On the Stokes phenomenon for holomorphic solutions of integro-differential equations with irregular singularity

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Abstract

We study the Stokes phenomenon for sectorial holomorphic solutions of linear integro-differential equations in two variables t, z with irregular singularity at $t = 0$. More precisely, we give sufficient conditions on the coefficients of the equations and initial conditions under which the Stokes transition functions can be expressed as generalized Laplace transform of convergent series of hyperfunctions defined on half-lines in $\mathbb{C}$.

Key words: Cauchy problem, partial integro-differential equation, Gevrey estimates, distribution, hyperfunction, analytic continuation. 2000 MSC: 35C10, 35C20.

1 Introduction

We consider partial integro-differential Cauchy problems of the form

$$4t^2\partial_t\partial_z^S\mathbf{X}_j(t,z) + (2t+2)\partial_z^S\mathbf{X}_j(t,z) = \sum_{k_1=0}^{S-1}\int_0^t \mathbf{A}_{k_1,j}(t-\tau,z)\partial_z^{k_1}\mathbf{X}_j(\tau,z)d\tau \tag{1}$$

with initial conditions

$$(\partial_z^q\mathbf{X}_j)(t,0) = \mathbf{X}_{q,j}(t) \quad , \quad 0 \le q \le S-1. \tag{2}$$

where $S \geq 1$ is a positive integer, $j \in \{1,2\}$ and the coefficients $\mathbf{A}_{k_1,j}(t,z)$ and initial data $\mathbf{X}_{q,j}(t)$ are holomorphic functions on domains $S_{d_j,\rho} \times D_R$ where $S_{d_j,\rho}$ are open sectors with opening larger than π in direction $d_j \in [0,2\pi)$ with radius $\rho > 0$ and D_R is a ball centered at 0 with radius $R > 0$.

These equations (1) belong to a class of partial differential equations with irregular singularity at $t = 0$.

In this paper, we investigate the existence and unicity of holomorphic solutions $\mathbf{X}_j(t,z)$, $j \in \{1,2\}$ of (1) on the domains $S_{d_j,\rho} \times D_R$.

Our main goal is to study the Stokes transitions functions $\mathbf{X}_2(t,z) - \mathbf{X}_1(t,z)$ in terms of the Stokes transitions $\mathbf{A}_{k_1,2}(t,z) - \mathbf{A}_{k_1,1}(t,z)$ and $\mathbf{X}_{q,2}(t) - \mathbf{X}_{q,1}(t)$, $0 \leq q, k_1 \leq S-1$ on $S_{d_1,\rho} \cap S_{d_2,\rho} \times D_R$.

More precisely, we will give sufficient conditions on the coefficients $\mathbf{A}_{k_1,j}(t,z)$ and on the initial conditions $\mathbf{X}_{q,j}(t)$ under which the Stokes transitions functions $\mathbf{X}_2(t,z)-\mathbf{X}_1(t,z)$ can be expressed as generalized Laplace transform of convergent series in $z \in D_R$ with coefficients be hyperfunctions defined on a half-line $\mathbb{R}_+e^{i\theta_1}$ where $\theta_1 \in (d_1,d_2)$, see Theorem 2.

This kind of expressions has been obtained in the context of resurgence analysis to study the Stokes phenomenon for difference and differential equations with irregular singularity, see [4], [8], [11]. In a more general setting, the concept of hyperfunctions and Fourier or Laplace transforms of hyperfunctions are crucial tools for the study of analytic continuation and the analysis of singularities of solutions of partial differential equations in the complex domain, see [7] [9], [10].

In the first part of the paper, we study weighted convolution Banach algebra of distributions on $\mathbb{R}_+$ known as staircase distributions $\mathcal{D}_\nu(\mathbb{R}_+)$ as introduced by O. Costin in [4]. In particular, we give conditions under which the multiplication by an analytic function is a bounded operator on $\mathcal{D}_\nu(\mathbb{R}_+)$. We also give sufficient conditions for difference of staircase distributions to be interpreted as hyperfunctions on half-lines $\mathbb{R}_+e^{i\theta}$. We introduce modified Laplace transforms defined on $\mathcal{D}_\nu(\mathbb{R}_+)$ which extend the generalized Laplace transform with respect to kernel functions (see [1]) defined on locally integrable $L^1_{loc}(\mathbb{R}_+)$ functions with at most exponential growth.

In a second part, we study linear integro-differential Cauchy problems in the Banach algebra $\mathcal{D}_\nu(\mathbb{R}_+)$ with norm estimates of the solutions in terms of the norms of the initial conditions. We also investigate sufficient conditions under which analytic continuation in infinite sectors of holomorphic solutions at the origin of these Cauchy problems can be performed near a half-line $\mathbb{R}_+e^{id}$ containing singularities using the method of majorazing series. The problem is then reduced to the classical study of analytic continuation of holomorphic solutions of linear differential equations with singularities in $\mathbb{C}$. The results obtained allow us to construct hyperfunctions solutions of the given linear integro-differential Cauchy problems for given hyperfunctions initial conditions.

In a third part, for given well-chosen initial conditions, we construct a $1-$summable formal solution $\hat{\mathbf{X}}(t,z)$ of (1) with respect to t in directions d_1,d_2 with the property that the analytic continuations in sectors of their modified Borel transforms satisfy integro-differential Cauchy problems studied in the previous part. We then show that each $1-$sum $\mathbf{X}_j(t,z)$, $j=1,2$, solution of (1), can be expressed as a modified Laplace transform of a convergent series in $z \in D_R$ with coefficients be staircase distributions on the half-line $\mathbb{R}_+e^{id}$ containing singularities where $d \in (d_1,d_2)$. So that the difference $\mathbf{X}_2(t,z) - \mathbf{X}_1(t,z)$ can be expressed as a modified Laplace transform of a convergent series in $z \in D_R$ with coefficients be differences of staircase distributions which, from the result of the second part, are shown to define hyperfunctions on the half-line $\mathbb{R}_+e^{id}$.

2 Weighted Banach spaces of distributions

Most of the results given in this section come from O. Costin's paper [4] and R. Kuik's thesis [11]. However, we will give some more specific results needed in the sequel.

We define $\mathcal{D}(\mathbb{R}_+)$ to be the space of complex valued $C^\infty-$functions with compact support in $\mathbb{R}_+$, where $\mathbb{R}_+$ is the set of the positive real numbers $x > 0$. We also denote by $\mathcal{D}'(\mathbb{R}_+)$ the space of distributions on $\mathbb{R}_+$. For $f \in \mathcal{D}'(\mathbb{R}_+)$, we write $f^{(k)}$ the $k-$derivative of f in the sense of distribution, for $k \geq 0$, with the convention $f^{(0)} = f$.

Definition 1 *A distribution $f \in \mathcal{D}'(\mathbb{R}_+)$ is called staircase if f can be written in the form*

$$f = \sum_{k=0}^{\infty} (\Delta_k(f))^{(k)}, \tag{3}$$

for unique integrable functions $\Delta_k(f) \in L^1(\mathbb{R}_+)$ such that the support $\mathrm{supp}(\Delta_k(f))$ *of $\Delta_k(f)$ is in $[k, k+1]$ for all $k \geq 0$.*

Remark: Given a compact set $K \in \mathbb{R}_+$, a general distribution $\Lambda \in \mathcal{D}'(\mathbb{R}_+)$ can always be written as a $k-$derivative of a continuous function on $\mathbb{R}_+$ restricted to the test functions with support in K, where the k depends on the K, see [15].

Definition 2 *Let $\nu > 0$. We denote by $L_\nu(\mathbb{R}_+)$ the space of locally integrable functions $f \in L^1_{loc}(\mathbb{R}_+)$ such that*

$$||f||_\nu = \int_0^{+\infty} e^{-\nu t} |f(t)| dt < \infty.$$

The space $L_\nu(\mathbb{R}_+)$ with norm $||.||_\nu$ is a Banach space. Moreover, $(L_\nu(\mathbb{R}_+), ||.||_\nu)$ is an algebra for the convolution product of functions. In particular, for all $f, \tilde{f} \in L_\nu(\mathbb{R}_+)$, we have $f * \tilde{f} \in L_\nu(\mathbb{R}_+)$ and $||f * \tilde{f}||_\nu \leq ||f||_\nu ||\tilde{f}||_\nu$.

Definition 3 *Let $\nu > 0$. We denote by $\mathcal{D}'_\nu$ the space of staircase distributions f such that*

$$||f||_{\nu,d} = \sum_{k=0}^{+\infty} \nu^k ||\Delta_k(f)||_\nu < \infty.$$

The space $\mathcal{D}'_\nu$ with the norm $||.||_{\nu,d}$ is a Banach space.

Proposition 1 *Let $\nu_0 > 1$. If $f \in \mathcal{D}'_{\nu_0}$, then $f \in \mathcal{D}'_\nu$ for all $\nu \geq \nu_0$ and we have that $\nu \mapsto ||f||_{\nu,d}$ is a decreasing function on $[\nu_0, +\infty)$.*

Let $\mathcal{H}$ be the Heaviside one step function defined by $\mathcal{H}(t) = 1$, if $t \geq 0$ and $\mathcal{H}(t) = 0$, if $t < 0$. Let $\mathcal{P}$ the operator defined on distributions $T \in \mathcal{D}'(\mathbb{R}_+)$ by $\mathcal{P}T = \mathcal{H} * T$. For a subset $A \subset \mathbb{R}$, we denote by 1_A the function which is equal to 1 on A and 0 elsewhere.

The proofs of the following lemma 1 and 2 and propositions 2,3 and 4 are given in the section 2.3 of [4].

Lemma 1 *Let $k \geq 0$ and $f = F^{(k)} \in \mathcal{D}'(\mathbb{R}_+)$, where $F \in L^1(\mathbb{R}_+)$ and* $\mathrm{supp}(F) \subset [k, +\infty)$. *Then f is a staircase distribution and the decomposition of f has the following terms $\Delta_0 = \Delta_1 = \ldots = \Delta_{k-1} = 0$, $\Delta_k = F 1_{[k,k+1]}$ and for $n \geq 1$, $\Delta_{k+n} = G_n 1_{[k+n,k+n+1]}$ where $G_n = \mathcal{P}(G_{n-1} 1_{[k+n,+\infty)})$ and $G_0 = F$.*

Lemma 2 *Let $\nu_0 > 1$ and let f be as in lemma 1. Then, for all $\nu \geq \nu_0$, we have $||\Delta_{k+n}||_\nu \leq \nu^{-n} ||F||_\nu$, if $n = 0, 1, 2$ and for $n \geq 3$,*

$$||\Delta_{k+n}||_\nu \leq e^{(2\nu - n\nu)} \frac{n^{n-1}}{(n-1)!} ||F||_\nu.$$

Proposition 2 *Take $\nu > \nu_0 > 1$ and let $f \in L_{\nu_0}(\mathbb{R}_+)$. Then f belongs to $\mathcal{D}'_\nu$ and the decomposition (3) of f has the following terms $\Delta_n = G_n 1_{[n,n+1]}$ with $G_n = \mathcal{P}(G_{n-1}1_{[n,+\infty)})$ and $G_0 = f$, for $n \geq 0$. Moreover, there exists a constant C which depends only on ν_0, ν such that $||f||_{\nu,d} \leq C||f||_{\nu_0}$.*

Proposition 3 *Let $\mathbb{N}$ be the set of positive integers $n \geq 1$. The set $\mathcal{D}(\mathbb{R}_+ \setminus \mathbb{N})$ of $C^\infty-$functions with compact support in $\mathbb{R}_+ \setminus \mathbb{N}$ is dense in $\mathcal{D}'_\nu$.*

Proposition 4 *For all $f, \tilde{f} \in \mathcal{D}'_\nu$, we have $f * \tilde{f} \in \mathcal{D}'_\nu$. Moreover, there exists $C > 0$ (depending on ν) such that*

$$||f * \tilde{f}||_{\nu,d} \leq C||f||_{\nu,d}||\tilde{f}||_{\nu,d},$$

for all $f, \tilde{f} \in \mathcal{D}'_\nu$.

In the paper, we will sometimes denote by $\partial_r^{-1} f(r)$ the convolution $\mathcal{H} * f$ for all $f \in \mathcal{D}'_\nu$.

Proposition 5 *Let $\nu > 1$ be large enough such that $\nu e^{1-\nu} < 1$ and let h be a $C^\infty-$function on $\mathbb{R}_+$ such that there exist constants $C_h > 0, \mu > 0$ and $\rho > 1/\nu$ such that*

$$|h^{(q)}(s)| \leq C_h \frac{q!}{(\rho(s+\mu))^{(q+1)}} \tag{4}$$

for all $s \in \mathbb{R}_+$. Then, for all $f \in \mathcal{D}'_\nu$, we have $h(t)f(t) \in \mathcal{D}'_\nu$. Moreover, there exists a constant $C_{\mu,\rho,\nu} > 0$ (dependent on μ, ρ, ν) such that $||h(t)f(t)||_{\nu,d} \leq C_h C_{\mu,\rho,\nu}||f(t)||_{\nu,d}$ for all $f \in \mathcal{D}'_\nu$.

Proof The proof is inspired from Lemma 2.9.1 in [11]. Without lost of generality, we can assume that f has the following form $f(t) = \Delta_k^{(k)}(t)$ where $\Delta_k \in L^1(\mathbb{R}_+)$ with $\mathrm{supp}(\Delta_k) \in [k, k+1]$. Put $g_{k,j}(t) = h^{(k-j)}(t)\Delta_k(t)$. Then, $\mathrm{supp}(g_{k,j}(t)) \subset [k, k+1]$.

From the Leibniz formula, we get the identity

$$h(t)\Delta_k^{(k)}(t) = \sum_{j=0}^{k} \frac{k!}{(k-j)!} g_{k,j}^{(j)}(t).$$

On the other hand, one can rewrite $g_{k,j}^{(j)}(t) = (\mathcal{P}^{[k-j]}g_{k,j})^{(k)}$, where $\mathrm{supp}(\mathcal{P}^{[k-j]}g_{k,j}) \in [k, +\infty)$ and $\mathcal{P}^{[q]}$ denotes the qth iteration of $\mathcal{P}$.

Due to Lemma 1, $g_{k,j}^{(j)}$ can be written $g_{k,j}^{(j)} = \sum_{l=k}^{+\infty}(\tilde{\Delta}_{l,j})^{(l)}$, with $\tilde{\Delta}_{l,j} = G_{l,j}1_{[l,l+1]}$, $G_{l,j} = \mathcal{P}(G_{l-1,j}1_{[l,+\infty)})$ and $G_{k,j} = \mathcal{P}^{(k-j)}g_{k,j}$.

So that we get the following identity

$$h(t)\Delta_k^{(k)}(t) = (h(t)\Delta_k(t))^{(k)} + \sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!}\tilde{\Delta}_{k,j}^{(k)} + \sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} \sum_{l=k+1}^{+\infty} \tilde{\Delta}_{l,j}^{(l)} \tag{5}$$

First of all, we have

$$||(h(t)\Delta_k(t))^{(k)}||_{\nu,d} = \nu^k \int_0^{+\infty} |h(t)\Delta_k(t)|e^{-\nu t}dt \leq \frac{C_h}{\rho\mu}\nu^k||\Delta_k(t)||_\nu, \tag{6}$$

where $C_h > 0$ is given in (4).

