ON THE EQUIVALENCE OF DIFFERENT LAX PAIRS FOR THE KAC–VAN MOERBEKE HIERARCHY

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Abstract. We give a simple algebraic proof that the two different Lax pairs for the Kac–van Moerbeke hierarchy, constructed from Jacobi respectively super-symmetric Dirac-type difference operators, give rise to the same hierarchy of evolution equations. As a byproduct we obtain some new recursions for computing these equations.

1. Introduction

There are two different Lax equations for the Kac–van Moerbeke equation: The original one of Kac and van Moerbeke [4] based on a Jacobi matrix with zero diagonal elements and its skew-symmetrized square and the second one based on super-symmetric Dirac-type matrices. Both approaches can be generalized to give corresponding hierarchies of evolution equations in the usual way and both reveal a close connection to the Toda hierarchy. In fact, the first approach shows that the Kac–van Moerbeke hierarchy (KM hierarchy) is contained in the Toda hierarchy by setting $b = 0$ in the odd equations. The second one relates both hierarchies via a Bäcklund transformation since the Dirac-type difference operator gives rise to two Jacobi operators by taking squares (respectively factorizing positive Jacobi operators to obtain the other direction). Both ways of introducing the KM hierarchy have its merits, however, tough it is *obvious* that both produce the same hierarchy by looking at the first few equations, we could not find a formal proof in the literature. The purpose of this short note is to give a simple algebraic proof for this fact. As a byproduct we will also obtain some new recursions for computing the equations in the KM hierarchy.

In Section 2 we review the recursive construction of the Toda hierarchy via Lax pairs involving Jacobi operators and obtain the KM hierarchy by setting $b = 0$ in the odd equations. In Section 3 we introduce the KM hierarchy via Lax pairs involving Dirac-type difference operators. In Section 4 we show that both constructions produce the same equations. Finally, we recall how to identify Jacobi operators with $b = 0$ in Section 5.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism following [2] (see also [7]).

We will only consider bounded solutions and hence require

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Hypothesis H. 2.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad b(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad a(n,t) \neq 0, \qquad (n,t) \in \mathbb{Z} \times \mathbb{R},$$

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t), b(t)$ is a Jacobi operator

$$H(t) = a(t)S^+ + a^-(t)S^- + b(t) \tag{2.1}$$

in $\ell^2(\mathbb{Z})$, where $S^\pm f(n) = f^\pm(n) = f(n \pm 1)$ are the usual shift operators and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. Moreover, choose constants $c_0 = 1$, c_j, $1 \leq j \leq r$, $c_{r+1} = 0$, and set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n \rangle, \\ h_j(n,t) &= 2a(n,t) \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle + c_{j+1}. \end{aligned} \tag{2.2}$$

The sequences g_j, h_j satisfy the recursion relations

$$\begin{aligned} g_0 = 1, \ h_0 &= c_1, \\ 2g_{j+1} - h_j - h_j^- - 2b g_j &= 0, \quad 0 \leq j \leq r, \\ h_{j+1} - h_{j+1}^- - 2(a^2 g_j^+ - (a^-)^2 g_j^-) - b(h_j - h_j^-) &= 0, \quad 0 \leq j < r. \end{aligned} \tag{2.3}$$

Introducing

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t) g_j(t) S^+ - h_j(t)) H(t)^{r-j} + g_{r+1}(t), \tag{2.4}$$

a straightforward computation shows that the Lax equation

$$\frac{d}{dt} H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{2.5}$$

is equivalent to

$$\mathrm{TL}_r(a(t), b(t)) = \begin{pmatrix} \dot{a}(t) - a(t)\Big(g_{r+1}^+(t) - g_{r+1}(t)\Big) \\ \dot{b}(t) - \Big(h_{r+1}(t) - h_{r+1}^-(t)\Big) \end{pmatrix} = 0, \tag{2.6}$$

where the dot denotes a derivative with respect to t. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy $\mathrm{TL}_r(a, b) = 0$. The corresponding homogeneous quantities obtained by taking all summation constants equal to zero, $c_\ell \equiv 0$, $\ell \in \mathbb{N}$, are denoted by $\hat{g}_j$, $\hat{h}_j$, etc., resp.

$$\widehat{\mathrm{TL}}_r(a, b) = \mathrm{TL}_r(a, b)\big|_{c_\ell \equiv 0, 1 \leq \ell \leq r}. \tag{2.7}$$

Next we show that we can set $b \equiv 0$ in the odd equations of the Toda hierarchy.

