Reply to "Comment on 'On the inconsistency of the Bohm-Gadella theory with quantum mechanics'"

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Abstract. In this reply, we show that when we apply standard distribution theory to the Lippmann-Schwinger equation, the resulting spaces of test functions would comply with the Hardy axiom only if classic results of Paley and Wiener, of Gelfand and Shilov, and of the theory of ultradistributions were wrong. As well, we point out several differences between the "standard method" of constructing rigged Hilbert spaces in quantum mechanics and the method used in Time Asymmetric Quantum Theory.

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1. Introduction

The authors of [1] allege to have shown that the conclusions of [2] regarding the inconsistency of Time Asymmetric Quantum Theory (TAQT) with quantum mechanics are false. In this reply, we will show that the arguments of [1] are missing essential aspects of [2], and that therefore the conclusions of [2] still stand.

The most important claims of [1] are the following:

1. There are many examples of TAQT, and the present author has inadvertently constructed another one.

2. The flaws of the Quantum Arrow of Time (QAT) pointed out in [2] are actually not flaws, because the original derivation of the QAT was misquoted from its source [3].

3. The crucial argument of [2] regarding the exponential blowup of the test functions $\widehat{\varphi}^{\pm}(z)$ does not prevent $\widehat{\varphi}^{\pm}(z)$ from being of Hardy class.

As we shall see, all these claims do not stand close scrutiny. In order to show why, in Sec. 2 we will outline the method to construct rigged Hilbert spaces in quantum mechanics based on the theory of distributions [4]. We shall refer to this method as the "standard method" and show that the resulting rigged Hilbert spaces are not of Hardy class. We shall also explain the meaning of the exponential blowup of $\widehat{\varphi}^{\pm}(z)$ and why it implies that the spaces of test functions are not of Hardy class. In Sec. 3, we briefly outline the method to introduce rigged Hilbert spaces of Hardy class in TAQT and compare such method with the "standard method." It will then be apparent that using the method of TAQT, one can introduce any arbitrary rigged Hilbert space for the

Gamow states. In order to address claim **2**, we show (again) in Sec. 4 that no matter how one introduces it, the Quantum Arrow of Time has little to do with the actual time evolution of a quantum system. To address claim **3**, in Sec. 5 we use classic results of Paley and Wiener and of Gelfand and Shilov to show that the "standard method" of dealing with the Lippmann-Schwinger equation leads to rigged Hilbert spaces that are *not* of Hardy class. Section 7 concludes that the arguments of [2] still stand.

2. The "standard method"

In this section, we illustrate the main features of the "standard method" to construct rigged Hilbert spaces in quantum mechanics [5]. Such "standard method" is based on the theory of distributions [4]. For the sake of clarity, we shall use the spherical shell potential of height V_0,

$$V(\vec{x}) = V(r) = \begin{cases} 0 & 0<r<a \\ V_0 & a<r<b \\ 0 & b<r<\infty\,. \end{cases} \tag{2.1}$$

For $l=0$, the Hamiltonian acts as (we take $\hbar^2/2m=1$)

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r)\,. \tag{2.2}$$

The regular solution is

$$\chi(r;E) = \begin{cases} \sin(\sqrt{E}\,r) & 0<r<a \\ \mathcal{J}_1(E)\mathrm{e}^{\mathrm{i}\sqrt{E-V_0}\,r} + \mathcal{J}_2(E)\mathrm{e}^{-\mathrm{i}\sqrt{E-V_0}\,r} & a<r<b \\ \mathcal{J}_3(E)\mathrm{e}^{\mathrm{i}\sqrt{E}\,r} + \mathcal{J}_4(E)\mathrm{e}^{-\mathrm{i}\sqrt{E}\,r} & b<r<\infty\,. \end{cases} \tag{2.3}$$

The Jost functions and the S matrix are given by

$$\mathcal{J}_+(E) = -2\mathrm{i}\mathcal{J}_4(E)\,, \quad \mathcal{J}_-(E) = 2\mathrm{i}\mathcal{J}_3(E)\,, \tag{2.4}$$

$$S(E) = \frac{\mathcal{J}_-(E)}{\mathcal{J}_+(E)}\,. \tag{2.5}$$

The solutions of the Lippmann-Schwinger equation can be written as

$$\langle r|E^\pm\rangle \equiv \chi^\pm(r;E) = \sqrt{\frac{1}{\pi}\frac{1}{\sqrt{E}}}\;\frac{\chi(r;E)}{\mathcal{J}_\pm(E)}\,. \tag{2.6}$$

When V tends to zero, these eigensolutions tend to the "free" eigensolution:

$$\langle r|E\rangle \equiv \chi_0(r;E) = \sqrt{\frac{1}{\pi}\frac{1}{\sqrt{E}}}\;\sin(\sqrt{E}\,r)\,. \tag{2.7}$$

These eigenfunctions are delta-normalized and therefore their associated unitary operators,

$$(U_\pm f)(E) = \int_0^\infty \mathrm{d}r\,\overline{\chi^\pm(r;E)}\,f(r) \equiv \widehat{f}_\pm(E)\,, \quad E\geq 0\,, \tag{2.8}$$

$$(U_0 f)(E) = \int_0^\infty \mathrm{d}r\, \overline{\chi_0(r;E)}\, f(r) \equiv \widehat{f}_0(E)\,, \quad E \geq 0\,, \tag{2.9}$$

transform from $L^2([0,\infty), \mathrm{d}r)$ onto $L^2([0,\infty), \mathrm{d}E)$.