From Lemma 2, we have the estimates

$$||\tilde{\Delta}_{k,j}||_\nu \leq ||\mathcal{P}^{[k-j]} g_{k,j}||_\nu \tag{7}$$

and

$$||\tilde{\Delta}_{l,j}||_\nu \leq e^{2\nu-(l-k)\nu} \frac{(l-k)^{(l-k-1)}}{(l-k-1)!} ||\mathcal{P}^{[k-j]} g_{k,j}||_\nu \tag{8}$$

for $l > k$. Now, we must estimate the quantity $||\mathcal{P}^{[k-j]} g_{k,j}||_\nu$. Using the Taylor formula with integral remainder and the hypothesis (4), we get

$$|\mathcal{P}^{[k-j]} g_{k,j}(t)| \leq C_h \frac{(k-j)!}{(k-j-1)!} \int_k^t \frac{(t-s)^{k-j-1}}{(\rho(s+\mu))^{1+(k-j)}} |\Delta_k(s)| ds.$$

So that, from the Fubini theorem and the identity

$$\int_s^{+\infty} e^{-\nu t} (t-s)^{k-j-1} dt = e^{-\nu s} \frac{(k-j)!}{\nu^{(k-j)}} \tag{9}$$

we deduce

$$\int_k^{+\infty} e^{-\nu t} |\mathcal{P}^{[k-j]} g_{k,j}(t)| dt \leq C_h \frac{(k-j)(k-j)!}{\nu^{(k-j)}} \int_k^{+\infty} e^{-\nu s} \frac{|\Delta_k(s)|}{(\rho(s+\mu))^{1+(k-j)}} ds$$

and hence

$$||\mathcal{P}^{[k-j]} g_{k,j}(t)||_\nu \leq \frac{C_h (k-j)(k-j)!}{\nu^{(k-j)} (\rho(k+\mu))^{1+k-j}} ||\Delta_k(s)||_\nu \tag{10}$$

From (7) and (10), we obtain

$$\sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} ||\tilde{\Delta}_{k,j}^{(k)}(t)||_{\nu,d} \leq C_h A_k \nu^k ||\Delta_k(s)||_\nu \tag{11}$$

where

$$A_k = \sum_{j=0}^{k-1} \frac{k!(k-j)}{j! \nu^{(k-j)} (\rho(k+\mu))^{1+k-j}}$$

Now, we have to estimate A_k. Due to the stirling formula, $k! \sim k^k e^{-k} (2\pi k)^{1/2}$ as k tends to infinity, there exists a universal constant $C_1 > 0$ such that

$$A_k \leq C_1 \frac{k^k}{(k+\mu)^k} \frac{1}{\rho(k+\mu)} (2\pi k)^{1/2} e^{-k} \sum_{j=0}^{k-1} \frac{(k-j)((k+\mu)\nu\rho)^j}{j!(\nu\rho)^k},$$

for all $k \geq 1$. Using the hypothesis $\nu\rho \geq 1$, we have

$$\sum_{j=0}^{k-1} (k-j) \frac{(k+\mu)^j}{j!} \frac{(\nu\rho)^j}{(\nu\rho)^k} \leq \sum_{j=0}^{k-1} (k-j) \frac{(k+\mu)^j}{j!} = k \frac{(k+\mu)^{k-1}}{(k-1)!} - \mu \sum_{j=0}^{k-2} \frac{(k+\mu)^j}{j!}$$

Using again the Stirling formula, we get a constant $C_{2,\mu} > 0$ (depending on μ) such that

$$k \frac{(k+\mu)^{k-1}}{(k-1)!} \leq C_{2,\mu} k^{1/2} e^k$$

for all $k \geq 1$. Moreover,

$$\mu \sum_{j=0}^{k-2} \frac{(k+\mu)^j}{j!} \leq \mu \sum_{j=0}^{+\infty} \frac{(k+\mu)^j}{j!} = \mu e^{k+\mu}$$

So that

$$\sum_{j=0}^{k-1} (k-j) \frac{(k+\mu)^j}{j!} \frac{(\nu\rho)^j}{(\nu\rho)^k} \leq (C_{2,\mu} k^{1/2} + \mu e^{\mu}) e^k,$$

for all $k \geq 1$. Finally, we obtain a constant $C_{1,\mu,\rho} > 0$ depending only on ρ, μ such that

$$A_k \leq C_{1,\mu,\rho}, \tag{12}$$

for all $k \geq 1$.

From (8) and (10), we have

$$\sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} \sum_{l=k+1}^{+\infty} ||\tilde{\Delta}_{l,j}^{(l)}||_{\nu,d} \leq C_h A_k ||\Delta_k(s)||_\nu \nu^k B_{k,\nu} \tag{13}$$

where

$$B_{k,\nu} = \sum_{l=k+1}^{+\infty} \nu^{l-k} e^{2\nu-(l-k)\nu} \frac{(l-k)^{(l-k-1)}}{(l-k-1)!} = \sum_{h=1}^{\infty} \nu^h e^{2\nu-h\nu} \frac{h^{(h-1)}}{(h-1)!}$$

Due to the Stirling formula, there exists a universal constant $C_3 > 0$ such that

$$B_{k,\nu} \leq C_3 \sum_{h=2}^{+\infty} \nu^h e^{h(1-\nu)+2\nu} (\frac{h}{h-1})^{h-1} \frac{1}{(2\pi(h-1))^{1/2}} \leq C_3 e^{2\nu} \sum_{h=2}^{+\infty} (\nu e^{1-\nu} \frac{h}{h-1})^h$$

and the latter series converges because of the hypothesis $\nu e^{1-\nu} < 1$. So that there exists $C_{1,\nu} > 0$ depending only on ν such that

$$B_{k,\nu} \leq C_{1,\nu}, \tag{14}$$

for all $k \geq 1$.

Finally, from the equality (5) and estimates (6), (11), (12), (13) and (14), we get a constant $C_{\mu,\rho,\nu} > 0$ depending only on μ, ρ, ν such that $||h(t)\Delta_k^{(k)}(t)||_\nu \leq C_h C_{\mu,\rho,\nu} ||\Delta_k^{(k)}(t)||_\nu$ for all $k \geq 0$. □

3 Analytic continuation of holomorphic functions across a half-line, hyperfunctions

In the following proposition we construct hyperfunctions as the jump on a half line L in $\mathbb{C}$ of holomorphic functions near L.

Proposition 6 *Let θ_1^-, θ_1 and θ_1^+ three real numbers such that $\theta_1^- < \theta_1 < \theta_1^+$. Consider the following sets in $\mathbb{C}$,*

$$\mathcal{W} = \{t \in \mathbb{C}^* / \arg(t) \in (\theta_1^-, \theta_1^+), \quad t \neq k e^{i\theta_1}, \quad \forall k \in \mathbb{N} \},$$

$$\mathcal{W}^+ = \{t \in \mathcal{W} / \arg(t) \in (\theta_1, \theta_1^+)\} \quad , \quad \mathcal{W}^- = \{t \in \mathcal{W} / \arg(t) \in (\theta_1^-, \theta_1)\}.$$

Let $D_1 \subset \mathbb{C}$ be the open disc with center 0 and radius 1. We consider holomorphic functions $f^- : \mathcal{W}^- \cup D_1 \to \mathbb{C}$ and $f^+ : \mathcal{W}^+ \cup D_1 \to \mathbb{C}$ such that $f^-(t) = f^+(t)$ for all $t \in D_1$. We make the following assumptions.

i) *Let $\nu > 1$. For all $\theta \in (\theta_1^-, \theta_1)$ the functions $r \mapsto f^-(re^{i\theta})$ belong to $\mathcal{D}'_\nu$. For all $\theta \in (\theta_1, \theta_1^+)$ the functions $r \mapsto f^+(re^{i\theta})$ belong to $\mathcal{D}'_\nu$*

ii) *There exists distributions $r \mapsto f^-(re^{i\theta_1})$ and $r \mapsto f^+(re^{i\theta_1})$ which belong to $\mathcal{D}'_\nu$ such that*
a)

$$\lim_{\theta \to \theta_1, \theta < \theta_1} ||f^-(re^{i\theta_1}) - f^-(re^{i\theta})||_{\nu,d} = 0$$

b)

$$\lim_{\theta \to \theta_1, \theta > \theta_1} ||f^+(re^{i\theta_1}) - f^+(re^{i\theta})||_{\nu,d} = 0.$$

Then, the staircase distribution

$$r \mapsto f^{+-}(re^{i\theta_1}) = f^+(re^{i\theta_1}) - f^-(re^{i\theta_1})$$

defines an hyperfunction on the half-line $\mathbb{R}_+ e^{i\theta_1} = \{t \in \mathbb{C}^* : t = re^{i\theta_1}, r > 0\}$. More precisely, if $f^{+-} = 0$ in $\mathcal{D}'_\nu$, then for all integers $l \geq 1$, the function $f^- : \mathcal{W}^- \cup D_1 \to \mathbb{C}$ (resp. $f^+ : \mathcal{W}^+ \cup D_1 \to \mathbb{C}$) can be analytically continued across $(le^{i\theta_1}, (l+1)e^{i\theta_1})$ into $\mathcal{W}^+ \cup D_1$ (resp. $\mathcal{W}^- \cup D_1$).

Proof The proof is inspired from Lemma 2.6.5 in [11], see also Theorem 3.1.12 in [7]. Assume that the hypotheses **i)** and **ii)** hold. Assume that $f^{+-} = 0$ in $\mathcal{D}'_\nu$. In the following, we denote by $f(re^{i\theta_1})$ the distribution $f^+(re^{i\theta_1}) = f^-(re^{i\theta_1})$.

First of all, we notice that for all $N \geq 1$, all $\theta \in (\theta_1^-, \theta_1)$,

$$\partial_t^{-N}(f^-(t)) = e^{iN\theta} \partial_r^{-N}(f^-(re^{i\theta})), \tag{15}$$

for all $t = re^{i\theta} \in \mathcal{W}^-$, $r > 0$. In the same way, for all $N \geq 1$, all $\theta \in (\theta_1, \theta_1^+)$,

$$\partial_t^{-N}(f^+(t)) = e^{iN\theta} \partial_r^{-N}(f^+(re^{i\theta})), \tag{16}$$

for all $t = re^{i\theta} \in \mathcal{W}^+$, $r > 0$.

In the first part of the proof, we will show that for $l \geq 1$, $N \geq l + 2$,

$$\lim_{\theta \to \theta_1, \theta < \theta_1} \sup_{r \in (l, l+1)} |e^{iN\theta} \partial_r^{-N}(f^-(re^{i\theta})) - e^{iN\theta_1} \partial_r^{-N}(f(re^{i\theta_1}))| = 0. \tag{17}$$

and

$$\lim_{\theta \to \theta_1, \theta > \theta_1} \sup_{r \in (l, l+1)} |e^{iN\theta} \partial_r^{-N}(f^+(re^{i\theta})) - e^{iN\theta_1} \partial_r^{-N}(f(re^{i\theta_1}))| = 0. \tag{18}$$

We will only give the proof for (17). The other limit (18) can be shown in the same manner. Indeed, from the decomposition (3), for $r \in (l, l+1)$ and $\theta \in (\theta_1^-, \theta_1)$, we write $f^-(re^{i\theta}) = (\Delta_{\theta,l}(r))^{(l)}$ and $f(re^{i\theta_1}) = (\Delta_{\theta_1,l}(r))^{(l)}$ where $\Delta_{\theta,l}(r), \Delta_{\theta_1,l}(r) \in L^1(\mathbb{R}_+)$ with $\mathrm{supp}(\Delta_{\theta,l})$ and $\mathrm{supp}(\Delta_{\theta_1,l})$ in $[l, l+1]$.

We deduce that for $N \geq l+2$, the functions $r \mapsto \partial_r^{-N}(f^-(re^{i\theta}))$ and $r \mapsto \partial_r^{-N}(f(re^{i\theta_1}))$ are continuous on $(l, l+1)$. Moreover, we have the estimates

$$|\partial_r^{-N}(f^-(re^{i\theta})) - \partial_r^{-N}(f(re^{i\theta_1}))| \leq e^{\nu(l+1)} \int_0^r e^{-\nu u} |\partial_u^{-N+1+l}(\Delta_{\theta,l}(u) - \Delta_{\theta_1,l}(u))| du$$

for all $r \in (l, l+1)$. So that, due to Definition 2 and the fact that $||\mathcal{H}||_\nu = 1/\nu$, we get

$$|\partial_r^{-N}(f^-(re^{i\theta})) - \partial_r^{-N}(f(re^{i\theta_1}))| \leq e^{\nu(l+1)} \nu^{1-N} ||\Delta_{\theta,l} - \Delta_{\theta_1,l}||_{\nu,d}. \tag{19}$$

In the same way, we also get

$$|\partial_r^{-N}(f^-(re^{i\theta}))| \leq e^{\nu(l+1)} \nu^{1-N} ||\Delta_{\theta,l}||_{\nu,d} \tag{20}$$

for all $r \in (l, l+1)$.

On the other hand, the following inequality holds

$$\begin{aligned} |e^{iN\theta}\partial_r^{-N}(f^-(re^{i\theta})) - e^{iN\theta_1}\partial_r^{-N}(f(re^{i\theta_1}))| \leq {} & |e^{iN\theta} - e^{iN\theta_1}||\partial_r^{-N}(f^-(re^{i\theta}))| \\ & + |\partial_r^{-N}(f^-(re^{i\theta})) - \partial_r^{-N}(f(re^{i\theta_1}))| \end{aligned} \tag{21}$$

for $N \geq l+2$ and $r \in (l, l+1)$.

From the hypothesis **ii),a)**, we have in particular that $||\Delta_{\theta,l} - \Delta_{\theta_1,l}||_{\nu,d}$ tends to 0 as $\theta \to \theta_1$ and $\theta < \theta_1$. So that, finally, the limit (17) is a consequence of (19), (20) and (21).

In the second part of the proof we will construct analytic continuations of f^- and f^+ on $\mathcal{W}$. Let $l \geq 1$ and $N \geq l+2$. We define a function h_N on the domain

$$\mathcal{D}_l = \mathcal{W}^- \cup D_1 \cup (le^{i\theta_1}, (l+1)e^{i\theta_1}) \cup \mathcal{W}^+$$

by

$$h_N(t) = \begin{cases} \partial_t^{-N}(f^-(t)) & \text{if } t \in \mathcal{W}^- \cup D_1 \\ e^{iN\theta_1}\partial_r^{-N}(f(re^{i\theta_1})) & \text{if } t = re^{i\theta_1} \in (le^{i\theta_1}, (l+1)e^{i\theta_1}) \\ \partial_t^{-N}(f^+(t)) & \text{if } t \in \mathcal{W}^- \cup \mathcal{W}^+ \cup D_1 \end{cases}$$

The function $h_N(t)$ is analytic on $D_1 \cup \mathcal{W}^- \cup \mathcal{W}^+$ as iterated integrals of the analytic functions f^- and f^+. Moreover, from the expressions (15), (16) and the limits (17), (18), we deduce that the function $h_N(t)$ is continuous for all $t \in (le^{i\theta_1}, (l+1)e^{i\theta_1})$ in $\mathcal{D}_l$.

Now we recall a well known lemma from complex analysis concerning analytic continuation across analytic arcs (see [12], p.280).

