Continuity of the four-point function of massive φ_4^4-theory above threshold

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Abstract

In this paper we prove that the four-point function of massive φ_4^4-theory is continuous as a function of its independent external momenta when posing the renormalization condition for the (physical) mass on-shell. The proof is based on integral representations derived inductively from the perturbative flow equations of the renormalization group. It closes a longstanding loophole in rigorous renormalization theory in so far as it shows the feasibility of a physical definition of the renormalized coupling.

1 Introduction

Analyticity and regularity of Feynman-amplitudes in quantum field theory have been a long-standing subject of research, as well for calculational aspects as for the mathematical structures lying behind. After the pioneering work of Landau [Lan] this area of research was particularly fruitful and active in the 1960ies [ELOP], [Nak], [Tod]. In the 1970ies the interest shifted somewhat away from these questions. With the advent of QCD, analyticity and dispersion relations were no more viewed as central for the understanding of the theory of strong interactions. Still there has been much progress, in particular on the calculational side of the subject, afterwards, progress which we are unable to review. See for example [tHV] where a general analysis of the singularity structure at one-loop level is achieved. A recent book on the state of the art in calculational techniques is [Smi].

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A mathematically rigorous analysis of analyticity and regularity properties is considerably complicated by the fact that the physically interesting theories need to be reparametrized and renormalized. This largely destroys the simple homogeneity properties of the bare Feynman amplitudes. As a consequence, analyticity studies were often performed on bare amplitudes, under the plausible asumption that the local counter terms introduced for renormalization, would not upset the results achieved for the bare theory. Historically one should note that a rigorous theory of renormalization was only at the disposal about a decade after Landau's paper. Some rigorous results taking into account renormalization are due to Chandler [Cha], who shows with the aid of analytical renormalization that renormalized Feynman amplitudes are holomorphic outside the Landau surfaces[1], and that they are distributions, which - under certain restrictions - are boundary values of holomorphic functions in the complexified momenta.

We also note that Minkowski space Green functions were much less studied in mathematical physics after the advent of the papers of Osterwalder and Schrader [OS] and related work which permit to conclude on the existence of a relativistic theory once its Euclidean counter part has been constructed and certain growth and regularity properties of its Schwinger functions have been verified.

The procedure of perturbative renormalization, as it is nowadays presented in text books, is as follows : One starts from a bare Lagrangian. This Lagrangian has to be complemented by counter terms to give meaningful results for perturbative calculations. The precise values of these counter terms are fixed through renormalization conditions, which express the free parameters appearing in the Lagrangian in such a way that the results of calculations agree with experiment. For example the fine structure constant in QED could be fixed such that the cross section for Compton scattering at some fixed values of energy-momenta agrees with experiment. In the theory of the massive self-interacting scalar field to which we will restrict in this paper, one has to determine correspondingly the renormalized coupling g by comparison with the experimental value of the boson-boson scattering cross section at some fixed physical energy-momenta. This means one has to fix the value of the four-point function at those values of the external energy-momenta.

But there is still a gap between this description and what we know : Renormalized Feynman-amplitudes are known to exist as distributions [Hep1], [Spe], [Zim], [EG]. This generally does not permit to prescribe their values at given external momenta on imposing a renormalization condition. It is also known that there are regions in momentum space where the renormalized Feynman-amplitudes exist as analytic functions. For the two-

[1] For high order graphs these surfaces are hard to visualize since their definition involves the momenta (loop and external), the Feynman parameters and the incidence and loop matrices at the same time.

point function, if properly renormalized[2], this region is known to include the mass-shell. In fact we know the 1PI two-point function to be analytic for $p^2 < 4m^2$ [Hep2], [Stei], [EG], see also [KKS]. This means that the mass and wave function renormalization can be performed at a physical point, namely the physical mass. *For the four-point function, the analyticity domain does not include physical values of the momenta* (where the external particles are on mass-shell). Already at one-loop, there is a cut starting at $s = 4m^2$ (s being the total energy in the centre-of-mass frame). On the other hand, knowing that the four-point function exists as a distribution, does not permit to define a physical renormalized coupling, i.e. a number. A reasonable minimal requirement for such a definition is the continuity of the four-point function in some region above threshold $s = 4m^2$, i.e. in the physical region. It is the aim of the present paper to show that the four-point function is a continuous function of the external momenta all over $\mathbb{R}^{12}$ (taking into account momentum conservation when counting the variables). With our methods one could go beyond, in the sense of proving Hölder continuity[3] of type η, $0 < \eta < 1/3$, w.r.t. the Lorentz invariant variables $p_i \cdot p_k$. We will also prove continuity of the two-point function in $\mathbb{R}^4$. Landau [Lan, ch.4], considered that the four-point function should be continuous above threshold, and that the degree of singularity of the Green functions increased with the number of external lines and decreased with the order of perturbation theory. While the first statement is for example confirmed by [tHV], the second one which is based on counting the number of integrations over Feynman parameters, seems to be too strong.

A first basic tool for the proof are the flow equations of the renormalization group which are presented in section 2. They permit to study properties of Green functions in an inductive framework. A second basic tool is the α-parametric representation of Feynman-amplitudes [Nak] as introduced by Schwinger, which has led to a representation of renormalized Feynman-amplitudes particularly suited for the study of analyticity properties [BZ], [IZ]. In section 3 we analyse integral representations for the Green functions w.r.t. those α-parameters which are obtained with the aid of the flow equations similarly as in [KKS]. With the aid of these integral representations we prove continuity of the four-point function in section 4.

[2] such that the 1PI two-point function vanishes on the mas-shell

[3] From explicit calculations one might suspect that the optimal value of η should be $1/2$.

2 The Flow Equations

For a general and pedagogical review on the renormalization theory based on flow equations we refer to [Mü], original papers are [Pol], [KKS1]. We consider the theory of the massive self-interacting scalar field, the Feynman-propagator of which is given by

$$\frac{i}{p_0^2 - \underline{p}^2 - m^2 + i\varepsilon} \; . \tag{1}$$

More precisely we will use the form

$$\frac{i}{p^2 - m^2 + i\varepsilon(\underline{p}^2 + m^2)} \; , \quad \varepsilon > 0 \; . \tag{2}$$

Using this form of the propagator [Zim] the power counting theorem for renormalized Feynman diagrams also holds in Minkowski space, in the sense that the Feynman amplitudes define Lorentz-invariant tempered distributions with a unique limit for $\varepsilon \to 0$, see also [GeSch], [Spe]. We use the notations

$$p = (p_0,\, p_1,\, p_2,\, p_3)\,, \;\; p^2 = p_0^2 - \underline{p}^2\,, \;\; \underline{p}^2 \;=\; p_1^2 + p_2^2 + p_3^2 \; . \tag{3}$$

The regularized flowing propagator for $0 \le \alpha_0 \le \alpha \le \infty$ is given by

$$C^{\alpha_0,\alpha}(p) \;=\; \int_{\alpha_0}^{\alpha} e^{i\alpha[p^2-m^2+i\varepsilon(\underline{p}^2+m^2)]}\; d\alpha \;=\; i\,\frac{e^{i\alpha_0[p^2-m^2+i\varepsilon(\underline{p}^2+m^2)]} - e^{i\alpha[p^2-m^2+i\varepsilon(\underline{p}^2+m^2)]}}{p^2 - m^2 + i\varepsilon(\underline{p}^2 + m^2)} \; . \tag{4}$$

Note that, for finite α, this propagator is an entire function of p. The full propagator is recovered by taking the regulator α_0 to 0 and the flow parameter α to ∞. The derivative of $C^{\alpha_0,\alpha}(p)$ also is an entire function of p, it takes the simple form

$$\dot{C}^{\alpha}(p) \;\equiv\; \partial_\alpha C^{\alpha,\alpha_0}(p) \;=\; e^{i\alpha[p^2-m^2+i\varepsilon(\underline{p}^2+m^2)]} \; .$$

The theory we want to study is massive φ_4^4-theory. This means that we start from the *bare action* at scale α_0

$$L_0(\varphi) = \frac{g}{4!}\int_x \varphi^4(x) \;+ \int_x \{\frac{1}{2}\,a_0\,\varphi^2(x) + \frac{1}{2}\,b_0\,(\partial_\mu\varphi)^2(x) + \frac{1}{4!}\,c_0\,\varphi^4(x)\} \; . \tag{5}$$

$$a_0\,,\; c_0 = O(\hbar)\,, \quad b_0 = O(\hbar^2) \; .$$

The parameter $\hbar$ is introduced as usual to obtain a systematic expansion in the number of loops. From the bare action and the flowing propagator we may define Wilson's *flowing* *effective action* $L^{\alpha_0,\alpha}$ by integrating out momenta in the region $\alpha_0^{-2} \le p^2 \le \alpha^{-2}$. In Minkowski space it can be defined through

$$e^{\frac{i}{\hbar}[L^{\alpha_0,\alpha}(\varphi)+I^{\alpha_0,\alpha}]} := e^{\hbar\Delta^{\alpha_0,\alpha}} \; e^{\frac{i}{\hbar}L_0(\varphi)} \tag{6}$$

and can be recognized to be the generating functional of the connected free propagator amputated Green functions (CAG) of the theory with propagator $C^{\alpha_0,\alpha}$ and bare action L_0. Here $\Delta^{\alpha_0,\alpha}$ is the functional Laplace operator $\langle\delta/\delta\varphi, C^{\alpha_0,\alpha}\,\delta/\delta\varphi\rangle$, where $\langle f,\,g\rangle$ denotes the standard (real) scalar product. For the multiplicative factor $e^{\frac{i}{\hbar}I^{\alpha_0,\alpha}}$ to be well defined, we have to restrict the theory to finite volume. All subsequent formulae are valid also in the thermodynamic limit since they do not involve any more the vacuum functional (or partition function) $I^{\alpha_0,\alpha}$.

