of the $Q_j^{(3,3)}$ we apply to (5.72) $h_B^kD^k$, with D the bosonic field derivative, $h_B=c\bar{g}^{-1/4}$ and use Lemma 5.1. Contributions for $k>4$ vanish. We use $g_n=O(\bar{g})$ from the domain hypothesis of Theorem 5.1. We estimate the kernel $w_n^{(1)}(x-y)$ using Lemma 5.9. As a result we get

$$
\|Q^{(3,3)}(\hat{X}_{\delta_n},\Phi;w_n^{(1)})\|_{\mathbf{h}} \le C_L\bar{g}^{-3/2}\int_{\hat{X}_{\delta_n}} dxdy\ \frac{1}{(|x-y|+\delta_n)^{7/4}}e^{\bar{g}/4\int_{X_{\delta_n}}dx|\varphi\bar{\varphi}(x)|^2}G_\kappa(X_{\delta_n},\varphi) \tag{5.73}
$$

The integral over $\hat{X}_{\delta_n}$ exists and is of $O(1)$ since X_{δ_n} is a small set in $(\delta_n\mathbb{Z})^3$. Therefore

$$
\|Q^{(3,3)}(\hat{X}_{\delta_n},\Phi;w_n^{(1)})\|_{\mathbf{h}} \le C_L\bar{g}^{-3/2}e^{\bar{g}/4\int_{X_{\delta_n}}dx|\varphi\bar{\varphi}(x)|^2}G_\kappa(X_{\delta_n},\varphi) \tag{5.74}
$$

Next turn to $Q^{(m,m)}$, $m=1,2$. From (4.6)

$$
\begin{aligned}
Q^{(2,2)}(\hat{X}_{\delta_n},\Phi;C,w_n^{(2)}) = -\int_{\hat{X}_{\delta_n}} dxdy\,\big[&:(\Phi(x)-\Phi(y))(\bar{\Phi}(x)-\bar{\Phi}(y))(\Phi(x)+\Phi(y))(\bar{\Phi}(x)+\bar{\Phi}(y)):_{C_n}+\\
&+:[(\Phi\bar{\Phi})(x)-(\Phi\bar{\Phi})(y)]^2:_{C_n}\big]\,w_n^{(2)}(x-y)
\end{aligned}
$$

We exhibit this as an element of the Grassmann algebra. This gives

$$
\begin{aligned}
&Q^{(2,2)}(\hat{X}_{\delta_n},\Phi;C_n,w_n^{(2)}) = -Q_0^{(2,2)}(\hat{X}_{\delta_n},\varphi;C_n,w_n^{(2)})-\int_{\hat{X}_{\delta_n}} dxdy\\
&\Big[Q_1^{(2,2)}(\hat{X}_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}):(\psi(x)+\psi(y))(\bar{\psi}(x)+\bar{\psi}(y)):_{C_n}+
\end{aligned}
$$

$$Q_2^{(2,2)}(\hat X_{\delta_n},\varphi,x,y;C_n,w_n^{(2)},):(\psi(x)-\psi(y))(\bar\psi(x)-\bar\psi(y)):_{C_n}+$$
$$+Q_3^{(2,2)}(\hat X_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}):(\psi\bar\psi(x)-\psi\bar\psi(y)):_{C_n}+$$
$$+Q_4^{(2,2)}(\hat X_{\delta_n},x,y,w_n^{(2)})\{(:\psi(x)-\psi(y))(\bar\psi(x)-\bar\psi(y))(\psi(x)+\psi(y))(\bar\psi(x)+\bar\psi(y)):_{C_n}+$$
$$:(\psi\bar\psi(x)-\psi\bar\psi(y))^2):_{C_n})\}\Big] \tag{5.75}$$

where

$$\begin{aligned}
Q_0^{(2,2)}(\hat X_{\delta_n},\varphi;C_n,w_n^{(2)}) &= \int_{\hat X_{\delta_n}} dxdy\, w_n^{(2)}(x-y)(:|\varphi(x)-\varphi(y)|^2|\varphi(x)+\varphi(y)|^2:_{C_n}\\
&\quad +:(|\varphi|^2(x)-|\varphi|^2(y))^2:_{C_n})\\
Q_1^{(2,2)}(\hat X_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}) &= w_n^{(2)}(x-y):|\varphi(x)-\varphi(y)|^2_{C_n}\\
Q_2^{(2,2)}(\hat X_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}) &= w_n^{(2)}(x-y):|\varphi(x)+\varphi(y)|^2:_{C_n}\\
Q_3^{(2,2)}(\hat X_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}) &= 2w_n^{(2)}(x-y):(|\varphi|^2(x)-|\varphi|^2(y)):_{C_n}\\
Q_4^{(2,2)}(\hat X_{\delta_n},x,y,w_n^{(2)}) &= w_n^{(2)}(x-y)
\end{aligned} \tag{5.76}$$

The $\|\cdot\|_{\mathbf{h},G_\kappa,\mathcal{A}_p}$ norm estimate for $Q^{(2,2)}(\hat X_{\delta_n},\Phi;C,w_n^{(2)})$ reposes on the following principles:
1. Undoing the Wick ordering produces lower order terms with Wick coefficients which are uniformly bounded independent of n by Corollary 1.2. Moreover by the domain hypothesis $g_n=O(\bar g)$
2. By Lemma 5.9, the kernel $w_n^{(2)}$ has the bound $|w_n^{(2)}(x-y)|\le k_L(|x-y|+\delta_n)^{-13/4}$ where the constant k_L is independent of n.
3. The fields $\varphi(x)$ are estimated by Lemma 5.1. Differences of fields $|\varphi(x)-\varphi(y)|$ are estimated by (5.18). This produces a factor $|x-y|$ which we retain, and majorise the Sobolev factor by the large field regulator. Diffferences of fields $\varphi\bar\varphi(x)-\varphi\bar\varphi(y)$ can also be expressed as in (5.17), substituting $\varphi\bar\varphi$ for φ. This requires estimating $(\partial_{\delta_n,e_j}\varphi\bar\varphi)(x+\cdot\cdot)$. We apply the lattice Leibniz which modifies the continuum rule by producing an extra term $\delta_n|\partial_{\delta_n,e_j}\varphi(x+\cdot\cdot)|^2$, (see equation (5.2),page 432 of [BGM]). We estimate the φ by Lemma 5.1, with $\kappa/2$ in the large field regulator. We estimate the gradient pieces by the Sobolev inequality as in (5.18), and then by the large field regulator with $\kappa/2$. We have also produced a factor $|x-y|$ as in (5.18).
Invoking the above principles we get the following bounds for the bosonic coefficients:

$$\frac{h_B^k}{k!}\|D^kQ_0^{(2,2)}(\hat X_{\delta_n},\varphi;C_n,w_n^{(2)})\|\le c_L\bar g^{-1/2}\int_{\hat X_{\delta_n}} dxdy\,(|x-y|+\delta_n)^{-(\frac{13}{4}+2)}e^{\bar g/4\int_{X_{\delta_n}}dx|\varphi\bar\varphi(x)|^2}G_\kappa(X_{\delta_n},\varphi) \tag{5.77}$$

and for $j=1,2,3$

$$\frac{h_B^k}{k!}\|D^kQ_j^{(2,2)}(\hat X_{\delta_n},\varphi;x,y,C_n,w_n^{(2)})\|\le c_L\bar g^{-1/2}\,(|x-y|+\delta_n)^{-13/4}f_j(x-y)e^{\bar g/4\int_{X_{\delta_n}}dx|\varphi\bar\varphi(x)|^2}G_\kappa(X_{\delta_n},\varphi) \tag{5.78}$$

where the maximum value of k which gives a nonvanising contribution is 4 and

$$f_1(x-y)=|x-y|^2,\ f_2(x-y)=1,\ f_3(x-y)=|x-y|, f_4(x-y)=1 \tag{5.79}$$

4. We must estimate the contribution of the fermionic pieces to the $\mathbf{h}$ norm. To this end denote by $F_j(\psi)$ the fermionic factor multiplying $Q_j^{(2,2)}$ in (5.75). Express the differences $\psi(x)-\psi(y)$ by the fermionic analogue of (5.14). We do the same also for $\psi\bar\psi(x)-\psi\bar\psi(y)$ and then apply the lattice Leibnitz rule to $\partial_{\delta_n,\varepsilon_j}\psi\bar\psi(x+\cdot)$. We

replace the fermionic pieces by the functions g_{2p} on $\hat{X}_{\delta_n} \cup \partial_2 \hat{X}_{\delta_n}$. and their lattice derivatives. Corresponding to $F_j(\psi)$ we get the contribution G_j which is a linear form on g_{2p_j}, where $p_j = 1$ for $j = 1,2,3$ and $p_4 = 2$. Let $\delta_n h_j$, $h_j \in \mathbf{Z}$ be the component of $y - x$ along the unit vector e_j. We have

$$G_1 = g_2(x,x) + g_2(x,y) + g_2(y,x) + g_2(y,y)$$

$$G_2 = \delta_n^2 \sum_{i_1,i_2=1}^{3} \sum_{0 \le s_{i_l} \le h_{i_l}-1,\, l=1,2} \partial^{(1)}_{\delta_n,e_{i_1}} \partial^{(2)}_{\delta_n,e_{i_2}} g_2(x + p_{i_1}(y-x, s_{i_1}), x + p_{i_2}(y-x, s_{i_2}))$$

$$G_3 = \delta_n \sum_{i=1}^{3} \sum_{s_i=0}^{h_i-1} \Big[\partial^{(1)}_{\delta_n,e_i} g_2(x + p_i(y-x,s_i), x + p_i(y-x,s_i)) + \partial^{(2)}_{\delta_n,e_i} g_2(x + p_i(y-x,s_i), x + p_i(y-x,s_i)) +$$

$$+ \delta_n \partial^{(1)}_{\delta_n,e_i} \partial^{(2)}_{\delta_n,e_i} g_2(x + p_i(y-x,s_i), x + p_i(y-x,s_i)) \Big]$$

$$G_4 = \delta_n^2 \sum_{i_1,i_2=1}^{3} \sum_{0 \le s_{i_l} \le h_{i_l}-1,\, l=1,2} \partial^{(1)}_{\delta_n,e_{i_1}} \partial^{(2)}_{\delta_n,e_{i_2}} \Big(g_4(x + p_{i_1}(y-x, s_{i_i}), x + p_{i_2}(y-x, s_{i_2}), x, x) +$$

$$+ g_4(x + \cdot\cdot, x + \cdot\cdot, x, y) + g_4(x + p_{i_1}(y-x, s_{i_1}), x + p_{i_2}(y-x, s_{i_2}), y, x) +$$

$$+ g_4(x + p_{i_1}(y-x, s_{i_1}), x + p_{i_2}(y-x, s_{i_2}), y, y) \Big) + \cdot\cdot$$

where the superscript on the lattice derivative denotes the argument on which it acts. The omitted terms $\cdot\cdot$ in G_4 comes from the square of the (first order) lattice Taylor expansion of $\psi\bar{\psi}(x) - \psi\bar{\psi}(y)$ and then replacing the product of 4 Grassmann fields by the test function g_4. For $j = 1,2,3$ we have the bounds

$$|G_j| \le O(1) \tilde{f}_j(x-y) \|g_2\|_{C^2(\hat{X}^2_{\delta_n})} \tag{5.80}$$

and for $j = 4$ we have

$$|G_4| \le O(1) \tilde{f}_4(x-y) \|g_4\|_{C^2(\hat{X}^4_{\delta_n})} \tag{5.81}$$

where

$$\tilde{f}_1 = |x-y|,\ \tilde{f}_2 = |x-y|^2,\ \tilde{f}_3 = |x-y|,\ \tilde{f}_4 = |x-y|^2 \tag{5.82}$$

.

On using the bounds (5.77)-(5.82) we get for $0 \le k \le 4$ and $0 \le p \le 2$

$$h_F^{2p} \frac{h_B^k}{k!} |D^{2p,n} Q^{(2,2)}(\hat{X}_{\delta_n}, \varphi, 0; f^{\times k}, g_{2p})| \le c_L \bar{g}^{-1/2} \int_{\hat{X}_{\delta_n}} dxdy\, (|x-y| + \delta_n)^{-(\frac{13}{4} - 2)}$$

$$\times e^{\bar{g}/4 \int_{\hat{X}_{\delta_n}} dx |\varphi\bar{\varphi}(x)|^2} G_\kappa(X_{\delta_n}, \varphi) \|f\|^{\times k}_{C^2(\hat{X}_{\delta_n})} \|g_{2p}\|_{C^2(\hat{X}^{2p}_{\delta_n})} \tag{5.83}$$

For $k > 4$ or $p > 2$ we have vanishing contribution. The integral over $\hat{X}_{\delta_n}$ exists and gives a contribution of $O(1)$ since X_{δ_n} is a small set in $(\delta_n \mathbb{Z})^3$. Therefore we obtain from the previous inequality

$$\|Q^{(2,2)}(\hat{X}_{\delta_n}, \varphi, 0)\|_{\mathbf{h}} \le c_L \bar{g}^{-1/2} e^{\bar{g}/4 \int_{\hat{X}_{\delta_n}} dx |\varphi\bar{\varphi}(x)|^2} G_\kappa(X_{\delta_n}, \varphi) \tag{5.84}$$

We can estimate in the same way the case $m = 1$. We have

$$\|Q^{(1,1)}(\hat{X}_{\delta_n}, \Phi; C, w^{(3)})\|_{\mathbf{h}} \leq c_L \bar{g}^{-1/2} \int_{\hat{X}_{\delta_n}} dxdy\ (|x-y| + \delta_n)^{-(\frac{19}{4}-2)} e^{\bar{g}/4 \int_{X_{\delta_n}} dx|\varphi\bar{\varphi}(x)|^2} G_\kappa(X_{\delta_n}, \varphi) \quad (5.85)$$

The integral over $\hat{X}_{\delta_n}$ exists and gives a contribution of $O(1)$. Therefore

$$\|Q^{(1,1)}(\hat{X}_{\delta_n}, \Phi; C, w^{(3)})\|_{\mathbf{h}} \leq c_L \bar{g}^{-1/2} e^{\bar{g}/4 \int_{X_{\delta_n}} dx|\varphi\bar{\varphi}(x)|^2} G_\kappa(X_{\delta_n}, \varphi) \tag{5.86}$$

Therefore from (5.69), Lemma 5.5 and the bounds (5.74), (5.84),(5.86) we get

$$\left\| Q_n(X_{\delta_n}) e^{-V_n(X_{\delta_n})} \right\|_{\mathbf{h},\mathbf{G}_\kappa} \leq c_L \bar{g}^{1/2}$$

and since Q_n is supported on small sets we get

$$\|Q_n e^{-V_n}\|_{\mathbf{h}, G_\kappa, \mathcal{A}_p} \leq C_{L,p} \bar{g}^{1/2} \tag{5.87}$$

which is (5.66). To prove (5.67) we estimate the r.h.s of (5.69) at $\Phi = 0$ after undoing the Wick ordering, set $\mathbf{h} = \mathbf{h}_*$, and use Lemma 5.5. ∎

In the following lemma we consider $Q_n(\Phi + \xi) e^{-V_n(\Phi+\xi)}$ as a function of $\varphi, \zeta, \psi, \eta$.

Lemma 5.11
Under the conditions of the domain $\mathcal{D}_n$ there exists constants $C_{L,p}$ independent of n such that

$$\|Q_n e^{-V_n}\|_{\mathbf{h}, \hat{G}_{\kappa,\rho}, \mathcal{A}_p, \delta_n} \leq C_{L,p} \bar{g}^{1/2} \tag{5.88}$$

$$\|Q_n e^{-V_n}\|_{\mathbf{h}_*, \tilde{G}_{\kappa,\rho}, \mathcal{A}_p, \delta_n} \leq C_{L,p} \bar{g}^2 \tag{5.89}$$

Proof
The bound (5.88) follows from (5.66) since $\hat{G}_{\kappa,\rho} > G_\kappa$. To prove (5.89) we first express $Q_n^{(m,m)}(X_{\delta_n}, \varphi + \zeta, \psi + \eta)$ in the Grassmann representation as in the proof of Lemma 5.10, substituting in the expressions there $\varphi \to \varphi + \zeta$, $\psi \to \psi + \eta$. Field derivatives are defined as in (5.26). For the bosonic coefficients we take derivatives at $\varphi = 0$. The resulting dependence on ζ is estimated by Lemma 5.2. The rest of the proof follows that of Lemma 5.10. We use Lemma 5.5 which implies that $\|e^{-V_n(X_{\delta_n}, \zeta, \eta)}\|_{\mathbf{h}_*} \leq 2^{|X_{\delta_n}|}$, and X_{δ_n} is a small set. ∎

We now prove a lemma to control the perturbative flow coefficients a_n, b_n given in (4.15) and (4.17). This lemma is independent of the domain $\mathcal{D}_n$.

Lemma 5.12
Let $v_{c*}^{(p)} = C_{c*,L}^p - C_{c*}^p$, with pointwise multiplication, where C_{c*} is the smooth continuum covariance in $\mathbb{R}^3$ of Corollary 1.2. Define

$$a_{c*} = 2 \int_{\mathbb{R}^3} dy\, v_{c*}^{(2)}(y), \quad b_{c*} = 4 \int_{\mathbb{R}^3} dy\, v_{c*}^{(3)}(y)$$

We have that a_n, b_n, a_{c*}, b_{c*} are strictly positive. Moreover there exist constants c_L independent of n such that

$$|a_n| \leq c_L, \quad |b_n| \leq c_L, \quad |a_{c*}| \leq c_L, \quad |b_{c*}| \leq c_L \tag{5.90}$$

and

$$|a_n - a_{c*}| \le c_L L^{-qn}, \quad |b_n - b_{c*}| \le c_L L^{-qn} \tag{5.91}$$

where $q>0$ is as in Therem 1.1.

Remark : The convergence rate estimates (5.91) play no role in the estimates of the present section. They are used in Section 6 for the existence proof of a global renormalization group trajectory.

Proof

From (4.16), for $p=2,3$, using $C_{n,L} = \Gamma_{n,L} + C_{n+1}$

$$v^{(p)}_{n+1} = C^p_{n,L} - C^p_{n+1} = \Gamma_{n,L}(\Gamma^{p-1}_{n,L} + p\Gamma^{p-2}_{n,L} C_{n+1} + \delta_{p,3} 3C^2_{n+1}) \tag{5.92}$$

with pointwise multiplication. The positivity in Fourier space of the integral kernels on the right hand side implies that $a_n>0$, $b_n>0$ as claimed. The common factor of $\Gamma_{n,L}(x)$ which has finite range 1 implies that $v^{(p)}_{n+1}(x)$ has support in the unit ball in $(\delta_{n+1}\mathbb{Z})^3$. From Theorem.1.1 and Corollary 1.2 we have that v^p_{n+1} above are uniformly bounded in $L^\infty((\delta_{n+1}\mathbf{Z})^{\mathbf{3}})$ by constants c_L. By the same arguments v_{c*} has finite range and belongs to $L^\infty(\mathbb{R}^3)$. The uniform bounds in the first part of the lemma now follow.

