# DECAY OF QUANTUM CORRELATIONS IN A LATTICE
## BY HEAT KERNEL METHODS.

**L. AMOUR, C. CANCELIER, P. LEVY-BRUHL and J. NOURRIGAT**

Département de Mathématiques, UMR CNRS 6056
Université de Reims. B.P. 1039. 51687 Reims Cedex 2. France

Abstract

We prove some estimations of the correlation of two local observables in quantum spin systems (with Schrödinger equations) at large temperature. For that, we describe the heat kernel of the Hamiltonian for a finite subset of the lattice, allowing the dimension to tend to infinity. This is an improved version of an earlier unpublished manuscript.

### 1. Introduction.

In the last years, many works were devoted to the estimates, or asymptotics, of the correlation of two local observables, (or Ursell functions of $n$ local observables), for classical spin systems, at large, or at small temperature. The aim of this work is a beginning of a similar study for quantum spin systems, related to the Schrödinger equation, (for two observables, at large temperature).

In classical statistical mechanics on a lattice $L$, an interaction is the definition, for each finite subset $\Lambda$ of $L$, of a function $V_\Lambda$ on $\mathbb{R}^\Lambda$, or $(\mathbb{R}^p)^\Lambda$ if there are $p$ degrees of freedom at each site. Then, for each $\beta > 0$ (the inverse of the temperature), the mean value of the local observable $f$, (i. e. of a function $f$ on $(\mathbb{R}^p)^\Lambda$), is:

$$E_{\Lambda,\beta}(f) = Z_\Lambda(\beta)^{-1} \int_{(\mathbb{R}^p)^\Lambda} e^{-\beta V_\Lambda(x)} f(x)dx \qquad Z_\Lambda(\beta) = \int_{(\mathbb{R}^p)^\Lambda} e^{-\beta V_\Lambda(x)} dx.$$

The bilinear analogue (correlation of two local observables), or the multilinear analogue (Ursell function of $n$ local observables), are defined in a standard way, (see, for example, D. Ruelle [21], B. Simon [22] or R.A. Minlos [18]). We say that $f$ is supported in a part $E$ of $\Lambda$ if $f$ depends only on the variables $x_\lambda$ corresponding to sites $\lambda$ which are in $E$. A classical problem is to estimate the decay of the correlation of two local observables $f$ and $g$ with disjoint supports $E$ and $F$ when the distance of $E$ and $F$ tends to $+\infty$. When $\beta$ is small enough (large temperature), this is a classical result of L. Gross [10]. For the study of Ursell functions at high temperature in classical mechanics, (tree decay), see, for example, Bertini, Cirillo and Oliveri [8]. For a small temperature, the problem is more complicated and more hypotheses are needed: see the works of Helffer, Sjöstrand, V. Bach, T. Jecko, J.S. Möller, O. Matte, ... [6],[7], [11] to [17], [24], [25].

In this work, we are interested to similar results in quantum statistical mechanics, for $\beta$ small enough. Let us consider a quantum $d$-dimensional lattice of particles, each of them moving in $\mathbb{R}^p$. To each finite subset $\Lambda$ of the lattice $L = \mathbb{Z}^d$, we shall define below, with more details, a potential $V_\Lambda$, which is still a real valued function on $(\mathbb{R}^p)^\Lambda$. We denote by $H_\Lambda$ the following differential operator in $(\mathbb{R}^p)^\Lambda$, depending on the Planck's constant $h$:

$$H_\Lambda = -\frac{h^2}{2} \sum_{\lambda\in\Lambda} \Delta_{x_\lambda} + V_\Lambda(x) \tag{1.1}$$

where $x = (x_\lambda)_{\lambda \in \Lambda}$ denotes the variable of $(\mathbb{R}^p)^\Lambda$, each variable $x_\lambda$ being in $\mathbb{R}^p$. With suitable hypotheses (see below), the exponential $e^{-\beta H_\Lambda}$ will be defined for $\beta > 0$, and will be of trace class. Then, a local observable is no more a function, but a bounded operator $A$ on the Hilbert space $\mathcal{H}_\Lambda = L^2((\mathbb{R}^p)^\Lambda)$. The mean value $E_{\Lambda,\beta}(A)$ is defined now, instead of the previous definition, by:

$$E_{\Lambda,\beta}(A) = Z_\Lambda(\beta)^{-1} \operatorname{Tr}(e^{-\beta H_\Lambda} A), \qquad Z_\Lambda(\beta) = \operatorname{Tr}(e^{-\beta H_\Lambda}). \tag{1.2}$$

If $E$ is a subset of $\Lambda$, we say that $A$ is supported in $E$ if $A$ can be seen as an operator on the Hilbert space $\mathcal{H}_E$. If $A$ and $B$ are two local observables, supported in two disjoint subsets $E_1$ and $E_2$ of $\Lambda$, a natural definition for the correlation is:

$$\operatorname{Cov}_{\Lambda,\beta}(A, B) = E_{\Lambda,\beta}(AB) - E_{\Lambda,\beta}(A) E_{\Lambda,\beta}(B). \tag{1.3}$$

One of the goals of this work is to give an analogue of the result of L. Gross in this situation, and to estimate the decay of $\operatorname{Cov}_{\Lambda,\beta}(A, B)$ when the distance of the supports $E_1$ and $E_2$ of $A$ and $B$ tends to $+\infty$ (see Theorem 1.3 below), assuming that $\beta$ is small enough. Perhaps such a bound can be obtained by probabilistic methods, (see [1], [18], [19], ...), but we want here to prove it by a careful study of the integral kernel of the operator $e^{-\beta H_\Lambda}$. Since $|\Lambda|$ tens sometimes to $+\infty$, we have to study a heat kernel in large dimension, like in Sjöstrand [23]. The bound of the correlations is applied to study the rate of convergence to some thermodynamic limits, and to prove that there is no phase transitions (discontinuity of the mean energy per site) for $\beta$ small enough (see Theorems 1.4 and 1.5 below).

Now, let us give more details on the *interaction*, i.e. on the family of functions $(V_\Lambda)_{\Lambda \subset \mathbb{Z}^d}$. We consider a function $A \in C^\infty(\mathbb{R}^p, \mathbb{R})$ and, for each pair of sites $\lambda$ and $\mu$ in the lattice, a function $B_{\lambda,\mu} \in C^\infty(\mathbb{R}^{2p}, \mathbb{R})$. For each finite subset $\Lambda$ of $\mathbb{Z}^d$, we denote by $V_\Lambda$ the following potential in $(\mathbb{R}^p)^\Lambda$:

$$V_\Lambda(x) \;=\; \sum_{\lambda \in \Lambda} A(x_\lambda) \;+\; \sum_{\substack{\lambda,\mu \in \Lambda \\ \lambda \neq \mu}} B_{\lambda,\mu}(x_\lambda, x_\mu) \qquad x = (x_\lambda)_{\lambda \in \Lambda}. \tag{1.4}$$

For sake of simplicity, we assume that $B_{\mu,\lambda} = B_{\lambda,\mu}$. When invariance by translation is needed, $B_{\lambda,\mu}$ will depend only on $\lambda - \mu$. We shall assume that $B_{\lambda,\mu}$ is small when $|\lambda - \mu|$ is large. More precisely, we assume that, for some $\varepsilon \in ]0, 1[$, the following hypothesis is satisfied:

$(H_\varepsilon)$ *For each $\alpha \geq 0$ and $\beta \geq 0$, there exists $C_{\alpha,\beta}(\varepsilon) > 0$ such that*

$$\sup_{\lambda \in \mathbb{Z}^d} \sum_{\mu \in \mathbb{Z}^d} \frac{\|\nabla_{x_\lambda}^\alpha \nabla_{x_\mu}^\beta B_{\lambda,\mu}\|}{\varepsilon^{|\lambda-\mu|}} \leq C_{\alpha,\beta}(\varepsilon). \tag{1.5}$$

*The function $A$ is bounded from below. All the derivatives of order $\geq 1$ of $A$ are bounded. For each $m > 0$ and $\beta > 0$, we have:*

$$\sup_{x \in \mathbb{R}^p} (1 + |x|)^m e^{-\beta A(x)} = C_m(\beta) < +\infty. \tag{1.6}$$

In all this paper, we denote by $\| \ \|$ the $L^\infty$ norm of a function. The hypothesis (1.6) is not needed for Theorems 1.1 and 1.2, but only for Theorem 1.3, 1.4 and 1.5. For the estimation of correlations (theorem 1.3), if the two observables are multiplications by bounded functions, we don't need (1.5) for all $\alpha$ and $\beta$, but for a finite number. It is the same if the supports of the observables are, for example, single points. If $(H_\varepsilon)$ is satisfied for some $\varepsilon \in ]0, 1[$, there exist $M_1(\varepsilon)$ and $M_2(\varepsilon)$, (independent of $\Lambda$), such that, for each finite set $\Lambda$, and for each point $\lambda \in \Lambda$:

$$\sup_{\lambda \in \Lambda} \|\nabla_{x_\lambda} V_\Lambda\| \leq M_1(\varepsilon) \qquad \sup_{\lambda \in \Lambda} \sum_{\mu \in \Lambda} \frac{\|\nabla_{x_\lambda} \nabla_{x_\mu} V_\Lambda\|}{\varepsilon^{|\lambda-\mu|}} \leq M_2(\varepsilon). \tag{1.7}$$

In the following, the parameter $\beta$ will be denoted by $t$, since we shall use evolution equations. All the results of this work, excepted the bounds for the first order derivatives, will be valid under the following condition:

$$ht = h\beta \leq T_0 \qquad T_0 = M_2(\varepsilon)^{-1/2}. \tag{1.8}$$

The first result is devoted to the description of the integral kernel of $e^{-tH_\Lambda}$, with inequalities where the constants are independent of $\Lambda$, if the condition $(H_\varepsilon)$ is satisfied. We denote by $\nabla_\lambda$ the differential, with respect to $(x_\lambda, y_\lambda) \in \mathbb{R}^{2p}$ $(\lambda \in \Lambda)$ of a $C^\infty$ function $f$ on $(\mathbb{R}^{2p})^\Lambda$. The norm of $\nabla_\lambda f(x)$ is its norm in $(\mathbb{R}^{2p})^\star$. We denote by $\mathrm{diam}(A)$ the diameter of a subset $A$ of $\mathbb{Z}^d$. (For the theorem 1.1, the norm in $\mathbb{Z}^d$ is arbitrary.)

**Theorem 1.1.** *Under the previous hypotheses, the integral kernel $U_\Lambda(x, y, t)$ of $e^{-tH_\Lambda}$ can be written in the form*

$$U_\Lambda(x, y, t) \;=\; (2\pi t h^2)^{-p|\Lambda|/2}\; e^{-\frac{|x-y|^2}{2th^2}}\; e^{-\psi_\Lambda(x,y,t)}, \tag{1.9}$$

*where $\psi_\Lambda$ is a $C^\infty$ function in $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0, +\infty[$, depending on the parameter $h > 0$. Moreover, if $(H_\varepsilon)$ is satisfied, for each finite subset $\Lambda$ of $\mathbb{Z}^d$, we have, for all $t > 0$*

$$\sup_{\lambda \in \Lambda} \|\nabla_\lambda \psi_\Lambda(., t)\| \leq t M_1(\varepsilon). \tag{1.10}$$

*For each $m \geq 2$, for all points $\lambda^{(1)}, ..., \lambda^{(m-1)}$ in $\Lambda$, we can write, if $ht \leq T_0$, (the constant of (1.4)):*

$$\sup_{(x,y)\in(\mathbb{R}^p)^\Lambda\times(\mathbb{R}^p)^\Lambda} \sum_{\mu\in\Lambda} \frac{|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m-1)}} \nabla_\mu \psi_\Lambda(x, y, t)|}{\varepsilon^{\mathrm{diam}(\{\lambda^{(1)},...\lambda^{(m-1)},\mu\})}} \leq t K_m(\varepsilon), \tag{1.11}$$

*where $K_m(\varepsilon)$ is independent of $\Lambda$.*

J. Sjöstrand proved in [23] that, near the diagonal, the integral kernel $U_\Lambda$ can be written in the form (1.9), and he proved that an approximation modulo $\mathcal{O}(h^\infty)$ of the function $\psi_\Lambda$ satisfies, near the diagonal, inequalities which are equivalent to (1.11). A family of functions satisfying the estimates (1.11) is called in [23] a $0-$*standard function with exponential weight*. Here, we study the function $\psi_\Lambda$ itself, not an approximation, and the estimation is global.

In the semiclassical limit, $\psi_\Lambda(x, y, t)$ is approximated by the product of $t$ by the average of $V_\Lambda$ on the segment $[x, y]$. By (1.4), this semiclassical approximation is written as a sum of terms, associated to points $\lambda$, or couples of points $(\lambda, \mu)$. In the next theorem, we shall describe $\psi_\Lambda$ itself, and not its approximation, in a similar way. In fact, there will be some difference: instead of a sum taken on the sites $\lambda$ or the couples of sites $(\lambda, \mu)$ like in (1.4), we shall need, for $\psi_\Lambda$, a sum of functions $T_Q\psi_\Lambda$, associated to all the boxes $Q$ contained in $\Lambda$. When it is restricted to the diagonal, the function associated to the box $Q$ will depend only on the variables $x_\lambda$ $(\lambda \in Q)$, and the function will decrease almost like $\varepsilon^{\mathrm{diam}(Q)}$ when $Q$ is large. In the literature on classical lattice spin systems, (without Schrödinger equation), the potential $V_\Lambda$ is often supposed to be a sum of such functions. A *box* of $\mathbb{Z}^d$ is a set of the following form:

$$\Lambda = \prod_{j=1}^{d} [a_j, b_j] \tag{1.12}$$

where $a_j$ and $b_j$ are in $\mathbb{Z}$ $(a_j \leq b_j)$. In Theorem 1.1, the choice of the norm in $\mathbb{Z}^d$ was irrelevant, but now it is the $\ell^\infty$ norm.

**Theorem 1.2.** *We can define, for each box $\Lambda$ of $\mathbb{Z}^d$, and for each box $Q \subseteq \Lambda$, ($Q$ may be a single point), a function $(T_Q\psi_\Lambda)(x, y, t)$ (where $\psi_\Lambda$ is the function of Theorem 1.1), such that:*

*1. The function $(T_Q\psi_\Lambda)(x,y,t)$ is $C^\infty$ and depends only on the variables $x_\lambda$ and $y_\lambda$ such that $\lambda\in Q$, and on the variables $x_\lambda - y_\lambda$ such that $\lambda\notin Q$, (restricted to the diagonal, this function is supported in $Q$).*

*2. We have:*

$$\psi_\Lambda(x,y,t) = \psi_\Lambda(0,y-x,t) + \sum_{Q\subseteq\Lambda}(T_Q\psi_\Lambda)(x,y,t) \tag{1.13}$$

*where the the sum is taken on all boxes $Q$ contained in $\Lambda$, including the points.*

*3. If $(H_\varepsilon)$ is satisfied $(0<\varepsilon<1)$, if $ht\le T_0$ (defined in (1.8) and (1.7)), for each integer $m\ge 1$, for each points $\lambda^{(1)}$, ..., $\lambda^{(m)}$ in $\Lambda$, we have, for some constant $K_m(\varepsilon)>0$ independent of $\Lambda$:*

$$\|\nabla_{\lambda^{(1)}}\dots\nabla_{\lambda^{(m)}}\big(T_Q\psi_\Lambda\big)(.,.,t,h)\| \le K_m(\varepsilon)t\varepsilon^{\mathrm{diam}(Q\cup\{\lambda^{(1)},\dots,\lambda^{(m)}\})}(1+\mathrm{diam}(Q))^{2d}. \tag{1.14}$$

*If $m=0$, this result is valid for boxes $Q$ not reduced to single points. If $m=0$ and $Q$ is a single point $\lambda$, we can write:*

$$\Big|T_{\{\lambda\}}\psi_\Lambda(x,y,t) - t\widetilde{A}(x_\lambda,y_\lambda) + t\widetilde{A}(0,y_\lambda - x_\lambda)\Big| \le K(\varepsilon)(t+h^2t^2). \tag{1.15}$$

*where $A$ is the function appearing in (1.4), and:*

$$\widetilde{A}(x_\lambda,y_\lambda) = \int_0^1 A(y_\lambda + \theta(x_\lambda - y_\lambda))d\theta. \tag{1.16}$$

Now, we shall apply the decomposition of Theorem 1.2 to the study of correlations. For each disjoint subsets $E_1$ and $E_2$ of $\mathbb{Z}^d$, let us denote by $K_{\mathrm{op,op}}(E_1,E_2,t,h)$ the smallest positive constant such that, for each $A\in\mathcal{L}(\mathcal{H}_{E_1})$ and $B\in\mathcal{L}(\mathcal{H}_{E_2})$, for each box $\Lambda$ containing $E_1$ and $E_2$, we have:

$$|\mathrm{Cov}_{\Lambda,t}(A,B)| \le\ K_{\mathrm{op,op}}(E_1,E_2,t,h)\ \|A\|\ \|B\|. \tag{1.17}$$

By the definitions (1.2) and (1.3), this constant exists, and is $\le 2$. Let $K_{\mathrm{fc,fc}}(E_1,E_2,t,h)$ the smallest positive constant such that, for each continuous, bounded functions $f$ and $g$, supported in $E_1$ and $E_2$, for each box $\Lambda$ containing $E_1$ and $E_2$, we have:

$$|\mathrm{Cov}_{\Lambda,t}(M_f,M_g)| \le\ K_{\mathrm{fc,fc}}(E_1,E_2,t,h)\ \|f\|\ \|g\|. \tag{1.18}$$

We define similarly $K_{\mathrm{fc,op}}(E_1,E_2,t,h)$ and $K_{\mathrm{op,fc}}(E_1,E_2,t,h)$.

**Theorem 1.3.** *Let $(V_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ be an interaction satisfying $(H_\varepsilon)$, $(0<\varepsilon<1)$. Then, for each $\delta$ such that $0<\varepsilon<\delta<1$, there exists $t_1(\varepsilon,\delta)$ and functions $M(N,t,h,\varepsilon,\delta)$ and $N(\varepsilon,\delta)$ with the following property. If $ht\le T_0$ (defined in (1.8)), and if $t\le t_1(\varepsilon,\delta)$, for each finite disjoint sets $E_1$ and $E_2$, we have:*

*a)* $$K_{\mathrm{op,op}}(E_1,E_2,t,h)\le M(|E_1\cup E_2|,t,h,\varepsilon,\delta)\ \delta^{\mathrm{dist}(E_1,E_2)}.$$

*b)* $$K_{\mathrm{fc,fc}}(E_1,E_2,t,h)\le\ t\ \inf\Big(|E_1|,|E_2|\Big)\ N(\varepsilon,\delta)\ \ \delta^{\mathrm{dist}(E_1,E_2)}.$$

*c)* $$K_{\mathrm{op,fc}}(E_1,E_2,t,h)\le\ M(|E_1|,t,h,\varepsilon,\delta)\ \delta^{\mathrm{dist}(E_1,E_2)}.$$

*If $f$ has bounded derivatives, and $g$ is bounded, we can write:*

*d)* $$|\mathrm{Cov}_{\Lambda,t}(M_f,M_g)|\le\ \|\nabla f\|_\infty\ \|g\|\ M(|E_1|,t,h,\varepsilon,\delta)\ \delta^{\mathrm{dist}(E_1,E_2)}.$$

*The functions denoted by $M$, as functions of $t$ and $h$, for fixed $\varepsilon$ and $\delta$, are bounded on each compact of the set $\{(t,h),\ h>0, 0<t<t_1(\varepsilon,\delta), ht<T_0\}$. The constant $t_1(\varepsilon,\delta)$, limiting the validity of the result, is independent of the three sets $E_1$, $E_2$ and $\Lambda$.*

We don't give precisely the behaviour of the constants as functions of $t=\beta$ when $t\to 0$, excepted in the case of multiplications by bounded functions. It depends of the behaviour of $C_m(t)$ in the hypothesis $(H_\varepsilon)$. For the proof of this theorem, we obtain, from Theorem 1.2, a decomposition of the integral kernel of $e^{-\beta H_\Lambda}$ which is similar to the Mayer decomposition in classical statistical mechanics (see B. Simon [22]).