The fundamental tool for our study of the renormalization problem is the functional *Flow Equation* (FE) [Mü]

$$\partial_\alpha\, L^{\alpha_0,\alpha} = \frac{\hbar}{2}\,\langle\frac{\delta}{\delta\varphi}, \dot{C}^\alpha\,\frac{\delta}{\delta\varphi}\rangle L^{\alpha_0,\alpha} - \frac{1}{2}\,\langle\frac{\delta L^{\alpha_0,\alpha}}{\delta\varphi}, \dot{C}^\alpha\,\frac{\delta L^{\alpha_0,\alpha}}{\delta\varphi}\rangle\ . \tag{7}$$

It is obtained by deriving both sides of (6) w.r.t. α. We then expand $L^{\alpha_0,\alpha}$ in moments w.r.t. φ

$$(2\pi)^{4(n-1)}\;\delta_{\varphi(p_1)}\ldots\delta_{\varphi(p_n)}L^{\alpha_0,\alpha}|_{\varphi\equiv 0} \;=\; \delta^{(4)}(p_1+\ldots+p_n)\,\mathcal{L}_n^{\alpha_0,\alpha}(p_1,\ldots,p_n)\ ,$$

and also in a formal powers series w.r.t. $\hbar$ to select the loop order l

$$\mathcal{L}_n^{\alpha_0,\alpha} = \sum_{l=0}^{\infty}\hbar^l\,\mathcal{L}_{n,l}^{\alpha_0,\alpha}\ .$$

From the functional FE (7) we then obtain the perturbative FEs for the n-point CAG by identifying coefficients

$$\partial_\alpha\mathcal{L}_{n,l}^{\alpha_0,\alpha} = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\;\mathcal{L}_{n+2,l-1}^{\alpha_0,\alpha}(\ldots,-p,p)\;\dot{C}^\alpha(p) - \sum_{l_i,n_i}\left[\mathcal{L}_{n_1,l_1}^{\alpha_0,\alpha}\;\dot{C}^\alpha\;\mathcal{L}_{n_2,l_2}^{\alpha_0,\alpha}\right]_{sym}\ , \tag{8}$$

$$l_1+l_2 = l\,,\quad n_1+n_2 = n+2\ .$$

Here *sym* means symmetrization - i.e. summing over all permutations of $(p_1,\ldots,p_n)$ *modulo those* which only rearrange the arguments of one factor.

The system of flow equations can be used to get control of the Green functions. To this end one first has to specify the boundary conditions. At $\alpha=\alpha_0$ they are determined through the form of the bare action $L_0 = L^{\alpha_0,\alpha_0}$ (5). The free constants appearing in (5), the so-called relevant parameters of the theory, are fixed by renormalization conditions on the IR side. For the proof of continuity properties of the Green functions, it is helpful to separate the UV or renormalizabilty problem from the large α-problem, the latter being directly related to the proof of continuity. We therefore impose renormalization conditions at some fixed positive intermediate scale $0 < \xi < \infty$:

$$\mathcal{L}^{\alpha_0,\xi}_{2,l}(p)|_{p^2=m^2} = a^{\xi}_l \ , \quad \partial_{p^2}\mathcal{L}^{\alpha_0,\xi}_{2,l}(p)|_{p^2=m^2} = b^{\xi}_l \ , \quad \mathcal{L}^{\alpha_0,\xi}_{4,l}(p^r_1,\ldots,p^r_4) = c^{\xi}_l \ , \quad l \geq 1 \tag{9}$$

for suitably chosen $p^r_1,\ldots,p^r_4$ with $(p^r_i)^2 = m^2$ and $\sum p^r_i = 0$, i.e. at physical values of the external momenta[4]. Once the boundary conditions are specified, the renormalization problem can be solved *inductively* by adopting an inductive scheme ascending in $n+2l$ and for fixed $n+2l$ ascending in l. For this scheme to work it is important to note that by definition there is no 0-loop two-point function in $L^{\alpha_0,\alpha}$.

To discuss analyticity and continuity properties it is preferable to work with one particle irreducible (1PI) Green functions, the generating functional of which is obtained from the one for connected Green functions by a Legendre transform. Starting from the generating functional of nonamputated connected Green functions $W^{\alpha_0,\alpha}$

$$W^{\alpha_0,\alpha}(J) \ = i\,L^{\alpha_0,\alpha}(C^{\alpha_0,\alpha}\,J) \ - \ \frac{1}{2}\,\langle J, C^{\alpha_0,\alpha}J\rangle \tag{10}$$

one defines

$$i\,\Gamma^{\alpha_0,\alpha}(\phi_c) \ = \ [W^{\alpha_0,\alpha}(J) \ - \ i\,\langle J,\phi_c\rangle]_{J=J(\phi_c)}\,, \quad \phi_c(p) \ = \ \frac{1}{i}(2\pi)^4\delta_{J(-p)}\,W^{\alpha_0,\alpha}(J) \tag{11}$$

with boundary terms

$$L_0(\varphi) = L^{\alpha_0,\alpha_0}(\varphi)\ , \quad \Gamma_0(\underline{\varphi}) = L_0(\varphi)|_{\underline{\varphi}\equiv\varphi}\ . \tag{12}$$

On taking in (11) a derivative w.r.t. α, and expressing the α-derivative of Γ through the one of L, using the FE for L and reexpressing L in terms of Γ, gives the flow equations (14), (15) for the perturbative 1PI Green functions $\Gamma^{\alpha_0,\alpha}_{n,l}$ [Mü].

For our purpose the most convenient procedure is to perform the Legendre transformation on the IR side only, i.e. w.r.t. the propagator $C^{\xi,\alpha}$, $\alpha \geq \xi$. By the renormalization group property we have

$$L^{\alpha_0,\alpha}(\varphi) \ = \ L^{\xi,\alpha}(\varphi)$$

[4]It is not possible to prove renormalizability on imposing renormalization conditions at a physical point without controlling the regularity of the Green functions at this point. This is due to the fact that the proof requires to perform Taylor expansions to go away from the renormalization point. When imposing conditions for *finite* ξ this poses no problem because, with our regularization, the propagator $C^{\alpha_0,\xi}(p)$, $\xi < \infty$, is analytic in p. This fact implies (as will be seen) the analyticity of the regularized Green functions at finite ξ.

for $\alpha_0 \le \xi \le \alpha$, understanding that the boundary value on the r.h.s. is

$$L^{\xi,\xi}(\varphi) \equiv L^{\alpha_0,\xi}(\varphi) \ .$$

Otherwise stated, $L^{\xi,\xi}$ now takes the role of the bare action. In analogy with (12) we then impose

$$\Gamma^{\xi,\xi}(\underline{\varphi}) = L^{\xi,\xi}(\varphi)|_{\underline{\varphi}\equiv\varphi} \ . \tag{13}$$

By performing the Legendre transformation w.r.t. the IR propagator $C^{\xi,\alpha}$ we obtain the generating functional $\Gamma^{\xi,\alpha}(\underline{\varphi})$ of the connected functions, irreducible w.r.t. $C^{\xi,\alpha}$. As indicated above we obtain the FE for these IR 1PI functions

$$\partial_\alpha \, \Gamma^{\xi,\alpha}_{n,l}(p_1,\ldots,p_{n-1}) = \int \frac{d^4p}{(2\pi)^4} \, \hat{\Gamma}^{\xi,\alpha}_{n+2,l-1}(p_1,\ldots,p_{n-1},-p,p) \, \dot{C}^{\alpha}(p) \ , \tag{14}$$

where $\Gamma^{\xi,\alpha}_{n,l}$ $(l \ge 1)$ is the regularized connected n-point function at loop order l in perturbation theory, one-particle irreducible w.r.t. the IR propagator $C^{\xi,\alpha}$. The $\hat{\Gamma}^{\xi,\alpha}_{n,l}$ are auxiliary functions, which can be expressed recursively in terms of the $\Gamma^{\xi,\alpha}_{n,l}$:

$$\hat{\Gamma}^{\xi,\alpha}_{n+2,l} = \sum_{c\ge 1}(-1)^{c+1} \sum_{l_k,n_k} \Big[\Big(\prod_{k=1}^{c-1} \Gamma^{\xi,\alpha}_{n_k+2,l_k} \, C^{\xi,\alpha}(q_k)\Big) \, \Gamma^{\xi,\alpha}_{n_c+2,l_c}\Big]_{sym} \ , \tag{15}$$

$$\sum_{k=1}^{c} l_k = l \ , \quad \sum_{k=1}^{c} n_k = n \ .$$

The momentum arguments q_k are determined by momentum conservation. They are given by the loop momentum p plus a subsum of incoming momenta p_i. All other momentum arguments have been suppressed. As in (8) one has to symmetrize w.r.t. the external momenta [5].

The CAG $\mathcal{L}^{\alpha_0,\alpha}_{n,l}$ can be expressed in terms of the $\Gamma^{\xi,\alpha}_{n,l}$ by connecting them via propagators $C^{\xi,\alpha}$ in all possible ways, as usual. One immediately realizes that an inductive scheme in the loop order l is viable for bounding the solutions of the 1PI FE.

The FE for 1PI Green functions (1PI w.r.t. the full propagator) was used in [KKS] to obtain an integral representation for these functions on successivley integrating the FE. This representation together with results from distribution theory [GeSch], [Spe] permits to obtain the following results, valid also for $\alpha_0 \to 0$:

1) The relativistic 1PI Green functions are *Lorentz-invariant tempered distributions.*

[5] By momentum conservation we write $\Gamma^{\xi,\alpha}_{n,l}(p_1,\ldots,p_{n-1})$ as a function of $n-1$ momenta though it has to be noted that they are symmetric functions of n momenta, where any one of them can be expressed in terms of the others by momentum conservation.