Lemma 2.2. *Let $b \equiv 0$. Then the homogeneous coefficients satisfy*

$$\hat{g}_{2j+1} = \hat{h}_{2j} = 0, \quad j \in \mathbb{N}_0.$$

Proof. We use induction on the recursion relations (2.3). The claim is true for $j = 0$. If $\hat{h}_{2j} = 0$ then $\hat{g}_{2j+1} = 0$, and $\hat{h}_{2j} = 0$ follows from the last equation in (2.3). □

In particular, if we choose $c_{2\ell} = 0$ in TL_{2r+1}, then we can set $b \equiv 0$ to obtain a hierarchy of evolution equations for a alone. In fact, set

$$G_j = \hat{g}_{2j}, \quad K_j = \hat{h}_{2j+1}, \tag{2.8}$$

in this case. Then they satisfy the recursion

$$\begin{aligned} &G_0 = 1, \quad K_0 = 2a^2, \\ &2G_{j+1} - K_j - K_j^- = 0, \quad 0 \le j \le r, \\ &K_{j+1} - K_{j+1}^- - 2(a^2 G_j^+ - (a^-)^2 G_j^-) = 0, \quad 0 \le j < r, \end{aligned} \tag{2.9}$$

and $\mathrm{TL}_{2r+1}(a, 0) = 0$ is equivalent to the KM hierarchy defined as

$$\mathrm{KM}_r(a) = \dot{a} - a(G_{r+1}^+ - G_{r+1}), \qquad r \in \mathbb{N}_0. \tag{2.10}$$

3. The Kac–van Moerbeke hierarchy as a modified Toda hierarchy

In this section we review the construction of the KM hierarchy as a modified Toda hierarchy. We refer to [2], [7] for further details.

Suppose $\rho(t)$ satisfies

Hypothesis H. 3.1. *Let*

$$\rho(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \quad \rho(n,t) \neq 0, \ (n,t) \in \mathbb{Z} \times \mathbb{R} \tag{3.1}$$

and let $t \mapsto \rho(t)$ be differentiable in $\ell^\infty(\mathbb{Z})$.

Define the "even" and "odd" parts of $\rho(t)$ by

$$\rho_e(n,t) = \rho(2n,t), \ \rho_o(n,t) = \rho(2n+1,t), \quad (n,t) \in \mathbb{Z} \times \mathbb{R}, \tag{3.2}$$

and consider the bounded operators (in $\ell^2(\mathbb{Z})$)

$$A(t) = \rho_o(t) S^+ + \rho_e(t), \ A(t)^* = \rho_o^-(t) S^- + \rho_e(t). \tag{3.3}$$

In addition, we set

$$H_1(t) = A(t)^* A(t), \quad H_2(t) = A(t) A(t)^*, \tag{3.4}$$

with

$$H_k(t) = a_k(t) S^+ + a_k^-(t) S^- + b_k(t), \qquad k = 1, 2, \tag{3.5}$$

and

$$a_1(t) = \rho_e(t)\rho_o(t), \qquad b_1(t) = \rho_e(t)^2 + \rho_o^-(t)^2, \tag{3.6}$$

$$a_2(t) = \rho_e^+(t)\rho_o(t), \qquad b_2(t) = \rho_e(t)^2 + \rho_o(t)^2. \tag{3.7}$$