Now, we are interested to thermodynamic limits, and to the rate of convergence to such limits. We say that a local observable $A\in\mathcal{L}(\mathcal{H}_\Lambda)$ is supported in a subset $Q\subset\Lambda$ if $A$ can be seen also as an element of $\mathcal{L}(\mathcal{H}_Q)$. The proof of the next theorems 1.4 and 1.5 relies on the estimations of correlations, (Theorem 1.3), and also, directly on the decomposition of $\psi_\Lambda$ (Theorem 1.2).

**Theorem 1.4.** *If the interaction satisfies $(H_\varepsilon)$, if $ht\leq T_0$ (defined in (1.8)), and if $t$ is small enough, for each local observable $A$, the following thermodynamic limit exists:*

$$\omega_t(A)=\lim_{n\to+\infty}E_{\Lambda_n,t}(A)\qquad \Lambda_n=\{-n,\dots,n\}^d. \tag{1.19}$$

*Moreover, if $\varepsilon<\delta<1$, there exists $t_1(\varepsilon,\delta)$ and a function $K(h,t,\varepsilon,\delta,N)$ such that, if $ht\leq T_0$, if $t\leq t_1(\varepsilon,\delta)$ and if $\Lambda_n$ contains the support of $A$,*

$$\Big|\omega_t(A)-E_{\Lambda_n,t}(A)\Big|\leq K(h,t,\varepsilon,\delta,|\mathrm{supp}(A)|)\ \|A\|\ \delta^{\mathrm{dist}(\mathrm{supp}(A),\Lambda_n^c)}. \tag{1.20}$$

Theorems 1.3 and 1.4 can be applied to prove some properties of a state of the $C^\star$ algebra $\mathcal{A}$ associated to the family of Hilbert spaces $\mathcal{H}_\Lambda$. Let us recall (see B. Simon [22], section II.1, or Bratteli-Robinson [9]), that, if $\Lambda_1\subset\Lambda_2$, we have a natural identification of $\mathcal{L}(\mathcal{H}_{\Lambda_1})$ as a subspace of $\mathcal{L}(\mathcal{H}_{\Lambda_2})$, and $\mathcal{A}$ is the closure of the union of (equivalence classes of) all the $\mathcal{L}(\mathcal{H}_\Lambda)$. Then, for each $h$ and $t$ such that Theorem 1.4 can be applied, the limit in (1.19) defines a state on $\mathcal{A}$, still denoted by $\omega_t$. Theorem 1.3 proves that this state has the *mixing property*, (quantum analogue of the definition III.1.21 of B. Simon [22]):

$$\lim_{|u|\to\infty}\Big[\omega_t(A\circ\tau_u B)-\omega_t(A)\omega_t(B)\Big]=0\qquad \forall A,B\in\mathcal{A} \tag{1.21}$$

if $ht<T_0$ and if $t$ is small enough. Here $\tau_u$ is the natural translation by a vector $u\in\mathbb{Z}^d$. For this application, it is useful that the condition of validity of Theorem 1.3 does not depend on the number of elements of the supports. In this application to the mixing property, invariance by translation is needed, and we assume that $B_{\lambda,\mu}$ depends only on $\lambda-\mu$.

In the second application, we consider the mean value, not of a local observable, but of the mean energy per site.

**Theorem 1.5.** *If $(H_\varepsilon)$ is satisfied, and if $ht\leq T_0$, the following limit exists:*

$$E(t)=\lim_{n\to+\infty}\frac{1}{|\Lambda_n|}X_{\Lambda_n}(t),\qquad X_\Lambda(t)=\frac{\partial}{\partial t}\ln Z_\Lambda(t), \tag{1.22}$$

*where $Z_\Lambda(t)$ is defined in (1.2) and $\Lambda_n$ in (1.19). We can write :*

$$\Big|E(t)-\frac{1}{|\Lambda_n|}X_{\Lambda_n}(t)\Big|\leq\frac{K(t,h)}{n} \tag{1.23}$$

*The constant $K(t,h)$ is bounded on each compact set of $\{(h,t),\ h>0, t>0,\ \ ht<T_0\}$.*

By the last statement, $E(t)$ is a continuous function of $t$ in the domain in which the theorem is applicable: in other words, there is no phase transition, if $t = \beta$ is small enough.

Theorem 1.1 is proved in section 2. The family of operators $T_Q$, which gives a decomposition of any function on $(\mathbb{R}^p)^\Lambda$, and that we use for the proof of theorem 1.2, is explained in section 3. The estimations of $T_Q \psi_\Lambda$ are proved in section 4. In sections 5 and 6, we see how to dissociate two disjoint sets $\Lambda_1$ and $\Lambda_2$ when $\Lambda$ is their union. Then, in sections 7 and 8, the decomposition of Theorem 1.2 will be applied to the correlations. For the bound of correlations in the multiplicative case, section 8 is not needed. Section 9 is devoted to the proof of Theorems 1.4 and 1.5.

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## 2. Proof of Theorem 1.1.

The heat kernel $U_\Lambda(x, y, t)$ must satisfy $\frac{\partial U_\Lambda}{\partial t} - \frac{h^2}{2} \Delta_x U_\Lambda + V_\Lambda(x) U_\Lambda = 0$ for $t > 0$. Therefore, if $U_\Lambda$ is written in the form (1.9), the function $\psi_\Lambda$ in $(\mathbb{R}^p)^\Lambda$ appearing in this expression must satisfy the Cauchy problem:

$$\frac{\partial \psi_\Lambda}{\partial t} + \frac{x-y}{t} \cdot \nabla_x \psi_\Lambda - \frac{h^2}{2} \Delta_x \psi_\Lambda = V_\Lambda(x) - \frac{h^2}{2} |\nabla_x \psi_\Lambda|^2 \tag{2.1}$$

$$\psi_\Lambda(x, y, 0, h) = 0 \tag{2.2}$$

This section is devoted to the study of this Cauchy problem. We shall use a maximum principle for the linearized of (2.1), and more generally, for operators $L_a$ in $(\mathbb{R}^p)^\Lambda \times [0, T]$ $(\Lambda \in \mathbb{Z}^d, T > 0)$ of the following form:

$$\Big(L_a u\Big) = \frac{\partial u}{\partial t} + \frac{x-y}{t} . \nabla u - \frac{h^2}{2} \Delta u + \sum_{\mu \in \Lambda} (a_\mu(x, t) \cdot \nabla_{x_\mu} u), \tag{2.3}$$

where $a = (a_\lambda)_{\lambda \in \Lambda}$ is a family of continuous and bounded functions in $(\mathbb{R}^p)^\Lambda \times [0, T]$, and $y \in (\mathbb{R}^p)^\Lambda$. Since there is a drift with unbounded coefficients, and since this maximum principle will be used again, it was perhaps useful to give a precise statement:

**Proposition 2.1.** *Let $y \in (\mathbb{R}^p)^\Lambda$, let $a = (a_\lambda)_{\lambda \in \Lambda}$ be a family of functions such that $a_\lambda(x, t)$ $(\lambda \in \Lambda)$ is continuous and bounded in $(\mathbb{R}^p)^\Lambda \times [0, T]$ $(T > 0)$, with values in $\mathbb{R}^p$, and $u$ be a function in $C((\mathbb{R}^p)^\Lambda \times [0, T]) \bigcap C^2((\mathbb{R}^p)^\Lambda \times ]0, T])$ such that $u$ and $\nabla_{x_\lambda} u$ $(\lambda \in \Lambda)$ are bounded in $(\mathbb{R}^p)^\Lambda \times ]0, T]$ and $u(x, 0) = 0$. Assume that the function $f = L_a u$, defined by (2.3), (where $h > 0$), is bounded. Then we have, for each $t_0$ and $t$ $(0 \leq t_0 \leq t \leq T)$,*

$$\|u(. , t)\| \leq \|u(. , t_0)\| + \int_{t_0}^t \|L_a u(., s)\| \, ds. \tag{2.4}$$

*Proof.* Let $\chi \in C^\infty((\mathbb{R}^p)^\Lambda)$ be a real-valued function with $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. For $R \geq 0$, set $\chi_R(x, y, t) = \chi(\frac{|x-y|}{Rt})$ , $x \in (\mathbb{R}^p)^\Lambda$, $t > 0$. If $0 < t_0 < t$, $y$ fixed, and $R > 0$, the standard maximum principle, applied in $(\mathbb{R}^p)^\Lambda \times [t_0, t]$ with bounded coefficients in the first order terms, gives

$$\|\chi_R u(., t)\| \leq \|\chi_R u(., t_0)\| + \int_{t_0}^t \|L_a(\chi_R u(., s)\| ds.$$

An explicit computation of $L_a(\chi_R u(.,s)$ with our cut-off function shows that, when $R \to +\infty$, $L_a(\chi_R u(.,s) \to L_a u$. The Proposition follows (we may also let $t_0 \to 0$).

Now, we give a result of global existence of the solution $\psi_\Lambda$ of the Cauchy problem (2.1), (2.2), with global bounds of all the derivatives of this function, but, at this step, all the bounds, excepted for the first order derivatives, may still depend on $\Lambda$, hence on the number of sites in the lattice, which will tend to infinity later.

**Proposition 2.2.** *Assuming only that $V_\Lambda$ is in $C^\infty((\mathbb{R}^p)^\Lambda)$, real-valued, and that all its derivatives of order $\geq 1$ are bounded, then, there exists a unique global classical solution $\psi_\Lambda(x,y,t)$ in $C^\infty((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0,+\infty[)$ to (2.1), (2.2). Moreover, we have the estimations (1.10) for the first order derivatives, with $M_1(\varepsilon)$ defined in (1.7). For each $T > 0$ and $h > 0$, all the derivatives of order $\geq 2$ with respect to $x$ and $y$ of $\psi_\Lambda$ are bounded on $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0,T]$ (with bounds which may depend, at this step, on $\Lambda$, $T$ and $h$).*

*Proof.* This result of existence is an adaptation, due to the presence of the term containing $\frac{x-y}{t}$, of a similar result about a non linear heat equation, proved in [2]. Let us only explain what is new here. In order to solve (2.1), (2.2) by a fixed point theorem, we use the explicit solution of the following Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{x-y}{t}\nabla_x u - \frac{h^2}{2}\Delta_x u = f(x,t) \quad (t > t_0), \qquad u(x,y,t_0) = g(x), \tag{2.5}$$

which is given by

$$u(x,y,t) = \int_{(\mathbb{R}^p)^\Lambda} G_h(x,x',y,t_0,t)g(x')dx' + \int_{\substack{x'\in(\mathbb{R}^p)^\Lambda \\ s\in[t_0,t]}} G_h(x,x',y,s,t)\ f(x',s)\ dsdx'$$

where we set, if $0 < s < t$:

$$G_h(x,x',y,s,t) = \left(\frac{a(s,t)}{2\pi h^2}\right)^{\frac{p}{2}|\Lambda|} e^{-\frac{a(s,t)}{2h^2}|x'-m(x,y,s,t)|^2} \tag{2.6}$$

$$a(s,t) = \frac{t}{s(t-s)} \qquad m(x,y,s,t) = \left(1-\frac{s}{t}\right)y + \frac{s}{t}x.$$

We shall use the following properties of this kernel, where $C > 0$ is independent of all the parameters:

$$\int_{(\mathbb{R}^p)^\Lambda} |\nabla_{x_\lambda} G_h(x,x',y,s,t)|dx' \leq \frac{C}{h}\sqrt{\frac{s}{t(t-s)}}$$

To simplify the notations, we assume here that $p = 1$, and we denote by $\mathbf{G}_h(y,s,t)$ (resp. $\mathbf{G}_h^{(\lambda)}(y,s,t)$) the operator with integral kernel $(x,x') \to G_h(x,x',y,s,t)$ (resp. $\partial_{x_\lambda} G_h(x,x',y,s,t)$). With these notations, we have:

$$\partial_{x_\lambda}\left(\mathbf{G}_h(y,s,t)f\right) = \frac{s}{t}\ \mathbf{G}_h(y,s,t)\ \partial x_\lambda f.$$

Using this operator, we study the integral equation satisfied, not by $\psi_\Lambda$ itself, but by its derivatives, if we want to find $\psi_\Lambda$ satisfying (2.1) and (2.2). For example, if $\psi_\Lambda$ satisfies (2.1) in an interval $[t_0,t_1]$, if $y$ is fixed, and if we are given $\varphi_{y,\lambda} = \nabla_{x_\lambda}\psi_\Lambda(.,y,t_0)$, we hope that the first order derivatives $u_{y,\lambda}(x,t) = \nabla_{x_\lambda}\psi_\Lambda(x,y,t)$ will satisfy, setting $u_y = (u_{y,\lambda})_{\lambda\in\Lambda}$ and $\varphi_y = (\varphi_{y,\lambda})_{\lambda\in\Lambda}$:

$$u_y = S_y\varphi_y + T_y(u_y) \qquad \big(S_y\varphi\big)_\lambda(.,t) = \frac{t_0}{t}\mathbf{G}_h(y,t_0,t)\varphi_\lambda,$$

$$\big(T_y(u)\big)_\lambda(.,t) = \int_{t_0}^t \frac{s}{t}\mathbf{G}_h(y,s,t)(\nabla_{x_\lambda}V_\Lambda)ds - \frac{h^2}{2}\int_{t_0}^t \mathbf{G}_h^{(\lambda)}(y,s,t)\sum_{\mu\in\Lambda}|u_\mu(.,s)|^2 ds$$

In order to solve such integral equations, we set, for each interval $I = [t_0, t_1]$ $(0 \le t_0 < t_1)$:

$$E_1(I) = \Big\{ u = (u_\lambda)_{\lambda\in\Lambda}, \qquad u_\lambda \in C^0((\mathbb{R}^p)^\Lambda \times I, \qquad \|u\|_{1,I} = \sup_{\substack{\lambda\in\Lambda \\ (x,t)\in(\mathbb{R}^p)^\Lambda\times I}} t^{-1}|u_\lambda(x,t)| < \infty \Big\},$$

and we denote by $B_{1,I}(r)$ the ball in $E_1(I)$ with radius $r$ and center at the origin. First, we solve $u_y = T_y(u_y)$ in an interval $I = [0, \tau]$, (the initial data vanishing at $t_0 = 0$). We can choose $t_1 > 0$ such that $T_y$ is a contraction from $B_{1,I}(2M_1(\varepsilon))$ into itself, where $I = [0, t_1]$ and $M_1(\varepsilon)$ is defined in (1.7). Let $u_y = (u_{y,\lambda})_{\lambda\in\Lambda}$ be the fixed point of $T_y$ in this interval. Then, we write the integral equation that the second, and third order derivatives must satisfy, and we solve them as the first one in an interval $[0, t_1]$. It follows that the functions $u_{y,\lambda}$ are $C^2$ in $(\mathbb{R}^p)^\Lambda \times ]0, t_1[$, and satisfy the equation:

$$\frac{\partial(tu_{y,\lambda})}{\partial t} + \frac{x-y}{t}.\nabla(tu_{y,\lambda}) - \frac{h^2}{2}\Delta(tu_{y,\lambda}) + \sum_{\mu\in\Lambda}(u_{y,\mu}(x,t)\ .\ \nabla_{x_\mu}(tu_{y,\lambda}) = t\partial_{x_\lambda}V_\Lambda(x,t). \tag{2.7}$$

The function $\psi_\Lambda$ defined by:

$$\psi_\Lambda(.,y,t) = \int_0^t \mathbf{G}_h(y,s,t)\Big[V_\Lambda(.) - \frac{h^2}{2}\sum_{\lambda\in\Lambda}|u_{y,\lambda}(.,s)|^2\Big]ds$$

satisfies (2.1) in $(\mathbb{R}^p)^\Lambda \times ]0, t_1[$. The maximum principle, (Proposition 2.1), applied to the equation (2.7), shows that $|u_{y,\lambda}(x,t)| \le \frac{t}{2}M_1(\varepsilon)$ if $0 \le t \le t_1$, which is a better estimation than which is given by the fixed point theorem. Then, we take $\varphi_{y,\lambda}(x) = u_{y,\lambda}(x, t_1)$ as an initial value for a problem in an interval $I = [t_1, t_2]$. We want a system of functions, again denoted by $u_y$, such that $u_y = S_y\varphi_y + T_yu_y$ in this interval. We can chose $t_2$ such that the map $u \to S_y\varphi + T_yu$ is a contraction of $B_{1,I}(2M_1(\varepsilon))$ into itself, where $I = [t_1, t_2]$. We prove again that $u$ is $C^2$ on $(\mathbb{R}^p)^\Lambda \times ]t_1, t_2[$, and satisfies (2.7), which implies again, by Proposition 2.1, that $|u_\lambda(x,t)| \le \frac{t}{2}M_1(\varepsilon)$ if $0 \le t \le t_2$. We can iterate this process, and prove the existence of the functions $u_{y,\lambda}$ in a sequence of intervals $[t_j, t_{j+1}]$, in which these functions are $C^2$, with bounded derivatives. At each time, Proposition 2.1, applied to the function $tu_{y,\lambda}$, to the equation (2.7), and to the interval $[0, t_j]$ proves that the initial data $\varphi_{y,\lambda}$ at time $t_j$ satisfies $\|\varphi_{y,\lambda}\| \le \frac{t_j}{2}M_1(\varepsilon)$. The length $t_{j+1} - t_j$ of the interval, allowing the fixed point theorem to work, depends only of the bound of this initial value, which is independent of $j$. Therefore $\tau = t_{j+1} - t_j$ may be chosen independent of $j$. Thus, we proved the global existence of $\psi_\Lambda$ of class $C^2$ satisfying (2.1), (2.2) and (1.10). Then we solve the integral equations satisfied by the higher order derivatives, including the derivatives with respect to $y$ also. For each order of derivation, we have to find a fixed point for an integral operator, and the length of the intervals, in which this operator may be contractive, depends only on the bounds of the first order derivatives, already obtained globally. Thus, we prove the global existence of the higher order derivatives, and that they are globally bounded in $(\mathbb{R}^p)^\Lambda \times [0, T]$ for each $T > 0$. This provides $\psi_\Lambda$ verifying the properties stated in proposition 2.2. $\square$

Then the function $U_\Lambda$ defined in (1.9), with this function $\psi_\Lambda$, is the integral kernel of the operator $e^{-tH_\Lambda}$. Excepted for the first order derivatives, the bounds given by the previous proof are not uniform with respect to the set $\Lambda$. The next step will be the proof of such uniform bounds for the second order derivatives. For that, we have to choose a norm $N_2$ on the second differential of any function $f$ on $(\mathbb{R}^p)^\Lambda$. We set, for $\varepsilon \in ]0, 1[$:

$$N_2\Big(d^2f(x), \varepsilon\Big) = \sup_{\lambda\in\Lambda}\sum_{\mu\in\Lambda}\frac{|\nabla_{x_\lambda}\nabla_{x_\mu}f(x)|}{\varepsilon^{|\lambda-\mu|}}, \tag{2.8}$$

(with the euclidian norm on $\mathbb{R}^p$.) We denote by $\| \ \|$ the $L^\infty$ norm. The norm $N_2$, and the norms $N_m$ of (2.10), appear in Sjöstrand [23] in the definition of 0-*standard functions with exponential weight*. The norm $\|N_2(\psi_\Lambda(.,y,t)\|$ is well defined for each $\Lambda$, $y$ and $t$, by Proposition 2.2. The norm $\|N_2(d^2V_\Lambda, \varepsilon)\|$ is bounded, independently of $\Lambda$, by the constant $M_2(\varepsilon)$ of (1.7).

