LONG-TIME ASYMPTOTICS FOR THE TODA LATTICE IN THE SOLITON REGION

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Abstract. We apply the method of nonlinear steepest descent to compute the long-time asymptotics of the Toda lattice for decaying initial data in the soliton region. In addition, we point out how to reduce the problem in the remaining region to the known case without solitons.

1. Introduction

In this paper we want to compute the long time asymptotics for the doubly infinite Toda lattice which reads in Flaschka's variables (see e.g. [16], [17], or [18])

$$\begin{aligned} \dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \tag{1.1}$$

$(n,t) \in \mathbb{Z} \times \mathbb{R}$. Here the dot denotes differentiation with respect to time. We will consider solutions (a,b) satisfying

$$\sum_n |n|^l (|a(n,t) - \frac{1}{2}| + |b(n,t)|) < \infty \tag{1.2}$$

for every $l \in \mathbb{N}$ for one (and hence for all, see [16]) $t \in \mathbb{R}$. It is well-known that this initial value problem has unique global solutions which can be computed via the inverse scattering transform [16].

The long-time asymptotics for this problem were first given by Novokshenov and Habibullin [12] and were later made rigorous by Kamvissis [7], however, only in the case without solitons. The purpose of the present paper is to finally fill this gap and show how to include solitons. As in [7], our approach is based on the nonlinear steepest descent analysis for oscillatory Riemann–Hilbert problems from Deift and Zhou [4]. It turns out that in the case of solitons, two new phenomena enter the scene which require significant adaptions to the original method of Deift and Zhou. Of course our technique also applies to other soliton equations, e.g., the Korteweg–de Vries equation.

First of all, it is well-known that there is a subtle nonuniqueness issue for the involved Riemann–Hilbert problems (see e.g. [2, Chap. 38]). In fact, in certain exceptional sets the corresponding vanishing Riemann–Hilbert problem has a nontrivial solution and hence by Fredholm theory the corresponding matrix Riemann–Hilbert problem has no solution at all. This problem does not affect the similarity region, since it is easy to see that there it does not happen for sufficiently large times. However, in the soliton region, this occurs precisely in the neighborhoods of the single

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solitons. To avoid this problem we will work directly with the vector Riemann–Hilbert problem and impose a symmetry condition in order to ensure uniqueness. This also has the advantage that it eliminates the step of going forth and back between the vector and matrix Riemann–Hilbert problem. It should be pointed out here that even the symmetry condition alone does not guarantee uniqueness, rather we will need existence of a certain solution with an additional property. We will also demonstrate that this additional property is in fact necessary.

Secondly, in the regions of the single solitons, the zeroth order asymptotics are not equal to zero but given by a one soliton solution. Hence for the usual perturbation argument based on the second resolvent identity to work, a uniform bound for the inverse of the singular integral equation associated with the one soliton solution is needed. Unfortunately, such a bound cannot be easily obtained. To overcome this problem we will shift the leading asymptotics from the one soliton solution to the inhomogeneous part of the singular integral equation and craft our Cauchy kernel in such a way that it preserves the pole conditions for this single soliton.

To state our main result, we begin by recalling that the sequences $a(n,t)$, $b(n,t)$, $n \in \mathbb{Z}$, for fixed $t \in \mathbb{R}$, are uniquely determined by their scattering data, that is, by the left reflection coefficient $R_+(z,t)$, $|z| = 1$ and the eigenvalues $\lambda_j \in (-\infty,-1) \cup (1,\infty)$, $j = 1,\dots,N$ together with the corresponding right norming constants $\gamma_{+,j}(t) > 0$, $j = 1,\dots,N$. Rather than in the complex plane, we will work on the unit disc using the usual Joukowski transformation

$$\lambda = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad z = \lambda - \sqrt{\lambda^2 - 1}, \qquad \lambda \in \mathbb{C},\ |z| \le 1. \tag{1.3}$$

In these new coordinates the eigenvalues $\lambda_j \in (-\infty,-1)\cup(1,\infty)$ will be denoted by $\zeta_j \in (-1,0) \cup (0,1)$. The continuous spectrum $[-1,1]$ is mapped to the unit circle $\mathbb{T}$. Moreover, the phase of the associated Riemann–Hilbert problem is given by

$$\Phi(z) = z - z^{-1} + 2\frac{n}{t}\log(z). \tag{1.4}$$

and the stationary phase points, $\Phi'(z) = 0$, are denoted by

$$z_0 = -\frac{n}{t} - \sqrt{(\frac{n}{t})^2 - 1}, \quad z_0^{-1} = -\frac{n}{t} + \sqrt{(\frac{n}{t})^2 - 1}, \qquad \lambda_0 = -\frac{n}{t}. \tag{1.5}$$

For $\frac{n}{t} < -1$ we have $z_0 \in (0,1)$, for $-1 \le \frac{n}{t} \le 1$ we have $z_0 \in \mathbb{T}$ (and hence $z_0^{-1} = \overline{z_0}$), and for $\frac{n}{t} > 1$ we have $z_0 \in (-1,0)$. For $|\frac{n}{t}| > 1$ we will also need the value $\zeta_0 \in (-1,0) \cup (0,1)$ defined via $\mathrm{Re}(\phi(\zeta_0)) = 0$, that is,

$$\frac{n}{t} = -\frac{\zeta_0 - \zeta_0^{-1}}{2\log(|\zeta_0|)}. \tag{1.6}$$

We will set $\zeta_0 = -1$ if $|\frac{n}{t}| \le 1$ for notational convenience. A simple analysis shows that for $\frac{n}{t} < -1$ we have $0 < \zeta_0 < z_0 < 1$ and for $\frac{n}{t} > 1$ we have $-1 < z_0 < \zeta_0 < 0$.

Furthermore, recall that the transmission coefficient $T(z)$, $|z| \le 1$, is time independent and can be reconstructed using the Poisson–Jensen formula. In particular, we define the partial transmission coefficient with respect to z_0 by

(1.7)
$$T(z,z_0)=\begin{cases}\prod\limits_{\zeta_k\in(\zeta_0,0)}|\zeta_k|\frac{z-\zeta_k^{-1}}{z-\zeta_k}, & z_0\in(-1,0),\\ \prod\limits_{\zeta_k\in(-1,0)}|\zeta_k|\frac{z-\zeta_k^{-1}}{z-\zeta_k}\cdot\exp\left(\frac{1}{2\pi\mathrm{i}}\int\limits_{z_0}^{\overline{z_0}}\log(|T(s)|)\frac{s+z}{s-z}\frac{ds}{s}\right), & |z_0|=1,\\ \prod\limits_{\zeta_k\in(-1,0)\cup(\zeta_0,1)}|\zeta_k|\frac{z-\zeta_k^{-1}}{z-\zeta_k}\cdot\exp\left(\frac{1}{2\pi\mathrm{i}}\int\limits_{\mathbb{T}}\log(|T(s)|)\frac{s+z}{s-z}\frac{ds}{s}\right), & z_0\in(0,1).\end{cases}$$

Here the integral is to be taken along the arc $\Sigma(z_0)=\{z\in\mathbb{T}|\mathrm{Re}(z)<\mathrm{Re}(z_0)\}$. Observe that $T(z,z_0)=T(z)$ once $z_0\in(0,1)$ and $(0,z_0)$ contains no eigenvalues. Moreover, $T(z,z_0)$ can be computed in terms of the scattering data since $|T(z)|^2=1-|R_+(z,t)|^2$.