Now we define operators $D(t)$, $Q_{2r+2}(t)$ in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ as follows,

$$D(t) = \begin{pmatrix} 0 & A(t)^* \\ A(t) & 0 \end{pmatrix}, \tag{3.8}$$

$$Q_{2r+2}(t) = \begin{pmatrix} P_{1,2r+2}(t) & 0 \\ 0 & P_{2,2r+2}(t) \end{pmatrix}, \quad r \in \mathbb{N}_0. \tag{3.9}$$

Here $P_{k,2r+2}(t)$, $k = 1, 2$ are defined as in (2.4), that is,

$$P_{k,2r+2}(t) = -H_k(t)^{r+1} + \sum_{j=0}^{r} (2a_k(t) g_{k,j}(t) S^+ - h_{k,j}(t)) H_k(t)^j + g_{k,r+1}, \tag{3.10}$$

$\{g_{k,j}(n,t)\}_{0\le j\le r}$, $\{h_{k,j}(n,t)\}_{0\le j\le r+1}$ are defined as in (2.2). Moreover, we choose the same integration constants in $P_{1,2r+2}(t)$ and $P_{2,2r+2}(t)$ (i.e., $c_{1,\ell}=c_{2,\ell}\equiv c_\ell$, $1\le \ell\le r$).

Analogous to equation (2.5) one obtains that

$$\frac{d}{dt}D(t)-[Q_{2r+2}(t),D(t)]=0 \tag{3.11}$$

is equivalent to

$$\begin{aligned}\mathrm{KM}_r(\rho) &= (\mathrm{KM}_r(\rho)_e,\ \mathrm{KM}_r(\rho)_o)\\ &= \begin{pmatrix}\dot\rho_e-\rho_e(g_{2,r+1}-g_{1,r+1})\\ \dot\rho_o+\rho_o(g_{2,r+1}-g^+_{1,r+1})\end{pmatrix}=0.\end{aligned} \tag{3.12}$$

As in the Toda context (2.6), varying $r\in\mathbb{N}_0$ yields the KM hierarchy which we denote by

$$\mathrm{KM}_r(\rho)=0,\quad r\in\mathbb{N}_0. \tag{3.13}$$

The homogeneous KM hierarchy is denoted by

$$\widehat{\mathrm{KM}}_r(\rho)=\mathrm{KM}_r(\rho)\big|_{c_\ell\equiv0,1\le\ell\le r}. \tag{3.14}$$

One look at the transformations (3.6), (3.7) verifies that the equations for ρ_o, ρ_e are in fact one equation for ρ. More explicitly, combining $g_{k,j}$, resp. $h_{k,j}$, into one sequence

$$\begin{array}{lcl}G_j(2n)&=&g_{1,j}(n)\\ G_j(2n+1)&=&g_{2,j}(n)\end{array},\ \text{resp.}\ \begin{array}{lcl}H_j(2n)&=&h_{1,j}(n)\\ H_j(2n+1)&=&h_{2,j}(n)\end{array}, \tag{3.15}$$

we can rewrite (3.12) as

$$\mathrm{KM}_r(\rho)=\dot\rho-\rho(G^+_{r+1}-G_{r+1}). \tag{3.16}$$

From (2.3) we see that G_j, H_j satisfy the recursions

$$\begin{aligned}&G_0=1,\ H_0=c_1,\\ &2G_{j+1}-H_j-H_j^{--}-2(\rho^2+(\rho^-)^2)G_j=0,\quad 0\le j\le r,\\ &H_{j+1}-H_{j+1}^{--}-2((\rho\rho^+)^2G_j^+-(\rho^-\rho)^2G_j^{--})\\ &\qquad -(\rho^2+(\rho^-)^2)(H_j-H_j^{--})=0,\quad 0\le j<r.\end{aligned} \tag{3.17}$$

The homogeneous quantities are denoted by $\hat G_j$, $\hat H_j$, etc., as before.