By the same arguments using $C_{*,L} = \Gamma_{c*} + C_{c*}$ we have that $a_{c*}>0$, $b_{c*}>0$ and $v^{(p)}_{c*}(x)$ has support in the unit ball in $\mathbb{R}^3$. Moreover using Corollary 1.2 we have $\|v^{(p)}_{c*}\|_{C^k(\mathbb{R}^3)} \le c_{k,L}$ for all $\kappa \ge 0$.

Define

$$a^{(p)}_n = \int_{(\delta_{n+1}\mathbb{Z})^3} dy\, v^{(p)}_{n+1}(y), \quad a^{(p)}_{c*} = \int_{\mathbb{R}^3} dy\, v^{(p)}_{c*}(y) \tag{5.93}$$

Then using the compact support property of $v^{(p)}_{n+1}$ and $v^{(p)}_{c*}$ we get

$$|a^{(p)}_n - a^{(p)}_{c*}| \le \|v^{(p)}_{n+1} - v^{(p)}_{c*}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} + \left| \int_{(\delta_{n+1}\mathbb{Z})^3} dy\, v^{(p)}_{c*}(y) - \int_{\mathbb{R}^3} dy\, v^{(p)}_{c*}(y) \right| \tag{5.94}$$

We estimate the first term on the right hand side of (5.94). We have

$$\|v^{(p)}_{n+1} - v^{(p)}_{c*}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} \le \|C^p_{c*,L} - C^p_{n,L}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} + \|C^p_{c*} - C^p_{n+1}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} \tag{5.95}$$

In the first term on the right in (5.95) we factor out $C_{c*,L} - C_{n,L}$ and in the second term we factor out $C_{c*} - C_{n+1}$. Then use of the bounds in Corollary 1.2 gives

$$\|v^{(p)}_{n+1} - v^{(p)}_{c*}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} \le c_{L,p} L^{-qn} \tag{5.96}$$

We estimate the second term on the right in (5.94) using Lemma 6.6 of [BGM] and the compact support of v_{c*}. This gives

$$\left| \int_{(\delta_{n+1}\mathbb{Z})^3} dy\, v^{(p)}_{c*}(y) - \int_{\mathbb{R}^3} dy\, v^{(p)}_{c*}(y) \right| \le O(1)\delta_{n+1} \|v^{(p)}_{c*}\|_{C^1(\mathbb{R}^3)} \le c_{L,p} L^{-(n+1)} \tag{5.97}$$

From (5.94), (5.96) and (5.97) we get with q that of Theorem 1.1

$$|a^{(p)}_n - a^{(p)}_{c*}| \le c_{L,p} L^{-qn} \tag{5.98}$$

which completes the proof of the lemma. ∎

Lemma 5.13

Under the conditions of the domain $\mathcal{D}_n$ there exist constants $C_{p,L}$ independent of n such that

$$\|Q_n(e^{-V_n} - e^{-\tilde{V}_n})\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A}_p,\delta_n} \leq C_{L,p}\bar{g}^{3/4} \tag{5.99}$$

$$\|Q_n(e^{-V_n} - e^{-\tilde{V}_n})\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A}_p,\delta_n} \leq C_{L,p}\bar{g}^{3} \tag{5.100}$$

Proof: The proof is on the same lines as that of the corresponding Lemma 5.13 in [BMS]. It follows from Lemmas 5.11, 5.6, 5.10, and 5.5 which are lattice equivalents of the corresponding lemmas in [BMS].

Lemma 5.14
Under the conditions of the domain $\mathcal{D}_n$ there exists constants C_L independent of n such that $K_n(\lambda)$ given by (4.10) satisfies the bounds

$$\|K_n(\lambda)\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A},\delta_n} \leq C_L|\lambda\bar{g}^{1/4-\eta/3}|^2 \quad \text{for } |\lambda\bar{g}^{1/4-\eta/3}| < 1 \tag{5.101}$$

$$\|K(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n} \leq C_L|\lambda\bar{g}^{11/12-\eta/3}|^2 \quad \text{for } |\lambda\bar{g}^{11/12-\eta/3}| < 1 \tag{5.102}$$

Proof: This follows from Lemmas 5.11 and 5.13 and the hypothesis (5.5) on R_n.

The following lemma generalizes Lemma 5.15 of [BMS] to the lattice and the additional presence of Grassmann fields. It will play a key role later in obtaining contractive estimates.

Lemma 5.15
For any polymer activity $\tilde{K}(X_{\delta_n}, \phi+\zeta, \psi+\eta)$:

$$\|\tilde{K}(X_{\delta_n},\zeta,0)\|_{\mathbf{h}_*} \leq O(1)\tilde{G}_{\rho,\kappa}(X_{\delta_n},\zeta)\left[|\tilde{K}(X_{\delta_n})|_{\mathbf{h}_*} + h_B^{-m_0}h_{B*}^{m_0}\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa}\right] \tag{5.103}$$

$$\|\tilde{K}(Y_{\delta_n},\phi,0)\|_{\mathbf{h}} \leq O(1)e^{\gamma\bar{g}\int_{Z_{\delta_n}\setminus Y_{\delta_n}} dy(\phi\bar{\phi}(y))^2} G_\kappa(Z_{\delta_n},\phi)\left[|\tilde{K}(Y_{\delta_n})|_{\mathbf{h}} + L^{-m_0 d_s}\|\tilde{K}(Y_{\delta_n})\|_{h_F,L^{d_s}h_B,G_\kappa}\right] \tag{5.104}$$

$$|\tilde{K}^\sharp(X_{\delta_n})|_{\hat{\mathbf{h}}_*} \leq O(1)2^{|X_{\delta_n}|}\left[|\tilde{K}(X_{\delta_n})|_{\mathbf{h}_*} + h_B^{-m_0}h_{B*}^{m_0}\|vv\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa}\right] \tag{5.105}$$

$$\|\tilde{K}^\sharp(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{2\kappa}} \leq 2^{|X_{\delta_n}|}\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa} \tag{5.106}$$

where $\tilde{G}_{\rho,\kappa}$ is as defined in (5.19), and $m_0 = 9$ is the maximum number of derivatives appearing in the definition of Kernel and h norms. In (5.104), $Y_{\delta_n}, Z_{\delta_n}, \gamma$ are as described in Lemma 5.1. Moreover in the above norms $\mathbf{h}_* = (h_F, h_{B*})$, and $\hat{\mathbf{h}}_* = (\frac{h_F}{2}, h_{B*})$ where $h_{B*} = (\rho\kappa)^{-1/2}$, $\mathbf{h} = (h_F, h_B)$, $\hat{\mathbf{h}} = (\frac{h_F}{2}, h_B)$, $h_B = c\bar{g}^{-1/4}$, c=O(1) very small, and h_F is taken to be sufficiently large depending on L. The last condition plays a role in the proofs of (5.105) and (5.106).

The superscript $\sharp$ stands for $d\mu_{\Gamma_n}(\zeta)$ integration. ρ is chosen as in Lemma 5.3, and κ as in Lemma 2.1. Note that we have that the constant $C(\rho,\kappa,j)$ appearing in Lemma 5.2 (this bounds ζ^j) satisfies

$$C(\rho,\kappa,j) = h_{B*}^j O(1)^j \tag{5.107}$$

.

Proof

We will first prove (5.103) following the lines of the proof of lemma 5.15 of [BMS] where the Grassman fields were absent. Recall from the definition in (5.35) that

$$\|\tilde{K}(X_{\delta_n}, 0, 0, \zeta, 0, 0)\|_{\mathbf{h}_*} = \sum_{m=0}^{m_0} \frac{h_{B*}^m}{m!} A_m \tag{5.108}$$

where

$$A_m = \sum_{p\geq 0} h_F^{2p} \frac{h_{B*}^m}{m!} \|D^{2p,m}\tilde{K}(X_{\delta_n}, 0, 0, \zeta, 0, 0)\| \tag{5.109}$$

First conside the case $m = m_0$. Then

$$A_{m_0} \leq h_{B*}^{m_0} h_B^{-m_0} \|\tilde{K}(X_{\delta_n}\|_{\mathbf{h},G_\kappa} \tilde{G}_{\rho,\kappa} \tag{5.110}$$

since $G_\kappa \leq \tilde{G}_{\rho,\kappa}$. Now let $m < m_0$.
We expand in ζ in Taylor series with remainder

$$\begin{aligned}(D^{2p,m}\tilde{K})(X_{\delta_n}, \zeta, 0; f^{\times m}, g_{2n}) = \sum_{j=0}^{m_0-m-1} \frac{1}{j!} (D^{2p,j+m}\tilde{K})(X_{\delta_n}, 0, 0; f^{\times m}, \zeta^{\times j}, g_{2p}) + \\ + \frac{1}{(m_0-m-1)!} \int_0^1 ds (1-s)^{m_0-m-1} (D^{2p,m}\tilde{K})(X_{\delta_n}, s\zeta, 0; f^{\times m}, \zeta^{\times m_0-m}, g_{2p})\end{aligned} \tag{5.111}$$

Therefore

$$\begin{aligned}\|(D^{2p,m}\tilde{K})(X_{\delta_n}, \zeta, 0)\| \leq \sum_{j=0}^{m_0-m-1} \frac{1}{j!} \|(D^{2p,j+m}\tilde{K})(X_{\delta_n}, 0, 0)\| \|\zeta\|^j_{C^2(X_{\delta_n})} + \\ + \frac{1}{(m_0-m-1)!} \int_0^1 ds (1-s)^{m_0-m-1} \|\zeta\|^{m_0-m}_{C^2(X_{\delta_n})} \|(D^{2p,m_0}\tilde{K})(X_{\delta_n}, s\zeta, 0)\|\end{aligned}$$

Hence

$$\begin{aligned}A_m \leq \sum_{j=0}^{m_0-m-1} \frac{(j+m)!}{j!m!} h_{B*}^{-j} \|\zeta\|^j_{C^2(X_{\delta_n})} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}_*} + \\ + \frac{m_0! h_{B*}^{m_0} h_B^{-m_0}}{m!(m_0-m-1)!} \int_0^1 ds (1-s)^{m_0-m-1} h_{B*}^{-(m_0-m)} \|\zeta\|^{m_0-m}_{C^2(X_{\delta_n})} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa} \tilde{G}_{\rho,\kappa}(X_{\delta_n}, s\zeta)\end{aligned}$$

By Lemma 5.2, with ζ replaced by $\sqrt{1-s^2}\zeta$, and (5.107),

$$h_{B*}^{-j} \|\zeta\|^j_{C^2(X_{\delta_n})} \leq O(1)^j \tilde{G}_{\rho,\kappa}(X_{\delta_n}, \sqrt{1-s^2}\,\zeta) \frac{1}{(1-s^2)^{j/2}} \tag{5.112}$$

where $0 \leq s < 1$. With $s = 0$ this bound is applied to the terms in the sum over j. For the Taylor remainder term take $j = m_0 - m$ and note that $(1-s)^{(m_0-m)/2-1}$ is integrable since $m_0 > m$. Hence:

$$\begin{aligned}A_m &\leq O(1)\tilde{G}_{\rho,\kappa}(X,\zeta) \Big[\sum_{j=0}^{m_0-m-1} \frac{(j+m)!}{j!m!} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}_*} + \frac{m_0! h_{B*}^{m_0} h_B^{-m_0}}{m!(m_0-m-1)!} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa} \Big] \\ &\leq O(1)\tilde{G}_{\rho,\kappa}(X,\zeta) \Big[\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}_*} + h_{B*}^{m_0} h_B^{-m_0} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa} \Big]\end{aligned} \tag{5.113}$$

Summing (5.113) over $0 \le m \le m_0 - 1$ and adding (5.110) proves (5.103).

Inequality (5.104) is also proved in the same way as (5.103). We are estimating the $\mathbf{h}$ norm which is given by (5.108) with $\mathbf{h}_*$ replaced by $\mathbf{h}$, h_B* by h_B and ζ by φ. We replace $\tilde{G}_{\rho,\kappa}$ by G_κ. Then (5.110) remains true with ζ replaced by φ and $\tilde{G}_{\rho,\kappa}$ replaced by G_k. Subsequently for $m < m_0$ we expand in Taylor series as above but now in φ. We do the norm estimate as above but now using Lemma 5.1 in place of Lemma 5.2. For ε sufficiently small depending on L we have $\bar{g}$ sufficiently small and therefore h_B^{-1} is sufficiently small. Hence $h_B^{-j}C \le O(1)$ where $C = \kappa^{-j/2}O(1)$ is the constant appearing in Lemma 5.1. In the Taylor remainder term we replace h_B by $L^{d_s}h_B$. which leads to the factor $L^{-m_0 d_s}$.

Next we will prove (5.105) and (5.106). Recall that $\hat{\mathbf{h}}_* = (\frac{h_F}{2}, h_{B*})$ and $\hat{\mathbf{h}} = (\frac{h_F}{2}, h_B)$. We consider the fluctuation integration on $\tilde{K}(X, \varphi, \psi)$ carrying it out explicitly for the Grassman fields. Then we have

$$\tilde{K}^\sharp(X_{\delta_n}, \varphi, \psi) = \int d\mu_{\Gamma_n}(\zeta)\Big[\sum_{p\ge 0}\sum_{\mathbf{l},\mathbf{a}}\sum_{\substack{I\subset\{1,\dots,2n\}\\ |I|\text{even}}}\int_{X^{2p}_{\delta_n}} d\mathbf{x}\tilde{K}^{\mathbf{l},\mathbf{a}}_{2p}(X_{\delta_n}, \varphi+\zeta, \mathbf{x})\partial^{\mathbf{l}}_{\delta_n}\prod_{i\in I}\psi_{a_i}(x_i)\det \Gamma^{\mathbf{a}}_{n\,I^c}(\mathbf{x}_{I^c})\Big] \tag{5.114}$$

where $I^c = \{1, ..., 2p\}\setminus I$. The matrix $\Gamma^{\mathbf{a}}_{n\,I^c}$ is an $|I^c|/2 \times |I^c|/2$ square matrix whose entry $(\Gamma^{\mathbf{a}}_{n\,I^c})_{rs}$ is defined for $r, s \in I^c$ when $a_r = 1$ and $a_s = -1$ and is then given by

$$(\Gamma^{\mathbf{a}}_{n\,I^c})_{r,s} = \Gamma_n(x_r - x_s)$$

Therefore,

$$D^{2j,m}\tilde{K}^\sharp(X_{\delta_n}, \varphi, 0; f^{\times m}, g_{2j}) = \int d\mu_{\Gamma_n}(\zeta)\Big[\sum_{p\ge 0}\sum_{\substack{I\subset\{1,\dots,2p\}\\ |I|\text{even}}}\delta_{2j,|I|}\sum_{\mathbf{l},\mathbf{a}}\int_{X^{2p}_{\delta_n}} d\mathbf{x}D^m\tilde{K}^{\mathbf{l},\mathbf{a}}_{2p}(X_{\delta_n}, \varphi+\zeta, \mathbf{x}; f^{\times m})$$

$$\partial^{\mathbf{l}}_{\delta_n} g_{|I|}(x_{i_1}, ..., x_{i_{|I|}})\det \Gamma^{a}_{n\,I^c}(\mathbf{x})\Big] \tag{5.115}$$

Define the function $\tilde{g}_{2p,I}$ on $X^{2p}_{\delta_n}$ by

$$\tilde{g}_{2p,I}(x_1, ..., x_{2p}) = g_{|I|}(x_{i_1}, ..., x_{i_{|I|}})\det \Gamma^{\mathbf{a}}_{n\,I^c}(\mathbf{x}_{I^c})$$

Then we can write (5.115) as

$$D^{2j,m}\tilde{K}^\sharp(X_{\delta_n}, \varphi, 0; f^{\times m}, g_{2j}) = \int d\mu_{\Gamma_n}(\zeta)\Big[\sum_{p\ge 0}\sum_{\substack{I\subset\{1,\dots,2p\}\\ |I|\text{even}}}\delta_{2j,|I|}D^{2p,m}\tilde{K}(X_{\delta_n}, \varphi+\zeta, 0; f^{\times m}, \tilde{g}_{2p,I})\Big]$$

Let $\partial^k_{\delta_n}$ be the lattice forward derivative of order k in multi-index notation. By part (5a) of Theorem 1.1 we have

$$\max_{0\le k\le 4}\|\partial^k_{\delta_n}\Gamma_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \le C_L \tag{5.116}$$

where C_L is a constant independent of n. Let now $\partial^{k_r}_{\delta_n}$, $0 \le k \le 2$, be the lattice forward derivative of order k_r, $0 \le k_r \le 2$ with respect to the points $x_r \in I^c$. Let $\mathbf{k} = (k_1, ..., k_{|I^c|})$ and define $\partial^{\mathbf{k}}_{\delta_n} = \prod_{r=1}^{|I^c|}\partial^{k_r}_{\delta_n}$. Let $\partial^{\mathbf{k}}_{\delta_n}$ act on the determinant. This produces another determinant with derivatives acting on the matrix elements $\Gamma_n(x_r - x_s)$. Since Γ_n is positive definite these matrices can be written as Gram matrices by a standard argument. Thus the matrix $a_{ij} = \partial^{k_i}_{\delta_n}\partial^{k_j}_{\delta_n}\Gamma_n(x_i - x_j)$ with $i, j \in I^c$ can be written as $a_{i,j} = (f_i, f_j)_{L^2((\delta_n\mathbb{Z})^3)}$

where $f_i(\cdot) = \partial_{\delta_n}^{k_i}\Gamma_n^{\frac{1}{2}}(x_i,\cdot)$. Gram's inequality says $|\det a_{ij}| \le \prod_{i=1}^{|I^c|}\|f_i\|^2_{L^2((\delta_n\mathbb{Z})^3)}$. We have $\|f_i\|^2_{L^2((\delta_n\mathbb{Z})^3)} = \partial_{\delta_n}^{*k_i}\partial_{\delta_n}^{k_i}\Gamma_n(x_i-y)$ for $y=x_i$. $\partial^*_{\delta_n}$, the L^2 adjoint of ∂_{δ_n}, is the backward derivative. Since $2k_i\le 4$, we have by (5.116) the bound $\|f_i\|^2_{L^2((\delta_n\mathbb{Z})^3)} \le C_L$. We therefore get

$$|\partial_{\delta_n}^{\mathbf{k}}\det\Gamma^{\mathbf{a}}_{n\,I^c}(\mathbf{x}_{I^c})| \le C_L^{|I^c|} \tag{5.117}$$