Moreover, we set

$$\begin{aligned}T_0(z_0)&=T(0,z_0)\\ &=\begin{cases}\prod\limits_{\zeta_k\in(\zeta_0,0)}|\zeta_k|^{-1}, & z_0\in(-1,0),\\ \prod\limits_{\zeta_k\in(-1,0)}|\zeta_k|^{-1}\cdot\exp\left(\frac{1}{2\pi\mathrm{i}}\int\limits_{z_0}^{\overline{z_0}}\log(|T(s)|)\frac{ds}{s}\right), & |z_0|=1,\\ \prod\limits_{\zeta_k\in(-1,0)\cup(\zeta_0,1)}|\zeta_k|^{-1}\cdot\exp\left(\frac{1}{2\pi\mathrm{i}}\int\limits_{\mathbb{T}}\log(|T(s)|)\frac{ds}{s}\right), & z_0\in(0,1),\end{cases}\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}T_1(z_0)&=\frac{\partial}{\partial z}\log T(z,z_0)\Big|_{z=0}\\ &=\begin{cases}\sum\limits_{\zeta_k\in(\zeta_0,0)}(\zeta_k^{-1}-\zeta_k), & z_0\in(-1,0),\\ \sum\limits_{\zeta_k\in(-1,0)}(\zeta_k^{-1}-\zeta_k)+\frac{1}{\pi\mathrm{i}}\int\limits_{z_0}^{\overline{z_0}}\log(|T(s)|)\frac{ds}{s^2}, & |z_0|=1,\\ \sum\limits_{\zeta_k\in(-1,0)\cup(\zeta_0,1)}(\zeta_k^{-1}-\zeta_k)+\frac{1}{\pi\mathrm{i}}\int\limits_{\mathbb{T}}\log(|T(s)|)\frac{ds}{s^2}, & z_0\in(0,1).\end{cases}\end{aligned} \tag{1.9}$$

Theorem 1.1. *Assume (1.2) and abbreviate by* $c_k=-\frac{\zeta_k-\zeta_k^{-1}}{2\log(|\zeta_k|)}$ *the velocity of the k'th soliton determined by* $\mathrm{Re}(\Phi(\zeta_k))=0$. *Then the asymptotics in the soliton region,* $|n/t|\geq 1+C/t\log(t)^2$ *for some* $C>0$, *are as follows.*

Let $\varepsilon>0$ *sufficiently small such that the intervals* $[c_k-\varepsilon,c_k+\varepsilon]$, $1\leq k\leq N$, *are disjoint and lie inside* $(1,0)\cup(0,1)$.

If $|\frac{n}{t}-c_k|<\varepsilon$ *for some k, one has*

$$\begin{aligned}\prod_{j=n}^{\infty}(2a(j,t))&=T_0(z_0)\left(\sqrt{\frac{1-\zeta_k^2+\gamma_k(n,t)}{1-\zeta_k^2+\gamma_k(n,t)\zeta_k^2}}+O(t^{-l})\right),\\ \sum_{j=n+1}^{\infty}b(j,t)&=T_1(z_0)+\frac{\gamma_k(n,t)\zeta_k(\zeta_k^2-1)}{2((\gamma_k(n,t)-1)\zeta_k^2+1)}+O(t^{-l}),\end{aligned} \tag{1.10}$$

for any $l\geq 1$, *where*

$$\gamma_k(n,t)=\gamma_k T(\zeta_k,c_k)^2\mathrm{e}^{t(\zeta_k-\zeta_k^{-1})}\zeta_k^{2n}. \tag{1.11}$$

If $|\frac{n}{t} - c_k| \geq \varepsilon$*, for all* k*, one has*

$$\prod_{j=n}^{\infty}(2a(j,t)) = T_0(z_0)\left(1 + O(t^{-l})\right),$$
$$\sum_{j=n+1}^{\infty} b(j,t) = T_1(z_0) + O(t^{-l}), \tag{1.12}$$

for any $l \geq 1$.

The proof will be given at the end of Section 4.

In the remaining regions the analysis in Section 4 also shows that the Riemann–Hilbert problem reduces to one without solitons. In fact, away from the soliton region, the asymptotics are given by

$$\prod_{j=n}^{\infty}(2a(j,t)) = T_0(-1)\prod_{j=n}^{\infty}(2\tilde{a}(j,t))\left(1 + O(t^{-l})\right),$$
$$\sum_{j=n+1}^{\infty} b(j,t) = T_1(-1) + \sum_{j=n+1}^{\infty}\tilde{b}(j,t) + O(t^{-l}), \tag{1.13}$$

where $\tilde{a}(n,t)$, $\tilde{b}(n,t)$ are the solutions corresponding to the case without solitons and with $R_+(z,0)$ replaced by

$$\tilde{R}_+(z,0) = T(z,-1)^{-2}R_+(z,0). \tag{1.14}$$

Note that the Blaschke product

$$T(z,-1) = \prod_{\zeta_k\in(-1,0)} |\zeta_k| \frac{z - \zeta_k^{-1}}{z - \zeta_k}$$

satisfies $|T(z,-1)| = 1$ for $z \in \mathbb{T}$. Hence everything is reduced to the case studied in [7].

Finally we remark that the same method can be used to handle solitons on a periodic background [10] (cf. also [6], [8], [9]).

2. The Inverse scattering transform and the Riemann–Hilbert problem

In this section we want to derive the Riemann–Hilbert problem from scattering theory. The special case without eigenvalues was first given in Kamvissis [7]. The eigenvalues will be added by appropriate pole conditions which are then turned into jumps following Deift, Kamvissis, Kriecherbauer, and Zhou [5].

For the necessary results from scattering theory respectively the inverse scattering transform for the Toda lattice we refer to [14], [15], [16].

Associated with $a(t), b(t)$ is a self-adjoint Jacobi operator

$$H(t) = a(t)S^+ + a^-(t)S^- + b(t) \tag{2.1}$$

in $\ell^2(\mathbb{Z})$, where $S^\pm f(n) = f^\pm(n) = f(n\pm 1)$ are the usual shift operators and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. By our assumption (1.2) the spectrum of H consists of an absolutely continuous part

$[-1,1]$ plus a finite number of eigenvalues $\lambda_k \in \mathbb{R}\backslash[-1,1]$, $1 \le k \le N$. In addition, there exist two Jost functions $\psi_\pm(z,n,t)$ which solve the recurrence equation

$$H(t)\psi_\pm(z,n,t) = \frac{z+z^{-1}}{2}\psi_\pm(z,n,t), \qquad |z| \le 1, \tag{2.2}$$

and asymptotically look like the free solutions

$$\lim_{n\to\pm\infty} z^{\mp n}\psi_\pm(z,n,t) = 1. \tag{2.3}$$

Both $\psi_\pm(z,n,t)$ are analytic for $0 < |z| < 1$ with smooth boundary values for $|z| = 1$. The asymptotics of the two projections of the Jost function are

$$\psi_\pm(z,n,t) = \frac{z^{\mp n}}{A_\pm(n,t)}\Big(1 + 2B_\pm(n,t)z + O(z^2)\Big), \tag{2.4}$$

as $z \to 0$, where

$$\begin{aligned} A_+(n,t) &= \prod_{j=n}^{\infty} 2a(j,t), \quad B_+(n,t) = -\sum_{j=n+1}^{\infty} b(j,t), \\ A_-(n,t) &= \prod_{j=-\infty}^{n-1} 2a(j,t), \quad B_-(n,t) = -\sum_{j=-\infty}^{n-1} b(j,t). \end{aligned} \tag{2.5}$$