Lemma 3 *Let U^- and U^+ two open connected sets in $\mathbb{C}$ such that $\overline{U^-} \cap \overline{U^+}$ contains an open segment $I = (\alpha, \beta) \in \mathbb{C}$. Let f be a function defined on $U^- \cup I \cup U^+$. If f is holomorphic on U^+ and U^- and continuous on I, then the function f is holomorphic on $U^- \cup I \cup U^+$*

From this latter lemma, we conclude that the function $h_N(t)$ is in fact holomorphic on $\mathcal{D}_l$. As a result, the function $\partial_t^{-N}(f^-(t))$ (resp. $\partial_t^{-N}(f^+(t))$) can be analytically continued across the segment $(le^{i\theta_1}, (l+1)e^{i\theta_1})$ from $\mathcal{W}^- \cup D_1$ into $\mathcal{W}^+ \cup D_1$ (resp. from $\mathcal{W}^+ \cup D_1$ into $\mathcal{W}^- \cup D_1$). It follows that, for all integers $l \geq 1$, the function $f^- : \mathcal{W}^- \cup D_1 \to \mathbb{C}$ (resp. $f^+ : \mathcal{W}^+ \cup D_1 \to \mathbb{C}$) can be analytically continued across $(le^{i\theta_1}, (l+1)e^{i\theta_1})$ into $\mathcal{W}^+ \cup D_1$ (resp. $\mathcal{W}^- \cup D_1$). This gives the result. $\square$

4 Modified Laplace transform on the spaces $\mathcal{D}'_\nu$

Definition 4 *Let* $e(z)$ *be a holomorphic function on the sector* $S_{\pi/2} = \{z \in \mathbb{C}^* : |\arg(z)| < \pi/2\}$ *such that*
(e1) $\int_0^{x_0} |e(xe^{i\tau})|/x dx$ *exists, for all* $x_0 > 0$ *and* $|\tau| < \pi$.
(e2) For all $\delta > 0$*, there exist* $C, K > 0$ *such that* $|e(z)| \leq Ce^{-|z|/K}$ *for all* z *such that* $|\arg(z)| \leq (\pi - \delta)/2$.

Definition 5 *Let* $f(r) \in \mathcal{D}'_\nu$ *and consider its decomposition (3) as a staircase distribution. The Laplace transform of* f *in the direction* $\theta \in [-\pi, \pi)$ *is defined as*

$$\mathcal{L}_\theta(f)(t) = \sum_{k=0}^{+\infty} 2 \int_0^{+\infty} (-1)^k \Delta_k(r) \partial_r^k (\frac{e(r^2 e^{i\theta}/t)}{r}) dr$$

where ∂_r^k *is the* $k-$*derivative with respect to* r.

Proposition 7 *Let* $f(r) \in \mathcal{D}'_\nu$ *for* $\nu > 1$ *and* $\theta \in [-\pi, \pi)$*. Then, there exist sufficiently small* $\delta_\theta > 0$ *such that the function* $\mathcal{L}_\theta(f)(t)$ *is holomorphic on sectors* $S_{\theta,\delta_\theta,\rho} = \{t \in \mathbb{C}^* : |\theta - \arg(t)| < \delta_\theta, |t| < \rho\}$ *for all* $\rho > 0$*. Moreover, there exists* $C > 0$ *(depending on the sector* $S_{\theta,\delta_\theta,\rho}$*) such that for all* $t \in S_{\theta,\delta_\theta,\rho}$ *we have*

$$|\mathcal{L}_\theta(f)(t)| \leq C||f||_{\nu,d}, \tag{22}$$

for all $f \in \mathcal{D}'_\nu$.

Proof From the Cauchy formula, for all $k \geq 1$, we have

$$\partial_u^k (\frac{e(u^2/t)}{u}) = \frac{k!}{2i\pi} \int_{C_R(u)} \frac{e(v^2/t)}{v(v-u)^{k+1}} dv,$$

where $C_R(u)$ is a circle, centered at u with radius $R > 0$, which belong to the domain $\{v \in \mathbb{C}^* : 2\arg(v) - \arg(t) \in (-\pi/2, \pi/2)\}$. This latter integral does not depend on the radius $R > 0$. In polar coordinates, $u = re^{i\theta/2}$, we get the equality

$$\partial_r^k (\frac{e(r^2 e^{i\theta}/t)}{re^{i\theta/2}}) = (e^{i\theta/2})^k \partial_u^k (\frac{e(u^2/t)}{u}) = \frac{k!(e^{i\theta/2})^k}{2i\pi} \int_0^{2\pi} \frac{e((re^{i\theta/2} + Re^{i\tau})^2/t) Rie^{i\tau}}{(re^{i\theta/2} + Re^{i\tau})(Re^{i\tau})^{k+1}} d\tau$$

In the following, we show that there exist small enough constants $C_\theta > 0$ and $\delta_\theta > 0$ such that for all $t \in \mathbb{C}^*$ with $\arg(t) \in (\theta - \delta_\theta, \theta + \delta_\theta)$ the following property is satisfied

$$2\arg(re^{i\theta/2} + C_\theta k e^{i\tau}) - \arg(t) \in (-\pi/2, \pi/2) \tag{23}$$

for all $k \geq 1$, for all $r \in [k, k+1]$ and all $\tau \in [-\pi, \pi)$. Indeed, if one writes $r = \mu_1 k + \mu_2 (k+1)$, with $\mu_1 + \mu_2 = 1$ and $\mu_1, \mu_2 \geq 0$, one has

$$re^{i\theta/2} + C_\theta k e^{i\tau} = k(e^{i\theta/2} + C_\theta e^{\tau}) + \mu_2 e^{i\theta/2},$$

so that there exist $d_\theta > 0$ such that

$$\frac{\theta}{2} - d_\theta \leq \arg(re^{i\theta/2} + C_\theta k e^{i\tau}) \leq \frac{\theta}{2} + d_\theta$$

with $d_\theta < \pi/8$, if $C_\theta > 0$ is chosen small enough. Therefore, the property (23) follows.
In what follows, we choose the radius R to be of the form $R = R_\theta = C_\theta k$. We then have

$$\frac{1}{|re^{i\theta/2} + R_\theta e^{i\tau}|} \leq \frac{1}{|r - R_\theta|} \leq \frac{1}{(1-C_\theta)k}$$

and

$$|e((re^{i\theta/2} + R_\theta e^{i\tau})^2/t)| \leq Ce^{-|re^{i\theta/2}+R_\theta e^{i\tau}|^2/K|t|} \leq Ce^{-(1-C_\theta)^2k^2/K|t|}$$

for all $r \in [k, k+1]$. Finally, using the Stirling formula, we deduce that there exist a constant $C > 0$, small enough constants $C_\theta > 0$ and $\delta_\theta > 0$ such that for all $t \in \mathbb{C}^*$ with $\arg(t) \in (\theta - \delta_\theta, \theta + \delta_\theta)$ and $|t| < \rho$, $\rho > 0$, the following inequality is satisfied for all $k \geq 1$,

$$\begin{aligned} A_{t,k,\theta} &= |\int_0^{+\infty} \Delta_k(r)\partial_r^k(\frac{e(r^2e^{i\theta}/t)}{r})dr| \\ &\leq C\int_k^{k+1} e^{-\nu r}|\Delta_k(r)|e^{\nu(k+1)}k^k k^{1/2}e^{-k}\frac{1}{(1-C_\theta)kC_\theta^k k^k}e^{-(1-C_\theta)^2k^2/K|t|}dr \\ &\leq \frac{Ce^{\nu(k+1)}e^{-k}}{(1-C_\theta)k^{1/2}(\nu C_\theta)^k}e^{-(1-C_\theta)^2k^2/K|t|}||\Delta_k(r)||_\nu \nu^k \qquad (24) \end{aligned}$$

So that there exists a constant $C > 0$ depending on ρ and C_θ such that

$$A_{t,k,\theta} \leq C||\Delta_k(r)||_\nu \nu^k,$$

for all $k \geq 1$ and all $t \in S_{\theta,\delta_\theta,\rho}$. This latter inequality shows that the function $\mathcal{L}_\theta(f)(t)$ is holomorphic on the sector $S_{\theta,\delta_\theta,\rho}$, for all $\rho > 0$ and that the inequality (22) hold. $\square$

Proposition 8 *Let $\nu > \nu_0 > 1$ and $f \in L_{\nu_0}(\mathbb{R}_+)$. Then, $f \in \mathcal{D}'_\nu$ and the function $\mathcal{L}_\theta(f)(t)$ coincides on its domain of definition with the generalized Laplace transform of f in direction $\theta \in [-\pi, \pi)$, (see [1], section 5.5),*

$$T_\theta(f)(t) = 2\int_0^\infty f(r)\frac{e(r^2e^{i\theta}/t)}{r}dr,$$

which is defined on every sector $S_{\theta,\delta,\rho} = \{t \in \mathbb{C}^* : \theta - \arg(t) \in (-\delta, \delta), |t| < \rho\}$, for every $\rho > 0$ and $0 < \delta < \pi/2$.

Proof From Proposition 2, we have that $f \in L_{\nu_0}(\mathbb{R}_+)$ is a staircase distribution that belongs to $\mathcal{D}'_\nu$ whose decomposition (3) has the following form $\Delta_n = G_n 1_{[n,n+1]}$ with $G_n = \mathcal{P}(G_{n-1}1_{[n,+\infty)})$ and $G_0 = f$, for $n \geq 0$. In the following, we will put $\psi(r) = e(r^2e^{i\theta}/t)/r$. We have to compute the integral $(-1)^k\int_k^{k+1}\Delta_k(r)\partial_r^k\psi(r)dr$.
If $k = 0$, we get that $\int_0^1 \Delta_0(r)\psi(r)dr = \int_0^1 f(r)\psi(r)dr$.
If $k = 1$, integrating by parts, we get

$$-\int_1^2 \Delta_1(r)\partial_r\psi(r)dr = -\int_1^2 G_1(r)\partial_r\psi(r)dr = [G_1(r)\psi(r)]_1^2 + \int_1^2 f(r)\psi(r)dr,$$

and more generally, we obtain that

$$\begin{aligned} (25)\quad (-1)^k\int_k^{k+1}\Delta_k(r)\partial_r^k\psi(r)dr &= (-1)^k\int_k^{k+1}\Delta_k(r)\partial_r^k\psi(r)dr = (-1)^k[G_k(r)\partial_r^{k-1}\psi(r)]_k^{k+1} \\ &+ (-1)^{k-1}[G_{k-1}(r)\partial_r^{k-2}\psi(r)]_k^{k+1} + \ldots + (-1)[G_1(r)\psi(r)]_k^{k+1} + \int_k^{k+1} f(r)\psi(r)dr \end{aligned}$$

Moreover, as a telescopic sum, we notice that

$$\sum_{k=1}^{\infty}(-1)^k[G_k(r)\partial_r^{k-1}\psi(r)]_k^{k+1} = \sum_{k=1} G_k(k)(\partial_r^{k-1}\psi)(k). \tag{26}$$

But, by construction, we have that $G_k(k) = 0$, for all $k \geq 1$, so that this latter sum (26) is equal to zero.

In conclusion, from (25) and (26), we deduce that

$$\sum_{k=0}^{+\infty}(-1)^k \int_k^{k+1} \Delta_k(r)\partial_r^k\psi(r)dr = \int_0^{+\infty} f(r)\psi(r)dr.$$

□

5 Cauchy problems in Banach spaces of distributions $\mathcal{D}'_\nu$

We consider following the integro-differential equation

$$\partial_z^S W(r,z) = \sum_{k_1=0}^{S-1} h_{k_1}(r)\int_0^r\int_0^{r_1} H_{k_1}(r_1-r_2,z)\partial_z^{k_1}W(r_2,z)dr_2dr_1 + d(r,z) \tag{27}$$

with initial conditions

$$(\partial_z^j W)(r,0) = W_j(r) \quad , \quad 0 \leq j \leq S-1. \tag{28}$$

Let $\nu > 1$ be a real number such that $\nu e^{1-\nu} < 1$ and $S \geq 2$ an integer. *We assume that*

(d1) For all $0 \leq j \leq S-1$, $W_j(r) \in \mathcal{D}'_\nu$.
(d2) For all $0 \leq k_1 \leq S-1$,

$$H_{k_1}(r,z) = \sum_{q=0}^{+\infty} H_{k_1,q}(r)z^q, \quad , \quad d(r,z) = \sum_{q=0}^{+\infty} d_q(r)z^q$$

are holomorphic functions in z on a disc D_R, $R > 0$, with values in $\mathcal{D}'_\nu$.
(d3) There exist $\mu, \kappa > 1/\nu$, such that for all $0 \leq k_1 \leq S-1$, $h_{k_1}(r)$ are $C^\infty-$functions on $(0,+\infty)$ such that

$$|h_{k_1}^{(q)}(r)| \leq C_{k_1}\frac{q!}{(\kappa(r+\mu))^{1+q}},$$

for a constant $C_{k_1} > 0$, for all $q \geq 0$ and all $r \in (0,+\infty)$.

The following lemma is a consequence of the hypotheses **(d1)**, **(d2)**, **(d3)** and Propositions 4, 5.

Lemma 4 *Consider a formal series*

$$W(r,z) = \sum_{k\geq 0} W_k(r)\frac{z^k}{k!},$$

where $W_k(r) \in \mathcal{D}'_{\nu}$, for all $k \geq 0$. Then, the series $W(r,z)$ is a solution of (27), for initial data $W_j(r) \in \mathcal{D}'_{\nu}$, $0 \leq j \leq S-1$, if and only if the following recursion is satisfied

$$\frac{W_{k+S}(r)}{k!} = \sum_{k_1=0}^{S-1} \sum_{q_1+q_2=k} h_{k_1}(r)\partial_r^{-1}(H_{k_1,q_1}(r) * \frac{W_{q_2+k_1}(r)}{q_2!}) + d_k(r), \tag{29}$$

for all $k \geq 0$. In particular, for given initial data $W_j(r) \in \mathcal{D}'_{\nu}$, $0 \leq j \leq S-1$, there exists a unique formal series $W(r,z)$ of the latter form, solution of (27) and (28).