Hence $\det\Gamma^{\mathbf{a}}_{n\,I^c}(\mathbf{x}_{I^c}))$ belongs to $C^2(X^{I_c}_{\delta_n})$ and its C^2 norm is bounded by $C_L^{|I^c|}$. Since $g_{|I|}$ belongs to $C^2(X^{|I|}_{\delta_n})$ we have that the product function $\tilde g_{2p,I} = g_{2|I|}\det\Gamma^{\mathbf{a}}_{n\,I^c}$ belongs to $C^2(X^{2p}_{\delta_n})$ and its norm is bounded by the product of the two norms. Therefore on using the definition of the norm in (2.15) we get

$$\left|D^{2p,m}\tilde K(X_{\delta_n},\varphi+\zeta,0;f^{\times m},\tilde g_{2p,I})\right| \le \|D^{2p,m}\tilde K(X_{\delta_n},\varphi+z,0)\|C_L^{|I_c|}.\|g_{2j}\|_{C^2(X^{2j}_{\delta_n})}\|f\|^m_{C^2(X_{\delta_n})}$$

where we have used the constraint $|I|=2j$. Hence

$$|D^{2j,m}\tilde K^\sharp(X_{\delta_n},\varphi,0;f^{\times m},g_{2j})| \le \int d\mu_{\Gamma_n}(\zeta)\sum_{p\ge 0}\sum_{I\subset\{1,\ldots,2p\}}\delta_{2j,|I|}\|D^{2p,m}\tilde K(X_{\delta_n},\varphi+\zeta,0)\|\times$$

$$\times C_L^{|I_c|}\|g_{2j}\|_{C^2(X^{2j}_{\delta_n})}\|f\|^m_{C^2(X_{\delta_n})}$$

whence

$$\|D^{2j,m}\tilde K^\sharp(X_{\delta_n},\varphi,0)\| \le \int d\mu_{\Gamma_n}(\zeta)\sum_{p\ge 0}\sum_{I\subset\{1,\ldots,2p\}}\delta_{2j,|I|}C_L^{|I_c|}\|D^{2p,m}\tilde K(X_{\delta_n},\varphi+\zeta,0)\|$$

Multiplying both sides by $\hat h_F^{2j}$ and summing over j gives

$$\sum_{j\ge 0}\hat h_F^{2j}\|D^{2j,m}\tilde K^\sharp(X_{\delta_n},\varphi,0)\| \le \int d\mu_{\Gamma_n}(\zeta)\sum_{p\ge 0}\sum_{I\subset\{1,\ldots,2p\}}\hat h_F^{|I|}C_L^{|I_c|}\|D^{2p,m}\tilde K(X_{\delta_n},\varphi+\zeta,0)\|$$

Observe that since $\hat h_F = h_F/2$ and $2p = |I|+|I_c|$

$$\hat h_F^{|I|} = 2^{-2p}h_F^{2p}\hat h_F^{-|I_c|}$$

Choose $h_F > 2C_L$, so that $\hat h_F > C_L$. We then obtain

$$\sum_{j\ge 0}\hat h_F^{2j}\|D^{2j,m}\tilde K^\sharp(X_{\delta_n},\varphi,0)\| \le \int d\mu_{\Gamma_n}(\zeta)\sum_{p\ge 0}2^{-2p}h_F^{2p}\sum_{I\subset\{1,\ldots,2p\}}\|D^{2p,m}\tilde K(X_{\delta_n},\varphi+\zeta,0)\|$$

Now $\sum_{I\subset\{1,\ldots,2p\}}1 = 2^{2p}$. Therefore

$$\sum_{j\ge 0}\hat h_F^{2j}\|D^{2j,m}\tilde K^\sharp(X_{\delta_n},\varphi,0)\| \le \int d\mu_{\Gamma_n}(\zeta)\sum_{p\ge 0}h_F^{2p}\|D^{2p,m}\tilde K(X_{\delta_n},\varphi+\zeta,0)\| \tag{5.118}$$

Set $\varphi=0$ in (5.118), multiply both sides of the above inequality by $h_B *^m/m!$ and sum over $0\le m\le m_0$ which gives

$$|\tilde K^\sharp(X_{\delta_n},0,0)|_{\hat{\mathbf{h}}_*} \le \int d\mu_{\Gamma_n}(\zeta)\|\tilde K(X_{\delta_n},\zeta,0)\|_{\mathbf{h}_*} \tag{5.119}$$

The inequality (5.105) now follows on using (5.103) followed by Lemma 5.3. To prove (5.106) we multiply (5.118) by $h_B^m/m!$ and sum over m as before to obtain

$$\|\tilde K^\sharp(X_{\delta_n},\varphi,0)\|_{\hat{\mathbf{h}}} \le \int d\mu_{\Gamma_n}(\zeta)\|\tilde K(X_{\delta_n},\varphi+\zeta,0)\|_{\mathbf{h}}$$

(5.106) now follows on using the stability of the large field regulator G_κ. ∎

The next lemma extends lemma 5.16 of [BMS] to the case when Grassmann fields are also present.

Lemma 5.16

For any $q>0$, there exists constants c_L independent of n such that for L sufficiently large, ε sufficiently small and h_F sufficiently large depending on L,

$$\|\mathcal{S}(\lambda,K_n)^\natural\|_{\mathbf{h},G_\kappa,\mathcal{A}_p,\delta_{n+1}}\le q \qquad \text{when } |\lambda\bar g^{1/4-\eta/3}|\le c_L \tag{5.120}$$

$$|\mathcal{S}(\lambda,K_n)^\natural|_{\mathbf{h}_*,\mathcal{A}_p,\delta_{n+1}}\le q \qquad \text{when } |\lambda\bar g^{11/12-\eta/3}|\le c_L \tag{5.121}$$

where $\mathbf{h}_*=(h_{B*},h_F)$ and $\natural$ denotes integration with respect to $d\mu_{\Gamma_{n,L}}(\xi)$, $\Gamma_{n,L}=S_{L^{-1}}\Gamma_n$ being the rescaled fluctuation covariance.

When $R_n=0$ we may set $\eta=0$ in (5.120) and replace $\lambda\bar g^{11/12-\eta/3}$ by $\lambda\bar g^{1/2-\delta/2}$ in (5.121)

Proof

We suppress the dependence on λ which plays a passive role in most of the following and make the dependence explicit towards the end when necessary. By definition

$$(\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}},\Phi)=\int d\mu_{\Gamma_n}(\xi)\mathcal{B}K_n(LZ_{\delta_n},S_L\Phi,\xi)$$

where $\mathcal{B}$ is the reblocking operator defined in (3.10) and after the introduction of the λ parameter in (4.11). We write the above in the Grassmann representation (5.24)

$$\mathcal{B}K_n(LZ_{\delta_n},S_L\Phi,\xi)=\sum_{p\ge0}\sum_{\mathbf{l},\mathbf{a}}\sum_{I\subset\{1,..,2p\}}\int_{(LZ_{\delta_n})^{2p}}d\mathbf{x}\,\mathcal{BK}^{\mathbf{l},\mathbf{a},I}_{n,2p}(LZ_{\delta_n},S_L\varphi,\zeta,\mathbf{x})\partial^{\mathbf{l}}_{\delta_n}\prod_{k\in I}S_L\psi_{a_k}(x_k)\prod_{k\in I^c}\eta_{a_k}(x_k)$$

Therefore

$$(\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}},\varphi,\psi)=\int d\mu_{\Gamma_n}(\zeta)\Big[\sum_{p\ge0}\sum_{\mathbf{l},\mathbf{a}}\sum_{\substack{I\subset\{1,..,2p\}\\|I|\text{even}}}\int_{(LZ_{\delta_n})^{2p}}d\mathbf{x}\,\mathcal{BK}^{\mathbf{l},\mathbf{a},I}_{n,2p}(LZ_{\delta_n},S_L\varphi,\zeta,\mathbf{x})\times$$

$$\times\partial^{\mathbf{l}}_{\delta_n}\prod_{k\in I}S_L\psi_{a_k}(x_k)\mathrm{det}\Gamma_{I^c}(\mathbf{x}_{I^c})\Big]$$

where we have carried out the Grassmann integration explicitly as in (5.114) with the same notations for the determinant. We follow the arguments after (5.114). Thus we define the function $\tilde g_{2p,I}$ on $(LZ_{\delta_n})^{2p}$ by

$$\tilde g_{2p,I}(x_1,...,x_{2p})=S_Lg_{|I|}(x_{i_1},...,x_{i_{|I|}})\det\Gamma^{\mathbf{a}}_{n\,I^c}(\mathbf{x}_{I^c})$$

Note that $g_{|I|}$ scales in the same way as the product of $|I|$ Grassmann fields. Observe that

$$D^{2j,m}(\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}},\varphi,0;f^{\times m},g_{2j})=\int d\mu_{\Gamma_n}(\zeta)\Big[\sum_{p\ge0}\sum_{\substack{I\subset\{1,\dots,2p\}\\|I|\text{even}}}\delta_{2j,|I|}\times$$

$$\times D^{2p,m}(\mathcal{B}K_n)^I_{2p}(LZ_{\delta_n},\varphi,\zeta,0;f^{\times m},\tilde g_{2p,I})\Big] \tag{5.122}$$

where $(\mathcal{B}K_n)^I_{2p}$ is defined as after (5.27). By (5.117), the arguments following it, and the constraint $|I|=2j$

$$\|\tilde g_{2p,I}\|_{C^2((LZ_{\delta_n})^{2p})}\le C_L^{|I^c|}\|S_Lg_{2j}\|_{C^2((LZ_{\delta_n})^{2j})}\|S_Lf\|^{\times m}_{C^2(LZ_{\delta_n})}$$

$$\leq C_L^{|I^c|} L^{-|I|d_s} L^{-md_s} \|g_{2j}\|_{C^2(Z^{2j}_{\delta_{n+1}})} \|f\|^{\times m}_{C^2(Z_{\delta_{n+1}})}$$

Therefore

$$\left|D^{2p,m}(\mathcal{B}K_n)^I_{2p}(LZ_{\delta_n},\varphi,\zeta,0;f^{\times m},\tilde g_{2p,I})\right| \leq C_L^{|I^c|} L^{-|I|d_s} L^{-md_s} \|D^{2p,m}\mathcal{B}K_n(LZ_{\delta_n},S_L\varphi,\zeta,0)\|\times$$
$$\times \|g_{2j}\|_{C^2(Z^{2j}_{\delta_{n+1}})} \|f\|^{\times m}_{C^2(Z_{\delta_{n+1}})} \tag{5.123}$$

where we have used $\|D^{2p,m}(\mathcal{B}K_n)^I_{2p}\| \leq \|D^{2p,m}\mathcal{B}K_n\|$ as follows from (5.30). ¿From (5.122) and (5.123) we have

$$\|D^{2j,m}(\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}},\varphi,0)\| \leq \int d\mu_{\Gamma_n}(\zeta)\Big[\sum_{p\geq 0}\sum_{I\subset\{1,\dots,2p\}} \delta_{2j,|I|} C_L^{|I^c|} L^{-|I|d_s} L^{-md_s}\times$$
$$\times \|D^{2p,m}\mathcal{B}K_n(LZ_{\delta_n},S_L\varphi,\zeta,0)\|\Big]$$

Multipling the previous bound by h_F^{2j} and summing over j gives

$$\sum_{j\geq 0} h_F^{2j}\|D^{2j,m}(\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}},\varphi,0)\| \leq \int d\mu_{\Gamma_n}(\zeta)\Big[\sum_{p\geq 0}\sum_{I\subset\{1,\dots,2p\}} h_F^{|I|} C_L^{|I^c|} L^{-|I|d_s}\times$$
$$\times \|D^{2p,m}\mathcal{B}K_n(LZ_{\delta_n},S_L\varphi,\zeta,0)\|\Big]$$

Observe that

$$C_L^{|I^c|} L^{-|I|d_s} h_F^{|I|} = (\frac{C_L L^{d_s}}{h_F})^{|I^c|} L^{-2pd_s} h_F^{2p} \leq L^{-2pd_s} h_F^{2p}$$

for h_F sufficiently large depending on L. Moreover for L sufficiently large and ε small we have

$$L^{-2pd_s} \sum_{I\subset\{1,\dots,2p\}} 1 \;= (\frac{2}{L^{d_s}})^{2p} \leq 1$$

.

Therefore

$$\sum_{j\geq 0} h_F^{2j}\|D^{2j,m}(\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}},\varphi,0)\| \leq \int d\mu_{\Gamma_n}(\zeta)\sum_{p\geq 0} h_F^{2p}\|D^{2p,m}\mathcal{B}K_n(LZ_{\delta_n},S_L\varphi,\zeta,0)\| \tag{5.124}$$

Multiplying (5.124) by $\frac{h_B^m}{m!}$ and summing over $m:\ 0\leq m\leq m_0$ gives

$$\|\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}},\varphi,0)\|_{\mathbf{h}} \leq \int d\mu_{\Gamma_n}(\zeta)\|\mathcal{B}K_n(LZ_{\delta_n},S_L\varphi,\zeta,0)\|_{\mathbf{h}} \tag{5.125}$$

Evaluate (5.124) at $\varphi=0$. Then multiplying by $\frac{h_{B_*}^m}{m!}$ and summing over $m:\ 0\leq m\leq m_0$ gives

$$|\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}})|_{\mathbf{h}_*} \leq \int d\mu_{\Gamma_n}(\zeta)\|\mathcal{B}K_n(LZ_{\delta_n},0,\zeta,0)\|_{\mathbf{h}_*} \tag{5.126}$$

We now prove (5.120) starting from (5.125). Inserting the definition (4.11) in (5.125) and using the multiplicative property of the $\mathbf{h}$ norm we get

$$\|(\mathcal{S}(K_n)^\natural)(Z_{\delta_{n+1}},\varphi,0)\|_{\mathbf{h}} \le \sum_{N+M\ge 1}\frac{1}{N!M!}\sum\nolimits_{(X_{\delta_n,j}),(\Delta_{\delta_n,i})\to LZ_{\delta_{n+1}}} \|e^{-\tilde V_{n,L}(Z_{\delta_{n+1}}\setminus L^{-1}(\mathbf{X}_{\delta_{n+1}}\cup\mathbf{\Delta}_{\delta_{\mathbf{n+1}}}),\varphi)}\|_{\mathbf{h}}\times$$

$$\times\int d\mu_{\Gamma_n}(\zeta)\prod\nolimits_{j=1}^{N}\|K_n(X_{\delta_n,j},S_L\varphi,\zeta,0)\|_{\mathbf{h}}\prod\nolimits_{i=1}^{M}\|P_n(\Delta_{\delta_n,i},S_L\varphi,\zeta,0)\|_{\mathbf{h}}$$

$$\le 2^{|Z_{\delta_{n+1}}|}\sum_{N+M\ge 1}\frac{1}{N!M!}\sum\nolimits_{(X_j),(\Delta_{\delta_n,i})\to LZ_{\delta_n}}\int d\mu_{\Gamma_L}(\zeta)\hat G_{\kappa,\rho}(\mathbf{X}_{\delta_n}\cup\mathbf{\Delta}_{\delta_{\mathbf{n}}},S_L\varphi,\zeta)\prod\nolimits_{j=1}^{N}\|K_n(X_j)\|_{\mathbf{h},\hat G_{\kappa,\rho}}$$

$$\times\prod\nolimits_{i=1}^{M}\|P_n(\Delta_{\delta_n,i})\|_{\mathbf{h},\hat G_{\kappa,\rho}} \tag{5.127}$$

where $\mathbf{X}_{\delta_n}=\cup X_{\delta_n,j}$, $\mathbf{\Delta}_{\delta_n}=\cup\Delta_{\delta_n,i}$ and we have bounded $e^{-\tilde V_{n,L}}$ using lemma 5.5 which continues to apply. Lemma 5.4 bounds the ζ integral by

$$2^{|\mathbf{X}_{\delta_n}\cup\mathbf{\Delta}_{\delta_{\mathbf{n}}}|}G_{3\kappa}(\mathbf{X}_{\delta_n}\cup\mathbf{\Delta}_{\delta_{\mathbf{n}}},S_L\varphi)\le 2^{|\mathbf{X}_{\delta_n}\cup\mathbf{\Delta}_{\delta_{\mathbf{n}}}|}G_\kappa(L^{-1}(\mathbf{X}_{\delta_{n+1}}\cup\mathbf{\Delta}_{\delta_{n+1}}),\phi)$$

$$\le\prod_{j=1}^{M}2^{|X_{\delta_n,j}|}\prod_{i=1}^{N}2^{|\Delta_{\delta_n,i}|}G_\kappa(Z_{\delta_{n+1}},\phi)$$

since $L^{-1}(\mathbf{X}_{\delta_{n+1}}\cup\mathbf{\Delta}_{\delta_{\mathbf{n+1}}})\subset Z_{\delta_{n+1}}$. Therefore

$$\|(\mathcal{S}(K_n)^\natural)(Z_{\delta_{n+1}})\|_{\mathbf{h},\mathbf{G}_\kappa}\le 2^{|Z_{\delta_{n+1}}|}\sum_{N+M\ge 1}\frac{1}{N!M!}\sum\nolimits_{(X_{\delta_n,j}),(\Delta_{\delta_n,i})\to LZ_{\delta_n}}\prod\nolimits_{j=1}^{M}2^{|X_{\delta_n,j}|}\|K_n(X_j)\|_{\mathbf{h},\hat G_{\kappa,\rho}}\times$$

$$\prod\nolimits_{i=1}^{N}2^{|\Delta_{\delta_n,i}|}\|P_n(\Delta_{\delta_n,i})\|_{\mathbf{h},\hat G_{\kappa,\rho}}$$