**Proposition 2.3.** *If $ht \leq T_0 = \|N_2(d^2V_\Lambda(.), \varepsilon)\|^{-1/2}$, we have, for all $y \in (\mathbb{R}^p)^\Lambda$, with the notation (1.7),*

$$\|N_2(d^2\psi_\Lambda(.y,t), \varepsilon)\| \leq 2t\|N_2(d^2V_\Lambda(.), \varepsilon)\| = 2tM_2(\varepsilon). \tag{2.9}$$

*Proof.* For each $\lambda \in \Lambda$, for each $X$ in $\mathbb{R}^p$, for each sequence $(Y_\mu)_{\mu\in\Lambda}$ of vectors $Y_\mu$ in $\mathbb{R}^p$, and for each $y \in (\mathbb{R}^p)^\Lambda$, the function

$$\varphi(x,t) = \sum_{\mu\in\Lambda} \frac{(X.\nabla_{x_\lambda})(Y_\mu.\nabla_{x_\mu})\psi_\Lambda(.,.,t,h)}{\varepsilon^{|\lambda-\mu|}}$$

satisfies the equation $L_a(t^2\varphi) = \ t^2F - h^2t^2G$, where $L_a$ is defined in (2.3) with, here, $a_\lambda = h^2\nabla_{x_\lambda}\psi_\Lambda$ and:

$$F = \sum_{\mu\in\Lambda} \frac{(X.\nabla_{x_\lambda})(Y_\mu.\nabla_{x_\mu})V_\Lambda}{\varepsilon^{|\lambda-\mu|}} \qquad G(x) = \sum_{(\mu,\nu)\in\Lambda^2} \frac{< \nabla_{x_\nu}(X.\nabla_{x_\lambda})\psi_\Lambda(x,y,t)\ ,\ \nabla_{x_\nu}(Y_\mu.\nabla_{x_\mu})\psi_\Lambda(x,y,t) >}{\varepsilon^{|\lambda-\mu|}}$$

We have, by the definition (2.8) of the norm $N_2$:

$$|F(x)| \leq N_2(d^2V_\Lambda(x), \varepsilon)\ |X|\ \sup_{\mu\in\Lambda}|Y_\mu|, \qquad |G(x)| \leq N_2(d_x^2\psi_\Lambda(x,y,t), \varepsilon)^2|X|\ \sup_{\mu\in\Lambda}|Y_\mu|.$$

Therefore, by the maximum principle, (Proposition 2.1), applied to the function $u = t^2\varphi$, and to the operator $L_a$, with $t_0 = 0$, we have:

$$\|t^2\varphi(.,.,t)\| \leq \ |X|\ \sup_{\mu\in\Lambda}|Y_\mu|\ \int_0^t \left[s^2\|N_2(d^2V_\Lambda(.), \varepsilon)\| + h^2s^2\|N_2(d_x^2\psi_\Lambda(.,y,s), \varepsilon)\|^2\right] ds.$$

In other words, taking the sup on all vectors $X$ and sequences $Y_\mu$, and on all points $x$ and $y$,

$$t^2\|N_2(d_x^2\psi_\Lambda(.,y,t), \varepsilon)\| \leq \int_0^t \left[s^2\|N_2(d^2V_\Lambda(.), \varepsilon)\| + h^2s^2\|N_2(d_x^2\psi_\Lambda(.,y,s), \varepsilon)\|^2\right] ds.$$

Then Proposition 2.3 follows easily, by a kind of quadratic Gronwall lemma.

*End of the proof of theorem 1.1.* Now, we need a norm $N_m$ for the higher order differential $d^mf(x)$ of a function on $(\mathbb{R}^p)^\Lambda$. We set:

$$N_m(d^mf(x), \varepsilon) = \sup_{(\lambda_1,\dots,\lambda_{m-1})\in\Lambda^{m-1}} \sum_{\mu\in\Lambda} \frac{|\nabla_{x_{\lambda_1}}...\nabla_{x_{\lambda_{m-1}}}\nabla_{x_\mu}f(x)|}{\varepsilon^{\mathrm{diam}(\{\lambda_1,\dots,\lambda_{m-1},\mu\})}}. \tag{2.10}$$

In [23], a family of functions, depending on $\Lambda$, is called a $0-$standard function with exponential weight if $N_m(d^mf(x), \varepsilon)$ is bounded, with bounds independent of $\Lambda$. The definition of [23] is more complicated, since $\ell^p$ norms are used, not only $\ell^1$ and $\ell^\infty$ norms like here. Again, $\|N_m(d^m\psi_\Lambda(.,.,t), \varepsilon)\|$ is well defined by Proposition 2.2, and $\|N_m(d^mV_\Lambda, \varepsilon)\|$ is bounded, independently of $\Lambda$, if the hypothesis $(H_\varepsilon)$ is satisfied. Now, we can prove, by an induction on $m \geq 2$, that, if $(H_\varepsilon)$ is satisfied, we have

$$\|N_m(d^m\psi_\Lambda(.,.,t), \varepsilon)\| \leq tK_m(\varepsilon) \qquad \text{if} \quad ht \leq T_0, \tag{$P_m$}$$

where $K_m(\varepsilon)$ is independent of $\Lambda$. For each sequence $(\lambda^{(1)},...,\lambda^{(m-1)})$ of points in $\Lambda$, for each vectors $X_j$ $(1 \leq j \leq m)$ and $Y_\mu$ $(\mu \in \Lambda)$ in $\mathbb{R}^p$, we shall estimate the function $t^m\varphi$, where

$$\varphi(x,t) = \sum_{\mu\in\Lambda} \frac{(X_1.\nabla_{x_{\lambda_1}})...(X_{m-1}.\nabla_{x_{\lambda_{m-1}}})(Y_\mu.\nabla_{x_\mu})\psi_\Lambda}{\varepsilon^{\mathrm{diam}(\{\lambda_1,\dots,\lambda_{m-1},\mu\})}}.$$

By differentiating (2.1), we see that this his function satisfies an equation of the type $L_a(t^m\varphi) = t^m F$ where $L_a$ is defined in (2.3) with $a_\mu = h^2 \nabla_{x_\mu} \psi_\Lambda$, and $F$ satisfies

$$|F(x,y,t)| \leq \ \big[\sup_{\mu\in\Lambda} |Y_\mu|\big] \ \left[\prod_{j=1}^{m-1} |X_j|\right] \ \Bigg[ N_m(d^m V_\Lambda(x), \varepsilon) + \dots$$

$$\dots + Ch^2 \sum_{k=2}^{m} N_k(d_x^k(\psi_\Lambda(x,y,t),\varepsilon)) \ N_{m+2-k} d_x^{m+2-k}(\psi_\Lambda(x,y,t),\varepsilon))\Bigg].$$

We can apply the maximum principle (Proposition 2.1) to the operator $L_a$ as before. Taking the sup over all the sequences of vectors $X_j$ and $Y_\mu$, using the hypothesis $(H_\varepsilon)$ for $V_\Lambda$, the induction hypothesis $(P_{m-1})$ for the terms with $2 < k < m$, and Proposition 2.3 for $k = 2$ or $k = m$, we obtain, if $ht \leq T_0$:

$$\|N_m(d^m\psi_\Lambda(.,.,t),\varepsilon)\| \leq (t + h^2t^3)K_m(\varepsilon) + 4Cth^2 M_2(\varepsilon) \int_0^t N_m(d^m\psi_\Lambda(.,.,s),\varepsilon) ds.$$

Then property $(P_m)$ follows by the usual Gronwall's lemma, (without more conditions on $h$ and $t$).

Then, all derivatives with respect to $x$ only are bounded as claimed. Since $e^{-tH_\Lambda}$ is self-adjoint, we have also bounds for the derivatives with respect to $y$ only. For the mixed derivatives, we need a new induction. For example, to estimate the matrix $\nabla_{x_\lambda}\nabla_{y_\mu}\psi_\Lambda$, we see that, for $X$ and $Y$ in $\mathbb{R}^p$, the function $\varphi =< X.\nabla_{x_\lambda} >< Y.\nabla_{y_\mu} > \psi_\Lambda(x,y,t)$ satisfied $L_a(t\varphi) = F$, where $L_a$ is defined in (2.3) and

$$F =< X.\nabla_{x_\lambda} >< Y.\nabla_{x_\mu} > \psi_\Lambda - h^2 \sum_{\nu\in\Lambda} \Big\langle \nabla_{x_\nu} < Y.\nabla_{y_\mu} > \psi_\Lambda \ . \ \nabla_{x_\nu} < X.\nabla_{x_\lambda} > \psi_\Lambda \Big\rangle$$

We use the estimations, already proven, of the derivatives with respect to $x$ only, we apply Proposition 2.1, and we apply again the usual Gronwall Lemma. The new induction follows the same ideas, and leads to the proof of Theorem 1.1.

## 3. Decomposition of a function on the lattice.

In this section, we shall define a decomposition of any function $f$ on $(\mathbb{R}^p)^L$ (where $L = \mathbb{Z}^d$) as a sum of functions $T_Q f$ associated to the boxes of $L$. We shall give two variants of this algorithm: the first one is simpler and works only for the restriction of our integral kernel to the diagonal, and the second one is useful for stronger estimates (for example, for trace norms). In the next section, we shall apply this decomposition to the function $\psi_\Lambda$ of Theorem 1.1, and prove estimates for the terms of the decomposition.

We shall associate, to each function $f$ and to each box $Q$, a function $T_Q f$. If $Q$ is a box in $L = \mathbb{Z}^d$, we denote by $\mathrm{Int}(Q)$ the set of non empty boxes $Q'$ ($Q' \subseteq Q$) which are either $Q$ itself, or obtained by removing in $Q$ some faces. We denote by $m(Q,Q')$ the number of faces of $Q$ which are removed in $Q'$ ($0 \leq m(Q,Q') < 2d$). If $Q$ is any set of $L = \mathbb{Z}^d$, we define a map $\pi_Q : (\mathbb{R}^p)^L \to (\mathbb{R}^p)^L$ by:

$$(\pi_Q x)_\lambda = \begin{cases} x_\lambda & \text{if} \quad \lambda \in Q \\ 0 & \text{if} \quad \lambda \notin Q \end{cases} \tag{3.1}$$

Then, we define the function $T_Q f$ by:

$$(T_Q f)(x) = \sum_{Q'\in \mathrm{Int}(Q)} (-1)^{m(Q,Q')} \Big[ f(\pi_{Q'}x) - f(0) \Big]. \tag{3.2}$$

Obviously, $T_Q f$ is supported in $Q$. If $f$ has a finite support, let $\Lambda$ be a box containing the support of $f$. Then we have:

$$f(x) - f(0) = \sum_{Q\subseteq\Lambda} (T_Q f)(x), \tag{3.3}$$

where the sum is taken over all the non empty boxes $Q$ contained in $\Lambda$, including the points. This equality follows from the definition of $T_Q f$ and from the following remark, for each box $P$ contained in $\Lambda$:

$$\sum_{\substack{Q\subseteq\Lambda \\ P\in\mathrm{Int}(Q)}} (-1)^{m(Q,P)} = \begin{cases} 1 & \text{if} \quad P=\Lambda \\ 0 & \text{if} \quad P\neq\Lambda \end{cases}$$

In order to estimate $T_Q f$, we shall use, rather than the derivatives of $f$, some operators of translation, with the following notations. For each $u \in (\mathbb{R}^p)^L$, we set:

$$(S_u f)(x) = f(x+u) - f(x) \qquad \sigma(u) = \{\lambda \in \mathbb{Z}^d, \qquad u_\lambda \neq 0\}. \tag{3.4}$$

For each box $Q$, of the form $Q = \prod_{j=1}^d [a_j, b_j]$, with $a_j \leq b_j$, we denote the different "faces" of $Q$ by:

$$B_-^{(k)}(Q) = \{\lambda \in Q, \qquad \lambda_k = a_k\}, \qquad B_+^{(k)}(Q) = \{\lambda \in Q, \qquad \lambda_k = b_k\}$$

For each $k \leq d$ such that $a_k < b_k$, we can write $Q = [a_k, b_k] \times \widetilde{Q}$, where $\widetilde{Q}$ is a box in $\mathbb{Z}^{d-1}$. For each box $P \in \mathrm{Int}\widetilde{Q}$, let us set:

$$P_\pm = [a_k, b_k] \times P, \qquad P_+ = ]a_k, b_k] \times P, \qquad P_- = [a_k, b_k[ \times P, \qquad P_0 = ]a_k, b_k[ \times P.$$

With these notations, we can write:

$$T_Q f(x) = \sum_{P\in\mathrm{Int}(\widetilde{Q})} (-1)^{m(\widetilde{Q},P)} \Big[ f(\pi_{P_\pm} x) - f(\pi_{P_+} x) - f(\pi_{P_-} x) + f(\pi_{P_0} x) \Big].$$

We can write this equality, using the previous operators $S_u$. If we set: $u_P = -\pi_{B_+^{(k)}(Q)} \pi_{P_\pm} x$ and $v_P = -\pi_{B_-^{(k)}(Q)} \pi_{P_\pm} x$, we can write:

$$f(\pi_{P_\pm} x) - f(\pi_{P_+} x) - f(\pi_{P_-} x) + f(\pi_{P_0} x) = (S_{u_P} S_{v_P} f)(\pi_{P_\pm} x) \tag{3.5}$$

It follows that

$$\|T_Q f\| \leq 4^d \sup_{\substack{\sigma(u)\subset B_+^{(k)}(Q) \\ \sigma(v)\subset B_-^{(k)}(Q)}} \|S_u S_v f\|.$$

We have similar estimates for the derivatives $\nabla_\lambda T_Q f$, if $\lambda$ is neither in $B_+^{(k)}(Q)$, nor in $B_-^{(k)}(Q)$. If $\lambda$ is, for example, in $B_-^{(k)}(Q)$, we can use, instead of (3.5):

$$f(\pi_{P_\pm} x) - f(\pi_{P_+} x) - f(\pi_{P_-} x) + f(\pi_{P_0} x) = (S_{u_P} f)(\pi_{P_+} x) - (S_{u_P} f)(\pi_{P_\pm} x),$$

and we get, if $\lambda$ is not in $B_+^{(k)}(Q)$:

$$\|\nabla_{x_\lambda} T_Q f\| \leq 4^d \sup_{\sigma(u)\subset B_+^{(k)}(Q)} \|S_u \nabla_{x_\lambda} f\|.$$

If $a_k = b_k$, we cannot apply that, but we can write, in all the cases:

$$\|\nabla_{x_\lambda} T_Q f\| \leq 4^d \|\nabla_{x_\lambda} f\|.$$

It is possible to define similar projections by replacing in (3.2) $\pi_Q$ by other projections. The heat kernel and our function $\psi_\Lambda$ are defined in $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$, but we shall only use translations by vectors in the diagonal. Therefore, we shall replace $\pi_Q$ by the following projector:

$$\Big(\Pi_Q(x,y)\Big)_\lambda = \begin{cases} (x_\lambda, y_\lambda) & \text{if} \quad \lambda \in Q \\ (0, y_\lambda - x_\lambda) & \text{if} \quad \lambda \notin Q \end{cases} \tag{3.6}$$

(We could make another choice, preserving self-adjointness). For each function $f(x,y)$ on $(\mathbb{R}^p)^L \times (\mathbb{R}^p)^L$, we define $T_Q f$ as before, with a small change:

$$(T_Q f)(x,y) = \sum_{Q' \in \mathrm{Int}(Q)} (-1)^{m(Q,Q')} \Big[ f(\Pi_{Q'}(x,y)) - f(0, y-x) \Big]. \tag{3.7}$$

Then the operators $S_u$ are replaced by the following one:

$$(S_u f)(x,y) = f(x+u, y+u) - f(x,y) \qquad \forall (x,y) \in (\mathbb{R}^p)^L \times (\mathbb{R}^p)^L \tag{3.8}$$

We verify easily the properties listed for later use in the next proposition.

**Proposition 3.1.** *With the previous notations, for each function $f$ in $C^\infty((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$, we have*
*i) $T_Q f$ depends only on the variables $x_\lambda$ and $y_\lambda$ such that $\lambda \in Q$, and on the variables $x_\lambda - y_\lambda$ such that $\lambda \notin Q$. If $f$ is supported in a finite box $\Lambda$, we have:*

$$f(x,y) - f(0, y-x) = \sum_{Q \subseteq \Lambda} (T_Q f)(x,y) \tag{3.9}$$

*ii) If $f$ depends in a smooth way on a parameter $\theta$, we have $T_Q(\frac{\partial f}{\partial \theta}) = \frac{\partial (T_Q f)}{\partial \theta}$.*
*iii) If $\lambda^{(1)}$, ... $\lambda^{(m)}$ is a finite sequence of points in $\Lambda$, we have:*

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} T_Q f\| \leq 4^d \|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} f\|. \tag{3.10}$$

*iv) If moreover, none of the points $\lambda^{(j)}$ is in the face $B_+^{(k)}(Q)$, we have*

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} T_Q f\| \leq 4^d \sup_{\sigma(u) \subset B_+^{(k)}(Q)} \|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} S_u f\|. \tag{3.11}$$

*v) If $a_k \neq b_k$, and if the points $\lambda^{(j)}$ are neither in $B_+^{(k)}(Q)$, nor in $B_-^{(k)}(Q)$ we have*

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} T_Q f\| \leq 4^d \sup_{\substack{\sigma(u) \subset B_+^{(k)}(Q) \\ \sigma(v) \subset B_-^{(k)}(Q)}} \|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} S_u S_v f\|. \tag{3.12}$$

## 4. Proof of Theorem 1.2, excepted (1.15).

The functions $T_Q \psi_\Lambda$ are defined by (3.7), and the properties 1 and 2 of Theorem 1.2 follow from Proposition 3.1. It remains to prove the bounds. We shall modify the norm $N_m(d^m f(x), \varepsilon)$ of section 2, and now, it will depend also on one subset $E$ of $\mathbb{Z}^d$, or sometimes on two subsets $E$ and $F$. We set, for each $E_1$, ... , $E_m$ which are either subsets of $\mathbb{Z}^d$, or points of $\mathbb{Z}^d$,

$$D(E_1, \dots, E_m) = \sup_{j,k \leq m} \mathrm{dist}(E_j, E_k). \tag{4.1}$$

If $f$ is a $C^\infty$ function on $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$, we set, for each $m \geq 2$, for each point $(x,y)$, for each $\varepsilon$ in $]0,1[$, and for each subsets $E$ and $F$ of $\mathbb{Z}^d$ :

$$N_m(d^m f(x,y), \varepsilon, E, F) = \sup_{(\lambda^{(1)}, \dots, \lambda^{(m-1)}) \in \Lambda^{m-1}} \sum_{\mu \in \Lambda} \frac{|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m-1)}} \nabla_\mu f(x,y)|}{\varepsilon^{D(\lambda^{(1)}, \dots, \lambda^{(m-1)}, \mu, E, F)}}. \tag{4.2}$$

If $m = 1$, there is no sup in (4.2), and the sum is taken on all $\lambda \in \Lambda$. We set also, if $m \geq 1$:

$$N_m^\infty(d^m f(x,y), \varepsilon, E, F) = \sup_{(\lambda^{(1)},\dots,\lambda^{(m)})\in\Lambda^m} \frac{|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} f(x,y)|}{\varepsilon^{D(\lambda^{(1)},\dots,\lambda^{(m)},E,F)}}. \tag{4.3}$$

When no set $E$ or $F$ appear, the first norm $N_m$ is the same as in section 2 if $m \geq 2$. For $m = 1$, the sup in section 2 (without sets $E$ and $F$) is repaced by a sum here (when there is at least one set). The second norm $N_m^\infty$ will be used in section 6. We still denote by $\| \ \|$ the $L^\infty$ norm. With these notations, the estimation (1.14) of Theorem 1.2 will follow from the next Lemma.