2) For external momenta $(p_{01}, \underline{p}_1, \ldots, p_{0n}, \underline{p}_n)$ with $|\sum_{i\in J} p_{0i}| < 2m \ \forall J \subset \{1, \ldots, n\}$ they agree[6] with the Euclidean ones for $(ip_{01}, \underline{p}_1, \ldots, ip_{0n}, \underline{p}_n)$ and are *smooth functions* in the (image of the) corresponding domain (under the Lorentz group). For $|\sum_{i\in J} p_{0i}| < 2m$ they are analytic in each of the complex time-like momentum variables $p_{01}, \ldots, p_{0n}$. These results imply in particular that $\Gamma^{\alpha_0,\infty}_{2,l}(p)$ is analytic in a neighbourhood of the mass-shell.

It is our aim to show inductively that for arbitrarily chosen b^{ξ}_l, c^{ξ}_l, and with a^{ξ}_l chosen such that $\Gamma^{\xi,\infty}_{2,l}(p)|_{p^2=m^2} = 0$, the four-point function is a continuous function of $p_1, \ldots, p_4$ for $\alpha_0 \in [0, \infty)$. The same will be shown for the two-point function. Since the renormalization conditions at $\alpha = \xi$ and at $\alpha = \infty$ are in one-to-one relation, it is then evident that the four and two-point functions are continuous for arbitrary physical renormalization conditions respecting $\Gamma^{\xi,\infty}_{2,l}(p)|_{p^2=m^2} = 0$. We note in passing that $\Gamma^{\xi,\infty}_{2,l}(p)|_{p^2=m^2} = 0$ implies $\mathcal{L}^{\alpha_0,\infty}_{2,l}(p)|_{p^2=m^2} = 0$, since a general contribution to $\mathcal{L}^{\alpha_0,\infty}_{2,l}(p)|_{p^2=m^2}$ is obtained by joining together $(n+1)$ kernels $\Gamma^{\xi,\infty}_{2,l_i}(p)|_{p^2=m^2}$ via n propagators $C^{\xi,\infty}(p)|_{p^2=m^2}$.

The two-point function depends on p^2 only[7], therefore we can use Schlömilch's interpolation formula to decompose[8] it as

$$\Gamma^{\xi,\alpha}_{2,l}(p^2) \;=\; \Gamma^{\xi,\alpha}_{2,l}(m^2) \;+\; (p^2 \;-\; m^2) \int_0^1 d\tau \; \partial_{p^2} \; \Gamma^{\xi,\alpha}_{2,l}((1-\tau)m^2 + \tau p^2) \; . \tag{16}$$

We want to impose

$$\Gamma^{\xi,\infty}_{2,l}(m^2) \;=\; 0 \tag{17}$$

which implies

$$\Gamma^{\xi,\alpha}_{2,l}(m^2) \;=\; \int_\alpha^\infty d\alpha' \; \partial_{\alpha'} \Gamma^{\xi,\alpha'}_{2,l}(m^2) \; . \tag{18}$$

To guarantee (17), we write the two-point function as a solution of the FE

$$\Gamma^{\xi,\alpha}_{2,l}(p^2) \;=\; \int_\xi^\alpha d\alpha_s \; \partial_{\alpha_s} \Gamma^{\xi,\alpha_s}_{2,l}(p^2) \;-\; \int_\xi^\infty d\alpha_s \; \partial_{\alpha_s} \Gamma^{\xi,\alpha_s}_{2,l}(m^2) \tag{19}$$

$$= \int_\xi^\alpha d\alpha_s \; \partial_{\alpha_s} \Big(\Gamma^{\xi,\alpha_s}_{2,l}(p^2) \;-\; \Gamma^{\xi,\alpha_s}_{2,l}(m^2) \Big) - \int_\alpha^\infty d\alpha_s \; \partial_{\alpha_s} \Gamma^{\xi,\alpha_s}_{2,l}(m^2) \; , \tag{20}$$

where the second term on the r.h.s. of (19) is a constant w.r.t. α, chosen such that (17) holds. It will be shown to be finite in the inductive proof so that it gives an admissible

[6] up to a factor of i^{V-1}, V being the number of vertices

[7] In slightly abusive notation we will write subsequently $\Gamma_2(p^2)$ instead of $\Gamma_2(p)$.

[8] For $\alpha < \infty$ the two-point function is an analytic function of p^2, as will be seen in the subsequent inductive proof.

finite boundary term

$$a_l^\xi \;=\; \Gamma_{2,l}^{\xi,\xi}(m^2) \;=\; -\int_\xi^\infty d\alpha_s\; \partial_{\alpha_s}\Gamma_{2,l}^{\xi,\alpha_s}(m^2)\;.$$

In the next section we will apply the decomposition (20), whenever there appears a two-point function on the r.h.s. of the FE.

3 Integral representations and large α behaviour

The following *integral representation* was proven inductively with the aid of the FE together with the subsequent properties in [KKS][9]. The statements are valid for general renormalization conditions at $\alpha = \xi$, that means in particular for renormalization conditions of the form (9) with α_0-independent (or weakly α_0-dependent) renormalization constants a_l^ξ, b_l^ξ, c_l^ξ. We have :
The perturbative CAG $\mathcal{L}_{n,l}^{\alpha_0,\xi}$ can be written as finite sum of integrals of the form

$$\mathcal{L}_{n,l}^{\alpha_0,\xi}(\vec p) \;=\; \sum_j \int_0^1 d\lambda_1\ldots\lambda_{\sigma_j}\int_{\alpha_0}^{\xi} d\xi_1\ldots d\xi_{s_j}\; G_{n,l}^{\xi,(j)}(\xi_1,\ldots,\xi_{s_j},\lambda_1,\ldots,\lambda_\sigma,\vec p)\;. \qquad (21)$$

Here $\vec p = (p_1,\ldots,p_{n-1})$; s_j is the number of internal lines in the respective contribution. We shall set $\vec\xi = (\xi_1,\ldots,\xi_{s_j}),\;\; \vec\lambda = (\lambda_1,\ldots,\lambda_{\sigma_j}),\;\; d\vec\xi = d\xi_1\ldots d\xi_{s_j}\,,\;\; d\vec\lambda = d\lambda_1\ldots d\lambda_{\sigma_j}\,.$
The functions $G_{n,l}^{\xi,(j)}(\vec\xi,\vec\lambda,\vec p)$ can be written as

$$G_{n,l}^{\xi,(j)}(\vec\xi,\vec\lambda,\vec p) \;=\; V^{\xi,(j)}(\vec\xi)\,Q^{(j)}(\vec\xi,\vec\lambda)\,P_{\varepsilon,j}(\vec p)\,e^{i[(\vec p,\,A_j(\vec\xi,\vec\lambda)\vec p)_\varepsilon - m_\varepsilon^2\sum_{k=1}^{s_j}\xi_k]}\;. \qquad (22)$$

We denote by $(\vec p,\,A_j(\vec\xi,\vec\lambda)\,\vec p)_\varepsilon$ a sum of scalar products $\sum_{k,v}(A_j)_{kv}(\vec\xi,\vec\lambda)(\,p_k\cdot p_v)_\varepsilon$, where

$$(p_k\cdot p_v)_\varepsilon = p_{0,k}\,p_{0,v} - (1-i\varepsilon)\underline{p}_k\,\underline{p}_v\;,\qquad m_\varepsilon^2 = (1-i\varepsilon)m^2\;. \qquad (23)$$

[9] In fact this integral representation was proven in [KKS] for the one-particle irrreducible Green functions $\Gamma_{n,l}^{\alpha_0,\xi}(\vec p)$. It can be proven in the same way for the connected Green functions starting from the FE for those. It can also be deduced from the integral representation for the $\Gamma_{n,l}^{\alpha_0,\xi}(\vec p)$, noting that the $\mathcal{L}_{n,l}^{\alpha_0,\xi}(\vec p)$ are sums of products of the $\Gamma's$ joined by propagators $C_{n,l}^{\alpha_0,\xi}(\vec p)$ for which we use (4). The integral representation (21) then also holds for sums of products of terms of the type (21). In [KKS] the integral representation was written for the case of vanishing renormalization conditions. It is easily seen to be valid also for nonvanishing ones. One only has to be aware of the fact that in this case the number of internal lines is no more fixed in terms of the number of loops and of external lines since the renormalization constants may be of loop order ≥ 1 themselves, a fact which we have already taken into account in (21), (22).

The matrices A_j are positive-semidefinite symmetric $(n-1)\times(n-1)$-matrices which are rational functions, homogeneous of degree 1 in $\vec{\xi}$ and continuous w.r.t. $\vec{\xi}$, $\vec{\lambda}$ (within the support of the integral).
The $V^{\xi,(j)}$'s are products of θ-functions of arguments $(\xi_i-\xi_k)$ which constrain the $\vec{\xi}$-integration domain. They stem from successively integrating the FE.
The $P_{\varepsilon,j}$ are products of monomials in the scalar products $(p_k\cdot p_v)_\varepsilon$.
The $Q^{(j)}$ are rational functions in $\vec{\xi}$, $\vec{\lambda}$, homogeneous of degree $d_j\in\mathbb{Z}$ in $\vec{\xi}$, and absolutely integrable for $\xi_i\to 0$.
The proof of these statements is in [KKS]. There it is also shown that $d_j>-s_j$. This lower bound on d_j is at the origin of the absolute integrability of $G^{\xi,(j)}_{n,l}$ for $\alpha_0\to 0$. The λ-integrals stem from successive use of interpolation formulas, similarly as the τ-integral in (16). We do not comment further on the proof here, since the subsequent statements on the large α-behaviour of Green functions are proven with the aid of the same techniques. As a consequence of these facts one realizes that, for $0<\alpha_0<\xi<\infty$, the functions $\mathcal{L}^{\alpha_0,\xi}_{n,l}(\vec{p})$ are analytic functions of $\vec{p}$.

