STRUCTURE AND f-DEPENDENCE OF THE A.C.I.M.
FOR A UNIMODAL MAP f OF MISIURECWICZ TYPE.

by David Ruelle*.

Abstract. *By using a suitable Banach space on which we let the transfer operator act, we make a detailed study of the ergodic theory of a unimodal map f of the interval in the Misiurewicz case. We show in particular that the absolutely continuous invariant measure ρ can be written as the sum of 1/square root spikes along the critical orbit, plus a continuous background. We conclude by a discussion of the sense in which the map $f \mapsto \rho$ may be differentiable.*

* Mathematics Dept., Rutgers University, and IHES, 91440 Bures sur Yvette, France. <ruelle@ihes.fr>

0 Introduction.

This paper is part of an attempt to understand the smoothness of the map $f \mapsto \rho$ where (M, f) is a differentiable dynamical system and ρ an SRB measure. [For a general introduction to the problems involved, see for instance [2], [31]]. Smoothness has been established for uniformly hyperbolic systems (see [17], [21], [14], [22], [9]). In that case, one finds that the derivative of ρ with respect to f can be expressed in terms of the value at $\omega = 0$ of a *susceptibility function* $\Psi(e^{i\omega})$ which is holomorphic when the *complex frequency* ω satisfies $\text{Im}\,\omega > 0$, and meromorphic for $\text{Im}\,\omega >$ some negative constant. In the absence of uniform hyperbolicity, $f \mapsto \rho$ need not be continuous. Consider then a family $(f_\kappa)_{\kappa \in \mathbf{R}}$. A theorem of H. Whitney [29] gives general conditions under which, if ρ_κ is defined on $K \subset \mathbf{R}$, then $\kappa \mapsto \rho_\kappa$ extends to a differentiable function of κ on $\mathbf{R}$. Taking ρ_κ to be an SRB measure for f_κ, this gives a reasonable meaning to the differentiability of $\kappa \mapsto \rho_\kappa$ on K (as proposed in [24], see [20], [11] for a different application of Whitney's theorem), even though we start with a noncontinuous function $\kappa \mapsto \rho_\kappa$ on $\mathbf{R}$.

Using Whitney's theorem to study SRB states as proposed above is a delicate matter. A simple situation that one may try to analyze is when (M, f) is a unimodal map of the interval and ρ an absolutely continuous invariant measure (a.c.i.m.). [From the vast literature on this subject, let us mention [12], [13], [6], [7], [8], [28]]. A preliminary study of the Markovian case (*i.e.*, when the critical orbit is finite, see [23], [16]) shows that the susceptibility function $\Psi(\lambda)$ has poles for $|\lambda| < 0$, but is holomorphic at $\lambda = 1$. This study suggests that in non-Markovian situations Ψ may have a natural boundary separating $\lambda = 0$ (around which Ψ has a natural expansion) and $\lambda = 1$ (corresponding to $\omega = 0$). Misiurewicz [19] has studied a class of unimodal maps where the critical orbit stays away from the critical point, and he has proved the existence of an a.c.i.m. ρ for this class. This seems a good situation where one could study the dependence of ρ on f, as pointed out to the author by L.-S. Young.

A desirable starting point to study the dependence of the a.c.i.m. ρ on f is to have an operator $\mathcal{L}$ on a Banach space $\mathcal{A}$ such that $\mathcal{L}\rho = \rho$, and 1 is a simple isolated eigenvalue of $\mathcal{L}$. The main content of the present paper is the construction of $\mathcal{A}$ and $\mathcal{L}$ with the desired properties. Specifically we write $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_2$ consists of *spikes*, *i.e.*, 1/square root singularities at points of the critical orbit, which are known to be present in ρ. We are thus able to prove that the a.c.i.m. ρ is the sum of a continuous background, and of the spikes (see Theorem 9, and the Remarks 16). Note that the construction of an operator $\mathcal{L}$ with a spectral gap had been achieved earlier by G. Keller and T. Nowicki [18], and by L.-S. Young [30] (our construction, in a more restricted setting, leads to stronger results).

We start studying the smoothness of the map $f \mapsto \rho$ by an informal discussion in Section 17. Theorem 19 proves the differentiability along topological conjugacy classes (which are codimension 1) and relates the derivative to the value at $\lambda = 1$ of a modified susceptibility function $\Psi(X, \lambda)$. [Following an idea of Baladi and Smania [5], it is plausible that differentiability in the sense of Whitney holds in directions tangent to a conjugacy class, see below]. Transversally to topological conjugacy classes the map $f \mapsto \rho$ is continuous, but appears not to be differentiable. While this nondifferentiability is not rigorously proved, it seems to be an unavoidable consequence of the fact that the weight of the n-th spike is roughly $\sim \alpha^{n/2}$ (for some $\alpha \in (0,1)$) while its speed when f changes is $\sim \alpha^{-n}$. [See Section 16(c). In fact, for a smooth family (f_κ) restricted to values $\kappa \in K$ such that f_κ is in a suitable Misiurewicz class, the estimates just given for the weight and speed of the spikes suggest that $\kappa \to \rho_\kappa(A)$ for smooth A is $\frac{1}{2}$-Hölder, and nothing better, but we have not proved this]. Physically, let us remark that the spikes of high order n will be drowned in noise, so that discontinuities of the derivative of $f \mapsto \rho$ will be invisible.

Note that the susceptibility functions $\Psi(\lambda)$, $\Psi(X, \lambda)$ to be discussed may have singularities both for large $|\lambda|$ and small $|\lambda|$. [The latter singularities do not occur for uniformly hyperbolic systems, but show up for the unimodal maps of the interval in the Markovian case, as we have mentioned above. A computer search of such singularities is of interest [10]].

A study similar to that of the present paper has been made (Baladi [3], Baladi and Smania [5]) for piecewise expanding maps of the interval. In that case it is found that $f \mapsto \rho$ is not differentiable in general, but Baladi and Smania study the differentiability of $f \mapsto \rho$ along directions tangent to topological conjugacy classes (horizontal directions), not just for f restricted to a class. Note that our 1/square root spikes are replaced in the piecewise expanding case by jump discontinuities. This entails some serious differences, in particular, in the piecewise expanding case $\Psi(\lambda)$ is holomorphic for $|\lambda| < 1$.

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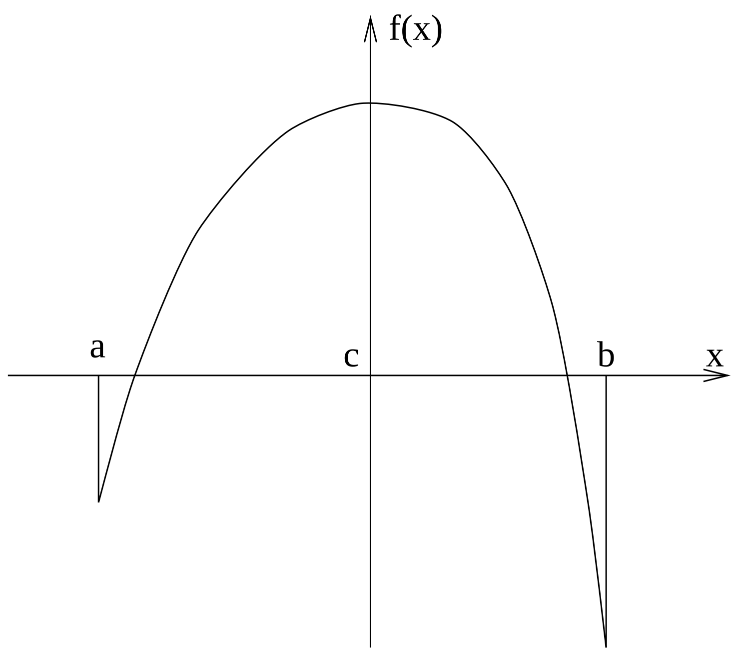


1 Setup.

Let I be a compact interval of $\mathbf{R}$, and $f : \mathbf{R} \to \mathbf{R}$ be real-analytic. We assume that there is c in the interior of I such that $f'(c) = 0$, $f'(x) > 0$ for $x < c$, $f'(x) < 0$ for $x > c$, and $f''(c) < 0$. Replacing I by a possibly smaller interval, we assume that $I = [a, b]$ where $b = fc$, $a = f^2c$, and $a < fa$.

We shall construct a *horseshoe* $H \subset (a, b)$, *i.e.*, a mixing compact hyperbolic set with a Markov partition for f. Following Misiurewicz [19] we shall assume that $fa \in H$.

Under natural conditions to be discussed below we shall study the existence of an a.c.i.m. $\rho(x)\,dx$ for f, and its dependence on f.

2 Construction of the set $H(u_1)$.

Let $u_1 \in [a, b]$ and define the closed set

$$H(u_1) = \{x \in [a, b] : f^n x \geq u_1 \text{ for all } n \geq 0\}$$

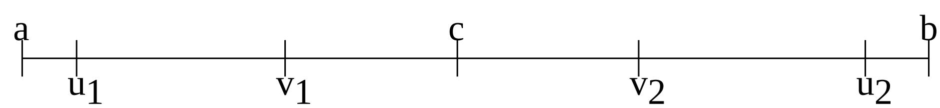
We have thus $fH(u_1) \subset H(u_1)$. Assuming that $H(u_1)$ is nonempty, let v be its minimum element, then $H(u_1) = H(v)$. [Since $v \in H(u_1)$ we have $v \geq u_1$, hence $H(v) \subset H(u_1)$. If $H(u_1)$ contained an element $w \notin H(v)$ we would have $H(u_1) \ni f^k w < v$ for some $k \geq 0$, in contradiction with the minimality of v]. Therefore we may (and shall) assume that $H(u_1) \ni u_1$. We shall also assume

$$a < u_1 < c, fa$$

(and $f^2u_1 \neq u_1$, which will later be replaced by a stronger condition). There is $u_2 \in [a, b]$ such that $fu_2 = u_1$ and, since $u_1 < fa$, it follows that u_2 is unique and satisfies $c < u_2 < b$. We have $u_2 \in H(u_1)$ [because $u_2 > c > u_1$ and $fu_2 \in H(u_1)$] and if $x \in H(u_1)$ then $x \leq u_2$ [because $x > u_2$ implies $fx < u_1$]. Therefore, u_2 is the maximum element of $H(u_1)$. Let

$$V_0 = \{x \in [a, b] : fx > u_2\}$$

then $u_1 < V_0$ [because $x \leq u_1$ implies $fx \leq fu_1 \in H(u_1) \leq u_2$] and $V_0 < u_2$ [because $x \geq u_2$ implies $fx \leq fu_2 = u_1 < u_2$]. Thus we may write $V_0 = (v_1, v_2)$, with $u_1 < v_1 < c < v_2 < u_2$ [$u_1 \neq v_1$ because $f^2u_1 \neq u_1$]. We have $v_1, v_2 \in H(u_1)$ [because $v_1, v_2 > u_1$ and $fv_1 = fv_2 = u_2 \in H(u_1)$].



Our assumptions ($H(u_1) \ni u_1$, $a < u_1 < c, fa$ and $f^2u_1 \neq u_1$) and definitions give thus

$$H(u_1) \subset [u_1, v_1] \cup [v_2, u_2]$$

$$f[u_1, v_1] \subset [u_1, u_2] \qquad , \qquad f[v_2, u_2] = [u_1, u_2]$$

and

$$H(u_1) = \{x \in [u_1, u_2] : f^n x \notin V_0 \text{ for all } n \geq 0\} = fH(u_1)$$

Let us say that the open interval $V_\alpha \subset [u_1, u_2]$ is of order n if f^n maps homeomorphically V_α onto $(v_1, v_2) = V_0$. We have thus

$$H(u_1) = [u_1, u_2] \backslash \cup \text{ all } V_\alpha$$

By induction on n we shall see that

$$[u_1, u_2] \backslash \cup \text{ the } V_\alpha \text{ of order} \leq n$$

is composed of disjoint closed intervals J, such that $f^n J \subset [u_1, v_1]$ or $[v_2, u_2]$ when $n > 0$, and the endpoints of $f^n J$ are u_1, u_2, v_1, v_2 or an image of these points by f^k with $k \leq n$. Assume that the induction assumption holds for n (the case of $n = 0$ is trivial) and let J be as indicated. Since $f^n J \subset [u_1, v_1]$ or $[v_2, u_2]$, f^{n+1} is monotone on J, and the endpoints of J are mapped by f^{n+1} outside of V_0 [because u_1, u_2, v_1, v_2 and their images by f^ℓ are in $H(u_1)$, hence $\notin (v_1, v_2)$]. The interval V_0 is thus either inside of $f^{n+1}J$ or disjoint from $f^{n+1}J$. Each V_α of order $n+1$ thus obtained is disjoint from other V_α of order $\leq n+1$, and the closed intervals $\tilde{J}$ in $[u_1, u_2] \backslash \cup$ the V_α of order $\leq n+1$, are such that the endpoints of $f^{n+1}\tilde{J}$ are u_1, u_2, v_1, v_2 or an image of these points by f^k with $k \leq n+1$, in agreement with our induction assumption.

We assume now that, for some $N \geq 0$, we have $f^{N+1}u_1 = u_1$ (take N smallest with this property), and we assume also that $(f^{N+1})'(u_1) > 0$. [$N = 0, 1$ cannot occur, in particular $f^2 u_1 \neq u_1$. Thus $N \geq 2$, with $f^N u_1 = u_2$, $f^{N-1}u_1 \in \{v_1, v_2\}$. Furthermore, $(f^{N-1})'(u_1) < 0$ if $f^{N-1}u_1 = v_1$, and $(f^{N-1})'(u_1) > 0$ if $f^{N-1}u_1 = v_2$, *i.e.*, $f^{N-1}(u_1+) = v_1-$ or v_2+].

Using the above assumption we now show that none of the intervals J in

$$[u_1, u_2] \backslash \cup \text{ the } V_\alpha \text{ of order} \leq n$$

is reduced to a point. We proceed by induction on n, assuming that $f^n J = [f^n x_1, f^n x_2]$, where $f^n x_1 < f^n x_2$ and $f^n x_1$ is of the form v_2, u_1 or $f^\ell u_1$ with $(f^\ell)'(u_1) > 0$ while $f^n x_2$ is of the form v_1, u_2 or $f^\ell u_2$ with $(f^\ell)'(u_2) > 0$. Therefore the lower limit of $f^{n+1}J$ is of the form $f^m u_1$ with $(f^m)'(u_1) > 0$ while the upper limit is of the form $f^m u_2$ with $(f^m)'(u_2) > 0$. If

$$f^{n+1}J \supset (v_1, v_2)$$

so that a new V_α of order $n+1$ is created, the set $f^{n+1}J \backslash (v_1, v_2)$ consists of two closed intervals, and one of them can be reduced to a point only if $f^m u_1 = v_1$ with $(f^m)'(u_1) > 0$ or if $f^m u_2 = v_2$ with $(f^m)'(u_2) > 0$. So, either $f^{m+2}u_1 = u_1$ with $(f^{m+2})'(u_1) < 0$, or $f^{m+1}u_2 = u_2$ with $(f^{m+1})'(u_2) < 0$ hence $f^{m+1}u_1 = u_1$ with $(f^{m+1})'(u_1) < 0$, in contradiction with our assumption that $(f^{N+1})'(u_1) > 0$.

3 Consequences.

(No isolated points)

$H(u_1)$ is obtained from $[u_1, u_2]$ by taking away successively intervals V_α of increasing order. A given $x \in H(u_1)$ will, at each step, belong to some small closed interval J, and the endpoints of J will not be removed in later steps, so that x cannot be an isolated point: *$H(u_1)$ has no isolated points.*

(Markov property)

Our assumption $f^{N+1}u_1 = u_1$ implies that, for $n = 1, \ldots, N-1$, the point $f^n u_1$ is one of the endpoints of an interval V_α of order $N-1-n$, which we call V_{N-1-n}. These open intervals V_k are disjoint, and their complement in $[u_1, u_2]$ consists of N intervals $U_1, \ldots, U_N$. Each U_i is closed, nonempty, and not reduced to a point. Furthermore, each U_i (for $i = 1, \ldots, N$) is mapped by f homeomorphically to a union of intervals U_j and V_k: this is what we call *Markov property*.

