Note on Norm and Pointwise Convergence of Exponential Products and their Integral Kernels for the Harmonic Oscillator

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Abstract. The aim of this paper is to study the exponential symmetric product formula for the semigroup of the one-dimensional harmonic oscillator to discuss its convergence pointwise of the integral kernels as well as in norm with sharp optimal error bound.

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1 Introduction and Results

The Trotter or Trotter–Kato product formula or the exponential product formula is a useful tool to approximate the semigroup for a generator being a sum of two operators ([15], [12]). One of the typical cases is: if A and B are nonnegative selfadjoint operators in a Hilbert space $\mathcal{H}$ and if the *operator sum* $H := A + B$ with domain $D[H] := D[A] \cap D[B]$ is selfadjoint in $\mathcal{H}$, it holds in *strong operator topology* that

$$\lim_{n\to\infty} [e^{-\frac{t}{n}A}e^{-\frac{t}{n}B}]^n = \lim_{n\to\infty} [e^{-\frac{t}{2n}B}e^{-\frac{t}{n}A}e^{-\frac{t}{2n}B}]^n = e^{-tH},$$

uniformly on each bounded t-interval in the closed half-line $[0, \infty)$.

It has been shown in [8], [11] that it holds even *in operator norm*, namely, that for $n = 1, 2, \ldots$,

$$\|[e^{-\frac{t}{n}A}e^{-\frac{t}{n}B}]^n - e^{-tH}\| \;=\; O(n^{-1}), \tag{1.1}$$

$$\|[e^{-\frac{t}{2n}B}e^{-\frac{t}{n}A}e^{-\frac{t}{2n}B}]^n - e^{-tH}\| \;=\; O(n^{-1}), \tag{1.2}$$

uniformly on each bounded t-interval in the closed half-line $[0, \infty)$. Further, if H is strictly positive, the convergence is uniform in the whole closed half-line $[0, \infty)$. Here the error bound $O(n^{-1})$ is optimal, in the sense that there exists a pair of unbounded selfadjoint operators A and B with their sum H being selfadjoint such that the left-hand side of the non-symmetric/symmetric product case (1.1)/(1.2) is bounded from below by $c(t)n^{-1}$ with a nonnegative continuous function $c(t)$ in $t \geq 0$, positive in $t > 0$ with $c(0) = 0$. Note that, for both A and B bounded operators, the error bound in the non-symmetric product case is $O(n^{-1})$, while in the symmetric product case $O(n^{-2})$.

This result applies, needless to say, to the Schrödinger operator $H = A+B$ in $L^2(\mathbf{R}^d)$ with $A := H_0 \equiv -\frac{1}{2}\Delta$ the Laplacian and $B := V(x)$ a potential function growing polynomially as $|x| \to \infty$, i.e. satisfying $V(x) \geq C(1 + |x|^2)^{\rho/2}$, $|\partial_x^\alpha V(x)| \leq C_\alpha(1 + |x|^2)^{(\rho-\delta|\alpha|)_+/2}$ for some constants C, $C_\alpha \geq 0$ and $\rho \geq 0$, $0 < \delta \leq 1$. In this case, Takanobu [14] showed the integral kernels also converge pointwise uniformly with error bound $O(n^{-\rho/2})$, using the Feynman–Kac formula. Recently we proved in [9] (cf. [10]) that the symmetric product formula (1.2) holds even with the error bound $O(n^{-2})$, sharper than the general optimal $O(n^{-1})$, both pointwise for the integral kernels and in operator norm, the convergence taking place uniformly on each *compact* t-interval in the *open* half-line $(0, \infty)$. As it had been anticipated that there be also a pair of unbounded selfadjoint operators A and B for which the error bound is $O(n^{-2})$, this problem was thus settled.

The aim of this paper is to specifically consider the one-dimensional harmonic oscillator $H := -\frac{1}{2}\Delta + \frac{1}{2}x^2$ with $H_0 := -\frac{1}{2}\Delta$, $V := \frac{1}{2}x^2$, in $L^2(\mathbf{R})$ to improve slightly these results in [9]. In fact, we show in the *symmetric* product case, apart from the *non-symmetric* case, that the integral kernels converge pointwise *uniformly on each bounded interval* $(0, T]$ in the *open* half-line $(0, \infty)$ *just* with the error bound $O(n^{-2})$. As a by-product, we also show that the error bound $O(n^{-2})$ for the symmetric product formula in norm is optimal not only from above but also from below. It is our main aim to settle this problem on optimal error bounds. We put the *symmetric* product as

$$K^{(n)}(t) := [e^{-\frac{t}{2n}V} e^{-\frac{t}{n}H_0} e^{-\frac{t}{2n}V}]^n, \tag{1.3}$$

and its integral kernel as $K^{(n)}(t, x, y)$. The integral kernel $K(t, x, y)$ of the semigroup $K(t) := e^{-tH}$ is known and *explicitly* given (e.g. [3]) by

$$K(t, x, y) = \frac{1}{\sqrt{\pi(e^t - e^{-t})}} \exp\Big[\frac{4xy - (e^t + e^{-t})(x^2 + y^2)}{2(e^t - e^{-t})}\Big]; \tag{1.4}$$

this fact is crucial in the process of this paper. The proof is made in an elementary way with Taylor's theorem, first obtaining an explicit expression of $K^{(n)}(t,x,y)$ and then estimating its difference from $K(t,x,y)$.

We give two theorems. The first theorem concerns the pointwise convergence of the integral kernels of the exponential symmetric product formula.

Theorem 1.1 *There exist uniformly bounded sequences* $\{R^{(n)}(t,x,y)\}_{n=1}^{\infty}$ *and* $\{Q^{(n)}(t,x,y)\}_{n=1}^{\infty}$ *of continuous functions in* $(0,\infty)\times\mathbf{R}\times\mathbf{R}$ *satisfing the following conditions: it holds for every fixed* $T>0$ *and for sufficiently large* n *that*

$$K^{(n)}(t,x,y) - K(t,x,y) = R^{(n)}(t,x,y)n^{-2}, \tag{1.5}$$

$$R^{(n)}(t,x,y) = R(t,x,y) + Q^{(n)}(t,x,y)n^{-1}, \tag{1.6}$$

in $(t,x,y)\in(0,T]\times\mathbf{R}\times\mathbf{R}$*, where* $R(t,x,y)$ *is a bounded continuous function in* $(0,\infty)\times\mathbf{R}\times\mathbf{R}$ *given by*

$$\begin{aligned} R(t,x,y) \;=\; K(t,x,y)\Big[&\frac{t^3}{12}\Big(\frac{1}{4}\frac{e^t+e^{-t}}{e^t-e^{-t}} + \frac{(e^t+e^{-t})xy-(x^2+y^2)}{(e^t-e^{-t})^2}\Big) \\ &+\frac{t^2}{16}\Big(1+\frac{4xy-(e^t+e^{-t})(x^2+y^2)}{e^t-e^{-t}}\Big)\Big]. \end{aligned} \tag{1.7}$$

If $t>0$*,* $R(t,x,y)$ *can become positive and negative.* $R(t,x,y)$*,* $Q^{(n)}(t,x,y)$ *and so* $R^{(n)}(t,x,y)$ *satisfy*

$$\lim_{t\to 0}\sup_{x,y}|R(t,x,y)| = \lim_{t\to\infty}\sup_{x,y}|R(t,x,y)| = 0, \tag{1.8}$$

$$\lim_{t\to 0}\sup_{n}\sup_{x,y}|Q^{(n)}(t,x,y)| = 0, \tag{1.9}$$

$$\lim_{t\to 0}\sup_{n}\sup_{x,y}|R^{(n)}(t,x,y)| = 0. \tag{1.10}$$

In this theorem it is interesting that $R(t,x,y)$ appearing in the main part of the error term can be identified as (1.7). Note that it determines $Q^{(n)}(t,x,y)$ as well as $R^{(n)}(t,x,y)$ through the relations (1.5) and (1.6).

Remark. As $t\to 0+$, the integral kernel $K(t,x,y)$ converges to $\delta(x-y)$ in the sense of distributions in $\mathbf{R}^2$. The approximation $K^{(n)}(t,x,y)$ can cancel out well the singularity of $K(t,x,y)$ in the limit when t approaches zero, since the right-hand side of (1.5) is uniformly bounded in $(0,T] \times \mathbf{R} \times \mathbf{R}$ and for sufficiently large n, i.e. $R(t,x,y)$ is uniformly bounded in $(0,\infty) \times \mathbf{R} \times \mathbf{R}$, and $Q^{(n)}(t,x,y)$ uniformly bounded in $(0,T] \times \mathbf{R} \times \mathbf{R}$ and for sufficiently large n, and satisfy (1.8) and (1.9), respectively, although both $K^{(n)}(t,x,y)$ and $K(t,x,y)$ are unbounded as functions in $\mathbf{R}^2$ and $t > 0$. We should like to emphasize that this point is clarified by the theorem. In this respect, Theorem 1.1 turns out to be an improvement of our previous result in [9].

From Theorem 1.1 we can also show the second theorem on the convergence in L^2- operator norm with sharp error bounds.

Theorem 1.2 *There exist nonnegative, bounded continuous functions $C(t)$ and $c(t)$ in $t \geq 0$, positive except $t = 0$ with $C(0) = c(0) = 0$, independent of n, such that, for every fixed $T > 0$ and for sufficiently large n,*

$$c(t)n^{-2} \leq \|[e^{-\frac{t}{2n}V} e^{-\frac{t}{n}H_0} e^{-\frac{t}{2n}V}]^n - e^{-tH}\| \leq C(t)n^{-2}, \tag{1.11}$$

in $t \in [0,T]$.

Of the two bounds in (1.11), the lower one and the upper with such t-dependent $C(t)$ are new. In [9] the upper bound was established, for more general potentials growing at infinity, as a sharp upper error bound Cn^{-2} uniformly on each compact t-interval inside the open half line $(0,\infty)$ with C being a positive constant possibly depending on this interval. We shall show Theorem 1.2 by exploiting the integral kernels to be obtained in the proof of Theorem 1.1.