We now regard $\alpha\geq\xi$ with the aim to analyse the behaviour for $\alpha\to\infty$. We call infrared lines those with propagators $C^{\xi,\alpha}$, and ultraviolet lines those with propagators $C^{\alpha_0,\xi}$. We want to prove the following

Proposition :

We have an integral representation for $\Gamma^{\xi,\alpha}_{n,l}(\vec{p})$ in terms of a finite sum of integrals, of the following type :

$$\Gamma^{\xi,\alpha}_{n,l}(\vec{p}) = \sum_j \int_\xi^\infty d\vec{\alpha}\int d\vec{\tau}\int d(\vec{\xi},\vec{\lambda})\; F_j(\vec{\xi},\vec{\lambda})\;\Theta^{\alpha,(j)}(\vec{\alpha})\;Q^{(j)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\;P_{j,\varepsilon}(\vec{p})\,\cdot \tag{24}$$

$$\cdot\; e^{i[(\vec{p},\,A_j(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\vec{p})_\varepsilon + m_\varepsilon^2\,A_j^{(m)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau}) - m_\varepsilon^2\,\sum_{ir}^{(j)}\alpha_k]}\;\prod_{f=1}^{c_j}\Gamma^{\xi,\alpha_{i_f}}_{2,l_f}(m^2)\;,\quad \sum l_f<l\;.$$

i) The factors $F_j(\vec{\xi},\vec{\lambda})$ are of the form

$$F_j(\vec{\xi},\vec{\lambda}) = V^{\xi,(j)}(\vec{\xi})\;Q^{(j)}(\vec{\xi},\vec{\lambda})\;e^{-im_\varepsilon^2\sum_{uv}^{(j)}\xi_i}\;, \tag{25}$$

and the properties of $V^{\xi,(j)}(\vec{\xi})$, $Q^{(j)}(\vec{\xi},\vec{\lambda})$, as well as those of the integration variables $\vec{\lambda}$, $\vec{\xi}$ are listed after (22), (23). The sum $\sum_{uv}^{(j)}\xi_i$ is over the internal UV lines, excluding those inside the $\Gamma^{\xi,\alpha_{i_f}}_{2,l_f}(m^2)$.
ii) The matrices $A_j(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})$ are positive-semidefinite symmetric $(n–1)\times(n–1)$-matrices. Their elements are rational functions, homogeneous of degree 1, in the variables $(\vec{\xi},\vec{\alpha})$:

$$A_j(\rho\,\vec{\xi},\vec{\lambda},\rho\,\vec{\alpha},\vec{\tau}) = \rho\,A_j(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\;. \tag{26}$$

For $\xi_i \in [0,\xi]$ and $\alpha_i \geq \xi$ they are continuous functions of $\vec{\xi}$ and smooth functions of $\vec{\alpha}$, $\vec{\lambda}$, $\vec{\tau}$. As functions of $\vec{\alpha}$ they are also rational functions. They obey the bounds

$$|A_j(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})| \;\leq\; O(1)\,\sup_i \alpha_i \tag{27}$$

uniformly in all other parameters (within the support of the integrals).
In the following we suppress the variables $(\vec{\lambda},\vec{\tau})$, since they are pure spectators. We also suppress the subscript j. The matrix elements A_{kv} of A admit the decomposition (suppressing also subscripts k,v)

$$A(\vec{\xi},\vec{\alpha}) = A_0(\vec{\xi},\vec{\alpha}) + A_1(\vec{\xi},\vec{\alpha}) + A_2(\vec{\xi},\vec{\alpha})\ . \tag{28}$$

Here the functions A_0, A_1, A_2 are are rational functions, homogeneous of degree 1, and they have the same continuity and smoothness properties as A above. Furthermore they have the following properties

$$A_0(\vec{\xi},\rho\vec{\alpha}) = \rho\, A_0(\vec{\xi},\vec{\alpha})\,,\ A_1(\vec{\xi},\rho\vec{\alpha}) = A_1(\vec{\xi},\vec{\alpha})\,,\ |\partial_\rho^n A_2(\vec{\xi},\rho\vec{\alpha})| \leq O(\rho^{-1-n})\ , \tag{29}$$

where $\rho > 0$ and $n \in \mathbb{N}$. The matrix (A_0) is also positive definite.
Finally $A_j^{(m)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})$ may be viewed as a 1×1-matrix with the same properties as the $A_j(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})$.
iii) The $Q^{(j)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})$ are rational functions[10] of $\vec{\xi}$, $\vec{\alpha}$, which are uniformly bounded for $\xi_i \in [0,\xi]$. They admit a similar decomposition as (29) (with the same notation)

$$Q(\vec{\xi},\vec{\alpha}) \;=\; Q_0(\vec{\xi},\vec{\alpha}) + Q_1(\vec{\xi},\vec{\alpha}) + Q_2(\vec{\xi},\vec{\alpha})\ ,$$

$$Q_0(\vec{\xi},\rho\vec{\alpha}) \;=\; \rho^k\, Q_0(\vec{\xi},\vec{\alpha})\,,\ \ Q_1(\vec{\xi},\rho\vec{\alpha}) \;=\; \rho^{k-1} Q_1(\vec{\xi},\vec{\alpha})\,,\ \ |\partial_\rho^n Q_2(\vec{\xi},\rho\vec{\alpha})| \;\leq\; O(\rho^{k-2-n}) \tag{30}$$

for suitable $k \in -\mathbb{N}$, and the Q_i have the same properties as those listed for Q.
For $\alpha = \sup_i \alpha_i \geq \xi$, the functions $Q(\vec{\xi},\alpha\vec{\beta})$, $\alpha\beta_i = \alpha_i$, are uniformly continuous in $\vec{\beta}$.
iv) The $P_{\varepsilon,j}$ are products of monomials in the scalar products $(p_k\cdot p_v)_\varepsilon$.
v) The τ-parameters are integrated each over the interval $[0,1]$. The sum $\sum_{ir}^{(j)} \alpha_k$ is over the internal IR lines, excluding those inside the $\Gamma_{2,l_f}^{\xi,\alpha_{i_f}}(m^2)$. Assuming their number to be s, we write $\vec{\alpha} = (\alpha_1,\ldots,\alpha_s)$. For $n \geq 4$ and for two-point functions of arbitrary momentum p^2, the $\Theta^{\alpha,(j)}(\vec{\alpha})$ are products of θ-functions of arguments $(\alpha_i - \alpha_k)$, and of one θ-function $\theta(\alpha-\alpha_s)$. In the expression for $\Gamma_{2,l}^{\xi;\alpha}(m^2)$, there appears one θ-function $\theta(\alpha_s-\alpha)$ instead of $\theta(\alpha-\alpha_s)$.

[10] they may also depend on m^2 which we view as constant, however

vi) For $n \geq 4$ we have the following bounds, uniformly in $\vec{\xi}, \vec{\alpha}, \vec{\tau}$

$$\int_{\xi}^{\infty} d\vec{\alpha}'' \; |\,\Theta^{\alpha_s,(j)}(\vec{\alpha}) \; Q^{(j)}(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) \prod_{f=1}^{c_j} \Gamma_{2,l_f}^{\xi,\alpha_{i_f}}(m^2)\,| \;\leq\; \alpha^{\frac{n-4}{2}+s''-s}\; \mathcal{P}_l \log \alpha \;. \tag{31}$$

Here $\mathcal{P}_l \log \alpha$ denotes a polynomial [11] *of degree $\leq l$ in α, and $\vec{\alpha}''$ is a subset of the α-parameters $(\alpha_1, \ldots, \alpha_s)$ which contains s'' elements.*
vii) The two-point functions satisfy the bound

$$|\,\Gamma_{2,l}^{\xi;\alpha}(p^2)| \;\leq\; O(1) \;. \tag{32}$$

The two-point functions on mass-shell satisfy

$$|\,\Gamma_{2,l}^{\xi;\alpha}(m^2)| \;\leq\; \alpha^{-1}\; \mathcal{P}_l \log \alpha \;. \tag{33}$$

Proof :
The proof is based on the standard inductive scheme which goes up in l. The statements of the Proposition then serve at the same time as an induction hypothesis, and the terms appearing on the r.h.s. of the FE (14), (15) satisfy (24) - (33) by induction. For readibility we again suppress the sub- or superscripts j and ε.
Starting the induction at $l = 0$ is trivial since we have $\Gamma_{n,0}^{\xi,\alpha}(\vec{p}) = \delta_{n,4}\, g$. For the boundary terms at $\alpha = \xi$ (13) the set of infrared lines with parameters $\{\vec{\alpha}\}$ is empty, as is the set $\{\vec{\tau}\}$. For them the proposition holds true due to (21), (22) and the subsequent statements.

i) The factors F_j, see (25), collect together all factorized ultraviolet contributions. Since these are not touched upon by the Gaussian integration in the FE, and since sums of products of terms of this kind still have the properties listed after (21) - (23), the confirmation of i) is then obvious.

Before verifying the other items we outline some apsects of the procedure to be followed. For $n \geq 4$ we will write the solutions of the FE as

$$\Gamma_{n,l}^{\xi,\alpha}(p_1, \ldots, p_{n-1}) \;=\; \Gamma_{n,l}^{\xi,\xi}(p_1, \ldots, p_{n-1}) \;+\; \int_{\xi}^{\alpha} d\alpha_s \;\partial_{\alpha_s} \Gamma_{n,l}^{\xi,\alpha_s}(p_1, \ldots, p_{n-1}) \;, \tag{34}$$

where the second term is obtained inductively from the r.h.s. of the FE (14), and the first term is obtained from (13).

[11] The coefficients of the polynomial may depend on the parameters $(\xi,\; m,\; n, l\,)$.