In the following, we give some notations. Let $w_j^{\nu,d} = ||W_j(r)||_{\nu,d}$, $0 \leq j \leq S-1$ and

$$|H|_{k_1}^{\nu,d}(z) = \sum_{q \geq 0} ||H_{k_1,q}(r)||_{\nu,d} z^q \quad , \quad |d|^{\nu,d}(z) = \sum_{q \geq 0} ||d_q(r)||_{\nu,d} z^q.$$

Proposition 9 *There exist two constants $R', C > 0$ (depending on C_{k_1}, κ, μ, ν, $|H|_{k_1}^{\nu,d}(z)$, $0 \leq k_1 \leq S-1$), such that*
i) *The formal series $W(r,z)$, solution of the problem (27) with (28), constructed in Lemma 4, is a convergent series in z on $D_{R'}$, with coefficients in $\mathcal{D}'_{\nu}$.*
ii) *The following estimates hold*

$$\sum_{k=0}^{+\infty} \frac{||W_k(r)||_{\nu,d}}{k!}(R')^k \leq C(\sum_{j=0}^{S-1} w_j^{\nu,d} + |d|^{\nu,d}(R)). \tag{30}$$

Proof From the recursion (29), the hypotheses **(d1)**, **(d2)**, **(d3)** and Propositions 4, 5, we get that the distributions $W_k(r)$, $k \geq 0$, defined in (29) all belong to $\mathcal{D}_{\nu}$. Moreover, if we put $w_k^{\nu,d} = ||W_k(r)||_{\nu,d}$, we get a constant $C^0 > 0$ (depending only on C_{k_1}, κ, μ, ν, $0 \leq k_1 \leq S-1$) such that the following inequalities hold

$$\frac{w_{k+S}^{\nu,d}}{k!} \leq \sum_{k_1=0}^{S-1} \sum_{q_1+q_2=k} C^0 ||H_{k_1,q_1}(r)||_{\nu,d} \frac{w_{q_2+k_1}^{\nu,d}}{q_2!} + ||d_k(r)||_{\nu,d} \tag{31}$$

On the other hand, consider the linear differential equation

$$\partial_z^S V(z) = \sum_{k_1=0}^{S-1} C^0 |H|_{k_1}^{\nu,d}(z) \partial_z^{k_1} V(z) + |d|^{\nu,d}(z) \tag{32}$$

with initial conditions

$$(\partial_z^j V)(0) = w_j^{\nu,d} \quad , \quad 0 \leq j \leq S-1. \tag{33}$$

From the classical *local* theory of linear differential equations in the complex domain, we know that the problem (32), (33) has a unique solution $V(z) = \sum_{p \geq 0} V_p z^p/p!$, holomorphic on a small disc $D_{R'}$, where $R' \leq R$ depends on the coefficients $C^0|H|_{k_1}^{\nu,d}(z)$, $0 \leq k_1 \leq S-1$. Moreover, there exists a constant $C > 0$, which depends on the coefficients $C^0|H|_{k_1}^{\nu,d}(z)$, $0 \leq k_1 \leq S-1$, such that the estimates hold,

$$\sum_{p \geq 0} \frac{|V_p| R'^p}{p!} \leq C(\sum_{j=0}^{S-1} w_j^{\nu,d} + |d|^{\nu,d}(R)) \tag{34}$$

Now, a formal series $V(z) = \sum_{k \geq 0} V_k z^k/k!$ is a solution of (32) and (33) if and only if the following equalities hold

$$\frac{V_{k+S}}{k!} = \sum_{k_1=0}^{S-1} \sum_{q_1+q_2=k} C^0 ||H_{k_1,q_1}(r)||_{\nu,d} \frac{V_{q_2+k_1}}{q_2!} + ||d_k(r)||_{\nu,d} \tag{35}$$

for all $k \geq 0$ and

$$V_j = w_j^{\nu,d} \quad , \quad 0 \leq j \leq S-1. \tag{36}$$

We deduce from (31) and (35) with (36), that

$$w_p^{\nu,d} \leq V_p,$$

for all $p \geq 0$. So that, for small enough $R' > 0$, we get that the series $\sum_{p \geq 0} w_p^{\nu,d} z^p/p!$ converges on a disc $D_{R'}$ and the formal solution $W(r,z)$ constructed in Lemma 4, is a convergent series in z on $D_{R'}$ with values in $\mathcal{D}'_\nu$. Moreover, from (34), we have that

$$\sum_{k=0}^{+\infty} \frac{||W_k(r)||_{\nu,d}}{k!} (R')^k = \sum_{p \geq 0} w_p^{\nu,d} \frac{R'^p}{p!} \leq \sum_{p \geq 0} V_p \frac{R'^p}{p!} \leq C(\sum_{j=0}^{S-1} w_j^{\nu,d} + |d|^{\nu,d}(R))$$

for a constant $C > 0$ which depends on the coefficients $C^0 |H|_{k_1}^{\nu,d}(z)$, $0 \leq k_1 \leq S-1$. □

6 Cauchy problems in weighted Banach spaces of holomorphic functions

We denote by $\bar{S}_{\tau,\delta} = \{t \in \mathbb{C}^* : |\tau - \arg(t)| \leq \delta\}$ a closed sector with opening $0 < 2\delta < \pi$ in $\mathbb{C}^*$ and by $\bar{D}_1$ the closed disc of radius 1 centered at 0. We denote by $\mathcal{A}_{\tau,\delta,\nu}$ the Banach space of holomorphic functions $\mathbf{G}(t)$ on $S_{\tau,\delta} \cup D_1$ continuous on $\bar{S}_{\tau,\delta} \cup \bar{D}_1$ such that

$$||\mathbf{G}(t)||_{\tau,\delta,\nu} := M_0 \sup_{t \in \bar{S}_{\tau,\delta} \cup \bar{D}_1} (1+|t|^2) e^{-\nu|t|} |\mathbf{G}(t)| < +\infty,$$

where $\nu > 1$ and $M_0 > 1$ is a constant such that

$$\int_0^{|t|} \frac{ds}{(1+s^2)(1+(|t|-s)^2)} \leq \frac{M_0}{(1+|t|^2)},$$

for all $t \in \bar{S}_{\tau,\delta} \cup \bar{D}_1$. In the following, we will need a lemma from [5], Corollary 4.9.

Lemma 5 *Let $\mathbf{F}(t), \mathbf{G}(t) \in \mathcal{A}_{\tau,\delta,\nu}$. Then, the function $t \mapsto \int_0^t \mathbf{F}(t-\tau)\mathbf{G}(\tau)d\tau$ belong to $\mathcal{A}_{\tau,\delta,\nu}$ and the following inequality holds*

$$|| \int_0^t \mathbf{F}(t-\tau)\mathbf{G}(\tau)d\tau ||_{\tau,\delta,\nu} \leq ||\mathbf{F}(t)||_{\tau,\delta,\nu} ||\mathbf{G}(t)||_{\tau,\delta,\nu}.$$

The following lemma will be usefull.

Lemma 6 *Let $\mathbf{F}(t)$ and $\mathbf{G}(t)$ be two holomorphic functions on the domain $S_{2\tau,2\delta} \cup D_1$ and continuous on $\bar{S}_{2\tau,2\delta} \cup \bar{D}_1$ such that the functions $t \mapsto \mathbf{F}(t^2)$ and $t \mapsto \mathbf{G}(t^2)$ belong to $\mathcal{A}_{\tau,\delta,\nu}$. Then, there exists a function $\mathbf{H}(t)$ holomorphic on $S_{2\tau,2\delta} \cup D_1$ and continuous on $\bar{S}_{2\tau,2\delta} \cup \bar{D}_1$ such that $\mathbf{H}(t^2) = \int_0^t \int_0^{t_1} \mathbf{F}((t_1-t_2)^2)\mathbf{G}(t_2^2)dt_2dt_1$ belong to $\mathcal{A}_{\tau,\delta,\nu}$.*

Proof Taking the parametrization $t_2 = s_2 t_1$ for $s_2 \in [0,1]$ and $t_1 = s_1 t$ for $s_1 \in [0,1]$, in the integral we get

$$\int_0^t \int_0^{t_1} \mathbf{F}((t_1-t_2)^2)\mathbf{G}(t_2^2)dt_2dt_1 = t^2 \int_0^1 \int_0^1 s_1 F(t^2 s_1^2(1-s_2)^2) G(t^2 s_1^2 s_2^2) ds_2 ds_1$$

So that we can choose the function $\mathbf{H}(t)$ to be

$$\mathbf{H}(t) = t \int_0^1 \int_0^1 s_1 F(t s_1^2(1-s_2)^2) G(t s_1^2 s_2^2) ds_2 ds_1.$$

The fact that $t \mapsto \mathbf{H}(t^2)$ belong to $\mathcal{A}_{\tau,\delta,\nu}$ follows from Lemma 5. □

We consider the following integro-differential equation

$$\partial_z^S \mathbf{W}(t^2,z) = \sum_{k_1=0}^{S-1} \mathbf{h}_{k_1}(t^2) \int_0^t \int_0^{t_1} \mathbf{H}_{k_1}((t_1-t_2)^2,z)\partial_z^{k_1}\mathbf{W}(t_2^2,z)dt_2dt_1 \tag{37}$$

with initial conditions

$$(\partial_z^j \mathbf{W})(t^2,0) = \mathbf{W}_j(t^2) \quad , \quad 0 \le j \le S-1. \tag{38}$$

We assume that

(h1) For all $0 \le j \le S-1$, the function $\mathbf{W}_j(t)$ is holomorphic on $S_{2\tau,2\delta} \cup D_1$ and continuous on $\bar{S}_{2\tau,2\delta} \cup \bar{D}_1$ such that the functions $t \mapsto \mathbf{W}_j(t^2)$ belong to $\mathcal{A}_{\tau,\delta,\nu}$.
(h2) For all $0 \le k_1 \le S-1$,

$$\mathbf{H}_{k_1}(t^2,z) = \sum_{q=0}^{+\infty} \mathbf{H}_{k_1,q}(t^2) z^q,$$

are convergent series in z on a disc D_R with coefficients $t \mapsto \mathbf{H}_{k_1,q}(t^2)$ that belong to $\mathcal{A}_{\tau,\delta,\nu}$ such that $\mathbf{H}_{k_1,q}(t)$ are holomorphic functions on $S_{2\tau,2\delta} \cup D_1$ and continuous on $\bar{S}_{2\tau,2\delta} \cup \bar{D}_1$.
(h3) For all $0 \le k_1 \le S-1$, the functions $\mathbf{h}_{k_1}(t)$ are holomorphic functions on $S_{2\tau,2\delta} \cup D_1$ and continuous on $\bar{S}_{2\tau,2\delta} \cup \bar{D}_1$ such that $t \mapsto \mathbf{h}_{k_1}(t^2)$ are bounded holomorphic functions on $\bar{S}_{\tau,\delta} \cup \bar{D}_1$.

Using Lemma 5 and lemma 6, the same proof as Lemma 4 yields

Lemma 7 *For given initial data $\mathbf{W}_j(t^2) \in \mathcal{A}_{\tau,\delta,\nu}$, $0 \le j \le S-1$, there exists a unique formal series*

$$\mathbf{W}(t^2,z) = \sum_{k=0}^{+\infty} \mathbf{W}_k(t^2) \frac{z^k}{k!}$$

solution of (37) and (38) where $\mathbf{W}_k(t)$, $k \ge 0$, are holomorphic functions on $S_{2\tau,2\delta} \cup D_1$ and continuous on $\bar{S}_{2\tau,2\delta} \cup \bar{D}_1$ such that $\mathbf{W}_k(t^2) \in \mathcal{A}_{\tau,\delta,\nu}$, for all $k \ge 0$.

In the following we introduce some notations. Let $\mathbf{w}_j^{\tau,\delta,\nu} = ||\mathbf{W}_j(t^2)||_{\tau,\delta,\nu}$, $0 \leq j \leq S-1$ and

$$|\mathbf{H}|_{k_1}^{\tau,\delta,\nu}(z) = \sum_{q\geq 0} ||\mathbf{H}_{k_1,q}(t^2)||_{\tau,\delta,\nu} z^q$$

The following proposition gives sufficient conditions for the contructed formal series to converge.

Proposition 10 *For every real number $R' > 0$ such that the disc $D_{R'}$ is contained in the open maximal domain $\mathcal{D}_{\tau,\delta,\nu}$ where the coefficients $|\mathbf{H}|_{k_1}^{\tau,\delta,\nu}(z)$, $0 \leq k_1 \leq S-1$, are holomorphic functions, the formal series $\mathbf{W}(t^2,z)$, solution of (37) with (38), constructed in Lemma 7, is a convergent series in z on $D_{R'}$ with coefficients in $\mathcal{A}_{\tau,\delta,\nu}$.*

Proof As in the proof of the proposition 9, from the hypotheses **(h1)** **(h2)**, **(h3)** and the lemma 6 we get that the functions $\mathbf{W}_k(t^2)$ defined as in the recursion (29) all belong to $\mathcal{A}_{\tau,\delta,\nu}$. Moreover, if we put $\mathbf{w}_k^{\tau,\delta,\nu} = ||\mathbf{W}_k(t^2)||_{\tau,\delta,\nu}$, we get a constant $C_0 > 0$ (one can choose $C_0 = \max_{0\leq k_1\leq S-1}\sup_{t\in\bar{S}_{\tau,\delta}\cup\bar{D}_1}|\mathbf{h}_{k_1}(t^2)|$), such that the following inequalities hold

$$\frac{\mathbf{w}_{k+S}^{\tau,\delta,\nu}}{k!} \leq \sum_{k_1=0}^{S-1}\sum_{q_1+q_2=k} C_0||\mathbf{H}_{k_1,q_1}(t^2)||_{\tau,\delta,\nu}\frac{\mathbf{w}_{q_2+k_1}^{\tau,\delta,\nu}}{q_2!} \tag{39}$$

On the other hand, consider the linear differential equation

$$\partial_z^S\mathbf{V}(z) = \sum_{k_1=0}^{S-1} C_0|\mathbf{H}|_{k_1}^{\tau,\delta,\nu}(z)\partial_z^{k_1}\mathbf{V}(z) \tag{40}$$

with initial conditions

$$(\partial_z^j\mathbf{V})(0) = \mathbf{w}_j^{\tau,\delta,\nu} \quad , \quad 0 \leq j \leq S-1. \tag{41}$$

From the classical *global* theory of linear differential equations in the complex domain, we know that the problem (40), (41) has a unique solution $\mathbf{V}(z) = \sum_{p\geq 0}\mathbf{V}_p z^p/p!$, holomorphic on every disc $D_{R'}$ contained in the open maximal domain $\mathcal{D}_{\tau,\delta,\nu}$ where the coefficients $C_0|\mathbf{H}|_{k_1}^{\tau,\delta,\nu}(z)$, $0 \leq k_1 \leq S-1$, are holomorphic functions.

Now, a formal series $\mathbf{V}(z) = \sum_{k\geq 0}\mathbf{V}_k z^k/k!$ is a solution of (40) and (41) if and only if the following equalities hold

$$\frac{\mathbf{V}_{k+S}}{k!} = \sum_{k_1=0}^{S-1}\sum_{q_1+q_2=k} C_0||\mathbf{H}_{k_1,q_1}(t^2)||_{\tau,\delta,\nu}\frac{\mathbf{V}_{q_2+k_1}}{q_2!} \tag{42}$$

for all $k \geq 0$ and

$$\mathbf{V}_j = \mathbf{w}_j^{\tau,\delta,\nu} \quad , \quad 0 \leq j \leq S-1. \tag{43}$$

We deduce from (39) and (42) with (43), that

$$\mathbf{w}_p^{\tau,\delta,\nu} \leq \mathbf{V}_p,$$

for all $p \geq 0$. So that, for every $R' > 0$, such that $D_{R'}$ is contained in the open maximal domain $\mathcal{D}_{\tau,\delta,\nu}$ where the coefficients $C_0|\mathbf{H}|_{k_1}^{\tau,\delta,\nu}(z)$, $0 \leq k_1 \leq S-1$, are holomorphic functions, we get that the series $\sum_{p\geq 0}\mathbf{w}_p^{\tau,\delta,\nu}z^p/p!$ converges on $D_{R'}$ and that the formal solution $\mathbf{W}(t^2,z)$ constructed in Lemma 7, is a convergent series in z on $D_{R'}$ with values in $\mathcal{A}_{\tau,\delta,\nu}$. □

7 Hyperfunctions solutions of Cauchy problems in $\mathcal{D}'_\nu$

7.1 Assumptions on the data of the Cauchy problem

Let θ_1^-, θ_1^+ be two real numbers such that $0 < \theta_1^+ - \theta_1^- < \pi$ and let $\theta_1 = (\theta_1^+ + \theta_1^-)/2$. We put $\tau^- = (\theta_1^- + \theta_1)/2$ and consider an increasing sequence $\delta_n^- > 0$, $n \geq 1$, such that $\tau^- - \delta_n^- > \theta_1^-$, $\tau^- + \delta_n^- < \theta_1$ and $\tau^- + \delta_n^- \to \theta_1$ as $n \to +\infty$. We also put $\tau^+ = (\theta_1 + \theta_1^+)/2$ and consider an increasing sequence $\delta_n^+ > 0$, $n \geq 1$, such that $\tau^+ + \delta_n^- < \theta_1^+$, $\tau^+ - \delta_n^+ > \theta_1$ and $\tau^+ - \delta_n^+ \to \theta_1$ as $n \to +\infty$. We also introduce the following sets in $\mathbb{C}$

$$\mathcal{W} = \{t \in \mathbb{C}^*/\arg(t) \in (\theta_1^-, \theta_1^+), \quad t \neq ke^{i\theta_1}, \quad \forall k \in \mathbb{N}\ \},$$

$$\mathcal{W}^+ = \{t \in \mathcal{W}/\arg(t) \in (\theta_1, \theta_1+)\} \quad , \quad \mathcal{W}^- = \{t \in \mathcal{W}/\arg(t) \in (\theta_1^-, \theta_1)\}.$$

We consider the following Cauchy problems

$$\partial_z^S \mathbf{W}^-(t^2, z) = \sum_{k_1=0}^{S-1} \mathbf{h}_{k_1}(t^2) \int_0^t \int_0^{t_1} \mathbf{H}_{k_1}^-((t_1 - t_2)^2, z) \partial_z^{k_1} \mathbf{W}^-(t_2^2, z) dt_2 dt_1 \tag{44}$$

with initial conditions

$$(\partial_z^j \mathbf{W}^-)(t^2, 0) = \mathbf{W}_j^-(t^2) \quad , \quad 0 \leq j \leq S-1. \tag{45}$$

and

$$\partial_z^S \mathbf{W}^+(t^2, z) = \sum_{k_1=0}^{S-1} \mathbf{h}_{k_1}(t^2) \int_0^t \int_0^{t_1} \mathbf{H}_{k_1}^+((t_1 - t_2)^2, z) \partial_z^{k_1} \mathbf{W}^+(t_2^2, z) dt_2 dt_1 \tag{46}$$

with initial conditions

$$(\partial_z^j \mathbf{W}^+)(t^2, 0) = \mathbf{W}_j^+(t^2) \quad , \quad 0 \leq j \leq S-1. \tag{47}$$

We make the following assumptions.