We impose now the following condition:

4 Hyperbolicity.

There are constants $A > 0, \alpha \in (0,1)$ such that if $x, fx, \ldots, f^{n-1}x \in [u_1, v_1] \cup [v_2, u_2]$, then

$$\Big|\frac{d}{dx} f^n x\Big|^{-1} < A\alpha^n$$

We label the intervals $U_1, \ldots, U_N$ from left to right, so that u_1 is the lower endpoint of U_1, and u_2 the upper endpoint of U_N. Define also an oriented graph with vertices U_j and edges $U_j \to U_k$ when $fU_j \supset U_k$. Write $U_{j_0} \stackrel{\ell}{\Longrightarrow} U_{j_\ell}$ if $U_{j_0} \to U_{j_1} \to \cdots \to U_{j_\ell}$, and $U_j \Longrightarrow U_k$ if $U_j \stackrel{\ell}{\Longrightarrow} U_k$ for some $\ell > 0$.

5 Lemma (mixing).

(a) For each U_j there is $r \geq 0$ such that $U_j \stackrel{r+3}{\Longrightarrow} U_1$.

(b) If there is $s > 0$ such that $U_1 \stackrel{s}{\Longrightarrow} U_1$ and $U_1 \stackrel{s}{\Longrightarrow} U_N$, then $U_1 \stackrel{s}{\Longrightarrow} U_k$ for $k = 1, \ldots, N$.

(c) If there is $s > 0$ such that $U_j \stackrel{s}{\Longrightarrow} U_k$ for all $U_j, U_k \in \{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\}$, then $U_j \stackrel{s}{\Longrightarrow} U_k$ for all $U_j, U_k \in \{U_1 \ldots, U_N\}$, and we say that $H(u_1)$ is mixing.

(d) In particular if $N+1$ is a prime, then $H(u_1)$ is mixing.

(e) Let $u_1 < \tilde{u}_1 < c, fa$, and suppose that $f^{\tilde{N}+1}\tilde{u}_1 = \tilde{u}_1$, $(f^{\tilde{N}+1})'(u_1) > 0$. Then if $H(u_1)$ is mixing, so is $H(\tilde{u}_1)$.

(a) The interval U_j is contained in either $[u_1, v_1]$ or $[v_2, u_2]$. Let the same hold for the successive images up to $f^r U_j$, but $f^{r+1}U_j \ni c$ [hyperbolicity and the fact that U_j is not reduced to a point imply that r is finite]. Then $U_j \stackrel{r+1}{\Longrightarrow} U_k$ with $U_k \ni v_1$ or v_2, hence $U_k \stackrel{2}{\Longrightarrow} U_1$ and $U_j \stackrel{r+3}{\Longrightarrow} U_1$.

(b) The U_j such that $U_1 \stackrel{s}{\Longrightarrow} U_j$ form a set of consecutive intervals and, since this set contains U_1 and U_N by assumption, it contains all U_j for $j = 1, \ldots, N$.

(c) By assumption, $U_1 \overset{s}{\Longrightarrow} U_1$ and $U_1 \overset{s}{\Longrightarrow} U_N$, so that $U_1 \overset{s}{\Longrightarrow} U_k$ for $k = 1, \ldots, N$ by (b). Therefore, $\{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\} = \{U_1, \ldots, U_N\}$ by (a), and thus $U_j \overset{s}{\Longrightarrow} U_k$ for all $U_j, U_k \in \{U_1 \ldots, U_N\}$.

(d) The *transitive* set $\{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\}$ decomposes into n disjoint subsets $S_0, \ldots, S_{n-1}$ such that $S_0 \overset{1}{\Longrightarrow} S_1 \overset{1}{\Longrightarrow} \cdots \overset{1}{\Longrightarrow} S_{n-1} \overset{1}{\Longrightarrow} S_0$ and there is $s > 0$ such that $U_j \overset{sn}{\Longrightarrow} U_k$ for all $U_j, U_k \in S_m$, where $m = 0, \ldots, n-1$. We may suppose that $U_1 \in S_0$, and therefore if $U_{(k)}$ denotes the interval containing $f^k u_1$ we have $U_{(k)} \in S_{(k)}$ where $(k) = k(\text{mod } n)$. Therefore $N+1$ is a multiple of n, where $n \leq N < N+1$. In particular, if $N+1$ is prime, then $n = 1$, and $U_j \overset{s}{\Longrightarrow} U_k$ for all $U_j, U_k \in \{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\}$, so that (c) can be applied.

(e) Since $H(\tilde{u}_1)$ is a compact subset of $H(u_1)$, without isolated points, the fact that $H(u_1)$ is mixing implies that $H(\tilde{u}_1)$ is mixing. $\square$

6 Horseshoes.

Note that we have

$$H(u_1) = \{x \in [u_1, u_2] : f^n x \notin V_0 \text{ for all } n \geq 0\} = \cap_{n \geq 0} f^{-n}([u_1, u_2] \backslash V_0)$$

The sets $U_i \cap H(u_1)$ form a *Markov partition* of $H(u_1)$, *i.e.*, $f(U_i \cap H(u_1))$ is a finite union of sets $U_j \cap H(u_1)$.

A set $H = H(u_1)$ as constructed in Section 2, with the hyperbolicity and mixing conditions will be called a *horseshoe*. A horseshoe is thus a mixing hyperbolic set with a Markov partition.

Remember that the open interval $V_\alpha \subset [u_1, u_2]$ is of order n if f^n maps V_α homeomorphically onto $V_0 = (v_1, v_2)$, and let $|V_\alpha|$ be the length of V_α. Hyperbolicity has the following consequence.

7 Lemma (a consequence of hyperbolicity).

There are constants $B > 0$, $\beta \in (0, 1)$ *such that*

$$\sum_{\alpha : \text{order } V_\alpha = n} |V_\alpha| \leq B \beta^n$$

It suffices to prove that

$$\text{Lebesgue meas. } ([u_1, u_2] \backslash \cup \text{ the } V_\alpha \text{ of order} \leq n) \leq G \beta^n$$

[incidentally, this shows that $H(u_1)$ has Lebesgue measure 0].

Let J denote one of the closed intervals in

$$[u_1, u_2] \backslash \cup \text{ the } V_\alpha \text{ of order} \leq n$$

and suppose that J is one of the two intervals adjacent to a given V_α of order n. There is $n' > n$ such that J contains no interval V of order $< n'$, but $J \supset V_{\alpha'}$ of order n'. We write $J = J_{nn'}(V_\alpha, V_{\alpha'})$ and note that J is entirely determined by V_α and $V_{\alpha'}$ (of orders n, n' respectively). The intervals in

$$[u_1, u_2] \backslash \cup \text{the } V_\alpha \text{ of order} \leq n$$

are all the $J_{n_1 n_2}$ with $n_1 \leq n$ and $n_2 > n$. There is a graph Γ with vertices V_α and oriented edges $J_{nn'}(V_\alpha, V_{\alpha'})$ such that for each V_α of order n two edges $J_{nn_1}(V_\alpha, V_{\alpha_1})$ come out of V_α and, if $n > 0$, one edge $J_{n_0 n}(V_{\alpha_0}, V_\alpha)$ goes in. The graph Γ is a tree, rooted at V_0.

We want to show that

$$\sum_{n_1 \leq n, n_2 > n} \sum_{\alpha_1 \alpha_2} |J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| \leq G\beta^n$$

In order to do this we shall introduce intervals $\tilde{J}^n_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2}) \supset J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$ such that, for fixed n, the $\tilde{J}^n_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$ are disjoint, and we shall find $\theta \in (0,1)$ and an integer $N > 0$ such that

$$\sum_{n_1 \leq n, n_2 > n} |\tilde{J}^{n+2N}_{n_1+2N, n_2+2N}(V_{\alpha'_1}, V_{\alpha'_2})| \leq \theta \sum_{n_1 \leq n, n_2 > n} |\tilde{J}^n_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})|$$

(where sums over α'_1, α'_2 and α_1, α_2 are implied). In fact, we shall prove that

$$\sum{}^* |\tilde{J}^{n+2N}_{n'_1, n'_2}(V_{\alpha'_1}, V_{\alpha'_2})| \leq \theta |\tilde{J}^n_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| \qquad (*)$$

for fixed $\tilde{J}^n_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$ such that $n_1 \leq n, n_2 > n$, where the sum $\sum^*$ extends over all $\tilde{J}^{n+2N}_{n'_1, n'_2}(V_{\alpha'_1}, V_{\alpha'_2})$ such that $J_{n'_1, n'_2}(V_{\alpha'_1}, V_{\alpha'_2})$ is above $J^n_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$ in the tree Γ, and that $n'_1 \leq n + 2N$, $n'_2 > n + 2N$. [This means that $\sum^*$ extends over $\tilde{J}^{n+2N}$ corresponding to the closed intervals J^* of

$$[u_1, u_2] \backslash \cup \text{the } V_{\alpha'} \text{ of order} \leq n + 2N$$

such that $J^* \subset J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$].

Note that $J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2}) \supset V_{\alpha_2}$ and that for some constant K_1 independent of n_1, n_2 we may write $|J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| \leq K_1 |V_{\alpha_2}|$ [otherwise $J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})$would contain a V_α of order $< n_2$]. We can also compare $|V_{\alpha_1}|$ and $|V_{\alpha_2}|$ because $f^{n_1} V_{\alpha_1} = f^{n_2} V_{\alpha_2} = V_0$: using hyperbolicity and the smoothness of f we find a constant K_2 such that $|V_{\alpha_2}| \leq K_2 \alpha^{n_2 - n_1} |V_{\alpha_1}|$. Thus

$$|J_{n_1 n_2}(V_{\alpha_1}, V_{\alpha_2})| \leq K_1 K_2 \alpha^{n_2 - n_1} |V_{\alpha_1}| \leq \alpha^{n_2 - n_1 - N} \frac{1}{3} |V_{\alpha_1}|$$

for suitable N. We also assume that $2\alpha^N < 1$.

If $n_2 - n_1 < 2N$ we define $\tilde{J}^n_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2}) = J_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})$. If $n_2 - n_1 \geq 2N$ we define $\tilde{J}^n_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})$ as the union of $J_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})$ and an adjacent subinterval $\tilde{V} \subset V_{\alpha_1}$ such that $|\tilde{V}| = \alpha^{\frac{1}{2}(n-n_1)}\frac{1}{3}|V_{\alpha_1}|$ and therefore (since $n < n_2$)

$$|\tilde{V}| > \alpha^{\frac{1}{2}(n_2-n_1)}\frac{1}{3}|V_{\alpha_1}| > \alpha^{n_2-n_1-N}\frac{1}{3}|V_{\alpha_1}| \geq |J_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})|$$

If $n + 2N < n_2$, there is only one term in the left-hand side of $(*)$, and this term is $\tilde{J}^{n+2N}_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})$, so that

$$\left|\frac{\tilde{J}^{n+2N}_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})}{\tilde{J}^{n}_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})}\right| \leq \frac{\alpha^{\frac{1}{2}(n-n_1+2N)}\frac{1}{3}|V_{\alpha_1}| + \alpha^{n_2-n_1-N}\frac{1}{3}|V_{\alpha_1}|}{\alpha^{\frac{1}{2}(n-n_1)}\frac{1}{3}|V_{\alpha_1}|}$$

$$= \alpha^N + \alpha^{n_2-\frac{1}{2}n_1-\frac{1}{2}n-N} \leq \alpha^N + \alpha^{n_2-n-N} \leq 2\alpha^N$$

If $n + 2N \geq n_2$ there are several terms in the left-hand side of $(*)$, obtained from the interval $J_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})$ from which at least a subinterval of length $\frac{1}{3}|V_{\alpha_2}|$ has been taken out. Therefore

$$\textstyle\sum^* \leq |J_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})| - \frac{1}{3}|V_{\alpha_2}|$$

and

$$\frac{\sum^*}{|\tilde{J}^n_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})|} \leq 1 - \frac{\frac{1}{3}|V_{\alpha_2}|}{|J_{n_1n_2}(V_{\alpha_1}, V_{\alpha_2})|} \leq 1 - \frac{\frac{1}{3}|V_{\alpha_2}|}{K_1|V_{\alpha_2}|} \leq 1 - \frac{1}{3K_1}$$

We have thus proved $(*)$ with $\theta = \max(2\alpha^N, 1 - 1/3K_1)$, and the lemma follows, with $\beta^N = \theta$. □

8 Remark (the set $\tilde{H}$).

Starting from the horseshoe $H = H(u_1)$ we can, by increasing u_1 to $\tilde{u}_1$ such that $\tilde{u}_1 < c, fa$, obtain a set $\tilde{H} = H(\tilde{u}_1) \subset H$ such that $\tilde{u}_1 \in \tilde{H}$ and the distance of $\tilde{H}$ to $\{u_1, u_2, v_1, v_2\}$ is $\geq \epsilon > 0$. [In fact, using our hyperbolicity assumption we can arrange that there is $\tilde{N}$ such that $f^{\tilde{N}+1}\tilde{u}_1 = \tilde{u}_1$, $(f^{\tilde{N}+1})'(\tilde{u}_1) > 0$. In that case $\tilde{H}$ is mixing (Lemma 5(e)) and therefore again a horseshoe].

9 Theorem.

Let $H = H(u_1)$ be a horseshoe, suppose that $fa = f^2b \in H$, and that $\{f^nb : n \geq 0\}$ has a distance $\geq \epsilon > 0$ from $\{u_1, u_2, v_1, v_2\}$. Then f has a unique a.c.i.m. $\rho(x)\,dx$. Furthermore

$$\rho(x) = \phi(x) + \sum_{n=0}^{\infty} C_n\psi_n(x)$$

The function ϕ is continuous on $[a, b]$, with $\phi(a) = \phi(b) = 0$. For $n \geq 0$ we shall choose $w_n \in \{u_1, u_2, v_1, v_2\}$ with $(w_n - c)(c - f^nb) < 0$ and let θ_n be the characteristic function *of* $\{x : (w_n - x)(x - f^n b) > 0\}$. *Then, the above constants* C_n *and* spikes ψ_n *are defined by*

$$C_n = \phi(c)|\frac{1}{2}f''(c)\prod_{k=0}^{n-1} f'(f^k b)|^{-1/2}$$

$$\psi_n(x) = \frac{w_n - x}{w_n - f^n b} \cdot |x - f^n b|^{-1/2}\,\theta_n(x)$$

[The condition that $\{f^n b : n \geq 0\}$ has distance $\geq \epsilon$ from $\{u_1, u_2, v_1, v_2\}$ is achieved, according to Remark 8, by taking $\epsilon \leq |u_1 - a|, |u_2 - b|$, and $f^2 b \in \tilde{H}$. Note also that $\psi_n(c) = 0$, so that $\phi(c) = \rho(c)$. Other choices of ψ_n can be useful, with the same singularity at $f^n b$, but greater smoothness at w_n and/or satisfying $\int dx\,\psi_n(x) = 0$].

10 Analysis.

We analyze the problem before starting the proof. Near c we have

$$y = fx = b - A(x-c)^2 + \text{h.o.}$$

with $A = -f''(c)/2 > 0$, hence $x - c = \pm\big((b-y)/A\big)^{1/2} + O(b-y)$. Therefore, writing $U = \rho(c)/\sqrt{A}$, the density of the image $f(\rho(x)dx)$ by f of $\rho(x)dx$ has, near b, a singularity

$$\frac{U}{\sqrt{(b-x)}} + O(\sqrt{b-x})$$

and, near a, a singularity

$$\frac{U}{\sqrt{-f'(b)(x-a)}} + O(\sqrt{x-a})$$

To deal with the general case of the singularity at $f^n b$, define $s_n = -\text{sgn}\prod_{k=0}^{n-1} f'(f^k b)$, so that

$$\prod_{k=0}^{n-1} f'(f^k b) = -s_n U^2 C_n^{-2}$$

The density of $f(\rho(x)dx)$ has then, near $f^n b$, a singularity given when $s_n(x - f^n b) > 0$ by

$$\frac{U}{\sqrt{(\prod_{k=0}^{n-1}|f'(f^k b)|)|x - f^n b|}} + O(\sqrt{|x - f^n b|})$$

$$= \frac{U}{\sqrt{-(x - f^n b)\prod_{k=0}^{n-1} f'(f^k b)}} + O(\sqrt{|x - f^n b|})$$

$$= \frac{C_n}{\sqrt{s_n(x - f^n b)}} + O(\sqrt{s_n(x - f^n b)})$$

and by 0 when $s_n(x - f^n b) < 0$.

