Hidden Structure in Tilings, Conjectured Asymptotic Expansion for λ_d in Multidimensional Dimer Problem

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Abstract

The dimer problem arose in a thermodynamic study of diatomic molecules, and was abstracted into one of the most basic and natural problems in both statistical mechanics and combinatoric mathematics. Given a rectangular lattice of volume V in d dimensions, the dimer problem loosely speaking is to count the number of different ways dimers (dominoes) may be laid down on the lattice to completely cover it. It is known that the number of such coverings is roughly $e^{\lambda_d V}$ for some number λ_d. The first terms in the expansion of λ_d have been known for about thirty years

$$\lambda_d \sim \frac{1}{2}\, ln(2d) - \frac{1}{2}\ .$$

Herein we present a mathematical argument for the next two terms in the expansion to be given as in

$$\lambda_d \sim \frac{1}{2}\, ln(2d) - \frac{1}{2} + \frac{1}{8}\frac{1}{d} + \frac{7}{48}\ \frac{1}{d^2}\ .$$

Although this is an expansion designed for large d, even at $d = 2$ it is remarkably accurate, in error by about .3%.

In a previous paper we worked with tiles for which there was a weighting function assigned to the shape of the tiles. The weighting function satisfied a normalization condition. We then associated a quantity we called the "pressure" to the tilings of a lattice with such tiles. Loosely speaking, the result of the paper was that if we varied the weighting function, letting it get smoother and smoother (that is, more slowly varying), the corresponding pressures approached a limit. The limit was, of course, the pressure associated to the "infinitely smooth weighting function" a constant function. Herein we continue the statistical mechanics approach one step further. We construct for any weighting function the *perturbation expansion* for the partition function that arises naturally from treating the difference between the actual weighting function and the infinitely smooth weighting function as a perturbation.

We construct a second "partition function" associated to the first partition function, which we develop in a "cluster expansion". Kernels $J_1, J_2, ...$ arise in a standard way from the cluster expansion development. We next specialize to the dimer problem. We relate our two partition functions and if $J_i \geq 0$ for $i \leq s+1$ we argue an asymptotic expansion for λ_d

$$\lambda_d \sim \frac{1}{2}\, ln(2d) - \frac{1}{2} + \frac{c_1}{d} + \frac{c_2}{d^2} + \cdots + \frac{c_s}{d^s}.$$

We know $s \geq 2$, $c_1 = 1/8$ and $c_2 = 7/48$. The c_i are computable from the $J's$. The leading term $\frac{1}{2}\, ln(2d) - \frac{1}{2}$ does indeed arise as the zeroth order term in the cluster expansion.

The present work was inspired by, and is a natural sequel to, our previous paper [1]. However it is essentially independent thereof. We write the paper so it may be read without reference to [1], some notation is modified, but we strive to keep consistency between the two papers.

We work with a d-dimensional lattice, Λ, taken to be periodic. We let N be the number of vertices of Λ, (hopefully without confusion, we also let Λ be the set of vertices). We consider tiles, t, of size, or area, n. (Our tiles need not be connected, and N/n must be an integer.) We have a weighting function, f, on tiles that is normalized so that

$$\sum_t f(t) = \frac{1}{n} \ . \tag{1}$$

A *located tile*, s_α, is a tile placed at a particular location on the lattice, and is exactly a subset of the lattice of cardinality n,

$$s_\alpha = \{x_1^\alpha, ..., x_n^\alpha\} \,. \tag{2}$$

We denote the corresponding tile as $\bar{s}_\alpha$. Equation (1) can be written as

$$\sum_{\substack{s_\alpha \\ x_0 \in s_\alpha}} f\left(\bar{s}_\alpha\right) = 1 \tag{3}$$

The sum in (3) is over subsets of Λ of cardinality n that contain a fixed point of the lattice x_0. This equation is easily identified with equation (3) of [1]; as naturally for any located tile s_α we identify

$$f\left(\bar{s}_\alpha\right) = f\left(x_1^\alpha, ..., x_n^\alpha\right) \tag{4}$$

where the left side f is of this paper, and the right side f is as in [1].

We now consider a tiling, T_i, of Λ by tiles of size n. T_i may be identified with a set of N/n located tiles

$$T_i = \left\{s_1^i, s_2^i, ..., s_{N/n}^i\right\} \tag{5}$$

Since T_i is a tiling

$$\bigcup_\alpha s_\alpha^i = \Lambda \tag{6}$$

and

$$s_\alpha^i \cap s_\beta^i = \phi, \quad \alpha \neq \beta \,. \tag{7}$$

The partition function, Z, and the quantity we call the pressure, p, are given by the equation

$$Z = e^{Np} = \sum_{T_i} \prod_{s_\alpha \in T_i} f(\bar{s}_\alpha). \tag{8}$$

The right side is the sum over tilings, each tiling weighted by the product of the weight functions of the tiles appearing in the tiling.