(h1') For all $0 \leq j \leq S-1$, the function $\mathbf{W}_j^-(t)$ (resp. $\mathbf{W}_j^+(t)$) is holomorphic on $S_{2\tau^-, 2\delta_n^-} \cup D_1$ and continuous on $\bar{S}_{2\tau^-, 2\delta_n^-} \cup \bar{D}_1$ (resp. holomorphic on $S_{2\tau^+, 2\delta_n^+} \cup D_1$ and continuous on $\bar{S}_{2\tau^+, 2\delta_n^+} \cup \bar{D}_1$) for all $n \geq 1$. Moreover, for $0 \leq j \leq S-1$, all $n \geq 1$, the function $t \mapsto \mathbf{W}_j^-(t^2)$ belong to $\mathcal{A}_{\tau^-, \delta_n^-, \nu}$ and the function $t \mapsto \mathbf{W}_j^+(t^2)$ belong to $\mathcal{A}_{\tau^+, \delta_n^+, \nu}$. We assume that $\mathbf{W}_j^-(t) = \mathbf{W}_j^+(t)$ for all $t \in D_1$, for all $0 \leq j \leq S-1$.

For every fixed $\theta \in (\theta_1^-, \theta_1)$, consider the function $W_{j,\theta}^-(r) = \mathbf{W}_j^-(r^2 e^{2i\theta})$ defined on $[0, +\infty)$. As a result, the function $r \mapsto W_{j,\theta}^-(r)$ is a continuous function from $[0, +\infty)$ into $\mathbb{C}$. Moreover, we have

$$\sup_{r \in [0, \infty)} |W_{j,\theta}^-(r)| e^{-\nu r} < +\infty.$$

In particular, from the proposition 2, the function $r \mapsto W_{j,\theta}^-(r)$ belong to the Banach space $\mathcal{D}'_{\nu'}$ for any $\nu' > \nu$.

For every fixed $\theta \in (\theta_1, \theta_1^+)$, consider the function $W_{j,\theta}^+(r) = \mathbf{W}_j^+(r^2 e^{i2\theta})$ defined on $[0, +\infty)$. As a result, the function $r \mapsto W_{j,\theta}^+(r)$ is a continuous function from $[0, +\infty)$ into $\mathbb{C}$. Moreover, we have

$$\sup_{r \in [0, \infty)} |W_{j,\theta}^+(r)| e^{-\nu r} < +\infty.$$

In particular, from the proposition 2, the function $r \mapsto W^{+}_{j,\theta}(r)$ belong to the Banach space $\mathcal{D}'_{\nu'}$ for any $\nu' > \nu$. We make the following assumption.

We take $\nu' > \nu$ such that $\nu' e^{1-\nu'} < 1$. There exist staircase distributions $W^{-}_{j,\theta_1}(r), W^{+}_{j,\theta_1}(r) \in \mathcal{D}'_{\nu'}$ such that

a)

$$\lim_{\theta \to \theta_1, \theta < \theta_1} ||W^{-}_{j,\theta}(r) - W^{-}_{j,\theta_1}(r)||_{\nu',d} = 0$$

b)

$$\lim_{\theta \to \theta_1, \theta > \theta_1} ||W^{+}_{j,\theta}(r) - W^{+}_{j,\theta_1}(r)||_{\nu',d} = 0.$$

In particular, from the proposition 6, for $0 \leq j \leq S-1$, the distribution $W^{+}_{j,\theta_1}(r) - W^{-}_{j,\theta_1}(r)$ defines a hyperfunction on the half-line $\mathbb{R}_+ e^{i\theta_1} = \{t \in \mathbb{C}^* : t = re^{i\theta_1}, r > 0\}$.

(h2') For all $0 \leq k_1 \leq S-1$,

$$\mathbf{H}^{-}_{k_1}(t^2, z) = \sum_{q \geq 0} \mathbf{H}^{-}_{k_1,q}(t^2) z^q$$

are convergent series in z on a disc D_{R_1} with coefficients $t \mapsto \mathbf{H}^{-}_{k_1,q}(t^2)$, $q \geq 0$, that belong to $\mathcal{A}_{\tau^-,\delta_n^-,\nu}$ such that the function $\mathbf{H}^{-}_{k_1,q}(t)$ is holomorphic on $S_{2\tau^-,2\delta_n^-} \cup D_1$ and continuous on $\bar{S}_{2\tau^-,2\delta_n^-} \cup \bar{D}_1$, for all $n \geq 1$. For all $0 \leq k_1 \leq S-1$,

$$\mathbf{H}^{+}_{k_1}(t^2, z) = \sum_{q \geq 0} \mathbf{H}^{+}_{k_1,q}(t^2) z^q$$

are convergent series in z on a disc D_{R_1} with coefficients $t \mapsto \mathbf{H}^{+}_{k_1,q}(t^2)$, $q \geq 0$, that belong to $\mathcal{A}_{\tau^+,\delta_n^+,\nu}$ such that the function $\mathbf{H}^{+}_{k_1,q}(t)$ is holomorphic on $S_{2\tau^+,2\delta_n^+} \cup D_1$ and continuous on $\bar{S}_{2\tau^+,2\delta_n^+} \cup \bar{D}_1$, for all $n \geq 1$. We assume that $\mathbf{H}^{-}_{k_1}(t,z) = \mathbf{H}^{+}_{k_1}(t,z)$ for all $(t,z) \in D_1 \times D_{R_1}$, for all $0 \leq k_1 \leq S-1$.

In particular, for $0 \leq k_1 \leq S-1$, the function $(t,z) \mapsto \mathbf{H}^{-}_{k_1}(t^2,z)$ (resp. $(t,z) \mapsto \mathbf{H}^{+}_{k_1}(t^2,z)$) is holomorphic on the domain $\mathcal{W}^- \times D_{R_1}$ (resp. $\mathcal{W}^+ \times D_{R_1}$).

Let $\mathcal{D}_{\tau^-,\tau^+,\delta_n^-,\delta_n^+,\nu}$ be the open maximal domain on which the functions $|\mathbf{H}|_{k_1}^{\tau^-,\delta_n^-,\nu}(z)$ and $|\mathbf{H}|_{k_1}^{\tau^+,\delta_n^+,\nu}(z)$, $0 \leq k_1 \leq S-1$, are holomorphic. We make the following assumption

$$D_{R_1} \subset \bigcap_{n \geq 1} \mathcal{D}_{\tau^-,\tau^+,\delta_n^-,\delta_n^+,\nu}.$$

For every fixed $\theta \in (\theta_1^-, \theta_1)$, consider the function $H^{-}_{k_1,\theta}(r,z) = \mathbf{H}^{-}_{k_1}(r^2 e^{2i\theta}, z)$ with expansion

$$H^{-}_{k_1,\theta}(r,z) = \sum_{q \geq 0} H^{-}_{k_1,\theta,q}(r) z^q.$$

As a result, from the proposition 2, $H^{-}_{k_1,\theta}(r,z)$ is a convergent series in the variable $z \in D_{R_1}$ with values in $\mathcal{D}'_{\nu'}$.

For every fixed $\theta \in (\theta_1, \theta_1^+)$, consider the function $H^{+}_{k_1,\theta}(r,z) = \mathbf{H}^{+}_{k_1}(r^2 e^{2i\theta}, z)$ with expansion

$$H^{+}_{k_1,\theta}(r,z) = \sum_{q \geq 0} H^{+}_{k_1,\theta,q}(r) z^q.$$

As a result, from the proposition 2, $H^{+}_{k_1,\theta}(r,z)$ is a convergent series in the variable $z \in D_{R_1}$ with values in $\mathcal{D}'_{\nu'}$.

We make the following assumption. There exist staircase distributions $H^{-}_{k_1,\theta_1,q}(r)$ and $H^{+}_{k_1,\theta_1,q}(r)$ in $\mathcal{D}'_{\nu'}$, $q \geq 0$, such that, if we put

$$H^{-}_{k_1,\theta_1}(r,z) = \sum_{q\geq 0} H^{-}_{k_1,\theta_1,q}(r) z^q \quad , \quad H^{+}_{k_1,\theta_1}(r,z) = \sum_{q\geq 0} H^{+}_{k_1,\theta_1,q}(r) z^q$$

we have

a) For $\theta \in (\theta_1^{-}, \theta_1)$, the series

$$|H^{-}_{k_1,\theta} - H^{-}_{k_1,\theta_1}|^{\nu',d}(R_2) = \sum_{q\geq 0} ||H^{-}_{k_1,\theta,q}(r) - H^{-}_{k_1,\theta_1,q}(r)||_{\nu',d} R_2^q$$

satisfies

$$\lim_{\theta\to\theta_1,\theta<\theta_1} |H^{-}_{k_1,\theta} - H^{-}_{k_1,\theta_1}|^{\nu',d}(R_2) = 0.$$

b) For $\theta \in (\theta_1, \theta_1^{+})$, the series

$$|H^{+}_{k_1,\theta} - H^{+}_{k_1,\theta_1}|^{\nu',d}(R_2) = \sum_{q\geq 0} ||H^{+}_{k_1,\theta,q}(r) - H^{+}_{k_1,\theta_1,q}(r)||_{\nu',d} R_2^q$$

satisfies

$$\lim_{\theta\to\theta_1,\theta<\theta_1} |H^{+}_{k_1,\theta} - H^{+}_{k_1,\theta_1}|^{\nu',d}(R_2) = 0.$$

(h3') For $0 \leq k_1 \leq S-1$, the function $t \mapsto \mathbf{h}_{k_1}(t)$ is holomorphic and bounded on $\bar{S}_{2\theta_1,2(\theta_1^{+}-\theta_1)} \cup \bar{D}_1$, so that the function $t \mapsto \mathbf{h}_{k_1}(t^2)$ is holomorphic and bounded on $\bar{S}_{\theta_1,(\theta_1^{+}-\theta_1)} \cup \bar{D}_1$. For every fixed $\theta \in (\theta_1^{-}, \theta_1^{+})$, we consider the function $h_{k_1,\theta}(r) = \mathbf{h}_{k_1}(r^2 e^{2i\theta})$ which is in particular a bounded $C^{\infty}-$function on $\mathbb{R}_+$ with values in $\mathbb{C}$. We make the following assumptions.

i) There exist $\mu, \kappa > 1/\nu'$ such that for all $0 \leq k_1 \leq S-1$, $h_{k_1,\theta}(r)$ are $C^{\infty}-$functions on $\mathbb{R}_+$ such that

$$|h^{(q)}_{k_1,\theta}(r)| \leq C_{k_1} \frac{q!}{(\kappa(r+\mu))^{1+q}},$$

for a $C_{k_1} > 0$, for all $q \geq 0$, all $r \in \mathbb{R}_+$ and all $\theta \in (\theta_1^{-}, \theta_1^{+})$.

ii) For all $\theta \in (\theta_1^{-}, \theta_1^{+}))$ and $0 \leq k_1 \leq S-1$, one can choose a constant $C_{k_1,\theta_1,\theta} > 0$ such that

$$|h^{(q)}_{k_1,\theta_1}(r) - h^{(q)}_{k_1,\theta}(r)| \leq C_{k_1,\theta_1,\theta} \frac{q!}{(\kappa(r+\mu))^{1+q}},$$

for all $q \geq 0$ and all $r \in \mathbb{R}_+$, with the additional property that

$$\lim_{\theta\to\theta_1} C_{k_1,\theta_1,\theta} = 0.$$

7.2 Hyperfunctions solutions for the Cauchy problems in $\mathcal{D}'_{\nu}$

From the hypotheses **(h1')**, **(h2')**, **(h3')**, the proposition 10 shows that the unique solution $(t,z) \mapsto \mathbf{W}^{-}(t^2,z) = \sum_{k\geq 0} \mathbf{W}^{-}_{k}(t^2) z^k/k!$ of the Cauchy problem (44), (45) and the unique solution $(t,z) \mapsto \mathbf{W}^{+}(t^2,z) = \sum_{k\geq 0} \mathbf{W}^{+}_{k}(t^2) z^k/k!$ of the Cauchy problem (46), (47) are convergent series in $z \in D_{R_1}$ with coefficients $t \mapsto \mathbf{W}^{-}_{k}(t^2)$ (resp. $t \mapsto \mathbf{W}^{+}_{k}(t^2)$) belonging to the spaces $\mathcal{A}_{\tau^{-},\delta_n^{-},\nu}$ (resp. $\mathcal{A}_{\tau^{+},\delta_n^{+},\nu}$), for all $n \geq 1$.

In particular, the function $(t,z) \mapsto \mathbf{W}^{-}(t^2,z)$ is holomorphic on the domain $\mathcal{W}^{-} \times D_{R_1}$ and the function $(t,z) \mapsto \mathbf{W}^{+}(t^2,z)$ is holomorphic on the domain $\mathcal{W}^{+} \times D_{R_1}$.