We note that the first results on the norm convergence were obtained by Rogava [13] for a more restricted pair of selfadjoint operators A and B with error bound $O(n^{-1/2}\log n)$, and by B. Helffer [4] for the Schrödinger operator $H = H_0 + V(x)$ as above with $\rho = 2, \delta = 0$ with error bound $O(n^{-1})$. The latter result was extended in [7] and [2] to the general ρ and δ. For some result on the *form sum* of two selfadjoint operators we refer to [6], and for further extensive references on related subjects to [5], [8] and [16].

The present paper is a slightly enlarged version of the results obtained by the first author [1].

Section 2 is devoted to derivation of the expression of the integral kernel of the product $K^{(n)}(t)$ through an recursion relation. In Section 3 we find $R(t,x,y)$, and prove the results described in Section 1. Section 4 briefly remarks another derivation of $R(t,x,y)$ by a sophisticated method of commutators as used in [9].

2 The integral kernel of $K^{(n)}(t)$

Let $t > 0$. Since the semigroup e^{-tH_0} in (1.3) has the integral kernel

$$e^{-tH_0}(x,y) = e^{\frac{t}{2}\Delta}(x,y) = (2\pi t)^{-\frac{1}{2}} e^{-(x-y)^2/2t}, \tag{2.1}$$

the integral kernel $K^{(n)}(t,x,y)$ of $K^{(n)}$ can be written as the following integral:

$$\begin{aligned} &K^{(n)}(t,x,y) \qquad (2.2) \\ &= \left(\frac{2\pi t}{n}\right)^{-\frac{n}{2}} \overbrace{\int_{\mathbf{R}} \cdots \int_{\mathbf{R}}}^{(n-1)\text{ times}} \prod_{j=1}^{n} \left[e^{-\frac{t}{4n}x_j^2} e^{-\frac{(x_j-x_{j-1})^2}{2t/n}} e^{-\frac{t}{4n}x_{j-1}^2} \right] dx_1 \cdots dx_{n-1} \end{aligned}$$

with $x_0 = y\text{C}x_n = x$. The $(n-1)$-ple integral on the right of (2.2) is calculated in the following lemma.

Lemma 2.1 *The integral kernel of $K^{(n)}$ is given by*

$$\begin{aligned} &K^{(n)}(t,x,y) \\ &= \frac{1}{\sqrt{\pi}} \left(\frac{\sqrt{1+\frac{t^2}{4n^2}}}{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n - \left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n} \right)^{1/2} \\ &\times \exp\left[\frac{2\sqrt{1+\frac{t^2}{4n^2}}}{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n - \left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n} xy \right] \qquad (2.3) \\ &\times \exp\Bigg\{ \Bigg[-\frac{t}{4n} - \frac{n}{2t}\Bigg(1 - \\ &\frac{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^{n-1} - \left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^{n-1}}{\left(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n - \left(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\right)^n} \Bigg) \Bigg] (x^2+y^2) \Bigg\}. \end{aligned}$$

Proof. For any integer k with $2 \le k \le n$, put

$$\begin{aligned} &K_k^{(n)}(t,x,y) \qquad (2.4) \\ &= \left(\frac{2\pi t}{n}\right)^{-\frac{k}{2}} \overbrace{\int_{\mathbf{R}} \cdots \int_{\mathbf{R}}}^{(k-1)\text{ times}} \prod_{j=1}^{k} \left[e^{-\frac{t}{4n}x_j^2} e^{-\frac{(x_j-x_{j-1})^2}{2t/n}} e^{-\frac{t}{4n}x_{j-1}^2} \right] dx_1 \cdots dx_{k-1}, \end{aligned}$$

with $x_0 = y$C$x_k = x$.

We are going to derive an explicit expression of $K_k^{(n)}(t,x,y)$ by induction on k. Put $a = \frac{t}{4n}$, $b = \frac{n}{2t}$. Though they satisfy $b = \frac{1}{8a}$, we shall not use this relation for a while. We use the formula for the Gaussian integral $\int_{\mathbf{R}} e^{-Ax^2+Bx}dx = \sqrt{\frac{\pi}{A}}e^{B^2/4A}$ to perform calculation.

For $k = 2$, we have with $x_0 = y$ and $x_2 = x$C

$$\begin{aligned}
&K_2^{(n)}(t,x,y) \\
= &\Big(\frac{2\pi t}{n}\Big)^{-1} \int \Big(e^{-\frac{t}{4n}x_2^2} e^{-\frac{(x_2-x_1)^2}{2t/n}} e^{-\frac{t}{4n}x_1^2}\Big)\Big(e^{-\frac{t}{4n}x_1^2} e^{-\frac{(x_1-x_0)^2}{2t/n}} e^{-\frac{t}{4n}x_0^2}\Big)\, dx_1 \\
= &\Big(\frac{2\pi t}{n}\Big)^{-1} \sqrt{\frac{\pi}{2(a+b)}} \exp\Big[-\Big(a+b-\frac{b^2}{2(a+b)}\Big)(x_0^2+x_2^2)\Big] \exp\Big[\frac{b^2}{a+b}x_0x_2\Big].
\end{aligned}$$

If $k = 3$, we have with $x_0 = y$ and $x_3 = x$C@

$$\begin{aligned}
K_3^{(n)}(t,x,y) &= \Big(\frac{2\pi t}{n}\Big)^{-3/2} \int\int \prod_{j=1}^{3} \Big[e^{-\frac{t}{4n}x_j^2} e^{-\frac{(x_j-x_{j-1})^2}{2t/n}} e^{-\frac{t}{4n}x_{j-1}^2}\Big] dx_1dx_2 \\
&= \Big(\frac{2\pi t}{n}\Big)^{-3/2} \sqrt{\frac{\pi}{2(a+b)}} \sqrt{\frac{\pi}{2(a+b)-\frac{b^2}{2(a+b)}}} \\
&\quad \times \exp\Big[-\Big(a+b-\frac{b^2}{2(a+b)-\frac{b^2}{2(a+b)}}\Big)x_3^2 \\
&\quad -\Big(a+b-\frac{b^2}{2(a+b)}-\frac{(\frac{b^2}{a+b})^2}{4(2(a+b)-\frac{b^2}{2(a+b)})}\Big)x_0^2\Big] \\
&\quad \times \exp\Big[\frac{\frac{b^3}{a+b}}{2(a+b)-\frac{b^2}{2(a+b)}}x_0x_3\Big].
\end{aligned}$$

We note, in the expression for $k = 3$ above, that the coefficient of x_3^2 coincides with that of x_0^2, because

$$\frac{b^2}{2(a+b)} + \frac{(\frac{b^2}{a+b})^2}{4(2(a+b)-\frac{b^2}{2(a+b)})} = \frac{b^2}{2(a+b)-\frac{b^2}{2(a+b)}}.$$

Now define the finite continued fractions A_k by the recursion relation

$$A_k = A_2 - \frac{b^2}{A_{k-1}}, \quad k \geq 3, \quad A_2 = 2(a+b). \tag{2.5}$$

Note that $A_3 = A_2 - \frac{b^2}{A_2} = 2(a+b) - \frac{b^2}{2(a+b)}$ and $A_4 = A_2 - \frac{b^2}{A_3}$. As a result, we turn to have

$$K_2^{(n)}(t,x,y) = \Big(\frac{2\pi t}{n}\Big)^{-1}\Big(\frac{\pi}{A_2}\Big)^{1/2}\exp\Big[-(\frac{A_2}{2}-\frac{b^2}{A_2})(x^2+y^2)\Big]\exp\Big[\frac{2b^2}{A_2}xy\Big],$$
$$K_3^{(n)}(t,x,y) = \Big(\frac{2\pi t}{n}\Big)^{-3/2}\Big(\frac{\pi^2}{A_2A_3}\Big)^{1/2}\exp\Big[-(\frac{A_2}{2}-\frac{b^2}{A_3})(x^2+y^2)\Big]\exp\Big[\frac{2b^3xy}{A_2A_3}\Big].$$

Then we want to prove that for general k with $2 \le k \le n$ with $x_0 = y$ and $x_k = x$C

$$\begin{aligned} & K_k^{(n)}(t,x,y) \\ = & \Big(\frac{2\pi t}{n}\Big)^{-k/2}\int\cdots\int\prod_{j=1}^{k}\Big[e^{-\frac{t}{4n}x_j^2}e^{-\frac{(x_j-x_{j-1})^2}{2t/n}}e^{-\frac{t}{4n}x_{j-1}^2}\Big]dx_1\cdots dx_{k-1} \\ = & \Big(\frac{2\pi t}{n}\Big)^{-k/2}\Big(\frac{\pi^{k-1}}{A_2A_3\cdots A_k}\Big)^{1/2} \\ & \qquad \times\exp\Big[(-\frac{A_2}{2}+\frac{b^2}{A_k})(x_k^2+x_0^2)\Big]\exp\Big[\frac{2b^k}{A_2A_3\cdots A_k}x_0x_k\Big]. \end{aligned} \tag{2.6}$$

We have seen above that this is valid for $k=2$ and also for $k=3$.