**Lemma 4.1.** *If $(H_\varepsilon)$ is satisfied $(0 < \varepsilon < 1)$, for each integer $m \geq 1$, there exists $K_m(\varepsilon) > 0$ such that, for each finite box $\Lambda$ of $\mathbb{Z}^d$, for each vectors $u$ and $v$ in $(\mathbb{R}^p)^\Lambda$, the functions $S_u\psi_\Lambda$ and $S_uS_v\psi_\Lambda$ defined like in (3.8), (with the function $\psi_\Lambda$ of Theorem 1.1), satisfy, if $ht \leq T_0$, (defined in (1.7) and (1.8)):*

$$\|N_m(d^m\big(S_u\psi_\Lambda\big)(.,.,t), \varepsilon, \sigma(u))\| \leq tK_m(\varepsilon)|\sigma(u)|, \tag{4.4}$$

$$\|N_m(d^m\big(S_uS_v\psi_\Lambda\big)(.,.,t), \varepsilon, \sigma(u), \sigma(v))\| \leq tK_m(\varepsilon)|\sigma(u)||\sigma(v)|, \tag{4.5}$$

*where the support $\sigma(u)$ is defined in (3.4). If $\sigma(u) \cap \sigma(v) = \emptyset$, we can write also:*

$$|S_uS_v\psi_\Lambda(x,y,t)| \leq tK_0(\varepsilon)\varepsilon^{\mathrm{dist}(\sigma(u),\sigma(v))}|\sigma(u)||\sigma(v)|. \tag{4.6}$$

Since one argument of the proof will be used again twice in section 6, we state it as a Lemma.

**Lemma 4.2.** *We consider a real valued function $A(x,y,t)$, $C^\infty$ on $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0,\infty[$. We assume that there is $\varepsilon$ in $]0,1[$ such that, for each $m \geq 1$, we can write, for some constant $K_m > 0$:*

$$\|N_m(d^m A(.,t), \varepsilon)\| \leq tK_m, \tag{4.7}$$

*where $N_m$ is the norm of section 2. We denote $L_a$ is the differential operator of (2.3), with $a_\lambda = h^2\nabla_{x_\lambda} A$. We consider also a smooth, real valued function $\varphi(x,y,t)$, such that $\varphi(y,x,t) = \varphi(x,y,t)$ and $\varphi(x,y,0) = 0$. We assume that there are two subsets $E$ and $F$ of $\mathbb{Z}^d$ such that, for each $m \geq 1$,*

$$\|N_m(d^m(L_a\varphi)(.,.,t), \varepsilon, E, F)\| \leq K_m. \tag{4.8}$$

*Then, if $ht$ is bounded, we can write, with another $K_m$, for each $m \geq 1$:*

$$\|N_m(d^m\varphi(.,.,t), \varepsilon, E, F)\| \leq tK_m. \tag{4.9}$$

*We have the same result with only one set $E$. We have also the same result if we replace, for all $m$, the norm $N_m$ by the norm $N_m^{(\infty)}$, both in the hypothesis (4.8) and in the conclusion (4.9).*

The proof is exactly like in section 2, but simpler since we have no more quadratic Gronwall Lemma, but the usual one. (In section 2, we had $A = \psi_\Lambda$ and $\varphi$ was a derivative of $A$. We had to prove, in the same time, bounds for $A$ and for $\varphi$, but now the bounds for $A$ are already available.)

*Proof of Lemma 4.1.* Since $\psi_\Lambda$ satisfies (2.1), if we apply to it the operator $S_u$ ( $u$ in $(\mathbb{R}^p)^\Lambda$) defined in (3.8), the function $f = S_u\psi_\Lambda$ satisfies, as a function of $x$ and $t$, while $y$ and $u$ are fixed, an equation of the form $L_a(f) = F$, where $L_a$ is the differential operator defined in (2.3), with:

$$a_\lambda(x,y,t) = \frac{h^2}{2}\nabla_{x_\lambda}\left[\psi_\Lambda(x+u, y+u, t) + \psi_\Lambda(x,y,t)\right] \qquad F(x) = S_uV_\Lambda(x). \tag{4.10}$$

By the hypothesis $(H_\varepsilon)$, we can write, for each $m \geq 1$, with some constant $K_m(\varepsilon)$ independent of $u$ and $\Lambda$:

$$\|N_m(d^m(S_u V_\Lambda)(.,.,t), \varepsilon, \sigma(u))\| \leq K_m(\varepsilon)|\sigma(u)|.$$

Then, (4.4) follows from Lemma 4.2. For the proof of (4.5), we remark that, if $f = S_u\psi_\Lambda$ satisfies $L_a f = F$, where $a$ and $F$ are defined in (4.10), the new function $g = S_u S_v \psi_\Lambda = S_v f$ satisfies $L_b g = G$, with:

$$b_\lambda(x,y,t) = a_\lambda(x+v, y+v, t) \qquad\qquad G = S_u S_v V_\Lambda - \sum_{\lambda\in\Lambda} (S_v a_\lambda).(\nabla_{x_\lambda} S_u \psi_\Lambda).$$

By the hypothesis $(H_\varepsilon)$, and by (4.4), for each $m \geq 1$, we can write, if$ht \leq T_0$:

$$\|N_m(d^m(G)(.,.,t), \varepsilon, \sigma(u), \sigma(v))\| \leq K_m(\varepsilon)\ |\sigma(u)|\ |\sigma(v)|.$$

Then (4.5) follows from Lemma 4.2. If $m = 0$, we remark that, if $\sigma(u) \cap \sigma(v) = \emptyset$, the unbounded self-interaction term disappear in $S_u S_v \psi_\Lambda$, and therefore $\|S_u S_v V_\Lambda\| \leq K(\varepsilon)\varepsilon^{\mathrm{dist}(\sigma(u),\sigma(v))} \inf(|\sigma(u)|, |\sigma(v)|)$. Then (4.6) follows from Proposition 2.1.

*Proof of the estimations of Theorem 1.2.* If $m = 0$ and $\mathrm{diam}(Q) \neq 0$, the bound (1.) follows from the point v) of Proposition 3.1 and (4.6), remarking that $|B_\pm^{(j)}(Q)| \leq (1 + \mathrm{diam}(Q))^{d-1}$. If $m \geq 1$, according to the geometric situation of the points $\lambda^{(j)}$ and the box $Q$, we apply either the point v) of Proposition 3.1 and (4.5), or the point iv) and (4.4), or the point iii) and Theorem 1.1.

When $Q$ is reduced to a single point $\lambda$, $T_Q\psi_\Lambda$ is not bounded, and we got bounds only for its derivatives. Some information on $T_{\{\lambda\}}\psi_\Lambda$ itself will be given in the next section.

## 5. Splitting the set $\Lambda$: case of a small subset.

In applications to the decay of correlations, (section 8), the set $\Lambda$ will be a large box, containing a smaller set $E$, (the union of the two supports of the observables), and we have to dissociate the two sets $E$ and $\Lambda \setminus E$. If $Z_\Lambda(t)$ is the partition function of (1.2), the next proposition will be useful, for example, to compare $Z_\Lambda(t)$ and $Z_{\Lambda\setminus E}(t)$, and to estimate the trace norm of the partial trace (with respect to $E$), of some operators.

For each operator $K$ with an integral kernel $K(x,y)$ in the Schwartz space $\mathcal{S}((\mathbb{R}^{2p})^\Lambda \times (\mathbb{R}^{2p})^\Lambda)$, for each subset $E \subset \Lambda$, for each $m$, $m'$ and $\mu$, we set:

$$\|K\|_{m,m',\mu} = \sup_{\substack{\lambda^{(1)},\dots,\lambda^{(k)}\in E^k \\ k\leq m}} I_{m',\mu}(\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(k)}} K), \tag{5.1}$$

where

$$I_{m',\mu}(K) = \int (1 + |x_E|_1 + |y_E|_1)^{m'}\ e^{\mu|x_E - y_E|_1}\ \Big|K(x_E, x_{\Lambda\setminus E}, y_E, x_{\Lambda\setminus E})\Big| dx_E dy_E dx_{\Lambda\setminus E}.$$

**Proposition 5.1.** *With these notations, we can write, for each $m$, $m'$ and $\mu$, for some constant $M(|E|, t, h, \varepsilon)$,*

$$\|U_\Lambda(t)\|_{m,m',\mu} \leq M(|E|, t, h, \varepsilon)\ Z_\Lambda(t).$$

The proof relies on the next Proposition, also used for the point (1.15) of Theorem 1.2. If $E \subset \Lambda \subset \mathbb{Z}^d$, we denote by $\psi_{\Lambda\setminus E}$ the function defined by Theorem 1.1, for the set $\Lambda \setminus E$. It can be seen also as a function on $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times \mathbb{R}_+$, depending only on $x_{\Lambda\setminus E}$, $y_{\Lambda\setminus E}$, and $t$. For each point $\lambda \in E$, we set $A_\lambda(x) = A(x_\lambda)$, where $A$ is the self-interaction term in the definition (1.4), and we denote by $\widetilde{A}_\lambda(x,y)$ the mean value of $A_\lambda$ in the segment between $x$ and $y$, as in (1.16).

**Proposition 5.2.** *With these notations, there exists $K(\varepsilon) > 0$, independent of the two sets $E$ and $\Lambda$ ($E \subset \Lambda \subset \mathbb{Z}^d$), such that:*

$$\Big\|\psi_\Lambda(.,.,t) - \Big[t\sum_{\lambda\in E} \widetilde{A}_\lambda + \psi_{\Lambda\setminus E}(.,.,t)\Big]\Big\| \leq |E|K(\varepsilon)(t+h^2t^2). \tag{5.2}$$

*Proof. First step.* Let us denote by $V_{\rm disc}$ the potential defined as if all the points of $E$ had no interaction with the others points in $\Lambda$, and no interactions between themselves:

$$V_{\rm disc} = \sum_{\lambda\in\Lambda} A_\lambda \ + \sum_{\lambda,\mu\in\Lambda\setminus E} B_{\lambda,\mu}.$$

For each $\theta \in [0,1]$, let us set $V_{\Lambda,E,\theta} = V_{\rm disc} + \theta(V_\Lambda - V_{\rm disc})$. Let us denote by $\psi_{\Lambda,E,\theta}$ the solution of the Cauchy problem (2.1), (2.2), where $V_\Lambda$ is replaced by $V_{\Lambda,E,\theta}$. We see easily that:

$$\psi_{\Lambda,E,0} = \sum_{\lambda\in E} \psi_{\{\lambda\}}(x_\lambda,y_\lambda,t) + \psi_{\Lambda\setminus E}(x_{\Lambda\setminus E}, y_{\Lambda\setminus E}, t) \qquad \psi_{\Lambda,E,1} = \psi_\Lambda, \tag{5.3}$$

where $\psi_{\{\lambda\}}$ is the function defined by Theorem 1.1 for the set reduced to the point $\{\lambda\}$, and to the potential $V_{\{\lambda\}} = A_\lambda$ reduced to the self-interaction term, according to (1.4). If we differentiate with respect to $\theta$ the non linear equation like (2.1), satisfied by $\psi_{\Lambda,E,\theta}$ (with $V_{\Lambda,E,\theta}$ in the RHS) , we see that $L_a\frac{\partial\psi_{\Lambda,E,\theta}}{\partial\theta} = F$, where $L_a$ is the differential operator of (2.3), with $a_\mu = h^2\nabla_{x_\mu}\psi_{\Lambda,E,\theta}$ and with $F = V_{\rm disc} - V_\Lambda$. By the hypothesis $(H_\varepsilon)$, we can write : $\|V_\Lambda - V_{\rm disc}\| \leq K(\varepsilon)|E|$. Therefore, by Proposition 2.1, we can write $\|\frac{\partial\psi_{\Lambda,E,\theta}}{\partial\theta}(.,.,t)\| \leq tK(\varepsilon)|E|$. Therefore we have, with another $K(\varepsilon)$,

$$\|\psi_\Lambda - \psi_{\Lambda,E,0}\| \leq t\ |E|\ K(\varepsilon). \tag{5.4}$$

*Second step.* Now, we shall compare $\psi_{\{\lambda\}}$, associated by Theorem 1.1 to the single set $\{\lambda\}$, and its semi-classical approximation $\psi^{(0)}_{\{\lambda\}}(x,y,t) = t\widetilde{A}(x_\lambda,y_\lambda)$. By a direct computation, we see that

$$\frac{\partial\psi^{(0)}_{\{\lambda\}}}{\partial t} + \frac{x-y}{t}\cdot\nabla_x\psi^{(0)}_{\{\lambda\}} = A_\lambda$$

and therefore that the function $\varphi = \psi_{\{\lambda\}} - \psi^{(0)}_{\{\lambda\}}$ satisfies the equation $L_0(\varphi) = F$ where $L_0$ is the operator defined in (2.3, with $a_\mu = 0$ and $F = \frac{h^2}{2}(\Delta_x\psi^{(0)}_{\{\lambda\}} - |\nabla_x\psi_{\{\lambda\}}|^2)$. By Theorem 1.1, we can write, if $ht \leq T_0$, $\|F(.,.,t)\| \leq K(\varepsilon)h^2(t+t^2)$. By Proposition 2.1, it follows that:

$$\|\psi_{\{\lambda\}} - \psi^{(0)}_{\{\lambda\}}(.,.,t)\| \leq K(\varepsilon)h^2(t^2+t^3) \qquad \text{if} \quad ht \leq T_0 \tag{5.5}$$

where $T_0$ is the constant of (1.8) and (1.7). Proposition 5.2 follows from (5.3), (5.4) and (5.5).

*Proof of the point (1.15) of Theorem 1.2.* We apply Proposition 5.2 with $E$ reduced to a single point $\lambda$. Then, we apply (3.10) with $m = 0$, $Q = \{\lambda\}$, and $f$ being the function in the LHS of (5.2). Thus we obtain:

$$\|T_{\{\lambda\}}\psi_\Lambda(.,.,t) - tT_{\{\lambda\}}\widetilde{A}_\lambda - T_{\{\lambda\}}\psi_{\Lambda\setminus\{\lambda\}}(.,.,t)\| \leq K(\varepsilon)(t+h^2t^2).$$

Using the definition (3.7) of $T_{\{\lambda\}}$, we see that

$$T_{\{\lambda\}}\psi_{\Lambda\setminus\{\lambda\}} = 0 \qquad T_{\{\lambda\}}\widetilde{A}_\lambda(x,y) = \widetilde{A}(x_\lambda,y_\lambda) - \widetilde{A}(0,y_\lambda - x_\lambda),$$

and the bound (1.15) of Theorem 1.2 follows.

We may also apply Proposition 5.2 to $E = \Lambda$, and we obtain that, if $ht \leq T_0$, the integral kernel $U_\Lambda(x, y, t)$ is in $\mathcal{S}((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$, but of course, in this way, all the bounds will depend on $\Lambda$.

**6. Splitting the box $\Lambda$ by an hyperplane.**

This section will be used in the proof of Theorems 1.4 and 1.5. Let $\Lambda \subset \mathbb{Z}^d$ be the union of two disjoints subsets $\Lambda_1$ and $\Lambda_2$, which are separated by an hyperplane $\Sigma$, orthogonal to one of the vectors of the canonical basis. We may assume that, for some $j \leq d$ and $\sigma \in \mathbb{Z}$:

$$\Sigma = \{\lambda \in \mathbb{Z}^d, \quad \lambda_j = \sigma\}, \qquad \Lambda_1 \subset \{\lambda \in \mathbb{Z}^d, \quad \lambda_j \leq \sigma\}, \qquad \Lambda_2 \subset \{\lambda \in \mathbb{Z}^d, \quad \lambda_j \geq \sigma\}. \tag{6.1}$$

We have again, like in Section 5, to dissociate the two subsets, but now, we need, in the situation (6.1), estimations which are uniform with respect to both of them. We denote by $V_{\rm inter}$ the sum of the interactions between a point of $\Lambda_1$ and a point of $\Lambda_2$ in the definition (1.4) of $V_\Lambda$:

$$V_{\rm Inter}(x) = \sum_{\lambda \in \Lambda_1, \mu \in \Lambda_2} B_{\lambda,\mu}(x_\lambda, x_\mu). \tag{6.2}$$

For each $\theta \in [0, 1]$, we set $V_{\Lambda,\theta} = V_\Lambda - \theta V_{\rm Inter}$, and we denote by $H_{\Lambda,\theta}$ the Hamiltonian defined as in (1.1), with $V_\Lambda$ replaced by $V_{\Lambda,\theta}$, and by $\psi_{\Lambda,\theta}$ the function associated to this Hamiltonian by Theorem 1.1. We are now interested to the derivative of this function with respect to $\theta$, and to the decomposition $T_Q \frac{\partial \psi_{\Lambda,\theta}}{\partial\theta}$ defined in section 3. For each box $Q$ of $\mathbb{Z}^d$, let $\pi_\Sigma(Q)$ be the orthogonal projection of $Q$ on $\Sigma$.

**Proposition 6.1.** *With the previous notations, we can write, for some constant $C_m(\varepsilon)$ independent of $\Lambda$, $\Lambda_1$ and $\Lambda_2$, for each points $\lambda^{(1)}$, ... $\lambda^{(m)}$ in $\Lambda$, for each box $Q \subseteq \Lambda$, if $ht \leq T_0$ and $\theta \in [0, 1]$:*

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} \Big( T_Q \frac{\partial \psi_{\Lambda,\theta}}{\partial\theta}(.,.,t)\Big)\| \leq t C_m(\varepsilon) \varepsilon^{{\rm diam}(Q \cup \pi_\Sigma(Q) \cup \{\lambda^{(1)},...,\lambda^{(m)}\})} \big(1 + {\rm diam}(Q)\big)^d \tag{6.3}$$

Since $V_{\Lambda,\theta}$ is of the same type that $V_\Lambda$, and satisfies the same hypotheses, with bounds independent of $\theta$, (excepted the hypothesis (1.6), which is not needed for theorems 1.1 and 1.2), the bounds given by Theorem 1.1, 1.2, and Lemma 4.1 for the function $\psi_{\Lambda,\theta}$ are also uniform in $\theta$. With this remark, the proof of Proposition 6.1 relies on the following Lemma.