The Lippmann-Schwinger and the "free" eigenfunctions can be analytically continued from the scattering spectrum into the whole complex plane. We shall denote such analytically continued eigenfunctions by $\chi^{\pm}(r;z)$ and $\chi_0(r;z)$. Whenever they exist, the analytic continuations of (2.8) and (2.9) are denoted by

$$\widehat{f}_{\pm}(z) = \int_0^\infty \mathrm{d}r\, \overline{\chi^{\pm}(r;\overline{z})}\, f(r)\,, \tag{2.10}$$

$$\widehat{f}_0(z) = \int_0^\infty \mathrm{d}r\, \overline{\chi_0(r;\overline{z})}\, f(r)\,, \tag{2.11}$$

where here and in the following z belongs to a two-sheeted Riemann surface.

The resonant energies are given by the poles z_n of the S matrix, and their associated Gamow states are

$$u(r;z_n) = N_n \begin{cases} \frac{1}{\mathcal{J}_3(z_n)} \sin(\sqrt{z_n}\, r) & 0 < r < a \\ \frac{\mathcal{J}_1(z_n)}{\mathcal{J}_3(z_n)} \mathrm{e}^{\mathrm{i}\sqrt{z_n - V_0}\, r} + \frac{\mathcal{J}_2(z_n)}{\mathcal{J}_3(z_n)} \mathrm{e}^{-\mathrm{i}\sqrt{z_n - V_0}\, r} & a < r < b \\ \mathrm{e}^{\mathrm{i}\sqrt{z_n}\, r} & b < r < \infty\,, \end{cases} \tag{2.12}$$

where N_n is a normalization factor.

The theory of distributions [4] says that a test function $\varphi(r)$ on which a distribution $d(r)$ acts is such that the following integral is finite:‡

$$\langle \varphi | d \rangle \equiv \int_0^\infty \mathrm{d}r\, \overline{\varphi(r)} d(r) < \infty\,, \tag{2.13}$$

where $\langle \varphi | d \rangle$ represents the action of the functional $|d\rangle$ on the test function φ. With some variations, this is the "standard method" followed by [7–15] to introduce spaces of test functions in quantum mechanics. Thus, contrary to what the authors of [1] assert, the method followed by the present author runs (somewhat) parallel to [8], not to TAQT.

In order to use (2.13) to construct the rigged Hilbert spaces for the analytically continued Lippmann-Schwinger eigenfunctions and for the Gamow states, we need to obtain the growth of $\chi^{\pm}(r;z)$, $\chi_0(r;z)$ and $u(r;z_n)$. Because the regular solution blows up exponentially [16],

$$|\chi(r;z)| \leq C\, \frac{|z|^{1/2}\, r}{1 + |z|^{1/2}\, r}\, \mathrm{e}^{|\mathrm{Im}\sqrt{z}|r}\,, \tag{2.14}$$

the growth of the eigenfunctions (2.6), (2.7) and (2.12) blows up exponentially:

$$|\chi^{\pm}(r;z)| \leq C\, \frac{1}{\mathcal{J}_{\pm}(z)}\, \frac{|z|^{1/4}\, r}{1 + |z|^{1/2}\, r}\, \mathrm{e}^{|\mathrm{Im}\sqrt{z}|r}\,, \tag{2.15}$$

‡ In quantum mechanics, we need to impose a few more requirements, but we will not need to go into such details here.

$$|\chi_0(r;z)| \leq C\,\frac{|z|^{1/4}\,r}{1+|z|^{1/2}\,r}\,\mathrm{e}^{|\mathrm{Im}\sqrt{z}|r}\,, \tag{2.16}$$

$$|u(r;z_n)| \leq C_n\,\frac{|z_n|^{1/2}\,r}{1+|z_n|^{1/2}\,r}\mathrm{e}^{|\mathrm{Im}\sqrt{z_n}|r}\,. \tag{2.17}$$

When we plug this exponential blowup into the basic requirement (2.13) of the "standard method," we see that the test functions on which those distributions act must fall off at least exponentially.