Now assuming the validity for k, we shall show the case $k+1$. Then we have by induction hypothesis for k with $x_0 = y$ and $x_{k+1} = x$,

$$\begin{aligned} & K_{k+1}^{(n)}(t,x,y) \\ = & \Big(\frac{2\pi t}{n}\Big)^{-\frac{k}{2}}\int_{\mathbf{R}}\cdots\int_{\mathbf{R}}\prod_{j=1}^{k}\Big[e^{-\frac{t}{4n}x_j^2}e^{-\frac{(x_j-x_{j-1})^2}{2t/n}}e^{-\frac{t}{4n}x_{j-1}^2}\Big]dx_1\cdots dx_{k-1} \\ = & \Big(\frac{2\pi t}{n}\Big)^{-(k+1)/2}\int(e^{-ax_{k+1}^2}e^{-b(x_{k+1}-x_k)^2}e^{-ax_k^2})\Big(\frac{\pi^{k-1}}{A_2A_3\cdots A_k}\Big)^{1/2} \\ & \quad\times\exp\Big[-(\frac{A_2}{2}-\frac{b^2}{A_k})(x_k^2+x_0^2)\Big]\exp\Big[\frac{2b^k}{A_2A_3\cdots A_k}x_0x_k\Big]dx_k \\ = & \Big(\frac{2\pi t}{n}\Big)^{-(k+1)/2}\Big(\frac{\pi^{k-1}}{A_2A_3\cdots A_k}\Big)^{1/2} \\ & \quad\times\int\exp\Big[-A_{k+1}x_k^2+(2bx_{k+1}+\frac{2b^k}{A_1\cdots A_k}x_0)x_k\Big]dx_k \\ & \quad\times\exp\Big[-\frac{A_2}{2}x_{k+1}^2-(\frac{A_2}{2}-\frac{b^2}{A_k})x_0^2\Big] \end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{2\pi t}{n}\Big)^{-(k+1)/2}\Big(\frac{\pi^k}{A_2A_3\cdots A_{k+1}}\Big)^{1/2}\exp\Big[\frac{2b^{k+1}}{A_2A_3\cdots A_{k+1}}x_0x_{k+1}\Big] \\
&\quad \times\exp\Big[-(\frac{A_2}{2}-\frac{b^2}{A_{k+1}})x_{k+1}^2-(\frac{A_2}{2}-\frac{b^2}{A_k}-\frac{b^{2k}}{(A_2A_3\cdots A_k)^2A_{k+1}})x_0^2\Big].
\end{aligned}
$$

Here we can see that the coefficients of x_0^2 and x_{k+1}^2 coincide. In fact, it holds that

$$
\frac{b^2}{A_k}+\frac{b^{2k}}{(A_2A_3\cdots A_k)^2A_{k+1}}=\frac{b^2}{A_{k+1}}, \quad n\geq 3, \tag{2.7}
$$

where we recall to have put $A_k = A_2 - \frac{b^2}{A_{k-1}}$, $A_2 = 2(a+b)$ in (2.5). To show it, since the left-hand side of (2.7) is equal to

$$
\frac{(A_2\cdots A_{k-1})^2A_kA_{k+1}b^2+b^{2k}}{(A_2A_3\cdots A_k)^2A_{k+1}}=\frac{b^2}{A_{k+1}}\frac{(A_2\cdots A_{k-1})^2A_kA_{k+1}+b^{2(k-1)}}{(A_2A_3\cdots A_k)^2}.
$$

we need to confirm that

$$
\frac{(A_2\cdots A_{k-1})^2A_kA_{k+1}+b^{2(k-1)}}{(A_2A_3\cdots A_k)^2}=1, \tag{2.8}
$$

or

$$
(A_2\cdots A_{k-1})^2A_kA_{k+1}+b^{2(k-1)}=(A_2A_3\cdots A_k)^2.
$$

In view of the recursion relation, it is equivalent to

$$
\begin{aligned}
&A_2(A_2\cdots A_{k-1})^2A_k-(A_2\cdots A_{k-1})^2b^2+b^{2(k-1)} \\
= \ &A_2(A_2\cdots A_{k-1})^2A_k-(A_2\cdots A_{k-2})^2A_{k-1}A_kb^2.
\end{aligned}
$$

In the same way, it is equivalent to

$$
\begin{aligned}
&A_2(A_2\cdots A_{k-2})^2A_{k-1}b^2-(A_2\cdots A_{k-3})^2A_{k-2}A_{k-1}b^4+b^{2(k-1)} \\
= \ &A_2(A_2\cdots A_{k-2})^2A_{k-1}b^2-(A_2\cdots A_{k-2})^2b^4,
\end{aligned}
$$

and again to

$$
(A_2\cdots A_{k-3})^2b^6+b^{2(k-1)}=(A_2\cdots A_{k-4})^2A_{k-3}A_{k-2}b^6,
$$

and further again to

$$
(A_2\cdots A_{k-5})^2A_{k-4}A_{k-3}b^8+b^{2(k-1)}=(A_2\cdots A_{k-4})^2b^8.
$$

Thus at last we reach its equivalent

$$(A_2)^2 A_3 A_4 b^{2(k-2)} + b^{2(k-1)} = (A_2 A_3)^2 b^{2(k-3)},$$

whence

$$(A_2)^2 b^{2(k-2)} - b^{2(k-1)} = A_2 A_3 b^{2(k-2)}.$$

The last equation holds, because $A_3 = A_2 - \frac{b^2}{A_2}$. Therefore we have proved (2.8) to yield the expression of $K_{k+1}^{(n)}(t,x,y)$, thus establishing the desired expression (2.6) for any $k \leq n$.

Finally, we determine A_n from their recursion relation (2.5). Let p and q be the two solutions of

$$z^2 - 2(a+b)z + b^2 = 0.$$

We have

$$p = (a+b) + \sqrt{a^2+2ab}, \quad q = (a+b) - \sqrt{a^2+2ab},$$

so that $p+q = 2(a+b)$, $pq = b^2$. It follows that

$$\begin{aligned} A_{n+1} - p &= \frac{2(a+b)A_n - b^2 - pA_n}{A_n} = \frac{qA_n - b^2}{A_n} \\ &= \frac{q(A_n - b^2/q)}{A_n} = \frac{q(A_n - p)}{A_n}, \end{aligned}$$

and similarly,

$$A_{n+1} - q = \frac{p(A_n - q)}{A_n}.$$

Therefore

$$\frac{A_n - p}{A_n - q} = \frac{q}{p}\frac{A_n - p}{A_n - q} = \left(\frac{q}{p}\right)^{n-2} \frac{A_2 - p}{A_2 - q}.$$

Hence we obtain

$$\begin{aligned} A_n &= \frac{p^n - q^n}{p^{n-1} - q^{n-1}} \\ &= \frac{1}{2} \frac{\left(\frac{t}{2n} + \frac{n}{t} + \sqrt{\frac{t^2}{4n^2} + 1}\right)^n - \left(\frac{t}{2n} + \frac{n}{t} - \sqrt{\frac{t^2}{4n^2} + 1}\right)^n}{\left(\frac{t}{2n} + \frac{n}{t} + \sqrt{\frac{t^2}{4n^2} + 1}\right)^{n-1} - \left(\frac{t}{2n} + \frac{n}{t} - \sqrt{\frac{t^2}{4n^2} + 1}\right)^{n-1}}. \end{aligned}$$

Substituting this A_n into $K_k^{(n)}(t,x,y)$, we have

$$\begin{aligned}
&\Big(\frac{2\pi t}{n}\Big)^{-k/2}\Big(\frac{\pi^{k-1}}{A_2A_3\cdots A_k}\Big)^{1/2}\\
=\ &\left[\frac{n^k}{2^k\pi^k t^k}\frac{\pi^{k-1}2^k\sqrt{1+\frac{t^2}{4n^2}}}{\Big(\frac{t}{2n}+\frac{n}{t}+\sqrt{\frac{t^2}{n^2}+1}\Big)^k-\Big(\frac{t}{2n}+\frac{n}{t}-\sqrt{\frac{t^2}{4n^2}+1}\Big)^k}\right]^{1/2}\\
=\ &\left[\frac{1}{\pi}\frac{\sqrt{1+\frac{t^2}{4n^2}}}{\Big(1+\frac{t^2}{2n^2}+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^k-\Big(1+\frac{t^2}{2n^2}-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^k}\right]^{1/2}.
\end{aligned}$$

We have also

$$\begin{aligned}
\frac{2b^k}{A_2A_3\cdots A_k}\ &=\ 2\Big(\frac{n}{2t}\Big)^k\frac{2^k\sqrt{1+\frac{t^2}{4n^2}}}{\Big(\frac{t}{2n}+\frac{n}{t}+\sqrt{\frac{t^2}{n^2}+1}\Big)^k-\Big(\frac{t}{2n}+\frac{n}{t}-\sqrt{\frac{t^2}{4n^2}+1}\Big)^k}\\
&=\ \frac{2\sqrt{1+\frac{t^2}{4n^2}}}{\Big(1+\frac{t^2}{2n^2}+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^k-\Big(1+\frac{t^2}{2n^2}-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^k},
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{A_2}{2}+\frac{b^2}{A_k}\\
=\ &-(a+b)+\frac{b^2}{A_k}\\
=\ &-\frac{t}{4n}-\frac{n}{2t}+(\frac{n}{2t})^2 2\frac{\Big(\frac{t}{2n}+\frac{n}{t}+\sqrt{\frac{t^2}{4n^2}+1}\Big)^{k-1}-\Big(\frac{t}{2n}+\frac{n}{t}-\sqrt{\frac{t^2}{4n^2}+1}\Big)^{k-1}}{\Big(\frac{t}{2n}+\frac{n}{t}+\sqrt{\frac{t^2}{4n^2}+1}\Big)^k-\Big(\frac{t}{2n}+\frac{n}{t}-\sqrt{\frac{t^2}{4n^2}+1}\Big)^k}\\
=\ &-\frac{t}{4n}-\frac{n}{2t}+\frac{n}{2t}\frac{\Big(1+\frac{t^2}{2n^2}+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^{k-1}-\Big(1+\frac{t^2}{2n^2}-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^{k-1}}{\Big(1+\frac{t^2}{2n^2}+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^k-\Big(1+\frac{t^2}{2n^2}-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^k}\\
=\ &-\frac{t}{4n}
\end{aligned}$$

$$-\frac{n}{2t}\Bigg(1-\frac{\Big(1+\frac{t^2}{2n^2}+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^{k-1}-\Big(1+\frac{t^2}{2n^2}-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^{k-1}}{\Big(1+\frac{t^2}{2n^2}+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^{k}-\Big(1+\frac{t^2}{2n^2}-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}\Big)^{k}}\Bigg).$$

Hence

$$\begin{aligned}
&K_k^{(n)}(t,x,y)\\
=\ &\frac{1}{\sqrt{\pi}}\Bigg(\frac{\sqrt{1+\frac{t^2}{4n^2}}}{\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k}-\Big(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k}}\Bigg)^{1/2}\\
&\times\exp\Bigg[\frac{2\sqrt{1+\frac{t^2}{4n^2}}}{\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k}-\Big(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k}}xy\Bigg]\\
&\times\exp\Bigg\{\Bigg[-\frac{t}{4n}-\frac{n}{2t}\Bigg(1-\\
&\frac{\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k-1}-\Big(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k-1}}{\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k}-\Big(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{k}}\Bigg)\Bigg](x^2+y^2)\Bigg\}.
\end{aligned}$$

Putting $k=n$Cso that $x_k=x_n=x$ and $x_0=y$, we have obtained $K^{(n)}(t,x,y)$ in (2.3) in Lemma 2.1. □

3 Proof of the theorems

We shall prove first Theorem 1.1 and next Theorem 1.2.