One has the scattering relations

$$T(z)\psi_\mp(z,n,t) = \overline{\psi_\pm(z,n,t)} + R_\pm(z,t)\psi_\pm(z,n,t), \qquad |z| = 1, \tag{2.6}$$

where $T(z)$, $R_\pm(z,t)$ are the transmission respectively reflection coefficients. The transmission and reflection coefficients have the following well-known properties:

Lemma 2.1. *The transmission coefficient $T(z)$ has a meromorphic extension to the interior of the unit circle with simple poles at the images of the eigenvalues ζ_j. The residues of $T(z)$ are given by*

$$\operatorname*{Res}_{z=\zeta_k} T(z) = -\zeta_k \frac{\gamma_{+,k}(t)}{\mu_k(t)} = -\zeta_k \gamma_{-,k}(t)\mu_k(t), \tag{2.7}$$

where

$$\gamma_{\pm,k}(t)^{-1} = \sum_{m\in\mathbb{Z}} |\psi_\pm(\zeta_k,n,t)|^2 \tag{2.8}$$

and $\psi_-(\zeta_k,n,t) = \mu_k(t)\psi_+(\zeta_k,n,t)$.

Moreover,

$$T(z)\overline{R_+(z,t)} + \overline{T(z)}R_-(z,t) = 0, \qquad |T(z)|^2 + |R_\pm(z,t)|^2 = 1. \tag{2.9}$$

In particular one reflection coefficient, say $R(z,t) = R_+(z,t)$, and one set of norming constants, say $\gamma_k(t) = \gamma_{+,k}(t)$, suffices. Moreover, the time dependence is given by:

Lemma 2.2. *The time evolutions of the quantities $R_+(z,t)$, $\gamma_{+,k}(t)$ are given by*

$$R(z,t) = R(z)\mathrm{e}^{t(z-z^{-1})} \tag{2.10}$$

$$\gamma_k(t) = \gamma_k \mathrm{e}^{t(\zeta_k - \zeta_k^{-1})}, \tag{2.11}$$

where $R(z) = R(z,0)$ and $\gamma_k = \gamma_k(0)$.

We will define a Riemann–Hilbert problem as follows:

$$m(z,n,t) = \begin{cases} \begin{pmatrix} T(z)\psi_-(z,n,t)z^n & \psi_+(z,n,t)z^{-n} \end{pmatrix}, & |z|<1, \\ \begin{pmatrix} \psi_+(z^{-1},n,t)z^n & T(z^{-1})\psi_-(z^{-1},n,t)z^{-n} \end{pmatrix}, & |z|>1. \end{cases} \tag{2.12}$$

We are interested in the jump condition of $m(z,n,t)$ on the unit circle $\mathbb{T}$ (oriented counterclockwise). To formulate our jump condition we use the following convention: When representing functions on $\mathbb{T}$, the lower subscript denotes the non-tangential limit from different sides,

$$m_\pm(z) = \lim_{\zeta\to z,\ |\zeta|^{\pm 1}<1} m(\zeta), \qquad |z|=1. \tag{2.13}$$

Using the notation above implicitly assumes that these limits exist in the sense that $m(z)$ extends to a continuous function on the boundary away from ± 1.

Theorem 2.3 (Vector Riemann–Hilbert problem)**.** *Let $\mathcal{S}_+(H(0)) = \{R(z),\ |z|=1;\ (\zeta_k,\gamma_k),\ 1\le k\le N\}$ the left scattering data of the operator $H(0)$. Then $m(z)=m(z,n,t)$ defined in (2.12) is meromorphic away from the unit circle with simple poles at ζ_k and satisfies:*

(i) *The jump condition*

$$m_+(z) = m_-(z)v(z), \qquad v(z) = \begin{pmatrix} 1-|R(z)|^2 & -\overline{R(z)}\mathrm{e}^{-t\Phi(z)} \\ R(z)\mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}, \tag{2.14}$$

for $z\in\mathbb{T}$,

(ii) *the pole conditions*

$$\begin{aligned} \operatorname*{Res}_{\zeta_k} m(z) &= \lim_{z\to\zeta_k} m(z) \begin{pmatrix} 0 & 0 \\ -\zeta_k\gamma_k\mathrm{e}^{t\Phi(\zeta_k)} & 0 \end{pmatrix} \\ \operatorname*{Res}_{\zeta_k^{-1}} m(z) &= \lim_{z\to\zeta_k^{-1}} m(z) \begin{pmatrix} 0 & \zeta_k^{-1}\gamma_k\mathrm{e}^{t\Phi(\zeta_k)} \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.15}$$

(iii) *the symmetry condition*

$$m(z^{-1}) = m(z)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.16}$$

(iv) *and the normalization*

$$m(0) = (m_1 \quad m_2), \quad m_1\cdot m_2 = 1 \quad m_1>0. \tag{2.17}$$

Here the phase is given by

$$\Phi(z) = z - z^{-1} + 2\frac{n}{t}\log z. \tag{2.18}$$

Proof. The jump condition (2.14) is a simple calculation using the scattering relations (2.6) plus (2.9). The pole conditions follow since $T(z)$ is meromorphic in $|z|<1$ with simple poles at ζ_k and residues given by (2.7). The symmetry condition holds by construction and the normalization (2.17) is immediate from the following lemma. □

Observe that the pole condition at ζ_k is sufficient since the one at ζ_k^{-1} follows by symmetry.

Moreover, we have the following asymptotic behaviour near $z=0$:

Lemma 2.4. *The function $m(z,n,t)$ defined in (2.12) satisfies*

$$m(z,n,t) = \begin{pmatrix} A(n,t)(1+2B(n-1,t)z) & \frac{1}{A(n,t)}(1-2B(n,t)z) \end{pmatrix} + O(z^2). \tag{2.19}$$

Here $A(n,t) = A_+(n,t)$ and $B(n,t) = B_+(n,t)$ are defined in (2.5).

Proof. This follows from (2.4) and $T(z) = A_+A_-(1-2(B_+ + b + B_-)z + O(z^2)$. □

For our further analysis it will be convenient to rewrite the pole condition as a jump condition and hence turn our meromorphic Riemann–Hilbert problem into a holomorphic Riemann–Hilbert problem following [5]. Choose ε so small that the discs $|z-\zeta_k| < \varepsilon$ are inside the unit circle and do not intersect. Then redefine m in a neighborhood of ζ_k respectively ζ_k^{-1} according to

$$m(z) = \begin{cases} m(z)\begin{pmatrix} 1 & 0 \\ \frac{\zeta_k\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} & 1 \end{pmatrix}, & |z-\zeta_k| < \varepsilon, \\ m(z)\begin{pmatrix} 1 & -\frac{z\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k^{-1}} \\ 0 & 1 \end{pmatrix}, & |z^{-1}-\zeta_k| < \varepsilon, \\ m(z), & \text{else}. \end{cases} \tag{2.20}$$

and for $|z| > 1$ by symmetry (2.16). Then a straightforward calculation using $\mathrm{Res}_\zeta\, m = \lim_{z\to\zeta}(z-\zeta)m(z)$ shows

Lemma 2.5. *Suppose $m(z)$ is redefined as in (2.20). Then $m(z)$ is holomorphic away from the unit circle and satisfies (2.14), (2.16), (2.17) and the pole condition is replaced by the jump condition*

$$\begin{aligned} m_+(z) &= m_-(z)\begin{pmatrix} 1 & 0 \\ \frac{\zeta_k\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} & 1 \end{pmatrix}, \quad |z-\zeta_k| = \varepsilon, \\ m_+(z) &= m_-(z)\begin{pmatrix} 1 & -\frac{z\gamma_k \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} \\ 0 & 1 \end{pmatrix}, \quad |z^{-1}-\zeta_k| = \varepsilon, \end{aligned} \tag{2.21}$$

where the small circle around ζ_k is oriented counterclockwise.