By using the Gelfand-Shilov theory of M an Ω functions [4], it was shown in [15] that when a and b are positive real numbers satisfying

$$\frac{1}{a}+\frac{1}{b}=1\,, \tag{2.18}$$

and when $\varphi^+(r)$ is an infinitely differentiable function whose tails fall off like $\mathrm{e}^{-r^a/a}$, then $\varphi^+(z)$ grows like $\mathrm{e}^{|\mathrm{Im}(\sqrt{z})|^b/b}$ in the infinite arc of the lower half-plane of the Riemann surface:

$$\text{If } |\varphi^+(r)| < C\mathrm{e}^{-\frac{r^a}{a}} \text{ as } r\to\infty, \text{ then } |\widehat{\varphi}^+(z)| \leq C\mathrm{e}^{\frac{|\mathrm{Im}(\sqrt{z})|^b}{b}} \text{ as } |z|\to\infty\,. \tag{2.19}$$

It was shown in [2] that when $\varphi^+(r)\in C_0^\infty$, $\widehat{\varphi}^+(z)$ blows up exponentially in the infinite arc of the lower half-plane of the Riemann surface:

$$\text{If } |\varphi^+(r)| = 0 \text{ when } r>A, \text{ then } |\widehat{\varphi}^+(z)| \leq C\mathrm{e}^{A|\mathrm{Im}\sqrt{z}|} \text{ as } |z|\to\infty\,. \tag{2.20}$$

From the above estimates, we concluded in [2] that the φ^+'s obtained from the "standard method" cannot be Hardy functions, since $\widehat{\varphi}^+(z)$ does not tend to zero as $|z|$ tends to infinity.

The authors of [1] argue that one cannot draw any conclusion on the limit $|z|\to\infty$ from estimates such as (2.19) or (2.20), and therefore they conclude that nothing prevents $\widehat{\varphi}^+(z)$ from tending to zero and therefore from being Hardy functions. Their conclusion is not true, because their argument does not take the nature of (2.19) and (2.20) into account. After we explain the meaning of those estimates, it will be clear why they prevent $\widehat{\varphi}^\pm(z)$ from tending to zero in any infinite arc of the Riemann surface.

In order to understand what (2.19) and (2.20) mean, we start with the simple sine function $\sin(\sqrt{E}\,r)$. When $E\geq 0$, the sine function oscillates between $+1$ and -1:

$$|\sin(\sqrt{E}\,r)| \leq 1\,, \quad E\geq 0\,. \tag{2.21}$$

As E tends to infinity, such oscillatory behavior remains, and in such limit the sine function does not tend to zero. When we analytically continue the sine function,

$$\sin(\sqrt{z}\,r)\,, \tag{2.22}$$

the oscillations are bounded by

$$|\sin(\sqrt{z}\,r)| \leq C\,\frac{|z|^{1/2}\,r}{1+|z|^{1/2}\,r}\,\mathrm{e}^{|\mathrm{Im}\sqrt{z}|r}\,. \tag{2.23}$$

Thus, as $|z|$ tends to infinity, $\sin(\sqrt{z}\,r)$ oscillates wildly, and the magnitude of its oscillation is tightly bounded by the exponential function. It is certain that as $|z|$

tends to infinity, $\sin(\sqrt{z}\,r)$ does not tend to zero, even though the function vanishes when $\sqrt{z}\,r = \pm n\pi$, $n = 0, 1, \ldots$

It just happens that the solutions of the Lippmann-Schwinger equation follow the same pattern. When E is positive, the eigensolutions are oscillatory and bounded by

$$|\chi^{\pm}(r;E)| \leq C\, \frac{1}{\mathcal{J}_{\pm}(E)}\, \frac{|E|^{1/4}\, r}{1+|E|^{1/2}\, r}\,. \tag{2.24}$$

When the energy is complex, their oscillations get wild and are bounded by Eq. (2.15).§ Thus, the analytic continuations of the Lippmann-Schwinger eigenfunctions oscillate wildly, and the magnitude of their oscillation is tightly bounded by an exponential function (multiplied by factors that don't cancel the exponential blowup when $|z| \to \infty$).

Because in Eqs. (2.10) and (2.11) we are integrating over r, the exponentially-bounded oscillations of $\chi^{\pm}(r;z)$ get transmitted into $\widehat{\varphi}^{\pm}(z)$. The estimates (2.19) and (2.20) bound the oscillation of the test functions of the "standard method," except for factors that don't cancel the exponential blowup. It is the exponentially-bounded oscillations of $\widehat{\varphi}^{\pm}(z)$ what prevent $\widehat{\varphi}^{\pm}(z)$ from tending to zero in any infinite arc of the Riemann surface and therefore from being of Hardy class.