The “infinitely smooth weight function” is constant on tiles. We let f_0 be its value on any tile, easily calculated to be

$$f_0 = \frac{(n-1)!(N-n)!}{(N-1)!} \,. \tag{9}$$

We write

$$\begin{aligned} f(\bar{s}_\alpha) &= f_0 + \Big(f(\bar{s}_\alpha) - f_0\Big) \\ &= f_0 + v(\bar{s}_\alpha) \end{aligned} \tag{10}$$

where we will treat v as a perturbation. Equation (8) is now

$$Z = e^{Np} = \sum_{T_i} \prod_{s_\alpha \in T_i} \Big(f_0 + v(\bar{s}_\alpha)\Big). \tag{11}$$

We expand Z in terms of powers of v

$$Z = Z_0 + Z_1 + Z_2 + ... \tag{12}$$

Z_i homogeneous of degree i in the $v's$. Singling out Z_0 first

$$Z_0 = \sum_{T_i} \prod_{s_\alpha \in T_i} f_0 = \sum_{T_i} f_0^{N/n} \tag{13}$$

and one calculates

$$\sum_{T_i} = \frac{N!}{(\frac{N}{n})!(n!)^{N/n}}. \tag{14}$$

We put together (9), (13), and (14) to define

$$Z_0 = e^{N\hat{p}^0(N)} \tag{15}$$

where

$$\lim_{N\to\infty} \hat{p}^0(N) = \frac{1-n}{n}. \tag{16}$$

(These are equations (8) and (9) of [1].)

We turn for a moment to the dimer problem.[2] We relate the quantity λ_d of the dimer problem to our variables (if our f is set equal a constant on dimers and zero on other tiles).

$$e^{N\lambda_d} = (2d)^{N/2}\, Z \tag{17}$$

or

$$\lambda_d = \frac{1}{2}\, ln(2d) + \frac{1}{N}\, ln\ Z \tag{18}$$

the $(2d)^{N/2}$ in equation (17) arises since our normalization condition (1) requires our f to be $\frac{1}{2d}$ on each dimer, whereas to just count tilings (with no weighting) would correspond to $f = 1$ on each dimer. Replacing Z by Z_0 in (18) and taking the limit $N \to \infty$ one gets

$$\lambda_d \cong \frac{1}{2}\, ln(2d) - \frac{1}{2}\ . \tag{19}$$

Our expressions for λ_d will all be in the infinite volume limit. Thus taking our zeroth order approximation for $Z, Z = Z_0$, we get the approximation for λ_d of (19), the result of [2]! In [2] there are bounds on the error of this approximation.

Returning to the general case we factor out Z_0 from Z in (12)

$$Z = Z_0 Z^* \tag{20}$$

$$Z^* = 1 + Z_1^* + Z_2^* + ... \tag{21}$$

$$Z_i^* = Z_i \big/ Z_0 \tag{22}$$

There is a natural factoring of Z_i into a contribution from the factors of v in (11) which we call $\bar{Z}_i^*$ and the factors of f_0 in (11) which we call $\beta(N,i)Z_0$ so that

$$Z_i = \beta(N,i) Z_0 \bar{Z}_i^* \tag{23}$$

and thus

$$Z_i^* = \beta(N,i)\bar{Z}_i^* \tag{24}$$

$\bar{Z}_i^*$ is defined below, and it is a tedious calculation to show the surprising result that $\beta(N,i) \to 1$ as $N \to \infty$. We let $\tilde{Z}^*$ be Z^* with $\beta(N,i)$ replaced by 1.

$$\tilde{Z}^* = 1 + \bar{Z}_1^* + \bar{Z}_2^* + ... \tag{25}$$

We turn to the detailed specification of $\bar{Z}_i^*$

$$\bar{Z}_i^* = \frac{1}{i!} \sum_{\substack{s_1, s_2, \ldots, s_i \\ \text{disjoint}}} \prod_{\alpha=1}^{i} v(\bar{s}_\alpha) \tag{26}$$

We are trying to put our expression for $\tilde{Z}^*$ into a form that can be identified with equation (2.5a) from [3]. Some knowledge of the structure of cluster expansions becomes necessary. Article [3] is a standard reference, but a knowledgeable patient friend may be more helpful.