The condition on the sum over the polymers above implies that $Z_{\delta_n}=(\cup L^{-1}\bar X^L_{\delta_n,j})\cup(\cup L^{-1}\bar\Delta^L_{\delta_n,i})$. This also implies that $Z_{\delta_{n+1}}=(\cup L^{-1}\bar X^L_{\delta_{n+1},j})\cup(\cup L^{-1}\bar\Delta^L_{\delta_{n+1},i})$.
Multiply both sides by $\mathcal{A}_p(Z_{\delta_{n+1}})$ and observe on the right hand side

$$\mathcal{A}_{p+1}(Z_{\delta_{n+1}})\le\prod\nolimits_{j=1}^{M}\mathcal{A}_{p+1}(L^{-1}\bar X_{j,\delta_{n+1}})\prod\nolimits_{i=1}^{N}\mathcal{A}_{p+1}(L^{-1}\bar\Delta_{\delta_{n+1},i})$$

$$\le O(1)^{N+M}\prod\nolimits_{j=1}^{M}\mathcal{A}_{-2}(X_{j,\delta_{n+1}})\prod\nolimits_{i=1}^{N}\mathcal{A}_{-2}(\Delta_{\delta_{n+1},i})$$

where we have first used the fact that the L-closures of the polymers are connected by definition of the reblocking operation, then Lemma 2.2 together with $|X_{j,\delta_{n+1}}|=|X_{j,\delta_n}|$ and $|\Delta_{\delta_{n+1},i}|=|\Delta_{\delta_n,i}|$. The last observation follows from our definition of polymers in section 1.3 and (1.75). Therefore

$$\|(\mathcal{S}(K_n)^\natural)(Z_{\delta_{n+1}})\|_{\mathbf{h},\mathbf{G}_\kappa}\mathcal{A}_p(Z_{\delta_{n+1}})\le\sum_{N+M\ge 1}\frac{1}{N!M!}O(1)^{N+M}\sum\nolimits_{(X_j),(\Delta_{\delta_n,i})\to LZ}$$

$$\prod\nolimits_{j=1}^{M}\|K_n(X_{j,\delta_n})\|_{\mathbf{h},\hat G_{\kappa,\rho}}\mathcal{A}_{-1}(X_{j,\delta_n})\prod\nolimits_{i=1}^{N}\|P_n(\Delta_{\delta_n,i})\|_{\mathbf{h},\hat G_{\kappa,\rho}}\mathcal{A}_{-1}(\Delta_{\delta_n,i})$$

Fix any $\Delta_{\delta_{n+1}}$ and sum over $Z_{\delta_{n+1}}\ni\Delta_{\delta_{n+1}}$. This fixes on the right hand side the sum over $Z_{\delta_n}\ni\Delta_{\delta_n}$ with Δ_{δ_n} fixed by restriction. The spanning tree argument of Lemma 7.1 of [BY] controls the sums over $N,M,Z_{\delta_n},(X_{j,\delta_n}),(\Delta_{i,\delta_n})\to LZ_{\delta_n}$ with the result (we have now made explicit the dependence on λ)

$$\|(\mathcal{S}(\lambda,K_n)^\natural)\|_{\mathbf{h},\mathbf{G}_\kappa,\mathcal{A}_{\mathbf{p}},\delta_{\mathbf{n+1}}}\le O(1)\sum_{N\ge 1}O(1)^N L^{3N}\Big(\|K_n(\lambda)\|_{\mathbf{h},\hat G_{\kappa,\rho},\mathcal{A},\delta_n}+\|P_n(\lambda)\|_{\mathbf{h},\hat G_{\kappa,\rho},\mathcal{A},\delta_n}\Big)^N$$

The proof of (5.120) is completed by Lemmas 5.8 and 5.14. When $R=0$ we can use Lemma 5.8 and replace Lemma 5.14 by Lemma 5.11.
To prove (5.121) we start from (5.126) and proceed as before. We replace $\hat{G}_{\kappa,\rho}$ by $\tilde{G}_{\kappa,\rho}$ and then use Lemma 5.3 to estimate the ζ integral. Proceeding as before now leads to

$$|(\mathcal{S}(\lambda,K_n)^{\natural})|_{\mathbf{h}_*,\mathcal{A}_p,\delta_{n+1}} \le O(1)\sum_{N\ge 1} O(1)^N L^{3N}\Big(\|K_n(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n} + \|P(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n}\Big)^N$$

Now use Lemmas 5.8 and 5.14 to complete the proof of (5.121). Finally when $R=0$ use Lemmas 5.8 and 5.11 as before. ∎

Estimates on relevant parts and flow coefficients from the remainder

Let $(\tilde{\alpha}_{n,P})$ be the coefficients $(\tilde{\alpha}_{n,2,0},\tilde{\alpha}_{n,2,1},\tilde{\alpha}_{n,2,\bar{1}}\tilde{\alpha}_{n,4})$ defined in (4.43) and (4.61). The flow coefficients $\xi_{n,R}$, $\rho_{n,R}$ are given in (4.53).

Lemma 5.17 : Under the conditions of the domain $\mathcal{D}_n$ we have

$$\|R_n^{\sharp}\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A}_{-1},\delta_n} \le \bar{g}^{3/4-\eta} \tag{5.128}$$

$$|R_n^{\sharp}|_{\hat{\mathbf{h}}_*,\mathcal{A}_{-1},\delta_n} \le O(1)\bar{g}^{11/4-\eta} \tag{5.129}$$

$$|\tilde{\alpha}_{n,P}|_{\mathcal{A},\delta_n} \le O(1)\bar{g}^{11/4-\eta} \tag{5.130}$$

$$|\xi_n| \le C_L\bar{g}^{11/4-\eta} \tag{5.131}$$

$$|\rho_n| \le C_L\bar{g}^{11/4-\eta} \tag{5.132}$$

where the constants C_L are independent of n and ε

Proof
(5.128) follows from (5.7) and Lemma 5.15, (5.106). (5.129) follow from (5.8) and lemma 5.15 with $m_0=9$ and ε sufficiently small depending on L so that $\bar{g}$ is sufficiently small. In fact in lemma 5.15 (with $\tilde{K}=R_n$) the first term has the desired bound by (5.8). By (5.7) together with $h_B^{-1}=c\bar{g}^{\frac{1}{4}}$ and $h_{B*}=h_{B*}(L)$ we see that the second term is bounded by $O(1)\bar{g}^{\frac{1}{4}}h_{B*}^9\bar{g}^{11/4-\eta}\le\bar{g}^{11/4-\eta}$ for $\bar{g}$ sufficiently small.
Recall that $\tilde{\alpha}_{n,P}(X_{\delta_n})$, are supported on small sets. Then (5.130) follows from (4.61) and (5.129). In fact the dominant contribution comes by setting $\tilde{V}_n=0$ because the difference gives additional powers of $\bar{g}$. Then we have

$$\|\tilde{\alpha}_{n,P}\|_{\mathcal{A},\delta_n} \le O(1)n(P)!\hat{\mathbf{h}}_*^{-n(P)}|1_S R_n^{\sharp}|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n}$$

where $n(P)$ is the number of fields in the monomial P, we have used the shorthand notation $\hat{\mathbf{h}}_*^{-n(P)} = \max_{n(P)_F+n(P)_B=n(P)}(h_{*B}^{-n(P)_B}\hat{h}_F^{-n(P)_F})$ and 1_S is the indicator function on small sets. Now use (5.129) to get (5.130). (5.131), (5.132) follow from (5.130), the definitions (4.53), (4.51) and Wick coefficients $C_n(0)$ are uniformly bounded by a L dependent constant by Corollary 1.2. ∎

Lemma 5.18
Under the conditions of $\mathcal{D}_n$ and ε sufficiently small depending on L, there exists a constant C_L independent of ε and n such that

$$|g_{n+1}-\bar{g}| < 2\nu\bar{g}^{3/2}, \quad |\mu_{n+1}| < C_L\bar{g}^{2-\delta} \tag{5.133}$$

Proof :
It is convenient to define

$$\tilde{g}_n = g_n - \bar{g}$$

Then from the flow equation (4.39) for g_n and the definition of $\bar{g}$ in (5.2) we get

$$\tilde{g}_{n+1} = (2 - L^{\varepsilon})\tilde{g}_n + \tilde{\xi}_n \tag{5.134}$$

where

$$\tilde{\xi}_n = -L^{2\varepsilon} a_{c*}\tilde{g}_n^2 - L^{2\varepsilon}(a_n - a_{c*})g_n^2 + \xi_n \tag{5.135}$$

From Lemma 5.12, Lemma 5.17 and $g_n \in \mathcal{D}_n$ we get for ε sufficiently small depending on L the bound

$$|\tilde{\xi}_n| \le C_L \bar{g}^2 \tag{5.136}$$

Therefore

$$|\tilde{g}_{n+1}| < \nu\bar{g}((2 - L^{\varepsilon}) + \frac{C_L}{\nu}\bar{g}) \tag{5.137}$$

For ε sufficiently small depending on L we get

$$|(1 - L^{\varepsilon}) + \frac{C_L}{\nu}\bar{g}| \le 1 \tag{5.138}$$

Therefore $|\tilde{g}_{n+1}| \le 2\nu\bar{g}$ which proves the first inequality of (5.133).
The bound on μ_{n+1} follows from the second of the flow equations (4.15), on using μ_n, g_n belong to $\mathcal{D}_n$, Lemma 5.12 and the bound (5.132) on ρ_n. ■

As stated in Theorem 3.1 borrowed from [BDH] the assumption of stability of the local potential with respect to perturbation by relevant parts (see (3.18) ensures the extraction estimate of (3.19). The following lemma proves the stability for the case at hand, namely that of $\tilde{V}_{n,L}(\Delta_{n+1})$ with respect to the relevant part F_n defined in section 4.

Recall from (4.12) that $F_n(\lambda) = \lambda^2 F_{Q_n} + \lambda^3 F_{R_n}$ and from (3.15) that (each part of) F_n decomposes: $F_n(X_{\delta_{n+1}}) = \sum_{\Delta_{\delta_{n+1}} \subset X_{\delta_{n+1}}} F_n(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}})$.

Lemma 5.19
For any $R > 0$ and $\xi := R\max(|\lambda^2|\bar{g}, |\lambda^3|\bar{g}^{7/4-\eta})$ sufficiently small,

$$\|e^{-\tilde{V}_{n,L}(\Delta_{n+1}) - \sum_{X_{\delta_{n+1}} \supset \Delta_{n+1}} z(X_{\delta_{n+1}})F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})}\|_{\mathbf{h},G_\kappa} \le 2^2 \tag{5.139}$$

where $z(X_{\delta_{n+1}})$ are complex parameters with $|z(X_{\delta_{n+1}})| \le R$.

Proof
It is easy to see that Lemma 5.5 still holds if we replace $\tilde{V}_n$ by $\tilde{V}_{n,L}$ provided ε is sufficiently small. This implies that $\bar{g}$ is sufficiently small. We then have

$$\|e^{-\tilde{V}_{n,L}(\Delta_{n+1}) - \sum_{X_{\delta_{n+1}} \supset \Delta_{n+1}} z(X_{\delta_{n+1}})F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})}\|_{\mathbf{h}} \le$$

$$2\, e^{-\bar{g}/4 \int_{\Delta_{n+1}} dx (|\varphi(x)|^2)^2 + \sum_{X_{\delta_{n+1}} \supset \Delta_{n+1}} R\|F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}}} \tag{5.140}$$

Recall that the relevant parts $F(X_{\delta_{n+1}}, \Delta_{n+1})$ are supported on small sets $X_{\delta_{n+1}}$. The proof now follows easily from the following

Claim: For ε sufficiently small

$$|z(X_{\delta_{n+1}})|\|F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \le C_L\xi \left(\bar{g}\int_{\Delta_{n+1}} d^3x(|\varphi(x)|^2)^2 + \bar{g}^{1/2}\||\varphi|^2\|_{\Delta_{n+1},1,5} + 1\right) \tag{5.141}$$

where $\|\phi\|^2_{\Delta_{n+1},1,5}$ is the square of the lattice Sobolev norm defined in (2.1).

Proof of the Claim:

We have $\|F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \le |\lambda|^2\|F_Q(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} + |\lambda|^3\|F_R(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}}$.

Consider (4.27)-(4.31). Undo the Wick ordering on the superfield field and note that, by virtue of supersymmetry, no field independent terms arise. The $m = 1$ term in (4.30) remains unchanged and in the $m = 2$ case there results an additional contribution $-2C_{n+1}(0)\Phi\bar{\Phi}$. The Wick constant $C_{n+1}(0)$ has a uniform bound C which depends only on L by Corollary 1.2. We write this in the Grassmann representation and notice that for the $m = 2$ case the $\psi\bar{\psi}(x)^2$ contribution vanishes by statistics. From the definition of the $\mathbf{h}$ norm with $h_B = c\bar{g}^{-1/4}$ and $h_F = h_F(L)$ we get the bound

$$\|F_Q(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \le C_L\bar{g}^2 \left(\int_{\Delta_{n+1}} d^3x\|(|\varphi|^2)^2(x)\|_{h_B} \sup_{x\in\Delta_{n+1}} |f_Q^{(2)}(X_{\delta_{n+1}}, x, \Delta_{n+1})| + \right.$$
$$\left.\left(\int_{\Delta_{n+1}} d^3x\|(|\varphi|^2)(x)\|_{h_B} + 1\right)\sum_{m=1}^{2} \sup_{x\in\Delta_{n+1}} |f_Q^{(m)}(X_{\delta_{n+1}}, x, \Delta_{n+1})|\right)$$

Now for $m = 1, 2$

$$\bar{g}\int_{\Delta_{n+1}} d^3x\|(|\varphi|^2)^m(x)\|_{h_B} \le O(1)\left(\bar{g}\int_{\Delta_{n+1}} d^3x(|\varphi|^2)^2(x) + 1\right)$$

From the definition (4.31) and the estimates obtained in the course of proving Lemma 5.12, we have

$$\sup_{x\in\Delta_{n+1}} |f_Q^{(m)}(X_{\delta_{n+1}}, x, \Delta_{n+1})| \le C_L$$

Therefore

$$|\lambda^2||z(X_{\delta_{n+1}})|\ \|F_Q(X_{\delta_{n+1}}, \Delta_{n+1})\|_h \le C_L R|\lambda|^2\bar{g}\left(\bar{g}\int_{\Delta_{n+1}} dx(|\varphi|^2)^2(x) + 1\right) \tag{5.142}$$

Next consider F_{R_n}, supported on small sets, defined in (4.43), (4.46). Recall (4.48),

$$F_R(X_{\delta_{n+1}}, \mathbf{\Phi}) = \sum_P \int_{\Delta_{n+1}} dx\ \alpha_P(X_{\delta_{n+1}}, x)P(\mathbf{\Phi}(x), \partial_{\delta_{n+1}}\mathbf{\Phi}(x))$$

By Lemma 5.17 and (4.49) we have $|\alpha_P(X_{\delta_{n+1}}, x)| \le C_L\bar{g}^{11/4-\eta}$, so that

$$\begin{aligned}|\lambda|^3|z(X_{\delta_{n+1}})|\|F_R(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \le& C_L R|\lambda|^3\bar{g}^{11/4-\eta}\sum_P\int_{\Delta_{n+1}} dx\ \|P(\Phi(x), \partial_{\delta_{n+1}}\Phi(x))\|_{\mathbf{h}}\\ \le& C_L R|\lambda|^3\bar{g}^{7/4-\eta}\left(\bar{g}\int_{\Delta_{n+1}} dx\,(|\varphi|^2)^2(x) + \bar{g}^{1/2}\||\varphi|^2\|_{\Delta_{n+1},1,5} + 1\right)\end{aligned}$$

The claim follows by combining this with (5.142). In the above inequality the Sobolev norm when estimating the term giving arise to $\phi\partial_{\delta_n,\mu}\bar{\phi}$. We bound $|\phi\partial_{\delta_n,\mu}\bar{\phi}| \le 1/2(||\varphi|^2|+|\partial_{\delta_n,\mu}\phi|^2)$ and then use the lattice Sobolev embedding inequality. ∎

Lemma 5.20
For any $R > 0$ and $\xi := R\max(|\lambda^2|\bar{g}^2, |\lambda^3|\bar{g}^{11/4-\eta})$ sufficiently small,

$$|e^{-\tilde{V}_{n,L}(\Delta_{n+1})-\sum_{X_{\delta_{n+1}}\supset\Delta_{n+1}} z(X_{\delta_{n+1}})F(\lambda,X_{\delta_{n+1}},\Delta_{n+1})}|_{\mathbf{h}_*} \le 2^2 \tag{5.143}$$

where $z(X)$ are complex parameters with $|z(X)| \le R$.

Proof
The proof is similar to the previous one except that we can use the estimate $|F(\lambda, X, \Delta)|_{\mathbf{h}_*} \le C_L\xi$ in place of (5.141) since the $\mathbf{h}_*$ norm is computed with field derivatives at $\Phi = 0$. ∎

We will now bound the remainder R_{n+1} given in (4.41). It consist of a sum of four contributions, namely $R_{n+1,\mathrm{main}}$, $R_{n+1,linear}$, $R_{n+1,3}$ and $R_{n+1,4}$ which we will estimate in turn. These estimates parallel those obtained for the continuum bosonic theory in [BMS].