A somewhat simpler way to understand the above estimates is by looking at the "free" incoming and outgoing wave functions $\varphi^{\rm in}$ and $\varphi^{\rm out}$. Because in the energy representation such wave functions are the same as the "in" and "out" wave functions,

$$\widehat{\varphi}^{\rm in}(E) = \langle E|\varphi^{\rm in}\rangle = \langle^{+}E|\varphi^{+}\rangle = \widehat{\varphi}^{+}(E)\,, \tag{2.25}$$

$$\widehat{\varphi}^{\rm out}(E) = \langle E|\varphi^{\rm out}\rangle = \langle^{-}E|\varphi^{-}\rangle = \widehat{\varphi}^{-}(E)\,, \tag{2.26}$$

in TAQT the analytic continuation of $\widehat{\varphi}^{\rm in}(E)$ and $\widehat{\varphi}^{\rm out}(E)$ are also of Hardy class. Since

$$\widehat{\varphi}^{\rm in,out}(z) = \int_0^{\infty} {\rm d}r\, \frac{1}{\sqrt{\pi}}\frac{1}{z^{1/4}}\, \sin(\sqrt{z}\,r)\, \varphi^{\rm in,out}(r)\,, \tag{2.27}$$

it is evident that the exponential blowup (2.23) of $\sin(\sqrt{z}\,r)$ will prevent $\widehat{\varphi}^{\rm in,out}(z)$ from tending to zero as $|z| \to \infty$ in any half-plane of the Riemann surface. Thus, $\widehat{\varphi}^{\rm in,out}(z)$ are not of Hardy class, contrary to TAQT.

Strictly speaking, the bounds (2.19) and (2.20) are not the tightest ones. We should include polynomial corrections, see Eq. (B.15) in [15], and the effect of $\frac{|z|^{1/4} r}{1+|z|^{1/2} r}$ and $\frac{1}{\mathcal{J}_{\pm}(z)}$ to obtain the tightest bounds. We shall not obtain those corrections here, because they do not cancel the exponential blowup at infinity, and because in this reply we shall use instead other classic bounds, see Sec. 5.

Let us summarize this section. In standard quantum mechanics, once the Lippmann-Schwinger equation is solved, the properties of $\widehat{\varphi}^{\pm}(z)$ are already determined by Eqs. (2.10) and (2.11), and there is no room for any extra assumption on their properties. This means, in particular, that the Hardy axiom cannot be simply assumed. Rather, the Hardy axiom must be proved using Eqs. (2.10) and (2.11).‖ It simply

§ The points at which $\mathcal{J}_{\pm}(z) = 0$ do not affect the essence of the argument.

‖ This is what in [2] it was meant by the assertion that the Hardy axiom is not a matter of assumption but a matter of proof.

happens that the "standard method" yields $\widehat{\varphi}^{\pm}(z)$ and $\widehat{\varphi}^{\rm in,out}(z)$ that oscillate wildly. Because these oscillations are bounded by exponential functions, $\widehat{\varphi}^{\pm}(z)$ and $\widehat{\varphi}^{\rm in,out}(z)$ do not tend to zero as $|z|$ tends to infinity in any half-plane of the Riemann surface—hence they are not of Hardy class.

3. TAQT vs. the "standard method"

In TAQT, one doesn't solve the Lippmann-Schwinger equation in order to afterward obtain the properties of $\widehat{\varphi}^{\pm}(z)$ using Eq. (2.10). Instead, one transforms into the energy representation (using $U_{\pm}$ in our example) and then imposes the Hardy axiom. If $\mathcal{H}^2_{\pm}$ denotes the spaces of Hardy functions from above (+) and below (−), $\mathcal{S}$ denotes the Schwartz space, and $\tilde{\Phi}_{\pm}$ denote their intersection restricted to the positive real line,

$$\tilde{\Phi}_{\pm} = \mathcal{H}^2_{\pm} \cap \mathcal{S}|_{\mathbb{R}^+} \, , \tag{3.1}$$

then the Hardy axiom states that the functions $\widehat{\varphi}^{\pm}(z)$ belong to $\tilde{\Phi}_{\mp}$:

$$\widehat{\varphi}^{\pm}(z) \in \tilde{\Phi}_{\mp} \, . \tag{3.2}$$

This means that in the position representation, the Gamow states and the analytic continuation of the Lippmann-Schwinger eigenfunctions act on the following spaces:

$$\Phi_{{\rm BG}\mp} = U_{\pm}^{-1}\tilde{\Phi}_{\mp} \, . \tag{3.3}$$

It is obvious that the choices (3.2)-(3.3) are arbitrary. One may as well choose another dense subset of $L^2([0,\infty), {\rm d}E)$ with different properties and obtain a different space of test functions for the Gamow states. What is more, $\Phi_{{\rm BG}\pm}$ are different from the spaces of test functions obtained through the "standard method," because the functions $\widehat{\varphi}^{\pm}(z)$ of the "standard method" are not of Hardy class.