For $n = 2$, once the integral representation has been proven, the boundary condition (17) is implemented as follows. Starting from (24) we have terms of the form

$$\int_\xi^\alpha d\alpha_s \int_\xi^\infty d\vec{\alpha}' \int d(\vec{\tau}, \vec{\xi}, \vec{\lambda}) \; F(\vec{\xi}, \vec{\lambda}) \; \Theta^{\alpha_s}(\vec{\alpha}') \; Q(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) \quad \cdot$$

$$\cdot \; P(p^2) \; e^{i[p^2 \, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau}) + m^2 \, A^{(m)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau}) - m^2 \, \sum_{ir'} \alpha_k]} \prod_{f=1}^{c^{(i)}} \Gamma^{\xi, \alpha_f^{(i)}}_{2, l_f^{(i)}}(m^2) \; , \quad (\vec{\alpha}', \alpha_s) = \vec{\alpha} \; .$$

Note that we replaced $\Theta^\alpha(\vec{\alpha}) \to \Theta^{\alpha_s}(\vec{\alpha}')$ since the last integration over α_s is the new one of the induction step. Inserting this representation into (20) we get

$$\begin{aligned} &\int_\xi^\alpha d\alpha_s \int d\vec{w} \; \mathcal{F}(\vec{w}) \; Q(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) \prod_{f=1}^{c^{(i)}} \Gamma^{\xi, \alpha_f^{(i)}}_{2, l_f^{(i)}}(m^2) \; e^{i\, m^2 \, (A^{(m)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau}) - \sum_{ir} \alpha_k)} \; \cdot \\ &\qquad \cdot \; \Big(P(p^2) \; e^{ip^2 \, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})} - P(m^2) \; e^{im^2 \, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})} \Big) \\ &- \int_\alpha^\infty d\alpha_s \int d\vec{w} \;\; \mathcal{F}(\vec{w}) \; P(m^2) \prod_{f=1}^{c^{(i)}} \Gamma^{\xi, \alpha_f^{(i)}}_{2, l_f^{(i)}}(m^2) \; e^{i\, m^2 \, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})} \end{aligned} \tag{35}$$

with

$$\vec{w} = (\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) \; , \quad \mathcal{F}(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) = F(\vec{\xi}, \vec{\lambda}) \; \Theta^{\alpha_s}(\vec{\alpha}) \; Q(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) \; .$$

The difference appearing in the first term can be reexpressed (cf. (17)) as

$$(p^2 - m^2) \int_0^1 d\tau \; e^{i((1-\tau)m^2 + \tau p^2) \, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})} \; \{[iA(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) \, + \, \partial_{p^2}] \, P\}((1-\tau)m^2 + \tau p^2) \; . \tag{36}$$

Contributions from the r.h.s. of the FE containing the first term in (35) are taken together with the propagator

$$C^{\xi,\alpha}(p) \; = \; i \, \frac{e^{i\xi(p^2-m^2)} - e^{i\alpha(p^2-m^2)}}{p^2 - m^2}$$

to give the two contributions

$$i \, \Big(e^{i\xi(p^2-m^2)} - e^{i\alpha(p^2-m^2)} \Big) \int_0^1 d\tau \; e^{i((1-\tau)m^2 + \tau p^2) \, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})} \; \ldots \tag{37}$$

The terms $\{[iA(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau}) \, + \, \partial_{p^2}] \, P\}((1-\tau)m^2 + \tau p^2)$ have to be absorbed in the new $Q(\vec{\xi}, \vec{\lambda}, \vec{\alpha}, \vec{\tau})$ resp. in the new $P(p)$. The term $e^{i(1-\tau)m^2 \, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})}$ contributes to the terms $e^{im^2 \, A^{(m)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})}$ in the integral representation. This means that the $A^{(m)}$-terms are A-terms of two-point functions, multiplied by factors of $(1-\tau)$. They therefore have the properties claimed for the A-terms.

The integral representation (24) is verified inductively starting from (14), (15). We thus use the integral representations for the terms $\Gamma^{\xi,\alpha_s}_{n_k+2,l_k}$ on the r.h.s. of (15), applying the special treatment of two-point functions indicated previously. For all propagators $C^{\xi,\alpha_s}(q_k)$, which do not multiply a term of the type of the second term on the r.h.s. of (16), reexpressed as in (36), we use the integral representation from (4). We then have to perform the Gaussian integral over p in (14) and afterwards the integral over α_s from ξ to α to pass from $\partial_{\alpha_s}\Gamma^{\xi,\alpha_s}_{n,l}$ to $\Gamma^{\xi,\alpha}_{n,l}$. Since all contributions to the exponent of the Gaussian integral satisfy ii) by the induction assumption, and since sums over matrices with the properties from ii) again satisfy ii), this integral has an exponent of the form $i\alpha_s p^2 + i\sum_{k,v=1}^{n+1}\tilde{A}_{kv}\, p_k\, p_v$, where the matrix $(\tilde{A}_{kv})$ satisfies ii). Here we denote $p_{n+1} = -p_n = p$, and α_s is the α-parameter of the derived line $\dot{C}^{\alpha_s}$ in (14), it is the largest one in the set of α-parameters; $\tilde{A}$ can be realized to be independent of α_s inductively on inspection of the FE [12]. The exponent previously given can be rearranged in a form suitable for integration over p

$$
\begin{aligned}
i\alpha_s p^2 + i\sum_{k,v=1}^{n+1}\tilde{A}_{kv}\, p_k\, p_v \;=\; i\sum_{k,v=1}^{n-1}\left[\tilde{A}_{kv} - \frac{(\tilde{A}_{kn} - \tilde{A}_{kn+1})(\tilde{A}_{vn} - \tilde{A}_{vn+1})}{\tilde{A}_{nn} + \tilde{A}_{n+1n+1} - 2\tilde{A}_{nn+1} + \alpha_s}\right] p_k\, p_v \,+ \qquad (38)\\
+\, i(\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} + \alpha_s)\,\left(p + \sum_{k=1}^{n-1}\frac{\tilde{A}_{kn} - \tilde{A}_{kn+1}}{\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} + \alpha_s}\, p_k\right)^2 .
\end{aligned}
$$

Since $\tilde{A}$ is positive semi-definite we have

$$
\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} \geq 0\ . \tag{39}
$$

On performing the Gaussian integral, in the absence of polynomials $P(\vec{p})$, we obtain a factor of

$$
(\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} + \alpha_s)^{-2} \leq\ \alpha_s^{-2}\ , \tag{40}
$$

and a new quadratic form with matrix elements

$$
A_{kv} \;=\; \tilde{A}_{kv} - \frac{(\tilde{A}_{kn} - \tilde{A}_{kn+1})(\tilde{A}_{vn} - \tilde{A}_{vn+1})}{\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} + \alpha_s}\ ,\quad 1 \leq k, v \leq n-1\ . \tag{41}
$$

We are now ready to verify the remaining items of the induction step :

ii) The positive semi-definiteness, homogeneity, continuity and smoothness properties of the matrix A_{kv} are verified from those of $\tilde{A}_{kv}$, for which they hold by induction, with the

[12] Note that α-parameters larger than α_s only appear inside the expressions of the terms $\Gamma^{\xi,\alpha_s}_{2,l_f}(m^2)$, due to the integrals $\int_\alpha^\infty$ in (20). These evidently do not appear in the matrix $\tilde{A}$.

aid of the explicit formula (41), using (39). In particular the positive (semi-)definiteness follows by noting that the second term on the r.h.s. of (38) can be made vanish by suitable choice of p, so that the first term is nonnegative since the l.h.s. is (on dividing by i). Assuming by induction the decomposition (28) to hold for the matrix elements of $\tilde{A}_{kv}$, the contributions in the decomposition for the matrix elements of A_{kv} are defined as follows

$$A_{0,kv}(\vec{\xi},\vec{\alpha}) = \tilde{A}_{0,kv}(\vec{\xi},\vec{\alpha}) - \frac{(\tilde{A}_{0,kn} - \tilde{A}_{0,kn+1})(\tilde{A}_{0,vn} - \tilde{A}_{0,vn+1})}{\tilde{A}_{0,n+1n+1} + \tilde{A}_{0,nn} - 2\tilde{A}_{0,nn+1} + \alpha_s} , \tag{42}$$

$$A_{1,kv}(\vec{\xi},\vec{\alpha}) \;=\; \tilde{A}_{1,kv}(\vec{\xi},\vec{\alpha}) \;-\; \frac{d_0\, e_1 + d_1\, e_0}{f_0 + \alpha_s} + \frac{d_0\, e_0\, f_1}{(f_0 + \alpha_s)^2} , \tag{43}$$

$$\begin{aligned} A_{2,kv}(\vec{\xi},\vec{\alpha}) \;&=\; \tilde{A}_{2,kv}(\vec{\xi},\vec{\alpha}) \;-\; \frac{d_2\, e + e_2\, d + d_1\, e_1}{f + \alpha_s} \\ &+ \frac{(d_0\, e_1 + d_1\, e_0)(f_1 + f_2)}{(f_0 + \alpha_s)^2} \;-\; \frac{d_0\, e_0}{f_0 + \alpha_s} \left\{ \frac{f_1^2}{(f_0 + \alpha_s)^2} - \frac{f_2}{f_0 + \alpha_s} \;+\; \frac{f_1\, f_2}{(f_0 + \alpha_s)^2} \right\} \end{aligned} \tag{44}$$

with the shorthands

$$d = (\tilde{A}_{kn} - \tilde{A}_{kn+1})(\vec{\xi},\vec{\alpha}) , \quad e = (\tilde{A}_{vn} - \tilde{A}_{vn+1})(\vec{\xi},\vec{\alpha}) , \quad f = (\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1})(\vec{\xi},\vec{\alpha}) ,$$

$$d_i = (\tilde{A}_{i,kn} - \tilde{A}_{i,kn+1})(\vec{\xi},\vec{\alpha}) , \quad e_i = (\tilde{A}_{i,vn} - \tilde{A}_{i,vn+1})(\vec{\xi},\vec{\alpha}) ,$$
$$f_i = (\tilde{A}_{i,n+1n+1} + \tilde{A}_{i,nn} - 2\tilde{A}_{i,nn+1})(\vec{\xi},\vec{\alpha}) , \quad i \in \{0, 1, 2\} .$$