3.1 Proof of Theorem 1.1

Introduce

$$E_n(t):=\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{n}-\Big(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{n}, \quad (3.1)$$

$$F_n(t):=\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{n-1}-\Big(1-\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{n-1}(3.2)$$

Then we may write $K^{(n)}(t,x,y)$ in (2.3) in Lemma 2.1 as the product of the following three factors $a(t,x,y)$, $b(t,x,y)$ and $c(t,x,y)$:

$$
\begin{aligned}
a(t,x,y) &= \frac{1}{\sqrt{\pi}}\Big[\frac{(1+\frac{t^2}{4n^2})^{1/2}}{E_n(t)}\Big]^{1/2}, &(3.3)\\
b(t,x,y) &= \exp\Big[\frac{2(1+\frac{t^2}{4n^2})^{1/2}}{E_n(t)}\,xy\Big], &(3.4)\\
c(t,x,y) &= \exp\Big[-\Big(\frac{t}{4n}+\frac{n}{2t}(1-\frac{F_n(t)}{E_n(t)})\Big)(x^2+y^2)\Big]. &(3.5)
\end{aligned}
$$

We want to estimate the difference between $K^{(n)}(t,x,y)$ and $K(t,x,y)$. To this end, put with $R(t,x,y)$ in (1.7)

$$
\begin{aligned}
R^{(n)}(t,x,y) &= n^2(K^{(n)}(t,x,y)-K(t,x,y)), &(3.6)\\
Q^{(n)}(t,x,y) &= n(R^{(n)}(t,x,y)-R(t,x,y)). &(3.7)
\end{aligned}
$$

First we shall study the property of $R(t,x,y)$, accepting its explicit expression as in (1.7) in Proposition 3.1, and next derive this expression in the proof of Proposition 3.2. It will turn out to be given through

$$\frac{1}{2!}R(t,x,y)=\frac{\partial^2}{\partial\varepsilon^2}K^\varepsilon(t,x,y)\Big|_{\varepsilon=0},$$

where $K^\varepsilon(t,x,y)$ is defined through

$$K^\varepsilon(t,x,y):=K^{(n)}(t,x,y)$$

first by putting $\varepsilon=1/n$ and then by letting ε be an arbitrary positive number. We can also see that $K^\varepsilon(t,x,y)$ makes sense for $\varepsilon<0$. Defining $K^\varepsilon(t,x,y)=K(t,x,y)$ for $\varepsilon=0$, because $\lim_{\varepsilon\to 0}K^\varepsilon(t,x,y)=K(t,x,y)$, we see further that $K^\varepsilon(t,x,y)$ is real-analytic in $\varepsilon\in\mathbf{R}$. Here note that the identity

$$1-\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}=\Big(1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}\Big)^{-1} \tag{3.8}$$

holds for τ real, and

$$\lim_{\varepsilon\to 0}\Big[\Big(1+t\varepsilon\sqrt{1+\frac{t^2\varepsilon^2}{4}}+\frac{t^2\varepsilon^2}{2}\Big)^{1/\varepsilon}-\Big(1-t\varepsilon\sqrt{1+\frac{t^2\varepsilon^2}{4}}+\frac{t^2\varepsilon^2}{2}\Big)^{1/\varepsilon}\Big]=e^t-e^{-t},$$

so that $\lim_{n\to\infty}E_n(t)=\lim_{n\to\infty}F_n(t)=e^t-e^{-t}$. Another derivation of $R(t,x,y)$ through the commutator method is given in Section 4.

Proposition 3.1 *If $t > 0$, $R(t,x,y)$ takes positive and negative values, and satisfies $|R(t,x,y)| \leq C_1(t)$ in $(0,\infty) \times \mathbf{R} \times \mathbf{R}$, where $C_1(t)$ is a bounded continuous function in $t \geq 0$ such that $\lim_{t\to 0+} C_1(t) = \lim_{t\to\infty} C_1(t) = 0$. In particular, $R(t,x,y)$ satisfies (1.8).*

Proof. We may rewrite the expression of $R(t,x,y)$ in (1.7) as

$$R(t,x,y) = K(t,x,y)s(t,x,y), \tag{3.9}$$

with

$$\begin{aligned} s(t,x,y) :=& \frac{t^3}{12}\Big(\frac{1}{4}\frac{e^t+e^{-t}}{e^t-e^{-t}} + \frac{(e^t+e^{-t})xy-(x^2+y^2)}{(e^t-e^{-t})^2}\Big) \\ & \qquad\qquad + \frac{t^2}{16}\Big(1 + \frac{4xy-(e^t+e^{-t})(x^2+y^2)}{e^t-e^{-t}}\Big) \qquad (3.10) \\ =& \frac{t^2}{2}\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big) \\ &+ \frac{4xy-(e^t+e^{-t})(x^2+y^2)}{2(e^t-e^{-t})}t^2\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big) + \frac{t^3}{48}(x^2+y^2)\,. \end{aligned}$$
$$(3.11)$$

In this section we use the second expression (3.11). Put

$$r(t,x,y) := -\frac{4xy-(e^t+e^{-t})(x^2+y^2)}{2(e^t-e^{-t})}, \tag{3.12}$$

which is nonnegative for all x, y and every $t > 0$, so that $K(t,x,y) = \frac{e^{-r(t,x,y)}}{\sqrt{\pi(e^t-e^{-t})}}$. Put also

$$\begin{aligned} q(t,x,y) \;:=&\; -\frac{4xy-(e^t+e^{-t})(x^2+y^2)}{2(e^t-e^{-t})}t^2\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big) - \frac{t^3}{48}(x^2+y^2) \\ =&\; t^2\Big[r(t,x,y)\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big) - \frac{t}{48}(x^2+y^2)\Big], \end{aligned} \tag{3.13}$$

so that from (3.11)

$$s(t,x,y) = \frac{t^2}{2}\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big) - q(t,x,y)\,. \tag{3.14}$$

The coefficients of x^2 and y^2 in $q(t,x,y)$ in (3.13) are nonnegative, because they are equal and given by

$$\frac{(e^t+e^{-t})t^2\left(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\right)-\frac{t^3}{24}(e^t-e^{-t})}{2(e^t-e^{-t})}=\frac{t^2(3(e^{2t}-e^{-2t})+4t)}{48(e^t-e^{-t})^2}.$$

Further, $q(t,x,y)$ is nonnegative. To see it, note that for $t>0$ and $(x,y)\neq(0,0)$,

$$\frac{q(t,x,y)}{t^2 r(t,x,y)}=\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}+\frac{t}{24}\frac{e^t-e^{-t}}{\frac{4xy}{x^2+y^2}-(e^t+e^{-t})}.$$

Since $-2\le\frac{4xy}{x^2+y^2}\le 2$ for $(x,y)\neq(0,0)$, it follows that

$$\begin{aligned}\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}-\frac{t}{24}\frac{e^t-e^{-t}}{(e^t+e^{-t})-2}\ &\le\ \frac{q(t,x,y)}{t^2 r(t,x,y)}\\ &\le\ \frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}-\frac{t}{24}\frac{e^t-e^{-t}}{(e^t+e^{-t})+2},\end{aligned}$$

which is equivalent to that

$$\frac{1}{8}-\frac{t}{12}\frac{1}{e^t-e^{-t}}\le\frac{q(t,x,y)}{t^2 r(t,x,y)}\le\frac{1}{8}+\frac{t}{12}\frac{1}{e^t-e^{-t}}.$$

Then, since the first member of the above inequality is equal to $\frac{1}{12}+\frac{(e^t-e^{-t})-2t}{24(e^t-e^{-t})}\ge\frac{1}{12}$, and the last member bounded by $\frac{1}{6}$, we obtain

$$\frac{1}{12}\le\frac{q(t,x,y)}{t^2 r(t,x,y)}\le\frac{1}{6}. \tag{3.15}$$

In particular, we have shown that $q(t,x,y)$ is nonnegative, since $r(t,x,y)$ is nonnegative. Thus, in view of (3.14), we see that if $t>0$, $s(t,x,y)$ takes positive and negative values, and so does $R(t,x,y)$.

Next, to see that $R(t,x,y)$ is bounded in $(0,\infty)\times\mathbf{R}\times\mathbf{R}$ and satisfies (1.8), we note that for fixed $m\ge 1>\alpha>0$,

$$\frac{t^m}{(e^t-e^{-t})^\alpha}\le c(\alpha,m)[t^{m-\alpha}\wedge 1],\qquad t>0, \tag{3.16}$$

$$t^m\frac{e^t+e^{-t}}{e^t-e^{-t}}\le\frac{e+e^{-1}}{e-e^{-1}}[1\vee t]t^{m-1}\le\frac{e+e^{-1}}{e-e^{-1}}[1\vee t^m],\qquad t>0, \tag{3.17}$$

with a postive constant $c(\alpha, m)$ depending on α and m, where we write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ for positive numbers a and b. Indeed, for (3.16) we have used the fact that $\tau^a e^{-\tau} \leq (a/e)^a$ for all $\tau > 0$ and $a > 0$.