The authors of [1] claim that the present author has inadvertently constructed an example of TAQT. That such is not the case can be seen not only from the differences between the "standard method" and the method used in TAQT to introduce rigged Hilbert spaces, but also from the outcomes. For example, whereas in the position representation the "standard method" calls for just *one* rigged Hilbert space for the Gamow states and for the analytically continued Lippmann-Schwinger eigenfunctions [15], TAQT uses *two* rigged Hilbert spaces

$$\Phi_{{\rm BG}\pm} \subset L^2([0,\infty), {\rm d}r) \subset \Phi_{{\rm BG}\pm}^{\times} \, . \tag{3.4}$$

One of the rigged Hilbert spaces is used for the "in" solutions and for the anti-resonant states, whereas the other one is used for the "out" solutions and for the resonant states. Another difference is that in TAQT, the solutions of the Lippmann-Schwinger equation for scattering energies have a time asymmetric evolution [17], whereas the "standard method" yields that such time evolution runs from $t=-\infty$ to $t=+\infty$, see [14]. Incidentally, this is an instance where TAQT differs not only mathematically but also physically from standard quantum mechanics, because in standard scattering theory, the time evolution of a scattering process goes from the asymptotically remote past ($t \to -\infty$) to the asymptotically far future ($t \to +\infty$). This is not so in TAQT [17].

It seems hardly necessary to clarify what the present author means by "standard quantum mechanics." Standard quantum mechanics means the Schrödinger equation, and standard scattering theory means the Lippmann-Schwinger equation. In standard quantum mechanics, one assumes that these equations describe the physics and then solves them. Because of the scattering and resonant spectra, their solutions lie within rigged Hilbert spaces. The construction of such rigged Hilbert spaces follows by application of the "standard method." By contrast, TAQT simply assumes that the solutions of the Schrödinger and the Lippmann-Schwinger equations comply with the Hardy axiom, without ever showing that the actual solutions of those equations comply with such axiom.

It was claimed in [2] that there is no example of TAQT. The authors of [1] dispute such claim and assert that there are many examples. The present author disagrees with their assertion, because *assuming* that for a large class of potentials the solutions of the Lippmann-Schwinger equation comply with the Hardy axiom is not the same as having an example where it is shown that the *actual* solutions of the Lippmann-Schwinger equation comply with the Hardy axiom. In fact, to the best of the present author's knowledge, no advocate of TAQT has ever used Eq. (2.10) to discuss the analytic properties of $\widehat{\varphi}^{\pm}(E) = \langle ^{\pm}E|\varphi^{\pm}\rangle$ in terms of the actual solutions $\chi^{\pm}(r;E)$ of the Lippmann-Schwinger equation.

The authors of [1] inadvertently acknowledge that there is no example of TAQT when they say that they still need "*to identify the form and properties*" of the functions of (3.3), see the last paragraph in section 2 of [1]. By saying so, they are acknowledging that they don't know whether the standard Gamow states defined in the position representation are well defined as functionals acting on $\Phi_{\mathrm{BG}\pm}$. If TAQT had an example, it would be known.

4. The Quantum Arrow of Time (QAT)

Advocates of TAQT argue that their choice (3.3) is not arbitrary but rather is rooted on a causality principle. Such causality principle is the "preparation-registration arrow of time," sometimes referred to as the "Quantum Arrow of Time" (QAT). For the "in" states φ^+, the causal statement of the QAT is written as

$$\widetilde{\varphi}^{+}(t) \equiv \int_{-\infty}^{+\infty} \mathrm{d}E \, \mathrm{e}^{-\mathrm{i}Et}\widehat{\varphi}^{+}(E) = 0 \, , \quad \text{for } t > 0 \, . \tag{4.1}$$

By one of the Paley-Wiener theorems, Eq. (4.1) is equivalent to assuming that $\widehat{\varphi}^{+}(E)$ is of Hardy class from below. The corresponding causal statement for the "out" wave functions φ^- implies that φ^- is of Hardy class from above. Hence, in TAQT, the choice (3.3) is not arbitrary but a consequence of causality.

It was pointed out in [2] that the QAT is flawed. The argument was twofold. First, it was pointed out that the original derivation [3] of Eq. (4.1) made use of the following flawed assumption:

$$0 = \langle E|\varphi^{\mathrm{in}}(t)\rangle = \langle ^{+}E|\varphi^{+}(t)\rangle = \mathrm{e}^{-\mathrm{i}Et}\widehat{\varphi}^{+}(E) \, , \quad \text{for all energies,} \tag{4.2}$$

which can happen only when φ^+ and $\varphi^{\rm in}$ are identically 0. It was then pointed out that even though one may simply assume the causal statement (4.1) and forget about how it was derived, such causal statement says little about the actual time evolution of a quantum system, because the quantum mechanical time evolution of φ^+ is not given by Eq. (4.1):

$$\varphi^+(t) = {\rm e}^{-{\rm i}Ht}\varphi^+ \ \neq \ \widetilde{\varphi}^+(t)\,. \tag{4.3}$$

To counter this argument, the authors of [1] claim that the derivation of the QAT was misquoted from the original source [3], and that the flawed assumption (4.2) was never used to derive the QAT (4.1). It seems therefore necessary to quote the original derivation (see [3], page 2597):¶