Recall from (4.36) that

$$R_{n+1,\mathrm{main}} = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4}\mathcal{E}\Big(\mathcal{S}(\lambda, Q_n e^{-V_n})^\natural, F_{Q_n}(\lambda)\Big) \tag{5.144}$$

Lemma 5.21

$$\|R_{n+1,\mathrm{main}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le C_L\bar{g}^{3/4} \tag{5.145}$$

$$|R_{n+1,\mathrm{main}}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \le C_L\bar{g}^{3-3\delta/2} \tag{5.146}$$

Proof
Choose the contour γ to be a circle of radius $|\lambda| = c_L\bar{g}^{-1/4}$. Then from Lemma 5.16, we have that for any $q > 0$, $\|\mathcal{S}(\lambda, Q_n e^{-V_n})^\natural\|_{\mathbf{h},G_\kappa,\mathcal{A}_1,\delta_{n+1}} \le q$. For $\lambda \in \gamma$, $R^{-1} = O(1)$ and $|z(X_{\delta_{n+1}})| = R$ the stability estimate of Lemma 5.19 holds since for ε sufficiently small, $\bar{g}$ is sufficiently small and the ξ in Lemma 5.16 is sufficiently small. The $f(X_{\delta_{n+1}})$ which figures in the extraction estimate below is supported on small sets and $f(X_{\delta_{n+1}}) \le 2|z(X_{\delta_{n+1}})|^{-1}$. Therefore by the extraction estimate (3.19) of Theorem 3.1 we have

$$\begin{aligned}\|\mathcal{E}(\mathcal{S}(\lambda, Q_n e^{-V_n})^\natural, F_{Q_n}(\lambda))\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} &\le \|\mathcal{S}(\lambda, Q_n e^{-V_n})^\natural\|_{\mathbf{h},G_\kappa,\mathcal{A}_1,\delta_{n+1}} + \|f\|_{\mathcal{A}_3,\delta_{n+1}}\\ &\le q + c'_L R^{-1} \le c_L\end{aligned}$$

The bound (5.145) now follows from (5.144) by a Cauchy estimate since the contour is of radius $c_L\bar{g}^{-1/4}$. To prove (5.146) we choose the contour γ to be a circle of radius $c_L\bar{g}^{-1+\delta/2}$ so that ξ in Lemma 5.20 is small and the hypothesis of Lemma 5.16 is satisfied. $z(X_{\delta_{n+1}})$ and $f(X_{\delta_{n+1}})$ are chosen as bedore. Then (5.146) follows from the extraction estimate (3.20), (5.121) of Lemma 5.16 and a Cauchy bound. ∎

Recall from (4.38) that

$$R_{n+1,3} = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4(\lambda-1)}\mathcal{E}\Big(\mathcal{S}(\lambda, K_n)^\natural, F_n(\lambda)\Big) \tag{5.147}$$

Lemma 5.22

$$\|R_{n+1,3}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le C_L\bar{g}^{1-4\eta/3} \tag{5.148}$$

$$|R_{n+1,3}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \le C_L \bar{g}^{10/3-4\eta/3} \tag{5.149}$$

Proof
The proof follows the lines of that of Lemma 5.21. To prove (5.148) we take the contour γ to of radius $|\lambda| = c_L \bar{g}^{-(1/4-\eta/3)}$. This ensures that the hypothesis of Lemma 5.16, (5.121) is satisfied and ξ of Lemma 5.19 is sufficiently small so that stability holds. (5.148) now follows from the extraction estimate (3.19) and the Cauchy bound as before. To prove (5.148) we take γ to be of radius $|\lambda| = c_L \bar{g}^{-(5/6-\eta/3)}$. Then for $\bar{g}$ sufficiently small the hypothesis for (5.121) of Lemma 5.16 is satisfied. Moreover then ξ of Lemma 5.20 is sufficiently small and and Lemma 5.20 holds. (5.148) now follows from the extraction estimate (3.20) and the Cauchy bound as before. ∎

From the definition of $R_{n+1,4}$ in (4.40) we have

$$R_{n+1,4} = \left(e^{-V_{n+1}} - e^{-\tilde{V}_{n,L}}\right) Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) + e^{-\tilde{V}_{n,L}} Q(C_{n+1}, \mathbf{w}_{n+1}, (g_{n+1}^2 - g_{n,L}^2))$$

Lemma 5.23

$$\|R_{n+1,4}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le C_L \bar{g}^{3/2} \qquad |R_{n+1,4}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \le C_L \bar{g}^3$$

Proof
We follow the lines of the proof of Lemma 5.23 in [BMS]. From the flow equations (4.39), together with the domain hypothesis $\mathcal{D}_n$ and Lemmas 5.12, 5.17, 5.9 we obtain the bounds

$$|g_{n+1} - g_{n,L}| \le C_L \bar{g}^2 \tag{5.150}$$

$$|\mu_{n+1} - \mu_{n,L}| \le C_L \bar{g}^2 \tag{5.151}$$

$$\|\mathbf{w}_{n+1}\|_{\delta_{n+1}} \le k_l \tag{5.152}$$

From (5.150) and Lemma 5.18 we get

$$|g_{n+1}^2 - g_{n,L}^2| \le C_L \bar{g}^3 \tag{5.153}$$

We estimate the two terms in the expression for $R_{n+1,4}$ above. Because of Q each term is supported on small sets. We write the first term as

$$\left(e^{-V_{n+1}} - e^{-\tilde{V}_{n,L}}\right) Q(C_{n+1}, \mathbf{w_{n+1}}, g_{n+1}) = Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) e^{-\frac{1}{2}V_{n+1}} \times$$

$$\times \int_0^1 ds (\tilde{V}_{n,L} - V_{n+1}) e^{-(\frac{1}{2}-s)V_{n+1} - sV_{n,L}}$$

evaluated on a small set $X_{\delta_{n+1}}$. We bound

$$\| \left(e^{-V_{n+1}} - e^{-\tilde{V}_{n,L}}\right) Q(C_{n+1}, \mathbf{w_{n+1}}, g_{n+1}) \|_{\mathbf{h},G_\kappa} \le \|Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) e^{-\frac{1}{2}V_{n+1}}\|_{\mathbf{h},G_\kappa} \times$$

$$\times \int_0^1 ds \|\tilde{V}_{n,L} - V_{n+1}\|_{\mathbf{h}} \|e^{-(\frac{1}{2}-s)V_{n+1}}\|_{\mathbf{h}} \|e^{-sV_{n,L}}\|_{\mathbf{h}} \tag{5.154}$$

We get by undoing the Wick orderings

$$\tilde V_{n,L}(X_{\delta_{n+1}})-V_{n+1}(X_{\delta_{n+1}})=\int_{X_{\delta_{n+1}}}dx\Big((g_{n,L}-g_{n+1})(|\varphi|^2)^2(x)+2(g_{n,L}-g_{n+1})\varphi\bar\varphi(x)\psi\bar\psi(x)+$$

$$+(\tilde\mu_{n,L}-\tilde\mu_{n+1})\psi\bar\psi(x)\Big)$$

where $\tilde\mu_{n+1}=\mu_{n+1}-2C_{n+1}(0)g_{n+1}$ and $\tilde\mu_{n,L}=\mu_{n,L}-2C_{n+1}(0)g_{n,L}$. Using (5.150), (5.151) and the uniform boundedness of the Wick coefficient $C_{n+1}(0)$ we can bound

$$\|\tilde V_{n,L}(X_{\delta_{n+1}})-V_{n+1}(X_{\delta_{n+1}})\|_{\mathbf h}\le C_L\frac{\bar g}{\gamma}e^{\gamma\bar g\int_{X_{\delta_{n+1}}}dx(\phi\bar\phi)^2(x)} \tag{5.155}$$

for any $\gamma=O(1)>0$. By Lemma 5.5 we can bound

$$\|e^{-(\frac12-s)V_{n+1}(X_{\delta_{n+1}})}\|_{\mathbf h}\le 2^{|X_{\delta_{n+1}}|}e^{-(\frac12-s)\frac{\bar g}{4}\int_{X_{\delta_{n+1}}}dx(\phi\bar\phi)^2(x)} \tag{5.156}$$

$$\|e^{-sV_{n,L}(X_{\delta_{n+1}})}\|_{\mathbf h}\le 2^{|X_{\delta_{n+1}}|}e^{-s\frac{\bar g}{4}\int_{X_{\delta_{n+1}}}dx(\phi\bar\phi)^2(x)} \tag{5.157}$$

We now plug into (5.154) the bounds (5.156) and (5.157) and choose $\gamma=1/2$. Then the s-integral in (5.154) is bounded by $C_L\bar g$. Lemma 5.10 continues to hold if we replace n by $n+1$ by virtue of Lemma 5.9 and (5.133) of Lemma 5.18. Therefore the first factor in (5.154) is bounded by $C_L\bar g^{\frac12}$. Putting these bounds together and recalling that the Q are supported on small sets we obtain

$$\|\Big(e^{-V_{n+1}}-e^{-\tilde V_{n,L}}\Big)Q(C_{n+1},\mathbf w_{n+1},g_{n+1})\|_{\mathbf h,G_\kappa,\mathcal A,\delta_{n+1}}\le C_L\bar g^{\frac32} \tag{5.158}$$

The second term in the expression for R_4 is bounded using (5.153) and Lemma 5.10 with n replaced by $n+1$ and g_n^2 by $g_{n+1}^2-g_{n,L}^2$. We get

$$\|e^{-\tilde V_{n,L}}Q(C_{n+1},\mathbf w_{n+1},(g_{n+1}^2-g_{n,L}^2))\|_{\mathbf h,G_\kappa,\mathcal A,\delta_{n+1}}\le C_L\bar g^{\frac32} \tag{5.159}$$

Adding together (5.158) and (5.159) we get the first bound of the Lemma. The second bound is proved similarly taking into account that all derivatives in the $\mathbf h_*$ norm are at $\Phi=0$. ∎

Lemma 5.24
Let X_{δ_n} be a small set and let $J(X_{\delta_n},\Phi)$ be normalized as in (4.59). Recall that the rescaled activity J_L is defined by $J_L(L^{-1}X_{\delta_{n+1}},\varphi,\psi)=J(X_{\delta_n},\varphi_{L^{-1}},\psi_{L^{-1}})$ where $\varphi_{L^{-1}}=S_L\varphi$ and $\psi_{L^{-1}}=S_L\psi$. Then we have
1. For $2p+m=2$ so that $(p,m)=(1,0),\ (0,2)$

$$\|D^{2p,m}J_L(L^{-1}X_{\delta_{n+1}},0,0)\|\le O(1)L^{-(7-\varepsilon)/2}\|D^{2p,m}J(X_{\delta_n},0,0)\| \tag{5.160}$$

2. For $2p+m=4$ so that $(p,m)=(2,0),(1,2),(0,4)$

$$\|D^{2p,m}J_L(L^{-1}X_{\delta_{n+1}},0,0)\|\le O(1)L^{-(4-\varepsilon)}\|D^{2p,m}J(X_{\delta_n},0,0)\| \tag{5.161}$$

Proof: In the proof of this lemma we will need to use the lattice Taylor expansion introduced in (5.13),(5.14) and (5.15) with a particular choice of a lattice path joining two points. The polymer X_{δ_n} being a small set is connected. It can be represented as $X_{\delta_n}=X\cap(\delta_n\mathbb Z)^3$ where X is a continuum connected polymer which is

a small set. As in the proof of Lemma 5.1 we say that X_{δ_n} is convex if X is a convex set. By the argument in the proof of Lemma 5.1 it suffices to consider the case when X_{δ_n} is convex.
Let u_k be a function in $C^2(X^k_{\delta_n})$ for $k \geq 2$. In the following u will represent one of the functions f_j in $C^2(X)$ giving a direction for a bosonic derivative or a function g_{2p} in $C^2(X^{2n})$ associated with a fermionic derivative of order $2p$. Let $e_1, e_2, ..., e_{3k}$ be the basis vectors of $(\delta_n\mathbb{Z})^{3k}$. Let $x = \sum_{i=1}^{3k}(x,e_i)e_i$ denote a point in $X^k_{\delta_n}$.
We recall the definition of the rescaled function

$$u_{k,L^{-1}}(x) = L^{-kd_s} u_k(\frac{x}{L})$$

Observe that

$$\|u_{k,L^{-1}}\|_{C^2(X^k_{\delta_n})} \leq L^{-kd_s}\|u_k\|_{C^2(L^{-1}X^k_{\delta_{n+1}})} \tag{5.162}$$

Fix a point $x_0 \in X^k_{\delta_n}$. We write $x - x_0 = \delta_n \sum_{i=1}^{3k} h_i e_i$ where the h_i are integers.
From (5.15) we have

$$u_{k,L^{-1}}(x) = u_{k,L^{-1}}(x_0) + \sum_{i=1}^{3k}((x-x_0),e_i)\partial_{\delta_n,e_i} u_{k,L^{-1}}(x_0) + \delta_n^2 \sum_{i,j=1}^{3k}\sum_{s_i=0}^{h_i-1}\sum_{s_j=0}^{h_j-1}$$

$$\partial_{\delta_n,e_i}\partial_{\delta_n,e_j} u_{k,L^{-1}}(x_0 + p_j(p_i(x-x_0,s_i),s_j))$$

The argument of $u_{k,L^{-1}}$ in the last term lies entirely in $X^k_{\delta_n}$ since X_{δ_n} is convex.
Define

$$\delta u_{k,L^{-1}}(x) = u_{k,L^{-1}}(x) - u_{k,L^{-1}}(x_0) = \delta_n \sum_{j=1}^{3k}\sum_{s_j=0}^{h_j-1}\partial_{\delta_n,e_j} u_{k,L^{-1}}(x_0 + p_j(x-x_0,s_j)) \tag{5.163}$$

and

$$\delta^2 u_{k,L^{-1}}(x) = h_{k,L^{-1}}(x) - u_{k,L^{-1}}(x_0) - \sum_{i=1}^{3k}((x-x_0),e_i)\partial_{\delta_n,e_i} u_{k,L^{-1}}(x_0)$$

$$= \delta_n^2 \sum_{i,j=1}^{3k}\sum_{s_i=0}^{h_i-1}\sum_{s_j=0}^{h_j-1}\partial_{\delta_n,e_i}\partial_{\delta_n,e_j} u_{k,L^{-1}}(x_0 + p_j(p_i(x-x_0,s_i),s_j)) \tag{5.164}$$

Now from (5.163), (5.164) we have using the definition of the rescaled function $u_{k,L^{-1}}$

$$\delta u_{k,L^{-1}}(x) = L^{-(1+k\frac{3-\varepsilon}{4})}\delta_n \sum_{j=1}^{3k}\sum_{s_j=0}^{h_j-1}(\partial_{\delta_n,e_j} u_k)_{L^{-1}}(x_0 + p_j(x-x_0,s_j))$$

and for $l \geq 1$

$$\partial_{\delta_n}{}^l \delta u_{k,L^{-1}}(x) = L^{-(l+k\frac{3-\varepsilon}{4})}(\partial_{\delta_n}{}^l u_k)(L^{-1}x)$$

where for $\partial_{\delta_n}{}^l$ a multi-index convention is implicit. This implies that

$$\|\delta u_{k,L^{-1}}\|_{C^2(X^k_{\delta_n})} \leq c_1 L^{-(1+k\frac{3-\varepsilon}{4})}\|h_k\|_{C^2(L^{-1}X^k_{\delta_{n+1}})} \tag{5.165}$$

where $c_1 = O(1)$ since X is a small set. In the same way starting from (5.164) a little bit of work shows that

$$\|\delta^2 h_{k,L^{-1}}\|_{C^2(X^k_{\delta_n})} \leq c_2 L^{-(2+k\frac{3-\varepsilon}{4})}\|h_k\|_{C^2(L^{-1}X_{\delta^k_{n+1}})} \tag{5.166}$$

where $c_2 = O(1)$.

We will first prove the bounds of (5.160).
Consider first the case $(p, m) = (1, 0)$. We have

$$D^{2,0} J_L(L^{-1} X_{\delta_{n+1}}, 0, 0; g_2) = D^{2,0} J(X_{\delta_n}, 0, 0; g_{2,L^{-1}}) = D^{2,0} J(X_{\delta_n}, 0, 0; \delta^2 g_{2,L^{-1}}) \tag{5.167}$$

where we have Taylor expanded the function $g_{2,L^{-1}}$ as in (5.164) and then used the first and second normalization conditions in (4.59). Therefore

$$\begin{aligned} |D^{2,0} J_L(L^{-1} X_{\delta_{n+1}}, 0, 0; g_2)| &\le \|D^{2,0} J(X_{\delta_n}, , 0, 0)\| \, \|\delta^2 g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})} \\ &\le O(1) L^{-(7-\varepsilon)/2} \|D^{2,0} J(X_{\delta_n}, 0)\| \, \|g_2\|_{C^2(L^{-1} X^2_{\delta_{n+1}})} \end{aligned}$$

where in the last step we have used the bound in (5.166) for $k = 2$ This proves the case $(p, m) = (1, 0)$ of the lemma.
To prove the case $(p, m) = (0, 2)$ we Taylor expand the function $f_{L^{-1}}(x)$ to second order, then use the first and the third conditions in (4.59) to get

$$\begin{aligned} D^{0,2} J(X_{\delta_n}, 0, 0; f^{\times 2}_{L^{-1}}) = D^{0,2} J(X_{\delta_n}, 0, 0; f_{1,L^{-1}}(x_0), \delta^2 f_{2,L^{-1}}) + D^{0,2} J(X_{\delta_n}, 0, 0; \delta^2 f_{1,L^{-1}}, f_{2,L^{-1}}(x_0)) \\ + D^{0,2} J(X_{\delta_n}, 0, 0; \delta f_{1,L^{-1}}, \delta f_{2,L^{-1}}) \end{aligned} \tag{5.168}$$

Therefore

$$\begin{aligned} |D^{0,2} J(X_{\delta_n}, 0, 0; f^{\times 2}_{L^{-1}})| \le \|D^{0,2} J(X_{\delta_n}, 0, 0)\| \Big(&\|f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta^2 f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \\ &+ \|f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta^2 f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \\ &+ \|\delta f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta f_{2,L^{-1}}\|_{C^2(X_{\delta_n})}) \Big) \\ &\le O(1) L^{-(7-\varepsilon)/2} \|D^{0,2} J(X_{\delta_n}, 0, 0)\| \prod_{j=1}^{2} \|f_j\|_{C^2(L^{-1} X_{\delta_{n+1}})} \end{aligned}$$

where we have used the bounds (5.162), (5.165) and (5.166) for the case $k = 1$. This proves the case $(p, m) = (0, 2)$.
Next we prove the bounds (5.161) . For this case $(p, m) = (2, 0), (1, 2), (0, 4)$ so that $2p + m = 4$. Taylor expand test functions around the fixed point $x_0 \in X_{\delta_n}$ to first order with remainder. We get for $(p, m) = (2, 0)$

$$D^{4,0} J_L(L^{-1} X_{\delta_{n+1}}, 0; g_4) = D^{4,0} J(X_{\delta_n}, 0; g_{4,L^{-1}}) = D^{4,0} J(X_{\delta_n}, 0; \delta g_{4,L^{-1}}) \tag{5.169}$$

where we have Taylor expanded the function $g_{4,L^{-1}}$ as in (5.163) and then used (4.58).
Therefore exploiting the bound (5.165) for $k = 4$ we get

$$|D^{4,0} J_L(L^{-1} X_{\delta_{n+1}}, 0, 0; g_4)| \le O(1) L^{-(4-\varepsilon)} \|D^{4,0} J(X_{\delta_n}, 0)\| \|g_4\|_{C^2(L^{-1} X^4_{\delta_{n+1}})}$$

which proves the case $(p, m) = (2, 0)$.
Next we turn to the case $(p, m) = (1, 2)$. We have

$$\begin{aligned} D^{2,2} J_L(L^{-1} X_{\delta_{n+1}}, 0; f^{\times 2}, g_2) = D^{2,2} J(X_{\delta_n}, 0; f^{\times 2}_{L^{-1}}, g_{2,L^{-1}}) = D^{2,2} J(X_{\delta_n}, 0; \delta f^{(1)}_{L^{-1}}, f^{(2)}_{L^{-1}}, g_{2,L^{-1}}) + \\ + D^{2,2} J(X_{\delta_n}, 0; f^{(1)}_{L^{-1}}(x_0), \delta f^{(2)}_{L^{-1}}, g_{2,L^{-1}}) + D^{2,2} J(X_{\delta_n}, 0; f^{(1)}_{L^{-1}}(x_0), f^{(2)}_{L^{-1}}(x_0), \delta g_{2,L^{-1}}) \end{aligned} \tag{5.170}$$

where we have used the fourth condition in (4.59). Therefore

$$|D^{2,2}J_L(L^{-1}X_{\delta_{n+1}},0,0;f^{\times 2},g_2)| \leq \|D^{2,2}J(X_{\delta_n},0,0)\|\Big(\|\delta f^{(1)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|f^{(2)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})}+$$

$$+\|\delta f^{(2)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|f^{(1)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})}+\|\delta g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})}\|f^{(1)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|f^{(2)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\Big)$$

Then using the bounds (5.162), (5.165) for $k=1,2$ and (5.166) for $k=1$ we get

$$|D^{2,2}J_L(L^{-1}X_{\delta_{n+1}},0,0;f^{\times 2},g_2)| \leq O(1)L^{-(4-\varepsilon)}\|D^{2,2}J(X_{\delta_n},0)\|\prod_{j=1}^{2}\|f^{(j)}\|_{C^2(L^{-1}X_{\delta_n})}\|g_2\|_{C^2(L^{-1}X^2_{\delta_{n+1}})}$$

which proves the case $(p,m)=(1,2)$.
Finally we treat the case $(n,m)=(0,4)$. Let $\mathcal{N}_4=(1,2,3,4)$. Then using the fourth condition of (4.59) we get

$$D^{0,4}J_L(L^{-1}X_{\delta_{n+1}},0,0;f^{\times 4}) = D^{0,4}J(X_{\delta_n},0;f^{\times 4}_{L^{-1}}) = \sum_{I\subset\mathcal{N}_4,|I|\neq 4} D^{0,4}J(X_{\delta_n},0;f_{L^{-1}}(x_0)^{\times|I|},\delta f^{\times|I_c|}_{L^{-1}})$$

where $f^{\times|I|}=(f_j)|_{j\in I}$.