Next we turn to uniqueness of the solution of this vector Riemann–Hilbert problem. This will also explain the reason for our symmetry condition. We begin by observing that if there is a point $z_1 \in \mathbb{C}$, such that $m(z_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, then $n(z) = \frac{1}{z-z_1} m(z)$ satisfies the same jump and pole conditions as $m(z)$. However, it will clearly violate the symmetry condition! Hence, without the symmetry condition, the solution of our vector Riemann–Hilbert problem will not be unique in such a situation. Moreover, a look at the one soliton solution verifies that this case indeed can happen.

Lemma 2.6 (One soliton solution)**.** *Suppose there is only one eigenvalue and a vanishing reflection coefficient, that is, $\mathcal{S}_+(H(t)) = \{R(z) \equiv 0,\ |z| = 1;\ (\zeta,\gamma)\}$. Then the solution of the Riemann–Hilbert problem (2.14)–(2.17) is given by*

$$\begin{aligned} m_0(z) &= \begin{pmatrix} f(z) & f(1/z) \end{pmatrix} \\ f(z) &= \frac{1}{\sqrt{1-\zeta^2+\gamma}\sqrt{1-\zeta^2+\zeta^2\gamma}} \left(\gamma\zeta^2 \frac{z-\zeta^{-1}}{z-\zeta} + 1 - \zeta^2 \right). \end{aligned} \tag{2.22}$$

In particular,

$$A_+ = \sqrt{\frac{1-\zeta^2+\gamma}{1-\zeta^2+\gamma\zeta^2}}, \qquad B_+ = \frac{\gamma\zeta(\zeta^2-1)}{2((\gamma-1)\zeta^2+1)}. \tag{2.23}$$

In fact, observe $f(z_1) = f(z_1^{-1}) = 0$ if and only if $z_1 = \pm 1$ and $\gamma = \pm(\zeta^{-1} - \zeta)$. Furthermore, even in the general case $m(z_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ can only occur at $z_1 = \pm 1$ as the following lemma shows.

Lemma 2.7. *If $m(z_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ for m defined as in (2.12), then $z_1 = \pm 1$. Moreover, the zero of at least one component is simple in this case.*

Proof. By (2.12) the condition $m(z_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ implies that the Jost solutions $\psi_-(z,n)$ and $\psi_+(z,n)$ are linearly dependent. This can only happen, at a band edge, $z_1 = \pm 1$, or at an eigenvalues $z_1 = \zeta_j$.

We begin with the case $z_1 = \zeta_j$. In this case the derivative of the Wronskian $W(z) = a(n)(\psi_+(z,n)\psi_-(z,n+1) - \psi_+(z,n+1)\psi_-(z,n))$ does not vanish $\frac{d}{dz}W(z)|_{z=z_1} \neq 0$ ([16, Chap. 10]). Moreover, the diagonal Green's function $g(\lambda,n) = W(z)^{-1}\psi_+(z,n)\psi_-(z,n)$ is Herglotz and hence can have at most a simple zero at $z = z_1$. Hence, if $\psi_+(\zeta_j,n) = \psi_-(\zeta_j,n) = 0$, both can have at most a simple zeros at $z = \zeta_j$. But $T(z)$ has a simple pole at ζ_j and hence $T(z)\psi_-(z,n)$ cannot vanish at $z = \zeta_j$, a contradiction.

It remains to show that one zero is simple in this case $z_1 = \pm 1$. In fact, one can show that $\frac{d}{dz}W(z)|_{z=z_1} \neq 0$ in this case as follows: First of all note that $\psi'_\pm(z)$ (where $'$ denotes the derivative with respect to z) again solves $H\psi'_\pm(z_1) = \lambda_1\psi'_\pm(z_1)$ if $z_1 = \pm 1$. Moreover, by $W(z_1) = 0$ we have $\psi_+(z_1) = c\psi_-(z_1)$ for some constant c (independent of n). Thus we can compute

$$\begin{aligned} W'(z_1) &= W(\psi'_+(z_1), \psi_-(z_1)) + W(\psi_+(z_1), \psi'_-(z_1)) \\ &= c^{-1}W(\psi'_+(z_1), \psi_+(z_1)) + cW(\psi_-(z_1), \psi'_-(z_1)) \end{aligned}$$

by letting $n \to +\infty$ for the first and $n \to -\infty$ for the second Wronskian (in which case we can replace $\psi_\pm(z_1)$ by $z_1^{\pm n}$), which gives

$$W'(z_1) = \frac{c + c^{-1}}{2}.$$

Hence the Wronskian has a simple zero. But if both functions had more than simple zeros, so would the Wronskian, a contradiction. □

3. A uniqueness result for symmetric vector Riemann–Hilbert problems

In this section we want to investigate uniqueness for the vector Riemann–Hilbert problem

$$\begin{aligned} & m_+(z) = m_-(z)v(z), \qquad z \in \Sigma, \\ & m(z^{-1}) = m(z)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ & m(0) = \begin{pmatrix} 1 & m_2 \end{pmatrix}. \end{aligned} \tag{3.1}$$

The normalization used here will be more convenient than (2.17). In fact, (2.17) will be satisfied by $m_2^{-1/2}m(z)$.

Hypothesis H. 3.1. *Let Σ consist of a finite number of smooth oriented finite curves in $\mathbb{C}$. Assume that the contour Σ does not contain 0 and is invariant under $z \mapsto z^{-1}$. Moreover, suppose the jump matrix v can be factorized according to $v = b_-^{-1} b_+ = (\mathbb{I} - w_-)^{-1}(\mathbb{I} + w_+)$, where $w_\pm$ are continuous and satisfy*

$$w_\pm(z^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w_\mp(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in \Sigma. \tag{3.2}$$

Now we are ready to show that the symmetry condition in fact guarantees uniqueness.

Theorem 3.2. *Suppose there exists a solution $m(z)$ of the Riemann–Hilbert problem (3.1) for which $m(z) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ can happen at most for $z = \pm 1$ in which case $\limsup_{z\to\pm 1} \frac{z\mp 1}{m_j(z)}$ is bounded from any direction for $j = 1$ or $j = 2$.*

Then the Riemann–Hilbert problem (3.1) with norming condition replaced by

$$m(0) = \begin{pmatrix} \alpha & m_2 \end{pmatrix} \tag{3.3}$$

for given $\alpha \in \mathbb{C}$, has a unique solution $m_\alpha(z) = \alpha\, m(z)$.