As a simple consequence of (3.11) we have

$$\frac{d}{dt}D(t)^2-[Q_{2r+2}(t),D(t)^2]=0 \tag{3.18}$$

and observing

$$D(t)^2=\begin{pmatrix}H_1(t)&0\\0&H_2(t)\end{pmatrix} \tag{3.19}$$

yields the implication

$$\mathrm{KM}_r(\rho)=0\Rightarrow \mathrm{TL}_r(a_k,b_k)=0,\quad k=1,2, \tag{3.20}$$

that is, given a solution ρ of the KM_r equation (3.13), one obtains two solutions, (a_1,b_1) and (a_2,b_2), of the TL_r equations (2.6) related to each other by the Miura-type transformations (3.6), (3.7). For more information we refer to [3], [6], [7], and [8].

4. Equivalence of both constructions

In this section we want to show that the constructions of the KM hierarchy outlined in the previous two sections yield in fact the same set of evolution equations. This will follow once we show that G_j defined in (2.8) is the same as G_j defined in (3.15). It will be sufficient to consider the homogeneous quantities, however, we will omit the additional hats for notational simplicity. Moreover, we will denote the sequence G_j defined in (2.8) by $\tilde{G}_j$ to distinguish it from the one defined in (3.15). Since both are defined recursively via the recursions (2.9) for $\tilde{G}_j, K_j$ respectively (3.17) for G_j, H_j our first aim is to eliminate the additional sequences K_j respectively H_j and to get a recursion for $\tilde{G}_j$ respectively G_j alone.

Lemma 4.1. *The coefficients $g_j(n)$ satisfy the following linear recursion*

$$\begin{aligned} g_{j+3}^+ - g_{j+3} = {} & (b+2b^+)g_{j+2}^+ - (2b+b^+)g_{j+2} \\ & - (2b+b^+)b^+ g_{j+1}^+ + b(2b^+ + b)g_{j+1} + k_{j+1}^+ + k_{j+1} \\ & + b(b^+)^2 g_j^+ - b^+ b^2 g_j - b k_j^+ - b^+ k_j, \end{aligned} \tag{4.1}$$

where

$$k_j = a^2 g_j^+ - (a^-)^2 g_j^-, \quad j \in \mathbb{N}. \tag{4.2}$$

Proof. It suffices to consider the homogeneous case $g_j(n) = \langle \delta_n, H^j \delta_n \rangle$. Then (compare [7, Sect 6.1])

$$g(z,n) = \langle \delta_n, (H-z)^{-1}\delta_n \rangle = -\sum_{j=0}^{\infty} \frac{g_j(n)}{z^{j+1}}$$

satisfies [7, (1.109)]

$$\frac{(a^+)^2 g^{++} - a^2 g}{z - b^+} + \frac{a^2 g^+ - (a^-)^2 g^-}{z-b} = (z-b^+)g^+ - (z-b)g,$$

and the claim follows after comparing coefficients. □

Corollary 4.2. *For $j \in \mathbb{N}_0$, the sequences $\tilde{G}_j$, defined by* (2.8) *and corresponding to the* TL *hierarchy with $b \equiv 0$, satisfy*

$$\tilde{G}_{j+1}^+ - \tilde{G}_{j+1} = (a^+)^2 \tilde{G}_j^{++} + a^2(\tilde{G}_j^+ - \tilde{G}_j) - (a^-)^2 \tilde{G}_j^-. \tag{4.3}$$

The corresponding sequences G_j for the KM *hierarchy defined in* (3.15) *satisfy*