For every fixed $\theta \in (\theta_1^-, \theta_1)$, we consider the function $V_\theta^-(r,z)$ defined on $[0,+\infty) \times D_{R_1}$ by $V_\theta^-(r,z) = \mathbf{W}^-(r^2 e^{2i\theta}, z)$. We expand $V_\theta^-(r,z) = \sum_{k\geq 0} V_{k,\theta}^-(r) z^k/k!$. As a result, the functions $r \mapsto V_{k,\theta}^-(r)$ are continuous functions from $[0,+\infty)$ into $\mathbb{C}$. Moreover, we have

$$\sup_{r\in[0,\infty)} |V_{k,\theta}^-(r)| e^{-\nu r} < +\infty.$$

For every fixed $\theta \in (\theta_1, \theta_1^+)$, we consider the function $V_\theta^+(r,z)$ defined on $[0,+\infty) \times D_{R_1}$ by $V_\theta^+(r,z) = \mathbf{W}^+(r^2 e^{2i\theta}, z)$. We expand $V_\theta^+(r,z) = \sum_{k\geq 0} V_{k,\theta}^+(r) z^k/k!$. As a result, the functions $r \mapsto V_{k,\theta}^+(r)$ are continuous functions from $[0,+\infty)$ into $\mathbb{C}$. Moreover, we have

$$\sup_{r\in[0,\infty)} |V_{k,\theta}^+(r)| e^{-\nu r} < +\infty.$$

In particular, from the proposition 2, the functions $r \mapsto V_{k,\theta}^-(r)$ and $r \mapsto V_{k,\theta}^+(r)$, $k \geq 0$, belong to the Banach space $\mathcal{D}'_{\nu'}$ for any $\nu' > \nu$, and $V_\theta^-(r,z)$ and $V_\theta^+(r,z)$ are convergent series in the variable z with values in $\mathcal{D}'_{\nu'}$ for any $\nu' > \nu$.

In the following proposition, we study tangential limit properties in $\mathcal{D}'_{\nu'}$ of the solutions $\mathbf{W}^-(t^2,z)$ and $\mathbf{W}^+(t^2,z)$.

Proposition 11 *Let the assumptions* **(h1')**, **(h2')**, **(h3')** *hold. Then, there exist series*

$$V_{\theta_1}^-(r,z) = \sum_{k\geq 0} V_{k,\theta_1}^-(r) \frac{z^k}{k!} \quad , \quad V_{\theta_1}^+(r,z) = \sum_{k\geq 0} V_{k,\theta_1}^+(r) \frac{z^k}{k!}$$

which are convergent in z on a neighborhood of the origin in $\mathbb{C}$ with values in $\mathcal{D}'_\nu$ with the following property. There exists $R_3 > 0$ small enough such that

a)

$$\lim_{\theta\to\theta_1, \theta<\theta_1} \sum_{k=0}^{+\infty} ||V_{k,\theta}^-(r) - V_{k,\theta_1}^-(r)||_{\nu',d} \frac{R_3^k}{k!} = 0$$

and

b)

$$\lim_{\theta\to\theta_1, \theta>\theta_1} \sum_{k=0}^{+\infty} ||V_{k,\theta}^+(r) - V_{k,\theta_1}^+(r)||_{\nu',d} \frac{R_3^k}{k!} = 0.$$

Proof For every fixed $\theta \in (\theta_1^-, \theta_1)$, the series $V_\theta^-(r,z)$ satisfies the following Cauchy problem

$$\partial_z^S V_\theta^-(r,z) = \sum_{k_1=0}^{S-1} h_{k_1,\theta}(r) e^{2i\theta} \int_0^r \int_0^{r_1} H_{k_1,\theta}^-(r_1 - r_2, z) \partial_z^{k_1} V_\theta^-(r_2,z) dr_2 dr_1 \tag{48}$$

with initial conditions

$$(\partial_z^j V_\theta^-)(r,0) = W_{j,\theta}^-(r) \quad , \quad 0 \leq j \leq S-1. \tag{49}$$

and for every fixed $\theta \in (\theta_1, \theta_1^+)$, the series $V_\theta^+(r,z)$ satisfies the following Cauchy problem

$$\partial_z^S V_\theta^+(r,z) = \sum_{k_1=0}^{S-1} h_{k_1,\theta}(r) e^{2i\theta} \int_0^r \int_0^{r_1} H_{k_1,\theta}^+(r_1 - r_2, z) \partial_z^{k_1} V_\theta^+(r_2,z) dr_2 dr_1 \tag{50}$$

with initial conditions

$$(\partial_z^j V_\theta^+)(r,0) = W_{j,\theta}^+(r) \quad , \quad 0 \le j \le S-1. \tag{51}$$

Under the assumptions **(h1')**, **(h2')**, **(h3')**, the proposition 9 shows that the Cauchy problem (48) and (49) in the space $\mathcal{D}'_{\nu'}$ also make sense for $\theta = \theta_1$. Let

$$V_{\theta_1}^-(r,z) = \sum_{k\ge 0} V_{k,\theta_1}^-(r)\frac{z^k}{k!}$$

its solution which is convergent in the variable $z \in D_{R_2}$, $R_2 > 0$ small enough, with coefficients in $\mathcal{D}'_{\nu'}$. Likewise, the Cauchy problem (50) and (51) in the space $\mathcal{D}'_{\nu'}$ also make sense for $\theta = \theta_1$. Let

$$V_{\theta_1}^+(r,z) = \sum_{k\ge 0} V_{k,\theta_1}^+(r)\frac{z^k}{k!}$$

its solution which is convergent in the variable $z \in D_{R_2}$, $R_2 > 0$ small enough, with coefficients in $\mathcal{D}'_{\nu'}$.

In the following, we will show that there exists $R_3 > 0$ small enough such that

a)

$$\lim_{\theta\to\theta_1, \theta<\theta_1} \sum_{k=0}^{+\infty} ||V_{k,\theta}^-(r) - V_{k,\theta_1}^-(r)||_{\nu',d}\frac{R_3^k}{k!} = 0$$

and

b)

$$\lim_{\theta\to\theta_1, \theta>\theta_1} \sum_{k=0}^{+\infty} ||V_{k,\theta}^+(r) - V_{k,\theta_1}^+(r)||_{\nu',d}\frac{R_3^k}{k!} = 0.$$

We will make the proof only for the first limit **a)**. The proof of the limit **b)** goes the same lines and will not be detailed.

For every fixed $\theta \in (\theta_1^-, \theta_1)$, we denote by $V_{\theta_1,\theta}^-(r,z) = V_{\theta_1}^-(r,z) - V_\theta^-(r,z)$ and we will also use the notations

$$h_{k_1,\theta_1,\theta}(r) = e^{2i\theta_1}h_{k_1,\theta_1}(r) - e^{2i\theta}h_{k_1,\theta}(r) = (e^{2i\theta_1} - e^{2i\theta})h_{k_1,\theta_1}(r) + e^{2i\theta}(h_{k_1,\theta_1}(r) - h_{k_1,\theta}(r))$$

and

$$H_{k_1,\theta_1,\theta}^-(r,z) = H_{k_1,\theta_1}^-(r,z) - H_{k_1,\theta}^-(r,z).$$

The function $r \mapsto V_{\theta_1,\theta}^-(r,z)$ satisfies the following Cauchy problem

$$\partial_z^S V_{\theta_1,\theta}^-(r,z) = A_{r,z,\theta,\theta_1} + D_{r,z,\theta,\theta_1}^1 + D_{r,z,\theta,\theta_1}^2 \tag{52}$$

where

$$A_{r,z,\theta,\theta_1} = \sum_{k_1=0}^{S-1} h_{k_1,\theta}(r)e^{2i\theta}\int_0^r\int_0^{r_1} H_{k_1,\theta}^-(r_1-r_2,z)\partial_z^{k_1}V_{\theta_1,\theta}^-(r_2,z)dr_2dr_1,$$

$$D_{r,z,\theta,\theta_1}^1 = \sum_{k_1=0}^{S-1} h_{k_1,\theta_1,\theta}(r)\int_0^r\int_0^{r_1} H_{k_1,\theta_1}^-(r_1-r_2,z)\partial_z^{k_1}V_{\theta_1}^-(r_2,z)dr_2dr_1$$

and

$$D^2_{r,z,\theta,\theta_1} = \sum_{k_1=0}^{S-1} h_{k_1,\theta}(r)e^{2i\theta} \int_0^r \int_0^{r_1} H^-_{k_1,\theta_1,\theta}(r_1 - r_2, z)\partial_z^{k_1} V^-_{\theta_1}(r_2, z)dr_2 dr_1,$$

with initial conditions

$$(\partial_z^j V_{\theta_1,\theta})(r,0) = W^-_{j,\theta,\theta_1}(r) = W^-_{j,\theta_1}(r) - W^-_{j,\theta}(r) \quad , \quad 0 \leq j \leq S-1. \tag{53}$$

We consider now the Cauchy problem (52), (53) as a Cauchy problem of the form (27), (28). Indeed, from the assumptions **(h1')**, **(h2')**, **(h3')**, we see that the hypotheses **(d1)**, **(d2)**, **(d3)** are fulfilled for this Cauchy problem.
We put $d_{\theta,\theta_1}(r,z) = D^1_{r,z,\theta,\theta_1} + D^2_{r,z,\theta,\theta_1}$. We expand $d_{\theta,\theta_1}(r,z)$ and $H_{k_1,\theta}(r,z)$ in Taylor series in the variable z,

$$d_{\theta,\theta_1}(r,z) = \sum_{q \geq 0} d_{q,\theta,\theta_1}(r) z^q \quad , \quad H^-_{k_1,\theta}(r,z) = \sum_{q\geq 0} H^-_{k_1,\theta,q}(r) z^q.$$

We put

$$w^{\nu',d}_{j,\theta_1,\theta} = ||W^-_{j,\theta,\theta_1}(r)||_{\nu',d} \quad , \quad |d_{\theta,\theta_1}|^{\nu',d}(z) = \sum_{q\geq 0} ||d_{q,\theta,\theta_1}(r)||_{\nu',d} z^q,$$

$$|H|^{\nu',d}_{k_1,\theta}(z) = \sum_{q \geq 0} ||H^-_{k_1,\theta,q}(r)||_{\nu',d} z^q$$

and

$$|V|^{\nu',d}_{\theta}(z) = \sum_{q\geq 0} ||V^-_{q,\theta}(r)||_{\nu',d} \frac{z^q}{q!} \quad , \quad |H|^{\nu',d}_{k_1,\theta,\theta_1}(z) = \sum_{q\geq 0} ||H^-_{k_1,\theta_1,q}(r) - H^-_{k_1,\theta,q}(r)||_{\nu',d} z^q,$$

for all $0 \leq j \leq S-1$, $0 \leq k_1 \leq S-1$.

From the proposition 9, there exist $R_3, C > 0$ (depending on $\kappa, \mu, \nu', C_{k_1}$, $0 \leq k_1 \leq S-1$ and $\sup_{\theta \in (\tau^-,\theta_1)} |H|^{\nu',d}_{k_1,\theta}(z)$, $0 \leq k_1 \leq S-1$), such that the following estimates hold

$$\sum_{k=0}^{+\infty} ||V^-_{k,\theta}(r) - V^-_{k,\theta_1}(r)||_{\nu',d} \frac{R_3^k}{k!} \leq C(\sum_{j=0}^{S-1} w^{\nu',d}_{j,\theta_1,\theta} + |d_{\theta,\theta_1}|^{\nu',d}(R_2)).$$

for all $\theta \in (\tau^-, \theta_1)$. *Now, in the last part of the proof, we show that*

$$\lim_{\theta \to \theta_1, \theta < \theta_1} w^{\nu',d}_{j,\theta_1,\theta} + |d_{\theta,\theta_1}|^{\nu',d}(R_2) = 0, \tag{54}$$

for $0 \leq j \leq S-1$. First of all, the assumption **(h1')** says that

$$\lim_{\theta\to\theta_1,\theta<\theta_1} w^{\nu',d}_{j,\theta_1,\theta} = 0. \tag{55}$$

Moreover, for all $q \geq 0$, we write $d_{q,\theta,\theta_1}(r) = d^1_{q,\theta,\theta_1}(r) + d^2_{q,\theta,\theta_1}(r)$, where

$$d^1_{q,\theta,\theta_1}(r) = \sum_{k_1=0}^{S-1} \sum_{q_1+q_2=q} h_{k_1,\theta_1,\theta}(r) \partial_r^{-1}(H^-_{k_1,\theta_1,q_1}(r) * \frac{V^-_{q_2+k_1,\theta_1}(r)}{q_2!})$$

and

$$d^2_{q,\theta,\theta_1}(r) = \sum_{k_1=0}^{S-1} \sum_{q_1+q_2=q} h_{k_1,\theta}(r) e^{2i\theta} \partial_r^{-1}((H^-_{k_1,\theta_1,q_1}(r) - H^-_{k_1,\theta,q_1}(r)) * \frac{V^-_{q_2+k_1,\theta_1}(r)}{q_2!}).$$

From the propositions 4, 5 and the assumption **(h3')**, we deduce that there exists $C_{\mu,\kappa,\nu'} > 0$, such that

$$||d^1_{q,\theta,\theta_1}(r)||_{\nu',d} \leq \sum_{k_1=0}^{S-1} \sum_{q_1+q_2=q} C_{\mu,\kappa,\nu'}(|e^{i\theta_1} - e^{i\theta}|C_{k_1} + C_{k_1,\theta_1,\theta})||H^-_{k_1,\theta_1,q_1}(r)||_{\nu',d} \frac{||V^-_{q_2+k_1,\theta_1}(r)||_{\nu',d}}{q_2!}$$

and

$$||d^2_{q,\theta,\theta_1}(r)||_{\nu',d} \leq \sum_{k_1=0}^{S-1} \sum_{q_1+q_2=q} C_{\mu,\kappa,\nu'} C_{k_1} ||H^-_{k_1,\theta_1,q_1}(r) - H^-_{k_1,\theta,q_1}(r)||_{\nu',d} \frac{||V^-_{q_2+k_1,\theta_1}(r)||_{\nu',d}}{q_2!}$$

for all $q \geq 0$. We deduce that

$$(56) \quad |d^1_{\theta,\theta_1}|^{\nu',d}(R_2) = \sum_{q\geq 0} ||d^1_{q,\theta,\theta_1}(r)||_{\nu',d}(R_2)^q \leq \sum_{k_1=0}^{S-1} C_{\mu,\kappa,\nu'}(|e^{i\theta_1} - e^{i\theta}|C_{k_1} + C_{k_1,\theta_1,\theta})|H|^{\nu',d}_{k_1,\theta_1}(R_2)(\partial_z^{k_1}|V|^{\nu',d}_{\theta_1})(R_2)$$

and

$$(57) \quad |d^2_{\theta,\theta_1}|^{\nu',d}(R_2) = \sum_{q\geq 0} ||d^2_{q,\theta,\theta_1}(r)||_{\nu',d}(R_2)^q \leq \sum_{k_1=0}^{S-1} C_{\mu,\kappa,\nu'} C_{k_1} |H|^{\nu',d}_{k_1,\theta,\theta_1}(R_2)(\partial_z^{k_1}|V|^{\nu',d}_{\theta_1})(R_2).$$

From the assumption **(h2')**, we have

$$\lim_{\theta\to\theta_1,\theta<\theta_1} |H|^{\nu',d}_{k_1,\theta,\theta_1}(R_2) = 0.$$

From the assumption **(h3')**, we have

$$\lim_{\theta\to\theta_1,\theta<\theta_1} C_{k_1,\theta_1,\theta} = 0.$$

So that, from (56) and (57), we deduce that

$$\lim_{\theta\to\theta_1,\theta<\theta_1} |d^1_{\theta,\theta_1}|^{\nu',d}(R_2) + |d^2_{\theta,\theta_1}|^{\nu',d}(R_2) = 0.$$

From the inequality $|d_{\theta,\theta_1}|^{\nu',d}(R_2) \leq |d^1_{\theta,\theta_1}|^{\nu',d}(R_2) + |d^2_{\theta,\theta_1}|^{\nu',d}(R_2)$, we deduce that

$$(58) \quad \lim_{\theta\to\theta_1,\theta<\theta_1} |d_{\theta,\theta_1}|^{\nu',d}(R_2) = 0$$

Finally, from (55) and (58), we get the result (54) and the limit **a)** of proposition 11 follows. □

In the next theorem, we get hyperfunctions solutions for the Cauchy problems (44), (45), (46), (47).