Proof. Let $m_\alpha(z)$ be a solution of (3.1) normalized according to (3.3). Then we can construct a matrix valued solution via $M = (m, m_\alpha)$ and there are two possible cases: Either $\det M(z)$ is nonzero for some z or it vanishes identically.

We start with the first case. By the last lemma, we can rewrite all poles as jumps with determinant one. Hence, the determinant of this modified Riemann–Hilbert problem has no jump. Since, it is bounded at infinity, the determinant is constant. But taking determinants in

$$M(z^{-1}) = M(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

gives a contradiction.

It remains to investigate the case where $\det(M) \equiv 0$. In this case we have $m_\alpha(z) = \delta(z) m(z)$ with a scalar function δ. Moreover, $\delta(z)$ must be holomorphic for $z \in \mathbb{C}\backslash\Sigma$ and differentiable for $z \in \Sigma$ except possibly at the points where $m(z_0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$. Since it has no jump across Σ,

$$\delta_+(z) m_+(z) = m_{\alpha,+}(z) = m_{\alpha,-}(z) v(z) = \delta_-(z) m_-(z) v(z) = \delta_-(z) m_+(z),$$

it is even holomorphic in $\mathbb{C}\backslash\{\pm 1\}$ with at most simple poles at $z = \pm 1$. Hence it must be of the form

$$\delta(z) = A + \frac{B}{z-1} + \frac{C}{z+1}.$$

Since δ has to be symmetric, $\delta(z) = \delta(z^{-1})$, we obtain $B = C = 0$. Now, by the normalization we obtain $\delta(z) = A = \alpha$. This finishes the proof. □

Furthermore, that the requirements cannot be relaxed to allow (e.g.) second order zeros in stead of simple zeros. In fact, if $m(z)$ is a solution for which both components vanish of second order at, say, $z = +1$, then $\tilde{m}(z) = \frac{z}{(z-1)^2} m(z)$ is a nontrivial symmetric solution of the vanishing problem (i.e. for $\alpha = 0$).

By Lemma 2.7 we have

Corollary 3.3. *The function $m(z, n, t)$ defined in (2.12) is the only solution of the vector Riemann–Hilbert problem (2.14)–(2.17).*

Observe that there is nothing special about the point $z = 0$ where we normalize, any other point would do as well. However, observe that for the one soliton solution (2.22), $f(z)$ vanishes at

$$z = \zeta \frac{1 + \gamma - \zeta^2}{(\gamma - 1)\zeta^2 + 1}$$

and hence the Riemann–Hilbert problem normalized at this point has a nontrivial solution for $\alpha = 0$ and hence, by our uniqueness result, no solution for $\alpha = 1$. This shows that uniqueness and existence connected, a fact which is not surprising since our Riemann–Hilbert problem is equivalent to a singular integral equation which is Fredholm of index zero (see Appendix A).

4. Solitons and the soliton region

This section demonstrates the basic method of passing from a Riemann–Hilbert problem involving solitons to one without. Furthermore, the asymptotics inside the soliton region are computed. Solitons are represented in a Riemann–Hilbert problem by pole conditions, for this reason we will further study how poles can be dealt with in this section.

For easy reference we note the following result which can be checked by a straightforward calculation.

Lemma 4.1 (Conjugation)**.** *Assume that $\widetilde{\Sigma} \subseteq \Sigma$. Let D be a matrix of the form*

$$D(z) = \begin{pmatrix} d(z)^{-1} & 0 \\ 0 & d(z) \end{pmatrix}, \tag{4.1}$$

where $d : \mathbb{C}\backslash\widetilde{\Sigma} \to \mathbb{C}$ is a sectionally analytic function. Set

$$\tilde{m}(z) = m(z) D(z), \tag{4.2}$$

then the jump matrix transforms according to

$$\tilde{v}(z) = D_-(z)^{-1} v(z) D_+(z). \tag{4.3}$$

If d satisfies $d(z^{-1}) = d(z)^{-1}$ and $d(0) > 0$. Then the transformation $\tilde{m}(z) = m(z)D(z)$ respects our symmetry, that is, $\tilde{m}(z)$ satisfies (2.16) if and only if $m(z)$ does.

In particular, we obtain

$$\tilde{v} = \begin{pmatrix} v_{11} & v_{12} d^2 \\ v_{21} d^{-2} & v_{22} \end{pmatrix}, \qquad z \in \Sigma\backslash\widetilde{\Sigma}, \tag{4.4}$$

respectively

$$\tilde{v} = \begin{pmatrix} \frac{d_-}{d_+} v_{11} & v_{12} d_+ d_- \\ v_{21} d_+^{-1} d_-^{-1} & \frac{d_+}{d_-} v_{22} \end{pmatrix}, \qquad z \in \Sigma \cap \widetilde{\Sigma}. \tag{4.5}$$

In order to remove the poles there are two cases to distinguish. If $\lambda_k > \frac{1}{2}(\zeta_0 + \zeta_0^{-1})$ the jump is exponentially decaying and there is nothing to do.

Otherwise we use conjugation to turn the jumps into exponentially decaying ones. It turns out that we will have to handle the poles at ζ_k and ζ_k^{-1} in one step in order to preserve symmetry and in order to not add additional poles elsewhere.

Lemma 4.2. *Assume that the Riemann–Hilbert problem for m has jump conditions near ζ and ζ^{-1} given by*

$$\begin{aligned} m_+(z) &= m_-(z) \begin{pmatrix} 1 & 0 \\ \frac{\gamma\zeta}{z-\zeta} & 1 \end{pmatrix}, \qquad |z-\zeta| = \varepsilon, \\ m_+(z) &= m_-(z) \begin{pmatrix} 1 & -\frac{\gamma z}{z-\zeta^{-1}} \\ 0 & 1 \end{pmatrix}, \quad |z^{-1}-\zeta| = \varepsilon. \end{aligned} \tag{4.6}$$

Then this Riemann–Hilbert problem is equivalent to a Riemann–Hilbert problem for $\tilde{m}$ which has jump conditions near ζ and ζ^{-1} given by

$$\begin{aligned} \tilde{m}_+(z) &= \tilde{m}_-(z) \begin{pmatrix} 1 & \frac{(\zeta z-1)^2}{\zeta(z-\zeta)\gamma} \\ 0 & 1 \end{pmatrix}, \qquad |z-\zeta| = \varepsilon, \\ \tilde{m}_+(z) &= \tilde{m}_-(z) \begin{pmatrix} 1 & 0 \\ -\frac{(z-\zeta)^2}{\zeta z(\zeta z-1)\gamma} & 1 \end{pmatrix}, \qquad |z^{-1}-\zeta| = \varepsilon, \end{aligned}$$

and all remaining data conjugated (as in Lemma 4.1) by

$$D(z) = \begin{pmatrix} \frac{z-\zeta}{\zeta z-1} & 0 \\ 0 & \frac{\zeta z-1}{z-\zeta} \end{pmatrix}. \tag{4.7}$$