$$
\begin{aligned}
G_{j+3} - G^{++}_{j+3} = {}& \big((a^-)^2 + a^2\big)^2 \big((a^+)^2 + (a^{++})^2\big) G_j \\
& + (a^{--})^2 (a^-)^2 G^{--}_{j+1} + a^2 (a^+)^2 G_{j+1} \\
& + \big((a^+)^2 + (a^{++})^2\big)\big(2(a^-)^2 + 2a^2 + (a^+)^2 + (a^{++})^2\big) G^{++}_{j+1} \\
& + \big(2(a^-)^2 + 2a^2 + (a^+)^2 + (a^{++})^2\big) G_{j+2} \\
& - \big((a^-)^2 + a^2\big)\big((a^+)^2 + (a^{++})^2\big)^2 G^{++}_j \\
& - \big((a^+)^2 + (a^{++})^2\big)\big((a^{--})^2 (a^-)^2 G^{--}_j - a^2 (a^+)^2 G^{++}_j\big) \\
& - \big((a^-)^2 + a^2\big)\big(a^2 (a^+)^2 G_j - (a^{++})^2 (a^{+++})^2 G^{++++}_j\big) \\
& - \big((a^-)^2 + a^2\big)\big((a^-)^2 + a^2 + 2(a^+)^2 + 2(a^{++})^2\big) G_{j+1} \\
& - a^2 (a^+)^2 G^{++}_{j+1} - (a^{++})^2 (a^{+++})^2 G^{++++}_{j+1} \\
& - \big((a^-)^2 + a^2 + 2(a^+)^2 + 2(a^{++})^2\big) G^{++}_{j+2}.
\end{aligned} \tag{4.4}
$$

Proof. Use (4.1) with $b \equiv 0$ for (4.3) resp. (3.6), (3.7) with $a = \rho$ for (4.4). □

Lemma 4.3. *For all $n \in \mathbb{Z}$,*

$$\tilde{G}_j(n) = G_j(n), \quad j \in \mathbb{N}_0. \tag{4.5}$$

Proof. Our aim is to show that $\tilde{G}_j$ satisfy the linear recursion relation (4.4) for $\hat{G}_j$. We start with (4.3),

$$
\begin{aligned}
\tilde{G}_{j+3} - \tilde{G}^{+}_{j+3} + \tilde{G}^{+}_{j+3} - \tilde{G}^{++}_{j+3} = {}& -(a^+)^2 \tilde{G}^{++}_{j+2} + a^2 (\tilde{G}_{j+2} - \tilde{G}^{+}_{j+2}) + (a^-)^2 \tilde{G}^{-}_{j+2} \\
& - (a^{++})^2 \tilde{G}^{+++}_{j+2} + (a^+)^2 (\tilde{G}^{+}_{j+2} - \tilde{G}^{++}_{j+2}) + a^2 \tilde{G}_{j+2},
\end{aligned} \tag{4.6}
$$

and observe that the right hand side of (4.4) only involves even shifts of G_j. Hence we systematically replace in (4.6) odd shifts of $\tilde{G}_j$ by (4.3),

$$
\tilde{G}_j = \begin{cases} G_{1,j} := \tilde{G}^{+}_j - (a^+)^2 \tilde{G}^{++}_{j-1} + a^2 (\tilde{G}_{j-1} - \tilde{G}^{+}_{j-1}) + (a^-)^2 \tilde{G}^{-}_{j-1} \\ G_{2,j} := \tilde{G}^{-}_j + a^2 \tilde{G}^{+}_{j-1} + (a^-)^2 (\tilde{G}_{j-1} - \tilde{G}^{-}_{j-1}) - (a^{--})^2 \tilde{G}^{--}_{j-1} \end{cases},
$$

as follows:

$$\tilde{G}^{+++}_{j+2} \to G^{+++}_{2,j+2}, \quad \tilde{G}^{+}_{j+2} \to x G^{+}_{1,j+2} + (1-x) G^{+}_{2,j+2}, \quad \tilde{G}^{-}_{j+2} \to G^{-}_{1,j+2},$$

with

$$x = \frac{(a^-)^2 + a^2 + (a^{++})^2}{a^2 - (a^+)^2}.$$

In the resulting equation we replace

$$\tilde{G}^{+++}_{j+1} \to G^{+++}_{2,j+1}, \quad \tilde{G}^{+}_{j+1} \to y G^{+}_{1,j+1} + (1-y) G^{+}_{2,j+1}, \quad \tilde{G}^{-}_{j+1} \to G^{-}_{1,j+1},$$

where

$$y = \frac{(a^-)^2 (a^{++})^2 + a^2 (a^{++})^2}{a^2 (a^{++})^2 - (a^-)^2 (a^+)^2}.$$

This gives (4.4) for $\tilde{G}_j$. □

Hence both constructions for the KM hierarchy are equivalent and we have

Theorem 4.4. *Let $r \in \mathbb{N}_0$. Then*

$$\mathrm{TL}_{2r+1}(a, 0) = \mathrm{KM}_r(a). \tag{4.7}$$

provided $c^{\mathrm{TL}}_{2j+1} = c^{\mathrm{KM}}_j$ and $c^{\mathrm{TL}}_{2j} = 0$ for $j = 0, \dots, r$.