We let now $w_0 = u_2$ and, for $n \geq 0$, define $w_{n+1} \in \{u_1, u_2, v_1, v_2\}$ inductively by:

$$(w_{n+1} - c)(f^{n+1}b - c) > 0 \qquad , \qquad (w_{n+1} - f^{n+1}b)(fw_n - f^{n+1}b) > 0$$

We have thus $w_0 = u_2, w_1 = u_1$, and in general

$$w_n \in \{u_1, u_2, v_1, v_2\} \quad , \quad (w_n - c)(f^n b - c) > 0 \quad , \quad s_n(w_n - f^n b) > 0 \quad , \quad |w_n - f^n b| \geq \epsilon$$

The above considerations show that the singularity expected near $f^n b$ for the density of $f(\rho(x)dx)$ is also represented by

$$(1 - \frac{x - f^n b}{w_n - f^n b}) \cdot \frac{C_n}{\sqrt{s_n(x - f^n b)}} \, \theta_n(x)$$

$$= C_n \, \frac{w_n - x}{w_n - f^n b} \, |x - f^n b|^{-1/2} \, \theta_n(x) = C_n \psi_n(x)$$

in agreement with the claim of the theorem.

11 Lemma.

Write

$$f(\psi_n(x)dx) = \tilde{\psi}_{n+1}(x)dx \qquad , \qquad \tilde{\psi}_{n+1} = |f'(f^n b)|^{-1/2}\psi_{n+1} + \chi_n$$

Then, for $n \geq 0$, the χ_n are continuous of bounded variation on $[a, b]$, with $\chi_n(a) = \chi_n(b) = 0$, and the $\mathrm{Var}\,\chi_n = \int_a^b |d\chi_n/dx| dx$ *are bounded uniformly with respect to n. Furthermore, if $n \geq 1$ and $V_\alpha \subset \mathrm{supp}\chi_n$, then $\chi_n | V_\alpha$ extends to a holomorphic function $\chi_{n\alpha}$ in a complex neighborhood D_α of the closure of V_α in* **R** *(further specified in Section 12), with the $|\chi_{n\alpha}|$ uniformly bounded.*

The case $n = 0$ can be handled by inspection, and we shall assume $n \geq 1$. We let

$$I_n = \begin{cases} (fa, b) & \text{if} \quad f^n b \in [a, c) \\ (a, b) & \text{if} \quad f^n b \in (c, b) \end{cases}$$

And define $f_n^{-1} : I_n \mapsto (a, b)$ to be the inverse of f restricted respectively to (a, c) or (c, b) in the two cases above. We have then

$$\tilde{\psi}_{n+1}(x) = \frac{\psi_n(f_n^{-1}x)}{|f'(f_n^{-1}x)|}$$

Since $n \geq 1$, the region of interest $f\mathrm{supp}\psi_n \cup \mathrm{supp}\psi_{n+1}$ is $\subset [u_1, u_2] \subset (a, b)$, and we have

$$f_n^{-1}x - f^n b = (x - f^{n+1}b)A_n(x)$$

where A_n is real analytic and $A_n(f^{n+1}b) = (f'(f^n b))^{-1}$. Therefore we may write

$$\frac{1}{f_n^{-1}x - f^n b} = \frac{f'(f^n b)}{x - f^{n+1}b}(1 + (x - f^{n+1}b)\tilde{A}_n(x))$$

$$\frac{1}{f'(f_n^{-1}x)} = \frac{1}{f'(f^n b)}(1 + (x - f^{n+1}b))\tilde{B}_n(x)$$

$$\frac{w_n - f_n^{-1}x}{w_n - f^n b} = 1 + (x - f^{n+1}b)\tilde{C}_n(x)$$

and since

$$\psi_n(f_n^{-1}x) = \theta_n(f_n^{-1}x)\Big|\frac{w_n - f_n^{-1}x}{w_n - f^n b}\Big| \cdot |f_n^{-1}x - f^n b|^{-1/2}$$

we find

$$\tilde{\psi}_{n+1}(x) = \frac{\theta_n(f_n^{-1}x)|f'(f^n b)|^{-1/2}}{\sqrt{|x - f^{n+1}b|}}\big(1 + (x - f^{n+1}b)\tilde{D}_n(x)\big)$$

with $\tilde{D}_n$ real analytic. Note that $\tilde{\psi}_{n+1}$ and $|f'(f^n b)|^{-1/2}\psi_{n+1}$ have the same singularity at $f^{n+1}b$. It follows readily that $\tilde{\psi}_{n+1} - |f'(f^n b)|^{-1/2}\psi_{n+1}$ is a continuous function χ_n vanishing at the endpoints of its support, and bounded uniformly with respect to n. It is easy to see that $\mathrm{Var}\,\chi_n$ is bounded uniformly in n. The extension of $\chi_n|V_\alpha$ to holomorphic $\chi_{n\alpha}$ in D_α is also handled readily (see Section 12 for the description of the D_α). ∎

12 The operator $\mathcal{L}$ and the space $\mathcal{A}$.

We have $f(\rho(x)\,dx) = (\mathcal{L}_{(1)}\rho)(x)\,dx$, where the transfer operator $\mathcal{L}_{(1)}$ on $L^1(a,b)$ is defined by

$$\mathcal{L}_{(1)}\rho = \sum_{\pm} \frac{\rho \circ f_\pm^{-1}}{|f' \circ f_\pm^{-1}|}$$

and we have denoted by

$$f_-^{-1} : [fa, b] \mapsto [a, c] \qquad \text{and} \qquad f_+^{-1}[a, b] \mapsto [c, b]$$

the branches of the inverse of f. The invariance of $\rho(x)\,dx$ under f is thus expressed by

$$\rho = \mathcal{L}_{(1)}\rho$$

We shall look for a solution of this equation in a Banach space $\mathcal{A}$ defined below. Roughly speaking, $\mathcal{A}$ consists of functions

$$\phi + \sum_{n=0}^{\infty} c_n\psi_n$$

where the ψ_n are defined in the statement of Theorem 9, and $\phi : [a, b] \to \mathbf{C}$ is a less singular rest with certain analyticity properties.

Remember that we may write

$$[a,b] = H \cup [a,u_1) \cup (u_2,b] \cup \text{the } V_\alpha \text{ of all orders} \geq 0$$

We have (see Remark 8)

$$\operatorname{clos}[a,u_1) \subset [a,\tilde{u}_1) \quad , \quad \operatorname{clos}(u_2,b] \subset (\tilde{u}_2,b] \quad , \quad \operatorname{clos} V_0 \subset \tilde{V}_0$$

where $\tilde{u}_2$ and $\tilde{V}_0 = (\tilde{v}_1,\tilde{v}_2)$, are defined for $\tilde{H}$ as u_2 and V_0 were defined for H. It is convenient to define $V_{-1} = (u_2,b]$ and $V_{-2} = [a,u_1)$ (of order -1 and -2 respectively) so that

$$[a,b] = H \cup \text{the } V_\alpha \text{ of all orders} \geq -2$$

We also define $\tilde{V}_{-1} = (\tilde{u}_2,b], \tilde{V}_{-2} = [a,\tilde{u}_1)$. We let now $\tilde{V}_\alpha$ denote the unique interval in $[a,b]\backslash\tilde{H}$ such that $V_\alpha \subset \tilde{V}_\alpha$. Note that the map $V_\alpha \mapsto \tilde{V}_\alpha$ is not injective!

For each V_α of order ≥ 0 we may choose an open set $D_\alpha \subset \mathbf{C}$ such that

$$\tilde{V}_\alpha \supset D_\alpha \cap \mathbf{R} \supset \operatorname{clos} V_\alpha$$

and, if $fV_\beta = V_\alpha$ of order ≥ 0, $fD_\beta \supset \operatorname{clos} D_\alpha$ [we have here denoted by $\operatorname{clos} V_\alpha$ the closure of V_α in $\mathbf{R}$, and by $\operatorname{clos} D_\alpha$ the closure of D_α in $\mathbf{C}$]. Let also R_a, R_b be two-sheeted Riemann surfaces, branched respectively at a,b, with natural projections $\pi_a,\pi_b : R_a,R_b \to \mathbf{C}$. We may choose open sets $D_{-1},D_{-2} \subset \mathbf{C}$ such that, for $\alpha = -1,-2$,

$$\tilde{V}_\alpha \supset D_\alpha \cap \{x \in \mathbf{R} : a \leq x \leq b\} \supset \operatorname{clos} V_\alpha$$

and f extends to holomorphic maps $\tilde{f}_{-1} : D_0 \to R_b, \tilde{f}_{-2} : (\tilde{f}_{-1}D_0) \to R_a$ such that $\tilde{f}_{-1}D_0 \supset \pi_b^{-1}\operatorname{clos} D_{-1}, \tilde{f}_{-2}\pi_b^{-1}D_{-1} \supset \pi_a^{-1}\operatorname{clos} D_{-2}$. [We shall say that $\tilde{f}_{-1}$ sends (v_1,c) to the *upper* sheet of R_b and (c,v_2) to the *lower* sheet of R_b; $\tilde{f}_{-2}$ sends the upper (lower) sheet of R_b to the upper (lower) sheet of R_a].

We come now to a precise definition of the complex Banach space $\mathcal{A}$. We write $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ where the elements of $\mathcal{A}_1$ are of the form (ϕ_α) and the elements of $\mathcal{A}_2$ of the form (c_n). Here the index set of the ϕ_α is the same as the index set of the intervals V_α (of order ≥ -2); the index n of the $c_n \in \mathbf{C}$ takes the values $0,1,\ldots$ [the c_n should not be confused with the critical point c]. We assume that ϕ_α is a holomorphic function in D_α when V_α is of order ≥ 0, while ϕ_{-1},ϕ_{-2} are holomorphic on $\pi_b^{-1}D_{-1},\pi_a^{-1}D_{-2}$ and, for all α, $||\phi_\alpha|| = \sup_{z\in D_\alpha}|\phi_\alpha(z)| < \infty$.

[We shall later consider a function $\phi : [a,b] \to \mathbf{C}$ such that $\phi|V_\alpha = \phi_\alpha|V_\alpha$ when V_α is of order ≥ 0. For $x \in V_{-1}$ we shall require $\phi(x) = \Delta\phi(x) = \phi_{-1}(x^+) - \phi_{-1}(x^-)$ where $x^+(x^-)$ is the preimage of x by π_b on the upper (lower) sheet of $\pi_b^{-1}D_{-1}$; for $x \in V_{-2}$ we shall require $\phi(x) = \Delta\phi_{-2}(x) = \phi_{-2}(x^+) - \phi_{-2}(x^-)$ where $x^+(x^-)$ is the preimage of x by π_a on the upper (lower) sheet of $\pi_a^{-1}D_{-2}$. But at this point we discuss an operator $\mathcal{L}$ on $\mathcal{A}$ instead of the transfer operator $\mathcal{L}_{(1)}$ acting on functions $\phi + \sum_n c_n\psi_n$].

Let γ, δ be such that $1 < \gamma < \beta^{-1}, 1 < \delta < \alpha^{-1/2}$ with β as in Lemma 7 and α as in the definition of hyperbolicity (Section 4). We write

$$||(\phi_\alpha)||_1 = \sup_{n \geq -2} \gamma^n \sum_{\alpha:\text{order}\, V_\alpha = n} |V_\alpha|.||\phi_\alpha|| \qquad , \qquad ||(c_n)||_2 = \sup_{n \geq 0} \delta^n |c_n|$$

and, for $\Phi = ((\phi_\alpha), (c_n))$, we let $||\Phi|| = ||(\phi_\alpha)||_1 + ||(c_n)||_2$. We let then $\mathcal{A}_1, \mathcal{A}_2$ be the Banach spaces of sequences $(\phi_\alpha), (c_n)$ as above , such that the norms $||(\phi_\alpha)||_1, ||(c_n)||_2$ are finite. We shall define $\mathcal{L}$ on $\mathcal{A}$ such that $\mathcal{L}\Phi = \tilde{\Phi}$. We first describe what contribution each ϕ_α or c_n gives to $\tilde{\Phi}$ and then we shall check that this is a consistent description of an element $\tilde{\Phi}$ of $\mathcal{A}$.

(i) $\phi_\beta \Rightarrow \hat{\phi}_{\beta\alpha} = \dfrac{\phi_\beta}{|f'|} \circ (f|D_\beta)^{-1} \qquad$ in D_α if order $\beta > 0$ and $fV_\beta = V_\alpha$

[we have here denoted by $|f'|$ the holomorphic function $\pm f'$ such that $\pm f' > 0$ for real argument, we shall use the same notation in (ii)-(vi) below].

(ii) $\phi_0 \Rightarrow \Big(\hat{c}_0 = C_0\phi_0(c)\, , \, \hat{\phi}_{-1} = \pm\dfrac{\phi_0}{|f'|} \circ \tilde{f}_{-1}^{-1} - C_0\phi_0(c)(\pm\dfrac{1}{2}\psi_0 \circ \pi_b) \qquad$ in $\pi_b^{-1}D_{-1}\Big)$

where the signs $\pm$ correspond to the upper/lower sheet of $\pi_b^{-1}D_{-1}$. We claim that $\hat{\phi}_{-1}$ is holomorphic in $\pi_b^{-1}D_{-1}$ as the difference of two meromorphic functions with a simple pole at the branch point b, with the same residue. To see this we uniformize $\pi_b^{-1}D_{-1}$ by the map $u \mapsto b - u^2$. We have thus to express $\pm\dfrac{\phi_0}{|f'|}(c+x) = \dfrac{\phi_0}{f'}(c+x)$ in terms of u where $c + x = \tilde{f}_{-1}^{-1}(b - u^2)$ or $u = \sqrt{b - \tilde{f}_{-1}(c+x)}$ which gives a meromorphic function with a simple pole $1/2\sqrt{A}u$. Since $\pm C_0\phi_0(c)\psi_0(b - u^2)$ is meromorphic with the same simple pole, $\hat{\phi}_{-1}$ is holomorphic in $\pi_b^{-1}D_{-1}$.

(iii) $\phi_{-1} \Rightarrow \hat{\phi}_{-2} = \dfrac{\phi_{-1}}{|f'|} \circ \tilde{f}_{-2}^{-1} \qquad$ in $\pi_a^{-1}D_{-2}$.

(iv) $\phi_{-2} \Rightarrow \hat{\phi}_\alpha = \dfrac{\Delta\phi_{-2}}{f'} \circ f^{-1} \qquad$ in D_α if $f(a, u_1) \supset V_\alpha$,0 otherwise

[we have written $\Delta\phi_{-2}(x) = \phi_{-2}(x^+) - \phi_{-2}(x^-)$ where $x^+(x^-)$ is the preimage of x by π_a on the upper (lower) sheet of $\pi_a^{-1}D_{-2}$].

(v) $c_0 \Rightarrow \Big(\hat{c}_1 = |f'(b)|^{-1/2}c_0\, , \, \chi_0 = \pm\dfrac{1}{2}c_0\big(\dfrac{\psi_0}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-1/2}\psi_1 \circ \pi_a\big)$ in $\pi_a^{-1}D_{-2}\Big)$ where the sign $\pm$corresponds to the upper/lower sheet of $\pi_a^{-1}D_{-2}$.

(vi) $c_n \Rightarrow \Big(\hat{c}_{n+1} = |f'(f^n b)|^{-1/2}c_n\, , \, \chi_{n\alpha} = c_n\big[\dfrac{\psi_n}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-1/2}\psi_{n+1}\big]$

in D_α if $V_\alpha \subset \{x : \theta_n(f_n^{-1}x) > 0\}$, 0 otherwise$\Big)$

if $n \geq 1$.

We may now write

$$\tilde{\Phi} = ((\tilde{\phi}_\alpha), (\tilde{c}_n))$$

where

$\tilde{\phi}_{-2} = \hat{\phi}_{-2} + \chi_0$ (see (iii),(v))
$\tilde{\phi}_{-1} = \hat{\phi}_{-1}$ (see(ii))
$\tilde{\phi}_{\alpha} = \sum_{\beta : fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha} + \hat{\phi}_{\alpha} + \sum_{n\geq 1} \chi_{n\alpha}$ if order $\alpha \geq 0$ (see (i),(iv),(vi))
$\tilde{c}_0 = \hat{c}_0$ (see (ii))
$\tilde{c}_1 = \hat{c}_1$ (see (v))
$\tilde{c}_n = \hat{c}_n$ for $n > 1$ (see (vi))

Note that, corresponding to the decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, we have

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_0 + \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}$$

where

$\mathcal{L}_0(\phi_\alpha) = (\sum_{\beta : fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha})$
$\mathcal{L}_1(\phi_\alpha) = (\hat{\phi}_\alpha)$
$\mathcal{L}_2(c_n) = (\chi_0, (\sum_{n\geq 1} \chi_{n\alpha})_{\alpha > -1})$
$\mathcal{L}_3(\phi_\alpha) = (\hat{c}_0, (0)_{n>0})$
$\mathcal{L}_4(c_n) = (0, (\hat{c}_n)_{n>0})$

Holomorphic functions in D_α are defined by (i),(iv),(vi) when order $\alpha \geq 0$, and in $\pi_b^{-1} D_{-1}$, $\pi_a^{-1} D_{-2}$ by (ii),(iii),(v). Using Lemma 7, one sees that $\mathcal{L}_0, \mathcal{L}_1$ are bounded $\mathcal{A}_1 \to \mathcal{A}_1$. Using Lemma 11, one sees that $\mathcal{L}_3$ is bounded $\mathcal{A}_2 \to \mathcal{A}_1$. It is also readily seen that $\mathcal{L}_2, \mathcal{L}_4$ are bounded, so that $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is bounded.