Proof. To turn γ into γ^{-1}, introduce D by

$$D(z) = \begin{cases} \begin{pmatrix} \frac{z-\zeta}{\zeta} & \frac{1}{\gamma} \\ -\gamma & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{z-\zeta^{-1}} & 0 \\ 0 & z-\zeta^{-1} \end{pmatrix}, & |z-\zeta| < \varepsilon, \\ \begin{pmatrix} 0 & \gamma \\ -\frac{1}{\gamma} & \frac{\zeta z-1}{\zeta z} \end{pmatrix} \begin{pmatrix} \frac{z-\zeta}{\zeta z} & 0 \\ 0 & \frac{\zeta z}{z-\zeta} \end{pmatrix}, & |z^{-1}-\zeta| < \varepsilon, \\ \begin{pmatrix} \frac{z-\zeta}{\zeta z-1} & 0 \\ 0 & \frac{\zeta z-1}{z-\zeta} \end{pmatrix}, & \text{else}, \end{cases}$$

and set $\tilde{m}(z) = m(z)D(z)$. □

Now we are ready to prove our main results:

Proof of Theorem 1.1. We begin by observing that the partial transmission coefficient $T(z, z_0)$ introduced in (1.7) satisfies the following scalar meromorphic Riemann–Hilbert problem:

(i) $T_+(z, z_0)$ is meromorphic in $\mathbb{C}\backslash\Sigma(z_0)$, where $\Sigma(z_0)$ is the arc given by $\Sigma(z_0) = \{z \in \mathbb{T} | \mathrm{Re}(z) < \mathrm{Re}(z_0)\}$, with simple poles at ζ_k and ζ_k^{-1} for all k with $\lambda_k < \frac{1}{2}(\zeta_0 + \zeta_0^{-1})$,
(ii) $T_+(z, z_0) = T_-(z, z_0)(1 - |R(z)|^2)$ for $z \in \Sigma(z_0)$,
(iii) $T(z^{-1}, z_0) = T(z, z_0)^{-1}$ and $T(0, z_0) > 0$.

Note also $\overline{T(z, z_0)} = T(\overline{z}, z_0)$ and in particular $T(z, z_0)$ is real-valued for $z \in \mathbb{R}$.

Next introduce

$$D(z) = \begin{cases} \begin{pmatrix} 1 & \frac{1}{\gamma_k(t)} \frac{z-\zeta_k}{\zeta_k} \\ -\gamma_k(t) \frac{\zeta_k}{z-\zeta_k} & 0 \end{pmatrix} D_0(z), & |z-\zeta_k| < \varepsilon, \quad \lambda_k < \frac{1}{2}(\zeta_0 + \zeta_0^{-1}), \\ \begin{pmatrix} 0 & \gamma_k(t) \frac{z\zeta_k}{z\zeta_k - 1} \\ -\frac{1}{\gamma_k(t)} \frac{z\zeta_k - 1}{z\zeta_k} & 1 \end{pmatrix} D_0(z), & |z^{-1}-\zeta_k| < \varepsilon, \quad \lambda_k < \frac{1}{2}(\zeta_0 + \zeta_0^{-1}), \\ D_0(z), & \text{else}, \end{cases}$$

where

$$D_0(z) = \begin{pmatrix} T(z,z_0)^{-1} & 0 \\ 0 & T(z,z_0) \end{pmatrix}.$$

Note that $D(z)$ is analytic in $\mathbb{C}\backslash\Sigma(z_0)$ and that we have

$$D(z^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we conjugate our problem using $D(z)$ and observe that, since $T(z,z_0)$ has the same behaviour as $T(z)$ for $z = \pm 1 > z_0$, the new vector $\tilde{m}(z) = m(z)D(z)$ is again continuous near $z = \pm 1$ (even if $T(z)$ vanishes there).

Then using Lemma 4.1 and Lemma 4.2 the jump corresponding $\lambda_k < \frac{1}{2}(\zeta_0 + \zeta_0^{-1})$ (if any) is given by

$$\tilde{v}(z) = \begin{pmatrix} 1 & \frac{(\zeta_k z - 1)^2}{\zeta_k (z-\zeta_k)\gamma_k T(\zeta_k,z_0)^2 \mathrm{e}^{t\Phi(\zeta_k)}} \\ 0 & 1 \end{pmatrix}, \qquad |z-\zeta_k| = \varepsilon, \tag{4.8}$$

$$\tilde{v}(z) = \begin{pmatrix} 1 & 0 \\ -\frac{(z-\zeta_k)^2}{\zeta z(\zeta_k z - 1)\gamma_k T(\zeta_k,z_0)^2 \mathrm{e}^{t\Phi(\zeta_k)}} & 1 \end{pmatrix}, \qquad |z^{-1}-\zeta_k| = \varepsilon, \tag{4.9}$$

and corresponding $\lambda_k \geq \frac{1}{2}(\zeta_0 + \zeta_0^{-1})$ (if any) by

$$\tilde{v}(z) = \begin{pmatrix} 1 & 0 \\ \frac{\zeta_k \gamma_k T(\zeta_k,z_0)^2 \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} & 1 \end{pmatrix}, \qquad |z-\zeta_k| = \varepsilon, \tag{4.10}$$

$$\tilde{v}(z) = \begin{pmatrix} 1 & -\frac{z\gamma_k T(\zeta_k,z_0)^2 \mathrm{e}^{t\Phi(\zeta_k)}}{z-\zeta_k} \\ 0 & 1 \end{pmatrix}, \qquad |z^{-1}-\zeta_k| = \varepsilon. \tag{4.11}$$

In particular, all jumps corresponding to poles, except for possibly one if $\zeta_k = \zeta_0$, are exponentially decreasing. Furthermore, the jump along $\mathbb{T}$ is given by

$$\tilde{v}(z) = \begin{cases} \tilde{b}_-(z)^{-1}\tilde{b}_+(z), & \lambda(z) > \lambda_0, \\ \tilde{B}_-(z)^{-1}\tilde{B}_+(z), & \lambda(z) < \lambda_0, \end{cases} \tag{4.12}$$

where

$$\tilde{b}_-(z) = \begin{pmatrix} 1 & \frac{R(z^{-1})\mathrm{e}^{-t\Phi(z)}}{T(z^{-1},z_0)^2} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+(z) = \begin{pmatrix} 1 & 0 \\ \frac{R(z)\mathrm{e}^{t\Phi(z)}}{T(z,z_0)^2} & 1 \end{pmatrix},$$

and

$$\tilde{B}_-(z) = \begin{pmatrix} 1 & 0 \\ -\frac{T_-(z^{-1},z_0)}{T_-(z,z_0)} R(z)\mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}, \quad \tilde{B}_+(z) = \begin{pmatrix} 1 & -\frac{T_+(z,z_0)}{T_+(z^{-1},z_0)} R(z^{-1})\mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}.$$

Here we have used

$$T_\pm(z,z_0) = T_\mp(z^{-1},z_0)^{-1}, \qquad z \in \Sigma(z_0).$$

Note also $T(z^{-1},z_0) = T(\overline{z},z_0) = \overline{T(z,z_0)}$ and $R(z^{-1}) = R(\overline{z}) = \overline{R(z)}$ for $z\mathbb{T}$.

Now the jump along $\mathbb{T}$ can also be made arbitrarily small following the nonlinear steepest decent method developed by Deift and Zhou [4]: Split the Fourier transform of the reflection coefficient into a part which has an analytic extension to a neighborhood of the unit circle plus a small error. Move the analytic part away from the unit circle using the factorizations from above. Since the Fourier coefficients decay faster than any polynomial, the errors from both parts can be made $O(t^{-l})$ for any $l \in \mathbb{N}$. We refer to [4] respectively [7] for details. Hence we can apply Theorem A.5 as follows:

If $|\frac{n}{t} - c_k| > \varepsilon$ for all k we can choose $\gamma_0 = 0$ and w_0^t by removing all jumps corresponding to poles from w^t. In particular, the error between the solutions of w^t and w_0^t is exponentially small. This proves the second part of Theorem 1.1.