> *"We are now in the position to give a mathematical formulation of the QAT: we choose $t=0$ to be the time before which all preparations of $\phi^{\rm in}(t)$ are completed and after which the registration of $\psi^{\rm out}(t)$ begins. This means that for $t>0$ the energy distribution of the preparation apparatus must vanish: $\langle E,\eta|\phi^{\rm in}(t)\rangle = 0$ for all values of the quantum numbers E and η (η are the additional quantum numbers which we usually suppress). As the mathematical statement for 'no preparations for $t>0$' we therefore write (the slightly weaker condition)*
>
> $$0 = \int {\rm d}E\,\langle E|\phi^{\rm in}(t)\rangle = \int {\rm d}E\,\langle {}^+E|\phi^+(t)\rangle = \int {\rm d}E\,\langle {}^+E|{\rm e}^{-{\rm i}Ht}|\phi^+\rangle \tag{4.4}$$
>
> *or*
>
> $$0 = \int_{-\infty}^{+\infty} {\rm d}E\,\langle {}^+E|\phi^+\rangle {\rm e}^{-{\rm i}Et} \equiv \mathcal{F}(t) \quad \text{for } t>0\,. \tag{4.5}$$

The readers can decide whether or not the flawed hypothesis (4.2) was used to derive the QAT (4.5).

Nevertheless, it is actually not very relevant whether the authors of [3] used (4.2) to derive (4.1). As pointed out in [2], and as mentioned above, even though one can forget (4.2) and simply assume (4.1) as the causal condition to be satisfied by φ^+, such causal condition has little to do with the time evolution of a quantum system, see again Eq. (4.3). In particular, as even the author of [6] has asserted, the t that appears in Eq. (4.1) is not the same as the parametric time t that labels the evolution of a quantum system.+ Thus, as far as standard quantum mechanics is concerned, the causal content of the QAT is physically vacuous, and therefore, regardless of how one motivates it, there is no physical justification for the choice (3.3).

5. TAQT vs. the "classic results"

In this section, we are going to compare the Hardy axiom of TAQT with some classic results of Paley and Wiener, of Gelfand and Shilov and of the theory of ultradistributions, which we shall collectively refer to as the "classic results." More

¶ In this quote, $\phi^{\rm in}$, ϕ^+, $\mathcal{F}(t)$ and Eq. (4.5) correspond, respectively, to $\varphi^{\rm in}$, φ^+, $\widetilde{\varphi}^+(t)$ and Eq. (4.1).
+ All this shows that the new term TAQT is a misnomer. A better name is Bohm-Gadella theory, because it was these two authors who proposed the theory and summarized it in [18].

precisely, we will see that the spaces of test functions $\widehat{\varphi}^{\pm}$ obtained by the "standard method" would be of Hardy class only if the "classic results" were wrong.

The direct comparison with the "classic results" is more easily done in one dimension, and therefore we shall use the example of the one-dimensional rectangular barrier potential:

$$V(x) = \begin{cases} 0 & -\infty < x < a \\ V_0 & a < x < b \\ 0 & b < x < \infty\,. \end{cases} \tag{5.1}$$

For this potential, the "in" and "out" eigensolutions are well known and can be found for example in [12]. We shall denote them by $\chi^{\pm}_{\mathrm{l,r}}(x;E)$, where the labels l,r denote left and right incidence. When we analytically continue these eigenfunctions, or when we consider the Gamow states for this potential, the "standard method" calls for test functions $\varphi^{\pm}_{\mathrm{l,r}}(x)$ for which the following integrals are finite:

$$\widehat{\varphi}^{\pm}_{\mathrm{l,r}}(z) = \int_{-\infty}^{\infty} \mathrm{d}x\, \overline{\chi^{\pm}_{\mathrm{l,r}}(x;\overline{z})}\, \varphi(x)\,. \tag{5.2}$$

Just as in the example discussed in Sec. 2, the test functions $\varphi(x)$ must at least fall off faster than exponentials.

To further simplify the discussion, we need to recall that, because of Eqs. (2.25) and (2.26), the Hardy axiom assumes that the "free" wave functions $\widehat{\varphi}^{\mathrm{in}}_{\mathrm{l,r}}(E)$ and $\widehat{\varphi}^{\mathrm{out}}_{\mathrm{l,r}}(E)$ are also of Hardy class. These "free" functions are given by (hereafter, we just consider $\varphi^{\mathrm{in}}_{\mathrm{l,r}}$, since the analysis for $\varphi^{\mathrm{out}}_{\mathrm{l,r}}$ is the same)

$$\widehat{\varphi}^{\mathrm{in}}_{\mathrm{l}}(E) = \frac{1}{\sqrt{4\pi k}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, \varphi^{\mathrm{in}}(x)\,, \tag{5.3}$$

$$\widehat{\varphi}^{\mathrm{in}}_{\mathrm{r}}(E) = \frac{1}{\sqrt{4\pi k}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx}\, \varphi^{\mathrm{in}}(x)\,, \tag{5.4}$$

where $k = \sqrt{E}$ is the wave number. The total wave function is given by the sum of left and right components:

$$\widehat{\varphi}^{\mathrm{in}}(E) = \widehat{\varphi}^{\mathrm{in}}_{\mathrm{l}}(E) + \widehat{\varphi}^{\mathrm{in}}_{\mathrm{r}}(E)\,. \tag{5.5}$$