Remark 4.5. *As pointed out by M. Gekhtman to us, an alternate way of proving equivalence is by showing that (in the semi-infinite case, $n \in \mathbb{N}$) both constructions give rise to the same set of evolutions for the moments of the underlying spectral measure (compare* [1]*). Our purely algebraic approach has the advantage that it does neither require the semi-infinite case nor self-adjointness.*

5. Appendix: Jacobi operators with $b \equiv 0$

In order to get solutions for the Kac–van Moerbeke hierarchy out of solutions of the Toda hierarchy one clearly needs to identify those cases which lead to Jacobi operators with $b \equiv 0$. For the sake of completeness we recall some folklore results here.

Let H be a Jacobi operator associated with the sequences a, b as in (2.1). Recall that under the unitary operator $Uf(n) = (-1)^n f(n)$ our Jacobi operator transforms according to $U^{-1}H(a,b)U = H(-a,b)$, where we write $H(a,b)$ in order to display the dependence of H on the sequences a and b. Hence, in the special case $b \equiv 0$ we infer that H and $-H$ are unitarily equivalent, $U^{-1}HU = -H$. In particular, the spectrum is symmetric with respect to the reflection $z \to -z$ and it is not surprising, that this symmetry plays an important role.

Denote the diagonal and first off-diagonal of the Green's function of a Jacobi operator H by

$$\begin{aligned} g(z,n) &= \langle \delta_n, (H-z)^{-1}\delta_n \rangle, \\ h(z,n) &= 2a(n)\langle \delta_{n+1}, (H-z)^{-1}\delta_n \rangle - 1. \end{aligned} \tag{5.1}$$

Then we have

Theorem 5.1. *For a given Jacobi operator, $b \equiv 0$ is equivalent to $g(z,n) = -g(-z,n)$ and $h(z,n) = h(-z,n)$.*

Proof. Set $\tilde{H} = -U^{-1}HU$, then the corresponding diagonal and first off-diagonal elements are related via $\tilde{g}(z,n) = -g(-z,n)$ and $\tilde{h}(z,n) = h(-z,n)$. Hence the claim follows since $g(z,n)$ and $h(z,n)$ uniquely determine H (see [7, Sect. 2.7] respectively [5] for the unbounded case). □

Note that one could alternatively use recursions: Since $g_j(n)$ and $h_j(n)$ are just the coefficients in the asymptotic expansions of $g(z,n)$ respectively $h(z,n)$ around $z = \infty$ (see [7, Chap. 6]), our claim is equivalent to $g_{2j+1}(n) = 0$ and $h_{2j}(n) = 0$.

Similarly, $b \equiv 0$ is equivalent to $m_\pm(z,n) = -m_\pm(-z,n)$, where

$$m_\pm(z,n) = \langle \delta_{n\pm1}, (H_{\pm,n} - z)^{-1}\delta_{n\pm1} \rangle \tag{5.2}$$

are the Weyl m-functions. Here $H_{\pm,n}$ are the two half-line operators obtained from H by imposing an additional Dirichlet boundary condition at n. The corresponding spectral measures are of course symmetric in this case.

For a quasi-periodic algebro-geometric solution (see e.g. [7, Chap. 9]), this implies $b \equiv 0$ if and only if both the spectrum and the Dirichlet divisor are symmetric with respect to the reflection $z \to -z$. For an N soliton solution this implies $b \equiv 0$ if and only if the eigenvalues come in pairs, E and $-E$, and the norming constants associated with each eigenvalue pair are equal.

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