Then we have by (3.16) and (3.17)

$$\begin{aligned}
|R(t,x,y)| &\leq \frac{e^{-r(t,x,y)}}{\sqrt{\pi(e^t - e^{-t})}} \frac{t^2}{2}\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t + e^{-t}}{e^t - e^{-t}}\Big) \\
&\quad + \frac{1}{\sqrt{\pi(e^t - e^{-t})}}\Big(e^{-r(t,x,y)} r(t,x,y)\Big)\frac{q(t,x,y)}{t^2 r(t,x,y)}\, t^2 \\
&\leq \frac{1}{\sqrt{\pi(e^t - e^{-t})}}\Big(\frac{1}{16} + \frac{1}{48}\frac{e + e^{-1}}{e - e^{-1}}[1 \vee t] + \frac{1}{6e}\Big) t^2 \\
&=: C_1(t). \qquad (3.18)
\end{aligned}$$

Then $C_1(t)$ is bounded in $t > 0$, because by (3.16), we have $C_1(t) \leq C_1 \frac{t^2[1\vee t]}{(e^t - e^{-t})^{1/2}}$, with a positive constant C_1 independent t. It also satisfies $\lim_{t\to 0+} C_1(t) = 0$, and so we may define $C_1(0) = 0$. Here we have used (3.15). We can also see $\lim_{t\to\infty} C_1(t) = 0$. This shows that $R(t,x,y)$ is bounded in $(0,\infty) \times \mathbf{R} \times \mathbf{R}$, also satisfying (1.8). This proves Proposition 3.1. $\Box$

Now we are going to estimate $R^{(n)}(t,x,y)$ when $0 < t \leq n^{1/2}$. After proving it, we shall show it yields Theorem 1.1. We note that $0 < t \leq n^{1/2}$ if and only it $t(t/n) \leq 1$, which also implies $t(t/n)^j \leq n^{-(j-1)/2} \leq 1$ for $j \geq 2$.

Proposition 3.2 *Let $0 < t \leq n^{1/2}$. Then $Q^{(n)}(t,x,y)$ and $R^{(n)}(t,x,y)$ in (3.6)/(3.7) or (1.5)/(1.6) satisfy*

$$|Q^{(n)}(t,x,y)| \leq C_2(t), \qquad (3.19)$$
$$|R^{(n)}(t,x,y)| \leq C_1(t) + C_2(t), \qquad (3.20)$$

in $(x,y) \in \mathbf{R} \times \mathbf{R}$ for large n, where $C_1(t)$ is the same bounded continuous function in Proposition 3.1, i.e. (3.18), and $C_2(t)$ another bounded continuous function in $t \geq 0$ such that $\lim_{t\to 0+} C_2(t) = 0$.

Before proving this proposition, we provide two lemmas to get expansions of $E_n(t)$, $F_n(t)$ in (3.1), (3.2) with respect to the powers of t/n with $0 < t \leq n^{1/2}$.

Lemma 3.3 *Let τ be real. Then*

$$\text{(i)}\qquad \sqrt{1+\frac{\tau^2}{4}} = 1+\frac{\tau^2}{8}+O(|\tau|^4)\,;$$

$$\text{(ii)}\qquad 1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2} = 1+\tau+\frac{\tau^2}{2}+\frac{\tau^3}{8}+O(|\tau|^5),$$

$$\Big(1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}\Big)^{-1} = 1-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{8}+O(|\tau|^5).$$

Proof. The assertion (i) and the first of (ii) follow from Taylor's theorem

$$(1+x)^{1/2} = 1+\frac{x}{2}-\frac{x^2}{8}(1+\theta x)^{-3/2}, \quad x>-1,$$

for some $0<\theta<1$, with $x=\frac{1}{4}\tau^2$, and the second from (3.8). $\square$

Lemma 3.4 *Let $0<|t|\le n^{1/2}$. Then*

$$\text{(i)}\qquad \Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^n = e^t\Big[1-\frac{t}{24}(t/n)^2+(t+t^2)\,O(|t/n|^4)\Big]\,;$$

$$\text{(ii)}\qquad \Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{n-1}$$

$$= e^t\Big[1-(t/n)+\Big(\frac{1}{2}-\frac{t}{24}\Big)(t/n)^2+\Big(-\frac{1}{8}+\frac{t}{24}\Big)(t/n)^3+(t+t^2)\,O(|t/n|^4)\Big]\,.$$

Proof. (i) First note that by Taylor's theorem we have for $x>-1$

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}(1+\theta x)^{-5}$$

for some $0<\theta<1$, and next that if $\tau\ge -1$, then $\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}\ge -\frac{2}{1+\sqrt{5}} > -\frac{2}{3}$, so that $\Big(1+\theta(\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2})\Big)^{-1} \le \Big(1-\frac{2}{1+\sqrt{5}}\Big)^{-1} = \frac{3+\sqrt{5}}{2}$. Then, for $\tau\ge -1$ or, in particular, for $|\tau|\le 1$, we obtain

$$\log\Big(1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}\Big)$$

$$= \sum_{j=1}^{4}\frac{(-1)^{j-1}}{j}\Big(\tau+\frac{\tau^2}{2}+\frac{\tau^3}{8}+O(|\tau|^5)\Big)^j$$

$$+\frac{1}{5}\Big(\tau+\frac{\tau^2}{2}+\frac{\tau^3}{8}+O(|\tau|^5)\Big)^5\Big(1+\theta(\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2})\Big)^{-5},$$

so that $\log\left(1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}\right)=\tau-\frac{\tau^3}{24}+O(|\tau|^5)$. Hence

$$\frac{1}{\tau}\log\left(1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}\right)=1-\frac{\tau^2}{24}+O(|\tau|^4).$$

Then, by taking $\tau=t/n$, we obtain for $|t|\le n^{1/2}$ or $|t||t/n|\le 1$

$$\begin{aligned}\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^n &= \Big(1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2}\Big)^n\\
&= \exp\Big[\frac{t}{\tau}\log(1+\tau\sqrt{1+\frac{\tau^2}{4}}+\frac{\tau^2}{2})\Big]\\
&= \exp\Big[t(1-\frac{\tau^2}{24}+O(|\tau|^4))\Big]\\
&= e^t\exp\Big[t(-\frac{1}{24}(t/n)^2+O((t/n)^4))\Big]\\
&= e^t\Big[1-\frac{t}{24}(t/n)^2+(|t|+|t|^2)O((t/n)^4)\Big].\end{aligned}$$

In the last equality above we note that since $|t|\le n^{1/2}$, we have $|t||\frac{1}{24}(t/n)^2-O(|\frac{t}{n}|^4)|=n^{-1/2}O(1)$, which becomes less than 1 for n large.

(ii) In the same way as in (i) above with Lemma 3.3 (ii), we have

$$\begin{aligned}&\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{n-1}\\
=\ &\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^n\Big(1+\frac{t}{n}\sqrt{1+\frac{t^2}{4n^2}}+\frac{t^2}{2n^2}\Big)^{-1}\\
=\ &e^t\Big[1-\frac{t}{24}(t/n)^2+(|t|+|t|^2)O((t/n)^4)\Big]\\
&\quad\times\Big(1-\frac{t}{n}+\frac{1}{2}(t/n)^2-\frac{1}{8}(t/n)^3+O(|t/n|^5)\Big)\\
=\ &e^t\Big[1-(t/n)+\Big(\frac{1}{2}-\frac{t}{24}\Big)(t/n)^2+\Big(-\frac{1}{8}+\frac{t}{24}\Big)(t/n)^3+(|t|+|t|^2)O((t/n)^4)\Big],\end{aligned}$$

because for the coefficient of $O((t/n)^4)$ we have $|t|+|t|^2+|t/n|+|t||t/n|\le 2(|t|+|t|^2)$ when $|t|\le n^{1/2}$. This proves the desired expansions. □

Now we start the proof of Proposition 3.2.

Proof of Proposition 3.2. We need to seek, when $0<t\le n^{1/2}$, expansions of the factors $a(t,x,y)$, $b(t,x,y)$and $c(t,x,y)$ in (3.3), (3.4) and (3.5)

of $K^{(n)}(t,x,y)$. Since that $0 < t \le n^{1/2}$ implies that $t(t/n)^j \le n^{-(j-1)/2} \le 1$, $j \ge 2$, we see by use of Lemma 3.4 (i) and (ii) that $E_n(t)$ and $F_n(t)$ in (3.1) and (3.2) have the following expansions

$$\begin{aligned} E_n(t) &= e^t - e^{-t} - \frac{t}{24}(e^t + e^{-t})(t/n)^2 + (t+t^2)(e^t+e^{-t})O((t/n)^4)) \\ &= (e^t - e^{-t})\Big[1 - \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}(t/n)^2 + [1\vee t^2]O((t/n)^4)\Big], \qquad (3.21) \\ F_n(t) &= e^t - e^{-t} - (e^t+e^{-t})(t/n) + \Big(\frac{e^t-e^{-t}}{2} - t\frac{e^t+e^{-t}}{24}\Big)(t/n)^2 \\ &\quad -t\Big(\frac{e^t+e^{-t}}{8} - t\frac{e^t-e^{-t}}{24}\Big)(t/n)^3 + (t+t^2)(e^t+e^{-t})O((t/n)^4) \\ &= (e^t-e^{-t})\Big[1 - \frac{e^t+e^{-t}}{e^t-e^{-t}}(t/n) + \Big(\frac{1}{2} - \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2 \\ &\quad -t\Big(\frac{1}{8}\frac{e^t+e^{-t}}{e^t-e^{-t}} - \frac{t}{24}\Big)(t/n)^3 + [1\vee t^2]O((t/n)^4)\Big]. \qquad (3.22) \end{aligned}$$

Here note by (3.17) that $(t+t^2)\frac{e^t+e^{-t}}{e^t-e^{-t}} \le [1\vee t] + [1\vee t^2] \le 2[1\vee t^2]$.

By using these new expressions (3.21) and (3.22), we find expansions for $a(t,x,y)$, $b(t,x,y)$ and $c(t,x,y)$. As to $a(t,x,y)$, we obtain with Lemma 3.3 (i)

$$\begin{aligned} a(t,x,y) &= \frac{1}{\sqrt{\pi}}\Big[\frac{(1+\frac{t^2}{4n^2})^{1/2}}{E_n(t)}\Big]^{1/2} \\ &= \frac{1}{\sqrt{\pi}}\Big[\frac{1 + \frac{1}{8}(\frac{t}{n})^2 + O((\frac{t}{n})^4)}{(e^t-e^{-t})[1 - \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}(\frac{t}{n})^2 + [1\vee t^2]O((\frac{t}{n})^4)}\Big]^{1/2} \\ &= \frac{1}{\sqrt{\pi(e^t-e^{-t})}}\Big[1 + \frac{1}{2}\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2 + [1\vee t^2]O((t/n)^4)\Big], \end{aligned}$$

because the function $\sqrt{\frac{1+\tau}{1+\sigma}}$ is analytic in $|\tau| < 1$ and $|\sigma| < 1$, and has an expansion

$$\begin{aligned} \sqrt{\frac{1+\tau}{1+\sigma}} &= \Big(1 + \frac{\tau}{2} - \frac{\tau^2}{8} + \frac{\tau^3}{16} - \cdots\Big)\Big(1 - \frac{\sigma}{2} + \frac{3\sigma^2}{8} - \frac{5\sigma^3}{13} + \cdots\Big) \\ &= 1 + \frac{\tau}{2} - \frac{\sigma}{2} - \frac{1}{8}(\tau^2 - \sigma^2) - \frac{\tau\sigma}{4} + \cdots. \end{aligned}$$

The last equality in the equation for $a(t,x,y)$ above is due to that one can think of the inverse function of $E_n(t)$ in $t>0$ for large n, because, on the right of $E_n(t)$ in (3.21), the function inside the blacket $[\cdots]$ is invertible for large n, since $|-\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}(t/n)^2+[1\vee t^2]O((t/n)^4)|$ is less than 1 by (3.16) and (3.17) for $0<t\le n^{1/2}$, if n is large. The same note will apply to equations for $b(t,x,y)$ and $c(t,x,y)$ which come below.