It is simpler to work with k rather than with E and define

$$\widehat{\varphi}^{\mathrm{in}}_{\mathrm{l,r}}(k) \equiv \sqrt{2k}\, \widehat{\varphi}^{\mathrm{in}}_{\mathrm{l,r}}(E)\,; \tag{5.6}$$

that is,

$$\widehat{\varphi}^{\mathrm{in}}_{\mathrm{l}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, \varphi^{\mathrm{in}}(x)\,, \quad k \geq 0\,, \tag{5.7}$$

$$\widehat{\varphi}^{\mathrm{in}}_{\mathrm{r}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx}\, \varphi^{\mathrm{in}}(x)\,, \quad k \geq 0\,. \tag{5.8}$$

The "total" wave function in the wave-number representation, $\widehat{\varphi}^{\mathrm{in}}(k) = \widehat{\varphi}^{\mathrm{in}}_{\mathrm{l}}(k) + \widehat{\varphi}^{\mathrm{in}}_{\mathrm{r}}(k)$, is thus the Fourier transform of $\varphi(x)$,

$$\widehat{\varphi}^{\mathrm{in}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx}\, \varphi^{\mathrm{in}}(x)\,, \quad k \in \mathbb{R}\,. \tag{5.9}$$

Its analytic continuation will be denoted as

$$\widehat{\varphi}^{\rm in}(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} {\rm d}x \, {\rm e}^{-{\rm i}qx} \, \varphi^{\rm in}(x) \, , \quad q \in \mathbb{C} \, . \tag{5.10}$$

At this point, we are ready to introduce two classic theorems. The first one is due to Paley and Wiener (see Theorem IX.11 in [19]):

Theorem 1 (Paley-Wiener). *An entire analytic function $\widehat{\varphi}(q)$ is the Fourier transform of a $C_0^\infty(\mathbb{R})$ function $\varphi(x)$ with support in the segment $\{x \mid |x| < A\}$ if, and only if, for each N there is a C_N so that*

$$|\widehat{\varphi}(q)| \leq \frac{C_N \, {\rm e}^{A|{\rm Im}(q)|}}{(1+|q|)^N} \tag{5.11}$$

for all $q \in \mathbb{C}$.

This theorem says that the Fourier transform of a C_0^∞ function is an analytic function that grows exponentially, and that such exponential growth is mildly corrected (but not canceled) by a polynomial falloff.

The second theorem we shall use is due to Gelfand and Shilov [4]. Before stating it, we need some definitions. Let a and b denote two positive real numbers satisfying (2.18). Let us define $\Phi_{a,b}$ as the set of all differentiable functions $\varphi(x)$ $(-\infty < x < \infty)$ satisfying the inequalities

$$\left| \frac{{\rm d}^n \varphi(x)}{{\rm d}x^n} \right| \leq C_n {\rm e}^{-\alpha \frac{|x|^a}{a}} \tag{5.12}$$

with constants C_n and $\alpha > 0$ which may depend on the function φ. Let us define the space $\widehat{\Phi}_{a,b}$ as the set of entire analytic functions $\widehat{\varphi}(q)$, $q = {\rm Re}(q) + {\rm i}\,{\rm Im}(q)$, which satisfy the inequalities

$$|q^n \widehat{\varphi}(q)| \leq C_n {\rm e}^{+\beta \frac{|{\rm Im}(q)|^b}{b}} \, , \tag{5.13}$$

where the constants C_n and $\beta > 0$ depend on the function φ. It is obvious that the elements of $\Phi_{a,b}$ are functions that, together with their derivatives, decrease at infinity faster than ${\rm e}^{-\frac{|x|^a}{a}}$, whereas the elements of $\widehat{\Phi}_{a,b}$ are analytic functions that grow exponentially at infinity as ${\rm e}^{+\frac{|{\rm Im}(q)|^b}{b}}$, except for a polynomial correction that doesn't cancel the exponential blowup.

Theorem 2 (Gelfand-Shilov). *The space $\widehat{\Phi}_{a,b}$ is the Fourier transform of $\Phi_{a,b}$.*

This theorem means that the smooth functions that fall off at infinity faster than ${\rm e}^{-|x|^a/a}$ are, in Fourier space, analytic functions that grow exponentially like ${\rm e}^{+|{\rm Im}(q)|^b/b}$.

The bounds (5.11) and (5.13) are to be understood in the same way as the bounds (2.19) and (2.20). That is, the bounds (5.11) and (5.13) mean that $\widehat{\varphi}(q)$ is an oscillatory function that grows exponentially in the infinite arc of the q-plane, the oscillation being tightly bounded by Eqs. (5.11) and (5.13) when $\varphi(x)$ belongs to C_0^∞

and $\Phi_{a,b}$, respectively. Note that after the addition of the corresponding polynomial corrections, the bounds (2.19) and (2.20) are entirely analogous to the bounds (5.11) and (5.13)—the operators $U_\pm$ are after all Fourier-like transforms [12].