**Lemma 6.2.** *For each integer $m \geq 1$, there exists $K_m(\varepsilon) > 0$ such that, if $ht \leq T_0$, we have, for each points $\lambda^{(1)}$, ... $\lambda^{(m)}$ in $\Lambda$:*

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} \frac{\partial \psi_{\Lambda,\theta}}{\partial\theta}\| \leq K_m(\varepsilon)\ t\ \varepsilon^{D(\lambda^{(1)},...,\lambda^{(m)},\Sigma)}. \tag{6.4}$$

*Moreover, for each $u$ in $(\mathbb{R}^p)^\Lambda$ such that the support $\sigma(u)$ is contained, either in $\Lambda_1$, or in $\Lambda_2$, we have the following bound, valid also for $m = 0$ :*

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} S_u \frac{\partial \psi_{\Lambda,\theta}}{\partial\theta}\| \leq K_m(\varepsilon)\ t\ \varepsilon^{D(\lambda^{(1)},...,\lambda^{(m)},\sigma(u),\Sigma)}\ |\sigma(u)|, \tag{6.5}$$

*where $D(\lambda^{(1)}, ..., \lambda^{(m)}, \Sigma)$ is defined in (4.1).*

*Proof.* Since $\psi_{\Lambda,\theta}$ satisfies (2.1), with $V_\Lambda$ replaced by $V_{\Lambda,\theta}$, it follows that $f = \frac{\partial \psi_{\Lambda,\theta}}{\partial\theta}$ satisfies $L_a f = -V_{\rm Inter}$, where $L_a$ is the operator of (2.3), and $a_\lambda = h^2 \nabla_{x_\lambda} \psi_{\Lambda,\theta}$. We can write, for each $m \geq 1$:

$$\|N_m^\infty(d^m V_{\rm Inter}, \varepsilon, \Sigma)\| \leq K_m(\varepsilon).$$

Then (6.4) follows from Lemma 4.2, with the $N_m^\infty$ norm. Now, we apply the operator $S_u$ defined in (3.8) to $f = \partial_\theta \psi_{\Lambda,\theta}$. Since $L_a f = -V_{\rm Inter}$, it follows that the function $g = S_u f$ satisfies $L_a g = G$, where still $a_\lambda = h^2 \nabla_{x_\lambda} \psi_{\Lambda,\theta}$ and:

$$G(x,y,t) = -S_u V_{\rm Inter}(x) - h^2 \sum_{\lambda\in\Lambda} \Big(\nabla_{x_\lambda} S_u \psi_{\Lambda,\theta}\Big) \cdot \Big(\nabla_{x_\lambda} \partial_\theta \psi_{\Lambda,\theta}(x+u, y+u, t)\Big).$$

In order to estimate this function, we use the form (6.2) of $V_{\rm Inter}$, the hypothesis $(H_\varepsilon)$, the bound (6.4), and Lemma 4 (bound (4.4)), applied to $\psi_{\Lambda,\theta}$ instead of $\psi_\Lambda$. We obtain, if $ht \leq T_0$:

$$\|G(.,.,t)\| \leq K_0(\varepsilon) \varepsilon^{{\rm dist}(\sigma(u),\Sigma)} |\sigma(u)|.$$

By the same arguments, we can write, for each $m \geq 1$,

$$\|N_m^\infty(d^m G(.,.,t), \varepsilon, \sigma(u), \Sigma)\| \leq K_m(\varepsilon)|\sigma(u)|.$$

Then, for $m = 0$, the estimation (6.5) follows from Proposition 2.1 and, for $m \geq 1$, it follows from Lemma 4.2 (with the $N_m^\infty$ norm).

*End of the proof of Proposition 6.1.* We may assume that $\Lambda_1$, $\Lambda_2$ and $\Sigma$ satisfy (6.1), with $j = 1$. If the box $Q \subset \Lambda_1 \cup \Lambda_2$ is not disjoint from $\Sigma$, (6.3) follows from Theorem 1.2. Now, we may assume that $Q = \prod_{k=1}^d [a_k, b_k]$, with $\sigma < a_1 \leq b_1$ and $a_k \leq b_k$ for $k \geq 2$. Let $k \leq d$ such that the diameter of the set $Q \cup \pi_\Sigma(Q) \cup \{\lambda^{(1)}, ..., \lambda^{(m)}\}$ is the length of its $k-$th projection. If $k \neq 1$, the estimation (6.3) follows from Theorem 1.2 (with the potential $V_{\rm Inter}$). If $k = 1$, and $\lambda_1^{(k)} \neq b_1$ for all $k$, it follows from (6.5) and the point iv of Proposition 3.1, applied with the set $B_1^+(Q)$. In the other case, it follows from (6.4) and point iii) of Proposition 3.1.

## 7. Representation of the quantum correlations.

In this section, we shall give an expression of the correlation ${\rm Cov}_{\Lambda,t}(A,B)$, (see (7.4)- (7.5)). Then, we shall prove Theorem 1.3, point b, i.e. when the local observables $A$ and $B$ are multiplications. In section 8, we shall prove Theorem 1.3 in the general case. For the expression of the correlation, we shall use the tensor product of the heat kernel by itself (doubling of variables), and the action, on this kernel, of a finite group of symmetries.

*1. Doubling of variables.* We shall denote by $(x', x'')$ the variable of $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$. For each operator $A$ in $L^2((\mathbb{R}^p)^\Lambda)$, we denote by $A'$ (resp. $A''$ ) the operator $A$, seen as an operator in $L^2((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$, acting only on the variable $x'$ (resp. on $x''$). We denote by $\widetilde{H}_\Lambda$ the operator in $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$ defined by $\widetilde{H_\Lambda} = H'_\Lambda + H''_\Lambda$. We denote by ${\rm Tr}_\Lambda(A)$ the trace of an operator $A \in \mathcal{L}(\widetilde{\mathcal{H}}_\Lambda)$, where $\widetilde{\mathcal{H}}_\Lambda = L^2((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$. Then, an easy computation shows that, if $A \in \mathcal{L}(\mathcal{H}_{E_1})$ and $B \in \mathcal{L}(\mathcal{H}_{E_2})$, where $E_1$ and $E_2$ are disjoint subsets, contained in a same box $\Lambda$, we have:

$$\text{(7.1)} \qquad {\rm Cov}_{\Lambda,t}(A,B) = \frac{1}{2\widetilde{Z}_\Lambda(t)} \,{\rm Tr}_\Lambda\, \Big(e^{-t\widetilde{H}_\Lambda}(A'-A'')(B'-B'')\Big) \qquad \widetilde{Z}_\Lambda(t) \;=\; {\rm Tr}_\Lambda\, \Big(e^{-t\widetilde{H}_\Lambda}\Big).$$

*2. Group of symmetries, and averaging.* If $\psi_\Lambda$ is the function, associated by Theorem 1.1, to the operator $H_\Lambda$, and $\widetilde{\psi}_\Lambda$ to the operator $\widetilde{H}_\Lambda$, we have:

$$\text{(7.2)} \qquad \widetilde{\psi}_\Lambda(X,t) = \psi_\Lambda(x', y', t) + \psi_\Lambda(x'', y'', t) \qquad X = (x,y) = (x', x'', y', y'').$$

The function $\widetilde{\psi}_\Lambda$ is not changed when the variables $(x', y')$ and $(x'', y'')$ are permuted. It will be useful to make also the substitution of $(x'_\lambda, y'_\lambda)$ and $(x''_\lambda, y''_\lambda)$ not everywhere, but only for $\lambda$ in a subset of $\Lambda$. We denote by $G_\Lambda(E_1, E_2)$ the set of maps $\sigma : \lambda \to \sigma_\lambda$ of $\Lambda$ in the group of substitutions $S_2$, which are constant on $E_1$ and on $E_2$, where $E_1$ and $E_2$ are disjoint subsets of $\Lambda$. This group has a natural action $X \to \sigma X$ on $(\mathbb{R}^{2p})^\Lambda$ and on $(\mathbb{R}^{4p})^\Lambda$. If $X$ is like in (7.2), we may have $(\sigma X)_\lambda = X_\lambda$ or $(\sigma X)_\lambda = (x''_\lambda, x'_\lambda, y''_\lambda, y'_\lambda)$. If $T$ is in $\mathcal{L}(\widetilde{\mathcal{H}}_\Lambda)$, with integral kernel $K$, let us denote by $T^{(\sigma)}$ the transformed operator, with integral kernel $K^{(\sigma)}(X) = K(\sigma X)$. For each set $F \subset \Lambda$, we define an element $\tau_F$ of $G_\Lambda(E_1, E_2)$ by:

$$\Big(\tau_F(x', x'', y', y'')\Big)_\lambda = \begin{cases} (x''_\lambda, x'_\lambda, y''_\lambda, y'_\lambda) & \text{if} \quad \lambda \in F \\ (x'_\lambda, x''_\lambda, y'_\lambda, y''_\lambda) & \text{if} \quad \lambda \notin F \end{cases} \tag{7.3}$$

For each $\sigma \in G_\Lambda(E_1, E_2)$, we denote by $\mathrm{sgn}_j(\sigma)$ the signature (in $\{-1, 1\}$) of $\sigma(\lambda)$ when $\lambda$ is any point of $E_j$ ($j = 1, 2$), and we set $\mathrm{sgn}(\sigma) = \mathrm{sgn}_1(\sigma)\mathrm{sgn}_2(\sigma)$. With these notations, we see, by easy computations, that:

$$\mathrm{Cov}_{\Lambda,t}(A, B) = \frac{1}{2\widetilde{Z}_\Lambda(t)} \ \mathrm{Tr}_\Lambda \ \Big(W(t)(A' - A'')(B' - B'')\Big)$$

where

$$W(t) = \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{sgn}(\sigma) \left(e^{-t\widetilde{H}_\Lambda}\right)^{(\sigma)} \qquad\qquad G = G_\Lambda(E_1, E_2) \tag{7.4}$$

Here $(A' - A'')(B' - B'')$ is an operator in $\mathcal{L}(\widetilde{\mathcal{H}}_{E_1 \cup E_2})$, and we omit the tensor product with the identity on $\Lambda \setminus (E_1 \cup E_2)$.

With the notations of B. Simon [22], if $E \subset \Lambda \subset \mathbb{Z}^d$, we shall use the operator of *partial trace* $\mathrm{Tr}_E^\Lambda$, from $\mathcal{L}_1(\widetilde{\mathcal{H}}_\Lambda)$ (the space of trace norm operators) to $\mathcal{L}_1(\widetilde{\mathcal{H}}_E)$, such that, for each $W \in \mathcal{L}_1(\widetilde{\mathcal{H}}_\Lambda)$, and for each operator $T \in \mathcal{L}(\widetilde{\mathcal{H}}_E)$, we have:

$$\mathrm{Tr}_\Lambda(WT) = \mathrm{Tr}_E\Big(T \ \mathrm{Tr}_E^\Lambda(W)\Big).$$

With these notations, we have:

$$\mathrm{Cov}_{\Lambda,t}(A, B) = \frac{1}{2\widetilde{Z}_\Lambda(t)} \ \mathrm{Tr}_{E_1 \cup E_2}\Big((A' - A'')(B' - B'')\mathrm{Tr}_{E_1 \cup E_2}^\Lambda(W(t))\Big) \tag{7.5}$$

For each set $E \subset \mathbb{Z}^d$, let us denote by $\| \ \|_E^{\mathrm{Tr}}$ the trace norm of an operator in $\mathcal{H}_E$. Then Theorem 1.3 will follow from the next proposition, (where we write only what is needed for the points a) and b)):

**Proposition 7.1.** *With the notations and hypotheses of Theorem 1.3, if $ht \leq T_0$, $\varepsilon < \delta < 1$, and if $t$ is smaller than some constant $T_1(\varepsilon, \delta) > 0$, the operator $W(t)$ defined in (7.4) and its integral kernel $W(X, t)$ satisfy:*

$$\|\mathrm{Tr}_{E_1 \cup E_2}^\Lambda(W(t))\|_{E_1 \cup E_2}^{\mathrm{Tr}} \ \leq \ M(|E_1 \cup E_2|, t, h, \varepsilon, \delta) \ \widetilde{Z}_\Lambda(t) \ \delta^{\mathrm{dist}(E_1, E_2)}. \tag{7.6}$$

$$\int_{(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda} |W(x', x'', x', x'', t)| dx' dx'' \leq \ t \ \inf(|E_1|, |E_2|) \ N(\varepsilon, \delta) \ \widetilde{Z}_\Lambda(t) \ \delta^{\mathrm{dist}(E_1, E_2)}. \tag{7.7}$$

For the proof of this proposition, we shall use suitable decompositions of the heat kernel $\widetilde{U}_\Lambda(x, y, t)$ of $e^{-t\widetilde{H}_\Lambda}$, and, therefore, of the kernel $W(x, y, t)$. Each time a function $\varphi$ is written as a sum of terms $\varphi_Q$ associated, for example, to the boxes $Q$ contained in a box $\Lambda$, the Mayer decomposition is a simple way to write the exponential $e^{-\varphi}$ as a sum of terms associated to the set of boxes $\Gamma \subset \mathcal{P}(\mathrm{Box}(\Lambda))$. This technique is often used for classical spin systems in a lattice (see B. Simon [22]). Let us describe this technique in our situation.

*3. Mayer decomposition.* For each box $Q$, let $T_Q$ be the operator defined by (3.7) with $\mathbb{R}^p$ replaced by $\mathbb{R}^{2p}$. We apply this operator to the function $\widetilde{\psi}_\Lambda(.,t)$ defined in (7.2). For each box $Q \subset \Lambda$ which is not a single point, we set

$$M_Q(t) = \sup_{X \in (\mathbb{R}^{4p})^\Lambda} (T_Q \widetilde{\psi_\Lambda})(X,t) \qquad f_Q(X,t) \;=\; e^{M_Q(t) - (T_Q\widetilde{\psi_\Lambda})(X,t)} - 1. \tag{7.8}$$

For each box reduced to a single point $\lambda$, we use another notation, and we set:

$$f_{\{\lambda\}}(X,t) \;=\; e^{-(T_{\{\lambda\}}\widetilde{\psi_\Lambda})(X,t)}. \tag{7.9}$$

We denote by $\widetilde{U}^{(0)}_\Lambda(X,t)$ the heat kernel for the free Laplacian in $(\mathbb{R}^{2p})^\Lambda$, and we set

$$\Phi_0(X,t) = \widetilde{U}^{(0)}_\Lambda(X,t)\; e^{-\widetilde{\psi}_\Lambda(0,y-x,t)} e^{-\sum_{Q\in \mathrm{Box}(\Lambda)} M_Q(t)}, \tag{7.10}$$

where $\mathrm{Box}(\Lambda)$ is the set of boxes in $\Lambda$, not reduced to single points. With these notations, by Theorem 1.2, the kernel of $e^{-t\widetilde{H}_\Lambda}$ can be written

$$\widetilde{U}_\Lambda(X,t) = \Phi_0(X,t) \prod_{\lambda\in\Lambda} f_{\{\lambda\}}(X,t) \prod_{Q\in \mathrm{Box}(\Lambda)} (1 + f_Q(X,t)).$$

In order to develop the last product, we write, for each set of boxes $\Gamma \in \mathcal{P}(\mathrm{Box}(\Lambda))$:

$$K_\Gamma(X,t) = \; \Phi_0(X,t) \prod_{\lambda\in\Lambda} f_{\{\lambda\}}(X,t) \prod_{Q\in\Gamma} f_Q(X,t). \tag{7.11}$$

If $\Gamma$ is the empty set, the last product is 1, by convention. We have $K_\Gamma \geq 0$. We denote by $T_\Gamma(t)$ the operator with integral kernel $K_\Gamma(X,t)$. With these notations, we can write the heat kernel and our mean operator $W(t)$ in the following form:

$$\widetilde{U}_\Lambda(X,t) = \sum_{\Gamma\in\mathcal{P}(\mathrm{Box}(\Lambda))} K_\Gamma(X,t), \qquad e^{-t\widetilde{H}_\Lambda} = \sum_{\Gamma\subseteq \mathrm{Box}(\Lambda)} T_\Gamma(t) \tag{7.12}$$

$$W(X,t) = \frac{1}{|G|} \sum_{\substack{\Gamma\subseteq \mathrm{Box}(\Lambda) \\ \sigma\in G}} (\mathrm{sgn}(\sigma)) K_\Gamma^{(\sigma)}(X,t) \qquad G = G_\Lambda(E_1,E_2). \tag{7.13}$$

Among the set of boxes appearing in the Mayer decomposition, some of them, called polymers, play an important role.

*4. Polymers, and sets $\mathcal{C}(E_1,E_2)$ and $\mathcal{NC}(E_1,E_2)$.* According to the terminology of B. Simon [22], we call *polymer* a finite sequence of boxes $(Q_1,...,Q_k)$, not reduced to single points, such that $Q_j \cap Q_{j+1} \neq \emptyset$ $(1 \leq j \leq k-1)$. We say that a set $\Gamma \in \mathcal{P}(B(\Lambda))$ is in $\mathcal{C}(E_1,E_2)$ if $E_1$ and $E_2$ are connected by $\Gamma$, i.e. if $\Gamma$ contains a polymer, with its first box intersecting $E_1$, and its last box intersecting $E_2$. We say that $\Gamma$ is in $\mathcal{NC}(E_1,E_2)$ in the opposite case.

All these tools will be used together in Section 8 for the proof of Proposition 7.1, and therefore of Theorem 1.3. Since the case of two observables which are multiplications is much simpler, let us prove the point b) of Theorem 1.3 now. The next two lemmas will show the role of the sets $\mathcal{C}(E_1,E_2)$ and $\mathcal{NC}(E_1,E_2)$.