On inspection of these expressions one realizes that the properties (29) are verified for the matrix elements of A if they are true for those of $\tilde{A}$. It also follows that the $A_{i,kv}$ are rational functions, homogeneous of degree one. The positivity of A_0 follows in the same way as that of A. Note finally that all denominators are bounded below by α_s, as follows from the positivity of $\tilde{A}$ resp. $\tilde{A}_0$.
Noting that $\tilde{A}$ is independent of α_s, the bound (27) follows from the induction hypothesis, using (41) and the fact that $\alpha_s = \sup_i \alpha_i$.
iii) The Gaussian integral is performed with the aid of a change of variable $p \;\rightarrow\; \tilde{p} = p + \sum_{k=1}^{n-1} \frac{\tilde{A}_{kn} - \tilde{A}_{kn+1}}{\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} + \alpha_s}\, p_k$, see (38). Consequently the monomials from $P(\vec{p})$ [13] which contain the variables $\pm p$ will lead after Gaussian integration to terms

$$\frac{\tilde{A}_{kn} - \tilde{A}_{kn+1}}{\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} + \alpha_s}\; \frac{\tilde{A}_{vn} - \tilde{A}_{vn+1}}{\tilde{A}_{n+1n+1} + \tilde{A}_{nn} - 2\tilde{A}_{nn+1} + \alpha_s}\; p_k \cdot p_v . \tag{45}$$

Terms $\sim (p^2)^n$ will give rise to terms with exponents $-(2+n)$ instead of -2 in (40). All these contributions are rational functions respecting the properties claimed for $Q(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})$

[13] remember that the monomials stem initially from the ultraviolet boundary terms in (21)

and allowed for by the induction hypothesis. The decomposition into $Q_0,\ Q_1,\ Q_2$ is performed in analogy with (42). For the terms from (45) one proceeds as in (42)-(44), for those from (40) we decompose according to

$$\frac{1}{f+\alpha_s} = \frac{1}{f_0+\alpha_s} - \frac{f_1}{(f_0+\alpha_s)^2} + \left\{\frac{f_1(f_1+f_2)}{(f_0+\alpha_s)^2} - \frac{f_2}{f_0+\alpha_s}\right\}\frac{1}{f+\alpha_s} \tag{46}$$

wherefrom the dominant and subdominant scaling contributions to Q can be read easily on taking (46) to the power 2 or higher. For $\alpha_s \geq \xi$ the uniform continuity of $Q(\alpha_s\vec{\beta})$ is evident by induction since all denominators appearing in the new factors contributing to $Q(\alpha_s\vec{\beta})$ are bounded below by α_s.

iv) After the linear change of variables and Gaussian integration the monomials in external momenta obviously still have the required properties.

v) The τ-parameters stem from the interpolation formula (36) applied to the off-shell part of the two-point function. So there appear at most $(l-1)$ τ-parameters at loop-order l. Each IR-line contributes a factor $e^{-im^2\alpha_i}$ via (4). When performing the α-integral at loop-order l we integrate

$$\int_\xi^\alpha d\alpha_s \ldots = \int_\xi^\infty d\alpha_s\ \theta(\alpha-\alpha_s)$$

apart from the contributions stemming from terms as the second one in (20) where we integrate

$$\int_\alpha^\infty d\alpha_s \ldots = \int_\xi^\infty d\alpha_s\ \theta(\alpha_s-\alpha)\ .$$

This explains the successive generation of θ-functions.

vi) By induction we have for the terms $\Gamma^{\xi,\alpha_s}_{n_k+2,l_k}$ with $n_k+2\geq 4$, appearing on the r.h.s. of the FE

$$\int_\xi^{\alpha_s} d\vec{\alpha}_k''\ |\,\Theta^{\alpha_s}(\vec{\alpha}_k)\ Q_{n_k+2,l_k}(\vec{\xi}_k,\vec{\lambda}_k,\vec{\alpha}_k,\vec{\tau}_k)\prod_{f=1}^{c_{j_k}}\Gamma_{2,l_f}^{\xi,\alpha_{i_f}^{(k)}}(m^2)\,|\ \leq\ \alpha_s^{\frac{n_k+2-4}{2}+s_k''-s_k}\ \mathcal{P}_{l_k}\log\alpha_s\ . \tag{47}$$

In the presence of two-point functions ($n_k=0$) we note that the contributions from the last term in (35) - i.e. the on-shell two-point functions - are integrated from $\alpha_{i_f}^{(k)}$ to ∞ and can be bounded inductively by $(\alpha_{i_f}^{(k)})^{-1}\mathcal{P}_{l_k}\log\alpha_{i_f}^{(k)}$, the integrand being bounded inductively by $(\alpha_{i_f}^{(k)})^{-2}\mathcal{P}_{l_k}\log\alpha_{i_f}^{(k)}$. On the other hand terms of the form of the first one in (20), (35) are bounded uniformly in α_s, using the inductive bounds on the integrands in (20), which are of the form $\alpha_s^{-2}\ \mathcal{P}_l\log\alpha_s$. If we have a number c' of terms of this form in a contribution from the r.h.s. of the FE, we can associate with each of them an underived propagator with the same momentum q_k, cf. (15), and the factor of $\frac{1}{q_k^2-m^2}$ of this accompanying propagator compensates the corresponding factor in (36), see (37). In total we have $c-1$ underived propagators in with $c > c'$ [14]. For the remaining $c-c'-1$ ones we use the integral representation (4), which results in a contribution of $c-c'-1$ - equal to the number of α_i-integrations from (4) - to the exponent of α in the bound to be established, remembering $\alpha \geq \alpha_s \geq \alpha_i$. Adding all contributions to this exponent resulting by induction from the bounds on the various terms from (14), (15) - we get, supposing that all α-parameters are integrated over

$$\sum_{k=1}^{c-c'} \frac{n_k+2-4}{2} \;+\; (c-c'-1)-2+1 \;=\sum_{k=1}^{c} \frac{n_k}{2}-2=\frac{n-4}{2}\;. \tag{48}$$

Here the contribution -2 stems from the bound (40) on the factor produced by Gaussian integration, and the contribution $+1$ corresponds to the final α_s-integration in (34). For $n=4$ the α_s-integral is logarithmically divergent for $\alpha_s \to \infty$, which leads to the appearance of a logarithm. Similarly α_s-integrals over the terms from (36) are bounded logarithmically. By induction we then arrive at a polynomial in logarithms the degree of which is inductively bounded by the maximal number of divergent subintegrations, i.e. by the number of loops. If some of the α-parameters are not integrated over, the above counting rules result in the exponent from (31).

vii) The bounds on the two-point functions are established in the same way as the previous ones. To get the improved bound for the two-point functions on the mass-shell, we note that due to the boundary conditions they are given as integrals

$$\Gamma^{\xi,\alpha}_{2,l}(m^2) \;=\; \int_\alpha^\infty d\alpha' \;\; \partial_{\alpha'}\, \Gamma^{\xi,\alpha'}_{2,l}(m^2)\;. \tag{49}$$

The integrand is given by the r.h.s of the FE, and from (31) we find (by induction on lower loop orders)

$$|\,\partial_\alpha \Gamma^{\xi,\alpha}_{2,l}(m^2)| \;\leq\; \alpha^{-2}\; \mathcal{P}_l \log\alpha\;. \tag{50}$$

$\blacksquare$

4 Continuity

To verify the continuity of the four-point function $\Gamma^{\xi,\alpha}_{4,l}(p_1,\ldots,p_4)$ for $\alpha \to \infty$, we consider the integrals from (24). We will leave out the polynomials[15] in external momenta, which

[14] Note that there is at least one $\Gamma^{\xi,\alpha_s}_{n_k+2,l_k}$ with $n_k > 0$ in (15) so that always $c > c'$.

[15] multiplying a continuous function by a polynomial results again in a continuous function

will not be touched upon, and we suppress again indices j and ε. For shortness we will also suppress the factors $e^{im^2 A^{(m)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})}$ so that one should read

$$(\vec{p}, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\vec{p}) \;\to\; (\vec{p}, A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\vec{p}) \,+\, m^2 A^{(m)}(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau}) \;. \tag{51}$$

We write as before $\vec{\alpha} = (\vec{\alpha}\,', \alpha_s)$. The integral contributions to $\Gamma^{\xi,\alpha}_{4,l}(p_1,\ldots,p_4)$ can then be written as

$$\int_\xi^\alpha d\alpha_s \int d\vec{\alpha}\,' \int d\vec{\tau} \int d(\vec{\xi},\vec{\lambda})\; e^{i[(\vec{p},A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\vec{p})-m^2\sum_{ir}\alpha_k]} \;. \tag{52}$$

$$\cdot\; F(\vec{\xi},\vec{\lambda})\;\Theta^{\alpha_s}(\vec{\alpha})\;Q(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\;\prod_{f=1}^{c}\Gamma^{\xi,\alpha_{i_f}}_{2,l_f}(m^2)\;.$$