Theorem 1 *Let $V_{\theta_1}^{-}(r,z)$ and $V_{\theta_1}^{+}(r,z)$ be the series constructed in the proposition 11. Then, the staircase distribution*

$$V_{\theta_1}^{+-}(r,z) = V_{\theta_1}^{+}(r,z) - V_{\theta_1}^{-}(r,z)$$

defines a hyperfunction on the half-line $\mathbb{R}_+ e^{i\theta_1}$ for all $z \in D_{R_3}$. More precisely, if $V_{\theta_1}^{+-}(r,z) = 0$ as convergent series with coefficients in $\mathcal{D}'_{\nu}$, then for all $l \geq 1$, the function $(t,z) \mapsto \mathbf{W}^{-}(t,z)$ can be analytically continued from the domain $\mathcal{W}^{-} \times D_{R_3}$ into $\mathcal{W}^{+} \times D_{R_3}$ across the segment $(le^{i\theta_1},(l+1)e^{i\theta_1})$ and the function $(t,z) \mapsto \mathbf{W}^{+}(t,z)$ can be analytically continued from the domain $\mathcal{W}^{+} \times D_{R_3}$ into $\mathcal{W}^{-} \times D_{R_3}$ across the segment $(le^{i\theta_1},(l+1)e^{i\theta_1})$.

Proof Assume that $V_{\theta_1}^{+-}(r,z) = 0$ as convergent series with coefficients in $\mathcal{D}'_{\nu}$. We denote by

$$V_{\theta_1}(r,z) = \sum_{k \geq 0} V_{k,\theta_1}(r) \frac{z^k}{k!}$$

the series $V_{\theta_1}^{+}(r,z) = V_{\theta_1}^{-}(r,z)$. Following the same arguments as in Proposition 6, one shows the following estimates.

Let $l \geq 1$ and $N \geq l+2$. We have, for all $r \in (l,l+1)$, $\theta \in (\theta_1^{-},\theta_1)$, $k \geq 0$,

$$|\partial_r^{-N}(V_{k,\theta}^{-}(r)) - \partial_r^{-N}(V_{k,\theta_1}(r))| \leq e^{\nu'(l+1)} \nu'^{1-N} ||V_{k,\theta}^{-}(r) - V_{k,\theta_1}(r)||_{\nu',d} \tag{59}$$

$$|\partial_r^{-N}(V_{k,\theta}^{-}(r))| \leq e^{\nu'(l+1)} \nu'^{1-N} ||V_{k,\theta}^{-}(r)||_{\nu',d}. \tag{60}$$

and for all $\theta \in (\theta_1,\theta_1^{+})$, $k \geq 0$,

$$|\partial_r^{-N}(V_{k,\theta}^{+}(r)) - \partial_r^{-N}(V_{k,\theta_1}(r))| \leq e^{\nu'(l+1)} \nu'^{1-N} ||V_{k,\theta}^{+}(r) - V_{k,\theta_1}(r)||_{\nu',d} \tag{61}$$

$$|\partial_r^{-N}(V_{k,\theta}^{+}(r))| \leq e^{\nu'(l+1)} \nu'^{1-N} ||V_{k,\theta}^{+}(r)||_{\nu',d}. \tag{62}$$

Now, for $t = re^{i\theta} \in \mathcal{W}^{-} \cup D_1$, $z \in D_{R_3}$ and $N \geq 1$, we have the expansion

$$\partial_t^{-N}(\mathbf{W}^{-}(t,z)) = \sum_{k \geq 0} e^{iN\theta} \partial_r^{-N}(V_{k,\theta}^{-}(r)) \frac{z^k}{k!}$$

and $t = re^{i\theta} \in \mathcal{W}^{+} \cup D_1$, $z \in D_{R_3}$ and $N \geq 1$, we have the expansion

$$\partial_t^{-N}(\mathbf{W}^{+}(t,z)) = \sum_{k \geq 0} e^{iN\theta} \partial_r^{-N}(V_{k,\theta}^{+}(r)) \frac{z^k}{k!}.$$

We define a function $h_N(t,z)$ on the domain

$$\mathcal{D}_l \times D_{R_3} = \mathcal{W}^{-} \cup D_1 \cup (le^{i\theta_1},(l+1)e^{i\theta_1}) \cup \mathcal{W}^{+} \times D_{R_3}$$

by

$$h_N(t,z) = \left\{ \begin{array}{ll} \partial_t^{-N}(\mathbf{W}^{-}(t,z)) & \text{if } t \in \mathcal{W}^{-} \cup D_1,\ z \in D_{R_3} \\ \sum_{k \geq 0} e^{iN\theta_1} \partial_r^{-N}(V_{k,\theta_1}(r)) \frac{z^k}{k!} & \text{if } t = re^{i\theta_1} \in (le^{i\theta_1},(l+1)e^{i\theta_1}),\ z \in D_{R_3} \\ \partial_t^{-N}(\mathbf{W}^{+}(t,z)) & \text{if } t \in \mathcal{W}^{+} \cup D_1,\ z \in D_{R_3} \end{array} \right. \tag{63}$$

The function $h_N(t,z)$ is analytic on $\mathcal{W}^- \cup \mathcal{W}^+ \cup D_1 \times D_{R_3}$ as iterated integrals of the analytic functions $\mathbf{W}^-(t,z)$, $\mathbf{W}^+(t,z)$. From the estimates (59), (60), (61), (62) and the limits **a)** and **b)** of Proposition 11, we deduce that the function $h_N(t,z)$ is continuous for all $t \in (le^{i\theta_1},(l+1)e^{i\theta_1})$ in $\mathcal{D}_l$ and holomorphic in $z \in D_{R_3}$. From Lemma 3, we conclude that the function $h_N(t,z)$ is in fact holomorphic on $\mathcal{D}_l \times D_{R_3}$.

As a result, the function $(t,z) \mapsto \partial_t^{-N}(\mathbf{W}^-(t,z))$ (resp. $(t,z) \mapsto \partial_t^{-N}(\mathbf{W}^+(t,z))$) can be analytically continued across the segment $(le^{i\theta_1},(l+1)e^{i\theta_1})$ from $\mathcal{W}^- \cup D_1 \times D_{R_3}$ into $\mathcal{W}^+ \cup D_1 \times D_{R_3}$ (resp. from $\mathcal{W}^+ \cup D_1 \times D_{R_3}$ into $\mathcal{W}^- \cup D_1 \times D_{R_3}$).

It follows that, for all integers $l \geq 1$, the function $(t,z) \mapsto \mathbf{W}^-(t,z)$ (resp. $(t,z) \mapsto \mathbf{W}^+(t,z)$) can be analytically continued across the segment $(le^{i\theta_1},(l+1)e^{i\theta_1})$ from $\mathcal{W}^- \cup D_1 \times D_{R_3}$ into $\mathcal{W}^+ \cup D_1 \times D_{R_3}$ (resp. $\mathcal{W}^+ \cup D_1 \times D_{R_3}$ into $\mathcal{W}^- \cup D_1 \times D_{R_3}$). This gives the result. □

8 Summability of Banach spaces valued formal series

We recall the definition of Borel summability of formal series with coefficients in a Banach space, see [1].

Definition 6 *A formal series*

$$\hat{X}(t) = \sum_{j=0}^{\infty} a_j t^j$$

with coefficients in a Banach space $(\mathbb{E}, ||.||_{\mathbb{E}})$ *is said to be* $k-$*summable with respect to* t *in the direction* $d \in \mathbb{R}$ *if*

i) *There exist* $\rho \in \mathbb{R}_+$ *such that the following formal series, called formal Borel transform with respect to the variable* t *of order* k,

$$W(t) = \sum_{j=0}^{\infty} \frac{a_j t^j}{\Gamma(1+j/k)},$$

is absolutely convergent for $|t| < \rho$,

ii) *There exists* $\delta > 0$ *such that the series* $W(t)$ *can be analytically continued with respect to* t *in a sector* $S_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta\}$. *Moreover, there exist* $\delta_1 < \delta$, $C > 0$, *and* $K > 0$ *such that*

$$||W(t)||_{\mathbb{E}} \leq Ce^{K|t|^k}$$

for all $t \in S_{d,\delta_1}$.

If this is so, the vector valued Laplace transform of order k of $W(t)$,

$$X(t) = t^{-k} \int_0^{\infty(\gamma)} W(\tau) e^{-(\tau/t)^k} d\tau^k,$$

where the integration is made along the ray $\arg(\tau) = \gamma$ with $|d - \gamma| < \delta$, is called the $k-$sum of the formal series $\hat{X}(t)$.

In the following, we associate to every formal series

$$\hat{X}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} \chi_j(z),$$

where $\chi_j(z)$ are holomorphic functions on D_r and continuous on $\bar{D}_r$, a modified Borel transform of order 1,

$$(\mathcal{B}_1\hat{X})(t,z) = V(t,z) := \sum_{j\geq 0} \frac{t^j}{(2j)!}\chi_j(z)$$

constructed from the moment function $m(u) = \Gamma(1+2u)/\Gamma(1+u)$ for $\mathrm{Re}(u) \geq 0$, see section 5.5 of [1] for details.
The following proposition is an application of a general moment summability result, Theorem 38, page 108 in [1].

Proposition 12 *The formal series $\hat{X}(t,z)$ is $1-$summable with respect to t in direction $d \in \mathbb{R}$ with values in the Banach space of holomorphic functions on D_R and continuous on $\bar{D}_R$, if*

i) *There exist $\rho \in \mathbb{R}_+$ such that $(\mathcal{B}_1\hat{X})(t,z)$ is absolutely convergent for all $(t,z) \in D_\rho \times D_R$.*

ii) *There exists $\delta > 0$ such that the series $(\mathcal{B}_1\hat{X})(t,z)$ can be analytically continued with respect to t in a sector $S_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta\}$. Moreover, there exist $\delta_1 < \delta$, $C > 0$, and $K > 0$ such that*

$$\sup_{z\in\bar{D}_r} |(\mathcal{B}_1\hat{X})(t,z)| \leq Ce^{K|t|}$$

for all $t \in S_{d,\delta_1}$.

Remark 1 : From Theorem 38 in [1], the $1-$sum $X(t,z)$ of $\hat{X}(t,z)$, defined on some finite sector $S_{d,\delta,\rho}$ where $\delta > \pi/2$ and $\rho > 0$, can be expressed in terms of the Borel transform using a modified Laplace transform

$$X(t,z) = \int_0^{\infty(d)} e(u/t)\frac{(\mathcal{B}_1\hat{X})(u,z)}{u}du,$$

where the kernel function $e(z)$ satisfies the following properties :
i) The function $e(z)$ is holomorphic on $S_{0,\pi/2}$.
ii) The integral $\int_0^{x_0} x^{-1}|e(xe^{i\tau})|dx$ exists for all $x_0 > 0$ and $2|\tau| < \pi$.
iii) For all $\delta > 0$, there exist $c, K > 0$ such that $|e(z)| \leq C\exp(-|z|/K)$ for $2|\arg(z)| \leq \pi - \delta$.
iv) We have $m(u) = \int_0^\infty x^{u-1}e(x)dx$ for $\mathrm{Re}(u) \geq 0$.

9 Solutions of irregular integro-differential equations as Laplace transform of hyperfunctions

In this section, we will assume that $-\sqrt{2}i, \sqrt{2}i \notin \bar{S}_{\theta_1,\theta_1^+-\theta_1}$. For all $0 \leq j \leq S-1$, we consider holomorphic functions $t \mapsto \mathbf{W}_j^-(t)$ and $t \mapsto \mathbf{W}_j^+(t)$ which satisfy the hypotheses **(h1')**. For all $0 \leq k_1 \leq S-1$, we also consider holomorphic functions $(t,z) \mapsto \mathbf{H}_{k_1}^-(t,z)$ and $(t,z) \mapsto \mathbf{H}_{k_1}^+(t,z)$ satisfying the assumptions **(h2')**. For all $\xi^- \in (2\theta_1^-, 2\theta_1)$, we define the modified Laplace transform

$$\mathbf{X}_j^-(t) = \int_0^{\infty(\xi^-)} \mathbf{W}_j^-(v)\frac{e(v/t)}{v}dv \quad , \quad \mathbf{A}_{k_1}^-(t,z) = \int_0^{\infty(\xi^-)} \mathbf{H}_{k_1}^-(v,z)\frac{e(v/t)}{v}dv \tag{64}$$

which are holomorphic functions on every domain of the form $S_{\xi^-,\delta,\rho} \times D_{R_1}$ for $0 < \delta < \pi/2$ and any $\rho > 0$. But, by the Cauchy formula, these latter integrals are independent of the ray of integration $\xi^- \in (2\theta_1^-, 2\theta_1)$. So that $\mathbf{X}_j^-(t)$ and $\mathbf{A}_{k_1}^-(t,z)$ are holomorphic on the domain $S_\rho^- \times D_{R_1}$ where $S_\rho^- = \cup_{\xi^- \in (2\theta_1^-, 2\theta_1), 0<\delta<\pi/2} S_{\xi^-,\delta,\rho}$ for any $\rho > 0$. In the same way, for all $\xi^+ \in (2\theta_1, 2\theta_1^+)$, we define the modified Laplace transform

$$\mathbf{X}_j^+(t) = \int_0^{\infty(\xi^+)} \mathbf{W}_j^+(v) \frac{e(v/t)}{v} dv \quad , \quad \mathbf{A}_{k_1}^+(t,z) = \int_0^{\infty(\xi^+)} \mathbf{H}_{k_1}^+(v,z) \frac{e(v/t)}{v} dv \tag{65}$$

which are holomorphic on the domain $S_\rho^+ \times D_{R_1}$ where $S_\rho^+ = \cup_{\xi^+ \in (2\theta_1, 2\theta_1^+), 0<\delta<\pi/2} S_{\xi^+,\delta,\rho}$, for any $\rho > 0$.

We are now in position to state the main result of the paper.

Theorem 2 *We consider the following two integro-differential Cauchy problems with irregular singularity at $t = 0$,*

$$4t^2 \partial_t \partial_z^S \mathbf{X}^-(t,z) + (2t+2)\partial_z^S \mathbf{X}^-(t,z) = \sum_{k_1=0}^{S-1} \int_0^t \mathbf{A}_{k_1}^-(t-\tau,z) \partial_z^{k_1} \mathbf{X}^-(\tau,z) d\tau \tag{66}$$

with initial conditions

$$(\partial_z^j \mathbf{X}^-)(t,0) = \mathbf{X}_j^-(t) \quad , \quad 0 \le j \le S-1. \tag{67}$$

and

$$4t^2 \partial_t \partial_z^S \mathbf{X}^+(t,z) + (2t+2)\partial_z^S \mathbf{X}^+(t,z) = \sum_{k_1=0}^{S-1} \int_0^t \mathbf{A}_{k_1}^+(t-\tau,z) \partial_z^{k_1} \mathbf{X}^+(\tau,z) d\tau \tag{68}$$

with initial conditions

$$(\partial_z^j \mathbf{X}^+)(t,0) = \mathbf{X}_j^+(t) \quad , \quad 0 \le j \le S-1. \tag{69}$$

Then, the problem (66), (67) has a unique solution $\mathbf{X}^-(t,z)$ which is holomorphic on the domain $S_\rho^- \times D_{R_1}$ and the problem (68), (69) has a unique solution $\mathbf{X}^+(t,z)$ which is holomorphic on the domain $S_\rho^+ \times D_{R_1}$, for any $\rho > 0$.