13 Theorem (structure of $\mathcal{L}$).

With our definitions and assumptions, the bounded operator $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is a compact perturbation of $\mathcal{L}_0 \oplus \mathcal{L}_4$; its essential spectral radius is $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$.

Since $fa \in \tilde{H}$, we may assume that $f(a, u_1) \supset V_\alpha$ implies $f(D_{-2} \backslash \text{negative reals}) \supset \operatorname{clos} D_\alpha$. Therefore, $\phi_{-2} \mapsto \hat{\phi}_\alpha | D_\alpha$ is compact. For N positive integer, define the operator $\mathcal{L}_{N1}$ such that

$$\mathcal{L}_{N1}(\phi_\alpha) = \frac{\Delta\phi_{-2}}{f'} \circ f^{-1} \qquad \text{in } D_\alpha \text{ if } f(a, u_1) \supset V_\alpha \text{ and order } \alpha > N \text{ , 0 otherwise}$$

Then $\mathcal{L}_1$ is a perturbation of $\mathcal{L}_{N1}$ by a compact operator and, using Lemma 7, we see that

$$||\mathcal{L}_{N1}(\phi_\alpha)||_1 \leq C \sup_{n>N} \gamma^n \beta^n \to 0 \qquad \text{when } N \to \infty$$

We can write $\mathcal{L}_2 = \mathcal{L}_{N2} +$ finite range, where

$$\mathcal{L}_{N2}(c_n) = (0, 0, (\sum_{n \geq N} \chi_{n\alpha})_{\alpha \geq 0})$$

Using Lemma 11 we find a bound $||\sum_{n\geq N}\chi_{n\alpha}||\leq C'\delta^N$ and, using Lemma 7,

$$||\mathcal{L}_{N2}||_{\mathcal{A}_2\to\mathcal{A}_1}\leq C''\delta^N\to 0 \qquad \text{when } N\to\infty$$

The operator $\mathcal{L}_3$ has one-dimensional range. Therefore $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3$ are compact operators, and the essential spectral radius of $\mathcal{L}$ is the max of the essential spectral radius of $\mathcal{L}_0$ on $\mathcal{A}_1$ and $\mathcal{L}_4$ on $\mathcal{A}_2$.

The spectral radius of $\mathcal{L}_4$ is

$$\leq ||\mathcal{L}_4^N||^{1/N}\leq \left(\delta^N C'''\sup_{\ell\geq 0}\prod_{k=0}^{N-1}|f'(f^{k+\ell}b)|^{-1/2}\right)^{1/N} \qquad \text{with limit} < \delta\alpha^{1/2} \text{ when } N\to\infty$$

The essential spectral radius of $\mathcal{L}_0$ is

$$\leq \lim_{N\to\infty}\frac{\sup_{n\geq N}\gamma^n\sum_{\alpha:\mathrm{order}V_\alpha=n}|V_\alpha|.||\sum_{\beta:fV_\beta=V_\alpha}\hat{\phi}_{\beta\alpha}||}{\sup_{n\geq N}\gamma^{n+1}\sum_{\beta:\mathrm{order}V_\beta=n+1}|V_\beta|.||\phi_\beta||}$$

$$\leq \gamma^{-1}\lim_{\mathrm{order}V_\alpha\to\infty}\frac{|V_\alpha|.||\sum_{\beta:fV_\beta=V_\alpha}\hat{\phi}_{\beta\alpha}||}{\sum_{\beta:fV_\beta=V_\alpha}|V_\beta|.||\phi_\beta||}=\gamma^{-1}$$

In fact, no eigenvalue of $\mathcal{L}_0$ can be $>\gamma^{-1}$, so the spectral radius of $\mathcal{L}_0$ acting on $\mathcal{A}_1$ is $\leq\gamma^{-1}$. The essential spectral radius of $\mathcal{L}$ is thus $\leq\max(\gamma^{-1},\delta\alpha^{1/2})$. □

[Note also that when $\gamma\to\beta^{-1},\delta\to 1$, we have $\max(\gamma^{-1},\delta\alpha^{1/2})\to\max(\beta,\alpha^{1/2})$].

14 The eigenvalue 1 of $\mathcal{L}$.

Let the map $\Delta:\mathcal{A}_1\to L^1(a,b)$ be such that $\Delta(\phi_\alpha)|(a,u_1)=\Delta\phi_{-2}$, $\Delta(\phi_\alpha)|(u_2,b)=\Delta\phi_{-1}$, and $\Delta(\phi_\alpha)|V_\beta=\phi_\beta$ if order $\beta\geq 0$. We also define $w:\mathcal{A}\to L^1(a,b)$ by $w((\phi_\alpha),(c_n))=\Delta(\phi_\alpha)+\sum_{n=0}^\infty c_n\psi_n$ and check readily that

$$w\mathcal{L}\Phi=\mathcal{L}_{(1)}w\Phi$$

If $\lambda^0\neq 0$ is an eigenvalue of $\mathcal{L}$, and $\Phi^0=((\phi^0_\alpha),(c^0_n))$ is an eigenvector to this eigenvalue, we have $w\Phi^0\neq 0$ [because $w\Phi^0=0$ implies $\phi^0_0=0$, hence $\phi^0_{-1}=0,\phi^0_{-2}=0$, and $(c^0_n)=0$; then $\Delta(\phi^0_\alpha)=0$, so $\phi^0_\alpha=0$ when order $\alpha\geq 0$, *i.e.*, $\Phi_0=0$]. Therefore

$$\lambda^0 w\Phi^0=\mathcal{L}_{(1)}(w\Phi^0)$$

$$|\lambda^0|\int_a^b|w\Phi^0|=\int_a^b|\mathcal{L}_{(1)}(w\Phi^0)|\leq\int_a^b\mathcal{L}_{(1)}|w\Phi^0|=\int_a^b|w\Phi^0|$$

hence $|\lambda^0|\leq 1$.

If $c^0_0=0$, then $(c^0_n)=0$, and λ^0 is thus an eigenvalue of $\mathcal{L}_0$ acting on $\mathcal{A}_1$, so that $|\lambda^0|\leq\gamma^{-1}$ (see Section 13). Therefore $|\lambda^0|>\gamma^{-1}$ implies $c^0_0\neq 0,c^0_1\neq 0$, hence $\Delta\phi_{-1}+c_0\psi_0\neq 0$,

$\Delta\phi_{-2} + c_1\psi_1 \neq 0$. Note that, by analyticity, $\Delta\phi_{-2} + c_1\psi_1$ is nonzero almost everywhere in (a, u_1). The image $f(a, u_1)$ contains some (small) interval $U_{i_0} \cap f^{-1}(U_{i_1} \cap f^{-1}(U_{i_2} \ldots))$ on which the image of $\Delta\phi_{-2} + c_1\psi_1$ by $\mathcal{L}_{(1)}$ does not vanish, and therefore (by mixing),

$$\int_a^b |\mathcal{L}_{(1)} w\Phi^0| < \int_a^b \mathcal{L}_{(1)} |w\Phi^0|$$

when $w\Phi^0/|w\Phi^0|$ is not constant on (a, b). Thus either (after multiplication of Φ^0 by a suitable constant $\neq 0$), $w\Phi^0 \geq 0$, or

$$|\lambda^0| \int_a^b |w\Phi^0| < \int_a^b |w\Phi^0| \tag{$*$}$$

i.e., $|\lambda^0| < 1$. Thus 1 is the only possible eigenvalue λ^0 with $|\lambda^0| = 1$, but 1 is an eigenvalue, otherwise the spectral radius of $\mathcal{L}$ would be < 1 [contradicting the fact that $\int_a^b w\mathcal{L}^n\Phi = \int_a^b w\Phi > 0$ when $w\Phi > 0$]. $(*)$ also implies that if $\mathcal{L}\Phi^1 = \Phi^1$, then $w\Phi^1$ is proportional to $w\Phi^0$, hence ϕ_0^1 is proportional to ϕ_0^0, hence Φ^1 is proportional to Φ^0. Furthermore, the generalized eigenspace to the eigenvalue 1 contains only the multiples of Φ_0 [otherwise there would exist Φ^1 such that $\mathcal{L}^n\Phi^1 = \Phi^1 + n\Phi^0$, contradicting $\int_a^b w\mathcal{L}\Phi^1 = \int_a^b w\Phi^1$]. We have proved the first part of the following

15 Proposition.

(a) Apart from the simple eigenvalue 1*, the spectrum of* $\mathcal{L}$ *has radius* < 1*. The eigenvector* Φ^0 *to the eigenvalue* 1 *(after multiplication by a suitable constant* $\neq 0$*) satisfies* $w\Phi^0 \geq 0$.

(b) Write $\Phi^0 = ((\phi_\alpha^0), (c_n^0))$ *and* $\Delta(\phi_\alpha^0) = \phi^0$*, then* ϕ^0 *is continuous, of bounded variation, and* $\phi^0(a) = \phi^0(b) = 0$.

The interval $[u_1, u_2]$ is divided into N closed intervals $W_1, \ldots, W_N$ by the points $f^n u_1$ for $n = 1, \ldots, N-1$. The intervals $W_1, \ldots, W_N$ are ordered from left to right, by doubling the common endpoints we make the W_j disjoint. Define $\gamma^0 = (\gamma_j^0)_{j=1}^N$ by $\gamma_j^0 = \phi^0 | W_j \in L^1(W_j)$. Then, the equation $\Phi^0 = \mathcal{L}\Phi^0$ implies

$$\gamma^0 = \mathcal{L}_*\gamma^0 + \eta \tag{$*$}$$

or

$$\gamma_j^0 = \sum_k \mathcal{L}_{jk}\gamma_k^0 + \eta_j$$

where $\mathcal{L} = (\mathcal{L}_{jk})$ is a transfer operator defined as follows. Letting $(f^{-1})_{kj} : W_j \to W_k$ be such that $f \circ (f^{-1})_{kj}$ is the identity on W_j we write

$$\mathcal{L}_{jk}\gamma_k = \begin{cases} \frac{\gamma_k \circ (f^{-1})_{kj}}{|f' \circ (f^{-1})_{kj}|} & \text{if } fW_k \supset W_j \\ 0 & \text{otherwise} \end{cases}$$

[the term $\mathcal{L}_*\gamma^0$ in $(*)$ comes from (i) in Section 12]. We let

$$\eta_j = \sum_{n=0}^{\infty} \eta_{jn}$$

Here

$$\eta_{j0}(x) = \frac{\Delta\phi^0_{-2}(y)}{f'(y)}$$

if $f(a,u_1) \cap W_j$ contains more than one point, and $y \in (a, u_1)$, $fy = x \in W_j$; we let $\eta_{j0}(x) = 0$ otherwise [this term comes from (iv) in Section 12]. For $n \geq 1$, we let $\eta_{jn} = C_n\chi_n|W_j$ where $\chi_n = (\psi_n/|f'|) \circ f_n^{-1} - |f'(f^n b)|^{-1/2}\psi_{n+1}$ [this term comes from (vi) in Section 12].

Because fu_1 is one of the division points between the intervals W_j, the function η_{j0} is continuous on W_j; the η_{jn} for $n \geq 1$ are also continuous. Furthermore, η_{j0} and the η_{jn} for $n \geq 1$ are uniformly of bounded variation. If $\mathcal{H}_j$ denotes the Banach space of continuous functions of bounded variation on W_j we have thus $\eta_j \in \mathcal{H}_j$ for $j = 1, \ldots, N$. We shall now obtain an upper bound on the essential spectral radius of $\mathcal{L}_*$ acting on $\mathcal{H} = \oplus_1^N \mathcal{H}_j$ by studying $||\mathcal{L}_*^n - F_n||$, where F_n has finite-dimensional range (we use here a simple case of an argument due to Baladi and Keller [4]). Define

$$W_{i_n \cdots i_0} = \{x \in W_{i_n} : fx \in W_{i_{n-1}}, \ldots, f^n x \in W_{i_0}\}$$

when $fW_{i_k} \supset W_{i_{k-1}}$ for $k = n, \ldots, 1$. For $\eta = (\eta_j) \in \mathcal{H}$, we let $\pi_n\eta = (\pi_{jn}\eta_j)$ where $\pi_{jn}\eta_j$ is a piecewise affine function on W_j such that $(\pi_{jn}\eta_j)(x) = \eta_j(x)$ whenever x is an endpoint of W_j or of an interval $W_{ji_{n-1}\cdots i_0}$, and is affine between all such endpoints. Then $F_n = \mathcal{L}_*^n\pi_n$ has finite rank (*i.e.*, finite-dimensional range), and $\mathcal{L}_*^n - F_n = \mathcal{L}_*^n(1-\pi_n)$ maps $\mathcal{H}$ to $\mathcal{H}$. Let $\mathrm{Var}\,\gamma = \sum_1^N \mathrm{Var}_j\gamma_j$ where Var_j is the total variation on W_j. Let also $||\cdot||_0$ denote the sup-norm and $||\cdot|| = \max\{\mathrm{Var}\,\cdot, ||\cdot||_0\}$ be the bounded variation norm. We have

$$\mathrm{Var}(\gamma - \pi_n\gamma) \leq 2\mathrm{Var}\,\gamma$$

$$\sum_{i_0\cdots i_n} ||(\gamma - \pi_n\gamma)|W_{i_n\cdots i_0}||_0 \leq \mathrm{Var}\,\gamma$$

[the second inequality follows from the first because $\gamma - \pi_n\gamma$ vanishes at the endpoints of $W_{i_n\cdots i_0}$]. Since $\mathcal{L}_*^n(1-\pi_n)\gamma$ vanishes at the endpoints of the W_j, we have

$$||(\mathcal{L}_*^n - F_n)\gamma|| = \mathrm{Var}((\mathcal{L}_*^n - F_n)\gamma)$$

$$= \mathrm{Var} \sum_{i_0\cdots i_n} ((\gamma - \pi_n\gamma)_{i_n} \circ \tilde{f}_{i_n\cdots i_0})(\tilde{f}' \circ \tilde{f}_{i_n\cdots i_0}) \cdots (\tilde{f}' \circ \tilde{f}_{i_1 i_0})$$

where we have written

$$\tilde{f}_{i_\ell\cdots i_0} = (f^{-1})_{i_\ell i_{\ell-1}} \circ \cdots (f^{-1})_{i_1 i_0}$$

and

$$\tilde{f}' = \frac{1}{|f'|}$$

hence

$$||(\mathcal{L}_*^n - F_n)\gamma|| \leq \sum_{i_0 \cdots i_n} \mathrm{Var}[((\gamma - \pi_n\gamma)_{i_n} \circ \tilde{f}_{i_n \cdots i_0})(\tilde{f}' \circ \tilde{f}_{i_n \cdots i_0}) \cdots (\tilde{f}' \circ \tilde{f}_{i_1 i_0})]$$

$$= \sum_{i_0 \cdots i_n} \mathrm{Var}[((\gamma - \pi_n\gamma)|W_{i_n \cdots i_0}) \prod_{\ell=0}^{n-1} (\tilde{f}' \circ (f^\ell|W_{i_n \cdots i_0}))]$$