Otherwise, if $|\frac{n}{t} - c_k| < \varepsilon$ for some k, we choose $\gamma_0^t = \gamma_k(n,t)$ and $w_0^t \equiv 0$. Again we conclude that the error between the solutions of w^t and w_0^t is exponentially small. □

Appendix A. Singular integral equations

In this section we show how to transform a vector Riemann–Hilbert problem

$$\begin{aligned} & m_+(z) = m_-(z)v(z), \qquad z \in \Sigma, \\ & \operatorname*{Res}_{\zeta} m(z) = \lim_{z\to\zeta} m(z)\begin{pmatrix} 0 & 0 \\ -\zeta\gamma & 0 \end{pmatrix}, \quad \operatorname*{Res}_{\zeta^{-1}} m(z) = \lim_{z\to\zeta^{-1}} m(z)\begin{pmatrix} 0 & \zeta^{-1}\gamma \\ 0 & 0 \end{pmatrix}, \\ & m(z^{-1}) = m(z)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ & m(0) = \begin{pmatrix} 1 & m_2 \end{pmatrix}. \end{aligned} \tag{A.1}$$

into a singular integral equation. Since we require the symmetry condition (2.16) for our Riemann–Hilbert problems we need to adapt the usual Cauchy kernel to preserve this symmetry. Moreover, we keep the single soliton as an inhomogeneous term which will play the role of the leading asymptotics in our applications.

The classical Cauchy-transform of a function $f : \Sigma \to \mathbb{C}$ which is square integrable is the analytic function $Cf : \mathbb{C}\backslash\Sigma \to \mathbb{C}$ given by

$$Cf(z) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{f(s)}{s-z} ds, \qquad z \in \mathbb{C}\backslash\Sigma. \tag{A.2}$$

Denote the tangential boundary values from both sides (taken possibly in the L^2-sense — see e.g. [3, eq. (7.2)]) by C_+f respectively C_-f. Then it is well-known that C_+ and C_- are bounded operators $L^2(\Sigma) \to L^2(\Sigma)$, which satisfy $C_+ - C_- = \mathbb{I}$ and $C_+C_- = 0$ (see e.g. [1]). Moreover, one has the Plemelj–Sokhotsky formula ([11])

$$C_\pm = \frac{1}{2}(\mathrm{i}H \pm \mathbb{I}),$$

where

$$Hf(t) = \frac{1}{\pi} ⨍_\Sigma \frac{f(s)}{t-s} ds, \qquad t \in \Sigma, \tag{A.3}$$

is the Hilbert transform and $⨍$ denotes the principal value integral.

In order to respect the symmetry condition we will restrict our attention to the set $L^2_s(\Sigma)$ of square integrable functions $f : \Sigma \to \mathbb{C}^2$ such that

$$f(z^{-1}) = f(z)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.4}$$

Clearly this will only be possible if we require our jump data to be symmetric as well (i.e., Hypothesis 3.1 holds).

Next we introduce the Cauchy operator

$$(Cf)(z) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma f(s)\Omega_\zeta(s,z) \tag{A.5}$$

acting on vector-valued functions $f : \Sigma \to \mathbb{C}^2$. Here the Cauchy kernel is given by

$$\Omega_\zeta(s,z) = \begin{pmatrix} \frac{z-\zeta^{-1}}{s-\zeta^{-1}}\frac{1}{s-z} & 0 \\ 0 & \frac{z-\zeta}{s-\zeta}\frac{1}{s-z} \end{pmatrix} ds = \begin{pmatrix} \frac{1}{s-z} - \frac{1}{s-\zeta^{-1}} & 0 \\ 0 & \frac{1}{s-z} - \frac{1}{s-\zeta} \end{pmatrix} ds, \tag{A.6}$$

for some fixed $\zeta \notin \Sigma$. In the case $\zeta = \infty$ we set

$$\Omega_\infty(s,z) = \begin{pmatrix} \frac{1}{s-z} - \frac{1}{s} & 0 \\ 0 & \frac{1}{s-z} \end{pmatrix} ds. \tag{A.7}$$

and one easily checks the symmetry property:

$$\Omega_\zeta(1/s,1/z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Omega_\zeta(s,z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.8}$$

The properties of C are summarized in the next lemma.

Lemma A.1. *Assume Hypothesis 3.1. The Cauchy operator C has the properties, that the boundary values $C_\pm$ are bounded operators $L^2_s(\Sigma) \to L^2_s(\Sigma)$ which satisfy*

$$C_+ - C_- = \mathbb{I} \tag{A.9}$$

and

$$(Cf)(\zeta^{-1}) = (0 \quad *), \qquad (Cf)(\zeta) = (* \quad 0). \tag{A.10}$$

Furthermore, C restricts to $L^2_s(\Sigma)$, that is

$$(Cf)(z^{-1}) = (Cf)(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in \mathbb{C}\backslash\Sigma \tag{A.11}$$

for $f \in L^2_s(\Sigma)$ and if $w_\pm$ satisfy (H.3.1) we also have

$$C_\pm(fw_\mp)(1/z) = C_\mp(fw_\pm)(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in \Sigma. \tag{A.12}$$

Proof. Everything follows from (A.8) and the fact that C inherits all properties from the classical Cauchy operator. □

We have thus obtained a Cauchy transform with the required properties. Following Section 7 and 8 of [1], we can solve our Riemann–Hilbert problem using this Cauchy operator.

Introduce the operator $C_w : L^2_s(\Sigma) \to L^2_s(\Sigma)$ by

$$C_w f = C_+(fw_-) + C_-(fw_+), \quad f \in L^2_s(\Sigma). \tag{A.13}$$

and recall from Lemma 2.6 that the solution corresponding to $v \equiv 0$ is given by

$$m_0(z) = \begin{pmatrix} f(z) & f(\frac{1}{z}) \end{pmatrix}, \quad f(z) = \frac{1}{1-\zeta^2+\gamma}\left(\gamma\zeta^2 \frac{z-\zeta^{-1}}{z-\zeta} + 1 - \zeta^2\right)$$

Observe that for $\gamma = 0$ we have $f(z) = 1$ and for $\gamma = \infty$ we have $f(z) = \frac{z-\zeta^{-1}}{z-\zeta}$. In particular, $m_0(z)$ is uniformly bounded away from ζ for all $\gamma \in [0,\infty]$.

Then we have the next result.

Theorem A.2. *Assume Hypothesis 3.1.*

Suppose m solves the Riemann–Hilbert problem (A.1). Then

$$m(z) = (1+c_0)m_0(z) + \frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s))\Omega_\zeta(s,z). \tag{A.14}$$

where

$$\mu = m_+ b_+^{-1} = m_- b_-^{-1} \quad \textit{and} \quad c_0 = \left(\frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s))\Omega_\zeta(s,0) \right)_1$$

Furthermore, μ solves

$$(\mathbb{I} - C_w)\mu = (1 + c_0) m_0(z). \tag{A.15}$$

Conversely, suppose $\tilde{\mu}$ solves

$$(\mathbb{I} - C_w)\tilde{\mu} = m_0(z), \tag{A.16}$$

and

$$c_0 = \left(\frac{1}{2\pi\mathrm{i}} \int_\Sigma \tilde{\mu}(s)(w_+(s) + w_-(s))\Omega_\zeta(s,0) \right)_1 \neq 0,$$

then m defined via (A.14), with $(1 - c_0) = (1 - \tilde{c}_0)^{-1}$ and $\mu = (1 - \tilde{c}_0)^{-1}\tilde{\mu}$, solves the Riemann–Hilbert problem (A.1) and $\mu = m_\pm b_\pm^{-1}$.