As to $b(t,x,y)$, we have

$$\begin{aligned}
b(t,x,y) &= \exp\Big[\frac{2(1+\frac{t^2}{4n^2})^{1/2}}{E_n(t)}\,xy\Big] \\
&= \exp\Bigg[\frac{2xy}{e^t-e^{-t}}\bigg(\frac{1+\frac{1}{8}(\frac{t}{n})^2+O((\frac{t}{n})^4)}{1-\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}(\frac{t}{n})^2+[1\vee t^2]O((\frac{t}{n})^4)}\bigg)\Bigg] \\
&= \exp\Big(\frac{2xy}{e^t-e^{-t}}\Big)\exp\Big[\frac{2xy}{e^t-e^{-t}}\Big(\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2 \\
&\qquad\qquad +[1\vee t^2]O(t/n)^4)\Big].
\end{aligned}$$

As to $c(t,x,y)$, first we observe with (3.21) and (3.22)

$$\begin{aligned}
&\frac{n}{2t}\Big(1-\frac{F_n(t)}{E_n(t)}\Big) \\
=\ & \frac{e^t+e^{-t}}{2(e^t-e^{-t})}\frac{1-\frac{1}{2}\frac{e^t-e^{-t}}{e^t+e^{-t}}\frac{t}{n}+(\frac{1}{8}-\frac{t}{24}\frac{e^t-e^{-t}}{e^t+e^{-t}})(\frac{t}{n})^2+[1\vee t^2]O((\frac{t}{n})^3)}{1-\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}(\frac{t}{n})^2+[1\vee t^2]O((\frac{t}{n})^4)} \\
=\ & \frac{e^t+e^{-t}}{2(e^t-e^{-t})}\Big[1-\frac{1}{2}\frac{e^t-e^{-t}}{e^t+e^{-t}}(t/n) \\
&\qquad +\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}-\frac{t}{24}\frac{e^t-e^{-t}}{e^t+e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^3)\Big].
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{t}{4n}+\frac{n}{2t}\Big(1-\frac{F_n(t)}{E_n(t)}\Big) \\
=\ & \frac{e^t+e^{-t}}{2(e^t-e^{-t})} \\
&\quad\times\Big[1+\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}-\frac{t}{24}\frac{e^t-e^{-t}}{e^t+e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^3)\Big].
\end{aligned}$$

so that

$$
\begin{aligned}
&c(t,x,y)\\
&= \exp\Big(-\frac{e^t+e^{-t}}{2(e^t-e^{-t})}(x^2+y^2)\Big)\\
&\quad\times\exp\Big[-\frac{e^t+e^{-t}}{2(e^t-e^{-t})}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2(x^2+y^2)\\
&\qquad\qquad +\frac{t}{48}(t/n)^2(x^2+y^2)-[1\vee t^2]O((t/n)^3)(x^2+y^2)\Big].
\end{aligned}
$$

Therefore, from the expansions of $a(t,x,y)$, $b(t,x,y)$ and $c(t,x,y)$ obtained above with $K(t,x,y)$ in (1.4), we get to the following expansion for $K^{(n)}(t,x,y)$:

$$
\begin{aligned}
K^{(n)}(t,x,y) &= a(t,x,y)b(t,x,y)c(t,x,y)\\
&= \frac{e^{-r(t,x,y)}}{\sqrt{\pi(e^t-e^{-t})}}\Big[1+\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^4)\Big]\\
&\quad\times\exp\Big[\frac{4xy-(e^t+e^{-t})(x^2+y^2)}{2(e^t-e^{-t})}\Big(\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2\\
&\qquad\qquad +[1\vee t^2]O((t/n)^3)\Big)+\frac{t}{48}(t/n)^2(x^2+y^2)\Big]\\
&= K(t,x,y)e^{-q(t,x,y)/n^2}\exp\Big[-r(t,x,y)[1\vee t^2]O((t/n)^3)\Big]\\
&\quad\times\Big[1+\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^4)\Big]. \qquad (3.23)
\end{aligned}
$$

Here in the last equality we have used (3.13). Then

$$
\begin{aligned}
R^{(n)}(t,x,y) &= n^2(K^{(n)}(t,x,y)-K(t,x,y))\\
&= n^2e^{-q(t,x,y)/n^2}K(t,x,y)\Big[e^{-r(t,x,y)[1\vee t^2]O((t/n)^3)}-1\Big]\\
&\quad\times\Big[1+\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^4)\Big]\\
&\quad +n^2K(t,x,y)(e^{-q(t,x,y)/n^2}-1)\\
&\quad\times\Big[1+\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^4)\Big]\\
&\quad +n^2K(t,x,y)\Big[\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^4)\Big]
\end{aligned}
$$

$$=: R_1^{(n)}(t,x,y) + R_2^{(n)}(t,x,y) + R_3^{(n)}(t,x,y). \tag{3.24}$$

Recall we are under the condition $0 < t \le n^{1/2}$. For $R_1^{(n)}(t,x,y)$ we have

$$\begin{aligned}
& R_1^{(n)}(t,x,y) \\
= & \Big[1 + \frac{1}{2}\Big(\frac{1}{8} + \frac{t}{24}\frac{e^t + e^{-t}}{e^t - e^{-t}}\Big)(t/n)^2 + [1 \vee t^2]O((t/n)^4)\Big] \\
& \times \frac{n^2 e^{-q(t,x,y)/n^2}}{[\pi(e^t - e^{-t})]^{1/2}} \int_0^1 \frac{d}{d\theta} e^{-r(t,x,y)(1+\theta[1\vee t^2]O((t/n)^3)))} d\theta \\
= & \, O(n^{-1}) e^{-q(t,x,y)/n^2} \frac{[1 \vee t^2]t^3}{[\pi(e^t - e^{-t})]^{1/2}} \int_0^1 r(t,x,y) e^{-r(t,x,y)(1+\theta[1\vee t^2]O((t/n)^3)))} d\theta \, .
\end{aligned} \tag{3.25}$$

Here we have used (3.16) with $t(t/n)^2 \le n^{-1/2}$ in the second equality, and note that $[1 \vee t^2](t/n)^4 \le [1 \vee t^2](t/n)^3 \le n^{-1/2}$, so that we have $|[1 \vee t^2]O((t/n)^3)| < \frac{1}{2}$ for large n, and hence

$$e^{-r(t,x,y)(1+\theta[1\vee t^2]O((t/n)^3))} \le e^{-r(t,x,y)/2}.$$

It follows that for large n,

$$\begin{aligned}
|R_1^{(n)}(t,x,y)| \le & \; O(n^{-1}) \frac{[1 \vee t^2]t^3}{[\pi(e^t - e^{-t})]^{1/2}} 2[\frac{r(t,x,y)}{2} e^{-r(t,x,y)/2}] \\
\le & \; \frac{2}{e} \frac{[1 \vee t^2]t^3}{[\pi(e^t - e^{-t})]^{1/2}} O(n^{-1}) \, ,
\end{aligned} \tag{3.26}$$

where we have used that $\tau e^{-\tau} \le 1$ for $\tau \ge 0$, and note that the coefficient of $O(n^{-1})$ is bounded in $t > 0$, by (3.16).

For one of the factors in $R_2^{(n)}(t,x,y)$, we have

$$e^{-q(t,x,y)/n^2} - 1 = -q(t,x,y)n^{-2} + \frac{q(t,x,y)^2}{2} O(n^{-4}),$$

by Taylor's theorem that $e^{-x} = 1 - x + \frac{x^2}{2} e^{-\theta x}$ $0 < \theta < 1$. Then

$$\begin{aligned}
& R_2^{(n)}(t,x,y) + R_3^{(n)}(t,x,y) \\
= & \; K(t,x,y)[-q(t,x,y) + \frac{q(t,x,y)^2}{2t^2} O((t/n)^2)]
\end{aligned}$$

$$
\begin{aligned}
&\times\Big[1+\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)(t/n)^2+[1\vee t^2]O((t/n)^4)\Big]\\
&+K(t,x,y)\Big[\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)t^2+[1\vee t^2]t^2O((t/n)^2)\Big]\\
=\;& K(t,x,y)\Big[\frac{t^2}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)-q(t,x,y)\Big]\\
&\quad+K(t,x,y)\Big[[1\vee t^2]t^2O((t/n)^2)-\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)q(t,x,y)(t/n)^2\\
&+\frac{q(t,x,y)^2}{2t^2}O((t/n)^2)-[1\vee t^2]q(t,x,y)O((t/n)^4)\\
&+\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)\frac{q(t,x,y)^2}{2t^2}O((t/n)^4)+[1\vee t^2]\frac{q(t,x,y)^2}{2t^2}O((t/n)^6)\Big]\\
=:\;& R^{(n)}_{23,1}(t,x,y)+R^{(n)}_{23,2}(t,x,y)\,. \qquad (3.27)
\end{aligned}
$$