The right-hand side is bounded by a sum of $n+1$ terms where Var is applied to $(\gamma - \pi_n\gamma)|W_{i_n \cdots i_0}$ or a factor $\tilde{f}' \circ (f^\ell|W_{i_n \cdots i_0}))$, and the other factors are bounded by their $||\cdot||_0$-norm. Thus, using the hyperbolicity condition of Section 4, we have

$$||(\mathcal{L}_*^n - F_n)\gamma||$$

$$\leq \mathrm{Var}(\gamma - \pi_n\gamma).A\alpha^n + \sum_{\ell=0}^{n-1} \sum_{i_0 \cdots i_n} ||(\gamma - \pi_n\gamma)|W_{i_n \cdots i_0}||_0.A\alpha^\ell.\mathrm{Var}(\tilde{f}'|W_{i_{n-\ell} \cdots i_0}).A\alpha^{n-\ell-1}$$

$$\leq 2A\alpha^n \mathrm{Var}\,\gamma + nA^2\alpha^{n-1}\mathrm{Var}\,\tilde{f}' \sum_{i_0 \cdots i_n} ||(\gamma - \pi_n\gamma)|W_{i_n \cdots i_0}||_0$$

$$\leq (2A + nA^2\alpha^{-1}\mathrm{Var}\,\tilde{f}')\alpha^n \mathrm{Var}\,\gamma \leq (2A + nA^2\alpha^{-1}\mathrm{Var}\,\tilde{f}')\alpha^n ||\gamma||$$

so that

$$||\mathcal{L}_*^n - F_n|| \leq (2A + nA^2\alpha^{-1}\mathrm{Var}\,\tilde{f}')\alpha^n$$

and therefore $\mathcal{L}_*$ has essential spectral radius $\leq \alpha < 1$ on $\mathcal{H}$. Suppose that there existed an eigenfunction $\gamma \in \mathcal{H}$ to the eigenvalue 1 of $\mathcal{L}_*$; the fact that γ is continuous and $\neq 0$ on some W_j would imply

$$\int (\mathcal{L}_*^n|\gamma|)(x)\,dx < \int |\gamma|(x)\,dx$$

[because, for some n, $\mathcal{L}_*^n$ sends "mass" into V_0]. But this is in contradiction with

$$\int |\gamma|(x)\,dx = \int |\mathcal{L}_*^n\gamma|(x)\,dx \leq \int (\mathcal{L}_*^n|\gamma|)(x)\,dx$$

Therefore, 1 cannot be an eigenvalue of $\mathcal{L}_*$, and there is $\gamma = (1 - \mathcal{L}_*)^{-1}\eta \in H$ such that

$$\gamma = \mathcal{L}_*\gamma + \eta$$

Since γ^0 satisfies the same equation in L^1, we have $\gamma^0 - \gamma = \mathcal{L}_*(\gamma^0 - \gamma)$ hence $\gamma^0 - \gamma = 0$ by the same argument as above [$|\gamma^0 - \gamma|$ is in L^1, with "mass" in some V_α because $H(u_1)$ has measure 0, and this is sent to V_0 by $\mathcal{L}_*^n$ for some n]. Thus γ^0 is continuous of bounded variation on the intervals W_j for $j = 1, \ldots, N$, and ϕ^0 has bounded variation on $[a, b]$, with possible discontinuities only at $f^n u_1$ for $n = 0, \ldots, N$, and $\phi^0(a) = \phi^0(b) = 0$. We have

$$\mathcal{L}_{(1)}\phi^0 - c_0^0\psi_0 + \sum_{n=0}^{\infty} c_n^0\chi_n = \phi^0$$

Therefore, hyperbolicity along the periodic orbit of u_1 shows that ϕ^0 cannot have discontinuities, and this proves part (b) of Proposition 15. $\square$

This also concludes the proof of Theorem 9. $\square$

16 Remarks.

(a) Theorem 9 shows that the density $\rho(x)$ of the unique a.c.i.m. $\rho(x)\,dx$ for f can be written as the sum of spikes $\approx |x - f^n b|^{-1/2}\theta_n(x)$ (where θ_n vanishes unless $x > f^n b$ or $x < f^n b$) and a continuous background $\phi(x)$. In fact, one can also write $\rho(x)$ as the sum of singular terms $\approx |x - f^n b|^{-1/2}\theta_n(x), |x - f^n b|^{1/2}\theta_n(x)$ and a background $\phi(x)$ which is now differentiable. This result is discussed in Appendix A. It seems clear that one could write $\rho(x)$ as a sum of terms $|x - f^n b|^{k/2}\theta_n(x)$ with $k = -1, 1, \ldots, \frac{2\ell-1}{2}$ and a background $\phi(x)$ of class C^ℓ, but we have not written a proof of this.

(b) Let $u \in (-\infty, u_1) \cup (u_1, v_1) \cup (v_2, u_2) \cup (u_2, \infty)$ and choose $w \in \{u_1, u_2, v_1, v_2\}$ such that w is an endpoint of the interval containing u. If $\pm(w - u) > 0$ and $\theta_\pm$ is the characteristic function of $\{x : (w - x)(x - u) > 0\}$ we define

$$\psi_{(u\pm)}(x) = \frac{w - x}{w - u} \cdot |x - u|^{-1/2}\theta_\pm(x)$$

or a similar expression with the same singularity at u, greater smoothness at w, and/or $\int \psi_{(u\pm)} = 0$. [Note that the ψ_n are of this form]. Claim: if $u \in \tilde{H}$, there exists a unique $(\phi_\alpha) \in \mathcal{A}_1$ such that $\phi_\alpha = \psi_{(u\pm)}|V_\alpha$ for all α; furthermore $||(\phi_\alpha)||_1$ has a bound independent of $u\pm$. These results are proved in Appendix B (assuming $\gamma < \alpha^{-1/2}$).

Note that if $((\phi_\alpha), (c_n)) \in \mathcal{A}$ and $c_0 = c_1 = 0$, there is $(\tilde{\phi}_\alpha) \in \mathcal{A}_1$ such that $\Delta(\tilde{\phi}_\alpha) = w((\phi_\alpha), (c_n))$. It seems thus that we might have replaced $\mathcal{A}$ by $\mathcal{A}_1$ in our earlier discussions. However, separating the spikes (c_n) from the background (ϕ_α) was needed in the spectral study of $\mathcal{L}$.

(c) The eigenvector Φ^0 of $\mathcal{L}$ corresponding to the eigenvalue 1 (with $w\Phi^0 \geq 0, \int w\Phi^0 = 1$) depends continuously on f. To make sense of this statement we may consider a one-parameter family (f_κ) such that $f_0 = f$. We let $H_\kappa, \tilde{H}_\kappa$ (hyperbolic sets) and $\mathcal{A}_{1\kappa}$ (Banach space) reduce to $H, \tilde{H}$ and $\mathcal{A}_1$ when $\kappa = 0$. We restrict κ to a compact set K such that $f_\kappa^3 c_\kappa \in \tilde{H}_\kappa$ (where c_κ is the critical point of f_κ). The intervals $V_{\kappa\alpha}$ associated with H_κ can be mapped to the V_α associated with H, providing an identification $\eta_\kappa : \mathcal{A}_{\kappa 1} \to \mathcal{A}_1$. There are natural definitions of $\mathcal{L}_\kappa : \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2 \to \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2$ and the eigenvector Φ_κ^0 reducing to $\mathcal{L}$ and Φ^0 when $\kappa = 0$. We claim that $\kappa \mapsto \Phi_\kappa^\times = (\eta_\kappa, \mathbf{1})\Phi_\kappa^0$ is a continuous function $K \to \mathcal{A}_1 \oplus \mathcal{A}_2$. This result is proved in Appendix C. It implies that, if A is smooth, $\kappa \to \langle \Phi_{f_\kappa}^0, A \rangle$ is continuous on K. The weight of the n-th spike is $C_0 \prod_{k=1}^n |f_\kappa'(f_\kappa^{k-1} b_\kappa)|^{-1/2}$ and its speed is

$$\frac{d}{d\kappa} f_\kappa^n b_\kappa = \prod_{k=1}^n f_\kappa'(f_\kappa^{k-1} b_\kappa) \frac{db_\kappa}{d\kappa} + \sum_{\ell=1}^n \prod_{k=\ell+1}^n f_\kappa'(f_\kappa^{k-1} b_\kappa) f_\kappa^*(f_\kappa^{\ell-1} b_\kappa) \quad \text{with} \quad f_\kappa^* = \frac{df_\kappa}{d\kappa}$$

The weight may be roughly estimated as $\sim \alpha^{n/2}$ and the speed as $\sim \alpha^{-n}$ for some $\alpha \in (0,1)$, suggesting that $\kappa \to \langle \Phi^0_{f_\kappa}, A\rangle$ is $\frac{1}{2}$-Hölder on K.

17 Informal study of the differentiability of $f \mapsto \langle \Phi^0_f, A\rangle$.

Writing Φ^0_f instead of Φ^0 we want to study the change of $\langle \Phi^0_f, A\rangle = \int dx\,(w\Phi^0_f)(x)A(x)$ when f is replaced by $\hat f$ close to f (and the critical orbit $\hat f^k \hat c$ for $k \geq 3$ is in the perturbed hyperbolic set $\hat{\hat{H}}$). Writing $g = \text{id} - \hat f(\hat c) + f(c)$, we see that $\hat f$ is conjugate to $g \circ \hat f \circ g^{-1}$, which has maximum $f(c)$ at $g(\hat c)$. With proper choice of the inverse f^{-1} we have $f^{-1} \circ (g \circ \hat f \circ g^{-1}) = h$ close to id, hence $g \circ \hat f \circ g^{-1} = f \circ h$ and $(h \circ g) \circ \hat f \circ (h \circ g)^{-1} = h \circ f$, *i.e.*, $\hat f$ is conjugate to $h \circ f$ and we may write

$$\langle \Phi^0_{\hat f}, A\rangle = \langle \Phi^0_{h\circ f}, A \circ h \circ g\rangle$$

The differentiability of $\hat f \mapsto A \circ h \circ g$ is trivial, and we concentrate on the study of $h \mapsto \langle \Phi^0_{h\circ f}, A\rangle$. Writing $h = \text{id} + X$, where X is analytic, we see that the change $\delta(w\Phi^0_f)$ when f is replaced by $(\text{id} + X) \circ f$ is, to first order in X, formally

$$(1 - \mathcal{L})^{-1}\mathcal{D}(-X\Phi^0_f)$$

where $\mathcal{D}$ denotes differentiation. [The above formula is standard first order perturbation calculation, and we have omitted the w map from our formula].

Writing $\Phi^0_f = ((\phi^0_\alpha), (C_n))$, we can identify $\mathcal{D}(-X((\phi^0_\alpha), 0))$ with an element $\Phi^\times$ of $\mathcal{A}$ (so that $w\Phi^\times = \mathcal{D}(Xw((\phi^0_\alpha), 0))$ and $\int dx\, w\Phi^\times(x) = 0$, use Appendix A) which is easy to study, and we are left to analyze the singular part $\mathcal{D}(-X(0, (C_n)))$. To study this singular part we shall write $(0, (C_n)) = \sum_{n=0}^\infty C_n \psi_{(f^n b)}$, and use the equivalence $\sim$ modulo the elements of $\mathcal{A}$. We extend the domain of definition of $\mathcal{L}$ so that $\mathcal{L}\psi_{(u)} \sim |f'(u)|^{-1/2}\psi_{(fu)}$, where we use the notation $\psi_{(u\pm)}$ of Section 16(b), but omit the $\pm$, and we assume that $\int \psi_{(u)} = 0$. We have thus

$$\mathcal{D}(-X(0,(C_n))) \sim -\sum_{n=0}^\infty C_n X(f^n b)\mathcal{D}\psi_{(f^n b)} \sim \sum_{n=0}^\infty C_n X(f^n b)\frac{d}{du}\psi_{(u)}\Big|_{u=f^n b}$$

$$= \sum_{n=0}^\infty C_n X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\frac{d}{db}\psi_{(f^n b)} \sim \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\frac{d}{db}\mathcal{L}^n C_0\psi_{(b)}$$

We may thus write (introducing $(1 - \lambda\mathcal{L})^{-1}$ instead of $(1 - \mathcal{L})^{-1}$)

$$(1 - \lambda\mathcal{L})^{-1}\mathcal{D}(-X(0,(C_n))) \sim \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\lambda^{-n}\frac{d}{db}(1 - \lambda\mathcal{L})^{-1}(\lambda\mathcal{L})^n C_0\psi_{(b)}$$

$$= \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\lambda^{-n}\frac{d}{db}(1 - \lambda\mathcal{L})^{-1} C_0\psi_{(b)} - Z$$

where

$$Z = \sum_{n=0}^{\infty} X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\lambda^{-n}\frac{d}{db}\sum_{\ell=0}^{n-1}(\lambda\mathcal{L})^\ell C_0\psi_{(b)}$$

$$\sim \sum_{n=0}^{\infty} X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\sum_{\ell=0}^{n-1}\lambda^{-n+\ell}|\prod_{k=0}^{\ell-1} f'(f^k b)|^{-1/2}\frac{d}{db}C_0\psi_{(f^\ell b)}$$

$$= \sum_{n=0}^{\infty} X(f^n b)\sum_{\ell=0}^{n-1}\lambda^{-n+\ell}[\prod_{k=\ell}^{n-1} f'(f^k b)]^{-1}|\prod_{k=0}^{\ell-1} f'(f^k b)|^{-1/2}\frac{d}{du}C_0\psi_{(u)}\Big|_{u=f^\ell b}$$

$$\sim -\mathcal{D}\sum_{n=0}^{\infty} X(f^n b)\sum_{\ell=0}^{n-1}\lambda^{-n+\ell}[\prod_{k=\ell}^{n-1} f'(f^k b)]^{-1}C_\ell\psi_\ell$$

$$= -\mathcal{D}\sum_{r=1}^{\infty}\sum_{\ell=0}^{\infty} X(f^{\ell+r}b)\lambda^{-r}[\prod_{k=0}^{r-1} f'(f^{\ell+k}b)]^{-1}C_\ell\psi_\ell$$

$$= -\mathcal{D}\sum_{\ell=0}^{\infty} C_\ell\psi_\ell\sum_{r=1}^{\infty}\lambda^{-r}[\prod_{k=0}^{r-1} f'(f^{\ell+k}b)]^{-1}X(f^{\ell+r}b)$$

We have thus an (informal) proof of the following result

For $\ell = 0, 1, \ldots$, define

$$F_\ell(X) = \sum_{n=1}^{\infty}\lambda^{-n}[\prod_{k=0}^{n-1} f'(f^{k+\ell}b)]^{-1}X(f^{n+\ell}b)$$

which are holomorphic functions of λ when $|\lambda| > \alpha$. Then the susceptibility *function*

$$\Psi(\lambda) = \langle(1-\lambda\mathcal{L})^{-1}\mathcal{D}(-X\Phi_f^0), A\rangle$$

has the form

$$\Psi(\lambda) \sim (X(b) + F_0(X))\frac{d}{db}\langle(1-\lambda\mathcal{L})^{-1}C_0\psi_{(b)}, A\rangle - \sum_{\ell=0}^{\infty} F_\ell(X)C_\ell\langle\psi_\ell, \mathcal{D}A\rangle$$

The derivative $\frac{d}{db}\langle(1-\lambda\mathcal{L})^{-1}C_0\psi_{(b)}, A\rangle$ exists as a distribution, but is in principle a divergent quantity for given b. The corresponding term disappears however if $X(b) + F_0(X) = 0$, and we are then left with a finite expression, meromorphic in λ for $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$ and holomorphic when $\alpha < |\lambda| \leq 1$.

Note that in writing the equivalence $\sim$ we have omitted terms with the singularities of $(1-\lambda\mathcal{L})^{-1}$; this explains the meromorphic contributions for $|\lambda| > 1$. The condition $X(b) + F_0(X) = 0$ for $\lambda = 1$ is known as *horizontality* (see the discussion in Section 19 below).