Moreover, there exists $R_3 > 0$ and $\delta_{\theta_1} > 0$ such that for all $t \in S_{2\theta_1,\delta_{\theta_1},\rho}$, $\rho > 0$, and $z \in D_{R_3}$, the Stokes transition function $\mathbf{X}^+(t,z) - \mathbf{X}^-(t,z)$ can be expressed as generalized Laplace transform in direction $2\theta_1$ of an hyperfunction $V_{\theta_1}^{+-}(r,z)$ as constructed in Theorem 1,

$$\mathbf{X}^+(t,z) - \mathbf{X}^-(t,z) = \sum_{k \ge 0} \mathcal{L}_{2\theta_1}(V_{\theta_1,k}^+ - V_{\theta_1,k}^-)(t) \frac{z^k}{k!}.$$

Proof Let

$$\mathbf{W}_j^-(t) = \mathbf{W}_j^+(t) = \mathbf{W}_j(t) = \sum_{k=0}^{+\infty} \frac{\mathbf{w}_{j,k}}{(2k)!} t^k \quad , \quad \mathbf{H}_{k_1}^-(t,z) = \mathbf{H}_{k_1}^+(t,z) = \mathbf{H}_{k_1}(t,z) = \sum_{k=0}^{+\infty} \frac{\mathbf{h}_{k_1,k}(z)}{(2k)!} t^k$$

be the Taylor expansions of $\mathbf{W}_j^-(t)$, $\mathbf{W}_j^+(t)$ and $\mathbf{H}_{k_1}^-(t,z)$, $\mathbf{H}_{k_1}^+(t,z)$ for $|t| < 1$. We introduce the formal series

$$\hat{\mathbf{X}}_j(t) = \sum_{k=0}^{+\infty} \frac{\mathbf{w}_{j,k}}{k!} t^k \quad , \quad \hat{\mathbf{A}}_{k_1}(t,z) = \sum_{k=0}^{+\infty} \frac{\mathbf{h}_{k_1,k}(z)}{k!} t^k.$$

Lemma 8 *The formal series*

$$\hat{\mathbf{X}}(t,z) = \sum_{k \geq 0} \frac{\mathbf{w}_k(z)}{k!} t^k,$$

where $\mathbf{w}_k(z)$, $k \geq 0$, *are holomorphic functions on a disc* D_{R_1} *continuous on* $\bar{D}_{R_1}$, *satisfies the formal Cauchy problem*

$$4t^2 \partial_t \partial_z^S \hat{\mathbf{X}}(t,z) + (2t+2)\partial_z^S \hat{\mathbf{X}}(t,z) = \sum_{k_1=0}^{S-1} \int_0^t \hat{\mathbf{A}}_{k_1}(t-\tau,z)\partial_z^{k_1}\hat{\mathbf{X}}(\tau,z)d\tau \tag{70}$$

with initial conditions

$$(\partial_z^j \hat{\mathbf{X}})(t,0) = \hat{\mathbf{X}}_j(t) \quad , \quad 0 \leq j \leq S-1. \tag{71}$$

if and only if its associated modified Borel transform of order 1, $(t,z) \mapsto \mathbf{W}(t^2,z) = (\mathcal{B}_1\hat{\mathbf{X}})(t^2,z)$ *satisfies the Cauchy problem*

$$\partial_z^S \mathbf{W}(t^2,z) = \sum_{k_1=0}^{S-1} \frac{1}{t^2+2} \int_0^t \int_0^{t_1} \mathbf{H}_{k_1}((t_1-t_2)^2,z)\partial_z^{k_1}\mathbf{W}(t_2^2,z)dt_2dt_1, \tag{72}$$

with initial conditions

$$(\partial_z^j \mathbf{W})(t^2,0) = \mathbf{W}_j(t^2) = (\partial_z^j \mathcal{B}_1 \hat{\mathbf{X}})(t^2,0) \quad , \quad 0 \leq j \leq S-1. \tag{73}$$

for all $(t,z) \in D_1 \times D_{R_1}$.

Proof From the classical identity

$$\int_0^t (t-\tau)^{m_1}\tau^{m_2}d\tau = \frac{m_1!m_2!}{(1+m_1+m_2)!}t^{m_1+m_2+1},$$

for $m_1, m_2 \geq 0$, we get

$$\int_0^t \hat{\mathbf{A}}_{k_1}(t-\tau,z)\hat{\mathbf{X}}(\tau,z)d\tau = \sum_{k\geq 1}\Big(\sum_{q_1+q_2=k-1} \mathbf{h}_{k_1,q_1}(z)\mathbf{w}_{q_2}(z)\Big)\frac{t^k}{k!}$$

and also if $\mathbf{W}(t,z) = \sum_{k\geq 0} \mathbf{w}_k(z)t^k/(2k)!$,

$$\int_0^t \int_0^{t_1} \mathbf{H}_{k_1}((t_1-\tau)^2,z)\mathbf{W}(\tau^2,z)d\tau dt_1 = \sum_{k\geq 1}\Big(\sum_{q_1+q_2=k-1} \mathbf{h}_{k_1,q_1}(z)\mathbf{w}_{q_2}(z)\Big)\frac{t^{2k}}{(2k)!}.$$

On the other hand,

$$(4t^2\partial_t + 2t)\hat{\mathbf{X}}(t,z) = \sum_{k\geq 0}(2k)(2k-1)\mathbf{w}_{k-1}(z)\frac{t^k}{k!}$$

and

$$t^2\mathbf{W}(t^2,z) = \sum_{k\geq 0}(2k)(2k-1)\mathbf{w}_{k-1}(z)\frac{t^{2k}}{(2k)!},$$

which yields the result. $\square$

From the proposition 10, we see that the Cauchy problem

$$\partial_z^S \mathbf{W}^-(t^2,z) = \sum_{k_1=0}^{S-1} \frac{1}{t^2+2}\int_0^t \int_0^{t_1} \mathbf{H}_{k_1}^-((t_1-t_2)^2,z)\partial_z^{k_1}\mathbf{W}^-(t_2^2,z)dt_2dt_1, \tag{74}$$

with initial conditions

$$(\partial_z^j \mathbf{W}^-)(t^2,0) = \mathbf{W}_j^-(t^2) \quad , \quad 0 \leq j \leq S-1. \tag{75}$$

has a unique solution $(t,z) \mapsto \sum_{k\geq 0} \mathbf{W}^-(t^2,z) = \sum_{k\geq 0} \mathbf{W}_k^-(t^2)z^k/k!$ which is a convergent series in $z \in D_{R_1}$ with coefficients $t \mapsto \mathbf{W}_k^-(t^2)$ belonging to the spaces $\mathcal{A}_{\tau^-,\delta_n^-,\nu}$, for all $n \geq 1$. Likewise, from the proposition 10, the Cauchy problem

$$\partial_z^S \mathbf{W}^+(t^2,z) = \sum_{k_1=0}^{S-1} \frac{1}{t^2+2}\int_0^t \int_0^{t_1} \mathbf{H}_{k_1}^+((t_1-t_2)^2,z)\partial_z^{k_1}\mathbf{W}^+(t_2^2,z)dt_2dt_1, \tag{76}$$

with initial conditions

$$(\partial_z^j \mathbf{W}^+)(t^2,0) = \mathbf{W}_j^+(t^2) \quad , \quad 0 \leq j \leq S-1. \tag{77}$$

has a unique solution $(t,z) \mapsto \sum_{k\geq 0} \mathbf{W}^+(t^2,z) = \sum_{k\geq 0} \mathbf{W}_k^+(t^2)z^k/k!$ which is a convergent series in $z \in D_{R_1}$ with coefficients $t \mapsto \mathbf{W}_k^-(t^2)$ belonging to the spaces $\mathcal{A}_{\tau^-,\delta_n^-,\nu}$, for all $n \geq 1$.

We deduce that the condition **i)** of Proposition 12 is satisfied for the function $\mathbf{W}(u,z) = \mathbf{W}^-(u,z) = \mathbf{W}^+(u,z)$ for $(u,z) \in D_1 \times D_{R_1}$. Moreover, the condition **ii)** of Proposition 12 is satisfied for the analytic continuation $(u,z) \mapsto \mathbf{W}^-(u,z)$ (resp. $(u,z) \mapsto \mathbf{W}^+(u,z)$) of $\mathbf{W}(u,z)$ with respect to u in the direction $\xi^- \in (2\theta_1^-, 2\theta_1)$ (resp. $\xi^+ \in (2\theta_1, 2\theta_1^+)$).

So that the formal series $\hat{\mathbf{X}}(t,z) = \sum_{k\geq 0} \mathbf{w}_k(z)t^k/k!$ solution of the formal Cauchy problem (70), (71), is 1−summable in directions ξ^- and ξ^+ as formal series in the Banach space of holomorphic functions on D_{R_2}, $R_2 < R_1$ and continuous on $\bar{D}_{R_2}$. We denote by $\mathbf{X}^-(t,z)$ the 1−sum in the direction ξ^- and $\mathbf{X}^+(t,z)$ the 1−sum in the direction ξ^+. From the remark 1, the 1−sums can be expressed as integral operators with kernel $z \mapsto e(z)$, with the change of variable $v = u^2$,

$$\mathbf{X}^-(t,z) = 2\int_0^{\infty(\xi^-/2)} \mathbf{W}^-(u^2,z)\frac{e(u^2/t)}{u}du \quad , \quad \mathbf{X}^+(t,z) = 2\int_0^{\infty(\xi^+/2)} \mathbf{W}^+(u^2,z)\frac{e(u^2/t)}{u}du. \tag{78}$$

In the same manner, the condition **i)** of Proposition 12 is satisfied for the function $\mathbf{H}_{k_1}(u,z) = \mathbf{H}_{k_1}^-(u,z) = \mathbf{H}_{k_1}^+(u,z)$ for $(u,z) \in D_1 \times D_{R_1}$. Moreover, the condition **ii)** of Proposition 12 is satisfied for the analytic continuation $(u,z) \mapsto \mathbf{H}_{k_1}^-(u,z)$ (resp. $(u,z) \mapsto \mathbf{H}_{k_1}^+(u,z)$) of $\mathbf{H}_{k_1}(u,z)$ with respect to u in the direction ξ^- (resp. ξ^+).

So that the formal series $\hat{\mathbf{A}}_{k_1}(t,z) = \sum_{k\geq 0} \mathbf{h}_{k_1,k}(z)t^k/k!$ is 1−summable in directions ξ^- and ξ^+ as formal series in the Banach space of holomorphic functions on D_{R_2}, $R_2 < R_1$ and continuous on $\bar{D}_{R_2}$. We denote by $\mathbf{A}_{k_1}^-(t,z)$ the 1−sum in the direction ξ^- and $\mathbf{A}_{k_1}^+(t,z)$ the 1−sum in the direction ξ^+. From the remark 1, the 1−sums can be expressed as an integral operator involving the kernel function $z \mapsto e(z)$ as in (64) and (65).

From the general summability theory, see [1], chapter 6.3, we deduce that the 1−sums $\mathbf{X}^-(t,z)$ satisfy the problem (66), (67) and $\mathbf{X}^+(t,z)$ satisfy the problem (68), (69).

In the second part of the proof, we show that each 1−sums $\mathbf{X}^-(t,z)$ and $\mathbf{X}^+(t,z)$ can be expressed as generalized Laplace transform of staircase distributions.

For all $\xi^- \in (2\theta_1^-, 2\theta_1)$, $0 < \delta < \pi/2$, $\rho > 0$, $t \in S_{\xi^-,\delta,\rho}$, $z \in D_{R_2}$,

$$\mathbf{X}^-(t,z) = 2\int_0^{+\infty} \mathbf{W}^-(r^2 e^{i\xi^-}, z)\frac{e(r^2 e^{i\xi^-}/t)}{r}dr = \sum_{k\geq 0} \mathcal{L}_{\xi^-}(V^-_{k,\xi^-/2}(r))(t)\frac{z^k}{k!}.$$

We choose $0 < \delta_{\theta_1} < \delta$. Then, for all $T \in S_{2\theta_1,\delta_{\theta_1},\rho}$, $z \in D_{R_2}$,

$$\mathbf{X}^-(Te^{i(\xi^- - 2\theta_1)}, z) = \sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^-_{k,\xi^-/2}(r))(T)\frac{z^k}{k!}. \tag{79}$$

Now, for all $T \in S_{2\theta_1,\delta_{\theta_1},\rho}$, $z \in D_{R_3}$, $R_3 > 0$ and $\delta_{\theta_1} > 0$, small enough, we have

$$\lim_{\xi^- \to 2\theta_1, \xi^- < 2\theta_1} \sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^-_{k,\xi^-/2}(r))(T)\frac{z^k}{k!} = \sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^-_{k,\theta_1}(r))(T)\frac{z^k}{k!}. \tag{80}$$

Indeed, from the proposition 7, we have a constant $C > 0$ depending on $S_{2\theta_1,\delta_{\theta_1},\rho}$ such that

$$|\sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^-_{k,\xi^-/2}(r))(T)\frac{z^k}{k!} - \sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^-_{k,\theta_1}(r))(T)\frac{z^k}{k!}| \leq CM_{\xi^-,\theta_1,R_3}$$

where

$$M_{\xi^-,\theta_1,R_3} = \sum_{k\geq 0} ||V^-_{k,\xi^-/2}(r) - V^-_{k,\theta_1}(r)||_{\nu',d}\frac{(R_3)^k}{k!}.$$

On the other hand, the conditions **(h1')** and **(h2')** are satisfied for the Cauchy problems (74), (75) and (76), (77). Moreover, using the Cauchy formula, we see that the function $t \mapsto 1/(t^2+2)$ satisfies the condition **(h3')**. From the proposition 11 **a)**, we get that $\lim_{\xi^- \to 2\theta_1} M_{\xi^-,\theta_1,R_3} = 0$. So that (80) holds.

From (79) and (80), we deduce that

$$\mathbf{X}^-(T,z) = \sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^-_{k,\theta_1}(r))(T)\frac{z^k}{k!},$$

for all $T \in S_{2\theta_1,\delta_{\theta_1},\rho}$, $z \in D_{R_3}$. Likewise, one can show with similar arguments that

$$\mathbf{X}^+(T,z) = \sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^+_{k,\theta_1}(r))(T)\frac{z^k}{k!},$$

for all $T \in S_{2\theta_1,\delta_{\theta_1},\rho}$, $z \in D_{R_3}$. So that

$$\mathbf{X}^+(T,z) - \mathbf{X}^-(T,z) = \sum_{k\geq 0} \mathcal{L}_{2\theta_1}(V^+_{\theta_1,k} - V^-_{\theta_1,k})(T)\frac{z^k}{k!},$$

for all $T \in S_{2\theta_1,\delta_{\theta_1},\rho}$, $z \in D_{R_3}$.

From the theorem 1, we get moreover that the staircase distribution $V^{+-}_{\theta_1}(r,z) = V^+_{\theta_1}(r,z) - V^-_{\theta_1}(r,z)$ defines a hyperfunction on the half-line $\mathbb{R}_+ e^{i\theta_1}$ for all $z \in D_{R_3}$.

Finally, for all $t \in S_{2\theta_1,\delta_{\theta_1},\rho}$ and $z \in D_{R_3}$, the Stokes transition function $\mathbf{X}^+(t,z) - \mathbf{X}^-(t,z)$ can be expressed as generalized Laplace transform in direction $2\theta_1$ of the hyperfunction $V^{+-}_{\theta_1}(r,z)$, which gives the result. □

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