Using absolute integrability and the decomposition (28), we may rewrite (52) in the form

$$\int_\xi^\alpha d\alpha_s\;\alpha_s^{s-1}\int_{\xi/\alpha_s}^1 d\vec{\beta}\int d\vec{\tau}\;\int d(\vec{\xi},\vec{\lambda})\;F_j(\vec{\xi}\,\vec{\lambda})\;e^{i(\vec{p},A_1(\vec{\xi},\vec{\lambda},\vec{\beta},\vec{\tau})\vec{p})}\;e^{i\alpha_s[(\vec{p},A_0(\vec{\xi},\vec{\lambda},\vec{\beta},\vec{\tau})\vec{p})-m^2\sum_{ir}\beta_k]}\;.$$

$$\cdot\;\left(1+\sum_{r=1}^{\infty}\frac{[i\,(\vec{p},A_2(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\alpha_s,\vec{\tau})\vec{p})]^r}{r\;!}\right)\;\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;\;Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})\;\prod_{f=1}^{c}\Gamma^{\xi,\alpha_s\beta_{i_f}}_{2,l_f}(m^2)\;. \tag{53}$$

Here we denote for $i\le s-1$, $\beta_i=\alpha_i/\alpha_s$ and $d\vec{\alpha}\,'=d(\alpha_s\,\vec{\beta})$. Subsequently we will write $A_0(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})$ intead of $A_0(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},1,\vec{\tau})$ understanding that $\beta_s=1$, and similarly for Q. From the Proposition we have the bound for the four-point function integrand

$$\int_\xi^\alpha d\alpha_s\;\alpha_s^{s-1}\;\int_{\xi/\alpha_s}^1 d\vec{\beta}\;|\,\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})\;\prod_{f=1}^{c}\Gamma^{\xi,\alpha_s\beta_{i_f}}_{2,l_f}(m^2)\,|\;\le\;\;\mathcal{P}_l\log\alpha_s\;.$$

In the following considerations we will leave out the factor of $1+\sum_{r=1}^{\infty}\frac{[i\,(\vec{p},A_2(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\alpha_s,\vec{\tau})\vec{p})]^r}{r\;!}$ for shortness and readibility. It can be easily realized that due to the large α_s-fall-off of $A_2(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\alpha_s,\vec{\tau})$ we obtain the same large α_s-bounds as those subsequently given on reinserting this factor. The same remark holds for the α_s-independent term $e^{i(\vec{p},A_1(\vec{\xi},\vec{\lambda},\vec{\beta},\vec{\tau})\vec{p})}$. We will also suppress the variables $(\vec{\xi},\vec{\lambda},\vec{\tau})$, which are kept fixed. We thus consider the integral

$$\int_\xi^\alpha d\alpha_s\;\int_{\xi/\alpha_s}^1 d\vec{\beta}\;e^{i\alpha_s[(\vec{p},A_0(\vec{\beta})\vec{p})-m^2\sum_{ir}\beta_k]}\;\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;\alpha_s^{s-1}\;Q(\alpha_s\vec{\beta})\;\prod_{f=1}^{c}\Gamma^{\xi,\alpha_s\beta_{i_f}}_{2,l_f}(m^2)\;.$$

For α_s in the interval

$$I_\nu \;=\; [M^\nu,\,M^{\nu+1}]\;,\quad M>1$$

we split up the integration domain $\mathcal{I}$ of $\vec{\beta}$ such that

$$\mathcal{D}_1^{(\nu)}(\alpha_s) = \{\vec{\beta} \in \mathcal{I} \mid \; |(\vec{p}, A_0(\vec{\beta})\,\vec{p}) - m^2 \sum_{ir} \beta_k| \geq M^{-\frac{2\nu}{3}} \;\},$$

$$\mathcal{D}_2^{(\nu)}(\alpha_s) = \{\vec{\beta} \in \mathcal{I} \mid \; |(\vec{p}, A_0(\vec{\beta})\,\vec{p}) - m^2 \sum_{ir} \beta_k| < M^{-\frac{2\nu}{3}} \;\}.$$ [16,17]

We then use partial integration to obtain[18]

$$\int_{I_\nu} d\alpha_s \int_{\mathcal{D}_1^{(\nu)}} d\vec{\beta}\; e^{i\alpha_s[(\vec{p},A_0(\vec{\beta})\,\vec{p})-m^2\sum_{ir}\beta_k]}\;\alpha_s^{s-1}\;\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})\prod_{f=1}^{c}\Gamma_{2,l_f}^{\xi,\alpha_s\beta_{i_f}}(m^2)$$

$$= \int_{\mathcal{D}_1^{(\nu)}} d\vec{\beta}\left[\frac{e^{i\alpha_s[(\vec{p},A_0(\vec{\beta})\,\vec{p})-m^2\sum_{ir}\beta_k]}}{i[(\vec{p},A_0(\vec{\beta})\,\vec{p})-m^2\sum_{ir}\beta_k]}\;\alpha_s^{s-1}\;\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})\prod_{f=1}^{c}\Gamma_{2,l_f}^{\xi,\alpha_s\beta_{i_f}}(m^2)\right]_{M^\nu}^{M^{\nu+1}}$$

$$-\int_{I_\nu} d\alpha_s \int_{\mathcal{D}_1^{(\nu)}} d\vec{\beta}\;\frac{e^{i\alpha_s[(\vec{p},A_0(\vec{\beta})\vec{p})-m^2\sum_{ir}\beta_k]}}{i[(\vec{p},A_0(\vec{\beta})\vec{p})-m^2\sum_{ir}\beta_k]}\cdot$$

$$\cdot\left(\partial_{\alpha_s} - \frac{\xi}{\alpha_s^2}\sum_{i=1}^{s-1}\delta(\beta_i-\frac{\xi}{\alpha_s})\right)\;\alpha_s^{s-1}\;\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;\;Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})\;\prod_{f=1}^{c}\Gamma_{2,l_f}^{\xi,\alpha_s\beta_{i_f}}(m^2)\;. \quad (54)$$

By the Proposition each of the three terms on the r.h.s. of (54) is suppressed by one power of α_s or M^ν as compared to the original bound on the four-point function, without counting the denominator. For the first term (31) shows that suppression of the α_s-integration leads to this gain. Furthermore application of the derivative ∂_{α_s} results in such a gain when applying it to the θ-function $\Theta^{\alpha_s}(\alpha_s\vec{\beta})$, and also when applying it to $\alpha_s^{s-1}\;Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})$ by the established homogeneity properties of $Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})$. Finally $\partial_{\alpha_s}\;\Gamma_{2,l_f}^{\xi,\alpha_s\beta_{i_f}}(m^2)$ is bounded by $\alpha_s^{-1}\;\alpha_{i_f}^{-1}\;\mathcal{P}_{l_f}\log\alpha_{i_f}$ inductively from the r.h.s. of the FE, using also the chain rule. The terms involving the δ-functions give contributions suppressed by two powers of α_s.
The r.h.s. of (54) can therefore be bounded by

$$M^{\frac{2\nu}{3}}\;\cdot\;M^{-\nu}\;\mathcal{P}_{l-1}\log M^\nu\;\leq\;M^{-\frac{\nu}{3}}\;\cdot\;\mathcal{P}_l\log M^\nu\;. \quad (55)$$

Summing over $\nu \in \mathbb{N}$ we obtain a bound $O(1)$, i.e. a bound uniform in α.

[16]One can realize that the optimal value for splitting the domains is indeed $M^{-\frac{2\nu}{3}}$. In this case we are left with a margin $M^{\frac{\nu}{3}}$ in both bounds (55) and (57) below. Therefrom it should be possible to deduce Hölder continuity of type $\eta < 1/3$, as mentioned in the introduction.

[17]The domains depend on α_s through the lower bounds of the $\vec{\beta}$-integrals.

[18]The contribution with the sum of δ-functions stems from deriving the lower bound of the β-integrals.

In the region $\mathcal{D}_2^{(\nu)}$ we analyse further the term $(\vec{p}, A_0(\vec{\alpha})\,\vec{p}) - m^2 \sum_{ir} \alpha_k$. On inspection of (42), remembering (51), the dependence of this expression on α_s can be written as

$$\sum_{k,v} A_{0,kv}(\vec{\xi},\vec{\alpha})\, p_k\, p_v \;+\; m^2\, A_0^{(m)}(\vec{\xi},\vec{\alpha}\,') \;-\; m^2 \sum_{ir} \alpha_k \;=\; -m^2 \left(d + \frac{a}{b+\alpha_s} + \alpha_s \right) \;, \quad (56)$$

where

$$a = \sum_{k,v} (\tilde{A}_{0,kn} \;-\; \tilde{A}_{0,kn+1})(\tilde{A}_{0,vn} \;-\; \tilde{A}_{0,vn+1})\; \frac{p_k\, p_v}{m^2}$$

$$d = -\sum_{k,v} \tilde{A}_{0,kv}\, \frac{p_k\, p_v}{m^2} \;-\; A_0^{(m)} \;+\; \sum_{k=1}^{s-1} \alpha_k \;, \quad b = \tilde{A}_{0,n+1n+1} + \tilde{A}_{0,nn} - 2\tilde{A}_{0,nn+1} \geq 0 \;.$$