First we note that $R^{(n)}_{23,1}(t,x,y)$, in fact is independent of n, is nothing but $R(t,x,y)$ in (1.7) or (3.9). Next for $R^{(n)}_{23,2}(t,x,y)$, we have

$$
\begin{aligned}
R^{(n)}_{23,2}(t,x,y) \;=\;& \frac{1}{[\pi(e^t-e^{-t})]^{1/2}}\Big\{e^{-r(t,x,y)}[1\vee t^2]t^2O((t/n)^2)\\
&-[e^{-r(t,x,y)}r(t,x,y)]\frac{q(t,x,y)}{t^2r(t,x,y)}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)t^2(t/n)^2\\
&+[e^{-r(t,x,y)}r(t,x,y)^2]\Big(\frac{q(t,x,y)}{t^2r(t,x,y)}\Big)^2\frac{t^2}{2}O((t/n)^2)\\
&-[e^{-r(t,x,y)}r(t,x,y)]\frac{q(t,x,y)}{t^2r(t,x,y)}[1\vee t^2]t^2O((t/n)^4)\\
&+[e^{-r(t,x,y)}r(t,x,y)^2]\Big(\frac{q(t,x,y)}{t^2r(t,x,y)}\Big)^2\frac{t^2}{2}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)O((t/n)^4)\\
&+[e^{-r(t,x,y)}r(t,x,y)^2]\Big(\frac{q(t,x,y)}{t^2r(t,x,y)}\Big)^2\frac{t^2}{2}[1\vee t^2]O((t/n)^6)\Big\}\,. \qquad (3.28)
\end{aligned}
$$

Then, by (3.15), (3.16) and by the fact that $\tau^a e^{-\tau}\le (a/e)^a$ for all $\tau>0$ and $a>0$, we have

$$
\begin{aligned}
&|R^{(n)}_{23,2}(t,x,y)|\\
\le\;& \frac{1}{[\pi(e^t-e^{-t})]^{1/2}}\Big\{[1\vee t^2]t^2O((t/n)^2)+\frac{1}{6e}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)t^2(t/n)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{2}{6e}\Big)^2\frac{t^2}{2}O((t/n)^2)+\frac{1}{6e}[1\vee t^2]t^2O((t/n)^4)\\
&+\Big(\frac{2}{6e}\Big)^2\frac{t^2}{2}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)O((t/n)^4)+\Big(\frac{2}{6e}\Big)^2\frac{[1\vee t^2]t^2}{2}O((t/n)^6)\Big\}\\
\le\ & \frac{1}{[\pi(e^t-e^t)]^{1/2}}\Big\{\Big[[1\vee t^2]t^4+\frac{1}{6e}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)t^4+\Big(\frac{1}{3e}\Big)^2\frac{t^2}{2}\Big]O(n^{-2})\\
&\qquad+\Big[\frac{[1\vee t^2]t^6}{6e}+\Big(\frac{1}{3e}\Big)^2\frac{t^6}{2}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)\Big]O(n^{-4})\\
&\qquad\qquad+\Big(\frac{1}{3e}\Big)^2\frac{[1\vee t^2]t^8}{2}O(n^{-6})\Big\}\,. \qquad (3.29)
\end{aligned}
$$

Hence, taking into account the relation $R^{(n)}(t,x,y)=R(t,x,y)+Q^{(n)}(t,x,y)n^{-1}$, which we get from (3.6) and (3.7), we obtain

$$
Q^{(n)}(t,x,y)=n(R_1^{(n)}(t,x,y)+R_{23,2}^{(n)}(t,x,y)),
$$

and

$$
\begin{aligned}
|Q^{(n)}(t,x,y)|\le\ & \frac{1}{[\pi(e^t-e^{-t})]^{1/2}}\Big\{\frac{2}{e}[1\vee t^2]t^3O(1)\\
&+\Big[[1\vee t^2]t^4+\frac{1}{6e}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)t^4+\Big(\frac{1}{3e}\Big)^2\frac{t^4}{2}\Big]O(n^{-1})\\
&+\Big[\frac{[1\vee t^2]t^6}{6e}+\Big(\frac{1}{3e}\Big)^2\frac{t^6}{2}\frac{1}{2}\Big(\frac{1}{8}+\frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\Big)\Big]O(n^{-3})\\
&\qquad+\Big(\frac{1}{3e}\Big)^2\frac{[1\vee t^2]t^8}{2}O(n^{-5})\Big\}\,. \qquad (3.30)
\end{aligned}
$$

The right-hand side of (3.30) is independent of x, y, and bounded in $t>0$ and $n\ge 1$, by (3.16), and tends to zero as $t\to 0+$. Therefore we see there exists a continuous function $C_2(t)$ bounded in $t\ge 0$, independent of n, such that

$$
|Q^{(n)}(t,x,y)|\le C_2(t), \qquad (x,y)\in\mathbf{R}\times\mathbf{R}, \qquad (3.31)
$$

with $0<t\le n^{1/2}$. We may assume with (3.16) that this $C_2(t)$ is so taken as to satisfy $\lim_{t\to 0+}C_2(t)=0$. Hence and from (3.18) we can also get the bound for $R^{(n)}(t,x,y)$, as $|R^{(n)}(t,x,y)|\le|R(t,x,y)|+|Q^{(n)}(t,x,y)|n^{-1}\le C_1(t)+C_2(t)$, namely, (3.20). Thus we have shown the desired assertion of Proposition 3.2, ending the proof of Proposition 3.2.

We are now in a position to prove Theorem 1.1.

Completion of Proof of Theorem 1.1. Let $T > 0$. Then take a sufficiently large positive integer N such that $T \leq N^{1/2}$. Then Proposition 3.2 holds for all $n \geq N$ and for all $t \in (0, T]$. This is nothing but the assertion of Theorem 1.1, completing the proof of Theorem 1.1.

3.2 Proof of Theorem 1.2

We use the result in Theorem 1.1. Let $T > 0$. Denote by $R(t)$ and $Q^{(n)}(t)$ the bounded selfadjoint operators on $L^2(\mathbf{R})$ with integral kernel $R(t, x, y)$ and $Q^{(n)}(t, x, y)$, respectively, so that $R^{(n)}(t) = R(t) + Q^{(n)}(t)n^{-1}$ is a selfadjoint operator with integral kernel $R^{(n)}(t, x, y)$. We need to find a lower and upper bound of the norm of $R^{(n)}(t)$.

First, for a upper bound of the norm of $R^{(n)}(t)$, note from (3.18) and (3.19)

$$\begin{aligned} \|R(t)\| &= \sup_{\|f\|=1} |(f, R(t)f)| \leq \sup_{x,y} |R(t,x,y)| \leq C_1(t), && (3.32) \\ \|Q^{(n)}(t)\| &= \sup_{\|f\|=1} |(f, Q^{(n)}(t)f)| \leq \sup_{x,y} |Q^{(n)}(t,x,y)| \leq C_2(t), && (3.33) \end{aligned}$$

where $C_1(t)$ and $C_2(t)$ are the bounded functions in Propositions 3.1 and 3.2. It follows that for sufficiently large n and for all $t \in (0, T]$,

$$\|R^{(n)}(t)\| \leq \|R(t)\| + \|Q^{(n)}(t)\|n^{-1} \leq C_1(t) + C_2(t) =: C(t).$$

This $C(t)$ is a continuous function in $t \geq 0$, uniformly bounded, with $C(0) = 0$ and positive in $t > 0$. Thus we have shown the upper bound of the relation (1.11).

Next, we seek a lower bound of the norm of $R^{(n)}(t)$. Since the quadatic expression $q(t, x, y)$ in (3.13) is nonnegative, $s(t, x, y)$ in (3.11)/(3.14) vanishes for (t, x, y) such that $\frac{t^2}{2}\left(\frac{1}{8} + \frac{t}{24}\frac{e^t+e^{-t}}{e^t-e^{-t}}\right) = q(t, x, y)$. For $t \geq 0$, let $S_t := \{(x, y) \in \mathbf{R}^2\,;\, s(t, x, y) > 0\}$. If $t = 0$, it reduces to one point: $S_0 = \{(0, 0)\}$. But if $t > 0$, it is a bounded open set in $\mathbf{R}^2$ around the origin. It contains, for given $\delta(t) > 0$, which is to determined later, a disc $B_t := \{(x, y) \in \mathbf{R}^2\,;\, x^2 + y^2 \leq \delta(t)^2\}$ as well as a square $Q_t := \{(x, y) \in \mathbf{R}^2\,;\, |x| \leq \delta(t), |y| \leq \delta(t)\}$, so that $B_t \subseteq Q_t$. Let $f_0(x) := 1/\sqrt{2\delta(t)}$, $|x| \leq \delta(t)\,;\, f_0(x) = 0$, $|x| > \delta(t)$. Then f_0 belongs to $L^2(\mathbf{R})$

and has norm $\|f_0\| = (\int |f_0(x)|^2 dx)^{1/2} = 1$. Note that $R(t)f_0$ is nonnegative. Therefore we have

$$\|R(t)\| = \sup_{\|f\|=1} |(f, R(t)f)| \geq (f_0, R(t)f_0).$$

where

$$(f_0, R(t)f_0) = \int\int_{Q_t} R(t,x,y)dxdy = \int\int_{Q_t} s(t,x,y)K(t,x,y)dxdy.$$

If $(x,y) \in Q_t$, then $x^2 + y^2 \leq 2\delta(t)^2$, so that by (3.15) with (3.12)

$$q(t,x,y) \leq \frac{t^2}{6} r(t,x,y) = \frac{t^2}{6} \frac{(e^t + e^{-t})(x^2+y^2) - 4xy}{e^t - e^{-t}} \leq \delta(t)^2 \frac{t^2}{3} \frac{e^t + e^{-t} + 2}{e^t - e^{-t}}.$$

Therefore, for each fixed $t > 0$, we choose $\delta(t) > 0$ such that

$$\delta(t)^2 \frac{t^2}{3} \frac{e^t + e^{-t} + 2}{e^t - e^{-t}} = \frac{1}{2} \frac{t^2}{2} \Big(\frac{1}{8} + \frac{t}{24} \frac{e^t + e^{-t}}{e^t - e^{-t}} \Big),$$

namely,

$$\delta(t)^2 = \frac{3}{4} \frac{\frac{1}{8}(e^t - e^{-t}) + \frac{t}{24}(e^t + e^{-t})}{e^t + e^{-t} + 2}. \tag{3.34}$$