**Lemma 7.2.** *a) There exists a function $a(t,\varepsilon)$ of the form $a(t,\varepsilon) = tK(\varepsilon)e^{tK(\varepsilon)}$, such that, for each polymer $\Pi = (Q_1,\dots,Q_k)$, we have:*

$$\prod_{j=1}^{k} |f_{Q_j}(X,t)| \leq N(\Pi,\varepsilon,a(t,\varepsilon)), \tag{7.14}$$

*where, for each $T$, we set:*

$$N(\Pi,\varepsilon,T) = \prod_{j=1}^{k} T\varepsilon^{\mathrm{diam}(Q_j)} < Q_j >^{2d} \qquad < Q >= 1 + \mathrm{diam}(Q). \tag{7.15}$$

*b) For each disjoint finite sets $E_1$ and $E_2$, for each $\delta$ such that $0 < \varepsilon < \delta < 1$, there exists $T_1(\varepsilon,\delta)$ and $K(\varepsilon,\delta)$ such that, if $T < T_1(\varepsilon,\delta)$,*

$$\sum_{\Pi\in \mathrm{Pol}(E_1,E_2)} N(\Pi,\varepsilon,T) \leq T \inf(|E_1|,|E_2|)K(\varepsilon,\delta)\delta^{\mathrm{dist}(E_1,E_2)}, \tag{7.16}$$

*where $\mathrm{Pol}(E_1,E_2)$ is the set of polymers connecting $E_1$ and $E_2$, (the first box intersecting $E_1$, and the last one $E_2$).*

*Proof.* By (7.8) and by the point 3 of Theorem 1.2, we can write, if $ht \leq T_0$,

$$0 \leq f_Q(X,t) \leq tK_0(\varepsilon)\varepsilon^{\mathrm{diam}(Q)}(\mathrm{diam}(Q))^{2d}e^{tK_0(\varepsilon)}, \tag{7.17}$$

and (7.14) follows easily. Let us prove the last point, assuming that $|E_1| \leq |E_2|$. For each polymer $\Pi = (Q_1,\dots,Q_p)$, let $L(\Pi)$ be the sum of the diameters of the boxes $Q_j$. We remark that the number of boxes with diameter $R \geq 1$ intersecting a given box of diameter $R_0 \geq 0$ is at most $(R_0+1)^{2d}(R+1)^{2d}$. If we take the sum of $N(\Pi,\varepsilon,T)$ for all polymers starting from $E_1$, with a given total length $L$, it follows that:

$$\sum_{\substack{\Pi\in \mathrm{Pol}(E_1,\mathbb{Z}^d)\\ L(\Pi)=L}} N(\Pi,\varepsilon,T) \leq |E_1| \sum_{\substack{R_1+\dots+R_p=L\\ p\geq 1,\ R_j\geq 1}} (1+R_1)^{6d}\dots(1+R_p)^{6d}\varepsilon^L\, T^p.$$

We remember that the number of ordered sequences $(R_1,\dots,R_p)$ such that $R_1+\dots+R_p = L$, and $R_j \geq 1$, is $C_{L-1}^{p-1}$, and we set $\Phi(t) = \sup_{R>0}(1+R)^{6d}t^R$ for each $t \in ]0,1[$ . If $0 < \varepsilon < \gamma$, it follows that:

$$\sum_{\substack{\Pi\in \mathrm{Pol}(E_1,\mathbb{Z}^d)\\ L(\Pi)=L}} N(\Pi,\varepsilon,T) \leq |E_1|\ T\ \Phi(\varepsilon/\gamma)\ \gamma^L\ \Big(1+T\Phi(\varepsilon/\gamma)\Big)^{L-1}. \tag{7.18}$$

We remark that, if $\Pi$ is a polymer connecting $E_1$ and $E_2$, we have $L(\Pi) \geq \mathrm{dist}(E_1,E_2)$. If $\varepsilon < \delta < 1$, we apply (7.18) with $\gamma = \sqrt{\varepsilon\delta}$. There exists $T_1(\varepsilon,\delta)$ such that, if $T < T_1(\varepsilon,\delta)$, we have $\gamma(1+T\Phi(\varepsilon/\gamma)) \leq \delta$. With this condition, we have:

$$\sum_{\Pi\in \mathrm{Pol}(E_1,E_2)} N(\Pi,\varepsilon,T) \leq |E_1|\ T\ \Phi((\varepsilon/\delta)^{1/2}) \sum_{L=\mathrm{dist}(E_1,E_2)}^{\infty} \delta^L.$$

The Lemma is proved.

**Lemma 7.3.** *If $\Gamma \in \mathcal{NC}(E_1,E_2)$, if $ht \leq T_0$, we have*

$$\sum_{\sigma\in G_\Lambda(E_1,E_2)} (\mathrm{sgn}(\sigma))\ K_\Gamma^{(\sigma)}(X,t) = 0 \qquad \forall X = (x',x'',x',x'') \in \mathrm{Diag}(\Lambda)$$

*where* $\mathrm{Diag}(\Lambda)$ *is the diagonal of* $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$.

*Proof.* Let $\Gamma \in \mathcal{NC}(E_1, E_2)$. For $k = 1, 2$, let us denote by $\widehat{E_k}$ the set of points $\lambda \in \Lambda$ which are, either in $E_k$, or connected to $E_k$ by a polymer in $\Gamma$. By Theorem 1.2, the function $f_Q$, when it is restricted to the diagonal, depends only on the variables $x_Q$. By (7.2) and (7.9), it is also invariant when all the variables $(x', y')$ and $(x'', y'')$ are permuted. By a combination of these remarks, for each $Q \in \Gamma$, $f_Q$, restricted to the diagonal, is invariant by the map $\sigma_1 = \tau_{\widehat{E}_1}$ defined in (7.3), and therefore $K_\Gamma^{(\sigma_1)}(X, t) = K_\Gamma(X, t)$ if $X$ is in the diagonal, and the lemma follows easily.

*Proof of the point b) of Theorem 1.3.* We can use the sum (7.13) for $W(X, t)$. When $X$ is in the diagonal ($x' = y'$, $x'' = y''$), the contribution of the terms $\Gamma \in \mathcal{NC}(E_1, E_2)$ in this sum (7.13) vanishes by Lemma 7.3. For each $\Gamma$ in $\mathcal{C}(E_1, E_2)$, we can choose a polymer $\Pi_\Gamma \in \mathrm{Pol}(E_1, E_2)$ contained in $\Gamma$. Then, we have, by Lemma 7.2:

$$\sum_{\Gamma \in \mathcal{C}(E_1,E_2)} |K_\Gamma(X,t)| \leq \sum_{\Pi \in \mathrm{Pol}(E_1,E_2)} N(\Pi, \varepsilon, a(t,\varepsilon)) \sum_{\Gamma' \in \mathcal{P}(\mathrm{Box}(\Lambda))} |K_{\Gamma'}(X,t)| \tag{7.19}$$

where $a(t, \varepsilon) = tK_0(\varepsilon)e^{tK_0(\varepsilon)}$. Applying the positivity of $K_\Gamma$ and (7.12), we obtain:

$$\int_{(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda} |W(x', x'', x', x'', t)| dx' dx'' \leq \widetilde{Z}_\Lambda(t) \sum_{\Pi \in \mathrm{Pol}(E_1,E_2)} N(\Pi, \varepsilon, a(t, \varepsilon)).$$

By the last point of Lemma 7.2, the estimation (7.7), and the point b) of Theorem 1.3 follow.

## 8. Estimation of the quantum correlations in the general case.

Now, we shall prove (7.6), and therefore Theorem 1.3 in the general case. If the integral kernel of $W(t)$, (defined in (7.4)), is denoted by $W(x, y, t)$, the integral kernel of the partial trace $\mathrm{Tr}_E^\Lambda W(t)$ ($E = E_1 \cup E_2$) is

$$K(x_E, y_E, t) = \int_{(\mathbb{R}^{2p})^\Lambda} W(x_E, x_{\Lambda\setminus E}, y_E, x_{\Lambda\setminus E}, t) dx_{\Lambda\setminus E} \qquad E = E_1 \cup E_2. \tag{8.1}$$

We shall estimate the trace norm of this partial trace, using the norm $\| \ \|_{m,m',\mu}$ defined in (5.1).

**Proposition 8.1. (Main estimate).** *With the notations and hypotheses of Proposition 7.1, if* $0 < \varepsilon < \delta < 1$, *if* $ht \leq T_0$, *and if* $t$ *is smaller than some constant* $T_1(\varepsilon, \delta)$, *we have, for some constants* $m$, $m'$ *and for some function* $\mu(t)$*:*

$$\|\mathrm{Tr}_{E_1\cup E_2}^\Lambda (W(t))\|_{E_1\cup E_2}^{\mathrm{Tr}} \ \leq \ M(|E_1 \cup E_2|, t, h, \varepsilon, \delta) \ \delta^{\mathrm{dist}(E_1,E_2)} \sum_{\Gamma \subseteq \mathrm{Box}(\Lambda)} \|K_\Gamma(t)\|_{m,m',\mu(t)}. \tag{8.2}$$

Classically, there exist $C > 0$, $m > 0$ and $m' > 0$, depending on $|E|$, such that, for each operator $K$ with integral kernel in the Schwartz space, we have, using the norm $\| \| \|_{m,m',\mu}$ defined in (5.1):

$$\|\mathrm{Tr}_E^\Lambda(K)\|_E^{\mathrm{Tr}} \ \leq \ C\|K\|_{m,m',0}. \tag{8.3}$$

For each $\Gamma \in \mathcal{NC}(E_1, E_2)$, let us denote by by $H_\Gamma$ the subgroup of all $\sigma \in G_\Lambda(E_1, E_2)$ which are constant on each connected component of $\Gamma$. We define:

$$K_\Gamma^{\mathrm{Av}}(t) = \frac{1}{|H_\Gamma|} \sum_{\sigma \in H_\Gamma} \mathrm{sgn}(\sigma) K_\Gamma^{(\sigma)}(t), \tag{8.4}$$

and we denote by $K_\Gamma^{\rm Av}(X,t)$ the integral kernel. By (7.13) and (8.3), we can write:

$$\|{\rm Tr}^\Lambda_{E_1\cup E_2}(W(t))\|^{\rm Tr}_{E_1\cup E_2} \;\leq\; C\left[\sum_{\Gamma\in\mathcal{C}(E_1,E_2)}\|K_\Gamma(t)\|_{m,m',0}+\sum_{\Gamma\in\mathcal{NC}(E_1,E_2)}\|K_\Gamma^{\rm Av}(t)\|_{m,m',0}\right]. \tag{8.5}$$

For the first sum, we need only a modification of the estimates (7.19) and (7.16), because we need also bounds for the derivatives of the functions. Without writing all the details, we obtain:

$$\sum_{\Gamma\in\mathcal{C}(E_1,E_2)}\|K_\Gamma(t)\|_{m,m',0}\leq M(|E_1\cup E_2|,t,h,\varepsilon,\delta)\;\delta^{{\rm dist}(E_1,E_2)}\sum_{\Gamma\subseteq{\rm Box}(\Lambda)}\|K_\Gamma(t)\|_{m,m',\mu(t)}. \tag{8.6}$$

Now, we have to estimate $\|K_\Gamma^{\rm Av}(t)\|_{m,m',0}$ for $\Gamma\in\mathcal{NC}(E_1,E_2)$. Therefore, we need a study of $K_\Gamma^{\rm Av}(X,t)$ only in the following set:

$$\mathrm{Diag}(\Lambda,E_1\cup E_2)=\{(x,y)\in(\mathbb{R}^{2p})^\Lambda\times(\mathbb{R}^{2p})^\Lambda,\qquad x_\lambda=y_\lambda\quad\forall\lambda\notin E_1\cup E_2\}. \tag{8.7}$$

We shall write $f\sim g$ if $f(X,t)=g(X,t)$ for all $X$ in $\mathrm{Diag}(\Lambda,E_1\cup E_2)$. Proposition 8.1 will follow from the above inequalities and from:

**Proposition 8.2.** *For each $\Gamma$ and $\Gamma'$ such that $\Gamma\in\mathcal{NC}(E_1,E_2)$ and $\Gamma'\subseteq\Gamma$, we shall find a function $B_{\Gamma,\Gamma'}(X,t)$ such that, if $ht\leq T_0$:*

$$K_\Gamma^{\rm Av}(X,t)\sim\sum_{\Gamma'\subseteq\Gamma}B_{\Gamma,\Gamma'}(X,t)K_{\Gamma'}(X,t) \tag{8.8}$$

*For each $\Gamma'\in\mathcal{NC}(E_1,E_2)$, the function $S_{\Gamma'}$ defined by:*

$$S_{\Gamma'}(X,t)=\sum_{\substack{\Gamma\in\mathcal{NC}(E_1,E_2)\\ \Gamma'\subseteq\Gamma}}\Big|B_{\Gamma,\Gamma'}(X,t)\Big| \tag{8.9}$$

*satisfies, if $0<\varepsilon<\delta$, $ht\leq T_0$, if $t$ is smaller than some constant $T_1(\varepsilon,\delta)$, and if $X\in\mathrm{Diag}(\Lambda,E_1\cup E_2)$,*

$$|S_{\Gamma'}(X,t)|\leq K(t,h,\varepsilon,\delta,|E_1\cup E_2|)\;e^{\mu(t)|x_E-y_E|_1}\;\delta^{{\rm dist}(E_1,E_2)}, \tag{8.10}$$

*where $\mu(t)$ and $K$ are some function, independent of $\Lambda$ containing $E_1$ and $E_2$, and depending on $E_1$ and $E_2$ by the number $|E_1\cup E_2|$. We have similar estimates for the derivatives of $B_{\Gamma,\Gamma'}(X,t)$ with respect to $x_E$ and $y_E$. The constants in the inequalities depend on the order of derivation, but not the condition of validity $t<T_1(\varepsilon,\delta)$.*

The rest of this section will be devoted to the proof of this Proposition. Let us introduce some functions which are bounded like the LHS of (8.10). Then, we shall give an expression of $K_\Gamma^{\rm Av}(X,t)$ as a polynomial expression of such functions.

For each $\Gamma\in\mathcal{NC}(E_1,E_2)$, we denote by $\widehat{E_k}$ $(k=1,2)$, the set of points $\lambda\in\Lambda$ wich are, either in $E_k$, or connected to $E_k$ by a polymer in $\Gamma$. The maps $\tau_{\widehat{E_k}}$ defined in (7.5) will be denoted by $\sigma_k$. By applying the operators $\sigma_1$ and $\sigma_2$ to the functions $f_Q$ of (7.8), or to the functions $f_\lambda$ of (7.9), we define the following functions $U_Q$, $V_Q$, $W_Q$, $U_\lambda$, $V_\lambda$ and $W_\lambda$, by:

$$U_Q=f_Q^{(\sigma_1)}-f_Q,\qquad U_\lambda=\Big[f_\lambda^{(\sigma_1)}-f_\lambda\Big]f_\lambda^{-1}. \tag{8.11}$$

$$V_Q=f_Q^{(\sigma_2)}-f_Q\qquad V_\lambda=\Big[f_\lambda^{(\tau_2)}-f_\lambda\Big]f_\lambda^{-1}. \tag{8.12}$$

$$W_Q := f_Q - f_Q^{(\sigma_1)} - f_Q^{(\sigma_2)} + f_Q^{(\sigma_1\sigma_2)} \qquad W_\lambda := \Big[f_\lambda - f_\lambda^{(\sigma_1)} - f_\lambda^{(\sigma_2)} + f_\lambda^{(\sigma_1\sigma_2)}\Big] f_\lambda^{-1}. \tag{8.13}$$

The definition of these functions depends on the set $\Gamma$ since $\sigma_1$ and $\sigma_2$ depend on it. For the estimations of these functions, we shall use the following ones, where $X = (x, y) = (x', x'', y', y'')$, $Q$ is a box, and $T > 0$:

$$M(Q, X) =< Q >^{2d} \sum_{\alpha\in E_1\cup E_2} |x_\alpha - y_\alpha|\varepsilon^{\mathrm{diam}(Q\cup\{\alpha\})}, \qquad < Q >= 1 + \mathrm{diam}(Q) \tag{8.14}$$

$$N(Q, X, \varepsilon, T) = T \sum_{\substack{\alpha\in E_1\\ \beta\in E_2}} |x_\alpha - y_\alpha||x_\beta - y_\beta|\varepsilon^{\mathrm{diam}(Q\cup\{\alpha,\beta\})} < Q >^{4d} . \tag{8.15}$$

**Lemma 8.3.** *With these notations, we can write, for each $\Gamma \in \mathcal{NC}(E_1, E_2)$, for each boxes $P$ and $Q$ such that $P$ is disjoint from $\widehat{E}_1$ and $Q$ is disjoint from $\widehat{E}_2$, if $X \in \mathrm{Diag}(\Lambda, E_1 \cup E_2)$, and $ht \leq T_0$:*

$$|U_P(X, t)| \leq a(t, \varepsilon)M(P, X, \varepsilon) \qquad |V_Q(X, t)| \leq a(t, \varepsilon)M(Q, X, \varepsilon), \tag{8.16}$$

*where $a(t, \varepsilon) = tK(\varepsilon)e^{tK(\varepsilon)}$, ($K(\varepsilon)$ being independent of $\Lambda$, $E_1$ and $E_2$). If $Q$ is disjoint from $\widehat{E_1}$ and $\widehat{E_2}$, we have:*

$$|W_Q(X, t)| \leq N\Big(Q, X, \varepsilon, a(t, \varepsilon)\Big) \tag{8.17}$$

*We have also similar estimations for points. If $\Phi_0$ is defined in (7.10), there exists a function $\Delta_0$ such that:*

$$\Phi_0^{(\sigma_1)} - \Phi_0 \sim \Phi_0\Delta_0 \qquad |\Delta_0(X, t)| \leq tK(\varepsilon)\varepsilon^{\mathrm{dist}(E_1,E_2)}e^{tK(\varepsilon)|x_E-y_E|_1} \tag{8.18}$$

*There exists $K(\varepsilon)$ such that, for each $T > 0$, for each finite set $\Lambda$,*

$$\sum_{E\subset\Lambda}\prod_{\lambda\in E} TM(\lambda, X, \varepsilon) + \sum_{E\subset \mathrm{Box}(\Lambda)}\prod_{Q\in E} TM(Q, X, \varepsilon) \leq e^{TK(\varepsilon)|x_E-y_E|_1}. \tag{8.19}$$

*Proof.* When it is restricted to $\mathrm{Diag}(\Lambda, E_1 \cup E_2)$, $f_P$ depends only on $x_{E_1} - y_{E_1}$, $x_{E_2} - y_{E_2}$, and $x_P$. If $P$ is disjoint from $\widehat{E}_1$, the map $\sigma_1 = \tau_{\widehat{E}_1}$ has the same effect, on $f_P$ restricted to $\mathrm{Diag}(\Lambda, E_1 \cup E_2)$, that the permutation $\tau_{E_1}$. Then the estimation of $U_P$ follows from Theorem 1.2. For the functions associated to points, we need also the last statement of Theorem 1.2, which shows that the function $T_{\{\lambda\}}\widetilde{\psi}_\Lambda(., t)$, up to an error $\mathcal{O}((t + h^2t^2)$, is equal to a function which depends only on $x_\lambda$ and $y_\lambda$, and is invariant by $\sigma_1$ and $\sigma_2$. By the form (7.10) of $\Phi_0$, by the form (7.2) of $\widetilde{\psi}_\Lambda$, the equality in (8.18) will be satisfied if we choose $\Delta_0 = e^g - 1$, with

$$g(X, t) = \frac{1}{2}\Big[\widetilde{\psi}_\Lambda(0, y - x, t) + \widetilde{\psi}_\Lambda^{(\sigma_1\sigma_2)}(0, y - x, t) - \widetilde{\psi}_\Lambda^{(\sigma_1)}(0, y - x, t) - \widetilde{\psi}_\Lambda^{(\sigma_2)}(0, y - x, t)\Big].$$

The inequality of (8.18) follows from Theorem 1.1 if $ht \leq T_0$. The last inequality (8.19) is a consequence of the following:

$$\sum_{\lambda\in\mathbb{Z}^d} M(\lambda, X, \varepsilon) + \sum_{Q\in\mathrm{Box}(\mathbb{Z}^d)} M(Q, X, \varepsilon) \leq K(\varepsilon)|x_E - y_E|_1.$$

□

Among the functions defined in (8.11)-(8.13), only the functions $N(Q, X, \varepsilon, T)$ and its analogue for points have a good rate of decay, like in (8.10), when $\mathrm{dist}(E_1, E_2)$ is large. They are used to estimate the functions $W_Q$ and $W_\lambda$. Beside these functions, we need also other functions, associated to polymers. If

$\Pi = (Q_1, ..., Q_k)$ $(k \geq 2)$ is a polymer, which does not connect $E_1$ and $E_2$, we have no information on the sum of the lengths of its boxes, and the product $f_{Q_1} \dots f_{Q_k}$ has not necessarily the good rate of decay. However, we shall see that, if $U_Q$ and $V_Q$ are defined in (8.11) and (8.12), the functions $U_{Q_1} f_{Q_2} \dots f_{Q_{k-1}} V_{Q_k}$ has a good rate of decay, like in (8.10), in terms of $\mathrm{dist}(E_1, E_2)$. In order to make this idea more precise, let us define the functions, used for the estimations.