18 A modified susceptibility function $\Psi(X,\lambda)$.

At this point we extend the definition of the operator $\mathcal{L}$ to $\mathcal{L}^\sim$ acting on a larger space. Remember that $\mathcal{L}$ was obtained from the transfer operator $\mathcal{L}_{(1)}$ by separating the spikes ψ_n from the background in order to obtain better spectral properties. We now also introduce derivatives ψ'_n of spikes, so that the transfer operator sends ψ'_n to

$$\frac{f'(f^nb)}{|f'(f^nb)|^{1/2}}\psi'_{n+1} + \text{a term in } w(\mathcal{A}_1+\mathcal{A}_2)$$

The coefficients of ψ'_n form an element of $\mathcal{A}_3 = \{(Y_n) : ||(Y_n)||_3 = \sup_n \delta^n|Y_n| < \infty\}$. We define $\mathcal{L}^\sim$ on $\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3$ so that

$$\mathcal{L}^\sim = \begin{pmatrix} \mathcal{L}_0+\mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_5 \\ \mathcal{L}_3 & \mathcal{L}_4 & \mathcal{L}_6 \\ 0 & 0 & \mathcal{L}_7 \end{pmatrix}$$

where we omit the explicit definition of $\mathcal{L}_5$, $\mathcal{L}_6$, and let

$$\mathcal{L}_7\Big(\frac{Z_n}{\prod_{k=0}^{n-1}|f'(f^kb)|^{1/2}}\Big) = \Big(\frac{\tilde{Z}_n}{\prod_{k=0}^{n-1}|f'(f^kb)|^{1/2}}\Big)$$

with $\tilde{Z}_0 = 0$, $\tilde{Z}_n = f'(f^{n-1}b)Z_{n-1}$ for $n > 0$. Since

$$\begin{pmatrix} 0 & 0 & \mathcal{L}_5 \\ 0 & 0 & \mathcal{L}_6 \\ 0 & 0 & \mathcal{L}_7 \end{pmatrix}\mathcal{L} = 0$$

we have

$$\mathcal{L}^{\sim n} = \mathcal{L}^n + \sum_{k=1}^{n}\mathcal{L}^{k-1}(\mathcal{L}_5+\mathcal{L}_6)\mathcal{L}_7^{n-k} + \mathcal{L}_7^n$$

and formally

$$(\mathbf{1}-\lambda\mathcal{L}^\sim)^{-1} = (\mathbf{1}_{12}-\lambda\mathcal{L})^{-1} + (\mathbf{1}_3-\lambda\mathcal{L}_7)^{-1} + (\mathbf{1}_{12}-\lambda\mathcal{L})^{-1}\lambda(\mathcal{L}_5+\mathcal{L}_6)(\mathbf{1}_3-\lambda\mathcal{L}_7)^{-1}$$

where $\mathbf{1}_{12}$ and $\mathbf{1}_3$ denote the identity on $\mathcal{A}_1\oplus\mathcal{A}_2$ and $\mathcal{A}_3$ respectively.

For λ close to 1, $(\mathbf{1}_3-\lambda\mathcal{L}_7)^{-1}$ and thus $(\mathbf{1}-\lambda\mathcal{L}^\sim)^{-1}$ are not well defined. But there is a natural definition of a left inverse $\mathcal{L}_{7L}^{-1}$ of $\mathcal{L}_7$ where

$$\mathcal{L}_{7L}^{-1}\Big(\frac{Z_n}{\prod_{k=0}^{n-1}|f'(f^kb)|^{1/2}}\Big) = \Big(\frac{\tilde{Z}_n}{\prod_{k=0}^{n-1}|f'(f^kb)|^{1/2}}\Big)$$

with $\tilde{Z}_n = f'(f^nb)^{-1}Z_{n+1}$ for $n \geq 0$. The spectral radius of $\mathcal{L}_{7L}^{-1}$ is thus $\leq \alpha^{1/2}/\delta$. This gives natural left inverses

$$(\mathbf{1}_3-\lambda\mathcal{L}_7)_L^{-1} = -\sum_{n=1}^{\infty}\lambda^{-n}\mathcal{L}_{7L}^{-n}$$

for $|\lambda| > \alpha^{1/2}/\delta$, and

$$(\mathbf{1} - \lambda\mathcal{L}^{\sim})_L^{-1} = (\mathbf{1}_{12} - \lambda\mathcal{L})^{-1} + (\mathbf{1}_3 - \lambda\mathcal{L}_7)_L^{-1} + (\mathbf{1}_{12} - \lambda\mathcal{L})^{-1}\lambda(\mathcal{L}_5 + \mathcal{L}_6)(\mathbf{1}_3 - \lambda\mathcal{L}_7)_L^{-1}$$

when $|\lambda| > \alpha^{1/2}/\delta$ and $(\mathbf{1}_{12} - \lambda\mathcal{L})^{-1}$ exists. This gives a modified susceptibility function

$$\Psi_L(\lambda) = \langle(\mathbf{1} - \lambda\mathcal{L}^{\sim})_L^{-1}\mathcal{D}(-X\Phi_f^0), A\rangle$$

meromorphic in λ for $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$ and holomorphic for $\alpha < |\lambda| \leq 1$.

Note that the $\mathcal{A}_3$ part of $\mathcal{D}(-X\Phi_f^0)$ is

$$(Y_n) = \Big(\frac{-X(f^n b)}{\frac{1}{2}|f''(c)|^{1/2}\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big)_{n\geq 0}$$

where $\sup_n |X(f^n b)| < \infty$. Therefore, for small $|\lambda|$,

$$(\mathbf{1}_3 - \lambda\mathcal{L}_7)^{-1}(Y_n) = \Big(\frac{-\sum_{k=0}^{n}\lambda^k(\prod_{\ell=1}^{k} f'(f^{n-\ell}b))X(f^{n-k}b)}{\frac{1}{2}|f''(c)|^{1/2}\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big)_{n\geq 0}$$

because the right-hand side is in $\mathcal{A}_3$. Note that the right-hand side is also in $\mathcal{A}_3$ under the condition

$$\sum_{n=0}^{\infty}\lambda^{-n}(\prod_{k=0}^{n-1} f'(f^k b))^{-1}X(f^n b) = 0 \qquad (*)$$

because this condition implies

$$-\sum_{k=0}^{n}\lambda^{-k}(\prod_{\ell=0}^{k-1} f'(f^\ell b))^{-1}X(f^k b) = \sum_{k=n+1}^{\infty}\lambda^{-k}(\prod_{\ell=0}^{k-1} f'(f^\ell b))^{-1}X(f^k b)$$

hence, multiplying by $\lambda^n\prod_{\ell=0}^{n-1} f'(f^\ell b)$,

$$-\sum_{k=0}^{n}\lambda^{n-k}(\prod_{\ell=k}^{n-1} f'(f^\ell b))X(f^k b) = \sum_{k=n+1}^{\infty}\lambda^{n-k}(\prod_{\ell=n}^{k-1} f'(f^\ell b))^{-1}X(f^k b)$$

or

$$-\sum_{k=0}^{n}\lambda^{k}(\prod_{\ell=1}^{k} f'(f^{n-\ell} b))X(f^{n-k} b) = \sum_{k=1}^{\infty}\lambda^{-k}(\prod_{\ell=0}^{k-1} f'(f^{n+\ell} b))^{-1}X(f^{n+k} b)$$

for each n, provided $|\lambda| > \alpha$. We have proved that:

Under the condition $(*)$, *a resummation of the series defining*

$$\langle(\mathbf{1} - \lambda\mathcal{L}^{\sim})^{-1}\mathcal{D}(-X\Phi_f^0), A\rangle$$

yields $\Psi_L(\lambda)$.

It is then natural to define a modified susceptibility function $\Psi(X,\lambda)$ by

$$(X,\lambda)\mapsto\Psi(X,\lambda)=\Psi_L(\lambda)\quad\text{on}\quad\{(X,\lambda):(*)\text{ holds}\}$$

Note that the left-hand side of $(*)$ is equal to the quantity $X(b)+F_0(X)$ met in Section 17, and that $(*)$ with $\lambda=1$ reduces to the horizontality condition.

We conclude with a rigorous result agreeing in part with the informal study in Section 17, in part with a conjecture of Baladi [3], Baladi and Smania [5].

19 Theorem (differentiability along topological conjugacy classes).

Let $f_\kappa=h_\kappa\circ f$ *where the* h_κ *are real analytic, depend smoothly on* κ, *and* $f_\kappa^3c=\xi_\kappa f^3c$ *identically in* κ. *[This last condition expresses that* f_κ *belongs to a conjugacy class, and* $\xi_k:H\to H_\kappa$ *is the conjugacy defined in Appendix C]. Then, if* A *is smooth,* $\kappa\mapsto\langle\Phi^0_\kappa,A\rangle=\int dx\,(w\Phi^0_{f_k})(x)A(x)$ *is continuously differentiable. Furthermore*

$$\frac{d}{d\kappa}\langle\Phi^0_{f_k},A\rangle\Big|_{\kappa=0}=\Psi(X,1)$$

where $\Psi(X,\lambda)$ *is defined in Section 18 with* $X=\frac{d}{d\kappa}h_\kappa|_{\kappa=0}$, *and* $\Psi(X,\lambda)$ *is holomorphic for* $\alpha<|\lambda|\le 1$, *meromorphic for* $\alpha<|\lambda|<\min(\beta^{-1},\alpha^{-1/2})$.

[The value $\kappa=0$ plays no special role, and is chosen for notational simplicity in the formulation of the theorem].

Our notion of topological conjugacy class is a special case of that discussed in [1].

Note that $\xi_0=\text{id}$, and that ξ_κ depends differentiably on κ. Since $f_\kappa^3c=\xi_\kappa f^3c$ and $f_\kappa\xi_\kappa=\xi_\kappa f$ on H, we have $f_\kappa^nc=\xi_\kappa f^nc$ for $n\ge 3$ and by differentiation (writing $\xi'=\frac{d}{d\kappa}\xi_\kappa|_{\kappa=0}$):

$$\sum_{k=1}^{n}[\prod_{\ell=k}^{n-1}f'(f^\ell c)]X(f^kc)=\xi'(f^nc)$$

or

$$\sum_{k=1}^{n}[\prod_{\ell=1}^{k-1}f'(f^\ell c)]^{-1}X(f^kc)=[\prod_{\ell=1}^{n-1}f'(f^\ell c)]^{-1}\xi'(f^nc)$$

and letting $n\to\infty$:

$$\sum_{k=1}^{\infty}[\prod_{\ell=1}^{k-1}f'(f^\ell c)]^{-1}X(f^kc)=0\qquad\text{or}\qquad\sum_{n=0}^{\infty}[\prod_{k=0}^{n-1}f'(f^kb)]^{-1}X(f^nb)=0$$

This is the horizontality condition derived much more generally in [1].

The proof of the theorem will be based on Appendices A, B, C, and use particularly the notation of Appendix C. We write $\Phi^0_{f_\kappa} = \Phi^0_\kappa$ and recall that the expression

$$\langle\Phi^0_\kappa, A\rangle_\kappa = \int dx\,(w_\kappa\Phi^0_\kappa)(x)A(x) = \sum_\alpha\int_{V_{\kappa\alpha}}\phi^0_{\kappa\alpha}A(x)dx + \sum_n c^0_{\kappa n}\int\psi_{\kappa n}(x)A(x)dx$$

depends explicitly on the intervals $V_{\kappa\alpha}$ and the points $f^k_\kappa c$ for $k \geq 1$. We shall first prove the existence of $\frac{d}{d\kappa}\langle\Phi^0_\kappa, A\rangle_\kappa|_{\kappa=0} = \lim_{\kappa\to 0}\frac{1}{\kappa}\int(w_\kappa\Phi^0_\kappa - w\Phi^0)A$ and give an expression involving only the intervals V_α and the points $f^k c$ (corresponding to $\kappa = 0$). Then we shall transform the expression obtained to the form $\Psi(X,1)$.

Since the map $\xi_\kappa : H \to H_\kappa$ depends smoothly on κ (in particular $f'_\kappa(f^k_\kappa b_\kappa) = f'_\kappa(\xi_\kappa f^k b)$ is continuous uniformly in k), it is easily seen that the operator $\mathcal{L}^\times_\kappa$ defined in Appendix C now depends continuously and even differentiably on κ.

We may write

$$\langle\Phi^0_\kappa, A\rangle_\kappa = \sum_\alpha\int_{V_{\kappa\alpha}}\phi^0_{\kappa\alpha}(x)A(x)\,dx + \sum_n c^0_{\kappa n}\int\psi_{\kappa n}(x)A(x)\,dx$$

$$= \langle((\phi^0_{\kappa\alpha}),(c^0_{\kappa n})),((A|V_{\kappa\alpha}),A)\rangle_\kappa$$

$$= \langle\Phi^0_\kappa,((A|V_{\kappa\alpha}),0)\rangle_\kappa + \langle\Phi^0_\kappa,(0,(c^0_{\kappa n}))\rangle_\kappa$$

For notational simplicity we study the derivative of this quantity at $\kappa = 0$ but the proof will show that the derivative depends continuously on κ. We have

$$\frac{1}{\kappa}\Big[\langle\Phi^0_\kappa, A\rangle_\kappa - \langle\Phi^0, A\rangle\Big] = I + II$$

where

$$II = \frac{1}{\kappa}\sum_n\int[c^0_{\kappa n}\psi_{\kappa n}(x) - c^0_n\psi_n(x)]A(x)\,dx$$

$$\to \sum_n\int[\frac{dc^0_{\kappa n}}{d\kappa}\psi_n(x) + c^0_n\frac{d}{d\kappa}\psi_{\kappa n}(x)]A(x)\,dx\Big|_{\kappa=0}$$

[$\frac{d}{d\kappa}\psi_{\kappa n}$ is a distribution with singular part $\frac{d}{d\kappa}|x - f^n_\kappa b_\kappa|^{-1/2}$; integrating by part over x, and using $f^n_\kappa b_\kappa = \xi_\kappa f^n b$ for $k \geq 2$, we see that the right-hand side makes sense, and is the limit of the left-hand side when $\kappa \to 0$].

We also have

$$\langle\Phi^0_\kappa,((A|V_{\kappa\alpha}),0)\rangle_\kappa = \langle\Phi^\times_\kappa,((A_{\kappa\alpha}),0)\rangle$$

where $A_{\kappa\alpha} = (A|V_{\kappa\alpha})\circ\tilde\eta^{-1}_{\kappa\alpha}$, so that

$$I = \langle\frac{\Phi^\times_\kappa - \Phi^\times_0}{\kappa},((A_{\kappa\alpha}),0)\rangle + \langle\Phi^\times_0,((\frac{A_{\kappa\alpha} - A_{0\alpha}}{\kappa}),0)\rangle$$

and the second term is readily seen to tend to a limit when $\kappa \to 0$. In the first term remember that for $\kappa = 0$ we have $\Phi_\kappa^\times = \Phi_0^\times = \Phi^0$, and $\mathcal{L}_\kappa^\times = \mathcal{L}_0^\times = \mathcal{L}$. Also

$$(\mathbf{1} - \mathcal{L})(\Phi_\kappa^\times - \Phi_0^\times) = (\mathcal{L}_\kappa^\times - \mathcal{L}_0^\times)\Phi_\kappa^\times$$

hence

$$\Phi_\kappa^\times - \Phi_0^\times = (\mathbf{1} - \mathcal{L})^{-1}(\mathcal{L}_\kappa^\times - \mathcal{L}_0^\times)\Phi_\kappa^\times$$

Since $(\mathbf{1} - \mathcal{L})^{-1}$ is bounded and $\kappa \mapsto \mathcal{L}_\kappa^\times$ differentiable, we have

$$\langle \frac{\Phi_\kappa^\times - \Phi_0^\times}{\kappa}, ((A_{\kappa\alpha}), 0)\rangle \to \langle (\mathbf{1} - \mathcal{L})^{-1}(\frac{d}{d\kappa}\mathcal{L}_\kappa^\times\Big|_{\kappa=0})\Phi^0, ((A_{0\alpha}), 0)\rangle$$

when $\kappa \to 0$, proving that $\kappa \mapsto \langle \Phi_\kappa^0, A\rangle$ is differentiable.

If we replace in the above calculation the Banach space $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ by $\mathcal{A}' = \mathcal{A}'_1 \oplus \mathcal{A}'_2$ as in Appendix A, we obtain an expression of $\frac{d}{d\kappa}\langle \Phi_\kappa^0, A\rangle_\kappa|_{\kappa=0}$ that can be re-expressed in terms of the ψ'_n, ψ_n and an element of $\mathcal{A}_1$. We may thus write

$$\frac{d}{d\kappa}\langle \Phi_\kappa^0, A\rangle_\kappa\Big|_{\kappa=0} = \langle \tilde\Phi, A\rangle^\sim$$

where $\tilde\Phi \in \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$. The part $\tilde\Phi_3$ of $\tilde\Phi$ in $\mathcal{A}_3$ is uniquely determined by $A \mapsto \langle \tilde\Phi, A\rangle^\sim$; the calculation of II above shows that n-th component of $\tilde\Phi_3$ is

$$-\frac{d}{d\kappa}f_\kappa^n b_\kappa\Big|_{\kappa=0} c_n^0 = -\frac{d}{d\kappa}f_\kappa^{n+1} c\Big|_{\kappa=0} c_n^0$$

$$= -\sum_{k=1}^{n+1} X(f^k c)\Big(\prod_{\ell=k}^{n} f'(f^\ell c)\Big)c_n^0 = -\sum_{k=0}^{n} X(f^k b)\Big(\prod_{\ell=k}^{n-1} f'(f^\ell b)\Big)c_n^0$$

and as a result

$$(\mathbf{1} - \mathcal{L}_7)\tilde\Phi_3 = (-X(f^n b)C_n^0)_{n\ge 0}$$

$$\tilde\Phi_3 = (\mathbf{1} - \mathcal{L}_7)_L^{-1}(-X(f^n b)C_n^0)_{n\ge 0}$$

The part Φ^* of $\tilde\Phi$ in $\mathcal{A}_1 \oplus \mathcal{A}_2$ is not uniquely determined (because of the ambiguity discussed in Appendix B); this part satisfies $\int w\Phi^* = 0$.