Therefore

$$|D^{0,4}J_L(L^{-1}X_{\delta_n},0,0;f^{\times 4})| \leq \|D^{0,4}J(X_{\delta_n},0,0)\|\sum_{I\subset\mathcal{N}_4,|I|\neq 4}\|f_{L^{-1}}\|^{|I|}_{C^2(X_{\delta_n})}\|\delta f_{L^{-1}}\|^{|I_c|}_{C^2(X_{\delta_n})}$$

Because of the condition on I in the above sum $|I_c|\geq 1$. Therefore using the bounds (5.162) and (5.165) we get

$$|D^{0,4}J_L(L^{-1}X_{\delta_n},0,0;f^{\times 4})| \leq O(1)L^{-(4-\varepsilon)}\|D^{0,4}J(X_{\delta_n},0)\|\prod_{j=1}^{4}\|f_j\|_{C^2(L^{-1}X_{\delta_{n+1}})}$$

which proves the case $(p,m)=(0,4)$ and thus completes the proof of Lemma 5.24. ∎

Corollary 5.25
Let $Y_{\delta_{n+1}}=L^{-1}X_{\delta_{n+1}}$ *where* X *is a small set,* $Z_{\delta_{n+1}}=L^{-1}\bar{X}^L_{\delta_{n+1}}$ *and let* $J(X_{\delta_n},\Phi)$ *be normalized as in (4.59). By definition* $J_L(Y_{\delta_{n+1}},\Phi)=J(X_{\delta_n},S_L\Phi)$*. Then*

$$|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}} \leq O(1)L^{-(7-\varepsilon)/2}|J(X_{\delta_n})|_{\hat{\mathbf{h}}} \tag{5.171}$$

$$\|J_L(Y_{\delta_{n+1}})e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}})}\|_{\mathbf{h},G_\kappa} \leq O(1)L^{-(7-\varepsilon)/2}\Big[|J(X_{\delta_n})|_{\hat{\mathbf{h}}}+\|J(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{3\kappa}}\Big] \tag{5.172}$$

$$|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}_*} \leq O(1)L^{-(7-2\varepsilon)/2}|J(X_{\delta_n})|_{\hat{\mathbf{h}}*} \tag{5.173}$$

Proof
(5.171), (5.173) follow easily from Lemma 5.24 taking advantage of the scaling present to shift from $\mathbf{h}$ to $\hat{\mathbf{h}}$. To see this recall that $h_F=2\hat{h}_F$. Hence

$$|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}} = \sum_{n=0}^{\infty}\sum_{m=0}^{m_0} \hat{h}_F^{2n}\frac{h_B^m}{m!}2^{2n}\|D^{2n,m}J_L(Y_{\delta_{n+1}},0)\| \tag{5.174}$$

Only terms with $2n+m$ even contribute. For $2n+m\le 4$ use Lemma 5.24 and observe that $2^{2n}\le O(1)$. For $2n+m\ge 6$, we have for L sufficiently large and ε sufficiently small depending on L

$$\begin{aligned}
2^{2n}\|D^{2n,m}J_L(Y_{\delta_{n+1}},0)\| &\le 2^{2n}L^{-(2n+m)\frac{(3-\varepsilon)}{4}}\|D^{2n,m}J(X_{\delta_n},0)\| \\
&\le L^{-(2n+m)\frac{(3-\frac{1}{3}-\varepsilon)}{4}}\|D^{2n,m}J(X_{\delta_n},0)\| \\
&\le O(1)L^{(7-\varepsilon)/2}\|D^{2n,m}J(X_{\delta_n},0)\|
\end{aligned}$$

Putting the two case together in (5.174) gives (5.171). The proof of (5.173) is the same on replacing h_B by h_{B*}.

For (5.172) we write

$$\begin{aligned}
&\|J_L(Y_{\delta_{n+1}},\mathbf{\Phi})e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}},\mathbf{\Phi})}\|_{\mathbf{h}} \le \|J_L(Y_{\delta_{n+1}},\mathbf{\Phi})\|_{\mathbf{h}}\|e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}},\mathbf{\Phi})}\|_{\mathbf{h}} \le \\
&\le O(1)G_\kappa(Z_{\delta_{n+1}},\phi)\Big[|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}} + L^{-m_0 d_s}\|J_L(Y_{\delta_{n+1}})\|_{h_F,L^{[\phi]}h_B,G_\kappa}\Big]
\end{aligned}$$

where we used Lemmas 5.5 and Lemma 5.15. By (5.171), and rewriting the second term by moving the scaling from J to the norm,

$$\|J_L(Y_{\delta_{n+1}},\mathbf{\Phi})e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}},\mathbf{\Phi})}\|_{\mathbf{h}} \le O(1)G_\kappa(Z_{\delta_{n+1}},\varphi)\Big[L^{-(7-\varepsilon)/2}|J(X_{\delta_n})|_{\hat{\mathbf{h}}} + L^{-m_0 d_s}\|J(X_{\delta_n})\|_{\hat{\mathbf{h}},\mathbf{G_{3\kappa}}}\Big]$$

Recall that $m_0=9$ and the scaling dimension $d_s=(3-\varepsilon)/4$. (5.172) now follows by multiplying both sides by $G_\kappa^{-1}(Z_{\delta_{n+1}},\varphi)$, and taking the supremum over φ. ∎

Lemma 5.26

$$\|\tilde{F}_{R_n}e^{-\tilde{V}_n}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_n} \le O(1)\varepsilon^{3/4-\eta} \tag{5.175}$$

$$|\tilde{F}_{R_n}e^{-\tilde{V}_n}|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n} \le O(1)\varepsilon^{11/4-\eta} \tag{5.176}$$

and $J_n = R_n^\sharp - \tilde{F}_{R_n}e^{-\tilde{V}_n}$ satisfies on small sets the bounds

$$\|J_n\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A},\delta_n} \le O(1)\bar{g}^{\frac{3}{4}-\eta} \qquad |J_n|_{\hat{\mathbf{h}},\mathcal{A},\delta_n} \le O(1)\bar{g}^{\frac{3}{4}-\eta} \qquad |J_n|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n} \le O(1)\bar{g}^{\frac{11}{4}-\eta} \tag{5.177}$$

Proof

First we prove (5.175). $\tilde{F}_{R_n}$ is defined in (4.43) and (4.45), and is supported on small sets. We estimate its $\mathbf{h}$ norm as in the proof of Lemma 5.19.

$$\begin{aligned}
\|\tilde{F}_{R_n}(X_{\delta_n},\mathbf{\Phi})\|_{\mathbf{h}} &\le \sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})|\int_{X_{\delta_n}} dx\ \|P(\mathbf{\Phi}(x),\partial\mathbf{\Phi}(x))\|_{\mathbf{h}} \\
&\le C_L\sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})|\bar{g}^{-1}\Big(\bar{g}\int_{X_{\delta_n}} dx\,(|\varphi|^2)^2(x) + \bar{g}^{1/2}\|\varphi\|^2_{X_{\delta_n},1,\sigma}+1\Big) \\
&\le C_L\sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})|\bar{g}^{-1}G_\kappa(X_{\delta_n},\varphi)e^{\gamma\bar{g}\int_{X_{\delta_n}} dy(|\varphi|^2)^2(y)}
\end{aligned}$$

for any $\gamma = O(1) > 0$. Hence, using Lemma 5.5

$$\|\tilde{F}_{R_n}(X_{\delta_n}, \mathbf{\Phi})e^{-\tilde{V}_n(X_{\delta_n},\mathbf{\Phi})}\|_{\mathbf{h}} \leq \|\tilde{F}_{R_n}(X_{\delta_n}, \mathbf{\Phi})\|_{\mathbf{h}}\|e^{-\tilde{V}_n(X_{\delta_n},\mathbf{\Phi})}\|_{\mathbf{h}} \leq C_L \sum_P 2^{|X_{\delta_n}|}|\tilde{\alpha}_{n,P}(X_{\delta_n})|\bar{g}^{-1}G_\kappa(X_{\delta_n}, \varphi)$$

We thus obtain (remembering that $\tilde{\alpha}_{n,P}$ are supported on small sets) on using (5.130) of Lemma 5.17 for ε sufficiently small depending on L, implying $\bar{g}$ sufficiently small,

$$\|\tilde{F}_{R_n}(X_{\delta_n}, \mathbf{\Phi})e^{-\tilde{V}_n(X_{\delta_n},\mathbf{\Phi})}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_n} \leq C_L\bar{g}^{-1}\sum_P \|\tilde{\alpha}_{n,P}\|_{\mathcal{A},\delta_n} \leq O(1)\bar{g}^{3/4-\eta}$$

This proves (5.175). Now we turn to the proof of (5.176). As observed in the proof of Lemma 5.17, $\hat{\mathbf{h}}_*^{n_P}|\tilde{\alpha}_{n,P}|_{\mathcal{A},\delta_n} \leq n(P)!\ |1_S R_n^\sharp|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_\mathbf{n}} \leq O(1)\bar{g}^{11/4-\eta}$. We have from the definition of $\tilde{F}_{R_n}$ given in (4.43) $|\tilde{F}_{R_n}(X_{\delta_n})|_{\hat{\mathbf{h}}_*} \leq O(1)\sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})|\hat{\mathbf{h}}_*^{n_P}$, whence

$$|\tilde{F}_{R_n}|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n} \leq \sum_P |\tilde{\alpha}_{n,P}|_{\mathcal{A},\delta_n}\hat{\mathbf{h}}_*^{n_P} \leq O(1)\bar{g}^{11/4-\eta}$$

which proves (5.176).

To get these bounds for $J_n = R_n^\sharp - \tilde{F}_{R_n}e^{-\tilde{V}_n}$ we use (5.175) and (5.176) to bound $\tilde{F}_{R_n}e^{-\tilde{V}_n}$ part. We can substitute $\hat{\mathbf{h}}$ for $\mathbf{h}$ in (5.175) since the $\hat{\mathbf{h}}$ norm is smaller than the $\mathbf{h}$ norm. We bound $R_n^\sharp$ by Lemma 5.17. We have also used the trivial bound $|J|_{\hat{\mathbf{h}},\mathcal{A},\delta_n} \leq \|J\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A},\delta_n}$ to obtain the second inequality for J from the first. ∎

Lemma 5.27

$$\|R_{n+1,\mathrm{linear}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \leq O(1)L^{-(1-\varepsilon)/2}\bar{g}^{3/4-\eta} \tag{5.178}$$

$$|R_{n+1,\mathrm{linear}}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \leq O(1)L^{-(1-\varepsilon)/2}\bar{g}^{11/4-\eta} \tag{5.179}$$

Proof

Let $R_{n+1,\mathrm{linear}}$, given in (4.47) is the sum of two terms which represent contributions from small/large sets respectively. Let $R_{n+1,\mathrm{linear,s.s}}$ denote the first term:

$$R_{n+1,\mathrm{linear,s.s}}(Z_{\delta_{n+1}}) = \sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}} e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}})}J_{n,L}(Y_{\delta_{n+1}}) \tag{5.180}$$

where $Y_{\delta_{n+1}} = L^{-1}X_{\delta_{n+1}}$ and $J_n = R_n^\sharp - \tilde{F}_{R_n}e^{-\tilde{V}_n}$. By Corollary 5.25 we get

$$\|R_{n+1,\mathrm{linear,s.s}}(Z_{\delta_{n+1}})\|_{\mathbf{h},G_\kappa} \leq O(1)L^{-(7-\varepsilon)/2} \sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}} \left[|J_n(X_{\delta_n})|_{\hat{\mathbf{h}}} + \|J(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{3\kappa}}\right] \tag{5.181}$$

Note that $Z_{\delta_{n+1}}$ fixes Z_{δ_n} by restriction and the sum on the right hand side is the same as the sum over X_{δ_n} such that $L^{-1}\bar{X}^L_{\delta_n} = Z_{\delta_n}$. We multiply both sides by $\mathcal{A}(Z_{\delta_{n+1}})$. On the right hand side we have $\mathcal{A}(Z_{\delta_{n+1}}) = \mathcal{A}(Z_{\delta_n}) = \mathcal{A}(\bar{X}^L_{\delta_n}) \leq O(1)\mathcal{A}(X_{\delta_n})$ by (2.9). We fix a unit block Δ_{n+1} and sum over $Z_{\delta_{n+1}} \supset \Delta_{n+1}$. This fixes by restriction to the over $Z_{\delta_n} \supset \Delta_n$ on the right hand side. The argument on p.790 of [BDH-est] controls the constrained sum on X_{δ_n} such that $L^{-1}\bar{X}^L_{\delta_n} = Z_{\delta_n} \supset \Delta_n$ by L^3 times the sum over $X_{\delta_n} \supset \Delta_n$. Taking then the supremum over the fixed unit block gives

$$\|R_{n+1,\text{linear,s.s}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le O(1)L^{-(7-\varepsilon)/2}L^3\Big[|J_n|_{\hat{\mathbf{h}},\mathcal{A},\delta_n} + \|J_n\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A},\delta_n}\Big]$$
$$\le O(1)L^{-(1-\varepsilon)/2}\bar{g}^{3/4-\eta} \tag{5.182}$$

where for the second inequality we used Lemma 5.26.
The second term in (4.47)for $R_{n+1,\text{linear}}$ which gets contributions only from large sets is

$$R_{n+1,\text{linear,l.s.}}(Z_{\delta_{n+1}}) = \sum_{\substack{X_{\delta_{n+1}}:\text{large sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}} e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}})}R^\natural_{n,L}(L^{-1}X_{\delta_{n+1}}) \tag{5.183}$$

where we have used $J_n = R^\sharp_n$ since the relevant part $\tilde{F}_{R_n}$ is supported on small sets. We first bound in the $\mathbf{h}$ norm and observe that because of the rescaling involved and Lemma 5.17

$$\|R^\natural_{n,L}(L^{-1}X_{\delta_{n+1}})\|_{\mathbf{h}} \le \|R^\sharp_n(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{3\kappa}}G_\kappa(X_{\delta_{n+1}}) \le O(1)2^{|X_{\delta_n}|}\bar{g}^{3/4-\eta}G_\kappa(Z_{\delta_{n+1}})$$

so that on using Lemma 5.5 for $e^{-\tilde{V}_L}$

$$\|R_{n+1,\text{linear,l.s.}}(Z_{\delta_{n+1}})\|_{\mathbf{h},G_\kappa} \le O(1)\bar{g}^{3/4-\eta}\sum_{\substack{X_{\delta_{n+1}}:\text{large sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}} 2^{2|X_{\delta_{n+1}}|}$$

We estimate the $\mathcal{A}$ norm as before except that for large sets we use from (2.10)
$\mathcal{A}(L^{-1}\bar{X}^L_{\delta_{n+1}}) \le c_pL^{-4}\mathcal{A}_{-p}(X_{\delta_{n+1}})$ for any positive integer p with $c_p = O(1)$. Choose $p = 2$. Therefore

$$\|R_{n+1,\text{linear,l.s.}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le O(1)L^{-1}\bar{g}^{3/4-\eta} \tag{5.184}$$

Adding the contributions (5.180) and (5.184) we get (5.178). (5.179) can be proved in the same way. For the small set contribution we use the kernel bounds in Corollary 5.25 and Lemma 5.26. For the large set contribution we first use the rescaling involved to shift on the right hand side the $\mathbf{h}_*$ norm to the $\hat{\mathbf{h}}_*$ norm followed by the kernel bound in Lemma 5.17. ∎

Proof of Theorem 5.1

From (4.41), R_{n+1} is the sum of $R_{n+1,\text{main}}$, $R_{n+1,\text{linear}}$, $R_{n+1,3}$ and $R_{n+1,4}$. $R_{n+1,\text{main}}$ satisfies the bound given in Lemmas 5.21. For L large and ε small depending on L implying $\bar{g}$ sufficiently small $C_L\bar{g}^{3/4} \le L^{-1/2}\bar{g}^{3/4-\eta}$ with $\eta = 1/64$. Similarly $C_L\bar{g}^{3-3\delta/2} \le L^{-1/2}\bar{g}^{11/4-\eta}$ for $\delta = \eta$. Therefore $|||R_{n+1,\text{main}}|||_{n+1} \le L^{-1/2}\bar{g}^{11/4-\eta}$. Similarly from Lemmas 5.22 and 5.23 we get $|||R_{n+1,j}|||_{n+1} \le L^{-1/2}\bar{g}^{11/4-\eta}$, for $j = 3, 4$. Adding these bounds to that provided for $R_{n+1,\text{main}}$ by Lemma 5.27 we have that the sum satisfies the bound (5.11) for L sufficiently large. The bounds (5.9), (5.10) and (5.12) have been proved in Lemmas 5.17 and 5.18. ∎

6. EXISTENCE OF THE GLOBAL RENORMALIZATION GROUP TRAJECTORY AND THE STABLE MANIFOLD

In this section we will prove the existence of a global RG trajectory. The global trajectory is one that exists on every scale n no matter how large, in other words on every lattice $(\delta_n\mathbb{Z})^3$ no matter how fine, and the entire trajectory lies in a uniformly bounded domain. Our strategy will be to hold the variables of the initial unit lattice theory in a suitable domain. Then apply the RG map n_0 times where $n_0 \ge 0$ is any finite integer. The relevant variable (mass) will have expanded. Then restart the trajectory at this scale ($n \ge n_0$). We will show that for all variables at scale $n_0 \ge 0$ held in a suitable domain (arrived at by n_0 iterations from

the unit lattice domain) there exists a critical mass μ_{n_0} in this domain such that the entire trajectory for all $n \geq n_0$ is uniformly bounded. We will then prove that when n_0 is sufficiently large the critical mass is a Lipshitz continuous function of the other (contracting) variables in the domain. This proves the existence of the stable manifold at scale n_0. The main theorems are Theorems 6.2 and 6.4. One consequence is that the coupling constant g_n is bounded away from 0 uniformly in n.
The idea of proving the existence of the stable manifold after a sufficient number of RG iterations originated in discussions with David Brydges. Indeed the stable manifold has been established for various continuum models (see [BDH-eps], [BMS]). By moving forward sufficiently on the trajectory we move sufficiently close to the continuum, lattice RG artifacts become less important and the stable manifold can be estblished.