If $\Pi = (Q_1, ..., Q_k)$ $(k \geq 2)$ is a polymer, we set, for each points $\alpha \in E_1$ and $\beta \in E_2$, and for each $T > 0$:

$$N_{\alpha,\beta}(\Pi, \varepsilon, T) = T^k \varepsilon^{\mathrm{diam}(Q_1 \cup \{\alpha\}) + \mathrm{diam}(Q_k \cup \{\beta\})} \prod_{j=2}^{k-1} \varepsilon^{\mathrm{diam}(Q_j)} \prod_{j=1}^{k} < Q_j >^{3d} . \tag{8.20}$$

Then we set:

$$N(\Pi, X, \varepsilon, T) = \sum_{\alpha \in E_1, \beta \in E_2} (1 + |x_\alpha - y_\alpha|)\ (1 + |x_\beta - y_\beta|) N_{\alpha,\beta}(\Pi, \varepsilon, T). \tag{8.21}$$

**Lemma 8.4.** *If $\Pi = (Q_1, ..., Q_k)$ is a polymer, all its boxes belonging to $\Gamma \in \mathcal{NC}(E_1, E_2)$, we can write, with some function $a(t, \varepsilon)$:*
*a) If $\Pi$ is starting from $E_1$ (i.e. $Q_1 \cap E_1 \neq \emptyset$), and if $\mu \in Q_k$,*

$$|f_{Q_1} ... f_{Q_{k-1}} V_{Q_k}(X, t)| + |f_{Q_1} ... f_{Q_{k-1}} f_{Q_k}(X, t) V_\mu(X, t)| \leq N(\Pi, X, \varepsilon, a(t, \varepsilon)). \tag{8.22}$$

*b) We have similar results if $\Pi$ starts from $E_2$.*
*c) If all the boxes $Q_1, \dots, Q_k$ of $\Pi$ are disjoint from $\widehat{E}_1$ and $\widehat{E}_2$, if $\lambda \in Q_1$ and $\mu \in Q_k$, we can write:*

$$|U_{Q_1} ... V_{Q_k}(X, t)| + |U_\lambda f_{Q_1} ... V_{Q_k}(X, t)| \leq N(\Pi, X, \varepsilon, a(t, \varepsilon)), \tag{8.23}$$

$$|U_{Q_1} ... f_{Q_k} V_\mu(X, t)| + |U_\lambda f_{Q_1} ... f_{Q_k} V_\mu(X, t)| \leq N(\Pi, X, \varepsilon, a(t, \varepsilon)), \tag{8.24}$$

*The factors which are not written in these products are the $f_{Q_j}$. In the points a), b) and c), we have similar estimations for the derivatives.*
*d) We can write also, if $0 < \varepsilon < \delta < 1$ and $t \leq T_1(\varepsilon, \delta)$:*

$$\sum_\Pi N(\Pi, X, \varepsilon, T)\ \leq K(T, \varepsilon, \delta, |E|) \Big(1 + |x_E - y_E|_1\Big)^2\ \delta^{\mathrm{dist}(E_1, E_2)} \tag{8.25}$$

*where the sum is taken on all the polymers in $\mathbb{Z}^d$, and $N(\Pi, X, \varepsilon, T)$ is defined in (8.21) if $\Pi$ has at least two boxes, and in (8.15) if $\Pi$ is reduced to a single box $Q$.*

The points a), b) and c) follow from Lemma 8.2. With our hypotheses, if $\Pi$ is starting from $E_1$, all its boxes $Q$ satisfy $Q \cap \widehat{E}_2 = \emptyset$, and we can apply (8.16) for $V_{Q_k}$. For the last point, given two boxes $Q$ and $Q'$, we make first a summation on all polymers connecting $Q$ and $Q'$. Then we make a sum over the boxes $Q$ and $Q'$. For the first sum, we apply Lemma 7.2 with $E_1$ and $E_2$ replaced by $Q$ and $Q'$. For the second summation, we use the following inequality, if $0 < \varepsilon < \delta$ and $\delta_1 = \sqrt{\varepsilon\delta}$:

$$\sum_{Q, Q' \in \mathrm{Box}(\mathbb{Z}^d)} < Q >^{4d} < Q' >^{4d} \delta_1^{\mathrm{diam}(Q \cup \{\alpha\}) + \mathrm{diam}(Q' \cup \{\beta\}) + \mathrm{dist}(Q, Q')} \leq C(\varepsilon, \delta)\ \delta^{|\alpha - \beta|}. \tag{8.26}$$

$\square$

*Proof of Proposition 8.2. Step 1. Generators of $H_\Gamma$.* For each set of boxes $A$, let us denote by $A_{\rm pct}$ the corresponding set of points. We shall denote $\tau_A$ and $\tau_\lambda$ instead of $\tau_{A_{pct}}$ and $\tau_{\{\lambda\}}$ the operators defined like in (7.3). Let Comp($\Gamma$) be the set of connected components $A$ of $\Gamma$ such that $A_{\rm pct}$ is disjoint from $E_1 \cup E_2$, and therefore from $\widehat{E}_1 \cup \widehat{E}_2$. We set $\mathrm{Ext}(\Gamma) = \Lambda \setminus (\Gamma_{\rm pct} \cup E_1 \cup E_2)$. The group $H_\Gamma$ is generated by the $\tau_A$ ($A \in \mathrm{Comp}(\Gamma)$), by the $\tau_{\{\lambda\}}$ ($\lambda \in \mathrm{Ext}(\Gamma)$), and by the elements $\sigma_k = \tau_{\widehat{E}_k}$ ($k = 1, 2$) already introduced. Therefore

$$|H_\Gamma| = 2^{|\mathrm{Comp}(\Gamma)| + |\mathrm{Ext}(\Gamma)| + 2}$$

The generators of the group $H_\Gamma$, listed above, have a different action on the factors of the product (7.11) defining $K_\Gamma$. Let us distinguish them. For each connected component $A$ in Comp($\Gamma$), let:

$$F_A = \prod_{Q \in A} f_Q \prod_{\lambda \in A_{\rm pct}} f_\lambda.$$

For the connected components containing a box intersecting $E_1$, or $E_2$, we set:

$$\Phi_k(X, t) = \prod_{\substack{Q \in \Gamma \\ Q_{\rm pct} \subset \widehat{E}_k}} f_Q \prod_{\lambda \in \widehat{E}_k} f_\lambda \qquad (1 \le k \le 2).$$

Then, using also the function $\Phi_0$ of (7.10), we can write:

$$K_\Gamma = \Phi_0 \Phi_1 \Phi_2 \prod_{A \in \mathrm{Comp}(\Gamma)} F_A \prod_{\lambda \in \mathrm{Ext}(\Gamma)} f_\lambda.$$

By theorem 1.2, for each box $Q$, the function $f_Q(., t)$, restricted to $\mathrm{Diag}(\Lambda, E_1 \cup E_2)$, depends only on the variables $x_{E_1} - y_{E_1}$, $x_{E_2} - y_{E_2}$, $x_Q$ and $y_Q$. The equality (7.2) shows that this function is invariant when all the variables $(x', y')$ and $(x'', y'')$ are permuted. From these two remarks, some properties follow for the functions defined above. We can write, if $A$ and $B$ are in Comp($\Gamma$), $A \neq B$, if $\lambda$ and $\mu$ are in Ext($\Gamma$), $\lambda \neq \mu$ :

$$F_A^{(\tau_B)} \sim F_A \qquad F_A^{(\tau_A)} \sim F_A^{(\sigma_1\sigma_2)} \qquad F_A^{(\tau_\lambda)} \sim F_A \qquad f_\lambda^{(\tau_A)} \sim f_\lambda \tag{8.27}$$

$$f_\lambda^{(\tau_\mu)} \sim f_\lambda \qquad \Phi_k^{(\tau_A)} \sim \Phi_k^{(\tau_\lambda)} \sim \Phi_k \quad (0 \le k \le 2) \qquad \Phi_1^{(\sigma_1)} \sim \Phi_1^{(\sigma_2)}, etc\ldots$$

Let us denote by $H_\Gamma^+$ the subgroup of $H_\Gamma$ generated by the $\tau_A$ ($A \in \mathrm{Comp}(\Gamma)$) and $\tau_\lambda$ ($\lambda \in \mathrm{Ext}(\Gamma)$). By the above equalities and similar ones, we can write:

$$K_\Gamma^{\rm Av} \sim 2K_\Gamma^+ - 2\Big(K_\Gamma^+\Big)^{(\sigma_1)}, \qquad K_\Gamma^+(X, t) = \frac{1}{|H_\Gamma|} \sum_{\sigma \in H_\Gamma^+} K_\Gamma^{(\sigma)}. \tag{8.28}$$

We have also:

$$K_\Gamma^+ \sim \frac{1}{|H_\Gamma|} \Phi_0 \Phi_1 \Phi_2 \prod_{A \in \mathrm{Comp}(\Gamma)} \Big[F_A + F_A^{(\sigma_1\sigma_2)}\Big] \prod_{\lambda \in \mathrm{Ext}(\Gamma)} \Big[f_\lambda + f_\lambda^{(\sigma_1\sigma_2)}\Big], \tag{8.29}$$

$$(K_\Gamma^+)^{(\sigma_1)} \sim \frac{1}{|H_\Gamma|} \Phi_0^{(\sigma_1)} \Phi_1^{(\sigma_2)} \Phi_2^{(\sigma_1)} \prod_{A \in \mathrm{Comp}(\Gamma)} \Big[F_A^{(\sigma_1)} + F_A^{(\sigma_2)}\Big] \prod_{\lambda \in \mathrm{Ext}(\Gamma)} \Big[f_\lambda^{(\sigma_1)} + f_\lambda^{(\sigma_2)}\Big] \tag{8.30}$$

*Step 2. Polynomial expression of $K_\Gamma^{\rm Av}$.* Now, we shall write the difference between (8.29) and (8.30) as a polynomial expression of functions that are bounded like in (8.10). We remember that:

$$f_Q^{(\sigma_1)} = f_Q + U_Q, \qquad f_Q^{(\sigma_2)} = f_Q + V_Q \qquad f_Q^{(\sigma_1\sigma_2)} = f_Q + U_Q + V_Q + W_Q.$$

According to the notations (8.11)-(8.18), we have $f_\lambda^{(\sigma_1)} = f_\lambda(1+U_\lambda)$, etc... and $\Phi_0^{(\sigma_1)} \sim (1+\Delta_0)\Phi_0$. Thus we can write

$$K_\Gamma^{\mathrm{Av}} \sim G_\Gamma\ \Phi_0 \prod_{\lambda\in\Lambda} f_\lambda,$$

where $G_\Gamma$ is a polynomial expression of the functions $f_Q$, $U_Q$, $V_Q$ and $W_Q$ ($Q \in \mathrm{Box}(\Lambda)$), of the functions $U_\lambda$, $V_\lambda$ and $W_\lambda$ ($\lambda \in \Lambda$), and of the function $\Delta_0$. Let us describe more carefully this polynomial. Let $I$ be the set of partitions of $\Gamma$ in four subsets $\mathcal{F}, \mathcal{U}, \mathcal{V}, \mathcal{W}$. Let $I_p$ be the set of triples $(\mathcal{U}^p, \mathcal{V}^p, \mathcal{W}^p)$ such that $\mathcal{U}^p$, $\mathcal{V}^p$, $\mathcal{W}^p$ are disjoint subsets of $\Lambda$. Let $J = I \times I_p \times \{0,1\}$. Each element $j \in J$ will be written $j = (\mathcal{F}_j, ...., \mathcal{W}_j^p, m_j)$. Thus we can write:

$$K_\Gamma^{\mathrm{Av}} \sim \sum_{j\in J} c_j G_\Gamma^{[j]} \qquad G_\Gamma^{[j]} = \Phi_0\ \Delta_0^{m_j} \prod_{Q\in\mathcal{F}_j} f_Q \prod_{Q\in\mathcal{U}_j} U_Q ... \prod_{\lambda\in\mathcal{W}_j^p} W_\lambda \Big[\prod_{\lambda\in\Lambda} f_\lambda\Big] \tag{8.31}$$

where the coefficient $c_j$ are constant, and uniformly bounded. Moreover, we have $c_j = 0$ unless one of the following three conditions $(A)$, $(B)$ or $(C)$ is satisfied.

$$(A) \qquad \mathcal{W}_j \neq \emptyset \quad \text{or} \quad \mathcal{W}_j^p \neq \emptyset \quad \text{or} \quad m_j = 1$$

$$(B) \qquad (\mathcal{V}_j)_{\mathrm{pct}} \cap \widehat{E_1} \neq \emptyset \quad \text{or} \quad \mathcal{V}_j^p \cap \widehat{E_1} \neq \emptyset \quad \text{or} \quad (\mathcal{U}_j)_{\mathrm{pct}} \cap \widehat{E_2} \neq \emptyset \quad \text{or} \quad \mathcal{U}_j^p \cap \widehat{E_2} \neq \emptyset$$

(C) There exists a connected component $A \in \mathrm{Comp}(\Gamma)$ such that, in one hand, $A \cap \mathcal{U}_j \neq \emptyset$, or $A_{\mathrm{pct}} \cap \mathcal{U}_j^p \neq \emptyset$, and in the other hand, $A \cap \mathcal{V}_j \neq \emptyset$, or $A_{\mathrm{pct}} \cap \mathcal{V}_j^p \neq \emptyset$.

Moreover, we have also $c_j = 0$ unless all the following condition are satisfied, and also the similar ones for the sets of points:

$$(\mathcal{U}_j)_{\mathrm{pct}} \cap \widehat{E_1} = \emptyset \qquad (\mathcal{V}_j)_{\mathrm{pct}} \cap \widehat{E_2} = \emptyset \qquad (\mathcal{W}_j)_{\mathrm{pct}} \cap (\widehat{E_1} \cup \widehat{E_2}) = \emptyset.$$

*Step 3. Construction of $B_{\Gamma,\Gamma'}$.* For each $j \in J$ such that $c_j \neq 0$, we shall write the term $G_\Gamma^{[j]}$ in (8.31) in the form:

$$G_\Gamma^{[j]} = B_j K_{\Gamma_j} \tag{8.32}$$

where $K_{\Gamma_j}$ is defined as in (7.11), with $\Gamma$ replaced by some subset $\Gamma_j$. The function $B_j$ will have the rate of decay of (8.10) when $\mathrm{dist}(E_1, E_2)$ is large, because the product defining $B_j$ will contain, either $\Delta_0$, or a function $W_\lambda$, or a function, like those of Lemma 8.4, corresponding to a polymer $\Pi_j$. In order to apply Lemma 8.4, we shall need sometimes one point $\lambda_j$, or two. We shall denote by $\mathcal{X}_j$ the set of points needed for the application of Lemma 8.4: this set has 0, 1 or 2 points. Let us define $\Gamma_j$, $\Pi_j$ and $\mathcal{X}_j$ in all the cases A, B and C.

If $j$ satisfies (A), let $\Gamma_j = \mathcal{F}_j$, and let $B_j$ be defined by (8.32). In the first case of (A), the polymer $\Pi_j$ is reduced to single box $Q$ chosen in $\mathcal{W}_j$, and $\mathcal{X}_j = \emptyset$. In the second case, $\Pi_j = \emptyset$, and for $\mathcal{X}_j$, we choose one point in $\mathcal{W}_j^p$. In the third case, $\Pi_j = \emptyset$, and $\mathcal{X}_j = \emptyset$.

If $j$ satisfies the first condition of (B), let $Q$ be a box in $\mathcal{V}_j$ such that $Q \cap \widehat{E}_1 \neq 0$. There is a polymer $\Pi_j$ in $\Gamma$, starting in $E_1$, the last box of which being $Q$. Let $\Gamma_j = \mathcal{F}_j \setminus \Pi_j$ be the set of the boxes in $\mathcal{F}_j$, excepted those we took from $\mathcal{F}_j$ to construct the polymer $\Pi_j$. Let $B_j$ be defined by (8.32), and $\mathcal{X}_j = \emptyset$. If $\mathcal{V}_j^p \cap \widehat{E}_1 \neq \emptyset$, let $\mu_j$ be a point in this set. There is a polymer $\Pi_j$, starting from $E_1$, such that $\mu_j$ is in its last box. We define $\Gamma_j$ and $B_j$ as before, but $\mathcal{X}_j = \{\mu_j\}$. We proceed in the same way in the other cases of (B).

Now, let $j$ satisfying one of the conditions of (C), for example such that, for some connected component $A \in \mathrm{Comp}(\Gamma)$, we have $A \cap \mathcal{U}_j \neq \emptyset$ and $A_{\mathrm{pct}} \cap \mathcal{V}_j^p \neq \emptyset$. Let $P$ is a box in the first set, and $\mu_j$ be a point in the second one. Since they are in the same connected component, there is a polymer $\Pi_j$, whose first box is $P$, and such that $\mu_j$ is in its last box. We define $\Gamma_j$ and $B_j$ as before, and $\mathcal{X}_j = \{\mu_j\}$.

Now for all $j$, we have defined $\Gamma_j$ and $B_j$ such that (8.32) is satisfied. If, for some $j \in J$, we are in several cases, we make a choice. For each $\Gamma' \subseteq \Gamma$, we set

$$B_{\Gamma,\Gamma'} = \sum_{\substack{j\in J \\ \Gamma_j=\Gamma'}} c_j B_j \tag{8.33}$$

and (8.8) is satisfied.

*Step 4. Estimation of* $B_{\Gamma,\Gamma'}$. When a polymer $\Pi_j$ is used for the application of Lemma 8.4, with a set $\mathcal{X}_j \subset \Lambda$ with 0, 1 or 2 points, we can write, by Lemmas 8.3 and 8.4, :

$$|B_j(X,t)| \leq N(\Pi_j, X, \varepsilon, a(t,\varepsilon)) \prod_{Q\in(\mathcal{U}_j\cup\mathcal{V}_j\cup\mathcal{W}_j)\setminus\Pi_j} a(t,\varepsilon)M(Q,X,\varepsilon) \prod_{\lambda\in(\mathcal{U}_j^p\cup\mathcal{V}_j^p\cup\mathcal{W}_j^p)\setminus\mathcal{X}_j} a(t,\varepsilon)M(\lambda,X,\varepsilon)\ .$$

In order to estimate $B_{\Gamma,\Gamma'}$, following (8.33), we make first a summation on all the sets $\mathcal{X}_j$ corresponding to a same polymer $\Pi_j$: this gives only a change in the power of $< Q >$ in the definition of $N(\Pi_j, ...)$. Then we sum on all the sets $\mathcal{U}_j^p$, $\mathcal{V}_j^p$ and $\mathcal{W}_j^p$, applying the last point of Lemma 8.3. We obtain, proceeding in a similar way for the terms without polymer:

$$|B_{\Gamma,\Gamma'}(X,t)| \ \leq \ e^{T|x_E-y_E|_1} \sum_{\Pi\in\mathrm{Pol}_{\Gamma\setminus\Gamma'}} \left[ N(\Pi,X,\varepsilon,T) \prod_{Q\in\Gamma\setminus(\Gamma'\cup\Pi)} TM_Q(X,\varepsilon) \right] \qquad T = a(t,\varepsilon).$$

Therefore, applying the last points of Lemmas 8.3 and 8.4, we see that the function $S_{\Gamma'}$ defined in (8.9) satisfies (8.10). We shall not write the details for the derivatives of $B_{\Gamma,\Gamma'}$.