If $\mathcal{L}_{(1)\kappa}$ is the transfer operator corresponding to f_κ, we have $\mathcal{L}_{(1)\kappa} w_\kappa \Phi_\kappa^0 = w_\kappa \Phi_\kappa^0$, hence

$$(\mathbf{1} - \mathcal{L}_{(1)})(w_\kappa \Phi_\kappa^0 - w\Phi^0) = (\mathcal{L}_{(1)\kappa} - \mathcal{L}_{(1)})w_\kappa \Phi_\kappa^0$$

Therefore (using the fact that we may let $\mathcal{L}_{(1)}$ act on A) we have

$$\langle (\mathbf{1} - \mathcal{L}^\sim)\tilde\Phi, A\rangle^\sim = \lim_{\kappa\to 0}\int A\frac{1}{\kappa}(\mathbf{1} - \mathcal{L}_{(1)})(w_\kappa \Phi_\kappa^0 - w\Phi^0)$$

$$= \lim_{\kappa\to0}\int A\frac{1}{\kappa}(\mathcal{L}_{(1)\kappa}-\mathcal{L}_{(1)})w_\kappa\Phi^0_\kappa = \lim_{\kappa\to0}\int A\frac{1}{\kappa}(\mathrm{id}^*-h^*_{-\kappa})w_\kappa\Phi^0_\kappa = \lim_{\kappa\to0}\int A\frac{1}{\kappa}(h^*_\kappa-\mathrm{id}^*)w\Phi^0$$

where h^* denotes the direct image of a measure (here a L^1 function) under h, and the last equality uses the existence of a continuous derivative for $\kappa\mapsto\langle\Phi^0_\kappa,A\rangle_\kappa$. According to Appendix A we may write $w\Phi^0$ as a sum of terms $C_n^{(0)}\psi_n^{(0)}$, $C_n^{(1)}\psi_n^{(1)}$, and a differentiable background. Corresponding to this we may identify $\lim_{\kappa\to0}\frac{1}{\kappa}(h^*_\kappa-\mathrm{id}^*)\Phi^0$ with a naturally defined element $\mathcal{D}(-X\Phi^0)$ of $\mathcal{A}_1\oplus\mathcal{A}_2\oplus\mathcal{A}_3$, where $\mathcal{D}$ denotes differentiation. We write $\mathcal{D}(-X\Phi^0)=(D^*,D_3)$ with $D^*\in\mathcal{A}_1\oplus\mathcal{A}_2, D_3\in\mathcal{A}_3$. Since the coefficient of ψ'_n in $\mathcal{D}(-X\Phi^0)$ is $-X(f^nb)c^0_n$, we have $D_3=(\mathbf{1}-\mathcal{L}_7)\tilde\Phi_3$. With $\tilde\Phi=(\Phi^*,\tilde\Phi_3)$ we have thus

$$\langle(\mathbf{1}-\mathcal{L}^\sim)(\Phi^*,\tilde\Phi_3),A\rangle^\sim=\langle\mathcal{D}(-X\Phi^0),A\rangle^\sim$$

and

$$\langle(\mathbf{1}-\mathcal{L})\Phi^*,A\rangle=\langle\mathcal{D}(-X\Phi^0)-(\mathbf{1}-\mathcal{L}^\sim)(0,\tilde\Phi_3),A\rangle$$

In particular $\int w[\mathcal{D}(-X\Phi^0)-(\mathbf{1}-\mathcal{L}^\sim)(0,\tilde\Phi_3)]=0$ and we may define

$$\Phi=(\mathbf{1}-\mathcal{L})^{-1}[\mathcal{D}(-X\Phi^0)-(\mathbf{1}-\mathcal{L}^\sim)(0,\tilde\Phi_3)]\in\mathcal{A}$$

We have then $\langle(\mathbf{1}-\mathcal{L})(\Phi^*-\Phi),A\rangle=0$, hence

$$w(\mathbf{1}-\mathcal{L})(\Phi^*-\Phi)=0$$

hence

$$w(\Phi^*-\Phi)=\mathcal{L}_{(1)}w(\Phi^*-\Phi)$$

with $\int w(\Phi^*-\Phi)=0$, so that $w(\Phi^*-\Phi)=0$, and

$$\langle\Phi^*,A\rangle=\langle\Phi,A\rangle=\langle(\mathbf{1}-\mathcal{L})^{-1}[\mathcal{D}(-X\Phi^0)-(\mathbf{1}-\mathcal{L}^\sim)(0,\tilde\Phi_3)],A\rangle$$

$$=\langle(\mathbf{1}-\mathcal{L})^{-1}[D^*+\mathcal{L}_5+\mathcal{L}_6)\tilde\Phi_3],A\rangle=\langle(\mathbf{1}-\mathcal{L})^{-1}[D^*+\mathcal{L}_5+\mathcal{L}_6)(\mathbf{1}-\mathcal{L}_7)_L^{-1}D_3],A\rangle$$

$$=\langle(\mathbf{1}-\mathcal{L}^\sim)_L^{-1}(D^*,D_3),A\rangle-\langle(\mathbf{1}-\mathcal{L}_7)_L^{-1}D_3,A\rangle=\Psi(X,1)-\langle(0,\tilde\Phi_3),A\rangle^\sim$$

so that finally

$$\frac{d}{d\kappa}\langle\Phi^0_\kappa,A\rangle_\kappa|_{\kappa=0}=\langle\tilde\Phi,A\rangle^\sim=\Psi(X,1)$$

as announced. □

Note that in [5], Baladi and Smania study (in the case of piecewise expanding maps) the more difficult problem of differentiability in horizontal directions (*i.e.*, directions tangent to a topological class). It appears likely that this could be done here also (as conjectured in [5]) , but we have not tried to do so.

20 Discussion.

The codimension 1 condition $X(b) + F_0(X) = 0$ for $\lambda = 1$ expresses that X is a *horizontal* perturbation, which means that it is tangent to a topological class of unimodal maps (see [1] and references given there). In our case, a family (f_κ) is in a topological conjugacy class if $f_\kappa^3 c_\kappa = \xi_\kappa f^3 c$ in the notation of Appendix C. The informal result obtained in Section 17 and the formal proof of differentiability along a topological conjugacy class given by Theorem 19 support the conjecture by Baladi and Smania [5] that the map $f \mapsto \langle \Phi_f^0, A\rangle$ is differentiable (in the sense of Whitney) in horizontal directions, *i.e.*, along a curve tangent to a topological conjugacy class. Our theorem 19 also relates the derivative along a topological conjugacy class to a naturally defined susceptibility function. It seems unlikely that a derivative (in the sense of Whitney) exists in nonhorizontal directions. Note however that if $f \mapsto \langle \Phi_f^0, A\rangle$ is nondifferentiable, it will be in a mild way: the "nondifferentiable" contribution to $\Psi(\lambda)$ is, as we saw above, proportional to

$$\frac{d}{db}\langle (1-\lambda\mathcal{L})^{-1}\psi_{(b)}, A\rangle \sim \sum_n \lambda^n \frac{d}{db}\langle \mathcal{L}^n\psi_{(b)}, A\rangle$$

where $\langle \mathcal{L}^n\psi_{(b)}, A\rangle$ decreases exponentially with n, while $\frac{d}{db}\langle \mathcal{L}^n\psi_{(b)}, A\rangle$ increases exponentially. Therefore, if one does not see the small scale fluctuations of $b \mapsto \langle (1-\lambda\mathcal{L})^{-1}\psi_{(b)}, A\rangle$, this function will seem differentiable. But the singularities with respect to λ (with $|\lambda| < 1$) may remain visible. In conclusion, a physicist may see singularities with respect to λ of a derivative (with respect to f or b) while this derivative may not exist for a mathematician.

A Appendix (proof of Remark 16(a)).

We return to the analysis in Section 10, and note that by an analytic change of variable $x \mapsto \xi(x)$ we can get $y = fx = b - \xi^2$ [we have indeed $b - y = A(x-c)^2(1+\beta(x).(x-c))$ with β analytic, and we can take $\xi = (x-c)A^{1/2}(1+\beta(x).(x-c))^{\frac{1}{2}}$]. Write $\rho(x)\,dx = \tilde{\rho}(\xi)\,d\xi$ (where $\tilde{\rho}$ is analytic near 0). The density of the image $\delta(y)\,dy$ by f of $\rho(x)\,dx = \tilde{\rho}(\xi)\,d\xi$ is, near b,

$$\delta(y) = \frac{1}{2\sqrt{y-b}}(\tilde{\rho}(\sqrt{y-b}) + \tilde{\rho}(-\sqrt{y-b})) = \frac{\hat{\rho}(y-b)}{\sqrt{y-b}}$$

where $\hat{\rho}$ is analytic near 0. Therefore, near b,

$$\delta(x) = \frac{U}{\sqrt{b-x}} + U'\sqrt{b-x} + \ldots$$

where $U = \rho(c)/\sqrt{A}$, and U' is linear in $\rho(c), \rho'(c), \rho''(c)$ with coefficients depending on the derivatives of f at c. Near a we find

$$\delta(x) = U|f'(b)|^{-1/2}\frac{1}{\sqrt{x-a}} + (U'|f'(b)|^{-3/2} - \frac{3}{4}Uf''(b)|f'(b)|^{-5/2})\sqrt{x-a}$$

Writing $s_n = -\mathrm{sgn}\prod_{k=0}^{n-1} f'(f^k b)$, $t_n = |\prod_{k=0}^{n-1} f'(f^k b)|^{-1/2}$, we claim that near $f^n b$ we have a singularity given for $s_n(x - f^n b) < 0$ by 0, and for $s_n(x - f^n b) > 0$ by

$$\delta(x) = \frac{Ut_n}{\sqrt{s_n(x-f^n b)}} + (U't_n^3 - \frac{3}{4}Ut_n\sum_{k=0}^{n-1} s_{k+1}\frac{f''(f^k b)}{|f'(f^k b)|}\frac{t_n^2}{t_k^2})\sqrt{s_n(x-f^n b)}$$

[to prove this we use induction on n, and the fact that, when $f : x \mapsto y$ for x close to $f^n b$ we have:

$$s_n(x - f^n b) = \frac{s_{n+1}(y - f^{n+1}b)}{|f'(f^n b)|}[1 - \frac{f''(f^n b)}{2|f'(f^n b)|^2}(y - f^{n+1}b)]$$

$$dx = \frac{dy}{|f'(f^n b)|}[1 - \frac{f''(f^n b)}{|f'(f^n b)|^2}(y - f^{n+1}b)] \qquad].$$

Define now

$$\psi_n^{(0)}(x) = \big(1 - (\frac{x - f^n b}{w_n - f^n b})^2\big)\frac{\theta_n(x)}{\sqrt{s_n(x - f^n b)}}$$

$$\psi_n^{(1)}(x) = \big(1 - (\frac{x - f^n b}{w_n - f^n b})^2\big)\theta_n(x)\sqrt{s_n(x - f^n b)}$$

for $s_n(x - f^n b) > 0$, 0 otherwise. Then, the expected singularity of δ near $f^n b$ is given by

$$Ut_n\psi_n^{(0)} + (U't_n^3 - \frac{3}{4}Ut_n\sum_{k=0}^{n-1} s_{k+1}\frac{f''(f^k b)}{|f'(f^k b)|}\frac{t_n^2}{t_k^2})\psi_n^{(1)} = C_n^{(0)}\psi_n^{(0)} + C_n^{(1)}\psi_n^{(1)}$$

where $C_0^{(0)} = U$, $C_0^{(1)} = U'$, and

$$C_{n+1}^{(0)} = |f'(f^n b)|^{-1/2} C_n^{(0)}$$

$$C_{n+1}^{(1)} = |f'(f^n b)|^{-3/2} C_n^{(1)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) C_n^{(0)}$$

$$= |f'(f^n b)|^{-3/2} (C_n^{(1)} - \frac{3}{4} s_{n+1} \frac{f''(f^n b)}{|f'(f^n b)|} C_n^{(0)})$$

Let

$$f(\psi_n^{(0)}(x)\, dx) = \tilde{\psi}_{n+1}^{(0)}(x)\, dx \qquad , \qquad f(\psi_n^{(1)}(x)\, dx) = \tilde{\psi}_{n+1}^{(1)}(x)\, dx$$

and write

$$\tilde{\psi}_{n+1}^{(0)} = |f'(f^n b)|^{-1/2} \psi_{n+1}^{(0)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) \psi_{n+1}^{(1)} + \chi_n^{(0)}$$

$$\tilde{\psi}_{n+1}^{(1)} = |f'(f^n b)|^{-3/2} \psi_{n+1}^{(1)} + \chi_n^{(1)}$$

The density of $f(C_n^{(0)} \psi_n^{(0)}(x)\, dx + C_n^{(1)} \psi_n^{(1)}(x)\, dx)$ is then

$$C_{n+1}^{(0)} \psi_{n+1}^{(0)} + C_{n+1}^{(1)} \psi_{n+1}^{(1)} + C_n^{(0)} \chi_n^{(0)} + C_n^{(1)} \chi_n^{(1)}$$

The functions $\chi_n^{(0)}, \chi_n^{(1)}$ have been constructed such that they and their first derivatives $\chi_n^{(0)\prime}, \chi_n^{(1)\prime}$ have the properties of Lemma 11. Namely, $\chi_n^{(0)}, \chi_n^{(1)}, \chi_n^{(0)\prime}, \chi_n^{(1)\prime}$ are continuous with bounded variation on $[a, b]$ uniformly in n, they vanish at a, b, and if $n \geq 1$ they extend to holomorphic functions on the appropriate D_α, with uniform bounds.