6.1. Define $\tilde{g}_n = g_n - \bar{g}$ with $\bar{g}$ defined by (5.2) and $\tilde{\mathbf{w}}_n = \mathbf{w}_n - \mathbf{w}_*$. Here $\mathbf{w}_*$ is the function on $\cup_{n\geq 0}(\delta_n \mathbb{Z})^3$ defined in Lemma 5.9. According to Lemma 5.9, $\tilde{\mathbf{w}}_n \to 0$ geometrically fast in $\mathcal{W}_l$ for all $l \geq 0$.
We will use as coordinates of the RG trajectory

$$\upsilon_n = (\tilde{g}_n, \mu_n, R_n, \tilde{\mathbf{w}}_n) \tag{6.1}$$

The RG map

$$\upsilon_{n+1} = f_{n+1}(\upsilon_n) \tag{6.2}$$

can be written in components (see (4.39), (4.42), (4.16) and Lemma 5.9)

$$\tilde{g}_{n+1} = f_{n+1,g}(\upsilon_n) = \alpha(\epsilon)\tilde{g}_n + \tilde{\xi}_n(\upsilon_n) \tag{6.3}$$

$$\mu_{n+1} = f_{n+1,\mu}(\upsilon_n) = L^{\frac{3+\varepsilon}{2}}\mu_n + \tilde{\rho}_n(\upsilon_n) \tag{6.4}$$

$$R_{n+1} = f_{n+1,R}(\upsilon_n) =: U_{n+1}(\upsilon_n) \tag{6.5}$$

$$\tilde{\mathbf{w}}_{n+1} = f_{n+1,w}(\upsilon_n) = (\mathbf{v}_{n+1} - \mathbf{v}_{c,*}) + \tilde{\mathbf{w}}_{n,L} \tag{6.6}$$

with initial

$$\upsilon_0 = (\tilde{g}_0, \mu_0, R_0, \tilde{\mathbf{w}}_0) \tag{6.7}$$

$\alpha(\varepsilon)$, ξ_n, ρ_n are defined by

$$\alpha(\epsilon) = 2 - L^{\varepsilon} = 1 - O(\log L)\varepsilon$$

for ε sufficiently small depending on L, and

$$\tilde{\xi}_n(\upsilon_n) = -L^{2\varepsilon}a_{c*}\tilde{g}_n^2 - L^{2\varepsilon}(a_n - a_c*)(\tilde{g}_n + \bar{g})^2 + \xi_n(\upsilon_n) \tag{6.8}$$

$$\tilde{\rho}_n(\upsilon_n) = -L^{2\varepsilon}b_n(\bar{g} + \tilde{g}_n)^2 + \rho_n(\upsilon_n) \tag{6.9}$$

Let E_n be the Banach space consisting of elements υ_n with the norm

$$\|\upsilon_n\|_n = \max((\nu\bar{g})^{-1}|\tilde{g}_n|,\ \bar{g}^{-(2-\delta)}|\mu_n|,\ \bar{g}^{-(11/4-\eta)}|||R_n|||_n, \tilde{c}_L^{-1}\|\tilde{\mathbf{w}}_n\|_n) \tag{6.10}$$

where the norm $|||R|||_{\delta_n}$ of R_n is as defined in (5.6). ν is the $O(1)$ constant which figures in the specification of the domain $\mathcal{D}_n$, (5.4). We have $0 < \nu < 1$. We will take $\nu > 0$ sufficiently small depending on L and this will be specified in the proofs of Lemmas 6.3 and 6.4 below. The norm $\|\tilde{\mathbf{w}}_n\|_{\delta_n}$ and the constant $\tilde{c}_L$ are as specified in Lemma 5.9.

6.2. Domains and bounds

Let $E_n(r) \subset E_n$ be the open ball of radius r, centered at the origin:

$$E_n(r) = \{\upsilon_n \in E_n : \|\upsilon_n\|_n < r\} \tag{6.11}$$

Let $\mathcal{D}_n$ be the domain of (g_n, μ_n, R_n) defined in (5.4) and (5.5).

$$\upsilon_n \in E_n(1) \Rightarrow (g_n, \mu_n, R_n) \in \mathcal{D}_n \tag{6.12}$$

and then Theorem 5.1 holds.

Let $\upsilon_n \in E_n(1)$. Let $\varepsilon > 0$ be sufficiently small (depending on L). Then from Theorem 5.1, Lemma 5.12 , (6.8) and (6.9) we get the bounds

$$\begin{aligned} |\tilde{\xi}_n(\upsilon_n)| \leq& C_L\Big((\nu^2 + L^{-nq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big) \\ |\tilde{\rho}_n(\upsilon)| \leq& C_L\bar{g}^2 \\ |||U_{n+1}(\upsilon_n)|||_{n+1} \leq& L^{-1/4}\bar{g}^{11/4-\eta} \end{aligned} \tag{6.13}$$

We have the following Lipshitz bounds :

Lemma 6.1

Let $\upsilon_n, \upsilon'_n \in E_n(1/4)$. Then we have:

$$\begin{aligned} &\text{(i)} \quad |\tilde{\xi}_n(\upsilon_n) - \tilde{\xi}_n(\upsilon'_n)| \leq C_L\Big((\nu^2 + L^{-nq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big)\|\upsilon_n - \upsilon'_n\|_n \\ &\text{(ii)} \quad |\tilde{\rho}_n(\upsilon_n) - \tilde{\rho}_n(\upsilon'_n)| \leq C_L\bar{g}^2\|\upsilon_n - \upsilon'_n\|_n \\ &\text{(iii)} \quad |||U_{n+1}(\upsilon_n) - U_{n+1}(\upsilon'_n)|||_{n+1} \leq O(1)L^{-1/4}\bar{g}^{11/4-\eta}\|\upsilon_n - \upsilon'_n\|_n \\ &\text{(iv)} \quad \tilde{c}_L^{-1}\|f_{n+1,\mathbf{w}}(\upsilon_n) - f_{n+1,\mathbf{w}}(\upsilon'_n)\|_{n+1} \leq L^{-1/5}\|\upsilon_n - \upsilon'_n\|_n \end{aligned} \tag{6.14}$$

Proof:

$\tilde{\xi}_n, \tilde{\rho}_n, U_{n+1}$ are (norm) analytic functions in $\mathcal{D}_n$ and thus in $E_n(1)$. The analyticity follows from the algebraic operations in Section 4, the norm analyticity of the reblocking map together with the norm analyticity of the extraction map (Theorem 5 [BDH-est]). Therefore we can use Cauchy estimates exactly as in in the proof of Lemma 6.1 of [BMS] together with the bounds (6.13) to get (i), (ii) and (iii). To get (iv) note that from (6.6) and the definition of the norms in (5.61) we have

$$\begin{aligned} \|f_{n+1,\mathbf{w}}(\upsilon_n) - f_{n+1,\mathbf{w}}(\upsilon'_n)\|_{n+1} &\leq L^{2d_s} \max_{1\leq p\leq 3} \sup_{x\in(\delta_{n+1}\mathbf{Z})^3} \Big((|x| + \delta_{n+1})^{\frac{6p+1}{4}}|\tilde{w}_n^{(p)}(Lx) - \tilde{w}_n'^{(p)}(Lx)|\Big) \\ &\leq L^{2d_s} \max_{1\leq p\leq 3} \sup_{y\in(\delta_n\mathbf{Z})^3} \Big((\frac{|y|}{L} + \delta_{n+1})^{\frac{6p+1}{4}}|\tilde{w}_n^{(p)}(y) - \tilde{w}_n'^{(p)}(y)|\Big) \\ &\leq L^{2d_s-7/4} \max_{1\leq p\leq 3} \sup_{y\in(\delta_n\mathbf{Z})^3} \Big((|y| + \delta_n)^{\frac{6p+1}{4}}|\tilde{w}_n^{(p)}(y) - \tilde{w}_n'^{(p)}(y)|\Big) \\ &\leq L^{-1/5}\|\tilde{\mathbf{w}}_n - \tilde{\mathbf{w}}'_n\|_n \end{aligned}$$

where we have used $d_s = \frac{3-\varepsilon}{4}$ and then L large and ε sufficiently small. Now dividing both sides by $\tilde{c}_L$ we get (iv). ∎

6.3. Existence of the global RG trajectory

Let $\mathcal{D}_n$ be the domain specified by (5.4),(5.5), and (5.6). Let $(\tilde{g}_0, \mu_0, R_0)$ belong to $\tilde{\mathcal{D}}_0 \subset \mathcal{D}_0$ where $\tilde{\mathcal{D}}_0$ is specified by

$$|\tilde{g}_0| < 2^{-(n_0+5)}\nu\bar{g}, \quad |\mu_0| < 2^{-(n_0+5)}L^{-\frac{3+\varepsilon}{2}n_0}\bar{g}^{2-\delta}$$

$$|||R_0|||_0 \leq \bar{g}^{11/4-\eta}$$

Let n_0 be a positive integer. By iterating the RG map n_0 times using Theorem 5.1 and the flow equation (4.39) recursively we obtain for ε sufficiently small depending on L and n_0, $(g_{n_0}, \mu_{n_0}, R_{n_0}) \in \mathcal{D}_{n_0}(1/32)$ where $\mathcal{D}_{n_0}(1/32)$ is specified by

$$|\tilde{g}_{n_0}| < \frac{1}{32}\nu\bar{g}, \quad |\mu_{n_0}| < \frac{1}{32}\bar{g}^{2-\delta}$$

$$|||R_{n_0}|||_{n_0} < \frac{1}{32}\bar{g}^{11/4-\eta}$$

We will now prove the existence of a global solution to the discrete flow map (6.2):

$$\upsilon_{n+1} = f_{n+1}(\upsilon_n), \; \forall \, n \geq n_0$$

with initial condition

$$\upsilon_{n_0} = (\tilde{g}_{n_0}, \mu_{n_0}, \tilde{\mathbf{w}}_{n_0}, R_{n_0})$$

in a bounded domain. To this end we consider the Banach space $\mathbf{E}_{n_0}$ of sequences $\mathbf{s}_{n_0} = \{\upsilon_n\}_{n\geq n_0}$, each $\upsilon_n \in E_n$, with the norm

$$\|\mathbf{s}_{n_0}\| = \sup_{n\geq n_0} \|\upsilon_n\|_n \tag{6.15}$$

and the open ball $\mathbf{E}_{n_0}(r) \subset \mathbf{E}$

$$\mathbf{E}_{n_0}(r) = \{\mathbf{s}_{n_0} : \|\mathbf{s}_{n_0}\| < r\} \tag{6.16}$$

We will derive on the space of sequences $\mathbf{E}_{n_0}$ an equation that a global RG trajectory must solve and then prove for $\upsilon_{n_0} \in E_{n_0}(1/32)$ the existence of a unique solution in the ball $\mathbf{E}_{n_0}(1/4)$, for a suitable choice of r, by the contraction mapping principle. This adapts a standard method from the theory of hyperbolic dynamical systems in Banach spaces due to Irwin in [I]. Irwin's analysis is explained by Shub in Appendix 2, Chapter 5 of [S]. For earlier applications see section 5 of [BDH-eps] and section 6 of [BMS].

Theorem 6.2
Let L be large, ν be sufficiently small depending on L, then ε sufficiently small depending on L. Let $\upsilon_{n_0} \in E_{n_0}(1/32)$ for any integer $n_0 \geq 0$. Then there is a μ_{n_0} such that there exists a sequence $\mathbf{s}_{n_0} = \{\upsilon_n\}_{n\geq n_0}$ in $\mathbf{E}_{n_0}(1/4)$ satisfying $\upsilon_{n+1} = f_{n+1}(\upsilon_n)$ for all $n \geq n_0$.

Proof
Our initial data will be at scale n_0. Let $n_0 \leq n \leq N-1$. We iterate the map (6.3) forwards N times. We iterate the map (6.4) backwards $N - n_0$ times starting from a given μ_N. We then easily derive

$$\tilde{g}_{n+1} = \alpha(\epsilon)^{n+1-n_0}\tilde{g}_{n_0} + \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j}\tilde{\xi}_j(\upsilon_j), \quad n_0 \le n \le N-1$$

$$\mu_{n+1} = L^{-\frac{3+\varepsilon}{2}(N-n-1)}\mu_N - \sum_{j=n+1}^{N-1} L^{-\frac{3+\varepsilon}{2}(j-n)}\tilde{\rho}_j(\upsilon_j), \quad n_0 - 1 \le n \le N-2$$

Let us fix $\mu_N = \mu_f$ and take $N \to \infty$. In other words we assume the μ_n flow is bounded and then must show that such a flow exists. We have

$$\tilde{g}_{n+1} = \alpha(\epsilon)^{n+1-n_0}\tilde{g}_{n_0} + \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j}\tilde{\xi}_j(\upsilon_j), \quad n \ge n_0 \tag{6.17}$$

$$\mu_{n+1} = -\sum_{j=n+1}^{\infty} L^{-\frac{3+\varepsilon}{2}(j-n)}\tilde{\rho}_j(\upsilon_j), \quad n \ge n_0 - 1 \tag{6.18}$$

together with

$$R_{n+1} = U_{n+1}(\upsilon_n), \quad n \ge n_0 \tag{6.19}$$

$$\tilde{\mathbf{w}}_{n+1} = (\mathbf{v}_{n+1} - \mathbf{v}_{c*}) + \tilde{\mathbf{w}}_{n,L} \tag{6.20}$$

The solution of the autonomous flow $\tilde{\mathbf{w}}_n$ given by lemma 5.9 can be incorporated in the υ_n and $\tilde{\mathbf{w}}_n$ is then no longer a flow variable.
For ε sufficiently small (depending on L)

$$0 < \alpha(\epsilon) < 1 \tag{6.21}$$

Note that $\mathbf{s}_{n_0} \in \mathbf{E}_{n_0}(1/4)$ implies $\upsilon_j \in E_j(1/4)$ for all $j \ge n_0$. Then the infinite sum of (6.18) converges by (6.21) and (6.13). So μ_{n_0} has now been determined provided (6.17)-(6.19) has a solution in the afore mentioned ball. It is easy to verify that any solution of (6.17)- (6.18), together with the autonomous $\tilde{\mathbf{w}}$ flow, is a solution of the RG flow $\upsilon_{n+1} = f_{n+1}(\upsilon_n)$ for $n \ge n_0$.