With the notation in [3], the formal cluster expansion for $\tilde{Z}^*$ is given as a translation for equation (2.7) of [3].

$$ln\ \tilde{Z}^* = \sum_s \frac{1}{s!}\ J_s \tag{27}$$

$$J_s = \sum_{s_1, s_2, ..., s_s} f(\bar{s}_1)...f(\bar{s}_s)\psi_c'(s_1, s_2, ..., s_s) \tag{28}$$

The located tiles appearing in the sum for J_s are forced to overlap so that they cannot be divided into two disjoint sets of located tiles. ψ_c' is a numerical factor depending on the pattern of overlaps.

The J_s for the dimer problem rather easily satisfy

$$J_1 = 0 \tag{29}$$

and

$$|J_s| \leq \frac{C_s}{d^{(s-1)}}\ N, \quad s > 1. \tag{30}$$

We want the asymptotic behavior of Z^* as $N \to \infty$

$$Z^* = \sum_i \beta(N, i)\bar{Z}_i^* \tag{31}$$

and we know from (27) the asymptotic behavior as $N \to \infty$ of $\tilde{Z}^*$

$$\tilde{Z}^* = \sum_i \bar{Z}_i^* \tag{32}$$

We argue the asymptotic behavior of (31) arises from a largest term of the form

$$\beta\big(N, \Sigma\ i\ \alpha_i\ N\big)\bar{J}_1^{\alpha_1 N} \cdots \bar{J}_{s+1}^{\alpha_{s+1} N} \cdot \frac{N^{\Sigma\ \alpha_i N}}{(\alpha_1 N)! \cdots (\alpha_{s+1} N)!} \tag{33}$$

provided all these $J's$ are ≥ 0. Here $\bar{J}_i N = (1/i!) J_i$. We are finding that portion of the asymptotic behavior due to just $J_1, ..., J_{s+1}$. We choose the α_i to maximize (33). We will use

$$\beta(N, jN) \sim e^{N\left[\left(\frac{1-2j}{2}\right) ln(1-2j)+j\right]}$$

Differentiating (33) with respect to the α_i leads to the equations

$$ln\ \alpha_k = ln\ \bar{J}_k + \frac{\partial}{\partial \alpha_k}\left[\left(\frac{1-2\Sigma i\alpha_i}{2}\right) ln(1-2\Sigma\ i\alpha_i) + \Sigma\ i\alpha_i\right] \tag{34}$$

or

$$\alpha_k = \bar{J}_k e^{F_k(\alpha' s)} \tag{35}$$

where (34) and (35) define the F_k. Equation (35) is solved for α_k as a formal power series in the $\bar{J}$ by iterating starting from setting the $\alpha' s = 0$ on the right side of equation.

We substitute (35) into (33) to get

$$Z^* \ \sim \ e^{N\left\{-\Sigma\ \alpha_i\ F_i + \Sigma\ \bar{J}_i\ e^{F_i} + \frac{1-2\Sigma\ i\ \alpha_i}{2}\ ln\left(1-2\Sigma\ i\ \alpha_i\right)+\Sigma i\alpha_i\right\}}. \tag{36}$$

From (34), (35), and (31) we can expand the exponent in (36) into a formal power series in the $\bar{J}_i$, and get the asymptotic series

$$\lambda_d \sim \ \frac{1}{2}\ ln(2d) - \frac{1}{2} + \frac{c_1}{d} + \frac{c_2}{d^2} + \cdots + \frac{c_s}{d^s} \tag{37}$$

and from

$$\bar{J}_2 = \frac{1}{8}\ \frac{1}{d}, \bar{J}_3 = \frac{1}{12}\ \frac{1}{d^2} \tag{38}$$

get

$$c_1 = \frac{1}{8},\ c_2 = \frac{7}{48}. \tag{39}$$

Keeping c_1 and c_2 in (37) we also look at $d = 2$ and 3 to get approximations

$$\lambda_2 \cong\ .2921 \tag{40}$$

$$\lambda_3 \cong\ .4538 \tag{41}$$

with error of about .3% for $\lambda_2\,!$. The J_i are hard to compute but one should compute more of them.

References

[1] Paul Federbush, Tilings With Very Elastic Tiles, Math–ph 0707.2525.

[2] Henryk Minc, An Asymptotic Solution of the Multidimensional Dimer Problem, Linear and Multilinear Algebra, 1980, **8**, 235-239.

[3] David C. Brydges, "A Short Course in Cluster Expansions" in "Phenomenes Critiques, Systems Aleatoires, Theories de Gauge, Part I, II" (Les Houches, 1984), 129-183, North Holland, Amsterdam, 1986.