Introducing for shortness the variable $x = \alpha_s + b \geq \alpha_s \geq M^\nu$, analysis of the function

$$f(x) \;=\; \frac{a}{x} + x + d' \;, \quad d' = d - b \;,$$

shows that the measure $\mu(\mathcal{C}_\nu)$ of the set $\mathcal{C}_\nu$ of points x such that $|\,f(x)\,| \;\leq\; M \cdot\; M^{\nu/3}$ inside $I_\nu + b$ satisfies[19, 20]

$$\mu(\mathcal{C}_\nu) \;\leq\; O(1)\; M^{2\nu/3} \;.$$

From this we obtain

$$\begin{aligned}
&|\int_{I_\nu} d\alpha_s \int_{\mathcal{D}_2^{(\nu)}} d\vec{\beta}\;\; e^{i\alpha_s[(\vec{p},A_0(\vec{\beta})\,\vec{p}) - m^2\sum_{ir}\beta_k]}\; \alpha_s^{s-1}\; \Theta^{\alpha_s}(\alpha_s\vec{\beta})\;\; Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau}) \prod_{f=1}^{c} \Gamma_{2,l_f}^{\xi,\alpha_s\beta_{i_f}}(m^2)\,| \\
&\qquad \leq\; \int_{I_\nu} d\alpha_s \int_{\mathcal{D}_2^{(\nu)}} d\vec{\beta}\;\; |\,\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;\alpha_s^{s-1}\;\; Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})\; \prod_{f=1}^{c} \Gamma_{2,l_f}^{\xi,\alpha_{i_f}}(m^2)\,| \\
&\quad \leq\; [\sup_{\vec{\beta}\in\mathcal{D}_2^{(\nu)}} \;\mu(\mathcal{C}_\nu)]\; \sup_{\alpha_s\in I_\nu}\; \int_{\mathcal{D}_2^{(\nu)}} d\vec{\beta}\;\; |\,\Theta^{\alpha_s}(\alpha_s\vec{\beta})\;\alpha_s^{s-1}\;\; Q(\vec{\xi},\vec{\lambda},\alpha_s\vec{\beta},\vec{\tau})\; \prod_{f=1}^{c} \Gamma_{2,l_f}^{\xi,\alpha_{i_f}}(m^2)\,| \\
&\qquad\qquad\qquad \leq\; O(1)\; M^{2\nu/3}\; M^{-\nu}\; \mathcal{P}_l \log M^{\nu+1} \;, \qquad\qquad (57)
\end{aligned}$$

where in the last bound we used (31) with $s'' - s = -1$. From this expression we again deduce a bound uniform in α on summing over $\nu \in \mathbb{N}$.
The continuity properties of A and Q and the compactness of the remaining variables then give, on summing both bounds (55), (57) over ν

$$|\int_{\xi}^{\infty} d\alpha_s \int d\vec{\alpha}' \int d\vec{\tau} \int d(\vec{\xi},\vec{\lambda})\;\; e^{i[(\vec{p},A(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\vec{p}) - m^2\sum_{ir}\alpha_k]} \;.$$

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[19] this condition on α_s is necessary for $\mathcal{D}_2^{(\nu)}$ to be nonempty.

[20] In fact $\mathcal{C}_\nu$ is a set of at most two intervals, and the constant $O(1)$ can be taken as $2\sqrt{2}M$, the bound for this choice being saturated if $a + x^2 + d'x$ has 2 zeroes at distance $2\sqrt{M}\,M^{\frac{2\nu}{3}}$ inside $I_\nu + b$.

$$\cdot\, F(\vec{\xi},\vec{\lambda})\ \Theta^{\alpha_s}(\vec{\alpha})\ \Theta^{\alpha_s}(\alpha_s\vec{\beta})\ Q(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\ \prod_{f=1}^{c}\Gamma_{2,l_f}^{\xi,\alpha_{i_f}}(m^2)\,\Big|\ <\ \infty.$$

From this uniform bound in α we easily deduce the continuity of the four-point function. Since (55)[21], (57) hold uniformly in $\vec{p}\in\mathbb{R}^{12}$, we can choose $\nu_0\in\mathbb{N}$ for $\varepsilon>0$ such that $\forall\ \vec{p}\in\mathbb{R}^{12}$

$$\sum_{\nu\geq\nu_0}\Big|\int_{I_\nu}d\alpha_s\ \int d\vec{\alpha}'\ e^{i[(\vec{p},A(\vec{\alpha})\,\vec{p})-m^2\sum_{ir}\alpha_k]}\ \Theta^{\alpha_s}(\vec{\alpha})\ \ Q(\vec{\xi},\vec{\lambda},\vec{\alpha},\vec{\tau})\ \prod_{f=1}^{c}\Gamma_{2,l_f}^{\xi,\alpha_{i_f}}(m^2)\,\Big|\ \leq\ \varepsilon/3\ .$$

Calling $\Gamma_j(\vec{p})$ the contribution to the four-point function corresponding to the previous integral we can therefore split

$$\Gamma_j(\vec{p})-\Gamma_j(\vec{p}^{\,'})=\Gamma_j(\vec{p})-\Gamma_j^{(<\nu_0)}(\vec{p})\ +\ \Gamma_j^{(<\nu_0)}(\vec{p})-\Gamma_j^{(<\nu_0)}(\vec{p}^{\,'})\ +\ \Gamma_j^{(<\nu_0)}(\vec{p}^{\,'})-\Gamma_j(\vec{p}^{\,'})\ .$$

The first and last terms are then bounded in modulus by $\varepsilon/3$, and since $\Gamma_j^{(<\nu_0)}(\vec{p})$ is an analytic function of $\vec{p}$, the second one is bounded by $\varepsilon/3$, if we choose $|\vec{p}-\vec{p}^{\,'}|$ sufficiently small.

It is obvious from the present proof, that the two-point function is also continuous in the variable p^2. Since $\Gamma_{2,l}^{\xi,\alpha}$ is uniformly bounded in α by the previous section, its continuity follows without taking into account the oscillating exponential. With the same methods as used for the four-point function, one can show that the two-point function is (Hölder) continuously differentiable in the variable p^2. We do not further elaborate on this since analyticity of the two-point function up to $p^2=4m^2$ is well-known anyway.

Finally continuity of the IR-1PI four-point function implies also the continuity connected (amputated) four-point function $\mathcal{L}_{4,l}^{0,\infty}$. This follows from the fact that in (symmetric) φ_4^4-theory the only 1PI kernels appearing in the decomposition of the connected four-point function are the 1PI two- and four-point functions. For our renormalization conditions the IR-1PI two-point functions vanish on mass-shell and can be expanded around it by analyticity. The factors of (p^2-m^2) coming from this expansion cancel the denominators of the IR propagators joined to the IR-1PI two-point functions, so that after this cancellation the connected four-point function appears as a product of continuous functions, which is then continuous itself.

[21] The expressions appearing in the integrands from (54) are not uniformly bounded in $\vec{p}\in\mathbb{R}^{12}$, but parameter values for which the denominators appearing in those expressions fall (in modulus) below $M^{-\frac{2\nu}{3}}$ do not belong to $\mathcal{D}_1^{(\nu)}$.

Acknowledgement:
The author is indebted to Jacques Bros for instruction on analyticity properties of the four-point function and to Xavier Lacroze for numerous discussions.

References:

[BZ] M. Bergère and J.B. Zuber, Renormalization of Feynman Amplitudes and parametric integral representation, Commun. Math. Phys. **35**, 113-140 (1974).

[Cha] C. Chandler, Some physical region mass shell properties of renormalized Feynman Integrals, Commun. Math. Phys. **19**, 169-188 (1970).

[EG] H. Epstein and V. Glaser, The Role of Locality in Perturbation Theory, Ann. Inst. Poincaré **XIX**, 211-295 (1973), and : Adiabatic Limit in Perturbation Theory, Erice Adavanced study Institute 1975; G. Velo and A. Wightman (eds.), D. Reidel Publishing Company Utrecht (1976).

[ELOP] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, The Analytic S-Matrix, Cambridge University Press, Cambridge (1966).

[GeSch] I.M. Gelfand and G.E. Schilow, Generalized Functions (Distributions), Academic Press, New York (1964).

[Hep1] K. Hepp, Proof of the Bogoliubov-Parasiuk Theorem on Renormalization, Commun. Math. Phys. **2**, 301-326 (1966), and : Théorie de la Renormalisation, Lecture Notes in Physics 2, Springer Verlag, Heidelberg (1969).

[Hep2] K. Hepp, Renormalization Theory, in: Statistical Mechanics and Quantum Field Theory, 429-500; C. de Witt and R. Stora eds., Gordon and Breach, New York (1971).

[IZ] C. Itzykson and J.B. Zuber, Quantum Field Theory, Mc Graw Hill, New York (1980).

[KKS] G. Keller, Ch. Kopper and C. Schophaus, Perturbative Renormalization with Flow Equations in Minkowski Space, Helv. Phys. Acta **70**, 247-274 (1997).

[KKS1] G. Keller, Ch. Kopper and M. Salmhofer, Perturbative Renormalization and Effective Lagrangians in ϕ_4^4, Helv. Phys. Acta **65**, 33-52 (1992).

[Mü] V.F. Müller, Perturbative Renormalization by Flow Equations, Rev. Math. Phys. **15**, 491-558 (2003).

[Lan] L.D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys. **13**, 181-192 (1959).

[Nak] N. Nakanishi, Graph Theory and Feynman Integrals, Gordon and Breach, New York (1970).

[OS] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions I, Commun. Math. Phys. **31**, 83-112 (1973), Axioms for Euclidean Green's functions II, Commun. Math. Phys. **42**, 281-305 (1975).

[Pol] J. Polchinski, Renormalization and Effective Lagrangians, Nucl. Phys. **B231**, 269-295 (1984).

[Smi] V.A. Smirnov, Feynman Integral Calculus, Springer Verlag, Heidelberg (2006).

[Spe] E. Speer, J. Math. Phys. **9**, 1404 (1968), and : Generalized Feynman Amplitudes, Princeton Univ. Press, Princeton (1969).

[Stei] O. Steinmann, Perturbation Expansions in Axiomatic Field Theory, Lecture Notes in Physics 11, Springer Verlag, Heidelberg etc. (1971).

[tHV] G. 't Hooft and M. Veltman, Scalar One-Loop Integrals, Nucl. Phys. **B513**, 365-401 (1979).

[Tod] I.T. Todorov, Analyticity Properties of Feynman Diagrams in Quantum Field Theory, Pergamon Press, Oxford (1971).

[Zim] W. Zimmermann, The power counting theorem for Minkowski metric, Commun. Math. Phys. **11**, 1-8 (1968), and : Convergence of Bogoliubov's method of renormalization in momentum space, ibid. **15**, 208-234 (1969).