We write (6.17)-(6.19) in the form

$$\upsilon_{n+1} = F_{n+1}(\mathbf{s}_{n_0}), \quad n \ge n_0 \tag{6.22}$$

where $\mathbf{s}_{n_0} = (\upsilon_{n_0}, \upsilon_{n_0+1}, \upsilon_{n_0+2}, ...)$ and F_{n+1} has components $(F^{(g)}_{n+1}, F^{(\mu)}_{n+1}, F^{(R)}_{n+1})$ given by the r.h.s. of (6.17), (6.18), (6.19) respectively.
If we write

$$\mathbf{F}(\mathbf{s}_{n_0}) = (F_{n_0}(\mathbf{s}_{n_0}), F_{n_0+1}(\mathbf{s}_{n_0}), ...)$$

then (6.22) can be written as a fixed point equation

$$\mathbf{s}_{n_0} = \mathbf{F}(\mathbf{s}_{n_0}) \tag{6.23}$$

We seek a solution of (6.23) in the open ball $\mathbf{E}_{n_0}(1/4)$ with initial data $\upsilon_{n_0} = (\tilde{g}_{n_0}, \mu_{n_0}, R_{n_0}, \tilde{\mathbf{w}}_{n_0})$ in $E_{n_0}(1/32)$. The existence of a unique solution follows by the standard contraction mapping principle and the next Lemma. ■

Lemma 6.3

$$\mathbf{s}_{n_0} \in \mathbf{E}_{n_0}(1/32) \Rightarrow \mathbf{F}(\mathbf{s}_{n_0}) \in \mathbf{E}_{n_0}(1/16) \tag{6.24}$$

Moreover, for $\mathbf{s}_{n_0}, \mathbf{s}'_{n_0} \in \mathbf{E}_{n_0}(1/4)$

$$\|\mathbf{F}(\mathbf{s}_{n_0}) - \mathbf{F}(\mathbf{s}'_{n_0})\| \leq \frac{1}{2}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\| \tag{6.25}$$

Proof

First we prove (6.24), and thus take $\mathbf{s}_{n_0} \in \mathbf{E}_{n_0}(1/32)$. Then $\upsilon_n \in E_n(1/32)$ for every $n \geq n_0$ and we can use the estimates (6.13). From (6.17) and the estimates in (6.13) we have

$$(\nu\bar{g})^{-1}|F^{(g)}_{n+1}(\mathbf{s}_{n_0})| < \alpha(\epsilon)\frac{1}{32} + (\nu\bar{g})^{-1}\sum_{j=n_0}^{n}\alpha(\epsilon)^{n-j}C_L\Big((\nu^2 + L^{-jq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big)$$

$$< \frac{1}{32} + C_L\frac{\bar{g}^{7/4-\eta}}{\nu(1-\alpha(\epsilon)}) + C_L\frac{\bar{g}\nu}{1-\alpha(\epsilon)} + C_L\frac{\bar{g}}{\nu(1-\alpha(\varepsilon)^{-1}L^{-q})}$$

$$< \frac{1}{32} + \frac{C_L}{\nu\log L}\varepsilon^{1/4-\eta} + \frac{C_L}{\log L}\nu + \frac{C_L}{\nu(1-L^{-q})}\varepsilon < \frac{1}{16}$$

for L sufficiently large, ν sufficiently small depending on L so that $\frac{C_L}{\log L}\nu \leq 1/96$ and then ε sufficiently small depending on L so that $\bar{g} \leq C_L\varepsilon$ is sufficiently small, $\frac{C_L}{\nu(1-L^{-q})}\varepsilon \leq 1/96$ and $\frac{C_L}{\nu\log L}\varepsilon^{1/4-\eta} \leq 1/96$
Similarly from (6.18) and (6.13) we have

$$\bar{g}^{-(2-\delta)}|F^{(\mu)}_{n+1}(\mathbf{s}_{n_0})| \leq c_L\bar{g}^{\delta}\sum_{j=n+1}^{\infty}L^{-\frac{3+\varepsilon}{2}(j-n)} \leq L^{-\frac{3+\varepsilon}{2}}(1-L^{-\frac{3+\varepsilon}{2}})^{-1} < \frac{1}{16}$$

since $\delta = 1/64$, L sufficiently large, and ε sufficiently small depending on L.
Finally from (6.19) and (6.13)

$$\bar{g}^{-(11/4-\eta)}|||F^{(R)}_{n+1}(\mathbf{s}_{n_0})|||_{n+1} \leq L^{-1/4} < \frac{1}{16}$$

for L sufficiently large. This proves (6.24).
To prove (6.25), take $\mathbf{s}_{n_0}, \mathbf{s}'_{n_0} \in \mathbf{E}_{n_0}(1/4)$. This implies that $\upsilon_n, \upsilon'_n \in E_n(1/4)$ for every $n \geq n_0$ and we can use the Lipshitz estimates of lemma 6.1. Note that the initial coupling g_{n_0} is held fixed. Then we have

$$(\nu\bar{g})^{-1}|F^{(g)}_{n+1}(\mathbf{s}_{n_0}) - F^{(g)}_{n+1}(\mathbf{s}'_{n_0})| \leq \sum_{j=n_0}^{n}\alpha(\epsilon)^{n-j}(\nu\bar{g})^{-1}|\tilde{\xi}_j(\upsilon_j) - \tilde{\xi}_j(\upsilon'_j)|$$

$$\leq (\nu\bar{g})^{-1}\sum_{j=n_0}^{n}\alpha(\epsilon)^{n-j}C_L\Big((\nu^2 + L^{-jq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big)\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\|$$

$$\leq \Big(\frac{C_L}{\nu\log L}\varepsilon^{1/4-\eta} + C_L\frac{\bar{g}\nu}{1-\alpha(\epsilon)} + C_L\frac{\bar{g}}{\nu(1-\alpha(\varepsilon)^{-1}L^{-q})})\Big)\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\| \leq \frac{1}{2}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\|$$

by estimating as above in the bound for $F^{(g)}_{n+1}(\mathbf{s}_{n_0})$ with L sufficiently large, ν sufficiently small depending on L and ε sufficiently small depending on L. Similarly,

$$\bar{g}^{-(2-\delta)}|F^{(\mu)}_{n+1}(\mathbf{s}_{n_0}) - F^{(\mu)}_{n+1}(\mathbf{s}'_{n_0})| \leq \sum_{j=n+1}^{\infty}L^{-\frac{3+\varepsilon}{2}(j-n)}\bar{g}^{-(2-\delta)}|\tilde{\rho}_j(\upsilon_j) - \tilde{\rho}_j(\upsilon'_j)|$$

$$\leq L^{-\frac{3+\varepsilon}{2}}(1-L^{-\frac{3+\varepsilon}{2}})^{-1}C_L\bar{g}^{\delta}\|\mathbf{s}_{n_0}-\mathbf{s}'_{n_0}\| \leq \frac{1}{2}\|\mathbf{s}_{n_0}-\mathbf{s}'_{n_0}\|$$

for L sufficiently large and ε sufficiently small depending on L. Finally

$$\bar{g}^{-(11/4-\eta)}|||F^{(R)}_{n+1}(\mathbf{s}_{n_0})-F^{(R)}_{n+1}(\mathbf{s}'_{n_0})|||_{n+1} = \bar{g}^{-(11/4-\eta)}|||U_{n+1}(\upsilon_n)-U_{n+1}(\upsilon'_n)|||_{n+1}$$

$$\leq O(1)L^{-1/4}\|\mathbf{s}_{n_0}-\mathbf{s}'_{n_0}\| \leq \frac{1}{2}\|\mathbf{s}_{n_0}-\mathbf{s}'_{n_0}\|$$

for L sufficiently large. Thus (6.25) has been proved. This completes the proof of Theorem 6.2. ■

6.4. Theorem 6.2 says that if $\upsilon_{n_0}=(\tilde{g}_{n_0},\mu_{n_0},R_{n_0},\tilde{\mathbf{w}}_{n_0})\in E_{n_0}(1/32)$ for any $n_0\geq 0$ then there is an μ_{n_0} such that a uniformly bounded RG trajectory exists. We call this a critical mass. The next theorem proves the uniqueness of μ_{n_0} for n_0 sufficiently large : μ_{n_0} is a norm continuous function of $(\tilde{g}_{n_0},R_{n_0},\tilde{\mathbf{w}}_{n_0})$.

To this end we represent the Banach space E_n as a product of two Banach spaces $E_n=E_{n,1}\times E_{n,2}$. We write $\upsilon_n\in E_n$ as $\upsilon_n=(\upsilon_{n,1},\upsilon_{n,2})$ where $\upsilon_{n,1}=(\tilde{g}_n,R_n,\tilde{\mathbf{w}}_n)$ and $\upsilon_{n,2}=\mu_n$. $\upsilon_{n,2}$ is the expanding (relevant)variable. Let $p_i,\ i=1,2$, denote the projector onto $E_{n,i}$ and $f_{n,i}=p_i\circ f_n$. The norm $\|\cdot\|_n$ on E_n being a a box norm we have

$$\|\upsilon_n\|_n=\max(\|\upsilon_{n,1}\|_n,\|\upsilon_{n,2}\|_n)$$

where $\|\upsilon_{n,2}\|_n=\bar{g}^{-(2-\delta)}|\mu_n|$ and $\|\upsilon_{n,1}\|_n=\max((\nu\bar{g})^{-1}|\tilde{g}_n|,\ \bar{g}^{-(11/4-\eta)}|||R_n|||_{\delta_n},\tilde{c}_L^{-1}\|\tilde{\mathbf{w}}_n\|_{\delta_n})$.

In the following we continue to assume that L is sufficiently large, followed by ν sufficiently small depending on L, then ε sufficiently small depending on L. The last condition also implies that $\bar{g}\leq C_L\varepsilon$.

Theorem 6.4 : Let $\mathbf{s}_{n_0}=\{\upsilon_n:\upsilon_{n+1}=f_{n+1}(\upsilon_n)\}_{n\geq n_0}\in\mathbf{E}(1/4)$ *be the global RG trajectory of Theorem 6.2. Then for* n_0 *sufficiently large there exists a Lipshitz continuous function* $h:E_{n_0,1}\to\mathbb{R}$ *with Lipshitz constant* 1 *such that the stable manifold of the sequence of maps* $\{f_n\}_{n\geq n_0+1}$, $W^s_{n_0}=\{\upsilon_{n_0}\in E_{n_0}(1/32):\mathbf{s}_{n_0}\in\mathbf{E}(1/4)\}$ *is the graph*

$$W^s_{n_0}=\{\upsilon_{n_0,1},h(\upsilon_{n_0,1}\}$$

We will prove the theorem following the analysis of Shub in [S, Section 5]. Similar results were proved in [Section 5.3, BDH-eps] and [Section 6, BMS] for continuum models. The proof rests on the following lemma.

Lemma 6.5
Let $\upsilon_n,\upsilon'_n\in E_n(1/4)$. Then for $n\geq n_0$, n_0 sufficiently large

$$\|f_{n+1,1}(\upsilon_n)-f_{n+1,1}(\upsilon'_n)\|_{n+1}\leq(1-\varepsilon)\|\upsilon_n-\upsilon'_n\|_n \tag{6.26}$$

and, if $\|\upsilon_{n,2}-\upsilon'_{n,2}\|_n\geq\|\upsilon_{n,1}-\upsilon'_{n,1}\|_n$ then

$$\|f_{n+1,2}(\upsilon_n)-f_{n+1,2}(\upsilon'_n)\|_{n+1}\geq(1+\varepsilon)\|\upsilon_n-\upsilon'_n\|_n \tag{6.27}$$

Proof
First we prove (6.26). $f_{n+1,1}$ has components $f_{n+1,g}$, $f_{n+1,R}$ and $f_{n+1,w}$. From (6.3)

$$f_{n+1,g}(\upsilon_n)=\alpha(\epsilon)\tilde{g}_n+\tilde{\xi}_n(\upsilon_n)$$

Since $\upsilon_n,\upsilon'_n\in E_n(1/4)$ we can use lemma 6.1. Therefore for $n\geq n_0$

$$
\begin{aligned}
(\nu\bar{g})^{-1}|f_{n+1,g}(\upsilon_n) - f_{n+1,g}(\upsilon'_n)| &\leq \alpha(\epsilon)\|\upsilon_n - \upsilon'_n\|_n + (\nu\bar{g})^{-1}|\tilde{\xi}_n(\upsilon_n) - \tilde{\xi}_n(\upsilon'_n)| \\
&\leq \Big(1 - \varepsilon\log(L) + C_L\varepsilon(\nu + \frac{1}{\nu}L^{-n_0 q} + \frac{1}{\nu}\varepsilon^{3/4-\eta}\Big)\|\upsilon_n - \upsilon'_n\|_n
\end{aligned}
$$

Let L be large. Let ν be sufficiently small and n_0 sufficiently large so that $C_L L^{-n_0 q/2} \leq \nu \leq \frac{1}{C_L}$ and ε sufficiently small so that $\nu \geq C_L \varepsilon^{3/8-\eta}$. Then we have

$$
\begin{aligned}
1 - \varepsilon\log(L) + \varepsilon C_L(\nu + \frac{1}{\nu}L^{-n_0 q} + \frac{1}{\nu}\varepsilon^{3/4-\eta}) &\leq 1 + \varepsilon(-\log(L) + 1 + L^{-qn_0/2} + \varepsilon^{3/8}) \\
&\leq 1 + \varepsilon(-\log(L) + 3) \leq 1 - \varepsilon
\end{aligned}
$$

Therefore

$$
(\nu\bar{g})^{-1}|f_{n+1,g}(\upsilon_n) - f_{n+1,g}(\upsilon'_n)| \leq (1-\varepsilon)\|\upsilon_n - \upsilon'_n\|_n
$$

Since $f_{n+1,R}(\upsilon_n) = U_{n+1}(\upsilon_n)$, we have from lemma 6.1 for L sufficiently large

$$
\bar{g}^{-(11/4-\eta)}|||f_{n+1,R}(\upsilon_n) - f_{n+1,R}(\upsilon'_n)|||_{n+1} \leq (1-\varepsilon)\|\upsilon_n - \upsilon'_n\|_n
$$

as well as

$$
\tilde{c}_L^{-1}\|f_{n+1,\mathbf{w}}(\upsilon_n) - f_{n+1,\mathbf{w}}(\upsilon'_n)\|_{n+1} \leq (1-\varepsilon)\|\upsilon_n - \upsilon'_n\|_n
$$

These three inequalities prove (6.26).
Next we turn to (6.27). In this case by assumption $\|\upsilon_{n,2} - \upsilon'_{n,2}\| \geq \|\upsilon_{n,1} - \upsilon'_{n,1}\|$ and hence, since our norms are box norms, we have

$$
\|\upsilon_n - \upsilon'_n\|_n = \|\upsilon_{n,2} - \upsilon'_{n,2}\|_n = \bar{g}^{-(2-\delta)}|\mu_n - \mu'_n|
$$

From (6.4)

$$
f_{n+1,\mu}(\upsilon_n) = L^{\frac{3+\varepsilon}{2}}\mu_n + \tilde{\rho}_n(\upsilon_n)
$$

Then, using lemma 6.1, we have

$$
\begin{aligned}
\bar{g}^{-(2-\delta)}|f_\mu(\upsilon_n) - f_\mu(\upsilon'_n)| \geq& L^{\frac{3+\varepsilon}{2}}\|\upsilon_n - \upsilon'_n\|_n - \bar{g}^{-(2-\delta)}|\tilde{\rho}(\upsilon_n) - \tilde{\rho}(\upsilon'_n)| \\
&\geq (L^{\frac{3+\varepsilon}{2}} - C_L\bar{g}^{\delta})\|\upsilon_n - \upsilon'_n\|_n \qquad\qquad n \\
&\geq (1+\varepsilon)\|\upsilon_n - \upsilon'_n\|
\end{aligned}
$$

for ε sufficiently small depending on L. This proves (6.27). ■

Proof of Theorem 6.4
To prove that $W^s_{n_0}$ is given by a graph of a function $\upsilon_{n_0,2} = h(y_{n_0,1})$ it is enough to prove that if in $M_{n,0}$ we take two points $\upsilon_{n_0} = (\upsilon_{n_0,1}, \upsilon_{n_0,2})$ and $\upsilon'_{n_0} = (\upsilon'_{n_0,1}, \upsilon'_{n_0,2})$ then

$$
\|\upsilon_{n_0,2} - \upsilon'_{n_0,2}\|_{n_0} \leq \|\upsilon_{n_0,1} - \upsilon'_{n_0,1}\|_{n_0} \tag{6.28}
$$

because then for a given $\upsilon_{n_0,1}$ we would have at most one $\upsilon_{n_0,2}$, and by theorem 6.2 there exists such a $\upsilon_{n_0,2}$. This means that $W^s_{n_0}$ is the graph of a function h, $\upsilon_{n_0,2} = h(\upsilon_{n_0,1})$, and moreover

$$
\|h(\upsilon_{n_0,1}) - h(\upsilon'_{n_0,1})\|_{n_0} \leq \|\upsilon_{n_0,1} - \upsilon'_{n_0,1}\|_{n_0}
$$

Suppose (6.28) is not true. Then

$$\|\upsilon_{n_0,2}-\upsilon'_{n_0,2}\|_{n_0} > \|\upsilon_{n_0,1}-\upsilon'_{n_0,1}\|_{n_0} \tag{6.29}$$

(6.29) implies that (6.27) holds. The latter followed by (6.26) gives

$$\begin{aligned}\|f_{n_0+1,2}(\upsilon_{n_0})-f_{n_0+2}(\upsilon'_{n_0})\|_{n_0+1} &\geq (1+\varepsilon)\|\upsilon_{n_0}-\upsilon'_{n_0}\|_{n_0} > (1-\varepsilon)\|\upsilon_{n_0}-\upsilon'_{n_0}\|_{n_0}\\ &\geq \|f_{n_0+1,1}(\upsilon_{n_0})-f_{n_0+1,1}(\upsilon'_{n_0})\|_{n_0+1}\end{aligned} \tag{6.30}$$

and hence

$$\|f_{n_0+1}(\upsilon_{n_0})-f_{n_0+1}(\upsilon'_{n_0})\|_{n_0+1} = \|f_{n_0+1,2}(\upsilon_{n_0})-f_{n_0+2}(\upsilon'_{n_0})\|_{n_0+1} \geq (1+\varepsilon)\|\upsilon_{n_0}-\upsilon'_{n_0}\|_{n_0} \tag{6.31}$$

Define the composition of maps

$$\mathcal{P}^k_n \doteq f_{n+k}\circ .....\circ f_{n+2}\circ f_{n+1}$$

Now

$$\|\mathcal{P}^2_{n_0}(\upsilon_{n_0})-\mathcal{P}^2_{n_0}(\upsilon'_{n_0})\|_{n_0+2} = \|f_{n_0+2}(f_{n_0+1}(\upsilon_{n_0}))-f_{n_0+2}(f_{n_0+1}(\upsilon'_{n_0}))\|_{n_0+2}$$

By (6.30),the second part of Lemma 6.4 followed by (6.31) we get

$$\|\mathcal{P}^2_{n_0}(\upsilon_{n_0})-\mathcal{P}^2_{n_0}(\upsilon'_{n_0})\|_{n_0+2} \geq (1+\varepsilon)\|f_{n_0+1}(\upsilon_{n_0})-f_{n_0+1}(\upsilon'_{n_0})\|_{n_0+1} \geq (1+\varepsilon)^2\|\upsilon_{n_0}-\upsilon'_{n_0}\|_{n_0}$$

Repeating this k times we get for all $k\geq 0$

$$\|\mathcal{P}^k_{n_0}(\upsilon_{n_0})-\mathcal{P}^k_{n_0}(\upsilon'_{n_0})\|_{n_0+k} \geq (1+\varepsilon)^k\|\upsilon_{n_0}-\upsilon'_{n_0}\|_{n_0} \tag{6.32}$$

Now $\upsilon_{n_0},\upsilon'_{n_0}$ belong to $W^s_{n_0}$ and $\mathcal{P}^k_{n_0}(\upsilon_{n_0})$ is a member of the sequence $\mathbf{s}_{n_0}\in\mathbf{E}(1/4)$. Therefore $\|\mathcal{P}^k_{n_0}(\upsilon_{n_0})\|_{n_0+k}<1/4$. Therefore we have from (6.32) the bound $\frac{1}{2}>(1+\varepsilon)^k\|\upsilon_{n_0}-\upsilon'_{n_0}\|_{n_0}$. By making k arbitrarily large we get a contradiction because $\upsilon_{n_0}\neq\upsilon'_{n_0}$ under (6.29). Hence (6.28) is true and the theorem 6.4 has been proved ∎

As a consequence of theorem 6.4 we have $\upsilon_n\in E_n(1/4)$, $\forall n\geq n_0$. This implies that $|\tilde{g}_n|<\frac{1}{4}\nu\bar{g}$, $\forall n\geq n_0$. Whence for all $n\geq 0$

$$(1-\frac{1}{4}\nu)\bar{g}<g_n<(1+\frac{1}{4}\nu)\bar{g} \tag{6.33}$$

We have $0<\nu<1$. Therefore the effective coupling constant generated by the discrete RG flow is uniformly bounded away from 0 at all RG scales.

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