*End of the proof of Proposition 8.1.* It remains to estimate the second sum in (8.5). By Proposition 8.2, with the similar estimates for derivatives, we can write, for some function $\mu_1(t)$:

$$\sum_{\Gamma\in\mathcal{NC}(E_1,E_2)} \|K_\Gamma^{\mathrm{Av}}(t)\|_{m,m',0} \leq K(t,h,\varepsilon,\delta,|E|)\delta^{\mathrm{dist}(E_1,E_2)} \sum_{\Gamma\in\mathcal{NC}(E_1,E_2)} \|K_\Gamma(t)\|_{m_1,m_1',\mu_1(t)}.$$

If we look at the derivatives of the functions $K_\Gamma(X,t)$ defined in (7.11), we can write:

$$\sum_{\Gamma\subseteq\mathrm{Box}(\Lambda)} \|K_\Gamma(t)\|_{m_1,m_1',\mu_1(t)} \leq M(|E|,t,h,\varepsilon) \sum_{\Gamma\subseteq\mathrm{Box}(\Lambda)} \|K_\Gamma(t)\|_{0,m_2',\mu_2(t)}$$

By the positivity of $K_\Gamma(X,t)$ and by the equality (7.12), we can write:

$$\sum_{\Gamma\subseteq\mathrm{Box}(\Lambda)} \|K_\Gamma(t)\|_{0,m_2',\mu_2(t)} \leq \|\widetilde{U}_\Lambda(t)\|_{0,m_2',\mu_2(t)}$$

By Proposition 5.1, we can write, with some constant $M(|E|,t,h,\varepsilon)$,

$$\|\widetilde{U}_\Lambda(t)\|_{0,m_2',\mu_2(t)} \leq M(|E|,t,h,\varepsilon)\ \widetilde{Z}_\Lambda(t).$$

The main estimate of Proposition 8.1, and therefore the estimation (7.6) of Proposition 7.1, and therefore the point a) of Theorem 1.1, are proven. The points b) and d) were proved in section 7, and the point c), (estimation of $K_{\mathrm{op,fc}}(E_1,E_2,t,h)$), is an intermediate between the two proofs: we need a partial trace $\mathrm{Tr}_{E_1}^\Lambda$ instead of $\mathrm{Tr}_{E_1\cup E_2}^\Lambda$. Thus we obtain, in the estimation, a constant depending on $|E_1|$ instead of $|E_1\cup E_2|$. Theorem 1.3 is proved.

## 9. Proof of Theorems 1.4 and 1.5.

For the proof of Theorem 1.4, we consider a box $\Lambda$ in $\mathbb{Z}^d$, which is the union of two disjoint boxes $\Lambda_1$ and $\Lambda_2$, separated by an hyperplane $\Sigma$, like in (6.1). We consider a local observable $A$, supported in one of these sets, for example $\Lambda_1$. We shall estimate the difference between $E_{\Lambda,t}(A)$, (the mean value of $A$, defined by (1.2), when $A$ is seen as an operator in $\mathcal{H}_\Lambda$), and $E_{\Lambda_1,t}(A)$, (the analogue for $\Lambda_1$). Theorem 1.4 will be an easy consequence of the next Proposition, since $\Lambda_{m+n}$ is obtain from $\Lambda_n$ (defined in (1.19)) by applying $2d$ times this procedure of enlarging.

**Proposition 9.1.** *With the above notations, if the interaction satisfies $(H_\varepsilon)$, if $\varepsilon < \delta < 1$, there exists $T_1(\varepsilon,\delta)$ and a function $K(t,h,\varepsilon,\delta,N)$ such that, if $ht < T_0$ and $t < T_1(\varepsilon,\delta)$:*

$$\Big| E_{\Lambda,t}(A) - E_{\Lambda_1,t}(A) \Big| \leq K(t,h,\varepsilon,\delta,|\mathrm{supp}(A)|) \ \ \delta^{\mathrm{dist}(\mathrm{supp}(A),\Sigma)} \|A\|. \tag{9.1}$$

*Proof. First step.* For each $\theta\in[0,1]$, we shall use the potential $V_{\Lambda,\theta} = V_\Lambda - \theta V_{\mathrm{inter}}$, (where $V_{\mathrm{inter}}$ is defined in (6.2)), the corresponding Hamiltonian $H_{\Lambda,\theta}$, the heat kernel $U_{\Lambda,\theta}$, the function $\psi_{\Lambda,\theta}$ of Theorem 1.1. and the corresponding correlation $\mathrm{Cov}_{\Lambda,t,\theta}$ of two operators, and the mean value $E_{\Lambda,t,\theta}(A)$. We denote by $Z_{\Lambda,\theta}(t)$ the trace of the operator $e^{-tH_{\Lambda,\theta}}$. If $A$ is supported in $\Lambda_1$, we have, with these notations:

$$E_{\Lambda,t,0}(A) = E_{\Lambda,t}(A) \qquad\quad E_{\Lambda,t,1}(A) = E_{\Lambda_1,t}(A) \tag{9.2}$$

We set, for each $\theta\in[0,1]$,

$$R_\theta(x,y,t) = \partial_\theta\psi_{\Lambda,\theta}(x,y,t) \ - \ \partial_\theta\psi_{\Lambda,\theta}(x,x,t), \qquad\quad \varphi_\theta(x,t) = \psi_{\Lambda,\theta}(x,x,t), \tag{9.3}$$

For a given $K(x,y)$, in $\mathcal{S}((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$, let $Op(K)$ be the operator with integral kernel $K(x,y)$. We denote by $\mathrm{Cov}_{\Lambda,t,\theta}(A,f)$ the correlation between an operator $A$ and the multiplication by a function $f$. We use the operators $T_Q$ of Section 3, with the first definition (3.2), applied to functions depending only on $x$. With all these notations, we have:

$$\partial_\theta E_{\Lambda,t,\theta}(A) = \ Z_{\Lambda,\theta}(t)^{-1}\mathrm{Tr}\Big( Op\big(U_{\Lambda,\theta}(.,t)R_\theta(.,t)\big)\circ A\Big) \ + \ \frac{1}{2}\sum_{Q\subseteq\Lambda}\mathrm{Cov}_{\Lambda,t,\theta}(A, T_Q\partial_\theta\varphi_\theta(.,t)). \tag{9.4}$$

*Second step.* Using (8.3), with the norm defined in (5.1), we can write, with $E=\mathrm{supp}(A)$:

$$\Big|\mathrm{Tr}(Op(U_{\Lambda,\theta}(.,t)R_\theta)\circ A)\Big| \leq K(|E|)\ \|A\| \ \|Op(U_{\Lambda,\theta}(.,t)R_\theta)\|_{m,m',0}$$

with $m$ and $m'$ depending on $|E|$. By the point (6.4) of Lemma 6.2, we can write, for each points $\lambda^{(1)}$, ... $\lambda^{(m)}$ of $E$, for each $(x,y)$

$$|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(m)}}R_\theta(x,y,t)| \leq tK(\varepsilon)\ \varepsilon^{\mathrm{dist}(E,\Sigma)}.$$

Therefore,

$$\|Op(U_{\Lambda,\theta}(.,t)R_\theta)\|_{m,m',0} \leq \ tK(\varepsilon)\ \varepsilon^{\mathrm{dist}(E,\Sigma)}\|U_{\Lambda,\theta}(.,t)\|_{m,m'+1,0}.$$

By Proposition 5.1, applied to the subset $E$, we can write:

$$\|U_{\Lambda,\theta}(.,t)\|_{m,m'+1,0} \leq K(t,h,\varepsilon,|E|)Z_{\Lambda,\theta}(t),$$

Therefore:

$$|\mathrm{Tr}(Op(U_{\Lambda,\theta}(.,t)R_\theta)\circ A)| \leq \ K(t,h,\varepsilon,|E|)\ \|A\|\ Z_{\Lambda,\theta}(t)\ \varepsilon^{\mathrm{dist}(E,\Sigma)} \tag{9.5}$$

Now, we shall estimate the second term in (9.4). By Proposition 6.1, we can write:

$$\|T_Q \partial_\theta \varphi_\theta(.,t)\| \leq tK(\varepsilon) < Q >^d \varepsilon^{\mathrm{diam}(Q \cup \pi_\Sigma(Q))}, \tag{9.6}$$

where $\pi_\Sigma(Q)$ is the orthogonal projection of $Q$ on the hyperplane $\Sigma$, which separates $\Lambda_1$ and $\Lambda_2$. If $\varepsilon < \delta < 1$, we can apply Theorem 1.3, point c, for the correlation between an operator and a function, with $\delta$ replaced by $\delta_1 = \sqrt{\varepsilon\delta}$. Thus we can write:

$$\Big|\mathrm{Cov}_{\Lambda,t,\theta}(B_Q, A)\Big| \leq K(t,h,\varepsilon,\delta,|E|)\ \|A\|\ \|T_Q \partial_\theta \varphi_\theta(.,t)\|\ \delta_1^{\mathrm{dist}(E,Q)}. \tag{9.7}$$

With the same relations between $\varepsilon$, $\delta_1$ and $\delta$, we have:

$$\sum_{Q \subset \mathbb{Z}^d} \delta_1^{\mathrm{dist}(E,Q)+\mathrm{diam}(Q\cup\pi_\Sigma(Q))} \leq K(\varepsilon,\delta)\delta^{\mathrm{dist}(E,\Sigma)}. \tag{9.8}$$

By (9.5)-(9.8), Proposition 9.1 is proved, and Theorem 1.4 follows easily.

□

Theorem 1.5, about the mean energy per site, will follow from the next Proposition. The mean energy $X_\Lambda(t)$ for the set $\Lambda$ is defined in (1.22).

**Proposition 9.2.** *For each box $\Lambda$ of $\mathbb{Z}^d$, split into two boxes $\Lambda_1$ and $\Lambda_2$, separated by an hyperplane $\Sigma$ as in (6.1), for each $t > 0$ and $h > 0$ such that $ht < T_0$ and $t$ is small enough,*

$$|X_\Lambda(t)\ - X_{\Lambda_1}(t)\ - X_{\Lambda_2}(t)\ | \leq K(t)\ |\Lambda_\perp|. \tag{9.9}$$

*where $\Lambda_\perp = \pi_\Sigma(\Lambda)$, and $\pi_\Sigma$ is the orthogonal projection on $\Sigma$.*

*Proof.* For each $\theta \in [0,1]$, let $X_{\Lambda,\theta}(t)$ be the mean energy, for the set $\Lambda$, but with $V_\Lambda$ replaced by the potential $V_{\Lambda,\theta} = V_\Lambda - \theta V_{\mathrm{inter}}$, where $V_{\mathrm{inter}}$ is defined in (6.2). Thus, $X_{\Lambda,0}(t) = X_\Lambda(t)$ and $X_{\Lambda,1}(t) = X_{\Lambda_1}(t) + X_{\Lambda_2}(t)$. If $\psi_{\Lambda,\theta}(x,y,t)$ is the function of Theorem 1.1, associated to the potential $V_{\Lambda,\theta}$, we set $\varphi_\theta(x,t) = \psi_{\Lambda,\theta}(x,x,t)$. By computations, similar to those of Proposition 9.1, we find that

$$\partial_\theta X_{\Lambda,\theta}(t) = E_{\Lambda,t,\theta}\left(\frac{\partial^2 \phi_\theta(.,t)}{\partial t \partial\theta}\right) + \frac{1}{2}\mathrm{Cov}_{\Lambda,t,\theta}\left(\frac{\partial\phi_\theta}{\partial t}, \frac{\partial \phi_\theta}{\partial\theta}\right) = F_1(\theta,t) + F_2(\theta,t). \tag{9.10}$$

where $\mathrm{Cov}_{\Lambda,t,\theta}(f,g)$ is the correlation of the multiplications $M_f$ and $M_g$, defined as in (1.22) for the Hamiltonian $H_{\Lambda,\theta}$, and $E_{\Lambda,t,\theta}(f)$ is defined in the same way.

*Estimation of $F_1(\theta,t)$.* Following (2.1), with $V_\Lambda$ replaced by $V_{\Lambda,\theta}$, we have:

$$\frac{\partial\phi_\theta(x,t)}{\partial t} = \frac{h^2}{2}(\Delta_x \psi_{\Lambda,\theta})(x,x,t) - \frac{h^2}{2}|(\nabla_x \psi_{\Lambda,\theta}(x,x,t)|^2 + V_{\Lambda,\theta}(x).$$

By differentiating with respect to $\theta$, we obtain

$$|\partial_t\partial_\theta\phi_\theta(x,t)| \leq \frac{h^2}{2}\sum_{\lambda\in\Lambda}|\Big(\Delta_{x_\lambda}\partial_\theta\psi_\theta\Big)(x,x,t)| + h^2\sum_{\lambda\in\Lambda}|\nabla_{x_\lambda}\partial_\theta\psi_\theta(x,x,t)|\ |\nabla_{x_\lambda}\psi_\theta(x,x,t)| + |\partial_\theta V_{\Lambda,\theta}|.$$

We remark that

$$|\partial_\theta V_{\Lambda,\theta}(x)| = |V_{\mathrm{Inter}}(x)| \leq K\sum_{\lambda\in\Lambda_1,\mu\in\Lambda_2}\varepsilon^{|\lambda-\mu|} \leq\ K(\varepsilon)|\Lambda_\perp|.$$

By Lemma 6.2 for $m = 1$ and $m = 2$, we can write, if $ht \leq T_0$,

$$|\nabla_{x_\lambda}\partial_\theta\psi_\theta(x,x,t)| + |\Big(\Delta_{x_\lambda}\partial_\theta\psi_\theta\Big)(x,x,t)| \leq tK(\varepsilon)\varepsilon^{\mathrm{dist}(\lambda,\Sigma)},$$

and therefore,
$$|\partial_t\partial_\theta\phi_\theta(x,t)| \leq K(\varepsilon)h^2(t+t^2)\sum_{\lambda\in\Lambda}\varepsilon^{\mathrm{dist}(\lambda,\Sigma)} \leq K(\varepsilon)h^2(t+t^2)|\Lambda_\perp|.$$

and therefore $|F_1(\theta,t)| \leq K(\varepsilon)h^2(t+t^2)|\Lambda_\perp|$.

*Estimation of* $F_2(\theta,t)$. Now, we use the decomposition of Theorem 1.2, (definition (3.7)), for $\psi_{\Lambda,\theta}$. We apply (1.13) with $\psi_\Lambda$ replaced by this function. Now, we take the derivatives of this equality with respect to the variables $x_\lambda$ ($\lambda\in\Lambda$), we restrict then to the diagonal, and we report into (9.17), we take also the derivative with respect to $\theta$, and we report both in the expression of $F_2(\theta)$ defined in (9.10). We remark that, if a function does not depend on $x$, its correlation with any other one vanishes. In order to write more shortly what we obtain, we introduce the following notations:

$$v_\lambda(t) = -(\nabla_{x_\lambda}\psi_{\Lambda,\theta})(0,0,t) \qquad f_{\lambda,Q}(x,t) = (\Delta_{x_\lambda}T_Q\psi_{\Lambda,\theta})(x,x,t) \qquad g_{\lambda,Q}(x,t) = (\nabla_{x_\lambda}T_Q\psi_{\Lambda,\theta})(x,x,t)$$
$$h_Q(x,t) = (T_Q\frac{\partial}{\partial\theta}\psi_{\Lambda,\theta})(x,x,t).$$

Thus, we obtain:

$$F_2(\theta) = \frac{h^2}{4}\sum_{\substack{\lambda\in\Lambda\\ Q,Q'\subseteq\Lambda}}\mathrm{Cov}_{\Lambda,t,\theta}(f_{\lambda,Q},h_{Q'}) - h^2\sum_{\substack{\lambda\in\Lambda\\ Q,Q'\subseteq\Lambda}}v_\lambda(t)\ .\ \mathrm{Cov}_{\Lambda,t,\theta}(g_{\lambda,Q}\ ,\ h_{Q'}) -$$
$$-\frac{h^2}{2}\sum_{\substack{\lambda\in\Lambda\\ Q,Q',Q''\subseteq\Lambda}}\mathrm{Cov}_{\Lambda,t,\theta}\Big(g_{\lambda,Q}.g_{\lambda,Q'}\ ,\ h_{Q''}\Big) + \sum_{Q,Q'\subseteq\Lambda}\mathrm{Cov}_{\Lambda,t,\theta}(T_QV_{\Lambda,\theta},h_{Q'}). \tag{9.11}$$

Now, we shall estimate all the terms. By Theorem 1.2, $f_{\lambda,Q}$, $g_{\lambda,Q}$ and $h_Q$ depend only on $x_Q$ and, if $ht\leq T_0$:

$$|f_{\lambda,Q}(x,t)| + |g_{\lambda,Q}(x,t)| \leq tK(\varepsilon) < Q >^{2d}\ \varepsilon^{\mathrm{diam}(Q\cup\{\lambda\})}$$

By Theorem 1.1, we can write $|v_\lambda(t)|\leq Ct$. By Proposition 6.1, we can write:

$$|h_Q(x,t)| \leq tK(\varepsilon) < Q >^{2d}\ \varepsilon^{\mathrm{diam}(Q\cup\pi_\Sigma(Q))}.$$

We apply Theorem 1.3, in the multiplicative case, to estimate the four correlations in the expression of $F_2(\theta)$. If $\varepsilon<\delta<1$, we apply theorem 1.3 with $\delta$ replaced by $\delta_1=\sqrt{\varepsilon\delta}$. We obtain:

$$\frac{h^2}{4}|\mathrm{Cov}_{\Lambda,t,\theta}(f_{\lambda,Q},h_{Q'})| \leq h^2t^3K(\varepsilon,\delta)\delta_1^{\mathrm{diam}(Q\cup\{\lambda\})+\mathrm{dist}(Q,Q')+\mathrm{diam}(Q'\cup\pi_\Sigma(Q'))}$$

Therefore, using again (8.26), as in the proof of Lemma 8.4, we get:

$$\frac{h^2}{4}\sum_{\substack{\lambda\in\Lambda\\ Q,Q'\subseteq\Lambda}}|\mathrm{Cov}_{\Lambda,t,\theta}(f_{\lambda,Q},h_{Q'})| \leq h^2t^3K(\varepsilon,\delta)\sum_{\lambda\in\Lambda,\mu\in\Lambda_\perp}\delta^{\mathrm{dist}(\lambda,\mu)} \leq h^2t^3K(\varepsilon,\delta)|\Lambda_\perp|.$$

The other terms in the expression (9.11) of $F_2(\theta)$ are bounded in the same way, excepted the terms $\mathrm{Cov}_{\Lambda,t,\theta}(T_QV_\theta,h_{Q'})$ such that $Q$ is reduced to a single point $\lambda$, for which we need the point d) of Theorem 1.3:
$$|\mathrm{Cov}_{\Lambda,t,\theta}(T_{\{\lambda\}}V_\theta,h_{Q'})| \leq K(t,h,\varepsilon,\delta)\|\nabla T_QV_\theta\|\ \delta^{\mathrm{dist}(Q,Q')+\mathrm{diam}(Q'\cup\pi_\Sigma(Q'))}$$

which gives us:
$$\sum_{\lambda\in\Lambda,Q\subseteq\Lambda}|\mathrm{Cov}_{\Lambda,t,\theta}(T_{\{\lambda\}}V_{\Lambda,\theta},h_{Q'})| \leq K(t,h,\varepsilon,\delta)|\Lambda_\perp|.$$

The Proposition is proved, and Theorem 1.5 follows by the same arguments as Sjöstrand [23], Section 8, p.45-46.

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laurent.amour@univ-reims.fr

claudy.cancelier@univ-reims.fr

pierre.levy-bruhl@univ-reims.fr

jean.nourrigat@univ-reims.fr