It follows that $\delta(t)$ is of order $O(t^{1/2})$. We have then for $(x,y) \in Q_t$

$$s(t,x,y) = \frac{t^2}{2} \Big(\frac{1}{8} + \frac{t}{24} \frac{e^t + e^{-t}}{e^t - e^{-t}} \Big) - q(t,x,y) \geq \frac{1}{2} \frac{t^2}{2} \Big(\frac{1}{8} + \frac{t}{24} \frac{e^t + e^{-t}}{e^t - e^{-t}} \Big).$$

We use the polar coordinates to bound from below

$$\begin{aligned}
(f_0, R(t)f_0) \;&\geq\; \frac{1}{2} \frac{t^2}{2} \Big(\frac{1}{8} + \frac{t}{24} \frac{e^t + e^{-t}}{e^t - e^{-t}} \Big) \int\int_{B_t} K(t,x,y)dxdy \\
&= \frac{\frac{t^2}{4}\left(\frac{1}{8} + \frac{t}{24} \frac{e^t + e^{-t}}{e^t - e^{-t}} \right)}{\sqrt{\pi(e^t - e^{-t})}} \\
&\qquad \times \int_0^{\delta(t)} \int_0^{2\pi} \exp\Big[- \frac{(e^t + e^{-t} - 4\sin\theta\cos\theta)\rho^2}{2(e^t - e^{-t})} \Big] \rho d\rho d\theta \\
&\geq \frac{\frac{t^2}{4}\left(\frac{1}{8} + \frac{t}{24} \frac{e^t + e^{-t}}{e^t - e^{-t}} \right)}{\sqrt{\pi(e^t - e^{-t})}} \int_0^{\delta(t)} \int_0^{2\pi} \exp\Big[- \frac{(e^t + e^{-t} + 2)\rho^2}{2(e^t - e^{-t})} \Big] \rho d\rho d\theta.
\end{aligned}$$

Calculating the integral on the right, we get with (3.34)

$$\begin{aligned}
&(f_0, R(t)f_0) \\
&\geq \frac{\sqrt{\pi(e^t - e^{-t})}}{2} t^2 \Big(\frac{1}{8} + \frac{t}{24}\frac{e^t + e^{-t}}{e^t - e^{-t}}\Big) \frac{1 - \exp[-\frac{3}{8}\frac{\frac{1}{8}(e^t - e^{-t}) + \frac{t}{24}(e^t + e^{-t})}{e^t - e^{-t}}]}{e^t + e^{-t} + 2} \\
&=: c_1(t). \qquad (3.35)
\end{aligned}$$

Then $c_1(t)$ is a positive, continous function in $t > 0$, which we see with (3.16) is uniformly bounded. Since $\lim_{t \to 0+} c_1(t) = 0$, we may define $c_1(0) = 0$. So we have obtained a lower bound $\|R(t)\| \geq c_1(t)$. Next, we get from (3.30)/(3.33)

$$(f_0, R^{(n)}(t)f_0) = (f_0, R(t)f_0) + (f_0, Q^{(n)}(t)f_0)n^{-1} \geq (c_1(t) - C_2(t)n^{-1}).$$

We can see that $C_2(t)/c_1(t)$ is bounded in $t > 0$, because all the seven functions on the right of (3.30)

$$\tfrac{[1\vee t^2]t^3}{(e^t - e^{-t})^{1/2}}, \quad \tfrac{[1\vee t^2]t^4}{(e^t - e^{-t})^{1/2}}, \quad \tfrac{\left(\frac{1}{8} + \frac{t}{24}\frac{e^t + e^{-t}}{e^t - e^{-t}}\right)t^4}{(e^t - e^{-t})^{1/2}}, \quad \tfrac{t^4}{(e^t - e^{-t})^{1/2}},$$
$$\tfrac{[1\vee t^2]t^6}{(e^t - e^{-t})^{1/2}}, \quad \tfrac{\left(\frac{1}{8} + \frac{t}{24}\frac{e^t + e^{-t}}{e^t - e^{-t}}\right)t^6}{(e^t - e^{-t})^{1/2}}, \quad \tfrac{[1\vee t^2]t^8}{(e^t - e^{-t})^{1/2}}$$

divided by $c_1(t)$ are bounded there, by (3.16) and (3.17). It follows that $c_1(t) - C_2(t)n^{-1}$ is positive in $t > 0$ for n large. In fact, there exists a positive integer N, independent of t, such that $c_1(t) - C_2(t)n^{-1} \geq \frac{1}{2}c_1(t) > 0$ for all $t > 0$ and all $n \geq N$. Take $c(t) := \frac{1}{2}c_1(t)$ so that $(f_0, R^{(n)}(t)f_0) \geq c(t)$. The function $c(t)$ is nonnegative, continuous and uniformly bounded in $t \geq 0$ with $c(0) := \lim_{t\to 0+} c(t) = 0$. We can also see $\lim_{t\to\infty} c(t) = 0$. This yields that $\|R^{(n)}(t)\| \geq c(t)$ in $t > 0$ for $n \geq N$, and hence, as we may think, for all $n \geq 1$, showing the lower bound of the desired relation (1.11). This completes the proof of Theorem 1.2.

4 Concluding Remark

The results of the present paper can be in fact obtained in a more sophisticated way as in [9], by the commutator method. Among others, we shall briefly mention only how by this method the bounded operator $R(t)$ with integral kernel $K(t, x, y)$ in (1.7), Theorem 1.1, comes out.

By the Baker–Campbell–Hausdorff formula (e.g. [11]), one has, when A and B are *bounded* operators,

$$[e^{-tB/2n}e^{-tA/n}e^{-tB/2n}]^n - e^{-t(A+B)}$$

$$
\begin{aligned}
&= \exp\big(-t(A+B) - n^{-2}\frac{t^2}{24}[2A+B,[A,B]] - O_p(n^{-3})\big) \\
&= e^{-t(A+B)} - n^{-2}\frac{t^2}{24}\int_0^t e^{-(t-s)(A+B)}[2A+B,[A,B]]e^{-s(A+B)}ds + O_p(n^{-3}),
\end{aligned}
\tag{4.1}
$$

where $[A,B] := AB - BA$, and $O_p(n^{-3})$ stands for an operator with norm of order $O(n^{-3})$. We cannot in general take unbounded operators as A and B. However, in our case with $A := H_0 = -\frac{1}{2}\Delta$ and $B := V = \frac{1}{2}x^2$, though they are unbounded operators in $L^2(\mathbf{R})$, we can show that (4.1) is still valid. We omit the proof. Instead, we content ourselves to see that the second term in the last member of (4.1) makes sense to yield $R(t)n^{-2}$. Namely, putting

$$
\hat{R}(t) := -\frac{t^2}{24}\int_0^t e^{-(t-s)H}[2H_0+V,[H_0,V]]e^{-sH}ds, \tag{4.2}
$$

we have

Proposition 4.1 *For the operators $H_0 = -\frac{1}{2}\Delta$ and $V = \frac{1}{2}x^2$ in $L^2(\mathbf{R})$, $\hat{R}(t)$ has the same integral kernel $R(t,x,y)$ in (1.7) as $R(t)$, so that both the operators $\hat{R}(t)$ and $R(t)$ coincide with each other.*

In the following proof, we shall refer to (1.7) related to the first one of the two expressions of $s(t,x,y)$ in (3.10)/(3.11) with which $R(t,x,y)$ is rewritten as (3.9).

Proof. We have

$$
\begin{aligned}
&[H_0,V] = -\frac{1}{2}(1+2x\partial_x) \\
&[H_0,[H_0,V]] = \partial_x^2 = -2H_0, \quad [V,[H_0,V]] = x^2 = 2V,
\end{aligned}
$$

so that

$$
[2H_0+V,[H_0,V]] = -4H_0 + 2V = -4H + 6V.
$$

Put $C := \frac{1}{2}(p\cdot x + x\cdot p)$ with $p = -i\partial_x$. Then since $H = H_0 + V = (p^2+x^2)/2$, we have $i[H,C] = [(p^2+x^2)/2, iC] = (p^2-x^2)$, so that

$$
\frac{1}{2}x^2 = \frac{1}{4}((p^2+x^2)-(p^2-x^2)) = \frac{1}{4}(2H - i[H,C]).
$$

Then

$$\begin{aligned}\int_0^t e^{-(t-s)H} V e^{-sH} ds &= \int_0^t e^{-(t-s)H}(x^2/2)e^{-sH} ds \\ &= \frac{1}{4}\int_0^t e^{-(t-s)H}(2H - i[H,C])e^{-sH} ds \\ &= \frac{t}{2} He^{-tH} - \frac{i}{4}\int_0^t \frac{d}{ds}[e^{-(t-s)H} C e^{-sH}] ds \\ &= \frac{t}{2} He^{-tH} - \frac{1}{4}(e^{-tH}(iC) - (iC)e^{-tH}).\end{aligned}$$

It follows that

$$\begin{aligned}\hat{R}(t) &= \frac{t^2}{24}\int_0^t e^{-(t-s)H}(4H - 6V)e^{-sH} ds \\ &= -t^2\Big[(-\frac{4}{24} + \frac{3}{24})tHe^{-tH} + \frac{1}{16}(e^{-tH}(iC) - (iC)e^{-tH})\Big] \\ &= t^2\Big[\frac{1}{24} tHe^{-tH} - \frac{1}{16}(e^{-tH}(iC) - (iC)e^{-tH})\Big]. \qquad (4.3)\end{aligned}$$

Then, calculating both $-He^{-tH} = \frac{d}{dt}e^{-tH} = \frac{d}{dt}K(t)$ and

$$\begin{aligned}e^{-tH}(iC) - (iC)e^{-tH} &= \frac{1}{2}[-y(\partial_y e^{-tH}) - \partial_y(ye^{-tH}) - x(\partial_x e^{-tH}) - \partial_x(xe^{-tH})] \\ &= -e^{-tH} - y(\partial_y e^{-tH}) - x(\partial_x e^{-tH}) \\ &= -\Big[1 + \frac{4xy - (e^t + e^{-t})(x^2 + y^2)}{e^t - e^{-t}}\Big]e^{-tH}\end{aligned}$$

on the right of (4.3), we get the first and the second term on the right-hand side of (1.7). □

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