Proof. First of all note hat by (A.12) $(\mathbb{I} - C_w)$ satisfies the symmetry condition and hence so do $(\mathbb{I} - C_w)^{-1} m_0$ and m.

So if m solves (A.1) and we set $\mu = m_\pm b_\pm^{-1}$, then m satisfies an additive jump given by

$$m_+ - m_- = \mu(w_+ + w_-).$$

Hence, if we denote the left hand side of (A.14) by $\tilde{m}$, both functions satisfy the same additive jump. So $m - \tilde{m}$ has no jump and must thus solve (A.1) with $v \equiv 0$. By uniqueness (Corollary 3.3) $m - \tilde{m} = \alpha m_0$ for some $\alpha \in \mathbb{C}$ and by looking at the first component at $z = 0$ we see $\alpha = 0$, that is $m = \tilde{m}$.

Moreover, if m is given by (A.14), then (A.9) implies

$$\begin{aligned} m_\pm &= (1 - c_0) m_0 + C_\pm(\mu w_-) + C_\pm(\mu w_+) \\ &= (1 - c_0) m_0 + C_w(\mu) \pm \mu w_\pm \\ &= (1 - c_0) m_0 - (\mathbb{I} - C_w)\mu + \mu b_\pm. \end{aligned} \tag{A.17}$$

From this we conclude that $\mu = m_\pm b_\pm^{-1}$ solves (A.15).

Conversely, if $\tilde{\mu}$ solves (A.16) then (A.17) implies $m_\pm = \mu b_\pm$ which shows that m defined via (A.14) solves the Riemann–Hilbert problem (A.1). □

Note that in the special case $\gamma = 0$ we have $m_0(z) = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and we can choose ζ as we please, say $\zeta = \infty$ such that $c_0 = \tilde{c}_0 = 0$ in the above theorem.

Hence we have a formula for the solution of our Riemann–Hilbert problem $m(z)$ in terms of $(\mathbb{I} - C_w)^{-1} m_0$ and this clearly raises the question of bounded invertibility of $\mathbb{I} - C_w$. This follows from Fredholm theory (cf. e.g. [19]):

Lemma A.3. *Assume Hypothesis 3.1. The operator $\mathbb{I} - C_w$ is Fredholm of index zero,*

$$\operatorname{ind}(\mathbb{I} - C_w) = 0. \tag{A.18}$$

Proof. Without loss we can assume that w have an analytic extension to a neighborhood of Σ, since the set of Fredholm operators is open within the space of bounded operators ([13, Thm. 1.3.8]).

Since one can easily check

$$(\mathbb{I} - C_w)(\mathbb{I} - C_{-w}) = (\mathbb{I} - C_{-w})(\mathbb{I} - C_w) = \mathbb{I} - T_w, \tag{A.19}$$

where

$$T_w = T_{++} + T_{+-} + T_{-+} + T_{--}, \qquad T_{\sigma_1\sigma_2}(f) = C_{\sigma_1}[C_{\sigma_2}(fw_{-\sigma_2})w_{-\sigma_1}],$$

it suffices to check that the operators $T_{\sigma_1\sigma_2}$ are compact ([13, Thm. 1.4.3]).

Indeed, suppose $f_n \in L^2(\Sigma)$ converges weakly to zero. Without loss we can assume f_n to be continuous. We will show that $\|T_w f_n\|_{L^2} \to 0$.

Using the analyticity of w in a neighborhood of Σ and the definition of $C_\pm$, we can slightly deform the contour Σ to some contour $\Sigma_\pm$ close to Σ, on the left, and have, by Cauchy's theorem,

$$T_{++}f_n(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma_+} (C(f_n w_-)(s) w_-(s)) \Omega_\zeta(s,z).$$

Now $(C(f_n w_-)w_-)(z) \to 0$ as $n \to \infty$. Also

$$|(C(f_n w_-)w_-)(z)| < const\, \|f_n\|_{L^2} \|w_-\|_{L^\infty} < const$$

and thus, by the dominated convergence theorem, $\|T_{++}f_n\|_{L^2} \to 0$ as desired.

Moreover, considering $\mathbb{I} - \varepsilon C_w = \mathbb{I} - C_{\varepsilon w}$ for $0 \le \varepsilon \le 1$ we obtain $\mathrm{ind}(\mathbb{I} - C_w) = \mathrm{ind}(\mathbb{I}) = 0$ from homotopy invariance of the index. □

By the Fredholm alternative, it follows that to show the bounded invertibility of $\mathbb{I} - C_w$ we only need to show that $\ker(\mathbb{I} - C_w) = 0$. The latter being equivalent to unique solvability of the corresponding vanishing Riemann–Hilbert problem.

Corollary A.4. *Assume Hypothesis 3.1. A unique solution of the Riemann–Hilbert problem (A.1) exists if and only if the corresponding vanishing Riemann–Hilbert problem, where the normalization condition is replaced by* $m(0) = \begin{pmatrix} 0 & m_2 \end{pmatrix}$, *has at most one solution.*

We are interested in comparing two Riemann–Hilbert problems associated with respective jumps w_0 and w with $\|w - w_0\|_\infty$ small, where

$$\|w\|_\infty = \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)}. \tag{A.20}$$

For such a situation we have the following result:

Theorem A.5. *Assume that for some data* w_0^t *the operator*

$$\mathbb{I} - C_{w_0^t} : L^2_s(\Sigma) \to L^2_s(\Sigma) \tag{A.21}$$

has a bounded inverse, where the bound is independent of t, *and let* $\zeta = \zeta_0$, $\gamma^t = \gamma_0^t$.

Furthermore, assume w^t *satisfies*

$$\|w^t - w_0^t\|_\infty \le \alpha(t) \tag{A.22}$$

for some function $\alpha(t) \to 0$ *as* $t \to \infty$. *Then* $(\mathbb{I} - C_{w^t})^{-1} : L^2_s(\Sigma) \to L^2_s(\Sigma)$ *also exists for sufficiently large* t *and the associated solutions of the Riemann–Hilbert problems (A.1) only differ by* $O(\alpha(t))$.

Proof. By boundedness of the Cauchy transform, one has

$$\|(C_{w^t} - C_{w_0^t})\| \le const \|w\|_\infty.$$

Thus, by the second resolvent identity, we infer that $(\mathbb{I} - C_{w^t})^{-1}$ exists for large t and

$$\|(\mathbb{I} - C_{w^t})^{-1} - (\mathbb{I} - C_{w_0^t})^{-1}\| = O(\alpha(t)).$$

From which the claim follows since this implies $\|\mu^t - \mu_0^t\|_{L^2} = O(\alpha(t))$ and thus $m^t(z) - m_0^t(z) = O(\alpha(t))$ uniformly in z away from Σ. □

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