Let $\mathcal{A}_1' \subset \mathcal{A}_1$ consist of the (ϕ_α) such that the derivatives ϕ_{-1}', ϕ_{-2}' of ϕ_{-1}, ϕ_{-2} vanish at $\pi_b^{-1} b$ and $\pi_a^{-1} a$ respectively. Let also $\mathcal{A}_2'$ consist of the sequences $(c_n^{(0)}, c_n^{(1)})$, with $c_n^{(0)}, c_n^{(1)} \in \mathbf{C}$, $n = 0, 1, \ldots$ such that

$$||(c_n^{(0)}, c_n^{(1)})||_2' = \sup_{n \geq 0} \delta^n (|c_n^{(0)}| + |c_n^{(1)}|) < \infty$$

If $\Phi' = ((\phi_\alpha), (c_n^{(0)}, c_n^{(1)})) \in \mathcal{A}' = \mathcal{A}_1' \oplus \mathcal{A}_2'$ we let $||\Phi'||' = ||(\phi_\alpha)||_1 + ||(c_n^{(0)}, c_n^{(1)})||_2'$, making $\mathcal{A}'$ into a Banach space. We may now proceed as in Section 12, replacing $\mathcal{A}$ by $\mathcal{A}'$, and defining $\mathcal{L}' : \mathcal{A}' \mapsto \mathcal{A}'$ in a way similar to $\mathcal{L} : \mathcal{A} \mapsto \mathcal{A}$, but with (ii), (v), (vi) replaced as follows:

(ii) $\phi_0 \Rightarrow \Big((\hat{c}_0^{(0)}, \hat{c}_0^{(1)}) = (U, U')\, , \, \hat{\phi}_{-1} = \pm \frac{\phi_0}{|f'|} \circ \tilde{f}_{-1}^{-1} - U(\pm \frac{1}{2} \psi_0^{(0)} \circ \pi_b) - U'(\pm \frac{1}{2} \psi_0^{(1)} \circ \pi_b)\Big)$ so that $\hat{\phi}_{-1}$ is holomorphic in $\pi_b^{-1} D_{-1}$ with vanishing derivative at $\pi_b^{-1} b$

(v) $(c_0^{(0)}, c_0^{(1)}) \Rightarrow \Big((\hat{c}_1^{(0)}, \hat{c}_1^{(1)}) = (|f'(b)|^{-1/2} c_0^{(0)}, |f'(b)|^{-3/2} c_0^{(1)} - \frac{3}{4} |f'(b)|^{-5/2} f''(b) c_0^{(0)}),$
$\chi_0 = \pm \frac{1}{2} c_0^{(0)} (\frac{\psi_0^{(0)}}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-1/2} \psi_1^{(0)} \circ \pi_a + \frac{3}{4} |f'(b)|^{-5/2} f''(b) \psi_1^{(1)} \circ \pi_a)$
$\pm \frac{1}{2} c_0^{(1)} (\frac{\psi_0^{(1)}}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-3/2} \psi_1^{(1)} \circ \pi_a)$ in $\pi_a^{-1} D_{-2}\Big)$

(vi) $(c_n^{(0)}, c_n^{(1)}) \Rightarrow$
$\Big((\hat{c}_{n+1}^{(0)}, \hat{c}_{n+1}^{(1)}) = (|f'(f^n b)|^{-1/2} c_n^{(0)}, |f'(f^n b)|^{-3/2} c_n^{(1)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) c_n^{(0)}),$

$\chi_{n\alpha} = c_n^{(0)} [\frac{\psi_n^{(0)}}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-1/2} \psi_{n+1}^{(0)} + \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) |\psi_{n+1}^{(1)}]$
$+ c_n^{(1)} [\frac{\psi_n^{(1)}}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-3/2} \psi_{n+1}^{(1)}]$ in D_α if $V_\alpha \subset \{x : \theta_n(f_n^{-1} x) > 0\}$, 0 otherwise$\Big)$
if $n \geq 1$.
We write then

$$\mathcal{L}' \Phi' = \tilde{\Phi}' = ((\tilde{\phi}_\alpha), (\tilde{c}_n^{(0)}, \tilde{c}_n^{(1)}))$$

where

$$\tilde{\phi}_{-2} = \hat{\phi}_{-2} + \chi_0 \quad , \quad \tilde{\phi}_{-1} = \hat{\phi}_{-1}$$

$$\tilde{\phi}_\alpha = \sum_{\beta : f V_\beta = V_\alpha} \hat{\phi}_{\beta\alpha} + \hat{\phi}_\alpha + \sum_{n \geq 1} \chi_{n\alpha} \qquad \text{if order } \alpha \geq 0$$

$$(\tilde{c}_n^{(0)}, \tilde{c}_n^{(1)}) = (\hat{c}_n^{(0)}, \hat{c}_n^{(1)}) \qquad \text{for } n \geq 0$$

With the above definitions and assumptions we find, by analogy with Theorem 13, that $\mathcal{L}' : \mathcal{A}' \to \mathcal{A}'$ has essential spectral radius $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$. There is (see Proposition 15) a simple eigenvalue 1, and the rest of the spectrum has radius < 1. It is convenient to denote by $\Phi^0 = ((\phi_\alpha^0), (c_n^{0(0)}, c_n^{0(1)}))$ the eigenfunction to the eigenvalue 1. We find again that $\phi^0 = \Delta(\phi_\alpha^0)$ is continuous, of bounded variation, and satisfies $\phi^0(a) = \phi^0(b) = 0$, but we can say more. Using the notation in the proof of Proposition 15, we have again

$$\gamma_j^0 = \sum_k \mathcal{L}_{jk} \gamma_k^0 + \eta_j$$

with $\eta_j = \sum_{n=0}^\infty \eta_{jn}$, but now $\eta_{jn} = c_n^{0(0)} \chi_n^{(0)} + c_n^{0(1)} \chi_n^{(1)} | W_j$ for $n \geq 1$, so that the η_j have derivatives $\eta_j' \in \mathcal{H}_j$. The derivatives $\gamma_j^{0\prime}$of the γ_j^0 are measures satisfying

$$\gamma_j^{0\prime} = \sum_k \mathcal{L}_{jk}' \gamma_k^{0\prime} + \eta_j^*$$

The operator $\mathcal{L}_{jk}'$ has the same form as $\mathcal{L}_{jk}$, but with an extra denominator $f' \circ (f^{-1})_{kj}$, and therefore $\mathcal{L}_*' = (\mathcal{L}_{jk}')$ acting on measures has spectral radius $\leq \alpha < 1$. The term η_j^* is the sum of η_j' and a term$\sum_k \mathcal{L}_{kj}^\sim \gamma_k^0$ where $\mathcal{L}_{kj}^\sim$ involves the derivative of $|f' \circ (f^{-1})_{kj}|^{-1}$ so that $\eta_j^* \in \mathcal{H}_j$. The operator $\mathcal{L}_*'$ also maps $\mathcal{H}$ to $\mathcal{H}$ and, by the same argument as for $\mathcal{L}_*$, has essential spectral radius < 1 on $\mathcal{H}$. Furthermore, 1 cannot be an eigenvalue since $\mathcal{L}_*'$ has spectral radius < 1 on measures. It follows that $(\gamma^{0\prime}) = (\gamma_j^{0\prime}) = (1 - \mathcal{L}_*')^{-1} (\eta_j^*) \in \mathcal{H}$. Therefore, the derivative $\phi^{0\prime}$ of ϕ^0 may have discontinuities only on the orbit of u_1, and hyperbolicity again shows that this cannot happen. In conclusion, ϕ^0 and its derivative $\phi^{0\prime}$ are both of bounded variation, continuous, and vanishing at a, b.

A discussion similar to the above shows that the equation $\gamma = (1 - \mathcal{L}_*')^{-1} \eta^*$ also defines γ with finite norm in $\mathcal{A}_1$, and this γ must coincide with $(\gamma^{0\prime})$ as a measure. Therefore the family of derivatives $(\phi_\alpha^{0\prime})$ is an element of $\mathcal{A}_1$. [For simplicity, we have written $\phi_{-1}^{0\prime}$, $\phi_{-2}^{0\prime}$ for the functions which, under application of Δ, give the derivative of $\Delta\phi_{-1}^0$, $\Delta\phi_{-2}^0$]. $\square$

B Appendix (proof of Remark 16(b)).

If $u \in \tilde{H}$ and $\psi_{(u\pm)}$ is defined as in Remark 16(b), we want to show that there is a unique (ϕ_α) in $\mathcal{A}_1$ such that $\phi_\alpha = \psi_{(u\pm)}|V_\alpha$ for all α. Furthermore $||(\phi_\alpha)||_1$ is bounded uniformly for $u \in \tilde{H}$, provided we assume $1 < \gamma < \min(\beta^{-1}, \alpha^{-1/2})$.

Note that uniqueness is automatic, and that $\phi_\alpha = 0$ unless order $V_\alpha > 0$. Omitting the $\pm$ we let

$$f(\psi_{(f^n u)}(x)\,dx) = [|f'(f^n u)|^{-1/2}\psi_{(f^{n+1}u)}(x) + \chi_{(f^n u)}(x)]\,dx$$

For $n \geq 0$ there is a unique ω_{un} such that $f^{n+1}(\omega_{un}\,dx) = \prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2}\chi_{(f^n u)}\,dx$ and $[f^k u - c] \times [\mathrm{supp}\, f^k(\omega_{un}(x)\,dx) - c] > 0$ for $0 \leq k \leq n$. Furthermore $\psi_{(u)} = \sum_{n=0}^{\infty} \omega_{un}$ where the sum restricted to each V_α is finite. If $[\chi_{(f^n u)}]$ denotes the element of $\mathcal{A}_1$ corresponding to $\chi_{(f^n u)}$, we find that $||[\chi_{(f^n u)}]||_1$ is bounded uniformly in n and u. Also note that we obtain ω_{un} from $\prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2}\chi_{(f^n u)}$ by multiplying with $\prod_{k=0}^{n-1} |f'(f^k u)|$ (up to a factor bounded uniformly in n because of hyperbolicity) and composing with f^{n+1} (restricted to a small interval J such that $f^{n+1}|J$ is invertible). We have thus

$$||[\omega_{un}]||_1 \leq \mathrm{const}\, \gamma^n \prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2}$$

where $[\omega_{un}]$ is the element of $\mathcal{A}_1$ corresponding to ω_{un} [This is because the replacement of $|V_\alpha|$ by $|(f|J)^{-n-1}V_\alpha|$ in the definition of $||\cdot||_1$ is compensated up to a multiplicative constant by the factor $\prod_{k=0}^{n-1} |f'(f^k u)|$]. Thus

$$||[\omega_{un}]||_1 \leq \mathrm{const}\, (\gamma \alpha^{1/2})^n$$

Since $\gamma < \alpha^{-1/2}$ we find that $\sum_n ||[\omega_{un}]||_1 <$ constant independent of u. Therefore, since $(\phi_\alpha) = \sum_n [\omega_{un}]$, we see that $||(\phi_\alpha)||_1$ is bounded independently of u. $\square$

C Appendix (proof of Remark 16(c)).

We consider a one-parameter family (f_κ) of maps, reducing to $f = f_0$ for $\kappa = 0$. We assume that $(\kappa, x) \mapsto f_\kappa x$ is real-analytic. For κ close to 0, f_κ has a critical point c_κ and maps $[a_\kappa, b_\kappa]$ to itself, with $b_\kappa = f_\kappa c_\kappa, a_\kappa = f_\kappa^2 c_\kappa$. There is (by hyperbolicity of H with respect to f) a homeomorphism $\xi_\kappa : H \to H_\kappa$ where H_κ is an f_κ-invariant hyperbolic set for f_κ and $f_\kappa \circ \xi_\kappa = \xi_\kappa \circ f$ on H. We shall consider a compact set K of values of κ such that $f_\kappa a_\kappa \in \tilde{H}_\kappa$; we let $K \ni 0$, K of small diameter, and assume now $\kappa \in K$. We may in a natural way define a Banach space $\mathcal{A}_\kappa = \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2$ and an operator $\mathcal{L}_\kappa : \mathcal{A}_\kappa \to \mathcal{A}_\kappa$ associated with f_κ so that $\mathcal{A}_\kappa, \mathcal{L}_\kappa$ reduce to $\mathcal{A}, \mathcal{L}$ for $\kappa = 0$. Note that, since $\kappa \in K$ is close to 0, we may assume that the constants A, α in the definition (Section 4) of hyperbolicity, and the constants B, β (Section 7) are uniform in κ.

Let $\eta_{\kappa,-2}$ be a biholomorphic map of the complex neighborhood D_{-2} of $[a, u_1]$ to the complex neighborhood $D_{\kappa,-2}$ of the corresponding interval $[a_\kappa, u_{\kappa 1}]$, and lift $\eta_{\kappa,-2}$ to a holomorphic map $\tilde{\eta}_{\kappa,-2} : \pi_a^{-1} D_{-2} \to \pi_{a_\kappa}^{-1} D_{\kappa,-2}$. We also lift $\eta_{\kappa,-1} = f_\kappa^{-1} \circ \eta_{\kappa,-2} \circ f$ to

$$\tilde{\eta}_{\kappa,-1} = \tilde{f}_{\kappa,-2}^{-1} \circ \eta_{\kappa,-2} \circ \tilde{f}$$

where the notation is that of Section 12, with obvious modification. We write

$$\tilde{\eta}_{\kappa 0} = \tilde{f}_{\kappa,-1}^{-1} \circ \tilde{\eta}_{\kappa,-1} \circ \tilde{f}_{-1}$$

and

$$\tilde{\eta}_{\kappa\beta} = (f_\kappa | V_{\kappa\beta})^{-1} \circ \tilde{\eta}_{\kappa\alpha} \circ f | V_\beta$$

if order $\beta > 0$ and $f V_\beta = V_\alpha$. We have defined $\eta_{\kappa\alpha}$ above for $\alpha = -1, -2$, and we let $\eta_{\kappa\alpha} = \tilde{\eta}_{\kappa\alpha}$ when order $\alpha \geq 0$.

We introduce a map $\eta_\kappa : \mathcal{A}_{\kappa 1} \to \mathcal{A}_1$ by

$$\eta_\kappa(\phi_{\kappa\alpha}) = ((\phi_{\kappa\alpha} \circ \tilde{\eta}_{\kappa\alpha}).\eta'_{\kappa\alpha})$$

so that $\mathcal{L}_\kappa^\times = (\eta_\kappa, \mathbf{1}) \mathcal{L}_\kappa (\eta_\kappa^{-1}, \mathbf{1})$ acts on $\mathcal{A}$. Using the decomposition

$$\mathcal{L}_\kappa = \begin{pmatrix} \mathcal{L}_{\kappa 0} + \mathcal{L}_{\kappa 1} & \mathcal{L}_{\kappa 2} \\ \mathcal{L}_{\kappa 3} & \mathcal{L}_{\kappa 4} \end{pmatrix}$$

as in Section 12, we define $L_\kappa^\times$ on $\mathcal{A}_1$ by

$$L_\kappa^\times(\phi_\alpha) = \eta_\kappa(\mathcal{L}_{\kappa 0} + \mathcal{L}_{\kappa 1}) \eta_\kappa^{-1}(\phi_\alpha) + (\eta_\kappa^{-1}\phi_\alpha)_0(c_\kappa).\eta_\kappa \mathcal{L}_{\kappa 2}(|\frac{1}{2} f''_\kappa(c_\kappa) \prod_{k=0}^{n-1} f'_\kappa(f_\kappa^k b_k)|^{-1/2})$$

$$= \mathcal{L}_0(\phi_\alpha) + \eta_\kappa \mathcal{L}_{\kappa 1} \eta_\kappa^{-1}(\phi_\kappa) + \eta'_{\kappa 0}(c_\kappa)^{-1} \phi_0(c_\kappa).\eta_\kappa \mathcal{L}_{\kappa 2}(|\frac{1}{2} f''_\kappa(c_\kappa) \prod_{k=0}^{n-1} f'_\kappa(f_\kappa^k b_k)|^{-1/2})$$

$L^\times_\kappa$ is a compact perturbation of $\mathcal{L}_{\kappa 0}$, and has therefore essential spectral radius $\leq \gamma^{-1}$. If (ϕ_α) is a (generalized) eigenfunction of $L^\times_\kappa$ to the eigenvalue μ, then

$$((\phi_\alpha), \eta_{\kappa 0}(c_\kappa)^{-1}\phi_0(c_\kappa).(|\frac{1}{2}f''_\kappa(c_\kappa)\prod_{k=0}^{n-1} f'_\kappa(f^k_\kappa b_k)|^{-1/2}))$$

is a (generalized) eigenfunction of $\mathcal{L}^\times_\kappa$ to the same eigenvalue μ. We have thus a multiplicity-preserving bijection of the eigenvalues μ of $L^\times_\kappa$ and $\mathcal{L}^\times_\kappa$ when $|\mu| > \max(\gamma^{-1}, \delta\alpha^{1/2})$. In particular, 1 is a simple eigenvalue of $L^\times_\kappa$ for the values of κ considered (a compact neighborhood K of 0).

The operator $L^\times_\kappa$ acting on $\mathcal{A}_1$ depends continuously on κ. [This is because $\hat{\phi}_{\kappa\alpha}$, $\chi_{\kappa 0}$, $\chi_{\kappa n\alpha}$ depend continuously on κ (in particular, the $\chi_{\kappa n\alpha}$ for large n are uniformly small). Note however that $\mathcal{L}^\times_\kappa$ does not depend continuously on κ because the continuity of $f'_\kappa(f^k_\kappa b_\kappa)$ is not uniform in k]. There is $\epsilon > 0$ such that $L^\times_\kappa$ has no eigenvalue μ_κ with $|\mu_\kappa - 1| < \epsilon$ except the simple eigenvalue 1 [otherwise the continuity of $\kappa \to L^\times_\kappa$ would imply that 1 has multiplicity > 1 for some κ]. Therefore, the 1-dimensional projection corresponding to the eigenvalue 1 of $L^\times_\kappa$ depends continuously on κ, and so does the eigenvector $\Phi^\times_\kappa = (\eta_\kappa, 1)\Phi^0_\kappa$ of $\mathcal{L}^\times_\kappa$, where Φ^0_κ denotes the eigenvector the the eigenvalue 1 of $\mathcal{L}_\kappa$ normalized so that $w_\kappa\Phi^0_\kappa \geq 0$ and $\int w_\kappa\Phi^0_\kappa = 1$, with the obvious definition of w_κ (involving the spikes $\psi_{\kappa n}$ associated with f_κ).

Note that a number of results have been obtained earlier on the continuous dependence of the a.c.i.m. ρ on parameters. I am indebted to Viviane Baladi for communicating the references [25], [27], [15], and also [26].

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