Let us now apply the above theorems to the functions $\varphi^{\rm in}(x)$ obtained by the "standard method." In order for Eq. (5.10) to make sense, $\varphi^{\rm in}(x)$ must fall off faster than exponentials. If we choose $\varphi^{\rm in}(x)$ to fall off like $\mathrm{e}^{-|x|^a/a}$, then the Gelfand-Shilov theorem tells us that $\widehat{\varphi}^{\rm in}(q)$ grows like $\mathrm{e}^{+|{\rm Im}(q)|^b/b}$. Even when we impose that $\varphi^{\rm in}(x)$ is C_0^∞, which is already a very strict requirement, the Paley-Wiener theorem says that $\widehat{\varphi}^{\rm in}(q)$ grows exponentially. This means, in particular, that the $\widehat{\varphi}^{\rm in}(q)$ do in general *not* tend to zero in the infinite arc of the q-plane, because if they did, the Paley-Wiener and the Gelfand-Shilov theorems would be wrong. Because of Eq. (5.6), $\widehat{\varphi}^{\rm in}(z)$ does in general not tend to zero as $|z|$ tends to infinity in the lower half-plane of the second sheet. Hence the space of $\widehat{\varphi}^{\rm in}$'s is not of Hardy class from below.

The space of $\widehat{\varphi}^+$'s cannot be of Hardy class from below either, because if it were, then

$$\lim_{|z|\to\infty} \widehat{\varphi}^+(z) = 0 \, , \tag{5.14}$$

where the limit is taken in the lower half plane of the second sheet. By Eq. (2.25), this implies that also the space of $\widehat{\varphi}^{\rm in}$'s would be of Hardy class and comply with this limit, which we know is not possible due to the "classic results." Thus, the "standard method" yields spaces of test functions that do *not* comply with the Hardy axiom. This is precisely what it was meant in [2] by the assertion that TAQT is inconsistent with standard quantum mechanics.

To finish this section, we note that if we chose the test functions as in [8], then we would be dealing with ultradistributions. In Fourier space, the test functions for ultradistributions grow faster than any exponential as we follow the imaginary axis, see [8] and references therein. Thus, if the "standard method" yielded spaces of Hardy functions, that property of ultradistributions would be false.

6. Further remarks

The authors of [1] claim that it is inaccurate to state that the proponents of TAQT dispense with asymptotic completeness. This statement should be compared with the first quote in section 6 of [2].

The authors of [1] also claim that TAQT obtains the resonant states by solving the Schrödinger equation subject to purely outgoing boundary conditions. This claim should be compared with the second quote in section 6 of [2].

The authors of [1] also dispute the assertion of [2] that TAQT sometimes uses the whole real line as though it coincided with the scattering spectrum of the Hamiltonian. A glance at, for example, the QAT (4.1) seems to support such assertion.

7. Conclusions

In standard scattering theory, one assumes that the physics is described by the Lippmann-Schwinger equation. When one solves such equation, one finds that its solutions must be accommodated by a rigged Hilbert space, and that its time evolution runs from $t = -\infty$ till $t = +\infty$ [14]. When one analytically continues the solutions of the Lippmann-Schwinger equation, one finds that they must be accommodated by *one* rigged Hilbert space, which also accommodates the resonant (Gamow) states. The construction of such rigged Hilbert space is determined by standard distribution theory.

By contrast, TAQT assumes that the solutions of the Lippmann-Schwinger equations belong to *two* rigged Hilbert spaces of Hardy class. In TAQT, one never explicitly solves the Lippmann-Schwinger equation for specific potentials in the position representation. Instead, one assumes that its solutions satisfy the Hardy axiom. Unlike in standard scattering theory, in TAQT the time evolution of the solutions of the Lippmann-Schwinger equation does not run from $t = -\infty$ till $t = +\infty$.

By comparing the properties of the actual solutions of the Lippmann-Schwinger equation with the Hardy axiom, we have seen that such actual solutions would comply with the Hardy axiom only if classic results of Paley and Wiener, of Gelfand and Shilov, and of the theory of ultradistributions were wrong. We have (again) stressed the fact that the Quantum Arrow of Time, which is the justification for using the rigged Hilbert spaces of Hardy class, has little to do with the time evolution of a quantum system. We have stressed that using the method of TAQT to introduce rigged Hilbert spaces, we could accommodate the Gamow states in a landscape of arbitrary rigged Hilbert spaces, see also [5].

Our claim of inconsistency should not be taken as a claim that TAQT is mathematically inconsistent or that TAQT doesn't have a beautiful mathematical structure. What the present author claims is that TAQT is not applicable in quantum mechanics and is in fact a different theory.

To finish, we would like to mention that the "classic theorems" are not in conflict with using Hardy functions in quantum mechanics. They are in conflict only with the Hardy axiom. Thus, our results do not apply to other works that use Hardy functions in a different way [20].

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